

DETERMINATION OF THE REACTIONS OF A FIXED ENDED  
DOUBLE ARCH BY AN ANALYTICAL METHOD

A Master Thesis

in

Civil Engineering Department

Near East University

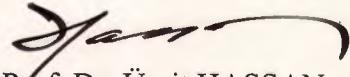
By

Mohamed Atwan

December, 1997



Approval of the Graduate School of Natural and Applied Sciences



Prof. Dr. Ümit HASSAN

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Director

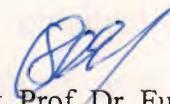
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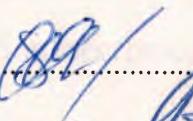
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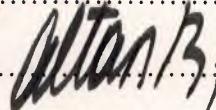
Asst. Prof. Dr. Fuad OKAY

-----  
Supervisor

Examining Committee in Charge:

Asst. Prof. Dr. Fuad OKAY.....

Assoc. Prof. Dr. Hüseyin GÖKÇEKUŞ.....

Prof. Dr. Burhanettin ALTAN (Committee Chairman).....

DETERMINATION OF THE REACTIONS OF A FIXED ENDED  
DOUBLE ARCH BY AN ANALYTICAL METHOD

ATWAN, Mohamed

Faculty of Engineering  
Department of Civil Engineering, M.S. Thesis  
Supervisor: Asst . Prof. Dr. Fuad Okay  
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ABSTRACT

The support reactions at the fixed ends of a space structure consisting of four, equally spaced, quarter circular arches are investigated; the structure is loaded by a single force perpendicular to the base of the structure.

18 equations for the 18 unknowns are obtained by Castigliano's theorem. A model problem is numerically solved with the help of the software Excel by Microsoft.

Key words : Castigliano's theorem, strain energy, matrix equations,  
superposition, reciprocal theorem.

ANKESTRE MESNETLİ ÇİFT KEMER PROBLEMİNİN  
ANALİTİK METODLA ÇÖZÜMÜ  
ATWAN, Mohamed

Mühendislik Fakültesi  
İnşaat Mühendisliği Bölümü, Yüksek Lisans Tezi  
Tez Yöneticisi: Yard. Doç. Dr. Fuad OKAY  
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ÖZET

Bu çalışmada dört çeyrek çembersel kemerin eşit olarak yerleştirilmesi ile elde edilen bir uzay taşıyıcı sistemin ankastre mesnetlerindeki tepkiler incelenmiştir. Yapı kemerlerin birine, yapının zeminine dik olarak etkiyen tekil bir kuvvetle yükülüdür.

Problemin 18 bilinmiyeni için gereken denklemler Castigliano teoremi ile elde edilmiştir. Örnek bir problem Microsoft firmasınca üretilmiş olan Excel isimli bir program yardımı ile nümerik olarak çözülmüştür.

Anahtar Kelimeleri: Castigliano Teoremi, Şekil Değiştirme Enerjisi, Matris Denklemleri, Karşılık Teoremi, süperpozisyon.

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## I. INTRODUCTION

In this study, the forces and the moments of a main frame of a dome where an external concentrated load is applied normal to it, are analyzed by Castigliano's theorem. The main frame is chosen as two arches coinciding each other perpendicularly.

Domes are frequently used as roofs over large circular floor areas for assembly halls, gymnasiums, field houses, mosques, and other buildings. They are very strong structures, since they are shells, and they are constructed of self-supporting reinforced concrete shells.

A dome exerts outward thrusts continuously around its perimeter. These are resisted by a tension ring. The dome and the tension ring are usually supported on columns spaced around the perimeter and braced to provide lateral stability for the structure. Also, a dome is an integral or self-contained unit. Its perimeter may be supported directly by the foundation, which carries the vertical and horizontal thrusts of the ribs. The simplest form of dome would be

generated by revolving a solid arch about a vertical axis through its center. The spaces between ribs may be spanned by purlins to support the roof deck [1].

Any plane-curved bar or rib, properly supported at its ends and so loaded that it acts primarily in direct compression, may be called an arch. It is assumed that the plane of curvature of the rib is also a plane of symmetry for each cross section and that external forces applied to the arch act only in this plane. Under such conditions, deformation will also take place in the plane of symmetry, and the problem of analysis becomes a two-dimensional one. If the cross sections of the rib are unsymmetrical with respect to the plane of curvature or if loads are applied normal to this plane, there will be twisting of the loaded rib, and under such conditions it is not properly considered to be an arch [2].

A ribbed arch is one of a series of arches providing structural support for the roof deck which spans the areas between arches and is continuous from one end of the roofed area to the other.

In contrast to rigid frames, arches are so proportioned that the stresses produced by the loads are primarily compressive and the shears and flexural stresses are relatively small. Flexural stresses must be considered, however, in the design of the arch section, splices, and supports.

A reinforced concrete roof arch, of constant dimensions in the longitudinal direction of a building, may be connected to heavy footings at the ends of the transverse span by means of dowels so that the arch itself maybe

considered to have two fixed supports .In the analysis of such an arch, only a typical slice of a unit width in the longitudinal direction needs to be considered [3].

In this study the problem of two arches coinciding each other perpendicularly, which are shown in figure 1, is solved by the help of Castigliano's theorem. Since the problem is a space (three dimensional) problem, each fixed support brings six unknown reactions.

There are four fixed supports in our structure, thus we have twenty four unknown reactions correspondingly. Any six of them can be obtained from the equations of equilibrium conditions. We choose these six unknowns to be the six reactions in one leg. The remaining 18 unknowns are obtained by the deformation conditions at the other three legs.

The displacement  $\delta_j$  at a coordinate  $j$  of a structural that behaves linear or nonlinear due to the effect of external applied loads and of temperature variation, shrinkage, or other environmental causes can be obtained from

$$\delta_j = \frac{\partial U^*}{\partial F_j} \quad (1.1)$$

where  $U^*$  is the complementary energy expressed in terms of the forces  $F_j$  [4].

In the special case where the structure is linear elastic and the displacements are caused by external forces only, the compatibility conditions can be used to relate the strain energy to the potential energy of energy and equilibrium.

*Two arches coinciding each other perpendicularly.*

As in the case of two beams meeting at a corner point under given boundary conditions as Castiglione's theorem, it will be proved that the

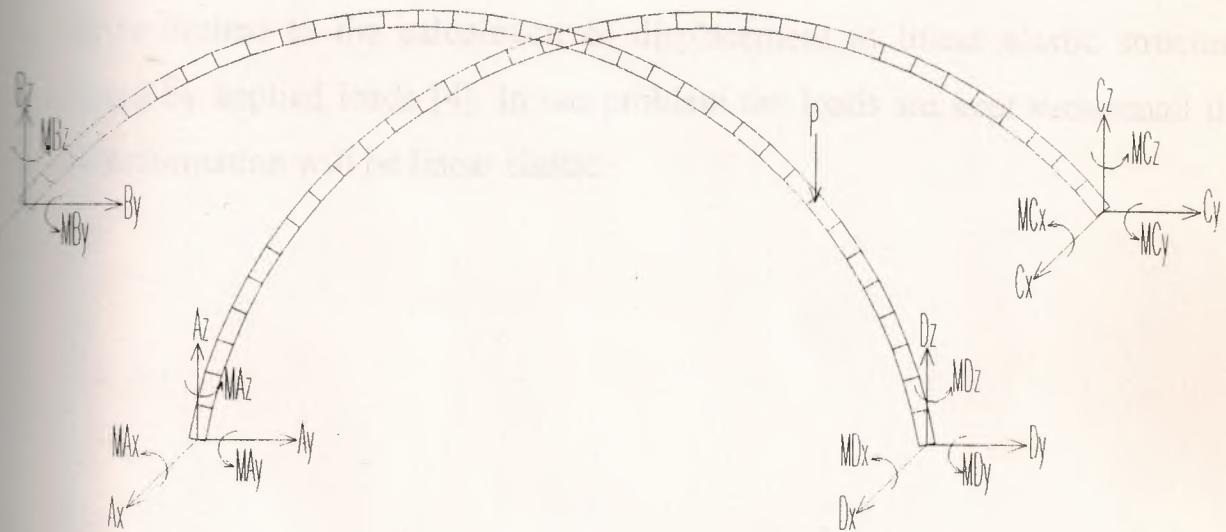


Figure 1. Two arches coinciding each other perpendicularly.

In the special case when the structure is linear elastic and the deformations are caused by external forces only, the complementary energy  $U^*$  is equal to the strain energy  $U$  by using the conservation of energy, and equation (1.1) becomes

$$\delta_j = \frac{\partial U}{\partial F_j} \quad (1.2)$$

This equation known as Castiglano's theorem. It must be remembered that its use is limited to the calculation of displacement in linear elastic structures caused by applied loads [4]. In our problem the loads are kept such small that the deformation will be linear elastic.

### 3. Equilibrium of the structure

From the equations of equilibrium conditions, we obtained well known equations from problem 2 is chosen as follows:

$$\sum F_i = 0$$

$$A + B + C + D = 0$$

$$D - A - B - C = 0$$

## II. DEFINITION OF THE PROBLEM

As it is shown in figure 1, all the forces and the moments are taken into consideration in analyzing the dome. It is known that, a dome is an integral or self-contained unit. Thus, the analysis will be focused on two arches coinciding each other perpendicularly, which they are represented by four legs, and each leg has a fixed support at its end. In the analysis of such an arch, only a typical slice of a unit width in the longitudinal direction needs to be considered.

### A. Equilibrium of the structure

From the equations of equilibrium conditions, six equations will be obtained at any leg. In our problem leg D is chosen as follows:

$$\sum F_x = 0$$

$$\begin{aligned} \frac{A}{x} + \frac{B}{x} + \frac{C}{x} + \frac{D}{x} &= 0 \\ \frac{D}{x} &= -\frac{A}{x} - \frac{B}{x} - \frac{C}{x} \end{aligned} \quad (2.1)$$

$$\sum F_y = 0$$

$$\begin{aligned} A_y + B_y + C_y + D_y &= 0 \\ D_y &= -A_y - B_y - C_y \end{aligned} \tag{2.2}$$

$$\sum F_z = 0$$

$$\begin{aligned} A_z + B_z + C_z + D_z - P &= 0 \\ D_z &= P - A_z - B_z - C_z \end{aligned} \tag{2.3}$$

Having the moment at joint D will result in:

$$\begin{aligned} \sum M_x &= 0 \\ M_{Dx} + M_{Ax} + M_{Bx} + M_{Cx} - A_z r + C_z r &= 0 \\ M_{Dx} &= -M_{Ax} - M_{Bx} - M_{Cx} + A_z r - C_z r \end{aligned} \tag{2.4}$$

$$\sum M_y = 0$$

$$\begin{aligned} M_{Dy} + M_{Ay} + M_{By} + M_{Cy} + A_z r + B_z 2r + C_z r - Pu &= 0 \\ M_{Dy} &= -M_{Ay} - M_{By} - M_{Cy} - A_z r - 2B_z r - C_z r + Pu \end{aligned} \tag{2.5}$$

$$\sum M_z = 0$$

$$\begin{aligned} M_{Dz} + M_{Az} + M_{Bz} + M_{Cz} + A_x r - C_x r - A_y r - B_y 2r - C_y r &= 0 \\ M_{Dz} &= -M_{Az} - M_{Bz} - M_{Cz} - A_x r + C_x r + A_y r + 2B_y r + C_y r \end{aligned} \tag{2.6}$$

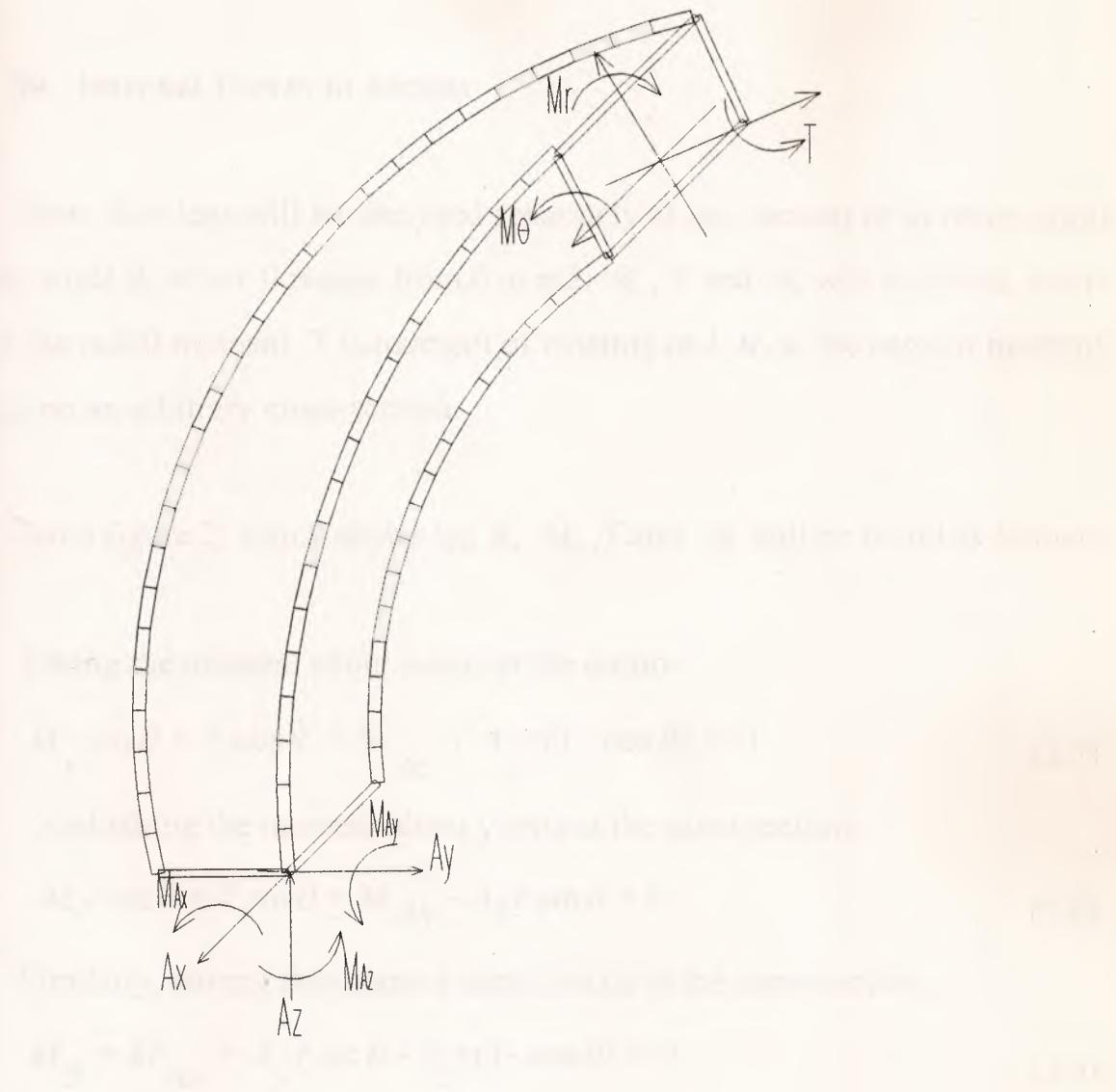


Figure 2. The reactions and the internal forces in leg A.

## B. Internal Forces in Arches

Now, four legs will be analyzed separately at any section or in other words at any angle  $\theta$ , where  $\theta$  ranges from 0 to  $\pi/2$ .  $M_r$ , T and  $M_\theta$  will be found, where  $M_r$  is the radial moment, T is moment of twisting and  $M_\theta$  is the angular moment, acting on an arbitrary cross-section.

From figure 2, which shows leg A,  $M_\theta$ , T and  $M_r$  will be found as follows:

Taking the moment about z-axis at the section

$$M_r \sin \theta + T \cos \theta + M_{Az} + A_x r(1 - \cos \theta) = 0 \quad (2.7)$$

And taking the moment about y-axis at the same section

$$-M_r \cos \theta + T \sin \theta + M_{Ay} - A_x r \sin \theta = 0 \quad (2.8)$$

Similarly, having the moment about x-axis at the same section

$$M_\theta + M_{Ax} + A_y r \sin \theta - A_z r(1 - \cos \theta) = 0 \quad (2.9)$$

Equation (2.9), will result in:

$$M_\theta = -M_{Ax} - A_y r \sin \theta + A_z r(1 - \cos \theta) \quad (2.10)$$

And Equation (2.7), will result in:

$$M_r \sin \theta = -T \cos \theta - M_{Az} - A_x r(1 - \cos \theta)$$

$$M_r = \frac{-T \cos \theta - M_{Az} - A_x r(1 - \cos \theta)}{\sin \theta} \quad (2.11)$$

Substituting  $M_r$  in equation (2.8), will result in:

$$T = -M_{Az} \cos \theta - M_{Ay} \sin \theta + A_x r(1 - \cos \theta) \quad (2.12)$$

Substituting the value of  $T$  in (2.11), will result in:

$$M_r = -M_{Az} \sin \theta + M_{Ay} \cos \theta - A_x r \sin \theta \quad (2.13)$$

Here, the same steps will be followed in leg B, which shown in figure 3 as follows:

$$z : M_r \sin \theta + T \cos \theta + M_{Bz} - B_y r(1 - \cos \theta) = 0$$

$$y : -M_\theta + M_{By} - B_x r \sin \theta + B_z r(1 - \cos \theta) = 0$$

$$x : -M_r \cos \theta + T \sin \theta + M_{Bx} + B_y r \sin \theta = 0$$

which will result in:

$$M_\theta = M_{By} - B_x r \sin \theta + B_z r(1 - \cos \theta) \quad (2.14)$$

$$T = -M_{Bz} \cos \theta - M_{Bx} \sin \theta - B_y r(1 - \cos \theta) \quad (2.15)$$

$$M_r = -M_{Bz} \sin \theta + M_{Bx} \cos \theta + B_y r \sin \theta \quad (2.16)$$

The same procedure will be done in leg C, which shown in figure 4 as follows:

$$z : M_r \sin \theta + T \cos \theta + M_{Cz} - C_x r(1 - \cos \theta) = 0$$

$$y : M_r \cos \theta - T \sin \theta + M_{Cy} - C_x r \sin \theta = 0$$

$$x : M_\theta + M_{Cx} + C_y r \sin \theta + C_z r(1 - \cos \theta) = 0$$

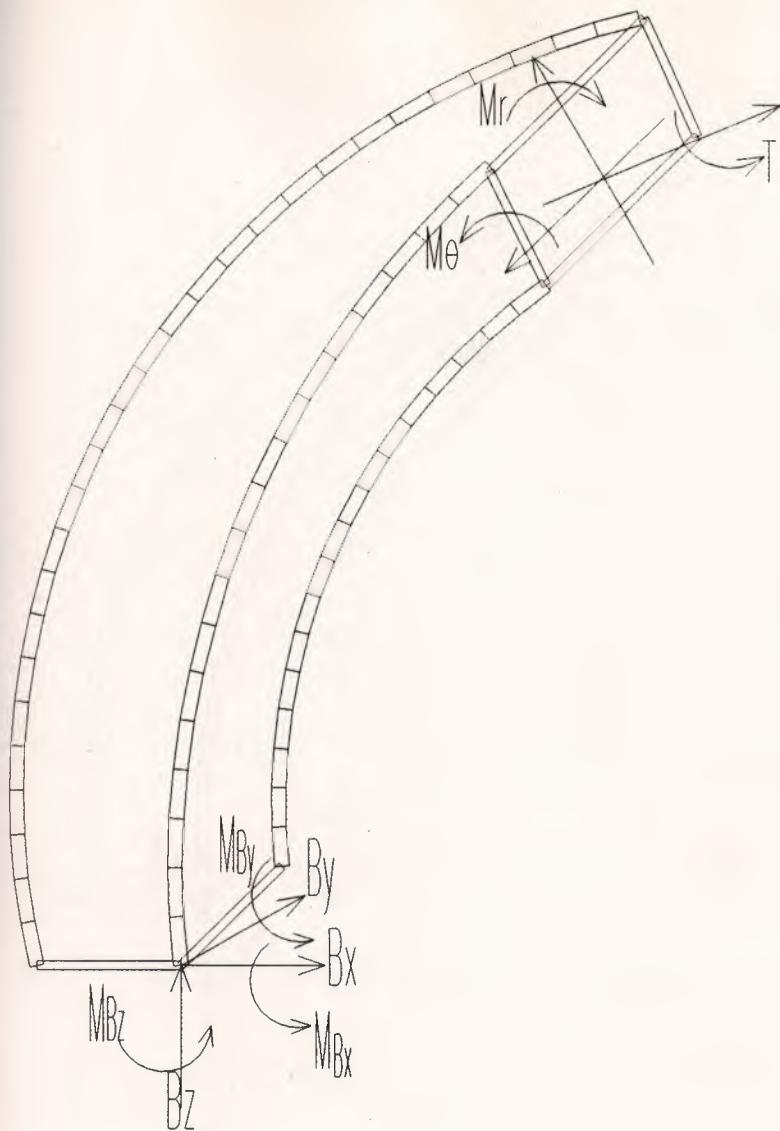
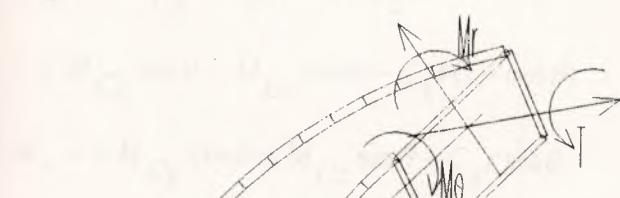


Figure 3. The reactions and the internal forces in leg B.

the equilibrium in

the vertical direction



and by the two forces  $P_x$  and  $P_y$  resulting from the joint  $O$ , which are perpendicular to the segments of the leg.

Let us now consider segment  $C$  alone, as shown in Figure 5, following the same reasoning we will result in

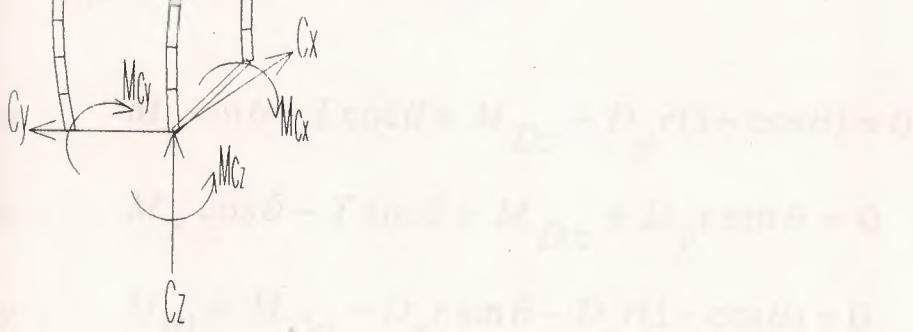


Figure 4. The reactions and the internal forces in leg  $C$ .

which will result in:

$$M_\theta = -M_{Cx} - C_y r \sin \theta - C_z r (1 - \cos \theta) \quad (2.17)$$

$$T = M_{Cy} \sin \theta - M_{Cz} \cos \theta - C_x r (1 - \cos \theta) \quad (2.18)$$

$$M_r = -M_{Cy} \cos \theta - M_{Cz} \sin \theta + C_x r \sin \theta \quad (2.19)$$

For leg D, two cases will be analyzed, because of the load P, which is applied normal to it at a distance u, from its edge.

I- When  $\theta$  ranges between 0 and  $\alpha$ , as shown in figure 5, following the same steps as before, it will result in:

$$z : M_r \sin \theta + T \cos \theta + M_{Dz} + D_y r (1 - \cos \theta) = 0$$

$$x : M_r \cos \theta - T \sin \theta + M_{Dx} + D_y r \sin \theta = 0$$

$$y : M_{\theta 1} + M_{Dy} - D_x r \sin \theta - D_z r (1 - \cos \theta) = 0$$

which results in:

$$M_{\theta 1} = -M_{Dy} + D_x r \sin \theta + D_z r (1 - \cos \theta) \quad (2.20)$$

where  $M_{\theta 1}$  is the angular moment ranging from 0 to  $\alpha$ .

$$T = -M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r (1 - \cos \theta) \quad (2.21)$$

$$M_r = -M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta \quad (2.22)$$

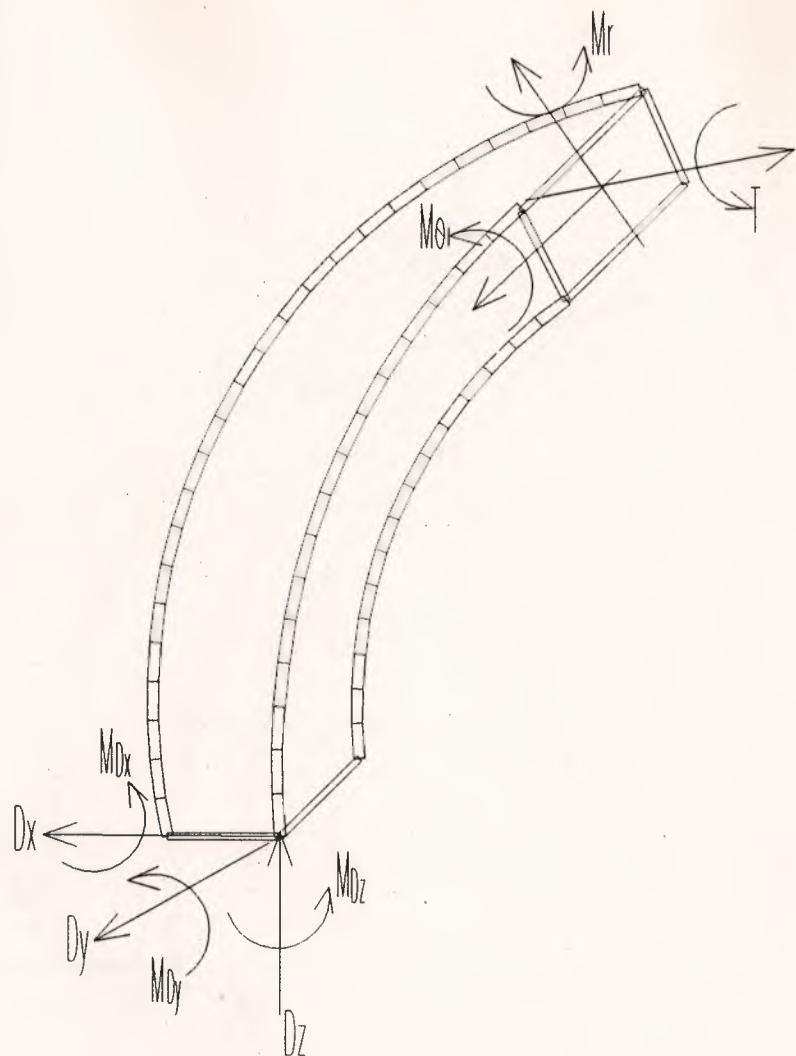


Figure 5. The reactions and the internal forces in leg D before the load  $P$ .

When it moves downwards and will which shows in figure 6  
with the next steps will result in

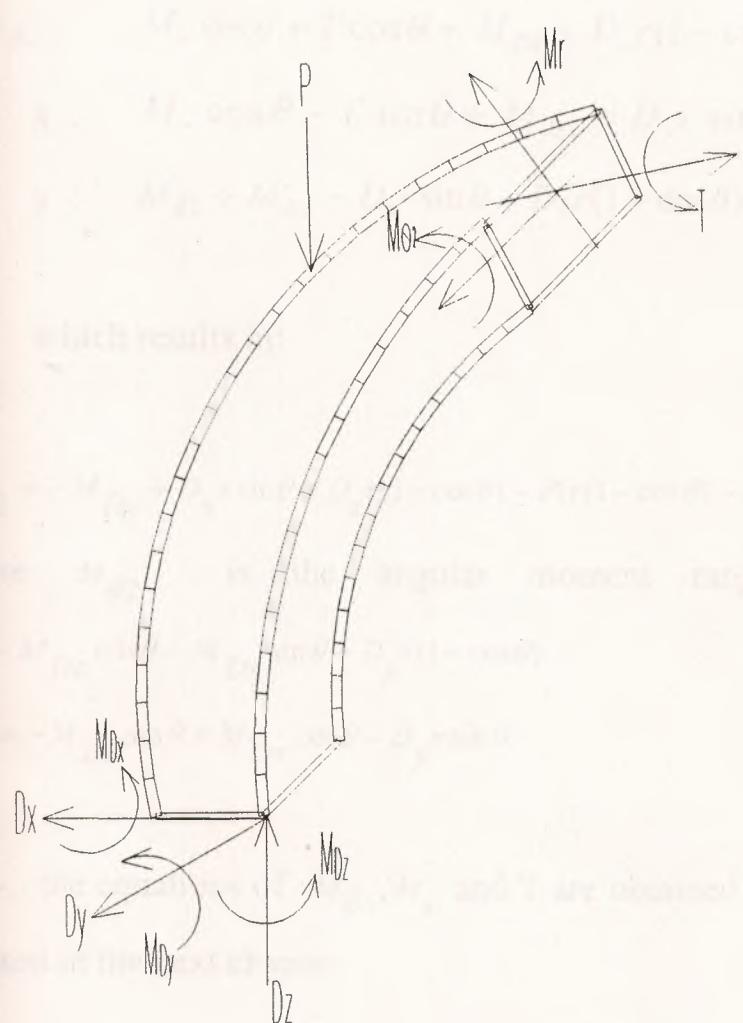


Figure 6. The reactions and the internal forces in leg D after the load  $P$ .

II- When  $\theta$  ranges between  $\alpha$  and  $\pi/2$ , which shown in figure 6, following the same steps will result in:

$$z : M_r \sin \theta + T \cos \theta + M_{Dz} + D_y r (1 - \cos \theta) = 0$$

$$x : M_r \cos \theta - T \sin \theta + M_{Dx} + D_y r \sin \theta = 0$$

$$y : M_{\theta 2} + M_{Dy} - D_x r \sin \theta - D_z r (1 - \cos \theta) + P[r(1 - \cos \theta) - u]$$

which results in:

$$M_{\theta 2} = -M_{Dy} + D_x r \sin \theta + D_z r (1 - \cos \theta) - P[r(1 - \cos \theta) - u] \quad (2.23)$$

where  $M_{\theta 2}$  is the angular moment ranging from  $\alpha$  to  $\pi/2$ .

$$T = -M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r (1 - \cos \theta) \quad (2.24)$$

$$M_r = -M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta \quad (2.25)$$

Now, the equations of  $M_{\theta}$ ,  $M_r$  and  $T$  are obtained for each leg, and they will be used in the next chapter.

Some of the integrations will be used in the following sections.

### III. CONSTRUCTING THE INTEGRATIONS

As it is explained in the first chapter, Castigliano's theorem will be used for solving this problem, and it is known that, the deflection  $\delta_j$  is found as:

$$\delta_j = \frac{\partial U}{\partial F_j} \quad (3.1)$$

And it is known that, the strain energy  $U$  is equal to:

$$U = \frac{1}{2} \sum_{i=1}^3 \frac{M_i^2}{EI_i} dx_i \quad (3.2)$$

The effect of normal forces and shear forces are small enough that we can easily neglect them.

By applying equation (3.1), and equation (3.2) the deflection in each leg will be found as:

$$\delta_i = \int \frac{M_i}{EI_i} \frac{\partial M_i}{\partial F_i} dx_i \quad (3.3)$$

where  $M_i$  is the moment on an arbitrary axis, and  $I_i$  is the corresponding moment of inertia.

Some of the integrals which will be used in the following equations:

$$\int \sin \theta \cos \theta d\theta = \frac{\sin^2 \theta}{2}$$

$$\int \cos \theta d\theta = \sin \theta$$

$$\int \sin \theta d\theta = -\cos \theta$$

$$\int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^2 \theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} = \frac{\pi}{4}$$

where

$$\sin 0 = 0$$

$$\sin \pi/2 = 1$$

$$\cos 0 = 1$$

$$\cos \pi/2 = 0$$

Since the distance traveled 's' is a circular arc, it is obvious that

$$s = r\theta, \text{ and } ds = r d\theta$$

By using equation (3.3), the deflection in leg A will be found in terms of one of the six reactions. And from equations (2.10), (2.12) and (2.13) we will get:

$$\frac{\partial M}{\partial A} \frac{\theta}{x} = 0$$

$$\frac{\partial T}{\partial A} \frac{\theta}{x} = r(1 - \cos \theta)$$

$$\frac{\partial M}{\partial A} \frac{r}{x} = -r \sin \theta$$

Applying each term in equation (3.3) will result in:

$$\int_A \frac{M}{E I} \frac{\theta}{\theta} \frac{\partial M}{\partial A} \frac{\theta}{x} ds = 0 \quad (3.4a)$$

$$\begin{aligned} & \int_A \frac{T}{GJ} \frac{\partial T}{\partial A} \frac{\theta}{x} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Az} \cos \theta - M_{Ay} \sin \theta + A_x r(1 - \cos \theta)]}{GJ} r(1 - \cos \theta) r d\theta \\ &= \frac{r^2}{GJ} [M_{Az} \left(\frac{\pi}{4} - 1\right) - \frac{M_{Ay}}{2} + A_x r \left(\frac{3\pi}{4} - 2\right)] \end{aligned} \quad (3.4b)$$

$$\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_x} ds$$

$$= \int_0^{\pi/2} \frac{[-M_{Az} \sin \theta + M_{Ay} \cos \theta - A_x r \sin \theta]}{EI_r} (-r \sin \theta) r d\theta$$

$$= \frac{r^2}{EI_r} [M_{Az} \frac{\pi}{4} - \frac{M_{Ay}}{2} + A_x \frac{r\pi}{4}] \quad (3.4c)$$

From equations (2.10), (2.12) and (2.13) we will get

$$\frac{\partial M}{\partial A} \frac{\theta}{y} = -r \sin \theta$$

$$\frac{\partial T}{\partial A} \frac{r}{y} = 0$$

$$\frac{\partial M}{\partial A} \frac{r}{y} = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial A_y} ds \\ &= \frac{\pi / 2 [-M_{Ax} - A_y r \sin \theta + A_z r (1 - \cos \theta)]}{EI_\theta} (-r \sin \theta) rd\theta \\ &= \frac{r^2}{EI_\theta} [M_{Ax} + A_y \frac{r\pi}{4} - A_z \frac{r}{2}] \end{aligned} \quad (3.5a)$$

$$\int_A \frac{T}{GJ} \frac{\partial T}{\partial A_y} ds = 0 \quad (3.5b)$$

$$\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_y} ds = 0 \quad (3.5c)$$

From equations (2.10), (2.12) and (2.13) we will get

$$\frac{\partial M}{\partial A_z} \theta = r(1 - \cos \theta)$$

$$\frac{\partial T}{\partial A_z} = 0$$

$$\frac{\partial M}{\partial A_z} r = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial A_z} ds \\ &= \int_0^{\pi/2} \frac{-M_{Ax} - A_y r \sin \theta + A_z r(1 - \cos \theta)}{EI_\theta} r(1 - \cos \theta) r d\theta \\ &= \frac{r^2}{EI_\theta} [M_{Ax}(1 - \frac{\pi}{2}) - A_y \frac{r}{2} + A_z r(\frac{3\pi}{4} - 2)] \end{aligned} \quad (3.6a)$$

$$\int_A \frac{T}{GJ} \frac{\partial T}{\partial A_z} ds = 0 \quad (3.6.b)$$

$$\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_z} ds = 0 \quad (3.6.c)$$

Equations (2.13), (2.13) and (2.13) will give

$$\frac{\partial M_\theta}{\partial M_{Ax}} = -1$$

$$\frac{\partial T}{\partial M_{Ax}} = 0$$

$$\frac{\partial M_r}{\partial M_{Ax}} = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Ax}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Ax} - A_y r \sin \theta + A_z r(1 - \cos \theta)]}{EI_\theta} (-1) r d\theta \\ &= \frac{r^2}{EI_\theta} [M_{Ax} \frac{\pi}{2r} + A_y + A_z (1 - \frac{\pi}{2})] \end{aligned} \quad (3.7a)$$

$$\int_A \frac{T}{GJ} \frac{\partial T}{\partial M_{Ax}} ds = 0 \quad (3.7b)$$

$$\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Ax}} ds = 0 \quad (3.7c)$$

From equations (2.10), (2.12) and (2.13) we will get

$$\begin{aligned}\frac{\partial M}{\partial M} \frac{\theta}{A_y} &= 0 \\ \frac{\partial T}{\partial M} \frac{\theta}{A_y} &= -\sin \theta \\ \frac{\partial M}{\partial M} \frac{r}{A_y} &= \cos \theta\end{aligned}$$

Applying each term in equation (3.3) will result in:

$$\int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Ay}} ds = 0 \quad (3.8a)$$

$$\begin{aligned}&\int_A \frac{T}{GJ} \frac{\partial T}{\partial M_{Ay}} ds \\&= \int_0^{\pi/2} \frac{[-M_{Az} \cos \theta - M_{Ay} \sin \theta + A_x r(1 - \cos \theta)]}{GJ} (-\sin \theta) r d\theta \\&= \frac{r^2}{GJ} \left[ \frac{M_{Az}}{2r} + M_{Ay} \frac{\pi}{4r} - \frac{A_x}{2} \right] \quad (3.8b)\end{aligned}$$

$$\begin{aligned}&\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Ay}} ds \\&= \int_0^{\pi/2} \frac{[-M_{Az} \sin \theta + M_{Ay} \cos \theta - A_x r \sin \theta]}{EI_r} (\cos \theta) r d\theta\end{aligned}$$

$$= \frac{r^2}{EI_r} \left[ \frac{-M_{Az}}{2r} + M_{Ay} \frac{\pi}{4r} - \frac{A_x}{2} \right] \quad (3.8c)$$

From equations (2.10), (2.12) and (2.13) we will get

$$\begin{aligned} \frac{\partial M_{\theta}}{\partial M_{Az}} &= 0 \\ \frac{\partial T}{\partial M_{Az}} &= -\cos \theta \\ \frac{\partial M_r}{\partial M_{Az}} &= -\sin \theta \end{aligned}$$

Applying each term in equation (3.3) will result in:

$$\int_A \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{Az}} ds = 0 \quad (3.9a)$$

$$\begin{aligned} &\int_A \frac{T}{GJ} \frac{\partial T}{\partial M_{Ay}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Az} \cos \theta - M_{Ay} \sin \theta + A_x r(1-\cos \theta)]}{GJ} (-\cos \theta) rd\theta \end{aligned}$$

$$= \frac{r^2}{GJ} \left[ M_{Az} \frac{\pi}{4r} + \frac{M_{Ay}}{2r} + A_x \left( \frac{\pi}{4} - 1 \right) \right] \quad (3.9b)$$

$$\begin{aligned} &\int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Az}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Az} \sin \theta + M_{Ay} \cos \theta - A_x r \sin \theta]}{EI_r} (-\sin \theta) rd\theta \end{aligned}$$

$$= \frac{r^2}{EI_r} \left[ M_{Az} \frac{\pi}{4r} - \frac{M_{Ay}}{2r} + A_x \frac{\pi}{4} \right] \quad (3.9c)$$

Similarly, The same steps will be followed to find the deflection in leg B. From equations (2.14), (2.15) and (2.16) we will get

$$\frac{\partial M_r}{\partial B_x} = 0$$

$$\frac{\partial T}{\partial B_x} = 0$$

$$\frac{\partial M_\theta}{\partial B_x} = -r \sin \theta$$

Applying each term in equation (3.3) will result in:

$$\int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_x} ds = 0 \quad (3.10a)$$

$$\int_B \frac{T}{GJ} \frac{\partial T}{\partial B_x} ds = 0 \quad (3.10b)$$

$$\int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial B_x} ds$$

$$= \int_0^{\pi/2} \frac{[M_{By} - B_x r \sin \theta + B_z r(1 - \cos \theta)]}{EI_\theta} (-r \sin \theta) r d\theta \\ = \frac{r^2}{EI_\theta} [-M_{By} + B_x \frac{r\pi}{4} - B_z \frac{r}{2}] \quad (3.10c)$$

From equations (2.14), (2.15) and (2.16) we will get

$$\begin{aligned}\frac{\partial M}{\partial B} \frac{r}{y} &= r \sin \theta \\ \frac{\partial T}{\partial B} \frac{y}{y} &= -r(1 - \cos \theta) \\ \frac{\partial M}{\partial B} \frac{\theta}{y} &= 0\end{aligned}$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned}&\int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_y} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Bz} \sin \theta + M_{Bx} \cos \theta + B_y r \sin \theta]}{EI_r} (r \sin \theta) rd\theta \\ &= \frac{r^2}{EI_r} \left[ -M_{Bz} \frac{\pi}{4} + \frac{M_{Bx}}{2} + B_y \frac{r\pi}{4} \right]\end{aligned}\tag{3.11a}$$

$$\begin{aligned}&\int_B \frac{T}{GJ} \frac{\partial T}{\partial B_y} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Bz} \cos \theta - M_{Bx} \sin \theta - B_y r(1 - \cos \theta)]}{GJ} (-r \{1 - \cos \theta\}) rd\theta \\ &= \frac{r^2}{GJ} \left[ M_{Bz} \left(1 - \frac{\pi}{4}\right) + \frac{M_{Bx}}{2} + B_y r \left(\frac{3\pi}{4} - 2\right) \right]\end{aligned}\tag{3.11b}$$

$$\int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial B_y} ds = 0\tag{3.11c}$$

From equations (2.14), (2.15) and (2.16) we will get

$$\frac{\partial M}{\partial B_z} r = 0$$

$$\frac{\partial T}{\partial B_z} = 0$$

$$\frac{\partial M}{\partial B_z} \theta = r(1 - \cos \theta)$$

Applying each term in equation (3.3) will result in:

$$\int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_z} ds = 0 \quad (3.12a)$$

$$\int_B \frac{T}{GJ} \frac{\partial T}{\partial B_z} ds = 0 \quad (3.12b)$$

$$\begin{aligned} & \int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial B_z} ds \\ &= \int_0^{\pi/2} \frac{[M_{By} - B_x r \sin \theta + B_z r(1 - \cos \theta)]}{EI_\theta} r(1 - \cos \theta) r d\theta \\ &= \frac{r^2}{EI_\theta} [M_{By} (\frac{\pi}{2} - 1) - B_x \frac{r}{2} + B_z r (\frac{3\pi}{4} - 2)] \end{aligned} \quad (3.12c)$$

From equations (2.14), (2.15) and (2.16) we will get

$$\frac{\partial M}{\partial M_{Bx}} r = \cos \theta$$

$$\frac{\partial T}{\partial M_{Bx}} = -\sin \theta$$

$$\frac{\partial M}{\partial M_{Bx}} \theta = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned}
 & \int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Bx}} ds \\
 &= \int_0^{\pi/2} \frac{[-M_{Bz} \sin \theta + M_{Bx} \cos \theta + B_y r \sin \theta]}{EI_r} (\cos \theta) r d\theta \\
 &= \frac{r^2}{EI_r} \left[ \frac{-M_{Bz}}{2r} + M_{Bx} \frac{\pi}{4r} + \frac{B_y}{2} \right] \quad (3.13a)
 \end{aligned}$$

$$\begin{aligned}
 & \int_B \frac{T}{GJ} \frac{\partial T}{\partial M_{Bx}} ds \\
 &= \int_0^{\pi/2} \frac{[-M_{Bz} \cos \theta - M_{Bx} \sin \theta - B_y r(1 - \cos \theta)]}{GJ} (-\sin \theta) r d\theta \\
 &= \frac{r^2}{GJ} \left[ \frac{M_{Bz}}{2r} + M_{Bx} \frac{\pi}{4r} + \frac{B_y}{2} \right] \quad (3.13b)
 \end{aligned}$$

$$\int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Bx}} ds = 0 \quad (3.13c)$$

From equations (2.14), (2.15) and (2.16) we will get

$$\frac{\partial M_r}{\partial M_{By}} = 0$$

$$\frac{\partial T}{\partial M_{By}} = 0$$

$$\frac{\partial M_\theta}{\partial M_{By}} = 1$$

Applying each term in equation (3.3) will result in:

$$\int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{By}} ds = 0 \quad (3.14a)$$

$$\int_B \frac{T}{GJ} \frac{\partial T}{\partial M_{By}} ds = 0 \quad (3.14b)$$

$$\begin{aligned} & \int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{By}} ds \\ &= \int_0^{\pi/2} \frac{[M_{By} - B_x r \sin \theta + B_z r(1 - \cos \theta)]}{EI_\theta} (1) r d\theta \\ &= \frac{r^2}{EI_\theta} [M_{By} \frac{\pi}{2r} - B_x + B_z (\frac{\pi}{2} - 1)] \end{aligned} \quad (3.14c)$$

From equations (2.14), (2.15) and (2.16) we will get

$$\begin{aligned} \frac{\partial M_r}{\partial M_{Bz}} &= -\sin \theta \\ \frac{\partial T}{\partial M_{Bz}} &= -\cos \theta \\ \frac{\partial M_\theta}{\partial M_{Bz}} &= 0 \end{aligned}$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_B \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Bz}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Bz} \sin \theta + M_{Bx} \cos \theta + B_y r \sin \theta]}{EI_r} (-\sin \theta) r d\theta \\ &= \frac{r^2}{EI_r} [M_{Bz} \frac{\pi}{4r} - \frac{M_{Bx}}{2r} - B_y \frac{\pi}{4}] \quad (3.15a) \\ & \int_B \frac{T}{GJ} \frac{\partial T}{\partial M_{Bz}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Bz} \cos \theta - M_{Bx} \sin \theta - B_y r(1 - \cos \theta)]}{GJ} (-\cos \theta) r d\theta \end{aligned}$$

$$= \frac{r^2}{GJ} [M_{Bz} \frac{\pi}{4r} + \frac{M_{Bx}}{2r} + B_y (1 - \frac{\pi}{4})] \quad (3.15b)$$

$$\int_B \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Bz}} ds = 0 \quad (3.15c)$$

By following the same steps, the same thing will be done for finding the deflection in leg C. From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M_r}{\partial C_x} = r \sin \theta$$

$$\frac{\partial T}{\partial C_x} = -r(1 - \cos \theta)$$

$$\frac{\partial M_\theta}{\partial C_x} = 0$$

Applying each term in equation (3.3), will result in:

$$\begin{aligned} & \int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_x} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cy} \cos \theta - M_{Cz} \sin \theta + C_x r \sin \theta]}{EI_r} (r \sin \theta) r d\theta \end{aligned}$$

$$= \frac{r^2}{EI_r} \left[ -\frac{M_{Cy}}{2} - M_{Cz} \frac{\pi}{4} + C_x \frac{r\pi}{4} \right] \quad (3.16a)$$

$$\begin{aligned} & \int_C \frac{T}{GJ} \frac{\partial T}{\partial C_x} ds \\ &= \int_0^{\pi/2} \frac{[M_{Cy} \sin \theta - M_{Cz} \cos \theta - C_x r(1 - \cos \theta)]}{GJ} [-r(1 - \cos \theta)] r d\theta \end{aligned}$$

$$= \frac{r^2}{GJ} \left[ -\frac{M_{Cy}}{2} + M_{Cz} \left(1 - \frac{\pi}{4}\right) + C_x r \left(\frac{3\pi}{4} - 2\right) \right] \quad (3.16b)$$

$$\int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial C_x} ds = 0 \quad (3.16c)$$

From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M_r}{\partial C_y} = 0$$

$$\frac{\partial T}{\partial C_y} = 0$$

$$\frac{\partial M_\theta}{\partial C_y} = -r \sin \theta$$

Applying each term in equation (3.3) will result in:

$$\int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_y} ds = 0 \quad (3.17a)$$

$$\int_C \frac{T}{GJ} \frac{\partial T}{\partial C_y} ds = 0 \quad (3.17b)$$

$$\begin{aligned} & \int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial C_y} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cx} - C_y r \sin \theta - C_z r(1 - \cos \theta)]}{EI_\theta} (-r \sin \theta) r d\theta \end{aligned}$$

$$= \frac{r^2}{EI_\theta} [M_{Cx} + C_y \frac{r\pi}{4} + C_z \frac{r}{2}] \quad (3.17c)$$

From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M_r}{\partial C_z} = 0$$

$$\frac{\partial T}{\partial C_z} = 0$$

$$\frac{\partial M_\theta}{\partial C_z} = -r(1 - \cos\theta)$$

Applying each term in equation (3.3) will result in:

$$\int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_z} ds = 0 \quad (3.18a)$$

$$\int_C \frac{T}{GJ} \frac{\partial T}{\partial C_z} ds = 0 \quad (3.18b)$$

$$\begin{aligned} & \int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial C_z} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cx} - C_y r \sin\theta - C_z r(1 - \cos\theta)]}{EI_\theta} \{-r(1 - \cos\theta)\} rd\theta \\ &= \frac{r^2}{EI_\theta} [M_{Cx} (\frac{\pi}{2} - 1) + C_y \frac{r}{2} + C_z r (\frac{3\pi}{4} - 2)] \end{aligned} \quad (3.18c)$$

From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M}{\partial M} \frac{r}{C_x} = 0$$

$$\frac{\partial T}{\partial M} \frac{r}{C_x} = 0$$

$$\frac{\partial M}{\partial M} \frac{\theta}{C_x} = -1$$

Applying each term in equation (3.3) will result in

$$\int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cx}} ds = 0 \quad (3.19a)$$

$$\int_C \frac{T}{GJ} \frac{\partial T}{\partial M_{Cx}} ds = 0 \quad (3.19b)$$

$$\begin{aligned} & \int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Cx}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cx} - C_y r \sin \theta - C_z r(1 - \cos \theta)]}{EI_\theta} (-1) r d\theta \\ &= \frac{r^2}{EI_\theta} [M_{Cx} \frac{\pi}{2r} + C_y + C_z (\frac{\pi}{2} - 1)] \end{aligned} \quad (3.19c)$$

From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M_r}{\partial M_{Cy}} = -\cos \theta$$

$$\frac{\partial T}{\partial M_{Cy}} = \sin \theta$$

$$\frac{\partial M_\theta}{\partial M_{Cy}} = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cy}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cy} \cos \theta - M_{Cz} \sin \theta + C_x r \sin \theta]}{EI_r} (-\cos \theta) rd\theta \\ &= \frac{r^2}{EI_r} [M_{Cy} \frac{\pi}{4r} + \frac{M_{Cz}}{2r} - \frac{C_x}{2}] \end{aligned} \quad (3.20a)$$

$$\begin{aligned} & \int_C \frac{T}{GJ} \frac{\partial T}{\partial M_{Cy}} ds \\ &= \int_0^{\pi/2} \frac{[M_{Cy} \sin \theta - M_{Cz} \cos \theta - C_x r(1 - \cos \theta)]}{GJ} (\sin \theta) rd\theta \\ &= \frac{r^2}{GJ} [M_{Cy} \frac{\pi}{4r} - \frac{M_{Cz}}{2r} - \frac{C_x}{2}] \end{aligned} \quad (3.20b)$$

$$\int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Cy}} ds = 0 \quad (3.20c)$$

From equations (2.17), (2.18) and (2.19) we will get

$$\frac{\partial M_r}{\partial M_{Cz}} = -\sin \theta$$

$$\frac{\partial T}{\partial M_{Cz}} = -\cos \theta$$

$$\frac{\partial M_\theta}{\partial M_{Cz}} = 0$$

Applying each term in equation (3.3) will result in:

$$\begin{aligned} & \int_C \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cz}} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Cy} \cos \theta - M_{Cz} \sin \theta + C_x r \sin \theta]}{EI_r} (-\sin \theta) r d\theta \\ &= \frac{r^2}{EI_r} \left[ \frac{M_{Cy}}{2r} + M_{CZ} \frac{\pi}{4r} - C_x \frac{\pi}{4} \right] \end{aligned} \quad (3.21a)$$

$$\begin{aligned} & \int_C \frac{T}{GJ} \frac{\partial T}{\partial M_{Cz}} ds \\ &= \int_0^{\pi/2} \frac{[M_{Cy} \sin \theta - M_{CZ} \cos \theta - C_x r(1-\cos \theta)]}{GJ} (-\cos \theta) r d\theta \\ &= \frac{r^2}{GJ} \left[ \frac{-M_{Cy}}{2r} + M_{Cz} \frac{\pi}{4r} + C_x \left(1 - \frac{\pi}{4}\right) \right] \end{aligned} \quad (3.21b)$$

$$\int_C \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Cz}} ds = 0 \quad (3.21c)$$

From leg D four equations are obtained as explained in the second chapter, and they are as follows:

$$\begin{aligned} M_r &= -M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta \\ T &= -M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta) \\ M_{\theta 1} &= -M_{Dy} + D_x r \sin \theta + D_z r(1 - \cos \theta) \\ M_{\theta 2} &= M_{\theta 1} - P[r - r \cos \theta - u] \end{aligned}$$

From figure (2.4) and figure (2.5) we will integrate  $M_r$  and T from 0 to  $\pi/2$ , but for  $M_{\theta 1}$  it will be integrated from 0 to  $\alpha$ , and for  $M_{\theta 2}$  it will be integrated from  $\alpha$  to  $\pi/2$ .

Substituting the values of  $D_x, D_y, D_z, M_{Dx}, M_{Dy}$  and  $M_{Dz}$  in the previous equations, we will have:

$$\begin{aligned} M_r &= -(M_{Az} - M_{Bz} - M_{Cz} - A_x r + C_x r + A_y r + 2rB_y + C_y r) \sin \theta \\ &\quad + (-M_{Ax} - M_{Bx} - M_{Cx} + A_Z r - C_Z r) \cos \theta \\ &\quad - (-A_y - B_y - C_y) r \sin \theta \end{aligned} \tag{3.22}$$

$$\begin{aligned} T &= -(M_{Az} - M_{Bz} - M_{Cz} - A_x r + C_x r + A_y r + 2rB_y + C_y r) \cos \theta \\ &\quad + (-M_{Ax} - M_{Bx} - M_{Cx} + A_Z r - C_Z r) \sin \theta \\ &\quad + (-A_y - B_y - C_y) r(1 - \cos \theta) \end{aligned} \tag{3.23}$$

$$\begin{aligned} M_{\theta 1} &= -(-M_{Ay} - M_{By} - M_{Cy} - A_z r - 2rB_z - C_z r + Pu) + (-A_x - B_x - C_x) r \sin \theta \\ &\quad + (P - A_z - B_z - C_z) r(1 - \cos \theta) \end{aligned} \tag{3.24}$$

$$M_{\theta 2} = M_{\theta 1} - P[r - r \cos \theta - u] \quad (3.25)$$

From equation (3.25) we will have

We know from leg D, that

$$\int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial \bullet} ds = \int_0^{\alpha} \frac{M_{\theta 1}}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial \bullet} ds + \int_{\alpha}^{\pi/2} \frac{M_{\theta 2}}{EI_\theta} \frac{\partial M_{\theta 2}}{\partial \bullet} ds \quad (3.26)$$

where  $\partial \bullet$  means that, leg D will be differentiated with respect to the unknown reaction  $\bullet$ .

From equation (3.25), it is given that:

$$M_{\theta 2} = M_{\theta 1} - P[r - r \cos \theta - u]$$

and

$$\frac{\partial M_{\theta 2}}{\partial \bullet} = \frac{\partial M_{\theta 1}}{\partial \bullet}$$

Thus, we have

$$\begin{aligned} \int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial \bullet} ds &= \int_0^{\alpha} \frac{M_{\theta 1}}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial \bullet} ds - \int_0^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial \bullet} ds \\ &= a - b \end{aligned} \quad (3.27)$$

### I - The equations in leg D containing the unknown reactions of leg A.

From equation (3.22), we will have

$$\begin{aligned}
 \frac{\partial M_r}{\partial A_x} &= r \sin \theta \\
 \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_x} ds & \\
 = \int_0^{\pi/2} \frac{[-M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta]}{EI_r} (r \sin \theta) rd\theta & \\
 = \frac{r^2}{EI_r} \left[ -M_{Dz} \frac{\pi}{4} + \frac{M_{Dx}}{2} - D_y \frac{r\pi}{4} \right] & \tag{3.28a}
 \end{aligned}$$

From equation (3.23) it is found that:

$$\begin{aligned}
 \frac{\partial T}{\partial A_x} &= r \cos \theta \\
 \int_D \frac{T}{GJ} \frac{\partial T}{\partial A_x} ds & \\
 = \int_0^{\pi/2} \frac{[-M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta)]}{GJ} (r \cos \theta) rd\theta & \\
 = \frac{r^2}{GJ} \left[ -M_{Dz} \frac{\pi}{4} + \frac{M_{Dx}}{2} + D_y r(1 - \frac{\pi}{4}) \right] & \tag{3.28b}
 \end{aligned}$$

From equation (3.24), we will get

$$\begin{aligned}
 \frac{\partial M_{\theta 1}}{\partial A_x} &= -r \sin \theta \\
 \int_0^{\pi/2} \frac{M_{\theta 1}}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial A_x} ds & \\
 = \int_0^{\pi/2} \frac{[-M_{Dy} + D_x r \sin \theta + D_z r(1 - \cos \theta)]}{EI_{\theta}} (-r \sin \theta) rd\theta & \\
 = \frac{r^2}{EI_{\theta}} [M_{Dy} - D_x r \frac{\pi}{4} - D_z \frac{r}{2}] & \quad (a)
 \end{aligned}$$

$$\begin{aligned}
 \int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial A_x} ds & \\
 = \int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_{\theta}} (-r \sin \theta) rd\theta & \\
 = \frac{-Pr^2 \cos \alpha}{EI_{\theta}} [r - \frac{r}{2} \cos \alpha - u] & \quad (b)
 \end{aligned}$$

$$\begin{aligned}
 \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial A_x} ds &= a - b \\
 = \frac{r^2}{EI_{\theta}} [M_{Dy} - D_x r \frac{\pi}{4} - D_z \frac{r}{2} + P \cos \alpha (r - \frac{r}{2} \cos \alpha - u)] & \quad (3.28c)
 \end{aligned}$$

It is found from equation (3.22) that:

$$\frac{\partial M_r}{\partial A_y} = 0$$

$$\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_y} ds = 0 \quad (3.29a)$$

From equation (3.23), we will get

$$\begin{aligned} & \frac{\partial T}{\partial A_y} = -r \\ & \int_D \frac{T}{GJ} \frac{\partial T}{\partial A_y} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta)]}{GJ} (-r) r d\theta \\ &= \frac{r^2}{GJ} [M_{Dz} - M_{Dx} - D_y r (\frac{\pi}{2} - 1)] \end{aligned} \quad (3.28b)$$

From equation (3.24), it is found that:

$$\begin{aligned} & \frac{\partial M_{\theta 1}}{\partial A_y} = 0 \\ & \int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial A_y} ds = 0 \end{aligned} \quad (3.29c)$$

From equation (3.22), we will get

$$\begin{aligned} & \frac{\partial M_r}{\partial A_z} = r \cos \theta \\ &= \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_z} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta]}{EI_r} (r \cos \theta) r d\theta \end{aligned}$$

$$\frac{r^2}{EI_r} \left[ -\frac{M_{Dz}}{2} + M_{Dx} \frac{\pi}{4} - D_y \frac{r}{2} \right] \quad (3.30a)$$

Similarly, from equation (3.23) it is found that:

$$\begin{aligned} \frac{\partial T}{\partial A_z} &= r \sin \theta \\ &= \int_D \frac{T}{GJ} \frac{\partial T}{\partial A_z} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta)]}{GJ} (r \sin \theta) rd\theta \\ &= \frac{r^2}{GJ} \left[ -\frac{M_{Dz}}{2} + M_{Dx} \frac{\pi}{4} + D_y \frac{r}{2} \right] \end{aligned} \quad (3.30b)$$

From equation (3.24), we will get

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial A_z} &= r \cos \theta \\ &= \int_0^{\pi/2} \frac{M_{\theta 1}}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial A_z} ds \\ &= \int_0^{\pi/2} \frac{[-M_{Dy} + D_x r \sin \theta + D_z r(1 - \cos \theta)]}{EI_\theta} (r \cos \theta) rd\theta \\ &= \frac{r^2}{EI_\theta} [-M_{Dy} + D_x \frac{r}{2} + D_z r(1 - \frac{\pi}{4})] \end{aligned} \quad (a)$$

$$\begin{aligned} &\int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial A_z} ds \\ &= \int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_\theta} (r \cos \theta) rd\theta \\ &= \frac{r^2 P}{EI_\theta} [r(1 - \sin \alpha) - r \left\{ \frac{\pi}{4} - \left( \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} \right) \right\} - u(1 - \sin \alpha)] \end{aligned} \quad (b)$$

$$\begin{aligned}
& \int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial A_z} ds = a - b \\
&= \frac{r^2}{EI_\theta} \left[ -M_{Dy} + D_x \frac{r}{2} + D_z r \left(1 - \frac{\pi}{4}\right) - P \left\{ r(1 - \sin \alpha) \right. \right. \\
&\quad \left. \left. - r \left\{ \frac{\pi}{4} - \left(\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right) \right\} - u(1 - \sin \alpha) \right\} \right] \tag{3.30c}
\end{aligned}$$

From equation (3.22), we will have

$$\begin{aligned}
& \frac{\partial M_r}{\partial M_{Ax}} = -\cos \theta \\
&= \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Ax}} ds \\
&= \int_0^{\pi/2} \frac{[-M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta]}{EI_r} (-\cos \theta) rd\theta \\
&= \frac{r^2}{EI_r} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} + \frac{D_y}{2} \right] \tag{3.31a}
\end{aligned}$$

From equation (3.23), it is found that:

$$\begin{aligned}
& \frac{\partial T}{\partial M_{Ax}} = -\sin \theta \\
&= \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Ax}} ds \\
&= \int_0^{\pi/2} \frac{[-M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta)]}{GJ} (-\sin \theta) rd\theta \\
&= \frac{r^2}{GJ} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} - \frac{D_y}{2} \right] \tag{3.31b}
\end{aligned}$$

Similarly, from equation (3.24) it is found that:

$$\frac{\partial M_{\theta 1}}{\partial M_{Ax}} = 0$$

$$\int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial M_{Ax}} ds = 0 \quad (3.31c)$$

From equation (3.22), we will get

$$\frac{\partial M_r}{\partial M_{Ay}} = 0$$

$$\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Ay}} ds = 0 \quad (3.32a)$$

And, from equation (3.23) it is given that:

$$\frac{\partial T}{\partial M_{Ay}} = 0$$

$$\int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Ay}} ds = 0 \quad (3.32b)$$

Similarly, from equation (3.24) it is found that:

$$\frac{\partial M_{\theta 1}}{\partial M_{Ay}} = 1$$

$$\int_0^{\pi/2} \frac{M_{\theta 1}}{EI_\theta} \frac{\partial M_{\theta 1}}{\partial M_{Ay}} ds$$

$$\begin{aligned}
 &= \int_0^{\pi/2} \frac{[-M_{Dy} + D_x r \sin \theta + D_z r(1 - \cos \theta)]}{EI_\theta} (1) rd\theta \\
 &= \frac{r^2}{EI_\theta} [-M_{Dy} \frac{\pi}{2r} + D_x + D_z (\frac{\pi}{2} - 1)] \quad (a)
 \end{aligned}$$

$$\begin{aligned}
 &\int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Ay}} ds \\
 &= \frac{r^2 P}{EI_\theta} [(\frac{\pi}{2} - \alpha) - (1 - \sin \alpha) - \frac{u}{r} (\frac{\pi}{2} - \alpha)] \quad (b)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{r^2}{EI_\theta} [-M_{Dy} \frac{\pi}{2r} + D_x + D_z (\frac{\pi}{2} - 1) \\
 &\quad - P \{(\frac{\pi}{2} - \alpha) - (1 - \sin \alpha) - \frac{u}{r} (\frac{\pi}{2} - \alpha)\}] \quad (3.32c)
 \end{aligned}$$

It is found from equation (3.22) that:

$$\begin{aligned}
 &\frac{\partial M_r}{\partial M_{Az}} = \sin \theta \\
 &\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Az}} ds \\
 &= \int_0^{\pi/2} \frac{[-M_{Dz} \sin \theta + M_{Dx} \cos \theta - D_y r \sin \theta]}{EI_r} (\sin \theta) rd\theta
 \end{aligned}$$

$$= \frac{r^2}{EI_r} \left[ -M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} - D_y \frac{\pi}{4} \right] \quad (3.33a)$$

Similarly from equation (3.23), we will get

$$\begin{aligned} \frac{\partial T}{\partial M_{Az}} &= \cos \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Az}} ds & \\ = \frac{\pi/2}{\int_0^{\pi/2}} \frac{[-M_{Dz} \cos \theta + M_{Dx} \sin \theta + D_y r(1 - \cos \theta)]}{GJ} (\cos \theta) rd\theta \\ &= \frac{r^2}{GJ} \left[ -M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} + D_y \left(1 - \frac{\pi}{4}\right) \right] \end{aligned} \quad (3.33b)$$

And from equation (3.24), it is found that:

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial M_{Az}} &= 0 \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{Az}} ds &= 0 \end{aligned} \quad (3.33c)$$

## II. The equations in leg D containing the unknown reactions of leg B.

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial B_x} &= 0 \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_x} ds &= 0 \end{aligned} \quad (3.34a)$$

Similarly, from equation (3.23), it is found that:

$$\frac{\partial T}{\partial B_x} = 0$$

$$\int_D \frac{T}{GJ} \frac{\partial T}{\partial B_x} ds = 0 \quad (3.34b)$$

And from equation (3.24), we will get

$$\frac{\partial M_{\theta 1}}{\partial B_x} = -r \sin \theta$$

$$\int_0^{\pi/2} \frac{M_{\theta 1}}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial B_x} ds$$

$$= \frac{r^2}{EI_{\theta}} [M_{Dy} - D_x \frac{r\pi}{4} - D_z \frac{r}{2}] \quad (a)$$

$$\int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial B_x} ds$$

$$= \frac{-r^2 P \cos \alpha}{EI_{\theta}} [r - \frac{r}{2} \cos \alpha - u] \quad (b)$$

$$\int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial B_x} ds = a - b$$

$$= \frac{r^2}{EI_{\theta}} [M_{Dy} - D_x \frac{r\pi}{4} - D_z \frac{r}{2} + P \cos \alpha (r - \frac{r}{2} \cos \alpha - u)] \quad (3.34c)$$

From equation (3.22), we will have

$$\frac{\partial M_r}{\partial B_y} = -r \sin \theta$$

$$\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_y} ds$$

$$= \frac{r^2}{EI_r} [M_{Dz} \frac{\pi}{4} - \frac{M_{Dx}}{2} + D_y \frac{r\pi}{4}] \quad (3.35a)$$

And from equation (3.23), we will get

$$\begin{aligned} \frac{\partial T}{\partial B_y} &= -r(\cos\theta + 1) \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial B_y} ds &= \\ = \frac{r^2}{GJ} [M_{Dz} (\frac{\pi}{4} + 1) - M_{Dx} \frac{3}{2} - D_y \frac{r\pi}{4}] \end{aligned} \quad (3.35b)$$

Similarly, from equation (3.24), it is found that:

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial B_y} &= 0 \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial B_y} ds &= 0 \end{aligned} \quad (3.35c)$$

From equation (3.22), we will have

$$\begin{aligned} \frac{\partial M_r}{\partial B_z} &= 0 \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial B_z} ds &= 0 \end{aligned} \quad (3.36a)$$

And from equation (3.23), it is found that:

$$\frac{\partial T}{\partial B_z} = 0$$

$$\int_D \frac{T}{GJ} \frac{\partial T}{\partial B_z} ds = 0 \quad (3.36b)$$

Similarly, from equation (3.24), it is found that:

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial B_z} &= r(1 + \cos \theta) \\ \int_0^{\pi/2} \frac{M_{\theta 1}}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial B_z} ds &= \frac{r^2}{EI_{\theta}} [-M_{Dy}(\frac{\pi}{2} + 1) + D_x \frac{3r}{2} + D_z \frac{r\pi}{4}] \end{aligned} \quad (a)$$

$$\begin{aligned} \int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial B_z} ds &= \frac{r^2 P}{EI_{\theta}} [r\{(\frac{\pi}{2} - \alpha) + (1 - \sin \alpha)\} - r\{(1 - \sin \alpha) + (\frac{\pi}{4} - \{\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\})\} \\ &\quad - u\{(\frac{\pi}{2} - \alpha) + (1 - \sin \alpha)\}] \end{aligned} \quad (b)$$

$$\begin{aligned} \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial B_z} ds &= a - b \\ \frac{r^2}{EI_{\theta}} [-M_{Dy}(\frac{\pi}{2} + 1) + D_x \frac{3r}{2} + D_z \frac{r\pi}{4} - P\{r\{(\frac{\pi}{2} - \alpha) + (1 - \sin \alpha)\} \\ &\quad - r\{(1 - \sin \alpha) + (\frac{\pi}{4} - \{\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\})\} - u\{(\frac{\pi}{2} - \alpha) + (1 - \sin \alpha)\}\}] \end{aligned} \quad (3.36c)$$

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial M_{Bx}} &= -\cos \theta \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Bx}} ds & \end{aligned}$$

$$= \frac{r^2}{EI_r} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} + \frac{D_y}{2} \right] \quad (3.37a)$$

From equation (3.23), we will get

$$\begin{aligned} \frac{\partial T}{\partial M_{Bx}} &= -\sin \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Bx}} ds & \\ = \frac{r^2}{GJ} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} - \frac{D_y}{2} \right] \end{aligned} \quad (3.37b)$$

Similarly, from equation (3.24) will be found

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial M_{Bx}} &= 0 \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{Bx}} ds &= 0 \end{aligned} \quad (3.37c)$$

And, equation (3.22) will result in:

$$\begin{aligned} \frac{\partial M_r}{\partial M_{By}} &= 0 \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{By}} ds &= 0 \end{aligned} \quad (3.38a)$$

Similarly, equation (3.23) will give

$$\begin{aligned} \frac{\partial T}{\partial M_{By}} &= 0 \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{By}} ds &= 0 \end{aligned} \quad (3.38b)$$

From equation (3.24), we will find that:

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial M_{By}} &= 1 \\ \int_0^{\pi/2} \frac{M_{\theta 1}}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial M_{By}} ds \\ &= \frac{r^2}{EI_{\theta}} [-M_{Dy} \frac{\pi}{2r} + D_x + D_z (\frac{\pi}{2} - 1)] \end{aligned} \quad (a)$$

$$\begin{aligned} \int_{\alpha}^{\pi/2} \frac{P[r - r \cos \theta - u]}{EI_{\theta}} \frac{\partial M_{\theta 1}}{\partial M_{By}} ds \\ &= \frac{r^2 P}{EI_{\theta}} [(\frac{\pi}{2} - \alpha) - (1 - \sin \alpha) - \frac{u}{r} (\frac{\pi}{2} - \alpha)] \end{aligned} \quad (b)$$

$$\begin{aligned} \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{By}} ds &= a - b \\ = \frac{r^2}{EI_{\theta}} &[-M_{Dy} \frac{\pi}{2r} + D_x + D_z (\frac{\pi}{2} - 1) - P \{(\frac{\pi}{2} - \alpha) \\ &- (1 - \sin \alpha) - \frac{u}{r} (\frac{\pi}{2} - \alpha)\}] \end{aligned} \quad (3.38c)$$

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial M_{Bz}} &= \sin \theta \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Bz}} ds \\ &= \frac{r^2}{EI_r} [-M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} - D_y \frac{\pi}{4}] \end{aligned} \quad (3.39a)$$

Similarly, equation (3.23) will result in:

$$\begin{aligned} \frac{\partial T}{\partial M_{Bz}} &= \cos \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Bz}} ds & \\ = \frac{r^2}{GJ} \left[ -M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} + D_y \left( 1 - \frac{\pi}{4} \right) \right] & \end{aligned} \quad (3.39b)$$

And, equation (3.24) will give

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial M_{Bz}} &= 0 \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{Bz}} ds &= 0 \end{aligned} \quad (3.39c)$$

### III. The equations in leg D containing the unknown reactions of leg C.

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial C_x} &= -r \sin \theta \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_x} ds & \\ = \frac{r^2}{EI_r} \left[ M_{Dz} \frac{\pi}{4} - \frac{M_{Dx}}{2} + D_y \frac{\pi r}{4} \right] & \end{aligned} \quad (3.40a)$$

And, equation (3.23) will result in:

$$\begin{aligned} \frac{\partial T}{\partial C_x} &= -r \cos \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial C_x} ds & \\ = \frac{r^2}{GJ} [M_{Dz} \frac{\pi}{4} - \frac{M_{Dx}}{2} - D_y r (1 - \frac{\pi}{4})] & \end{aligned} \quad (3.40b)$$

Similarly, equation (3.24) will give

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial C_x} &= -r \sin \theta \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial C_x} ds & \\ = \frac{r^2}{EI_{\theta}} [M_{Dy} - D_x \frac{\pi r}{4} - D_z \frac{r}{2} + P \cos \alpha (r - \frac{r}{2} \cos \alpha - u)] & \end{aligned} \quad (3.40c)$$

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial C_y} &= 0 \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_y} ds &= 0 \end{aligned} \quad (3.41a)$$

And, equation (3.23) will give

$$\begin{aligned} \frac{\partial T}{\partial C_y} &= -r \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial C_y} ds & \end{aligned}$$

$$= \frac{r^2}{GJ} [M_{Dz} - M_{Dx} - D_y r (\frac{\pi}{2} - 1)] \quad (3.41b)$$

Similarly, equation (3.24) will result in:

$$\frac{\partial M_{\theta 1}}{\partial C_y} = 0 \quad (3.41c)$$

$$\int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial C_y} ds = 0$$

From equation (3.22), we will get

$$\begin{aligned} \frac{\partial M_r}{\partial C_z} &= -r \cos \theta \\ \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial C_z} ds &= \\ &= \frac{r^2}{EI_r} \left[ \frac{M_{Dz}}{2} - M_{Dx} \frac{\pi}{4} + D_y \frac{r}{2} \right] \end{aligned} \quad (3.42a)$$

And, equation (3.23) will result in:

$$\begin{aligned} \frac{\partial T}{\partial C_z} &= -r \sin \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial C_z} ds &= \\ &= \frac{r^2}{GJ} \left[ \frac{M_{Dz}}{2} - M_{Dx} \frac{\pi}{4} - D_y \frac{r}{2} \right] \end{aligned} \quad (3.42b)$$

Similarly, equation (3.24) will give

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial C_z} &= r \cos \theta \\ \int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial C_z} ds &= \end{aligned}$$

**Y.D.Ü**  
**FEN BİLİMLERİ ENSTİTÜSÜ**

**TEZ SINAVI TUTANAK FORMU**

Gönderen:...**İnşaat Mühendisliği**...EABD Başkanlığı  
Gönderilen: Enstitü Müdürlüğü

Enstitü Anabilim Dalımız **YÜKSEK LİSANS/DOKTORA** Programı öğrencisi  
**Mohamed ATWAN**.....tez çalışmalarını sonuçlandırmış ve kurulan jüri  
önünde tezini savunmuştur. Sınav tutanağı aşağıdadır.

29/12/1997

Tarih

Enstitü Anabilim Dalı Başkanı

**SINAV TUTANAĞI:**

Jürimiz 29/12/1997 Tarihinde toplanmış, yukarıda adı geçen öğrencinin '**Determination of the Reactions of a Fixed Ended Double Arch by an Analytical Method**' konulu tezi incelenmiş ve yapılan sözlü sınav sonunda OYBİRLİĞİ/~~ÖÝÇOKLUĞU~~ ile aşağıdaki kararı almıştır.

Başarılı

Başarısız

Düzeltme (Yüksek Lisans 3 ay/Doktora 6 ay)

Tez Sınavı Jürisi	Unvanı, Adı Soyadı	İmza
Başkan	<b>Prof. Dr. Burhanettin ALTAN</b>	
Tez Danışmanı	<b>Asst. Prof. Dr. Fuad OKAY</b>	
Üye	<b>Assoc. Prof. Dr. Hüseyin GÖKÇEKUŞ</b>	
Üye		
Üye		

Uzatma alan veya başarısız olan öğrenciler için jüri raporu eklenmelidir. Jüri raporunu tüm jüri üyeleri ve/veya yalnızca jüri başkanı imzalamalıdır.

Gönderen: FBE

Gönderilen: ÖİDB

Yukarıda adı geçen öğrenci Sınav Tutanağında belirtildiği üzere mezun olmaya  
**HAK KAZANMIŞTIR/KAZANMAMMIŞTIR.**

Gereğini rica ederim.

Tarih

  
Enstitü Müdürü

$$\begin{aligned}
&= \frac{r^2}{EI_\theta} \left[ -M_{Dy} + D_x \frac{r}{2} + D_z r \left(1 - \frac{\pi}{4}\right) - \right. \\
&\quad \left. P \left\{ r(1 - \sin \alpha) - r \left(\frac{\pi}{4} - \left\{\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right\}\right) - u(1 - \sin \alpha) \right\} \right] \tag{3.42c}
\end{aligned}$$

From equation (3.22), we will get

$$\begin{aligned}
&\frac{\partial M_r}{\partial M_{Cx}} = -\cos \theta \\
&\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cx}} ds \\
&= \frac{r^2}{EI_r} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} + \frac{D_y}{2} \right] \tag{3.43a}
\end{aligned}$$

And, equation (3.23) will result in:

$$\begin{aligned}
&\frac{\partial T}{\partial M_{Cx}} = -\sin \theta \\
&\int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Cx}} ds \\
&= \frac{r^2}{GJ} \left[ \frac{M_{Dz}}{2r} - M_{Dx} \frac{\pi}{4r} - \frac{D_y}{2} \right] \tag{3.43b}
\end{aligned}$$

Similarly, equation (3.24) will give

$$\begin{aligned}
&\frac{\partial M_{\theta 1}}{\partial M_{Cx}} = 0 \\
&\int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial M_{Cx}} ds = 0 \tag{3.43c}
\end{aligned}$$

From equation (3.22), we will get

$$\frac{\partial M_r}{\partial M_{Cy}} = 0$$

$$\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cy}} ds = 0 \quad (3.44a)$$

And, equation (3.23) will result in:

$$\frac{\partial T}{\partial M_{Cy}} = 0$$

$$\int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Cy}} ds = 0 \quad (3.44b)$$

Similarly, equation (3.24) will give

$$\frac{\partial M_{\theta 1}}{\partial M_{Cy}} = 1$$

$$\int_D \frac{M_{\theta}}{EI_{\theta}} \frac{\partial M_{\theta}}{\partial M_{Cy}} ds$$

$$= \frac{r^2}{EI_{\theta}} [-M_{Dy} \frac{\pi}{2r} + D_x + D_z (\frac{\pi}{2} - 1) - P \{(\frac{\pi}{2} - \alpha) - (1 - \sin \alpha) - \frac{u}{r} (\frac{\pi}{2} - \alpha)\}] \quad (3.44c)$$

From equation (3.22), we will get

$$\frac{\partial M_r}{\partial M_{Cz}} = \sin \theta$$

$$\int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_{Cz}} ds$$

$$= \frac{r^2}{EI} \left[ -M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} - D_y \frac{\pi}{4} \right] \quad (3.45a)$$

And, equation (3.23) will result in:

$$\begin{aligned} \frac{\partial T}{\partial M_{Cz}} &= \cos \theta \\ \int_D \frac{T}{GJ} \frac{\partial T}{\partial M_{Cz}} ds & \\ = \frac{r^2}{GJ} \left[ -M_{Dz} \frac{\pi}{4r} + \frac{M_{Dx}}{2r} + D_y \left(1 - \frac{\pi}{4}\right) \right] \end{aligned} \quad (3.45b)$$

Finally, equation (3.24) will give

$$\begin{aligned} \frac{\partial M_{\theta 1}}{\partial M_{Cz}} &= 0 \\ \int_D \frac{M_{\theta}}{EI} \frac{\partial M_{\theta}}{\partial M_{Cz}} ds &= 0 \end{aligned} \quad (3.45c)$$

Thus, all the equations in leg D, containing the unknown reactions of the other three legs, are obtained.

#### IV. OBTAINING THE EQUATIONS AND SOLUTION

It is known from Castigliano's theorem and from the fact that deformations at fixed ends should be zero that:

$$\begin{aligned} & \int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial \bullet} ds + \int_A \frac{T}{GJ} \frac{\partial T}{\partial \bullet} ds + \int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial \bullet} ds + \\ & \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial \bullet} ds + \int_D \frac{T}{GJ} \frac{\partial T}{\partial \bullet} ds + \int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial \bullet} ds = 0 \end{aligned} \quad (4.1)$$

where  $\partial \bullet$  refers to the 6 unknowns of leg A. And it is worthy to mention that, equation (4.1) is also applicable for leg B and C.

By using equation (4.1) for leg A, B and C, 18 equations will be obtained. And we arrive that:

$$\begin{aligned} & \int_A \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_x} ds + \int_A \frac{T}{GJ} \frac{\partial T}{\partial A_x} ds + \int_A \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial A_x} ds \\ & + \int_D \frac{M_r}{EI_r} \frac{\partial M_r}{\partial A_x} ds + \int_D \frac{T}{GJ} \frac{\partial T}{\partial A_x} ds + \int_D \frac{M_\theta}{EI_\theta} \frac{\partial M_\theta}{\partial A_x} ds = 0 \end{aligned} \quad (4.2)$$

Substituting the value of each term, will give us:

$$\begin{aligned}
 & \frac{r^2}{EI_r} [M_{Az} \frac{\pi}{4} - \frac{M_{Ay}}{2} + A_x \frac{r\pi}{4}] + \frac{r^2}{GJ} [M_{Az} (\frac{\pi}{4} - 1) - \frac{M_{Ay}}{2} \\
 & + A_x r (\frac{3\pi}{4} - 2)] + 0 + \frac{r^2}{EI_r} [-M_{Dz} \frac{\pi}{4} + \frac{M_{Dx}}{2} - D_y r \frac{\pi}{4}] \\
 & + \frac{r^2}{GJ} [-M_{Dz} \frac{\pi}{4} + \frac{M_{Dx}}{2} + D_y r (1 - \frac{\pi}{4})] \\
 & + \frac{r^2}{EI_\theta} [M_{Dy} - D_x r \frac{\pi}{4} - D_z \frac{r}{2} + P \cos \alpha (r - \frac{r}{2} \cos \alpha - u)] = 0
 \end{aligned} \tag{4.3}$$

Substituting the values of  $D_x$ ,  $D_y$ ,  $D_Z$ ,  $M_{Dx}$ ,  $M_{Dy}$  and  $M_{Dz}$  in equation (4.3), it can be written as follows:

$$\begin{aligned}
 & M_{Az} [\frac{\pi/2 - 1}{GJ} + \frac{\pi/2}{EI_r}] + M_{Ay} [-\frac{1}{2GJ} - \frac{1}{2EI_r} - \frac{1}{EI_\theta}] \\
 & + A_x [r(\frac{\pi/2}{EI_r} + \frac{\pi - 2}{GJ} + \frac{\pi/4}{EI_\theta})] + M_{Bz} [\frac{\pi/4}{EI_r} + \frac{\pi/4}{GJ}] + \\
 & M_{Cz} [\frac{\pi/4}{EI_r} + \frac{\pi/4}{GJ}] + C_x [r(-\frac{\pi/4}{EI_r} - \frac{\pi/4}{GJ} + \frac{\pi/4}{EI_\theta})] \\
 & + A_y [\frac{-r}{GJ}] + B_y [r(-\frac{\pi/4}{EI_r} - \frac{\pi/4 + 1}{GJ})] + C_y [\frac{-r}{GJ}] + \\
 & M_{Ax} [-\frac{1}{2}(\frac{1}{EI_r} + \frac{1}{GJ})] + M_{Bx} [-\frac{1}{2}(\frac{1}{EI_r} + \frac{1}{GJ})] \\
 & + M_{Cx} [-\frac{1}{2}(\frac{1}{EI_r} + \frac{1}{GJ})] + A_z [\frac{r/2}{EI_r} + \frac{r/2}{GJ} - \frac{r/2}{EI_\theta}] + \\
 & C_z [-\frac{r/2}{EI_r} - \frac{r/2}{GJ} - \frac{r/2}{EI_\theta}] + M_{By} [\frac{-1}{EI_\theta}] + M_{Cy} [\frac{-1}{EI_\theta}] \\
 & + B_z [-\frac{3r/2}{EI_\theta}] + B_x [\frac{r\pi/4}{EI_\theta}] = \frac{P[-u + r/2 - \cos \alpha(r - r \cos \alpha/2 - u)]}{EI_\theta}
 \end{aligned} \tag{4.4}$$

To simplify this equation, some techniques are used as follows:

It is supposed that:  $\frac{h}{b} = \beta$ , where h and b are the length and the width of a cross sectional area of the leg respectively. So, the moment of inertia  $I_\theta$ , and the radial moment  $I_r$ , can be written as follows [5].

$$I_\theta = \frac{hb^3}{12} = \frac{\beta b^4}{12} \quad (4.4a)$$

$$I_r = \frac{bh^3}{12} = \frac{\beta^3 b^4}{12} = \beta^2 I_\theta \quad (4.4b)$$

Also, for the same cross sectional area, it is known that, the torsional constants J is given by the following formula [4].

$$J = bh^3 \left[ \frac{1}{3} - 0.21 \frac{h}{b} \left( 1 - \frac{h^4}{12b^4} \right) \right] \quad (4.4c)$$

$$J = bh^3 \left[ \frac{1}{3} - 0.21 \frac{h}{b} \left( 1 - \frac{h^4}{12b^4} \right) \right] \frac{12}{12} \quad (4.4d)$$

$$J = \beta^2 I_\theta \left[ \frac{1}{3} - 0.21 \beta \left( 1 - \frac{\beta^4}{12} \right) \right] 12 \quad (4.4e)$$

$$\text{Let } \gamma = 12\beta^2 \left[ \frac{1}{3} - 0.21 \beta \left( 1 - \frac{\beta^4}{12} \right) \right] \quad (4.4f)$$

$$\text{so } J = \gamma I_\theta \quad (4.4g)$$

And, modulus of rigidity is given by:

$$G = \frac{E}{2(1+\nu)} \quad (4.4h)$$

where, E is the modulus of elasticity, and  $\nu$  is poison ratio [5].

Substituting these values in equation (4.4), it will give the first equation.

$$\begin{aligned}
& A_x [r \left\{ \frac{\pi/2}{\beta^2} + \frac{2(\pi-2)(1+\nu)}{\gamma} + \frac{\pi}{4} \right\}] + A_y \left[ \frac{-2r(1+\nu)}{\gamma} \right] \\
& + A_z \left[ \frac{r}{2\beta^2} + \frac{r(1+\nu)}{\gamma} - \frac{r}{2} \right] + M_{Ax} \left[ -\frac{1}{2} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] \\
& + M_{Ay} \left[ \frac{-(1+\nu)}{\gamma} - \frac{1}{2\beta^2} - 1 \right] + M_{Az} \left[ \frac{2(\pi/2-1)(1+\nu)}{\gamma} + \frac{\pi}{2\beta^2} \right] \\
& + B_x \left[ \frac{r\pi}{4} \right] + B_y \left[ r \left\{ \frac{-\pi}{4\beta^2} - \frac{2(\pi/4+1)(1+\nu)}{\gamma} \right\} \right] + B_z \left[ \frac{-3r}{2} \right] \\
& + M_{Bx} \left[ -\frac{1}{2} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] + M_{By} \left[ -1 \right] + M_{Bz} \left[ \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] \\
& + C_x \left[ r \left\{ \frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} + \frac{\pi}{4} \right\} \right] + C_y \left[ \frac{-2r(1+\nu)}{\gamma} \right] \\
& + C_z \left[ \frac{-r}{2\beta^2} - \frac{r(1+\nu)}{\gamma} - \frac{r}{2} \right] + M_{Cx} \left[ -\frac{1}{2} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] \\
& + M_{Cy} \left[ -1 \right] + M_{Cz} \left[ \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] = \text{Pr} \left[ \frac{-1}{2} + \cos \alpha - \frac{1}{2} \cos^2 \alpha \right]
\end{aligned} \tag{4.5}$$

Following the same steps, the remaining equations will be obtained as in the same way.

$$\begin{aligned}
& A_x \left[ \frac{-2r(1+\nu)}{\gamma} \right] + A_y \left[ \frac{r\pi}{4} + \frac{r\pi(1+\nu)}{\gamma} \right] + A_z \left[ \frac{-r}{2} - \frac{2r(1+\nu)}{\gamma} \right] + \\
& M_{Ax} \left[ 1 + \frac{2(1+\nu)}{\gamma} \right] + M_{Az} \left[ -\frac{2(1+\nu)}{\gamma} \right] + B_y \left[ \frac{2r(1+\pi/2)(1+\nu)}{\gamma} \right] \\
& + M_{Bx} \left[ \frac{2(1+\nu)}{\gamma} \right] + M_{Bz} \left[ \frac{-2(1+\nu)}{\gamma} \right] + C_x \left[ \frac{2r(1+\nu)}{\gamma} \right] + C_y \left[ \frac{r\pi(1+\nu)}{\gamma} \right] \\
& + C_z \left[ \frac{2r(1+\nu)}{\gamma} \right] + M_{Cx} \left[ \frac{2(1+\nu)}{\gamma} \right] + M_{Cz} \left[ \frac{-2(1+\nu)}{\gamma} \right] = 0
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
& A_x \left[ \frac{r}{2\beta^2} + \frac{r(1+\nu)}{\gamma} - \frac{r}{2} \right] + A_y \left[ \frac{-r}{2} - \frac{2r(1+\nu)}{\gamma} \right] + A_z \left[ \frac{r\pi}{4\beta^2} + r(\pi-2) \right. \\
& \left. + \frac{r\pi(1+\nu)}{2\gamma} \right] + M_{Ax} \left[ 1 - \frac{\pi}{2} - \frac{\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + M_{Ay} [1] \\
& + M_{Az} \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] + B_x \left[ \frac{-r}{2} \right] + B_y \left[ \frac{-r}{2\beta^2} - \frac{3r(1+\nu)}{\gamma} \right] \\
& + B_z \left[ r(1+\frac{\pi}{4}) \right] + M_{Bx} \left[ -\frac{\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + M_{By} [1] \\
& M_{Bz} \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] + C_x \left[ -\frac{r}{2\beta^2} - \frac{r(1+\nu)}{\gamma} - \frac{r}{2} \right] + C_y \left[ -\frac{2r(1+\nu)}{\gamma} \right] \\
& + C_z \left[ \frac{-r\pi}{4\beta^2} + \frac{r\pi}{4} - \frac{r\pi(1+\nu)}{2\gamma} \right] + M_{Cx} \left[ \frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + M_{Cy} [1] \\
& + M_{Cz} \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] = \Pr \left[ \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} - \sin \alpha \cos \alpha \right]
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
& A_x \left[ \frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + A_y \left[ 1 + \frac{2(1+\nu)}{\gamma} \right] + A_z \left[ 1 - \frac{\pi}{2} - \frac{\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] \\
& + M_{Ax} \left[ \frac{\pi}{4r} \left\{ 2 + \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] + M_{Az} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] + \\
& B_y \left[ \frac{1}{2\beta^2} + \frac{3(1+\nu)}{\gamma} \right] + M_{Bx} \left[ \frac{\pi}{4r} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] + M_{Bz} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] \\
& + C_x \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] + C_y \left[ \frac{2(1+\nu)}{\gamma} \right] + C_z \left[ \frac{\pi}{4} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] + \\
& M_{Cx} \left[ \frac{\pi}{4r} \left\{ \frac{1}{\beta^2} + \frac{2(1+\nu)}{\gamma} \right\} \right] + M_{Cz} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] = 0
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
& A_x \left[ \frac{-(1+\nu)}{\gamma} - \frac{1}{2\beta^2} - 1 \right] + A_z [1] + M_{Ay} \left[ \frac{\pi(1+\nu)}{2r\gamma} + \frac{\pi}{4r\beta^2} + \frac{\pi}{2r} \right] \\
& + M_{Az} \left[ \frac{(1+\nu)}{r\gamma} - \frac{1}{2r\beta^2} \right] + B_x [-1] + B_z \left[ \frac{\pi}{2} + 1 \right] + M_{By} \left[ \frac{\pi}{2r} \right] \\
& + C_x [-1] + C_z [1] + M_{Cy} \left[ \frac{\pi}{2r} \right] = P[\sin \alpha - \alpha \cos \alpha]
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
& A_x \left[ \frac{\pi}{2\beta^2} + \frac{2(\pi/2-1)(1+\nu)}{\gamma} \right] + A_y \left[ \frac{-2(1+\nu)}{\gamma} \right] + A_z \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] \\
& + M_{Ax} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] + M_{Ay} \left[ \frac{-1}{2r\beta^2} + \frac{(1+\nu)}{r\gamma} \right] + M_{Az} \left[ \frac{\pi(1+\nu)}{r\gamma} + \frac{\pi}{2r\beta^2} \right] \\
& + B_y \left[ \frac{-\pi}{4\beta^2} - \frac{2(\pi/4+1)(1+\nu)}{\gamma} \right] + M_{Bx} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] \\
& + M_{Bz} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + C_x \left[ \frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + C_y \left[ \frac{-2(1+\nu)}{\gamma} \right] + \\
& C_z \left[ \frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + M_{Cx} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] + M_{Cz} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] = 0
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
& A_x \left[ \frac{r\pi}{4} \right] + A_z \left[ \frac{-r}{2} \right] + M_{Ay} [-1] + B_x \left[ \frac{r\pi}{2} \right] + B_z [-2r] + M_{By} [-2] + \\
& C_x \left[ \frac{r\pi}{4} \right] + C_z \left[ \frac{-r}{2} \right] + M_{Cy} [-1] = \Pr \left[ -\frac{1}{2} + \cos \alpha - \frac{1}{2} \cos^2 \alpha \right]
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
& A_x \left[ \frac{-r\pi}{4\beta^2} - \frac{2r(\pi/4+1)(1+\nu)}{\gamma} \right] + A_y \left[ \frac{2r(\pi/2+1)(1+\nu)}{\gamma} \right] + \\
& A_z \left[ \frac{-r}{2\beta^2} - \frac{3r(1+\nu)}{\gamma} \right] + M_{Ax} \left[ \frac{1}{2\beta^2} + \frac{3(1+\nu)}{\gamma} \right] + \\
& M_{Az} \left[ \frac{-\pi}{4\beta^2} - \frac{2(\pi/4+1)(1+\nu)}{\gamma} \right] + B_y \left[ \frac{r\pi}{2\beta^2} + \frac{3r\pi(1+\nu)}{\gamma} \right] + \\
& + M_{Bx} \left[ \frac{1}{\beta^2} + \frac{4(1+\nu)}{\gamma} \right] + M_{Bz} \left[ \frac{-\pi}{2\beta^2} - \frac{\pi(1+\nu)}{\gamma} \right] + \\
& C_x \left[ \frac{r\pi}{4\beta^2} + \frac{2r(\pi/4+1)(1+\nu)}{\gamma} \right] + C_y \left[ \frac{2r(\pi/2+1)(1+\nu)}{\gamma} \right] + \\
& C_z \left[ \frac{r}{2\beta^2} + \frac{3r(1+\nu)}{\gamma} \right] + M_{Cx} \left[ \frac{1}{2\beta^2} + \frac{3(1+\nu)}{\gamma} \right] \\
& + M_{Cz} \left[ \frac{-\pi}{4\beta^2} - \frac{2(\pi/4+1)(1+\nu)}{\gamma} \right] = 0
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
& A_x \left[ \frac{-3r}{2} \right] + A_z \left[ r \left( \frac{\pi}{4} + 1 \right) \right] + M_{Ay} \left[ \frac{\pi}{2} + 1 \right] + B_x \left[ -2r \right] + B_z \left[ \frac{3r\pi}{2} \right] + M_{By} \left[ \pi \right] \\
& + C_x \left[ \frac{-3r}{2} \right] + C_z \left[ r \left( \frac{\pi}{4} + 1 \right) \right] + M_{Cy} \left[ \frac{\pi}{2} + 1 \right] = Pu \left( \frac{\pi}{2} + 1 \right) - P \frac{r\pi}{4} + P \left\{ r \left\{ \left( \frac{\pi}{2} - \alpha \right) \right. \right. \\
& \left. \left. + (1 - \sin \alpha) \right\} - r \left\{ (1 - \sin \alpha) + \left( \frac{\pi}{4} - \left\{ \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} \right\} \right) \right\} - u \left\{ \left( \frac{\pi}{2} - \alpha \right) + (1 - \sin \alpha) \right\} \right]
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
& A_x \left[ \frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + A_y \left[ \frac{2(1+\nu)}{\gamma} \right] + A_z \left[ \frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + \\
& M_{Ax} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + M_{Az} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] \\
& + B_y \left[ \frac{1}{\beta^2} + \frac{4(1+\nu)}{\gamma} \right] + M_{Bx} \left[ \frac{\pi}{2r\beta^2} + \frac{\pi(1+\nu)}{r\gamma} \right] + M_{Bz} \left[ \frac{-1}{r\beta^2} \right] \\
& + C_x \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] + C_y \left[ \frac{2(1+\nu)}{\gamma} \right] + C_z \left[ \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] \\
& + M_{Cx} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + M_{Cz} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] = 0
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
& A_x[-1] + A_z[1] + M_{Ay}[\frac{\pi}{2r}] + B_x[-2] + B_z[\pi] + M_{By}[\frac{\pi}{r}] + C_x[-1] \\
& + C_z[1] + M_{Cy}[\frac{\pi}{2r}] = P[\sin \alpha - \alpha \cos \alpha]
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
& A_x[\frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma}] + A_y[-\frac{2(1+\nu)}{\gamma}] + A_y[\frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma}] + \\
& M_{Ax}[\frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] + M_{Az}[\frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma}] + B_y[\frac{-\pi}{2\beta^2} - \frac{\pi(1+\nu)}{\gamma}] \\
& + M_{Bx}[\frac{-1}{r\beta^2}] + M_{Bz}[\frac{\pi}{2r\beta^2} + \frac{\pi(1+\nu)}{r\gamma}] + C_x[\frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma}] + \\
& C_y[-\frac{2(1+\nu)}{\gamma}] + C_z[-\frac{1}{2\beta^2} - \frac{(1+\nu)}{\gamma}] + M_{Cx}[\frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] \\
& + M_{Cz}[\frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma}] = 0
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
& A_x[\frac{-r\pi}{4\beta^2} + \frac{r\pi}{4} - \frac{r\pi(1+\nu)}{2\gamma}] + A_y[\frac{2r(1+\nu)}{\gamma}] + A_z[\frac{-r}{2\beta^2} - \frac{r}{2} - \frac{r(1+\nu)}{\gamma}] \\
& + M_{Ax}[\frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma}] + M_{Ay}[-1] + M_{Az}[\frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma}] + \\
& B_x[\frac{r\pi}{4}] + B_y[\frac{r\pi}{4\beta^2} + \frac{2r(\pi/4+1)(1+\nu)}{\gamma}] + B_z[\frac{-3r}{2}] + M_{Bx}[\frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma}] \\
& + M_{By}[-1] + M_{Bz}[\frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma}] + C_x[\frac{r\pi}{2\beta^2} + \frac{r\pi}{4} + \frac{2r(\pi-2)(1+\nu)}{\gamma}] \\
& + C_y[\frac{2r(1+\nu)}{\gamma}] + C_z[\frac{r}{2\beta^2} - \frac{r}{2} + \frac{r(1+\nu)}{\gamma}] + M_{Cx}[\frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma}] + \\
& M_{Cy}[\frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma} - 1] + M_{Cz}[\frac{-\pi}{2\beta^2} + \frac{2(1-\pi/2)(1+\nu)}{\gamma}] \\
& = \text{Pr}[-\frac{1}{2} + \cos \alpha - \frac{1}{2} \cos^2 \alpha]
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
& A_x \left[ \frac{-2r(1+\nu)}{\gamma} \right] + A_y \left[ \frac{r\pi(1+\nu)}{\gamma} \right] + A_z \left[ \frac{-2r(1+\nu)}{\gamma} \right] + M_{Ax} \left[ \frac{2(1+\nu)}{\gamma} \right] \\
& + M_{Az} \left[ \frac{-2(1+\nu)}{\gamma} \right] + B_y \left[ \frac{2r(1+\pi/2)(1+\nu)}{\gamma} \right] + M_{Bx} \left[ \frac{2(1+\nu)}{\gamma} \right] + \\
& M_{Bz} \left[ \frac{-2(1+\nu)}{\gamma} \right] + C_x \left[ \frac{2r(1+\nu)}{\gamma} \right] + C_y \left[ \frac{r\pi}{4} + \frac{r\pi(1+\nu)}{\gamma} \right] + \\
& C_z \left[ \frac{r}{2} + \frac{2r(1+\nu)}{\gamma} \right] + M_{Cx} \left[ 1 + \frac{2(1+\nu)}{\gamma} \right] + M_{Cz} \left[ \frac{-2(1+\nu)}{\gamma} \right] = 0
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
& A_x \left[ -\frac{r}{2\beta^2} - \frac{r(1+\nu)}{\gamma} - \frac{r}{2} \right] + A_y \left[ \frac{2r(1+\nu)}{\gamma} \right] + A_z \left[ -\frac{r\pi}{4\beta^2} + \frac{r\pi}{4} - \frac{r\pi(1+\nu)}{2\gamma} \right] \\
& + M_{Ax} \left[ \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] + M_{Ay} [1] + M_{Az} \left[ -\frac{1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + B_x \left[ \frac{-r}{2} \right] \\
& + B_y \left[ \frac{r}{2\beta^2} + \frac{3r(1+\nu)}{\gamma} \right] + B_z [r(1+\pi/4)] + M_{Bx} \left[ \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] + M_{By} [1] \\
& + M_{Bz} \left[ -\frac{1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + C_x \left[ \frac{r}{2\beta^2} - \frac{r}{2} + \frac{r(1+\nu)}{\gamma} \right] + C_y \left[ \frac{r}{2} + \frac{2r(1+\nu)}{\gamma} \right] + \\
& C_z \left[ \frac{r\pi}{4\beta^2} + r(\pi-2) + \frac{r\pi(1+\nu)}{2\gamma} \right] + M_{Cx} \left[ \frac{\pi}{2} - 1 + \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] + M_{Cy} [1] \\
& + M_{Cz} \left[ -\frac{1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] = \Pr \left[ \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} - \sin \alpha \cos \alpha \right]
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
& A_x \left[ \frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma} \right] + A_y \left[ \frac{2(1+\nu)}{\gamma} \right] + A_z \left[ \frac{-\pi}{4\beta^2} - \frac{\pi(1+\nu)}{2\gamma} \right] + \\
& M_{Ax} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + M_{Az} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] + B_y \left[ \frac{1}{2\beta^2} + \frac{3(1+\nu)}{\gamma} \right] \\
& + M_{Bx} \left[ \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + M_{Bz} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] + C_x \left[ \frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma} \right] \\
& + C_y \left[ 1 + \frac{2(1+\nu)}{\gamma} \right] + C_z \left[ \frac{\pi}{2} - 1 + \frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma} \right] \\
& + M_{Cx} \left[ \frac{\pi}{2r} + \frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma} \right] + M_{Cz} \left[ \frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma} \right] = 0
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
& A_x[-1] + A_z[1] + M_{Ay}[\frac{\pi}{2r}] + B_x[-1] + B_z[\frac{\pi}{2} + 1] + M_{By}[\frac{\pi}{2r}] \\
& + C_x[\frac{-1}{2\beta^2} - 1 - \frac{(1+\nu)}{\gamma}] + C_z[1] + M_{Cy}[\frac{\pi}{4r\beta^2} + \frac{\pi}{2r} + \frac{\pi(1+\nu)}{2r\gamma}] \\
& + M_{Cz}[\frac{1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] = P[\sin \alpha - \alpha \cos \alpha]
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
& A_x[\frac{\pi}{4\beta^2} + \frac{\pi(1+\nu)}{2\gamma}] + A_y[\frac{-2(1+\nu)}{\gamma}] + A_z[\frac{1}{2\beta^2} + \frac{(1+\nu)}{\gamma}] + \\
& M_{Ax}[\frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] + M_{Az}[\frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma}] + \\
& B_y[\frac{-\pi}{4\beta^2} - \frac{2(\pi/4+1)(1+\nu)}{\gamma}] + M_{Bx}[\frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] \\
& + M_{Bz}[\frac{\pi}{4r\beta^2} + \frac{\pi(1+\nu)}{2r\gamma}] + C_x[\frac{-\pi}{2\beta^2} + \frac{2(1-\pi/2)(1+\nu)}{\gamma}] + \\
& C_y[\frac{-2(1+\nu)}{\gamma}] + C_z[\frac{-1}{2\beta^2} - \frac{(1+\nu)}{\gamma}] + M_{Cx}[\frac{-1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] \\
& + M_{Cy}[\frac{1}{2r\beta^2} - \frac{(1+\nu)}{r\gamma}] + M_{Cz}[\frac{\pi}{2r\beta^2} + \frac{\pi(1+\nu)}{r\gamma}] = 0
\end{aligned} \tag{4.22}$$

According to Maxwell's reciprocal theorem, in a linear elastic structure, the displacements at coordinate  $i$  due to a unit force at coordinate  $j$  is equal to the displacement at  $j$  due to a unit force acting at  $i$ , which is given by the equation

$$f_{ij} = f_{ji}$$

where  $f_{ij}$  is the displacement at  $i$  due to a unit force at  $j$ , and  $f_{ji}$  is the displacement at  $j$  due to a unit force at  $i$ .

The displacements in this equation are the flexibility coefficients. For a structure in which  $m$  coordinates are indicated, the flexibility coefficients, when arranged in a matrix of the order  $m \times m$ , give the flexibility matrix of the structure. This matrix must be symmetrical, by virtue of the previous equation [4].

So, from the 18 equations which are obtained, a symmetrical matrix of the order  $18 \times 18$  is arranged analytically in a general form. For solving these equations numerically, some numerical values must be given. And in our case we choose

$$P = 10.0 \text{ KN.}$$

$$R = 10.0 \text{ m.}$$

$$\alpha = 60^\circ = \frac{\pi}{3}$$

$$u = r(1 - \cos \alpha) = 5.0 \text{ m.}$$

$$\nu = 0.3$$

$$\beta = 2.0$$

Then equation (4.4f) becomes:

$$\gamma = 22.72$$

After applying these values to the equations, it can easily be solved as:

$$[A] [x] = [C]$$

where  $[A]$  is a symmetrical matrix of the order  $18 \times 18$ , which shows the elements of the equations as shown in the next page.

13,09	-1,144	-3,178	-0,182	-1,182	0,458	7,854	-4,007	-15	-0,182	-1	0,286	4,992	-1,144	-6,822	-0,182	-1	0,286
-1,144	9,652	-6,144	1,114	0	-0,114	0	2,942	0	0,114	0	-0,114	1,144	1,798	1,144	0,114	0	-0,114
-3,178	-6,144	14,28	-0,857	1	0,182	-5	-2,967	17,85	-0,286	1	0,182	-6,882	-1,144	4,992	-0,286	1	0,182
-0,182	1,114	-0,857	0,186	0	-0,018	0	0,297	0	0,029	0	-0,018	0,182	0,114	0,286	0,029	0	-0,018
-1,182	0	1	0	0,186	-0,007	-1	0	2,571	0	0,157	0	-1	0	0	0,157	0	0,029
0,458	-0,114	0,182	-0,018	-0,007	0,057	0	-0,401	0	-0,018	0	0,029	-0,286	-0,114	-0,182	-0,018	0	0,029
7,854	0	-5	0	-1	0	15,71	0	-20	0	-2	0	7,854	0	-5	0	-1	0
-4,007	2,942	-2,967	0,297	0	-0,401	0	9,32	0	0,479	0	-0,572	4,007	2,942	2,967	0,297	0	-0,401
-15	0	17,85	0	2,571	0	-20	0	47,12	0	3,142	0	-15	0	17,85	0	2,571	0
-0,182	0,114	-0,286	0,029	0	-0,018	0	0,479	0	0,057	0	-0,025	0,182	0,114	0,286	0,029	0	-0,018
-1	0	1	0	0,157	0	-2	0	3,142	0	0,314	0	-1	0	1	0	0,157	0
0,286	-0,114	0,182	-0,018	0	0,029	0	-0,572	0	-0,025	0	0,057	-0,286	-0,114	-0,182	-0,018	0	0,029
4,992	1,144	-6,822	0,182	-1	-0,286	7,854	4,007	-15	0,182	-1	-0,286	13,09	1,144	-3,178	0,182	-1,182	-0,458
-1,144	1,798	-1,144	0,114	0	-0,114	0	2,942	0	0,114	0	-0,114	1,144	9,652	6,144	1,114	0	-0,114
-6,822	1,144	4,992	0,286	1	-0,182	-5	2,967	17,85	0,286	1	-0,182	-3,178	6,144	14,28	0,857	1	-0,182
-0,182	0,114	-0,286	0,029	0	-0,018	0	0,297	0	0,029	0	-0,018	0,182	1,114	0,857	0,186	0	-0,018
-1	0	1	0	0,157	0	-1	0	2,571	0	0,157	0	-1,182	0	1	0	0,186	0,007
0,286	-0,114	0,182	-0,018	0	0,029	0	-0,401	0	-0,018	0	0,029	-0,458	-0,114	-0,182	-0,018	0,007	0,057

And  $[x]$  is a matrix of the order  $18 \times 1$  which shows the unknowns as:

$$x_1 = A_x$$

$$x_2 = A_y$$

$$x_3 = A_z$$

$$x_4 = M_{Ax}$$

$$x_5 = M_{Ay}$$

$$x_6 = M_{Az}$$

$$x_7 = B_x$$

$$x_8 = B_y$$

$$x_9 = B_z$$

$$x_{10} = M_{Bx}$$

$$x_{11} = M_{By}$$

$$x_{12} = M_{Bz}$$

$$x_{13} = C_x$$

$$x_{14} = C_y$$

$$x_{15} = C_z$$

$$x_{16} = M_{Cx}$$

$$x_{17} = M_{Cy}$$

$$x_{18} = M_{Cz}$$

Also [c] is a matrix of the order 18x1 which gives the constants as:

$$c_1 = -12.5$$

$$c_2 = 0.0$$

$$c_3 = 30.709$$

$$c_4 = 0.0$$

$$c_5 = 3.424$$

$$c_6 = 0.0$$

$$c_7 = -12.5$$

$$c_8 = 0.0$$

$$c_9 = 64.952$$

$$c_{10} = 0.0$$

$$c_{11} = 3.424$$

$$c_{12} = 0.0$$

$$c_{13} = -12.5$$

$$c_{14} = 0.0$$

$$c_{15} = 30.709$$

$$c_{16} = 0.0$$

$$c_{17} = 3.424$$

$$c_{18} = 0.0$$

Where the forces are given by KN, and the moments are given by KNm.

Then, by the help of a computer program (Microsoft Excel), the matrix is solved, and the results are found as:

$$A_x = 0.886596$$

$$A_y = 0.42503$$

$$A_z = 0.47057$$

$$M_{Ax} = -0.98242$$

$$M_{Ay} = 3.577007$$

$$M_{Az} = -4.57213$$

$$B_x = 1.816587$$

$$B_y = -0.00959$$

$$B_z = 1.606615$$

$$M_{Bx} = 0.125205$$

$$M_{By} = 5.564286$$

$$M_{Bz} = -0.04756$$

$$C_x = 0.895539$$

$$C_y = -0.41242$$

$$C_z = 0.441539$$

$$M_{Cx} = 1.034371$$

$$M_{Cy} = 3.6332$$

$$M_{Cz} = 4.572547$$

where the forces are given by KN, and the moments are given by KN.m.

By substituting the results in the equations, which obtained from statics in the second chapter, the forces and the moments of leg D can be found as:

$$D_x = -3.59869$$

$$D_y = -0.00302$$

$$D_z = 7.481275$$

$$M_{Dx} = 0.113159$$

$$M_{Dy} = -4.02789$$

$$M_{Dz} = 0.071212$$

Now, all the forces and the moments of the two arches are found, and the problem is solved.

These equations are calculated analytically in a very accurate way, that can be solved by applying any normal concentrated load normal to the beam. And it's worthy to note that all terms that concerned the deflections are taken into consideration.

By applying the principle of superposition, this problem can be solved for different external loads [2]. This can be done not only by concentrated loads, but also for distributed loads as well. This procedure can be performed by applying the loads as frequent as it is desired. Thus a satisfactory analysis can be obtained from engineering point of view.

## V. CONCLUSION

In this study, 18 equations by 18 unknowns are obtained for analyzing a dome problem, by using Castiglano's theorem.

These equations are constructed analytically in a very general way, that can be solved by applying any external concentrated load normal to the dome. And it is worthy to note that all terms that contributed the deformation are taken into consideration.

By applying the principle of superposition, this problem can be solved for different external loads [2]. This can be done not only for concentrated loads, but also for distributed loads as well. This procedure can be performed by applying the loads as frequent as it is desired. Thus a satisfactory analysis can be obtained from engineering point of view.

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