BESSEL DIFFERENTIAL EQUATION
AND
APPLICATIONS OF BESSEL FUNCTIONS

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ABSTRACT

This thesis consists of three chapters. In the first chapter, the historical progress of the subject is considered.Also, some essential definitions are given. In the second chapters, Bessel equation is obtained through the cylindrical coordinates of Laplace equation. In addition, Bessel functions which are the solutions of Bessel equation and their properties are studied. In the third chapter, applications of Bessel functions which are vibrations of circular membrane(vibration drum) and Schrödinger equation are examined. Two-dimensional wave equation is obtained. So, we transform the wave equation into polar coordinates and Cylindrical functions of first and second kind are obtained by solving radial eigenvalue problem. At the end Schrödinger equation in spherical coordinates is studied and Spherical functions of first kind and second kind are obtained by solving spherical Bessel differential equation which is radial Schrödinger equation.

Key words: Bessel equation, Bessel functions, vibrating membrane, wave equation, Schrödinger equation.

ÖZET

Bu tez üç bölümden oluşmaktadır. Birinci bölümde konunun tarihsel gelişimi verilmiştir ve bazı temel kavramlar verilmiştir. İkinci bölümde Laplace denkleminin silindirik koordinatlardaki ifadesinden yararlanarak Bessel denklemi elde edilmiştir. Bessel denkleminin çözümleri olan Bessel fonksiyonları ve onların özellikleri üzerinde durulmuştur. Üçüncü bölümde Bessel fonksiyonlarının uygulamaları olan dairesel zarın titreşimleri ve Schrödinger denklemi incelenmiştir. İki boyutlu dalga denklemi elde edilmiştir. Daha sonra dalga denklemini kutupsal koordinatları kullanarak (r, θ) türünden yazdık ve radyal (yarıçapa ait) eigenvalue problemini çözerek birinci ve ikinci tür silindirik Bessel fonksiyonlarını elde ettik. Bu bölümün en sonunda ise küresel koordinatlarda Schrödinger denklemini inceledik ve radyal Schrödinger denklemi olan küresel Bessel denklemini çözerek birinci ve ikinci tür küresel Bessel fonksiyonlarını elde ettik.

Anahtar sözcükler: Bessel denklemi, Bessel fonksiyonları, Titreşimli zar, dalga denklemi, Schrödinger denklemi.

CONTENTS

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Chapter 1

INTRODUCTION AND DEFINITIONS

Bessel functions are named after Wilhelm Bessel (1784 - 1846), however, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessels functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end. In 1764 Leonhard Euler employed Bessel functions of both zero and integral orders in an analysis of vibrations of a stretched membrane, an investigation which was further developed by Lord Rayleigh in 1878, where he demonstrated that Bessels functions are particular cases of Laplace's functions. Bessel, while receiving named credit for these functions, did not incorporate them into his work as an astronomer until 1817. The Bessel function was the result of Bessel's study of a problem of Kepler for determining the motion of three bodies moving under mutual gravitation. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, that is the motion of the Sun caused by the perturbing body. It was likely Lagrange's work on elliptical orbits that first suggested to Bessel to work on the Bessel functions. Subsequent studies of Bessel functions included the works of Mathews in 1895, "A treatise on Bessel functions and their applications to physics" written in collaboration with Andrew Gray. It was the first major treatise on Bessel functions in English and covered topics such as applications of Bessel functions to electricity, hydro dynamics and diraction. In 1922, Watson first published his comprehensive examination of Bessel functions "A Treatise on the Theory of Bessel Functions".

The given differential equation is named after the German mathematician Friedrich Wilhelm Bessel and Astronomer who studied this equation in detail and showed (in 1824) that its solutions are a special class of functions called through *expressed cylinder functions* or *Bessel functions*. Frobenius series method soluble Bessel equation of second order differential equations with variable coefficients has an important place. Laplace equation in polar coordinates with the Bessel equation is obtained using the expression. In mathematical physics, basic sciences and engineering sciences in the field of occupational many functions common to solving the problems of this equation. Bessel functions, first defined by Daniel Bernoulli and generalized by Friedrich Bessel differential equation canonical solutions. In addition, solutions of Bessel functions separable in polar coordinates with the Helmholtz equation. As a result associated with a particular wave propagation problems

Applications of Bessel functions to heat conduction theory, including dynamical and linked problems are very numerous. In elasticity theory the solutions in Bessel Functions are effective for all spatial problems, which are solved in spherical or cylindrical coordinates; also for different problems concerning the oscillations of plates and the equilibrium of plates on an elastic foundation; for a series of questions of theory of shells; for problems on the concentration of the stresses near cracks and others. In each of these areas there is wide range of different applications of Bessel functions. References where these functions are present are actually immense. Different parts of Bessel function theory are widely used when solving problems of acoustics, radio physics, hydrodynamics, atomic and nuclear physics, quantum physics and so on.

Bessel's differential equation in (2.9) is often encountered when solving boundary value problems, such as separable solutions to Laplace's equation or the Helmholtz equation, especially when working in cylindrical or spherical coordinates. Bessel functions made their first appearance by relating the angular position of a planet moving along a Keplerian ellipse to elapsed time. However the integral and power series shows up in other places, generally concerning the radial variable after separating Laplace's equation in polar or spherical polar coordinates. In many problems of mathematical physics, whose solution is connected with the application of cylindrical and spherical coordinates. The constant v, determines the order of the Bessel functions found in the solution to Bessel's differential equation and can take on any real numbered value. For cylindrical problems the order of the Bessel function is an integer value ($v = n$) while for spherical problems the order is of half integer value $v = n + 1/2$. Bessel functions are therefore especially important for many problems of wave propagation and static potentials and its applications are as:

Electromagnetic waves in a cylindrical waveguide, heat conduction in a cylindrical object, diffusion problems on lattice, modes of vibration of a thin circular or annular artificial membrane and solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates for a free particle). We are going to examine last two applications in these applications. Firstly we consider the solution of the two dimensional wave equation of the circular membrane and examine modes of vibration of circular membrane. Secondly, we consider solutions to the radial Schrödinger equation in spherical coordinates for a free particle.

In vibrating membrane, the problem is that find the frequencies of vibration of a circular drum when the modes of vibration are rotationally invariant. A kettledrum is a percussive instrument consisting of a circular drumhead (usually plastic, but in older times, an animal skin) that is tautly stretched over a metal bowl. The vibrations of the kettledrum's drumhead can be modelled by the wave equation in (3.13), where a is the speed of waves travelling on the drumhead. The constant a is directly related to the tension of the drumhead and the corresponding pitch that is generated by hitting the drumhead with a mallet, and can be adjusted using a foot pedal. The characteristic sound of the kettledrum is determined by its vibrational modes and their corresponding frequencies. Any kettledrum player will tell you that the proper place to strike the drumhead is not the center of the drumhead, but rather a spot somewhere about one-sixth of the diameter away from the edge of the drumhead. The most common drums have a diameter between 23 to 29 inches, so that means striking the timpani about 4 to 5 inches in from the edge of the drumhead. Striking the drumhead in the center produces a sound that is somewhat hollow.(Yong, 2006)

In application of vibration, we will give some mathematical explanations for why this occurs. We consider the vibrations of a circular membrane of radius c as shown in figure 3.3. Also, this section considers the solution of the two dimensional wave equation of the circular membrane. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary.

To fit the boundary condition of no displacement on other than rectangular boundaries requires the use of an appropriate two dimensional orthogonal curvilinear coordinate system such that the boundary of membrane coincides with coordinate lines in this system. Furthermore, it is necessary that the variables of the wave equation be seperable in the new system. It turns out that the choice of curvilinear coordinate systems is severely limited, and it is impossible, except in an approximate way, to analyze the vibrations of a membrane having an arbitrarily shaped boundary which is circular boundary given by $x^2 + y^2 = c^2$. It is more natural to use polar coordinates as indicated in figure 3.3. The solution of one of the seperated equations consists of Bessel functions. We shall transform the two dimensional Cartesian wave equation into its polar form in terms of r and θ using the parametric equations. So, the boundary condition is given $z(c, \theta, t) = 0$ for all $t > 0$ and $\theta \in [\pi, -\pi]$.

Definition 1 : *(Ordinary Points) If the coefficients$P(x)$ and $Q(x)$ of equation $y'' + P(x)y' + Q(x)y = 0$ are both analytic at the point x_0, then x_0 is called an ordinary point for the equation.*

Definition 2 : *(Wronskian determinant) Let the first derivative of the $f(x)$ and $g(x)$*

funtions be defined on the interval $|x - x_0| \leq a$ *. Under this condition* $W(f,g) = f(x)g\prime(x) - f\prime(x)g(x)$ *is called the wronskian of* $f(x)$ *and* $g(x)$*. (Marchenko, 1986).*

Definition 3 *: (Orthogonal functions) A function is orthogonal if a defined inner product vanishes between two unlike components of a particular inner product space (an inner product between a function* $\Psi(a)$ *and* $\Psi(b)$ *shall be depicted mathematically by* $\langle \Psi(a) \mid \Psi(b) \rangle$ *. It is common to use the following inner product for two functions* f *and* g*:*

$$\langle f, g \rangle_w = \int_a^b f(x)g(x)w(x)dx$$

Here we introduce a nonnegative weight function $w(x)$ *in the definition of this inner product. We say that those functions are orthogonal if that inner product is zero.*

$$\int_a^b f(x)g(x)w(x)dx = 0$$

Definition 4 *:(Norm of function)- The norm of the function defined* $||\, f||$ *which is equal*

$$\int_0^1 f^2(x)dx.$$

Definition 5 *:(The generating function for* $J_n(x)$*)-* $F(x,t)$ *be two variables function and its Taylor expantion for one of its variable could be as follows.*

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

The $F(x,t)$ *function with* $\{f_n(x)\}$, $n = 0, 1, 2, ...$ *called the generating function for* $J_n(x)$*. This series of funtions are not necessarily converge for all* x*'s and* t*'s . Let* I *be a closed interval and* r *be a positive constant and let* $|t| < r$ *and* $x \in I$ *is enough for convergence.*

Definition 6 :(Frequency) Frequency is the number of vibrations of a repeating event per unit time.

Definition 7 :(Vibration mode) A mode of vibration is characterized by a modal frequency and mode shape and is numbered according to the number of half waves in the vibration.

Definition 8 :(Superposition principle) For a linear homogeneous ordinary differential equation, if $y_1(x)$ and $y_2(x)$ are solutions, then so is $y_1(x) + y_2(x)$.

Definition 9 :(Heisenberg's Uncertainty Principle) Heisenberg's Uncertainty Principle is one of the fundamental concepts of Quantum Physics, and is the basis for the initial realization of fundamental uncertainties in the ability of an experimenter to measure more than one quantum variable at a time. Attempting to measure an elementary particle's position to the highest degree of accuracy, for example, leads to an increasing uncertainty in being able to measure the particle's momentum to an equally high degree of accuracy. Heisenberg's Principle is typically written mathematically in either of two forms:

$$\Delta E \Delta t \geq h/4\pi \qquad \text{and} \qquad \Delta x \Delta p \geq h/4\pi$$

In essence, the uncertainty in the energy (ΔE) times the uncertainty in the time (Δt) – or alternatively, the uncertainty in the position (Δx) multiplied times the uncertainty in the momentum (Δp) – is greater or equal to a constant ($h/4\pi$). The constant, h, is called Planck's Constant (where $h/4\pi = 0.527x10 - 34$ Joule-second)

Chapter 2

BESSEL DIFFERENTIAL EQUATION AND BESSEL FUNCTIONS

The given differential equation is named after the German mathematician Friedrich Wilhelm Bessel and Astronomer who studied this equation in detail and showed (in 1824) that its solutions are a special class of functions called through *expressed cylinder functions* or *Bessel functions*. Frobenius series method soluble Bessel equation of second order differential equations with variable coefficients has an important place. Laplace equation in polar coordinates with the Bessel equation is obtained using the expression. In mathematical physics, basic sciences and engineering sciences in the field of occupational many functions common to solving the problems of this equation. Bessel functions, first defined by Daniel Bernoulli and generalized by Friedrich Bessel differential equation canonical solutions. In addition, solutions of Bessel functions separable in polar coordinates with the Helmholtz equation. As a result associated with a particular wave propagation problems.

2.1 Bessel Differential Equation

Bessel equation is frequently occurrence in physical problems. For example, it arises in the determination of the solutions of Laplace's equation associated with the circular cylinder and so its solutions are called *cylinder functions*. For suppose we transform Laplace's equation

$$\begin{aligned} \nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (2.1) \\ x &= u\cos\phi \quad , \qquad y = u\sin\phi \quad , \qquad z = z \end{aligned}$$

this gives

$$\frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (2.2)$$

A typical technique used to solve many kinds of partial differential equations is a method known as separation of variables. We use separation of variables to solve this equation. To illustrate in the case of our example, namely we suppose the solution:

$$V(u, \phi, z) = U(u)\Phi(\phi)Z(z)$$

and taking derivatives appropriately, the following equations have been obtained:

$$\begin{aligned} \frac{\partial V}{\partial u} &= \frac{dU}{du}\Phi Z; & \frac{\partial^2 V}{\partial u^2} &= \frac{d^2 U}{du^2}\Phi Z; \\ \frac{\partial^2 V}{\partial \phi^2} &= \frac{d^2\Phi}{d\phi^2}UZ; & \frac{\partial^2 V}{\partial z^2} &= \frac{d^2 Z}{dz^2}U\Phi \end{aligned}$$

substitute these derivatives into (2.2), it has been obtained:

$$\frac{d^2U}{du^2}\Phi Z + \frac{1}{u}\frac{dU}{du}\Phi Z + \frac{1}{u^2}\frac{d^2\Phi}{d\phi^2}UZ + \frac{d^2 Z}{dz^2}U\Phi = 0$$

$U(u)\Phi(\phi)Z(z) \neq 0$, and divide the above equation by $U\Phi Z$ for the both sides, we get

$$
\begin{aligned}
\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{U}\frac{1}{u}\frac{dU}{du} + \frac{1}{u^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} &= 0 \\
\frac{U''}{U} + \frac{1}{u}\frac{U'}{U} + \frac{1}{u^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} &= 0 \\
\frac{U''}{U} + \frac{1}{u}\frac{U'}{U} + \frac{1}{u^2}\frac{\Phi''}{\Phi} &= -\frac{Z''}{Z}
\end{aligned} \tag{2.3}
$$

Right hand side of the equation (2.3) depends on z and the left hand side depends on u and ϕ. By using these facts the right hand side of the equation could be equalize $-\lambda^2$which is a constant. From here;

$$\frac{U''}{U} + \frac{1}{u}\frac{U'}{U} + \frac{1}{u^2}\frac{\Phi''}{\Phi} = -\lambda^2 \tag{2.4}$$

and since $\frac{Z''}{Z} = +\lambda^2$, the following equation is obtained

$$Z'' - \lambda^2 Z = 0$$

Multiply by u^2 for the both side of (2.4)

$$u^2\frac{U''}{U} + u\frac{U'}{U} + \frac{\Phi''}{\Phi} = -\lambda^2 u^2$$

After some operations we find,

$$u^2\frac{U''}{U} + u\frac{U'}{U} + \lambda^2 u^2 = -\frac{\Phi''}{\Phi} \tag{2.5}$$

Take the right hand side of the equation and equalize it to V^2 as a constant. Then the equation (2.5) could be written

$$u^2\frac{U''}{U} + u\frac{U'}{U} + \lambda^2 u^2 = -V^2 \tag{2.6}$$

Since $\frac{\Phi''}{\Phi} = -V^2$ we found

$$\ddot{o}l\Phi'' + V^2\Phi = 0$$

and the equation (2.6) finally becomes in the form

$$u^2U'' + uU' + (\lambda^2u^2 - v^2)U = 0 \tag{2.7}$$

By using $\lambda u = x$ transformation, we obtain $U(u)$. If we take first and second derivatives of $U(u)$ we found the following equations. Substitute these derivatives into the equation (2.7) you could found

$$\begin{aligned} \frac{x^2}{\lambda^2}\left(\lambda^2\frac{d^2y}{dx^2}\right) + \frac{x}{\lambda}\left(\lambda\frac{dy}{dx}\right) + \left(\lambda^2\frac{x^2}{\lambda^2} - v^2\right)y &= 0 \\ x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - v^2\right)y &= 0 \end{aligned} \tag{2.8}$$

and finally the following equation found

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0$$

which is called *the Bessel equation of index* v, and its solutions called *cylindrical or Bessel functions.*

2.2 Applications of Power Series and Laplace transform for the Bessel Differential Equation and Cylindrical Functions of the First Kind

Consider the Bessel differential equation with index v

$$x^2y'' + xy' + (x^2 - v^2)\, y = 0 \tag{2.9}$$

Equation (2.9) is a linear differential equation of second order, hence it is general integral can be expressed in the form

$$u(x) = C_1y_1(x) + C_2y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are linearly independent partial solutions of equation (2.9). We verify $x = 0$ is a regular singular point. In some applications of Bessel's equation the variable x will be distance of a point from the origin in polar coordinates. It will be very important to understand how the solution behaves when x is close to 0, and the point is close origin. So, we shall seek a solution of equation (2.9) in the form of a generalized power series which is Frobenious method in increasing powers of argument x.

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

where $a_0 \neq 0$.Differentiating the solution (2.10):

$$\begin{aligned} y' &= \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} \\ y'' &= \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2} \end{aligned}$$

Substituting with the series in equation (2.9):

$$x^2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_nx^{m+n-2}+x\sum_{n=0}^{\infty}(m+n)a_nx^{m+n-1}+(x^2-v^2)\sum_{n=0}^{\infty}a_nx^{m+n}=0$$

Shifting the index in the second sum and isolating the first two terms in the first sum, this becomes

$$(m^2-v^2)a_0x^m+\left[(m+1)^2-v^2\right]a_1x^{m+1}+\sum_{n=2}^{\infty}\left\{\left[(m+n)^2-v^2\right]a_n+a_{n-2}\right\}x^{m+n}=0 \tag{2.11}$$

Setting the coefficients of the first term in (2.11) equal to 0 because of $a_0\neq 0$, we get the indicial equation $I(m)=m^2-v^2=0$. Thus, the roots are $m=\pm v$. Setting the coefficient of the second term in (2.11) equal to 0, we get

$$\left[(m+1)^2-v^2\right]a_1=(m+1+v)(m+1-v)a_1=0$$

Since coefficient of a_1 can not be 0, this requires $a_1=0$.Setting the coefficient of the general term in (2.11) equal to 0, the recurrence formula becomes

$$a_n=-\frac{1}{(m+n)^2-v^2}a_{n-2}\qquad ,n\geq 2 \tag{2.12}$$

Since $a_1=0$, we immediately conclude that $a_{2n+1}=0$ for all n, and for the even-numbered terms, the even numbered coefficient becomes

$$a_{2n}=-\frac{1}{(m+2n)^2-v^2}a_{2n-2} \tag{2.13}$$

We substitute for value 2 to $2p$ respectively in (2.13), the following equation is obtained

$$a_{2p}=\frac{(-1)^p}{[(m+2)^2-v^2]\,[(m+4)^2-v^2]\ldots[(m+2p)^2-v^2]}a_0 \tag{2.14}$$

So we substitute in solution (2.10),

$$y = a_0 x^m \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{[(m+2)^2 - v^2]\,[(m+4)^2 - v^2] \ldots [(m+2p)^2 - v^2]} x^{2p} \right\} \tag{2.15}$$

Now we find the solutions for the roots $m_1 = v$ and $m_2 = -v$. In (2.9), equation has v^2 and we assume $v \geq 0$. Evaluating at $m = v$ in (2.15), the following is obtained

$$y = a_0 x^v \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{[(v+2)^2 - v^2]\,[(v+4)^2 - v^2] \ldots [(v+2p)^2 - v^2]} x^{2p} \right\}$$

or

$$y = a_0 x^v \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{2^{2p} p! (v+1)(v+2) \ldots (v+p)} x^{2p} \right\} \tag{2.16}$$

Using property of gamma functions in appendix 1, one can rewrite the series obtained in a more compact form. It becomes

$$\begin{aligned} \Gamma(v+p+1) &= (v+p)\Gamma(v+p) = (v+p)(v+p-1)\Gamma(v+p-1) \\ &= (v+p)(v+p-1) \ldots \ \Gamma(v+1) \end{aligned}$$

Using gamma function property, equation in (2.16) can be written

$$y = y_1(x) = \sum_{p=1}^{\infty} \frac{(-1)^p a_0 \Gamma(v+1)}{2^{2p} p! \Gamma(v+p+1)} x^{2p+v}$$

The coefficient a_0 can be assigned any non-zero value so it becomes

$$a_0 = \frac{1}{\Gamma(v+1) 2^v}$$

to simplify the notation. Then our solution denoted by

$$y_1(x) = J_v(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p! \Gamma(v+p+1)} \left(\frac{x}{2} \right)^{v+2p} \tag{2.17}$$

The first series $y_1(x)$ defines a function which is called *the Bessel or cylindrical function of the first kind, of index* v and denoted by $J_v(x)$. For $v > 0$, $\Gamma(v+p+1) = (v+p)!$. So, the solution becomes

$$J_v(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(v+p)!} \left(\frac{x}{2}\right)^{v+2p} \tag{2.18}$$

Now evaluating at $m = -v$ in (2.15), it has been obtained

$$y = a_0 x^{-v} \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{[(2-v)^2 - v^2]\,[(4-v)^2 - v^2] \ldots [(2p-v)^2 - v^2]} x^{2p} \right\}$$

or

$$y_2(x) = a_0 x^{-v} \left\{ 1 + \sum_{p=1}^{\infty} \frac{(-1)^p}{2^{2p} p!(1-v)(2-v)\ldots(p-v)} x^{2p} \right\} \tag{2.19}$$

The coefficient a_0 can be assigned any nonzero value so it becomes

$$a_0 = \frac{1}{\Gamma(1-v)2^{-v}}$$

to simplify the notation. Then our solution denoted by

$$J_{-v}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(-v+p+1)} \left(\frac{x}{2}\right)^{2p-v} \tag{2.20}$$

For $v < 0$, $\Gamma(-v+p+1) = (p-v)!$. So, the second solution becomes

$$y_2(x) = J_{-v}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p-v)!} \left(\frac{x}{2}\right)^{2p-v} \tag{2.21}$$

The second series $y_2(x)$ defines a function which is called *Bessel or cylindrical function of the first kind ,of index* $-v$ and denoted by $J_{-v}(x)$. As you can see in (2.17) and (2.20), if v is not an integer and $v \neq 0$, the solution becomes

$$y(x) = C_1 \ J_v(x) + C_2 \ J_{-v}(x) \quad , \quad v \notin \mathbb{Z} \tag{2.22}$$

2.2.1 Solution of Bessel Differential Equation for $v = 0$:

We can also use Laplace transform in appendix 4 to solve Bessel equation. Now we shall solve Bessel differential equation for $v = 0$ by using Laplace transform. Apply the Laplace transform to other sides of the equation. Thus

$$L[x^2y''] + L[xy'] + L\left[x^2\right] = L[0]$$

We can apply our new formulas for Laplace transform in appendix 4 to the first and third terms on the left. And of course we apply the usual formula for the Laplace transform of the derivative to the second term on the left. The result is

$$-\frac{d}{ds}[s^2Y - s] + \{sY - 1\} + \frac{-dY}{ds} = 0$$

we may simplify this equation to

$$(s^2 + 1)\frac{dY}{ds} = -sY$$

This is a new differential equation and we may solve it by separation of variables. Now

$$\frac{dY}{Y} = -\frac{sds}{s^2 + 1}$$

so

$$\ln Y = -\frac{1}{2}\ln(s^2 + 1) + C$$

Exponenting both sides gives

$$Y = D.\left(1+\frac{1}{s^2}\right)^{-1/2}$$

We call the Binomial expansion :

$$(1+z)^a = 1 + az + \frac{a(a-1)}{2!}z^2 + ... + \frac{a(a-1)...(a-n+1)}{n!}z^n + ... \tag{2.23}$$

We apply this formula to the second term on the right of equation (2.23). Thus

$$\begin{aligned} Y &= \frac{D}{s}.\left(1 - \frac{1}{2}.\frac{1}{s^2} + \frac{1}{2!}.\frac{1}{2}.\frac{3}{2}.\frac{1}{s^4} - \frac{1}{3!}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{1}{s^6} + ... + \frac{1.3.5...(2n-1)}{2^n n!}\frac{(-1)^n}{s^{2n}} + ...\right) \\ &= D.\sum_{j=0}^{\infty} \frac{(2j)!}{2^{2j}(j!)^2}.\frac{(-1)^j}{s^{2j+1}} \end{aligned}$$

The good news is that we can now calculate L^{-1} of Y (thus obtaining y) by just calculating the inverse Laplace transform of each term of this series. The result is

$$\begin{aligned} y(x) &= D.\sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j}(j!)^2}.x^{2j} \\ &= D.\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...\right) \end{aligned}$$

Also, if we use (2.17), the solution becomes

$$J_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ... + \frac{(-1)^s}{(s!)^2}$$

So, the series we have adjusted derived defines the celebrate and important Bessel function $J_0(x)$. And also we use again (2.17) , it becomes

$$J_1(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(2)}{s! \Gamma(s+2)} \left(\frac{x}{2}\right)^{1+2s} = \frac{x}{2} - \frac{1}{1!.2!} \frac{x^3}{2!} + \frac{1}{2!.3!} \frac{x^5}{2^5} - ... + \frac{(-1)^s}{s!(s+1)!} \frac{x^{2s+1}}{2^{2s+1}} + ...$$

The relation between the above equations could be summarize as follows,

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

The roots of these equations $J_0(x) = 0$ and $J_1(x) = 0$ could be found by equalizing them to zero. These are power series expansion and by using Strum theory. We can see that each equation involves infinitely many real roots. Since the difference between these roots getting bigger, the results converging to the number π. For this reason the functions $J_0(x)$ ve $J_1(x)$ are called *periodic functions.* Obviously, for a non-integer index the functions $J_v(x)$ and $J_{-v}(x)$ are linearly independent. If $v = n$ is an integer, then $\Gamma(n) = (n-1)!$ and the function $J_n(x)$ can be written in the following form:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+m)!} \left(\frac{x}{2}\right)^{2s+n} \tag{2.24}$$

vanishes; rewriting (2.20) starting from the $(n+1)$th term, the following equations are obtained

$$\begin{aligned}
J_{-n}(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(-n+s+1)} \left(\frac{x}{2}\right)^{2s-n} \\
&= \sum_{s=n}^{\infty} \frac{(-1)^s}{s!\Gamma(-n+s+1)} \left(\frac{x}{2}\right)^{2s-n} \\
&= \frac{(-1)^n}{n!\Gamma(-n+n+1)} \left(\frac{x}{2}\right)^{2n-n} + \frac{(-1)^{n+1}}{(n+1)!\Gamma(-n+n+2)} \left(\frac{x}{2}\right)^{-n+2n+2} + ... \\
&= (-1)^n \left[\frac{(x/2)^n}{0!n!} - \frac{(x/2)^{n+2}}{1!(n+1)!} + \frac{(x/2)^n}{2!(n+2)!} - ... \right] \\
&= (-1)^n \, J_n(x)
\end{aligned} \tag{2.25}$$

As you can see in (2.24); $J_v(x)$ and $J_{-v}(x)$ are linearly dependent when n is an integer. Indeed,

$$y(x) = c_1\ J_v(x) + c_2\ J_{-v}(x\ ) = [c_1 + (-1)^v c_2]\ J_v(x) = CJ_v(x)$$

In case $v = n$ an integer, we need second solution.

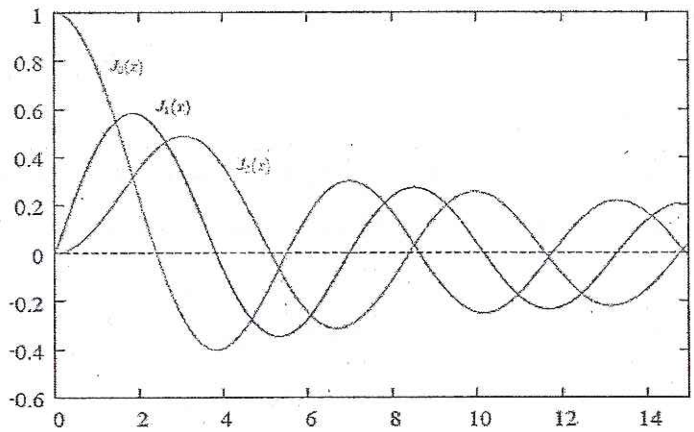

Figure 2.1 : *Bessel function of first kind.*

2.3 Cylindrical Functions of the Second Kind (Neumann Functions)

It has been shown that for a non-integer index the general solution of the Bessel equation can be written in the form

$$y(x) = c_1\ J_v(x) + c_2\ J_{-v}(x)$$

Obviously, in this case the function

$$y(x) = c_3\ J_v(x) + c_4 Z_v(x),$$

where

$$Z_v(x) = B_1 J_v(x) + B_2\ J_{-v}(x),$$

is also a solution of this equation. Here c_1, c_2, c_3, c_4, B_1 and B_2 do not depend the argument x; $B_1 \neq 0$. If we set $B_1 = \cot v\pi$ and $B_2 = -\csc v\pi$, then we obtain a function which has been introduced by Weber and is denoted by $Y_v(x)$ or $N_v(x)$:

$$N_v(x) = Y_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi} \tag{2.26}$$

In the literature this function is often called *the Neumann function* and is sometimes denoted by $N_v(x)$. The function $N_v(x)$ or $Y_v(x)$ is also called *the Bessel or cylindrical function of the second kind of index* v of argument x. For an integer value $v = n$ the right hand side of (2.25) is an indeterminacy of the type $\frac{0}{0}$. In order to find $Y_v(x)$ we remove the indeterminacy by the L'Hospital rule and we define

$$\begin{aligned} Y_n(x) &= \lim_{v\to n} \frac{\frac{\partial}{\partial v}\left[J_v(x)\cos v\pi - J_{-v}(x)\right]}{\frac{\partial}{\partial v}\sin v\pi} \\ &= \lim_{v\to n} \frac{\cos v\pi \frac{\partial J_v(x)}{\partial v} - \pi \sin v\pi J_v(x) - \frac{\partial J_{-v}(x)}{\partial v}}{\pi\cos v\pi} \\ &= \frac{1}{\pi}\lim_{v\to n}\left\{\frac{\partial J_v(x)}{\partial v} - (-1)^n \frac{\partial J_{-v}(x)}{\partial v}\right\} \end{aligned} \tag{2.27}$$

Denoting the derivative of $\ln\Gamma(t)$ by $\Psi(t)$ and using by (2.17), the followings are obtained

$$
\begin{aligned}
\frac{\partial J_v(x)}{\partial v} &= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{x}{2}\right)^{2p}\left[\frac{\left(\frac{x}{2}\right)^v \ln\frac{x}{2}\Gamma(v+p+1)-\left(\frac{x}{2}\right)^v\Gamma'(v+p+1)}{\Gamma(v+p+1)\Gamma(v+p+1)}\right] \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(v+p+1)}\left(\frac{x}{2}\right)^{2p+v}\left[\ln\frac{x}{2}-\frac{\Gamma'(v+p+1)}{\Gamma(v+p+1)}\right] \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(v+p+1)}\left(\frac{x}{2}\right)^{2p+v}\left[\ln\frac{x}{2}-\Psi(v+p+1)\right] \\
&\rightarrow \sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2p+n}\left[\ln\frac{x}{2}-\Psi(n+p+1)\right] \qquad (2.28)
\end{aligned}
$$

where $\frac{\Gamma'(v+p+1)}{\Gamma(v+p+1)} = \Psi(v+p+1)$.

$$
\frac{\Gamma'(v+p+1)}{\Gamma(v+p+1)} = \Psi(n+p+1)
$$

When n is positive, we have to consider separately the first n terms of the series for $\frac{\partial J_{-v}(x)}{\partial v}$ since $\Gamma(-v+p+1)$ and $\Psi(-v+p+1)$ have poles at $v=n$ when p takes the values $1,2,...,n-1$. This difficulty does not arise when n is zero when $p=0,1,2,...,n-1$, so the derivative of this equation becomes

$$
\frac{\partial}{\partial v}\left[(-1)^p\left(\frac{x}{2}\right)^{-v+2p}\frac{1}{p!\Gamma(-v+p+1)}\right] =
$$

By using (4.12) in appendix 1 ,

$$
\begin{aligned}
&= \frac{\partial}{\partial v}\left[\frac{\left(\frac{x}{2}\right)^{-v+2p}\sin v\pi\Gamma(v-p)}{\pi p!}\right] \\
&= \left(\frac{x}{2}\right)^{-v+2p}\frac{\Gamma(v-p)}{\pi p!}\left[-\sin v\pi\ln\frac{x}{2}+\pi\cos v\pi+\sin v\pi+\Psi(v-p)\right] \\
&\rightarrow (-1)^n\left(\frac{x}{2}\right)^{-n+2p}\frac{(n-p-1)!}{p!}
\end{aligned}
$$

as $v\rightarrow n$. Treating the terms for which $p\geq n$ in the straight forward way, we see that

$$(-1)^n \frac{\partial J_{-v}(x)}{\partial v} \to \sum_{p=0}^{n-1} \left(\frac{x}{2}\right)^{-n+2p} \frac{(n-p-1)!}{p!} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{2s+n} \left[\ln\frac{x}{2} - \Psi(s+1)\right]$$

as $v \to n$. Hence, when n is positive integer, Neumann function can be expanded as a series of the form

$$\begin{aligned} Y_n(x) &= \frac{1}{\pi}\sum_{p=0}^{\infty}(-1)^p(\frac{x}{2})^{2p+n}\left\{2\ln\frac{x}{2} - \Psi(p+1) - \Psi(n+p+1)\right\} - \\ &\quad -\frac{1}{\pi}\sum_{p=0}^{n-1}\left(\frac{x}{2}\right)^{-n+2p}\frac{(n-p-1)!}{p!} \end{aligned} \tag{2.29}$$

This formula also holding when $n = 0$, the terms in the second line being omitted. It is sometimes, useful to modify this formula by writing

$$\Psi(1) = -\gamma,\ \Psi(p+1) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p} - \gamma$$

and also we will use notation

$$H(p) = \sum_{k=1}^{\infty} \frac{1}{k} \tag{2.30}$$

where γ is Euler's constant, defined by

$$\begin{aligned} \gamma &= \lim_{p\to\infty} [H(p) - \ln p] \\ &= \lim_{p\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p} - \ln p\right] \\ &= 0.5772156 \end{aligned}$$

As a result, also we obtain the formula

$$
\begin{aligned}
Y_n(x) &= \frac{2}{\pi}(\ln\frac{x}{2}+\gamma)J_v(x) - \frac{2}{\pi}\sum_{p=0}^{n-1}\left(\frac{x}{2}\right)^{-n+2p}\frac{(n-p-1)!}{p!} + \\
&+\frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}[H(p)+H(p+n)]}{p!(v+p)!}\left(\frac{x}{2}\right)^{2p+n} \qquad (2.31)
\end{aligned}
$$

The Neumann function $Y_v(x)$ and the function $J_v(x)$ form a fundamental system of solutions to the Bessel equation for any, including integer, value of index. Namely we set the solutions for all values of v

$$
y(x) = C_1 J_v(x) + C_2 Y_v(x), \qquad (2.32)
$$

where $C_1$and C_2 are arbitrary constants.

Let us deduce formula (2.30). It is sufficient to obtain this formula for the special case $n = 0$. From (2.27), we have

$$
\begin{aligned}
Y_0(x) &= \frac{1}{\pi}\lim_{v\to 0}\left\{\frac{\partial J_v(x)}{\partial v} - \frac{\partial J_{-v}(x)}{\partial v}\right\}_{v=0} \\
Y_0(x) &= \frac{1}{\pi}\left[\frac{\partial J_v(x)}{\partial v}\right]_{v=0} - \left[\frac{\partial J_{-v}(x)}{\partial v}\right]_{v=0}
\end{aligned}
$$

while, because $J_v(x)$ is monogenic function of v at $v = 0$, we have

$$
\begin{aligned}
Y_0(x) &= \frac{2}{\pi}\left[\frac{\partial J_v(x)}{\partial v}\right]_{v=0} \\
\left[\frac{\partial J_{-v}(x)}{\partial v}\right]_{v=0} &= \left[\frac{\partial J_v(x)}{\partial(-v)}\right]_{v=0} = -\left[\frac{\partial J_v(x)}{\partial v}\right]_{v=0}
\end{aligned}
$$

and hence it follows that using by (2.28), we obtain

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p}\left[\ln\frac{x}{2}-\Psi(p+1)\right] \\ &= \frac{2}{\pi}\left[\ln\frac{x}{2}J_0(x)-\sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p}\Psi(p+1)\right] \\ &= \frac{2}{\pi}\left\{\gamma+\ln\frac{x}{2}\right\}J_0(x)-\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}\right] \end{aligned}$$

And using by (2.30), we obtain

$$Y_0(x)=\frac{2}{\pi}\left\{\gamma+\ln\frac{x}{2}\right\}J_0(x)-\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^pH(p)}{p!\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p} \qquad (2.33)$$

For small $x>0$, $Y_0(x)$ behaves like $\ln x$ and tends to $-\infty$ when x$\to 0$. $Y_0(x)$ is called *the Bessel function of the second kind of index* 0. $Y_0(x)(N_0(x))$ is plotted in figure 2.2.

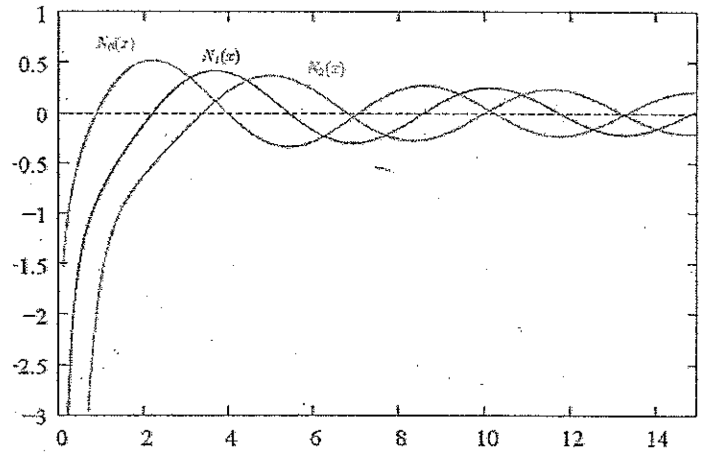

Figure 2.2 : Bessel functions of second kind.

2.4 Cylindrical Functions of the Third Kind (Hankel functions)

Any linear combination of the solutions obtained in section 2.2 is also an integral of the Bessel equation. Consider these functions

$$\left\{ \begin{array}{l} H_v^{(1)} = J_v(x) + iY_v(x) \\ H_v^{(2)} = J_v(x) - iY_v(x) \end{array} \right\} \quad (2.34)$$

which are called *cylindrical functions of the third kind.* They are also called *Hankel functions of the first and second kind*, respectively. Since the functions of the third kind are linear combination of

$$\begin{aligned} H_v^{(1)} &= J_v(x) + iY_v(x) = i\frac{e^{-v\pi i}J_v(x) - J_{-v}(x)}{\sin v\pi} \\ H_v^{(2)} &= J_v(x) - iY_v(x) = -i\frac{e^{-v\pi i}J_v(x) - J_{-v}(x)}{\sin v\pi} \end{aligned} \quad (2.35)$$

apparently,

$$H_v^{-(1)} = H_v^{(2)}$$

These functions are linearly independent solutions of the Bessel equations: Specially when ($x \to \infty$) and since the simplifications of the definitions of asymtotic behaviours it can be used in applied mathematics and most of its related fields. In the above equation v denotes the degrees of the Hankel functions. Adding $H_v^{(1)}$ and $H_v^{(2)}$ side by side, we found

$$\begin{aligned} H_v^{(1)} + H_v^{(2)} &= 2J_v(x) \\ J_v(x) &= \frac{1}{2}\left[H_v^{(1)} + H_v^{(2)}\right] \end{aligned} \quad (2.36)$$

Subtructing (2.35) side by side, we obtain

$$\begin{aligned} H_v^{(1)} + H_v^{(2)} &= 2J_v(x) \\ J_v(x) &= \frac{1}{2}\left[H_v^{(1)} + H_v^{(2)}\right] \end{aligned} \tag{2.37}$$

Hence the first and second kind Hankel Functions is multiplied by $e^{iv\pi}$ and $e^{-iv\pi}$ respectively and then adding them side by side,the followings were obtained

$$\begin{aligned} e^{iv\pi}\, H_v^{(1)} + e^{-iv\pi} H_v^{(2)} &= 2J_{-v}(x) \\ J_{-v}(x) &= \frac{1}{2}\left[\, e^{iv\pi}\, H_v^{(1)} + e^{-iv\pi} H_v^{(2)}\right] \end{aligned} \tag{2.38}$$

2.5 Cylindrical Functions of a Pure Imaginary Argument

Consider the Bessel equation

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0$$

After changing the variable by the formula $x = ix$, we obtain the equation

$$\begin{aligned} (ix)^2 \left(-\frac{d^2y}{dx^2}\right) + \frac{(ix)}{i}\left(\frac{dy}{dx}\right) + \left((ix)^2 - v^2\right) y &= 0 \\ -x^2 \left(-\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right) + (-x^2 - v^2)y &= 0 \\ x^2 \left(\frac{d^2y}{dx^2}\right) + x\frac{dy}{dx} - (x^2 + v^2)y &= 0 \\ x^2 y'' + xy' - \left(x^2 + v^2\right) y &= \end{aligned} \tag{2.39}$$

which is called *Modified Bessel Equation*. This equation has regular singular point at $x = 0$. So when we solve this equation by using Frobenious method, we find the solutions for the roots $m_1 = v$ and $m_2 = -v$. Firstly we obtain the solution for $m_1 = v$,

$$I_v(x) = \sum_{p=0}^{\infty} \frac{1}{p!\Gamma(v+p+1)} \left(\frac{x}{2}\right)^{v+2p}$$

If we modified Bessel Equations as follows

$$t = ix \quad (i = \sqrt{-1})$$

we obtain

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$

This shows us that the Bessel functions and Modifed Bessel functions are the same equations. This could be seen below,

$$\begin{aligned} J_v(t) &= J_v(ix) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{ix}{2}\right)^{v+2p} \\ &= i^v \sum_{p=0}^{\infty} \frac{1}{p!\Gamma(v+p+1)} \left(\frac{x}{2}\right)^{v+2p} \\ &= i^v I_v(x) \end{aligned}$$

or

$$I_v(x) = i^{-v} J_v(ix) \tag{2.40}$$

$$I_v(x) = e^{-v\pi i/2} J_v(ix) \tag{2.41}$$

Since v can not defined as integer $(v \notin \mathbb{Z})$

$$I_{-v}(x) = \sum_{p=0}^{\infty} \frac{1}{p!\Gamma(-v+p+1)} \left(\frac{x}{2}\right)^{-v+2p} \tag{2.42}$$

is the second linearly independent solution of the modified Bessel equation. When $x = 0$ this function becomes infinity. In general the Second kind of Bessel functions could be defined as follows:

$$K_v(x) = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin v\pi}, \qquad v \notin \mathbb{Z} \tag{2.43}$$

In particular as the second integral of equation (2.39), usually, the following function is taken

$$K_v(x) = \frac{1}{2}\pi i e^{v\pi i/2} H_v^{(1)}(ix), \tag{2.44}$$

This function is called *Macdonald function*; for integer $v = n$ the power series expansion of this function has the form

$$K_n(x) = \lim_{v \to n} K_v(x), \qquad n \in \mathbb{Z} \tag{2.45}$$

$$\begin{aligned} K_n(x) &= (-1)^{n+1} I_n(x) \ln\frac{x}{2} + \frac{1}{2}\sum_{p=0}^{n-1} (-1)^p \left(\frac{x}{2}\right)^{-n+2p} \frac{(n-p-1)!}{p!} + \\ &+ (-1)^{n+1} \frac{1}{2} \sum_{p=0}^{\infty} \frac{(x/2)^{n+2p}}{p(n+p)!} \left[2\gamma - \sum_{k=1}^{p+n} \frac{1}{k} - \sum_{k=1}^{p} \frac{1}{k}\right] \end{aligned} \tag{2.46}$$

where, as before γ is Euler constant. This formula can easily be obtained, if we introduce $J_v(x)$ and $Y_v(x)$ into (2.44) with the help of the first of formulae (2.34) and afterwards change the argument x by ix in formulae (2.17) and (2.31) and express $J_n(ix)$ via $I_n(x)$ using formula (2.41). Note that the formulae given above for the modified functions can be applied, in this case the modified function $I_v(x)$ is defined in the following way:

$$\begin{aligned} I_v(x) &= e^{-v\pi i/2} J_v(xe^{\pi i/2}), \qquad -\pi < \arg x \le \frac{\pi}{2}, \\ I_v(x) &= e^{3v\pi i/2} J_v(xe^{-3\pi i/2}), \qquad \frac{\pi}{2} < \arg x \le \pi. \end{aligned}$$

For large values of the argument x the functions $I_n(x)$ and $K_n(x)$ behave similarly to the exponential function of a real positive and real negative argument, respectively. Therefore, sometimes the functions $e^x K_n(x)$ and $e^{-x} I_n(x)$ are tabulated. Consequently, the general solution of the Modified Bessel equation is become when v is not integer

$$y(x) = c_1 I_v(x) + c_2 I_{-v}(x) \tag{2.47}$$

and the solution is become when v is integer

$$y(x) = c_1 I_v(x) + c_2 K_v(x) \tag{2.48}$$

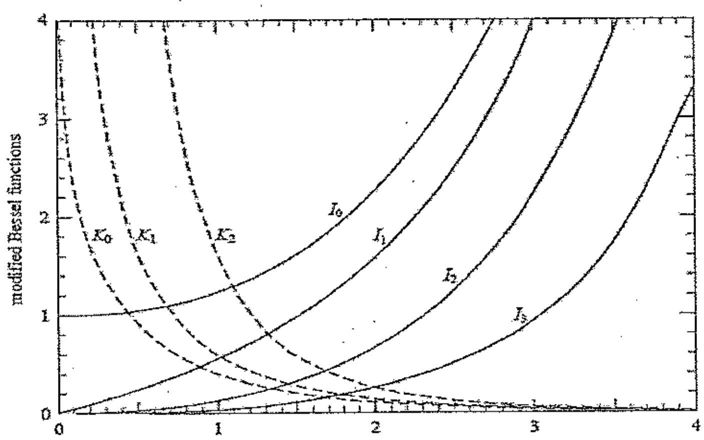

Figure 2.3 : *Modified Bessel functions* $I_0(x)$ *through* $I_3(x)$, $K_0(x)$ *through* $K_2(x)$.

2.6 Formulae of Differentiation, Recurrence Relations

Dividing (2.17) by x^v, we have

$$\frac{J_v(x)}{x^v} = \frac{1}{2^v} \sum_{p=0}^{\infty} \frac{(-1)^p)}{p!\Gamma(v+p+1)} \left(\frac{x}{2}\right)^{2p}$$

differentiating this equality with respect to the argument x, we obtain the relation

$$\frac{d}{dx}\frac{J_v(x)}{x^v} = \frac{1}{2^v}\sum_{p=1}^{\infty}\frac{(-1)^p}{(p-1)!\Gamma(v+p+1)}\left(\frac{x}{2}\right)^{2p-1} = -\frac{J_{v+1}(x)}{x^v}$$

which can be rewritten in the following form:

$$\frac{1}{x}\frac{d}{dx}\frac{J_v(x)}{x^v} = -\frac{J_{v+1}(x)}{x^{v+1}} \tag{2.49}$$

Similarly, we can obtain the formula

$$\frac{d}{x\,dx}\left[x^v J_v(x)\right] = x^{v-1}J_{v-1}(x) \tag{2.50}$$

After differentiating the left-hand side of formulae (2.49) and (2.50) and simplificating the expressions obtained, we have the equalities

$$\frac{d}{dx}J_v(x) = -J_{v+1}(x) + \frac{vJ_v(x)}{x} \tag{2.51}$$

$$\frac{d}{dx}J_v(x) = J_{v-1}(x) - \frac{vJ_v(x)}{x} \tag{2.52}$$

which imply the following recurrence relations:

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2vJ_v(x)}{x} \tag{2.53}$$

$$J_{v-1}(x) - J_{v+1}(x) = 2\frac{d}{dx}J_v(x) \tag{2.54}$$

In the equation (2.17) substituting x with mx, we obtain

$$J_v(mx) = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(v+p+1)}\left(\frac{mx}{2}\right)^{v+2p}$$

Hence multiply by x^v

$$
\begin{aligned}
x^v\ J_v(mx) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{mx}{2}\right)^{v+2p} x^v \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{m}{2}\right)^{v+2p} x^{2(p+v)}
\end{aligned}
$$

and then differentiate for the both side,

$$
\begin{aligned}
\frac{d}{dx}\left(x^v\ J_v(mx)\right) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{m}{2}\right)^{v+2p} 2(p+v)\ x^{2(p+v)-1} \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p 2(p+v)}{p!\Gamma(v+p)(p+v)} \left(\frac{m}{2}\right)^{v+2p} x^{2p+2v-1} \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p)} \left(\frac{mx}{2}\right)^{v+2p-1} x^v m \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma((v-1)+p+1)} \left(\frac{mx}{2}\right)^{(v-1)+2p}
\end{aligned}
$$

So, we obtain

$$
\frac{d}{dx}\left[x^v\ J_v(mx)\right] = mx^v J_{v-1}(mx) \qquad (2.55)
$$

$$
\frac{d}{dx}\left[x^{-v}\ J_v(mx)\right] = -mx^{-v} J_{v+1}(mx) \qquad (2.56)
$$

By using (2.55) and (2.56), we get

$$
\frac{d}{dx}\left[J_v(mx)\right] = mJ_{v-1}(mx) - \frac{v}{x} J_v(mx) \qquad (2.57)
$$

$$
\frac{d}{dx}\left[J_v(mx)\right] = -mJ_{v+1}(mx) + \frac{v}{x} J_v(mx) \qquad (2.58)
$$

One can replace $J_v(x)$ in all these formulae by any of the functions: $Y_v(x)$, $H_v^{(1)}(x)$, $H_v^{(2)}(x)$. Repeatedly differentiating formulae (2.49) and (2.50), one can obtain

$$\left(\frac{d}{x\,dx}\right)^m [x^v J_v(x)] = x^{v-m} J_{v-m}(x) \tag{2.59}$$

$$\left(\frac{d}{x\,dx}\right)^m [x^{-v} J_v(x)] = (-1)^m x^{-v-m} J_{v+m}(x) \tag{2.60}$$

For the modified cylindrical functions, we have the following formulae of differentiation, which are obtained as a result of the change of the argument x by ix and representation of the functions $J_v(x)$ and $H_v^{(1)}(x)$ via the functions $I_v(x)$ and $K_v(x)$:

$$\frac{d}{dx} I_v(x) = \frac{1}{2}[I_{v-1}(x) + I_{v+1}(x)] \tag{2.61}$$

$$\frac{d}{dx} K_v(x) = -\frac{1}{2}[K_{v-1}(x) + K_{v+1}(x)] \tag{2.62}$$

The corresponding recurrence relations have the form

$$I_{v-1}(x) - I_{v+1}(x) = \frac{2v}{x} I_v(x) \tag{2.63}$$

$$K_{v-1}(x) - K_{v+1}(x) = -\frac{2v}{x} K_v(x) \tag{2.64}$$

2.7 Cylindrical Functions with a Half-integer Index

Setting the index $v = 1/2$ in the expansion $J_v(x)$ in (2.17) we obtain

$$J_{1/2}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma\left(\frac{3}{2}+p\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2p}$$

Use the property of Gamma functions we obtain

$$\Gamma\left(p+1+\frac{1}{2}\right) = \Gamma\left(1+\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)...\left(\frac{1}{2}+p\right)$$

since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ the above Gamma Function becomes

$$\Gamma\left(p+1+\frac{1}{2}\right) = \sqrt{\pi}\frac{1.2.3...(2p+1)}{2.2.2...2}$$

If the numerator and denominator of the above equation multiplied by $2.4.6..(2p) = 2^p.p!$ we found the following equation;

$$\Gamma\left(p+1+\frac{1}{2}\right) = \sqrt{\pi}\frac{(2p+1)!}{2^{2p+1}p!}$$

If we substitute the above obtained function to $J_{1/2}(x)$ we obtain;

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{p=0}^{\infty}\frac{(-1)^p x^{2p+1}}{(2p+1)!} = \sqrt{\frac{2}{\pi x}}\sin x \tag{2.65}$$

Differentiating equation of (2.61), we obtain

$$J'_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x \tag{2.66}$$

The we use formula (2.52); setting $v = 1/2$, one can easily obtain

$$xJ'_{1/2}(x) + \frac{1}{2}J_{1/2}(x) = xJ_{-1/2}(x) \tag{2.67}$$

then we substitute (2.65) and (2.66) in (2.67), we obtain

$$\begin{aligned}
x\left(\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x\right) + \frac{1}{2}\left(\sqrt{\frac{2}{\pi x}}\sin x\right) &= xJ_{-1/2}(x) \\
\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x + \frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin x &= J_{-1/2}(x) \\
J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}}\cos x \qquad (2.68)
\end{aligned}$$

Using the recurrence relations one can find the Bessel function for any index of the form $n+1/2$, where n is integer, and prove that for any positive integer n the following formulae hold:

$$J_{n+1/2} = \frac{(-1)^n(2x)^{n+1/2}}{\sqrt{\pi}}\frac{d^n}{(dx^2)^n}\left(\frac{\sin x}{x}\right) \quad (2.69)$$

$$J_{-n-1/2} = \frac{(-1)^n(2x)^{n+1/2}}{\sqrt{\pi}}\frac{d^n}{(dx^2)^n}\left(\frac{\cos x}{x}\right) \quad (2.70)$$

In the same way one can obtain formulae similar to (2.65), (2.69) and (2.70) for the modified functions; in particular,

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, \qquad K_{1/2}(x) = \sqrt{\frac{2}{\pi x}}e^{-x} \quad (2.71)$$

2.8 Wronskian Determinant

If $y_1 = J_v(x)$ and $y_2 = J_{-v}(x)$ are linearly independent solutions of the Bessel equation, Wronskian determinant must be not zero. Let us find the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Hence the Wronskian for $J_v(x)$ and $J_{-v}(x)$ becomes

$$\begin{aligned} W(y_1, y_2) &= W\left[J_v(x), J_{-v}(x)\right] \\ &= \begin{vmatrix} J_v(x) & J_{-v}(x) \\ J_v'(x) & J_{-v}'(x) \end{vmatrix} \\ &= J_v(x)J_{-v}'(x) - J_{-v}(x)\, J_v'(x) \end{aligned} \quad (2.72)$$

The functions $J_v(x)$ ve $J_{-v}(x)$ are the solutions of the Bessel equation so they could be verified the (2.9) equation. If we substitute the functions $J_v(x)$ ve $J_{-v}(x)$ into the equation (2.9), we obtain

$$J''_{-v}(x) + \frac{1}{x}J'_{-v}(x) + \left(1 - \frac{v^2}{x^2}\right) J_{-v}(x) = 0 \tag{2.73}$$

$$J''_{v}(x) + \frac{1}{x}J'_{v}(x) + \left(1 - \frac{v^2}{x^2}\right) J_{v}(x) = 0 \tag{2.74}$$

If (2.73) and (2.74) multiplied by $J_v(x)$ and $J_{-v}(x)$ respectively, we obtain

$$J''_{-v}(x)J_v(x) + \frac{1}{x}J'_{-v}(x)J_v(x) + \left(1 - \frac{v^2}{x^2}\right) J_{-v}(x)J_v(x) = 0$$

$$J''_{v}(x)J_{-v}(x) + \frac{1}{x}J'_{v}(x)J_{-v}(x) + \left(1 - \frac{v^2}{x^2}\right) J_{v}(x)J_{-v}(x) = 0$$

If the above equations subtructed side by side, we obtain

$$J_v(x)J''_{-v}(x) - J_{-v}(x)\ J''_v(x) + \frac{1}{x}\left[J_v(x)J'_{-v}(x) - J_{-v}(x)\ J'_v(x)\right] = 0$$

$$\frac{d}{dx}\left[J_v(x)J'_{-v}(x) - J_{-v}(x)\ J'_v(x)\right] + \frac{1}{x}\left[J_v(x)J'_{-v}(x) - J_{-v}(x)\ J'_v(x)\right] = 0 \tag{2.75}$$

So we have by substituting

$$J_v(x)J'_{-v}(x) - J_{-v}(x)\ J'_v(x) = w,$$

$$\frac{dw}{dx} + \frac{w}{x} = 0$$

Using by separation of variables we get

$$w(x) = \frac{C(v)}{x} \tag{2.76}$$

Suppose that the equation has an non-integer index v and let us find the Wronskian

$$w(J_v(x), J_{-v}(x)) = \frac{C(v)}{x} \tag{2.77}$$

$$C(v) = x\left[J_v(x)J'_{-v}(x) - J_{-v}(x)\ J'_v(x)\right] \tag{2.78}$$

The value constant $C(v)$ can easily be determined, if we pass to the limit as $x \longrightarrow 0$ in formula (2.72) and use the expansions of the Bessel functions obtain section 2.2. Note that if v is non-integer, then by using equations (2.17) and (2.20) we obtain

$$\begin{aligned}
J_v(x) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{x}{2}\right)^{v+2p} \\
&= \left(\frac{x}{2}\right)^v \frac{1}{\Gamma(v+1)} + \sum_{p=1}^{\infty} \frac{(-1)^p}{p!\Gamma(v+p+1)} \left(\frac{x}{2}\right)^{v+2p} \\
J_{-v}(x) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(-v+p+1)} \left(\frac{x}{2}\right)^{2p-v} \\
&= \left(\frac{x}{2}\right)^{-v} \frac{1}{\Gamma(-v+1)} + \sum_{p=1}^{\infty} \frac{(-1)^p}{p!\Gamma(-v+p+1)} \left(\frac{x}{2}\right)^{2p-v}
\end{aligned}$$

and also we obtain

$$J_v(x) = \left(\frac{x}{2}\right)^v \frac{1}{\Gamma(v+1)} \left(1 + O(x^2)\right) \tag{2.79}$$

$$J_v'(x) = \left(\frac{x}{2}\right)^{v-1} \frac{1}{2\Gamma(v)} \left(1 + O(x^2)\right) \tag{2.80}$$

and similarly, we obtain

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \frac{1}{\Gamma(-v+1)} \left(1 + O(x^2)\right) \tag{2.81}$$

$$J_{-v}'(x) = \left(\frac{x}{2}\right)^{-v-1} \frac{1}{2\Gamma(-v)} \left(1 + O(x^2)\right) \tag{2.82}$$

as $x \longrightarrow 0$, where $O(x^2)$ denotes a quantity, whose ratio to x^2 is bounded as $x \longrightarrow 0$. If we substitute (2.79), (2.80), (2.81) and (2.82) in (2.78), we get

$$C(v) = x\left[\left(\frac{x}{2}\right)^{v}\frac{1}{\Gamma(v+1)}\left(1+O(x^2)\right)\left(\frac{x}{2}\right)^{-v-1}\frac{1}{2\Gamma(v)}\left(1+O(x^2)\right)\right]+$$
$$+x\left[-\left(\frac{x}{2}\right)^{-v}\frac{1}{\Gamma(-v+1)}\left(1+O(x^2)\right)\left(\frac{x}{2}\right)^{v-1}\frac{1}{2\Gamma(v)}\left(1+O(x^2)\right)\right]$$

as $x \to 0$ $O(x^2) = 0$, we get

$$C(v) = \left[\frac{1}{\Gamma(v+1)}\frac{1}{\Gamma(v)} - \frac{1}{\Gamma(-v+1)}\frac{1}{\Gamma(v)}\right] \tag{2.83}$$

By using the formula of the Gamma function in (2.83) which is $\Gamma(v)\Gamma(-v+1) = \frac{\pi}{\sin v\pi}$, we get

$$C(v) = -\frac{\sin v\pi}{\pi} - \frac{\sin v\pi}{\pi} = -\frac{2\sin v\pi}{\pi} \tag{2.84}$$

By substituting (2.84) in (2.77), we obtain

$$W\left[J_v(x), J_{-v}(x)\right] = -\frac{2\sin v\pi}{\pi x} \tag{2.85}$$

Since $\sin v\pi$ is not zero, because v is not integer. Therefore, $W\left[J_v(x), J_{-v}(x)\right] \neq 0$. Since the functions $J_v(x)$ and $J_{-v}(x)$ are linearly independent, the solution is fundamental system. (These functions are linearly independent and hence these can be create a solution system.) Just in the same way, using relations (2.26) between the Bessel functions of the first and second kind, one can obtain

$$J_v(x)Y_v'(x) - Y_v(x)\ J_v'(x) = \frac{2}{\pi x} \tag{2.86}$$

As we can see above wronskian never becomes zero. In addition the functions $J_v(x)$ ve $Y_v(x)$ are always linearly independent. For this reason they create a solution system. The Wronskian determinant for the functions $I_v(x)$ and $K_v(x)$ is equal to

$$I_v(x)K'_v(x) - K_v(x)\ I'_v(x) = -\frac{1}{x} \tag{2.87}$$

The functions $I_v(x)$ ve $K_v(x)$ are also linearly independent and they create the solution system too. And for $J_v(x)$, $H_v^{(1)}(x)$ and $H_v^{(1)}(x)$, $H_v^{(2)}(x)$ we have, respectively,

$$J_v(x)\frac{dH_v^{(1)}(x)}{dx} - H_v^{(1)}(x)\frac{d\ J_v(x)}{dx} = 2\frac{i}{\pi x} \tag{2.88}$$

$$H_v^{(1)}(x)\frac{dH_v^{(2)}(x)}{dx} - H_v^{(2)}(x)\frac{dH_v^{(1)}(x)}{dx} = -\frac{4i}{\pi x} \tag{2.89}$$

2.9 Orthogonality and Norm of Bessel Functions

Let α be a real constant and from $u = J_v(\alpha x) \Rightarrow u' = J'_v(\alpha x)$, $u'' = J''_v(\alpha x)$ we obtain,

$$x^2u'' + xu' + \left(\alpha^2x^2 - v^2\right)u = (\alpha x)^2J''_v(\alpha x) + (\alpha x)J'_v(\alpha x) + \left[(\alpha x)^2 - v^2\right]J_v(\alpha x) = 0 \tag{2.90}$$

The function $J_v(x)$ is a Bessel function, by using (αx) it satisfies the Bessel differential equation. Breifly since α be a real constant with the function $u = J_v(\alpha x)$ together satifies the equation

$$x^2u'' + xu' + \left(\alpha^2x^2 - v^2\right)u = 0.$$

Let $\alpha_1, \alpha_2, ..., \alpha_n, ...$ be the positive roots of the equation $J_v(\alpha) = 0$, the following function

$$f_n(x) = \sqrt{x}J_v(\alpha x) \quad , \quad n = 1, 2, ...$$

on the interval $(0, 1)$ represents the perpendicular system. Let us prove that:

$$(f_m, f_n) = \int_0^1 f_m(x)\, f_n(x)dx = \int_0^1 x J_v(\alpha_m x) J_v(\alpha_n x)dx = 0 \ , \quad m \neq n$$

Suppose $u = J_v(\alpha x)$, $w = J_v(\beta x)$. From the equation (2.90), the following equations could be obtained

$$\begin{aligned} x^2u'' + xu' + \left(\alpha^2x^2 - v^2\right) u &= 0 \\ x^2w'' + xw' + \left(\beta^2x^2 - v^2\right) w &= 0 \end{aligned}$$

then divide by x, we will obta in the followings

$$\begin{aligned} xu'' + u' + \left(\alpha^2 - \frac{v^2}{x^2}\right) u &= 0 \\ xw'' + w' + \left(\beta^2 - \frac{v^2}{x^2}\right) w &= 0 \end{aligned}$$

If the first equation above is multiply by w and the second one is multiplied by u and then subtract them from each other we obtain

$$x(uw'' - wu'') + (uw' - wu') + x(\beta^2 - \alpha^2)uw = 0$$

or

$$\frac{d}{dx}\left[x(uw' - wu')\right] + (\beta^2 - \alpha^2)xuw = 0$$

by integration on the interval $[0, 1]$

$$\begin{aligned}(\beta^2-\alpha^2)\int_0^1 xuwdx &= -x(uw'-wu')|_0^1 \\ &= x(wu'-uw')|_0^1\end{aligned}$$

and making substitution $u = J_v(\alpha x)$, $w = J_v(\beta x)$, $u' = \alpha J_v'(\alpha x)$ and $w' = \beta J_v'(\beta x)$ we obtain

$$(\beta^2-\alpha^2)\int_0^1 xJ_v(\alpha x)J_v(\beta x)dx = \alpha J_v'(\alpha)J_v(\beta) - \beta J_v'(\beta)J_v(\alpha) \tag{2.91}$$

This integral is the so-called *Lommel integral.* From (2.91) if we choose $\alpha = \alpha_m$, $\beta = \alpha_n$,for $m \neq n$ $\alpha_m^2 - \alpha_n^2 \neq 0$ and $J_v(\alpha x) = 0$, $J_v(\beta x) = 0$ namely, we obtain

$$\int_0^1 xJ_v(\alpha x)J_v(\beta x)dx = \frac{\alpha J_v'(\alpha_m)J_v(\alpha_n) - \beta J_v'(\alpha_n)J_v(\alpha_m)}{\alpha_n^2-\alpha_m^2} = 0 \tag{2.92}$$

Hence by differentiating both side of (2.92) by β, we obtain

$$\begin{aligned}&2\beta\int_0^1 xJ_v(\alpha x)J_v(\beta x)dx + \quad (\beta^2-\alpha^2)\frac{d}{d\beta}\int_0^1 xJ_v(\alpha x)J_v(\beta x)dx; ? \\ =& \ \alpha J_v'(\alpha x)J_v'(\beta x) - \beta J_v''(\beta)J_v(\alpha)\end{aligned}$$

From the last two equations above when we take $\beta = \alpha = \alpha_n$ and since $J_v(\alpha_n) = 0$,

$$\begin{aligned}\int_0^1 x\left[J_v(\alpha_n x)\right]^2 dx &= \frac{1}{2\alpha_n}\left\{\alpha_n\left[J_v'(\alpha_n)\right]^2 - \alpha_n J_v''(\alpha_n)J_v(\alpha_n) - J_v'(\alpha_n)J_v(\alpha_n)\right\} \\ &= \frac{1}{2}\left[J_v'(\alpha_n)\right]^2 \end{aligned} \tag{2.93}$$

On the other hand if we recall from (2.58)

$$\frac{d}{dx}\left[J_v(mx)\right] = -mJ_{v+1}(mx) + \frac{v}{x}J_v(mx)$$

and by recurrence relation $n = 1$, $x = \alpha_n$ and suppose $J_v(\alpha_n) = 0$, we obtain

$$J'_v(\alpha_n) = -J_{v+1}(\alpha_n)$$

If we substitute the final equation into (2.93), the following equation could be found

$$\int_0^1 x\left[J_v(\alpha_n x)\right]^2 dx = \frac{1}{2}\left[J'_{v+1}(\alpha_n)\right]^2$$

From the final equation since $f_n(x) = \sqrt{x}\, J_v(\alpha_n x)$ the norm of $\|f_n\|$ could be found as,

$$\begin{aligned} \|f_n\|^2 &= \int_0^1 f_n^2(x)dx = \int_0^1 \left[\sqrt{x}J_v(\alpha_n x)\right]^2 dx = \int_0^1 x\left[J_v(\alpha_n x)\right]^2 dx \\ &= \frac{1}{2}J_{v+1}^2(\alpha_n) \\ \|f_n\| &= \frac{1}{\sqrt{2}}J_{v+1}(\alpha_n) \end{aligned}$$

2.10 Bessel Integral and Jacobi Expansion

Now consider the power series expansions of the functions $e^{xt/2}$and $e^{-\frac{x}{2t}}$.Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We obtain

$$e^{xt/2} = \sum_{s=0}^{\infty} \frac{1}{s!}\left(\frac{xt}{2}\right)^s \quad and \;\; e^{-\frac{x}{2t}} = \sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{x}{2t}\right)^r$$

The product of these expansions gives

$$F(x,t) = e^{\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s x^{r+s} t^{r-s}}{r!s!2^{r+s}}$$

Setting $s = m$, $r = m+n$ and recalling that $\frac{1}{\Gamma(m+n+1)} = 0$ for $n < -m$, we rewrite in the following form

$$\begin{aligned} F(x,t) &= e^{\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]} = \sum_{m=0}^{\infty}\sum_{n=-m}^{\infty} \frac{(-1)^m x^{n+2m} t^n}{\Gamma(m+n+1)\Gamma(m+1)2^{n+2m}} \\ &= \sum_{m=1}^{\infty}\sum_{n=-m}^{-1} \frac{(-1)^m x^{n+2m} t^n}{\Gamma(m+n+1)\Gamma(m+1)2^{n+2m}} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m x^{n+2m} t^n}{\Gamma(m+n+1)\Gamma(m+1)2^{n+2m}} \\ &= \sum_{n=-\infty}^{-1}\sum_{m=-n}^{\infty} \frac{(-1)^m x^{n+2m} t^n}{\Gamma(m+n+1)\Gamma(m+1)2^{n+2m}} \\ &\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m} t^n}{\Gamma(m+n+1)\Gamma(m+1)2^{n+2m}} \end{aligned} \tag{2.94}$$

For $n \geq 0$, the coefficients of t^n

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+n+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{n+2m} = J_n(x) \quad , \quad n = 0,1,2,... \tag{2.95}$$

and also for $n \in \mathbb{Z}^-$, with location of $k = -n$ we obtain the coefficients of t^n

$$\begin{aligned} &\sum_{m=-n}^{\infty} \frac{(-1)^m}{\Gamma(m+n+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{n+2m} \\ &= \sum_{m=k}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(-k+m+1)} \left(\frac{x}{2}\right)^{2m-k} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+k}}{\Gamma(m+1)\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k} \\ &= (-1)^k J_k(x) \\ &= (-1)^{-n} J_{-n}(x) \\ &= (-1)^{-n}\frac{(-1)^{2n}}{(-1)^{2n}} J_{-n}(x) \end{aligned} \tag{2.96}$$

By substituting (2.95) and (2.96) in (2.94), we obtain

$$F(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{-1} J_n(x)t^n + \sum_{n=0}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{2.97}$$

which is called *the generating function of* $J_n(x)$. Let us study the integral:

$$A_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\theta)-in\theta}d\theta$$

To evaluate this integral, we use the Taylor expansion of the exponent:

$$e^{ix\sin(\theta)} = \sum_{p=0}^{\infty}\frac{1}{p!}(ix\sin(\theta))^p = \sum_{p=0}^{\infty}\frac{1}{p!}\left(\frac{x}{2}\right)^p\left(e^{i\theta}-e^{-i\theta}\right)^p$$

Now, notice that the integral:

$$I_{p,\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{i\theta}-e^{-i\theta}\right)^p e^{-in\theta}d\theta = 0 \;\; if \; p<0 \tag{2.98}$$

Then, we denote $p = n+q$. The integrand in (2.98) can be presented in the form:

$$\frac{1}{2\pi}\left(e^{i\theta}-e^{-i\theta}\right)^{n+q}e^{-in\theta} = \left(1-e^{-2i\theta}\right)^n\left(e^{i\theta}-e^{-i\theta}\right)^q$$

Suppose that q is odd ($q = 2k+1$). All terms in the first paretheses are even powers of $e^{-i\theta}$, while all terms in the second paretheses are odd powers (positive or negative) on $e^{-i\theta}$. As a result, the integrand is a linear combination of odd powers of $e^{-i\theta}$. Thus the integral is zero, and we can put $q = 2k$. We obtain the following intermediate result:

$$A_n(x) = \left(\frac{x}{2}\right)^n\sum_{n=0}^{\infty}\frac{1}{(n+2k)!}\left(\frac{x}{2}\right)^k I_{k,\pi} \tag{2.99}$$

Where

$$I_{k,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{i\theta} - e^{-i\theta}\right)^{n+2k} e^{-in\theta} d\theta$$

To calculate $I_{k,n}$ we use the binomial expansion in the paretheses. In this expansion, we are interested only in the single term proportional to $e^{in\theta}$. All other terms after multiplication to (2.99) and integration over θ are cancelled. Hence,

$$\begin{aligned} \left(e^{i\theta} - e^{-i\theta}\right)^{n+2k} &\approx \frac{(n+2k)!}{k!(n+k)!}\left(e^{in\theta}\right)^{n+k}\left(-e^{-i\theta}\right)^{k} = \frac{(-1)^k(n+2k)!}{k!(n+k)!}e^{in\theta} \\ \textit{and} & \\ I_{k,n} &= \frac{(-1)^k(n+2k)!}{k!(n+k)!} \end{aligned} \tag{2.100}$$

By plugging (2.100) into (2.98), we get finally:

$$A_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^k = J_n(x)$$

We obtained the integral representation for $J_n(x)$:

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\theta) - in\theta} d\theta \tag{2.101}$$

Setting $t = \pm e^{\,i\theta}$ here, we obtain

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\left(e^{i\theta}-\frac{1}{e^{i\theta}}\right)} = \sum_{n=-\infty}^{\infty} e^{in\theta}\, J_n(x)$$

By using $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $e^{\,in\theta} = \cos n\theta + i \sin n\theta$, we obtain

$$
\begin{aligned}
e^{ix\sin\theta} &= \sum_{n=-\infty}^{\infty} (\cos n\theta + i\sin n\theta)\, J_n(x) \\
&= \sum_{n=-\infty}^{-1} (\cos n\theta + i\sin n\theta)\, J_n(x) + J_0(x) + \sum_{n=1}^{\infty} (\cos n\theta + i\sin n\theta)\, J_n(x) \\
&= J_0(x) + \sum_{n=1}^{\infty} (\cos n\theta - i\sin n\theta)\, J_n(x) + \sum_{n=1}^{\infty} (\cos n\theta + i\sin n\theta)\, J_n(x)
\end{aligned}
$$

Using relation $J_{-n}(x) = (-1)^n J_n(x)$, we obtain

$$
\begin{aligned}
e^{ix\sin\theta} &= J_0(x) + \sum_{n=1}^{\infty} (\cos n\theta + (-1)^n \cos n\theta + i\sin n\theta - i(-1)^n \sin n\theta)\, J_n(x) \\
&= J_0(x) + \sum_{n=1}^{\infty} (2\cos(2n\theta) J_{2n}(x) + 2i\sin((2n+1)\theta))\, J_{2n+1}(x) \\
&= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\ \cos(2n\theta) \pm 2i \sum_{n=1}^{\infty} J_{2n+1}(x)\sin(2n+1)\theta \qquad (2.102)
\end{aligned}
$$

This equality implies by using $e^{\,ix} = \cos x + i\sin x$

$$
\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta \qquad (2.103)
$$

$$
\sin(x\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin(2n+1)\theta \qquad (2.104)
$$

If we replace θ by $\pi/2 - \eta$ in (2.103) and (2.104), then we obtain

$$
\cos(x\cos\eta) = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)\cos 2n\eta \qquad (2.105)
$$

$$
\sin(x\cos\eta) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)\cos(2n+1)\eta \qquad (2.106)
$$

These expansions have been obtained by *Jacobi* and are called by his name, *Jacobi expansions.* Replacing θ by φ in (2.103), multiplying the left- and right-hand sides by

$\cos n\varphi$ and integrating with respect to φ from 0 to π, we obtain

$$\int_0^\pi \cos(x\cos\varphi)\cos n\varphi d\varphi = \int_0^\pi \left[J_0(x) + 2\sum_{n=1}^\infty J_{2n}(x)\cos 2n\varphi\right]\cos n\varphi d\varphi$$

$$= \int_0^\pi J_0(x)\cos n\varphi d\varphi + 2\int_0^\pi \sum_{n=1}^\infty J_{2n}(x)\cos 2n\varphi\cos n\varphi d\varphi$$

$$= 2\int_0^\pi \sum_{n=1}^\infty J_{2n}(x)\cos 2n\varphi\cos n\varphi d\varphi$$

$$\int_0^\pi \cos(x\cos\varphi)\cos n\varphi d\varphi = \left\{\begin{matrix} \pi J_n(x) & for\ even\ n, \\ 0 & for\ odd\ n. \end{matrix}\right\} \tag{2.107}$$

Similarly, from (2.104) we obtain

$$\int_0^\pi \sin(x\sin\varphi)\sin n\varphi d\varphi = 2\int_0^\pi \sum_{n=1}^\infty J_{2n+1}(x)\cos(2n+1)\varphi\cos n\varphi d\varphi$$

$$\int_0^\pi \sin(x\sin\varphi)\sin n\varphi d\varphi = \left\{\begin{matrix} 0 & \text{for even n,} \\ \pi J_n(x) & \text{for odd n.} \end{matrix}\right\} \tag{2.108}$$

Adding (2.107) and (2.108), we obtain that for any integer n

$$\pi J_n(x) = \int_0^\pi [\cos(x\sin\varphi)\cos n\varphi + \sin(x\sin\varphi)\sin n\varphi]\, d\varphi$$

$$= \int_0^\pi \cos(x\sin\varphi - n\varphi)d\varphi$$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\varphi - n\varphi)d\varphi \tag{2.109}$$

The integral in the left-hand side of (2.109) is called the *Bessel integral*; Bessel took this equality (more precisely, slightly different) as a definition of the function $J_n(x)$. Note that for a non-integer index this integral does not give the Bessel function and is denoted by

$$J_v(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\varpi\theta - x\sin\theta)d\theta \tag{2.110}$$

where $J_v(x)$ is a function which is usually called the *Anguer function.* For $n = 0$ integral (2.109) is called *Parseval integral*

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)d\theta \tag{2.111}$$

The deduction of the formulae given above illustres in a sense the character of applications of the Bessel integral. It arises naturally in cases when one should pass from solution of the Helmholtz equation in the Cartesian coordinates to a solution in polar coordinates. Now we shall now that integrals of the Bessel integral type can also be obtained when passing from other coordinte systems to the polar system. Consider once again the Parseval integral. Let us take a partial solution of the Helmholtz equation (we shall see this equation in detail in applications of Bessel functions.)

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + \lambda^2 u = 0$$

in the form $u = \cos\lambda\xi = \cos(\lambda R\cos\theta)$; here R and θ are the polar coordinates of the point with Cartesian coordinates ξ and η.Let us form the superposition of such solutions for $0 \le \theta \le 2\pi$, assuming that the $\xi - axis$ rotates around the origin

$$u^*(R) = \int_0^{2\pi} \cos(\lambda R\cos\theta)d\theta$$

Obviously, $u^*(R)$ is a solution of the axially symmetric problem for the Helmholtz equation without a singularity at the origin; hence, $u^*(R) = BJ_0(\lambda R)$. Let us set $R = 0$. Since $J_0(0) = 1$, we have $B = \int_0^{2\pi} d\theta = 2\pi$ and

$$J_0(\lambda R) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\lambda R\cos\theta)d\theta \tag{2.112}$$

In the case when $u = \sin\lambda\xi$, we have

$$u^*(R) = \int_0^{2\pi} \sin(\lambda R\cos\theta)d\theta = BJ_0(\lambda R)$$

Setting here $R = 0$, we obtain $B = 0$. Setting $u = \cos\alpha\xi\cos\beta\eta$, where $\alpha^2+\beta^2 = \lambda^2$, we have

$$BJ_0(\lambda R) = \int_0^{2\pi} \cos(\alpha R\cos\theta)\cos(\beta R\sin\theta)d\theta$$

Obviously, $B = 2\pi$. Hence,

$$J_0(\sqrt{\alpha^2+\beta^2}R) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha R\cos\theta)\cos(\beta R\sin\theta)d\theta \tag{2.113}$$

If we replace the function $u(\xi,\eta)$ by an arbitrary function which is a solution of the Helmholtz equation and has no singularities in the finite domain under consideration, then rotating this function we also obtain up to a constant factor the Bessel function of zero index. So, if we use an elliptic coordinate system and set $u(\xi,\eta) = ce_0(\xi)Ce_0(\eta)$, then, obviously, we have

$$BJ_0(\lambda R) = \int_0^{2\pi} ce_0(R\cos\theta)Ce_0(R\sin\theta)d\theta \tag{2.114}$$

One can continue the generalizations connected with the fact that one can interchange the arguments of the functions ce_0 and Ce_0 as well as introduce other indices different zero. Thus, the Parseval integral (2.112) can be considered as the superposition of plane waves and integrals (2.113) and (2.114) can be considered as the superposition of waves of a more complicated structure. It should be noted that, using the Parseval integral, one can be expand the Bessel function of zero index into a power series. For this purpose it is sufficient to represent the cosine in (2.112) in the form of a power series, denote $\lambda R = r$ and change the order of the integration and summation.

2.11 Differential Equation Reducible to the Bessel Equation

We consider the problem to which differential equations can be reduced to the Bessel equation and how the necessary transformations should be performed. Many differential equations can be reduced to the Bessel equation using transformations of dependent variables. We begin with the simplest special case. Consider the equation

$$\frac{\partial^2 \eta}{dx^2} - c^2\eta = \frac{p(p+1)}{x^2}\eta$$

Let us set $y = x^{-1/2}\eta$. With respect to this new variable y the equation takes the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left[c^2x^2 + \left(p+\frac{1}{2}\right)^2\right]y = 0$$

Denoting $p + 1/2 = v$ and $icx = x$, we reduce this equation to the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - v^2\right)y = 0$$

The x and y variables in the above equation could be defined in terms of the variable of t and $u(t)$ function i.e;

$$x = \gamma t^{\beta} \qquad and \qquad y = t^{\alpha}u(t) \tag{2.115}$$

In (2.115) we assume $\beta, \gamma \neq 0$ and α, β and γ are constants. If we redefine the Bessel equation the following equation would be obtained,

$$\frac{d^2y}{dx^2} = \frac{1}{\beta^2\gamma^2} t^{1-\beta} \left[(1-\beta)\, t^{-\beta} \frac{dy}{dt} + t^{1-\beta} \frac{d^2y}{dt^2} \right] \tag{2.116}$$

From the second transformation of (2.115),

$$\begin{aligned} \frac{dy}{dt} &= t^{\alpha} \frac{du}{dt} + \alpha t^{\alpha-1} u(t) \\ \frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \alpha t^{\alpha-1} \frac{du}{dt} + t^{\alpha} \frac{d^2u}{dt^2} + \alpha(\alpha-1) t^{\alpha-2} u(t) + \alpha t^{\alpha-1} \frac{du}{dt} \end{aligned} \tag{2.117}$$

Namely,

$$\frac{d^2y}{dt^2} = t^{\alpha} \frac{d^2u}{dt^2} + 2\alpha t^{\alpha-1} \frac{du}{dt} + \alpha(\alpha-1) t^{\alpha-2} u(t) \tag{2.118}$$

If we substitute the the equations (2.115) and (2.116) into the canonical type of Bessel Differential Equation and rewrite them we obtain ;

$$\frac{t^{2\beta}}{\beta^2} t^{1-\beta} \left[(1-\beta)\, t^{-\beta} \frac{dy}{dt} + t^{1-\beta} \frac{d^2y}{dt^2} \right] + \frac{t^{\beta}}{\beta} t^{1-\beta} \frac{dy}{dt} + \left(\gamma^2 t^{2\beta} - \beta^2 v^2\right) y = 0$$

or,

$$t^2 \frac{d^2y}{dt^2} + t \frac{dy}{dt} + \left[\beta^2\gamma^2 t^{2\beta} - \beta^2 v^2\right] y = 0 \tag{2.119}$$

The equations (2.117) ve (2.118) substitute into the equation (2.119), we obtain

$$t^2 \left[t^{\alpha} \frac{d^2u}{dt^2} + 2\alpha t^{\alpha-1} \frac{du}{dt} + \alpha(\alpha-1) t^{\alpha-2} u \right] +$$

$$+ t \left[t^{\alpha} \frac{du}{dt} + \alpha t^{\alpha-1} u \right] + \left[\beta^2 t^2 t^{2\beta} - \beta^2 v^2\right] t^{\beta} u = 0$$

When we rewite the last equation, we obtain

$$t^2\frac{d^2u}{dt^2} + (2\alpha+1)t\frac{du}{dt} + (\alpha^2 - \beta^2v^2 + \beta^2\gamma^2t^{2\beta})u = 0 \tag{2.120}$$

Let us take $a = 2\alpha + 1$, $b = \alpha^2 - \beta^2v^2$, $c = \beta^2\gamma^2$, $m = 2\beta$ we would found,

$$t^2\frac{d^2u}{dt^2} + \alpha t\frac{du}{dt} + (b + ct^m)u = 0$$

In this last equation suposse $a \neq 0, b \neq 0, c \neq 0$. Finally the general solution of the canonical type of Bessel equation could be as follows

$$y(x) = c_1J_v(x) + c_2J_{-v}(x)$$

By using canonical type of Bessel Equation and the transformation that we used in (3.115) the Bessel Equation in (2.120) could have the following special solution

$$u(t) = t^{-\alpha}y(x) = c_1t^{-\alpha}J_v\left(\gamma t^\beta\right) + c_2t^{-\alpha}J_{-v}\left(\gamma t^\beta\right) \tag{2.121}$$

2.11.1 Some differential equations reducible to Bessel's Equation

1. One of the well-known equations tied with the Bessel's differential equation is the modified Bessel's equation in (2.39) that was obtained by replacing x to $-ix$. This equation has the form:

$$x^2y'' + xy' - \left(x^2 + v^2\right)y = 0 \tag{2.39}$$

The solution of this equation were expressed through the so-called *modified Bessel functions of the first and second kind:*

$$y(x) = C_1J_v(-ix) + C_2Y_v(-ix) = C_1I_v(x) + C_2K_v(x)$$

where $Iv(x)$ and $Kv(x)$ are modified Bessel functions of the 1st and 2nd kind, respectively.

2. The Airy differential equation known in astronomy and physics has the form:

$$y'' - xy = 0$$

It can be also reduced to the Bessel equation. Its solution is given by the Bessel functions of the fractional order 1/3:

$$y(x) = C_1\sqrt{x}J_{1/3}\left(\frac{2}{3}ix^{3/2}\right) + C_2\sqrt{x}J_{-1/3}\left(\frac{2}{3}ix^{3/2}\right)$$

Also, the one dimensional Schrödinger equation for a constant force are Airy functions which can be transformed into Bessel functions of order 1/3.

3.The differential equation of type

$$x^2y'' + xy' + \left(a^2x^2 - v^2\right)y = 0$$

differs from the Bessel equation only by a factor a^2 before x^2 and has the general solution in the form:

$$y(x) = C_1J_v(ax) + C_2Y_v(ax)$$

2.12 Asymptotic Expansion of Bessel Functions

As a rule, practical calculations connected with the application of Bessel functions are based on the use of tables of these functions. In some cases, one can perform calculations, based on using the generalized power series, which have been given at the beginning of the thesis, as well as series containing the factor $\ln x$ in some cases; these series in increasing powers of the argument are convenient for the calculations only for small values of the argument. If the argument is large enough, then a question arises concerning the construction of approximate solutions to the Bessel equation, which are available for large values of the argument. The problem formulated is connected with the construction of the asymptotic expansion; when solving it, one should operate with divergent series, for which, however, the property holds that they are convenien for the calculations in a certain domain of the values and admit a simple estimate of the error.

From the beginning, we should emphasize the difference in the solution to problems of different classes here, depending on the behavior of the argument and the index v of cylindrical function. To find the asymptotic behavior of the Bessel functions at $x \to \infty$, we will use the device similar to the one used for the derivation of the Stirling formula in appendix 1. We present integral (2.101) in the form:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\Phi(x,\theta)} d\theta \qquad (2.122)$$

$$\Phi(x,\theta) = x \sin\theta - n\theta \qquad (2.123)$$

If $x \to \infty$, the integrand is the fast oscillation function everywhere except the two points where $\frac{d\Phi}{d\theta} = 0$. These points are defined by the equation:

$$\begin{aligned} \frac{d\Phi}{d\theta} &= x\cos\theta - n = 0 \\ x\cos\theta &= n \qquad at \qquad x \to \infty \\ \cos\theta &\to 0 \quad , \qquad \theta \to \pm\frac{\pi}{2} \end{aligned}$$

The contributions of the points $\theta^{\pm} = \pm\frac{\pi}{2}$ give complex conjuated results. Hence, it is enough to study the neighbourhood of the point $\theta = \frac{\pi}{2}$. Let us introduced $\theta = \frac{\pi}{2} + \tau$. For small τ,

$$\Phi(x,\theta) \approx x - \frac{n\pi}{2} - \frac{1}{2}x\tau^2$$

Integral (2.122) can be replaced approximately by the following integral:

$$J_n(x) = \frac{1}{\pi} \operatorname{Re} e^{i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \int_{-\infty}^{\infty} e^{-\frac{ix}{2}\tau^2} d\tau, \qquad \operatorname{Re} = Realpart$$

Let us make the change of variables:

$$\tau = \sqrt{\frac{2}{ix}} y, \quad \frac{1}{\sqrt{i}} = e^{-\frac{\pi i}{4}}$$

Then,

$$J_n(x) = \frac{\sqrt{2}}{\pi\sqrt{x}} \operatorname{Re} e^{i\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)} \int_{-\frac{i\pi}{4}*\infty}^{\frac{i\pi}{4}*\infty} e^{-y^2} dy \tag{2.124}$$

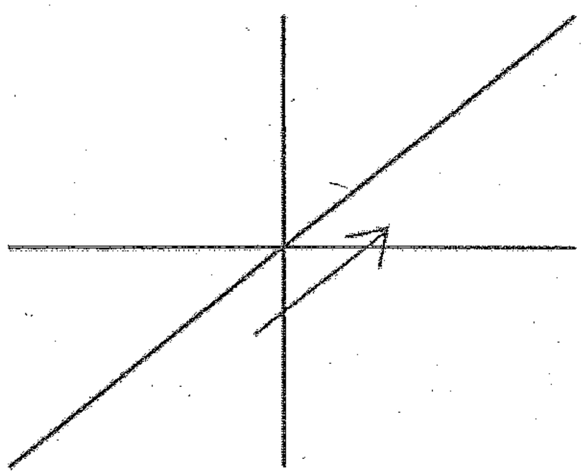

Figure 2.4 : *Contour of Integration.*

Integration is going in the complex plane along the straight line turned by 45° with respect to the real axis. This is demonstrated in figure 2.4. However, the contour of integration can be turned back and returned to the real axis. In other words, the integral in (2.124) can be replaced by the integral $\int_{-\infty}^{\infty} e^{-y^2} = \sqrt{\pi}$. We end up with the following result:

$$J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)$$

or

$$J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - (2n+1)\frac{\pi}{4}\right) \tag{2.125}$$

We derive this expression only for integral n. In fact, this is correct for all v. In general,

$$J_v(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - (2v+1)\frac{\pi}{4}\right) \tag{2.126}$$

In particular,

$$J_{\frac{1}{2}}(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\right) \to \sqrt{\frac{2}{\pi x}} \sin x$$

This is the unique Bessel function coinciding with its own asymptotic behavior. Also, Bessel functions of the second kind are defined as solutions of the Bessel equation with the following asymptotics:

$$Y_v(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - (2v+1)\frac{\pi}{4}\right) \tag{2.127}$$

So recalling (2.34), we get

$$\begin{aligned} H_v^{(1)} &= J_v(x) + iY_v(x) \\ H_v^{(2)} &= J_v(x) - iY_v(x) \end{aligned} \tag{2.128}$$

And substituting (2.126) and (2.127) in (2.34), we get

$$H_v^{(1)} \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - (2v+1)\frac{\pi}{4}\right) + i\sqrt{\frac{2}{\pi x}} \sin\left(x - (2v+1)\frac{\pi}{4}\right),$$

$$H_v^{(2)} \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - (2v+1)\frac{\pi}{4}\right) - i\sqrt{\frac{2}{\pi x}} \sin\left(x - (2v+1)\frac{\pi}{4}\right)$$

or

$$H_v^{(1)} \sim \sqrt{\frac{2}{\pi x}} e^{i(x-(2v+1)\frac{\pi}{4})}$$

$$H_v^{(2)} \sim \sqrt{\frac{2}{\pi x}} e^{-i\left(x-(2v+1)\frac{\pi}{4}\right)}$$

2.13 Roots of Bessel Function

The problems of the determination of the Bessel functions play a very important role in applications. As an example, we shall consider the displacements of a circular membrane in next part which performs free axi-asymmetrical oscillations: $w = AJ_0(\lambda r)\sin(wt+\varphi_0)$, where w is the circular frequency of oscillations; $\lambda = w\sqrt{p/T}$, p is the surface density; T is the tension.

This solution should vanish on the contour of the membrane: $w(R) = 0$. This implies that for the existence of a non-trivial solution it is necessary that

$$J_0(\lambda R) = 0. \tag{2.129}$$

It is well known that equation (2.129) has infinitely many simple real roots and has neither multiple, nor complex roots. From physical reasonings, it is obvious that, for sufficiently large numbers of the roots, the problem reduces to the consideration of the oscillations of a membrane which has a rather large number of a nodal curves representing concentric circumferences whose center coincides with the center of the contour. Therefore, we should expect some analogy with the problem of oscillations of a string. Hence, we have reasons to assume that the large roots of the Bessel function will be distant from one another by a distance which tends to a constant equal to π.

It is clear that from (2.126) that the Bessel function $J_v(x)$ has an infinite amount of zeros as a_N^v, where $N = 1, 2, ...\infty$. From (2.126), one can conclude that the distance between two neighboring zeros tends to π :

$$a_{N+1}^v - a_N^v \rightarrow \pi as \quad N \rightarrow \infty \tag{2.130}$$

The first five of each are presented in table 2.1. Notice that:

$$a_5^5 - a_4^5 = 3.2377$$

While:

$$a_5^0 - a_4^0 = 3.1394$$

Both values are closed to π. The derivatives of Bessel functions have the following asymptotic behavior:

$$J_v^{'}(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - (2v+1)\frac{\pi}{4}\right)$$

The derivative of $J_v^{'}(x)$ also have an infinite amount of zeros b_N^v. Again:

$$b_{N+1}^v - b_N^v \rightarrow \pi \; if \quad N \rightarrow \infty \tag{2.131}$$

Table 2.1 : *Roots of the function* $J_n(x)$*'s are given in the following table.*

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2.2 : *Roots of the function* $J_n(x)$*'s derivatives are given in the following table.*

zero	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Chapter 3

APPLICATIONS OF BESSEL FUNCTIONS: VIBRATIONS OF CİRCULAR MEMBRANE AND SCHRÖDINGER EQUATION

Bessel's differential equation in (2.9) is often encountered when solving boundary value problems, such as separable solutions to Laplace's equation or the Helmholtz equation, especially when working in cylindrical or spherical coordinates. Bessel functions made their first appearance by relating the angular position of a planet moving along a Keplerian ellipse to elapsed time. However the integral and power series shows up in other places, generally concerning the radial variable after separating Laplace's equation in polar or spherical polar coordinates. In many problems of mathematical physics, whose solution is connected with the application of cylindrical and spherical coordinates. The constant v, determines the order of the Bessel functions found in the solution to Bessel's differential equation and can take on any real numbered value. For cylindrical problems the order of the Bessel function is an integer value ($v = n$) while for spherical problems the order is of half integer value $v = n + 1/2$. Bessel functions are therefore especially important for many problems of wave propagation and static potentials and its applications are as:

Electromagnetic waves in a cylindrical waveguide, heat conduction in a cylindrical object, diffusion problems on lattice, modes of vibration of a thin circular or annular artificial membrane and solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates for a free particle). We are going to examine last two applications in these applications. Firstly we consider the solution of the two dimensional wave equation of the circular membrane and examine modes of vibration of circular membrane. Secondly, we consider solutions to the radial Schrödinger equation in spherical coordinates for a free particle.

3.1 Two Dimensional Wave Equation

We shall be concerned with the two dimensional geometry of the membrane in a plane. We shall be considering a uniformly thin sheet of flexible material. The sheet will be pulled taut into a state of uniform tension and clamped along a given closed curve (a circle, perhaps) in the $xy-$ plane. When the membrane is displaced slightly from its equlibrium position and then released, the restoring forces created by the deformation cause it to vibrate. For instance, this is how a drum work. To simplfy the mathematics, we shall consider only *small oscillations* of a *freely vibrating membrane.*

Before preparing a model for this problem, we describe a few assumptions concerning the material and behavior of the membrane:

1. The membrane is homogeneous. The density is constant.

2. The membrane is composed of a perfectly flexible material which offers no resistance to deformation perpendicular to the xy- plane. Motion of each element is perpendicular to the xy- plane.

3. The membrane is stretched and fixed along a boundary in the xy- plane.

4. The tension per unit length T due to stretching is the same in every direction and

is constant during the motion. Weight of the membrane is negligible.

5. The deflection $z(x, y, t)$ of the membrane whiles in motion is relatively.

To derive the differential equation which governs the motion of the membrane, we consider the forces acting on a small portion of the membrane as shown in figure 3.1 below:

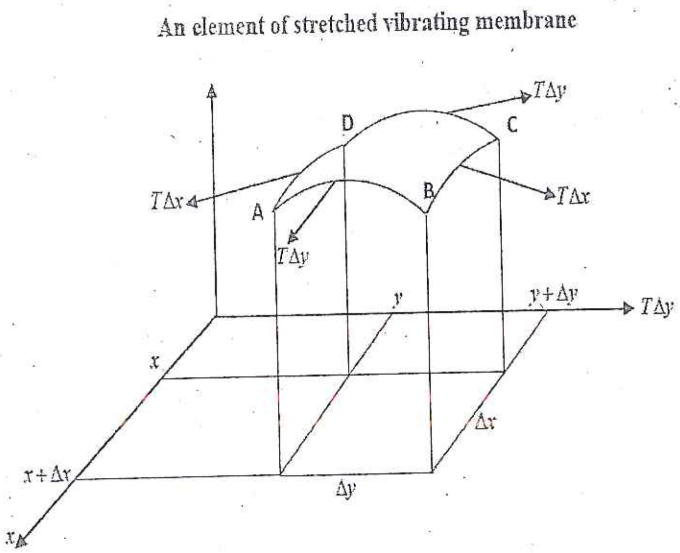

Figure 3.1 : *An element and projection of a stretched membrane.*

An element of the membrane ABCD in figure 3.1 is projected into a small rectangle with edges Δx and Δy parallel to the x and y axes. Deflections and angles of inclination are small enough so that the sides of the element are approximated by Δx and Δy. According to the assumption (4), the forces acting on the edges are approximately $T\Delta x$ and $T\Delta y$, and acts tangential to the membrane.

Horizontal components involve cosines of very small angles of inclination. Since these forces are directed in opposite direction they add to zero approximately. The sum of the horizontal forces in the x direction (See fig. 3.2) is

$$T\Delta y\,(\cos\beta - \cos\alpha) = 0 \tag{3.1}$$

And in the y direction the sum is

$$T\Delta x\,(\cos\delta - \cos\gamma) = 0 \tag{3.2}$$

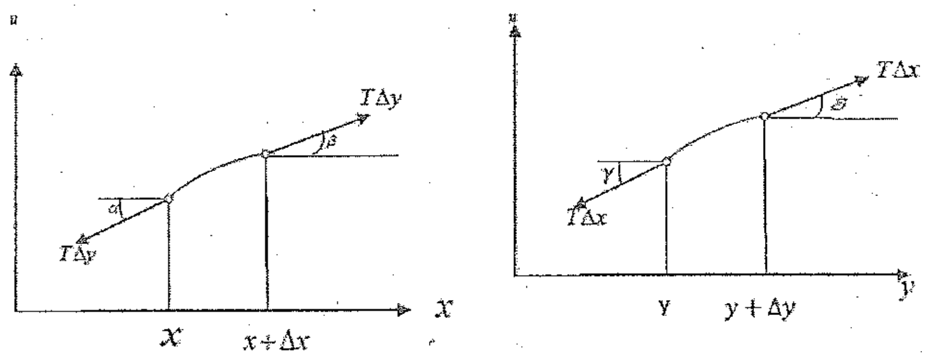

Figure 3.2 : *Cross sections in x and y plane of membrane showing angles of inclination.*

From the cross section in the x and y planes, if the horizontal component of $T\Delta y$ is T_{hx}, then from equation (3.1)

$$T_{hx} = T\Delta y\cos\beta = T\Delta y\cos\alpha \tag{3.3}$$

and that of $T\Delta y$ is T_{hx} then from equation (3.2) we have

$$T_{hy} = T\Delta x\cos\delta = T\Delta y\cos\gamma \tag{3.4}$$

Equations (3.3) and (3.4) becomes

$$T\Delta y = \frac{T_{hx}}{\cos\beta} = \frac{T_{hx}}{\cos\alpha} \tag{3.5}$$

and

$$T\Delta x = \frac{T_{hy}}{\cos\delta} = \frac{T_{hy}}{\cos\gamma} \tag{3.6}$$

If ρ is constant mass per unit area of the membrane, then the mass of the rectangular piece is $\rho\Delta x\Delta y$. Newton's second law of motion then tells us that

$$F = \rho\Delta x\Delta y\frac{\partial^2 z}{\partial t^2} \tag{3.7}$$

is the force acting on the membrane z-direction. Adding the forces in the vertical direction and using Newton's second law of motion, we obtain

$$T\Delta y(\sin\beta - \sin\alpha) + T\Delta x\,(\sin\delta - \sin\gamma) = \rho\Delta x\Delta y z_{tt} \tag{3.8}$$

If $T\Delta x$ and $T\Delta y$ in equation (3.8) are replaced by equations (3.5) and (3.6) then

$$T_{hx}\left[\tan\beta - \tan\alpha\right] + T_{hy}\left[\tan\delta - \tan\gamma\right] = \rho\Delta x\Delta y z_{tt} \tag{3.9}$$

Recognizing that

$$\tan\beta = z_x\,(x + \Delta x, y, t) \quad and \quad \tan\alpha = z_x(x, y, t)$$

$$tan\delta = z_y(x, y + \Delta y, t) \quad and \quad \tan\gamma = z_y(x, y, t)$$

Therefore equation (3.9) becomes

$$T_{hx}\left[z_x\,(x + \Delta x, y, t) - z_x(x, y, t)\right] + T_{hy}\left[z_y(x, y + \Delta y, t) - z_y(x, y, t)\right] = \rho\Delta x\Delta y z_{tt} \tag{3.10}$$

If the cosine of the inclinations is all approximately, then equation (3.10) yields

$$T\Delta y\left[z_x\left(x+\Delta x,y,t\right)-z_x(x,y,t)\right]+T\Delta x z_y(x,y+\Delta y,t)-z_y(x,y,t)=\rho\Delta x\Delta y z_{tt} \quad (3.11)$$

Division of equation (3.11) by $\rho\Delta x\Delta y$ permits the form

$$\frac{T}{\rho}\left[\frac{z_x(x+\Delta x,y,t)-z_x(x,y,t)}{\Delta x}+\frac{z_y(x,y+\Delta y,t)-z_x(x,y,t)}{\Delta y}\right]=z_{tt}(x,y,t)$$

as $\Delta x\to 0$ and $\Delta y\to 0$ in equation (3.12)

$$z_{tt}(x,y,t)=a^2\nabla^2 z(x,y,t)$$

where $\nabla^2=z_{xx}+z_{yy}$ and $a^2=T/\rho$.

$$a^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)=\frac{\partial^2 z}{\partial t^2} \quad (3.13)$$

This is the *two-dimensional wave equation.*

3.2 Vibrating Drum of an Arbitrarily Shaped Membrane

Vibrating drums are surfaces which vibrate, like flat drum-heads in the shape of a circle or square. A string is one-dimensional but a drum head is two-dimensional. A string only has a length, but a drum has a shape. How does the shape influence the sound of the drum? This is the famous problem, **"Can you hear the shape of a drum?"**

Let's consider the PDE (3.13) with initial conditions

$$
\begin{aligned}
z(x,y,0) &= f(x,y) \quad \textit{(initial displacement)} \\
\frac{\partial z}{\partial t}(x,y,0) &= g(x,y) \quad \textit{(initialvelocity)}
\end{aligned}
$$

We can not specify any boundary conditions at this point since of the domain is not given. For seperation of variables we start with the Ansatz

$$z(x,y,t) = w(t)\varphi(x,y) \tag{3.14}$$

so that the partial derivatives are

$$
\begin{aligned}
\frac{\partial^2 z}{\partial t^2}(x,y,t) &= w''(t)\varphi(x,y) \\
\frac{\partial^2 z}{\partial x^2}(x,y,t) &= w(t)\frac{\partial\varphi}{\partial x^2}(x,y) \\
\frac{\partial^2 z}{\partial y^2}(x,y,t) &= w(t)\frac{\partial\varphi}{\partial y^2}(x,y)
\end{aligned}
$$

and the wave equation turns into in (3.13)

$$w''(t)\varphi(x,y) = a^2 w(t)\left(\frac{\partial\varphi}{\partial x^2}(x,y) + \frac{\partial\varphi}{\partial y^2}(x,y)\right)$$

or

$$\frac{\frac{\partial\varphi}{\partial x^2}(x,y) + \frac{\partial\varphi}{\partial y^2}(x,y)}{\varphi(x,y)} = \frac{1}{a^2}\frac{w''(t)}{w(t)} \tag{3.15}$$

Now our analysis follows familiar lines: Now our analysis follows familiar lines (3.15) depends on x and y only and the right-hand side depends on t only, we conclude that both sides are equal to some constant λ. In order for the membrane to vibrate, $w(t)$ must be periodic. Thus the constant τ must be negative (if it is positive, then the solutions of $w'' - \tau a^2 w = 0$ will be real exponentials and hence not periodic). We thus equate both sides of equation (3.4) with $\tau = -\lambda^2$ and obtain

$$w''(t) = -\lambda^2 a^2 w(t) \tag{3.16}$$

which has oscillatory solution for $\lambda > 0$, and one PDE for the spatial part

$$\begin{aligned} \frac{\partial \varphi}{\partial x^2}(x,y) + \frac{\partial \varphi}{\partial y^2}(x,y) &= -\lambda^2 \varphi(x,y) \\ \Leftrightarrow \ \nabla^2 \varphi(x,y) &= -\lambda^2 \varphi(x,y) \end{aligned} \tag{3.17}$$

This PDE eigenvalue equation is known as the *Helmholtz equation.*

In order to attempt a solution of the Helmholtz equation with the help of seperation of variables we will need to have a "nice" region and appopriate boundary conditions. If the region is rectangular, then we seperate

$$\varphi(x,y) = X(x)Y(y)$$

If the region is circular, then

$$\varphi(x,y) = \varphi^{\sim}(r,\theta) = R(r)\Theta(\theta)$$

We will only study circular region because of related of Bessel functions:

3.3 Vibrations of Circular Membrane

In this subsection, the problem is that find the frequencies of vibration of a circular drum when the modes of vibration are rotationally invariant. A kettledrum is a percussive instrument consisting of a circular drumhead (usually plastic, but in older times, an animal skin) that is tautly stretched over a metal bowl. The vibrations of the kettledrum's drumhead can be modelled by the wave equation in (3.13), where a is the speed of waves travelling on the drumhead. The constant a is directly related to the tension of the drumhead and the corresponding pitch that is generated by hitting the drumhead with a mallet, and can be adjusted using a foot pedal. The characteristic sound of the kettledrum is determined by its vibrational modes and their corresponding frequencies. Any kettledrum player will tell you that the proper place to strike the drumhead is not the center of the drumhead, but rather a spot somewhere about one-sixth of the diameter away from the edge of the drumhead. The most common drums have a diameter between 23 to 29 inches, so that means striking the timpani about 4 to 5 inches in from the edge of the drumhead. Striking the drumhead in the center produces a sound that is somewhat hollow.(Yong, 2006)

In this subsection, we will give some mathematical explanations for why this occurs. We consider the vibrations of a circular membrane of radius c as shown in figure 3.3. Also, this section considers the solution of the two dimensional wave equation of the circular membrane. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary.

To fit the boundary condition of no displacement on other than rectangular boundaries requires the use of an appropriate two dimensional orthogonal curvilinear coordinate system such that the boundary of membrane coincides with coordinate lines in this system. Furthermore, it is necessary that the variables of the wave equation be seperable in the new system. It turns out that the choice of curvilinear coordinate systems is severely limited, and it is impossible, except in an approximate way, to analyze the vibrations of a membrane having an arbitrarily shaped boundary which is circular boundary given by $x^2 + y^2 = c^2$. It is more natural to use polar coordinates as indicated in figure 3.3. The solution of one of the seperated equations consists of Bessel functions. We shall transform the two dimensional Cartesian wave equation into its polar form in terms of r and θ using the parametric equations. So, the boundary condition is given $z(c, \theta, t) = 0$ for all $t > 0$ and $\theta \in [\pi, -\pi]$.

Now we shall consider the displacement of a circular membrane. So our study will be the model for a drum. Of course we shall use polar coordinates, with the origin at the center of the drum. The operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is called the *Laplacian* and the 2-dimensional wave equation may then be written in terms of the Laplacian simply as

$$\partial^2 z/\partial t^2 = a^2\nabla^2 z$$

Since the boundary of the circular membrane may be expressed by the simple equation $r = constant$, we first transform the Laplacian into polar coordinates (r, θ). Note that $x = rcos(\theta)$ and $y = rsin(\theta)$ and we denote partial derivatives by subscripts. By the chain rule, we get

$$z_x = z_r r_x + z_\theta \theta_x.$$

Differentiating again using the product rule, we get

$$\begin{aligned} z_{xx} &= z_{rx}r_x + z_r r_{xx} + z_{\theta x}\theta_x + z_\theta \theta_{xx}. \\ z_{rx} &= z_{rr}r_x + z_{r\theta}\theta_x \; and \; z_{\theta x} = z_{\theta r}r_x + z_{\theta\theta}\theta_x. \end{aligned} \tag{3.18}$$

Now $r = (x^2 + y^2)^{1/2}$ and $\theta = tan^{-1}(y/x)$, so $r_x = x/r$ and $\theta_x = -y/r^2$. Differentiating again, $r_{xx} = (r^2 - x^2)/r^3 = y^2/r^3$ and $\theta_{xx} = 2xy/r^4$. Substituting into (3.18) and assuming the continuity of first and second derivatives so that $z_{r\theta} = z_{\theta r}$, we obtain

$$z_{xx} = (x^2/r^2)z_{rr} - 2(xy/r^3)z_{r\theta} + (y^2/r^4)z_{\theta\theta} + (y^2/r^3)z_r + 2(xy/r4)z_\theta \tag{3.19}$$

Similarly,

$$z_{yy} = (y^2/r^2)z_{rr} + 2(xy/r^3)z_{r\theta} + (x^2/r^4)z_{\theta\theta} + (x^2/r^3)z_r - 2(xy/r^4)z_\theta \tag{3.20}$$

Now add (3.19) and (3.20) to find the Laplacian of z in polar coordinates:

$$\nabla^2 z = z_{xx} + z_{yy} = z_{rr} + (1/r)z_r + (1/r^2)z_{\theta\theta}$$
$$\nabla^2 z = \partial^2 z/\partial r^2 + (1/r)\partial z/\partial r + (1/r^2)\partial^2 z/\partial\theta^2$$

Using polar coordinates (r, θ) where $x = rcos(\theta)$ and $y = rsin(\theta)$, the wave equation becomes

$$\partial^2 z/\partial t^2 = a^2(\partial^2 z/\partial r^2 + (1/r)\partial z/\partial r + (1/r2)\partial^2 z/\partial\theta^2) \tag{3.21}$$

Here, naturally, $z(r, \theta, t)$ is a function of the polar coordinates r, θ and of time t. Since the drumhead is tautly held down, we impose Dirichlet conditions at the boundary of the drumhead:

$$z(r, \theta, t) = 0\,, \quad 0 \leq r \leq c, \quad -\pi \leq \theta \leq \pi, \quad t > 0 \tag{3.22}$$

and the initial conditions are the standard ones

$$\begin{aligned} z(r, \theta, 0) &= f(r, \theta) \\ \tfrac{\partial z}{\partial t}(r, \theta, 0) &= g(r, \theta) \end{aligned} \tag{3.23}$$

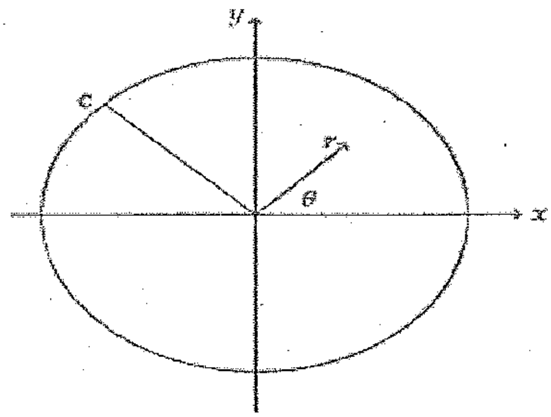

Figure 3.3 : *The circular membrane of radius is given by the dis*tan*ce the center r, and the angle, θ. There are fixed boundary conditions along the edge at c.*

We begin with a seperation of variables Ansatz:

$$z(r, \theta, t) = \varphi(r,\ \theta)w(t) \tag{3.24}$$

So that we get the ODE

$$w''(t) = -\lambda^2 a^2 w(t)$$

and the Helmholtz PDE in polar coordinates

$$\begin{aligned} \nabla^2\varphi + \lambda^2\varphi &= 0 \\ \textit{with BC}\ \varphi(c, \theta) &= 0 \end{aligned}$$

We can write the PDE eigenvalue problem as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \lambda^2 \varphi = 0, \qquad \varphi(c, \theta) = 0$$

Now we again apply seperation of variables for this polar coordinate problem using Ansatz $\varphi(r, \theta) = R(r)\Theta(\theta)$. This gives us

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[R(r)\Theta(\theta)]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[R(r)\Theta(\theta)] + \lambda^2 [R(r)\Theta(\theta)] = 0$$

or

$$\frac{\Theta(\theta)}{r}\frac{d}{dr}(rR'(r)) + \frac{R(r)}{r^2}\Theta''(\theta) + \lambda^2 R(r)\Theta(\theta) = 0$$

Multiplication by $\frac{r^2}{R(r)\Theta(\theta)}$ and a little rearranging gives

$$\frac{r}{R(r)}\frac{d}{dr}(rR'(r)) + \lambda^2 r^2 = \frac{-\Theta''(\theta)}{\Theta(\theta)} = \mu \tag{3.25}$$

which results in two additional Sturm-Liouville ODE eigenvalue problems. We can determine the hidden boundary conditions by making some observation. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$, noting that at these values for any $r < c$ we are at the same physical point. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = \pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This tells us

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi)$$

Such boundary conditions are called periodic boundary conditions. So, we get

$$\left\{ \begin{array}{c} \Theta''(\theta) = -\mu\Theta(\theta) \\ \textit{with periodic BC } \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi) \end{array} \right\} \tag{3.26}$$

and

$$\left\{ \begin{array}{c} r\frac{d}{dr}\left(rR'(r)\right) + \left(\lambda r^2 - \mu\right) R(r) = 0 \\ with\ singularity\ BC\ R(c) = 0, \quad |R(0)| < \infty \end{array} \right\} \tag{3.27}$$

The equation in (3.27) should be solved over the interval $0 \leq r \leq c$ subject to the conditions that $R(0)$ be finite and $R(c) = 0$. Because we are solving this equation on the interval $0 \leq r \leq c$ and the functions r and r^{-1} are not positive at the end points of this interval, this boundary-value problem is not a regular Sturm-Liouville eigenvalue problem but rather a singular Sturm-Liouville eigenvalue problem. Furthermore, Bessel's equation is linear and has a regular point at $r = 0$, so we expect that at least one of the linearly independent solutions to this equation has a singularity at $r = 0$.

Nevertheless, this eigenvalue problem still has a complete set of orthogonal eigenfunctions. From (3.27), we see that the weight function is r, so inner products should be computed as integrals over the interval $0 \leq r \leq c$ with an extra factor of r. (This extra factor accounts for the fact that the area of a circle is proportional to the square of its radius.)

The angular eigenvalue problem in (3.27) has eigenvalues and eigenfunctions

$$\mu_n = n^2, \quad \Theta_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \quad n = 0, 1, 2, 3, ...$$

Also, the radial eigenvalue problem in (3.27) becomes by using the product rule

$$0 = r\frac{d}{dr}\left(rR'(r)\right) + \left(\lambda^2 r^2 - \mu\right) R(r) = r^2 R''(r) + R'(r) + \left(\lambda^2 r^2 - \mu\right) R(r) \tag{3.28}$$

We know that $\lambda > 0$ and so we can do a variable substitution $u = \lambda r$. Note that, by the chain rule, we then have

$$\frac{dR}{dr} = \frac{dR}{du}\frac{dz}{dr} = \frac{dR}{du}\lambda \tag{3.29}$$

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\frac{dR}{dr} = \frac{d}{dr}\left[\frac{dR}{du}\lambda\right] = \frac{d^2R}{du^2}\lambda^2 \tag{3.30}$$

Therefore, if we apply the substitutions (3.29) and (3.30) and eigenvalues $\mu_n = n^2$ to the equation (3.27) we get

$$\begin{aligned} \frac{u^2}{\lambda^2}\lambda R''(u) + \frac{u}{\lambda}\sqrt{\lambda}R'(u) + \left(\lambda^2\frac{u^2}{\lambda^2} - n^2\right)R(u) &= 0 \\ u^2R''(u) + uR'(u) + \left(u^2 - n^2\right)R(u) &= 0, \quad n = 0, 1, 2, 3, \ldots \end{aligned} \tag{3.31}$$

This is known as *Bessel's equation of index n.* The general solution to (3.31) is

$$R(u) = CJ_n(u) + DY_n(u)$$

where $J_n(u)$ and $Y_n(u)$ are Bessel's functions of first and second kind, respectively. Since we substituted $u = \lambda r$ in Bessel's equation, the solution becomes

$$R(r) = CJ_n(\lambda r) + DY_n(\lambda r) \tag{3.32}$$

Bessel's functions of the second kind have singularities at $r = 0$, so for $R(0)$ to remain finite we must choose $D = 0$. Now we are left with

$$R(r) = J_n(\lambda r)$$

We have set $C = 1$ for simplicity. We can apply the vanishing condition at $r = c$.We will have the trivial solution and the boundary conditions tell us that the eigenvalues $\lambda_{n,m}$ are such that

$$R_n(c) = J_n(\lambda_{n,m} c) = 0$$

where $\lambda_{n,m}c$ is the m-th zero of the Bessel function J_n or,

$$\lambda_{n,m} = \frac{u_{n,m}}{c}$$

where $n = 0, 1, 2, 3, ...$ and $m = 1, 2, 3, ...$ and $u_{n,m}$ is the m-th zero of the Bessel function of order n.

$$J_n(u_{n,m}) = 0$$

Table 3.1 : *The zeros of Bessel functions* $J_n(u_{nm}) = 0$.

m	n= 0	n= 1	n= 2	n= 3	n= 4	n= 5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Moreover time equation becomes

$$w''(t) = -\lambda_{n,m}^2 a^2 w(t)$$

has general solution

$$w_{n,m}(t) = c_1 \cos(\lambda_{n,m} a t) + c_2 \sin(\lambda_{n,m} a t) \tag{3.33}$$

The product solutions that describe the vibrational modes of the circular drumhead and therefore

$$z_{n,m}(r,\theta,t) = J_n(\lambda_{n,m} r)\left[A_{n,m}\cos n\theta + B_{n,m}\sin n\theta\right]\left[\alpha_{n,m}\cos(a\lambda_{n,m}t) + \beta_{n,m}\sin(a\lambda_{n,m}t)\right]$$

To solve explicitly for z, one needs to create a proper superposition of product solutions and determine constants from initial conditions using orthogonality conditions. Therefore, superposition requires the solution to be of the form

$$z = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} J_n(\lambda_{n,m} r)\left[A_{n,m}\cos n\theta + B_{n,m}\sin n\theta\right]\left[\alpha_{n,m}\cos a\lambda_{n,m}t + \beta_{n,m}\sin a\lambda_{n,m}t\right] \tag{3.34}$$

and the Fourier coefficients can be found by using initial conditions. For example, let $g(r,\theta) = 0$ in (3.23), we find $\alpha_{n,m} = \beta_{n,m} = 0$. Then

$$h(r,\theta) = \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} A_{n,m} J_n(\lambda_{n,m} r)\right)\cos(n\theta) + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} A_{n,m} J_n(\lambda_{n,m} r)\right)\sin(n\theta)$$

where

$$A_{n,m} = \frac{\int_0^c \int_0^{2\pi} h(r,\theta) J_n(\lambda_{n,m} r) \cos(n\theta) r \, dr \, d\theta}{\int_0^c \int_0^{2\pi} J_n^2(\lambda_{n,m} r) \cos^2(n\theta) r \, dr \, d\theta}$$

$$B_{n,m} = \frac{\int_0^c \int_0^{2\pi} h(r,\theta) J_n(\lambda_{n,m} r) \sin(n\theta) r \, dr \, d\theta}{\int_0^c \int_0^{2\pi} J_n^2(\lambda_{n,m} r) \sin^2(n\theta) r \, dr \, d\theta}$$

Because of the trigonometric identitiy

$$\alpha \cos\phi + \beta \sin\phi = \sqrt{\alpha^2 + \beta^2} \cos(\phi + \phi_0)$$

where ϕ_0 is phase shift, we recognize that the product solution $z_{n,m}$ above can be written in a more compact form. If we are only after the qualitative behavior of each vibrational mode, we can ignore the angular and temporal phase shift by defining

$$z_{n,m}(r,\theta,t) = J_n(\lambda_{n,m} r) \cos n\theta \cos(a\lambda_{n,m} t) \tag{3.35}$$

for $n = 0, 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. Plots of these vibrational modes appear at the end of this handout. Each of these vibrational modes has a frequency of $2\pi c/au_{n,m}$. When the drum is struck, it is the combination of all of the vibrational modes and their corresponding frequencies that contributes to its characteristic sound (timbre).

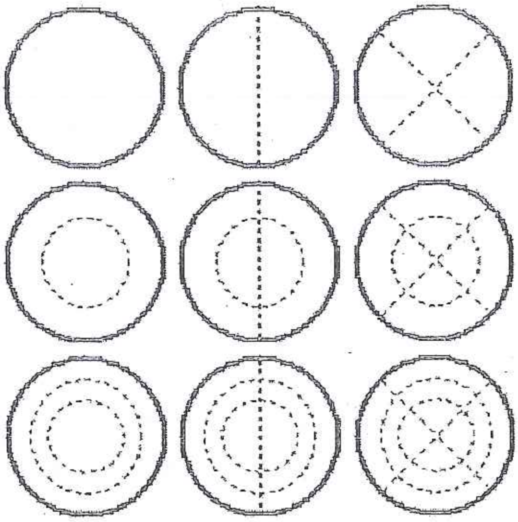

Figure 3.4 : *The first few modes of the vibrating circular membrane.*

The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in figure 3.5. We are interested in the shapes of the harmonics. So, we consider the spatial solution $(t = 0)$.

$$\phi(r, \theta) = \cos n\theta \left(J_n \frac{u_{n,m}}{c} r \right)$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r, \theta)$. This can be satisfied if

$$\cos n\theta = 0 \ or \ J_n \left(\frac{u_{n,m}}{c} r \right) = 0$$

The various nodal curves which result are shown in figure 3.4. For the angular part, we easily see that the nodal curves are radial lines, $\theta = const$. For $n = 0$, there are no solutions, since $\cos n\theta = 1$ $n = 0$. In figure 3.4 this is seen by the absence of radial lines in the first column.

For $n = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in figure 3.4. For $n = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}$, $\frac{3\pi}{4}$. This results in the two lines shown in the last column of figure 3.4.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{u_{n,m}}{c} r$ is a zero of the Bessel function and lies in the interval $[0, c]$. Thus, we have,

$$\frac{u_{n,m}}{c} r = u_{n,u}$$

These will give circles of this radius with $u_{n,u} < u_{n,m}$ *or* $u < m$. The zeros can be found in Table 3.1. For $n = 0$ and $m = 1$, there is only one zero and $r = c$. In fact, for all $m = 1$ modes, there is only one zero and $r = c$. Thus, the first row in figure 3.4 shows no interior nodal circles.

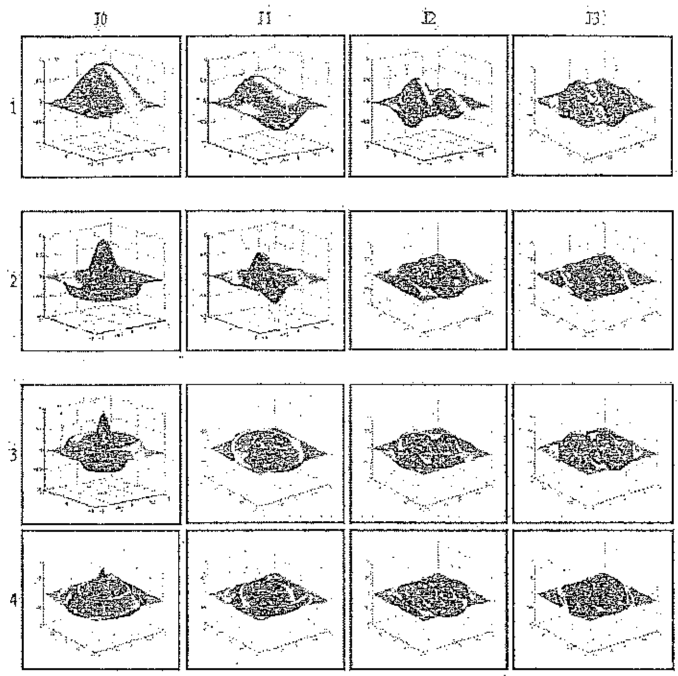

Figure 3.5 : *A three dimensional view of the vibrating circular membrane for the lowest mode. Compare these images with the nodal line plots in figure* 3.4.

Drums resonate in various ways, creating different vibrational patterns. A mathematical language has been developed to define the different resonant vibrational mode patterns in a circular membrane. For the modes (n, m) where n is the number of nodal diameters(0 means it goes in all directions) and m is the number of nodal circle.

$$r = \frac{u_{n,p}}{u_{n,m}} , \qquad p = 1, 2, ..., m-1$$

For $m = 2$ modes, we have two nodal circles,

$$r = c \ and \ r = \frac{u_{n,1}}{u_{n,2}}$$

as shown in the second row of figure 3.5. For $n = 0$,

$$r = \frac{2.405}{5.520}c \approx 0.436c$$

for the inner circle. For $n = 1$,

$$r = \frac{3.832}{7.018}c \approx 0.546c.$$

and for $n = 2$,

$$r = \frac{5.135}{8.147}c \approx 0.630c$$

where vibrational modes has a frequency of $2\pi c/au_{n,m}$.

$(0, 1)$ mode is the first or the fundamental mode of vibration of a circular membrane. This is the mode that is excited when the drum is struck in the center of the membrane. It sounds like a deep "thump". The vibrations occur at the lowest frequency of all of the drum vibrational modes. The lowest frequecy being $2.405\,(c/a)$.

The second mode of vibration of a circular membrane is the $(1, 1)$ mode. This is the most important vibrational mode in terms of the musical quality of the drum. The $(1, 1)$ mode vibrates at a frequency 1.593 times the frequency of the $(0, 1)$ mode, and the nodal point runs the diameter of the drum.

The third main mode of vibration of a circular membrane is the $(2, 1)$ mode. This is the second most important vibrational mode in terms of the musical quality of the drum.

The $(2, 1)$ mode vibrates at a frequency 2.135 times the frequency of the $(0, 1)$ mode. The nodal point run the diameter of the drum at right angles to each other, making a large x.

The $(0, 2)$ mode, shown in figure 3.5 does not have any diameter nodes, but has two circular nodes - one at the outside edge and one at a distance of $0.436c$ (c is the radius of the circular membrane) from the center of the membrane. The frequency of the $(0, 2)$ mode is 2.296 times the frequency of the $(0, 1)$ mode and decays faster than the $(1, 1)$

mode, so it does not contribute to the musical quality of the drum, but to the thump.

The $(1,2)$ mode vibrates at 2.917 times the frequency of the $(0,1)$ mode and does not contribute to the musical quality of a drum, even though it takes a relatively long time to decay.

The $(2,2)$ mode vibrates in a complex pattern that acts somewhat like two opposing 4 pole sound sources. It is the combination of a circular node intersecting two radial nodes which form a large x.

Now we will examine vibration of a circularly symmetric or radially symmetric drum with zero initial velocity. We will only try to examine the frequencies and modes of vibration for "radial mode i.e. mode which are only functions of the distance r from the center. They vibrate only in the radial direction from the center of the drum to the boundary circle. We are not going to consider modes of vibration which travel along the boundary. Therefore, because of circular symmetry there is no change in the angular variable and the wave equation (3.10) becomes

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) \tag{3.36}$$

$$z_{0,m}(r,t) = J_0(\lambda_{0,m} r)\cos(a\lambda_{0,m} t) \tag{3.37}$$

The frequencies $\lambda_{0,m}$ are determined by Dirichlet's boundary condition that the drum is still at $r = 1$(the circle) so $J_0(\lambda) = 0$. Thus, the zeros of the Bessel function J_0 give the frequencies of radial vibration of a circular drum. The base note of a drum is a radial mode. The associated frequency of vibration is the smallest zero $\lambda_{0,1}$ of the J_0 Bessel function. Thus the vibration of the drum is described by

$$J_0(\lambda_{0,1} r)\cos(\lambda_{0,1} a t)$$

This is what you hear most when you play the drum. $J_0(\lambda_{0,1} r)$ has one hump. $J_0(\lambda_{0,2} r)$ has two 'humps', one up and one down as seen in figure 3.5. In this second mode, the inner part of the drum moves up while the outer part moves down. The frequencies with which a drum vibrates radial are the zeros $\{u_{0,m}\}$ of the Bessel function J_0. For a drum, the frequencies $\{u_{0,m}\}$ are far from being an arithmetic progression. The radial frequencies $\{\lambda_{0,m}\}$, i.e. the zeros of J_0 are not an arithmetic progression. The totality of frequencies behaves like 'random numbers', i.e. numbers put out by a random number generator. We hear the bass note of a drum; the drum's muffled sound is due to the randomness of the higher frequencies.

3.4 Schrödinger Equation in Spherical Coordinates

The future state of a particle in classical mechanics the initial position and momentum of a particle is completely determined by forces acting on the quantum mechanics finds correlations between observable quantities, but the current position of a particle in the Heisenberg uncertainty principle, how well you know the location and therefore the degree of subsequent momentum obscured. Ervin Schrödinger, de Broglie wave on the basis of the accompanying article, adapted to different physical problems mathematically. Classical physics, according to a particle mass μ and space in three dimension, ie, x, y, z locations of potential energy V, the potential and kinetic energies $(p^2/2\mu)$, the sum of particle its energy gives: $E_{total} = V + p^2/2\mu$. Schrödinger advantage of this, the particle or particles system, showing to what extent in different places has created a wave function Ψ. Thus, the time-dependent Schrödinger equation was born in 1926. This equation is written so that the material carries all the characteristics of a wave equation that describes the motion of the object. Then the equation, the equation of quantum mechanics, has been widely accepted. In short, the problem of quantum mechanics, the free movement of an object is to find the wave function of the object are limited by external forces. Bessel's equation is due to review the application areas of the global coordinates (r, θ, φ) is the radial part of the solution of time-independent Schrödinger equation. Namely, Spherical Bessel functions are found to be the solution to the Schrödinger equation with spherical symmetry in a situation. Three are important postulates of quantum mechanics (Schiff, 1968)

(i) A system is completely specified by a state function, or wave function, $\Psi(r,t)$, with $\langle\Psi,\Psi\rangle = 1$. For example, if the system consists of a particle moving in an external potential, then $|\Psi(r,t)|^2\, d^3r$ is the probability of finding the particle in a small volume d^3r that surrounds the point r, and hence we need

$$\iiint_{\mathbb{R}^3} |\Psi(r)|^2\, d^3r = \langle\Psi,\Psi\rangle = 1.$$

(ii) For every system there exist a certain Hermitian operator, H, called the Hamiltonian operator, such that

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{3.38}$$

where $2\pi\hbar \approx 6.62\ x\ 10^{-34}\ Js^{-1}$ is Planck's constant.

(iii) To each obsevable property of the system there corresponds a linear, Hermitian operator, and any measurement of the property gives one of the eigenvalues. For example, the operators that correspond to momentum and energy are $-i\hbar\nabla(-i\hbar\partial/\partial r)$ and $i\hbar\partial/\partial t$.

The form of the Schrödinger equation depends on the physical situation. The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \tag{3.39}$$

where Ψ is the wave function of the quantum system, i is the imaginary unit, $\hbar$ is the reduced Planck constant, and H is the Hamiltonian operator, which characterizes the total energy of any given wavefunction and takes different forms depending on the situation. For a single particle of mass μ , we get

$$H = \frac{p^2}{2\mu} + V \tag{3.40}$$

where p is momentum of the particle. Using by momentum operator we get

$$\begin{aligned} H &= -\frac{\hbar^2}{2\mu}\nabla^2 + V(r,t), \\ &\quad \textit{where } V(r,t) \textit{ is potential energy} \end{aligned}$$

Schrödinger equation becomes

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}\Psi(r,t) + V(r,t)\Psi(r,t) \tag{3.41}$$

This is Schrödinger's equation, which governs the evolution of the wave function. Equation, $\Psi(r,t)$ is the wave function of the particle at time t, the probability amplitude to make the point r. Amplitude, the ripple of events is a concept born in the directory. Possibility is the two-time (past and future) can be explained as a confrontation between. $\Psi(r,t)$, any boundary conditions and is known at the time t_0 instant $\Psi(r,t)$, at any time in the past and future $\Psi(r,t)$ to calculate.

Let's look for a separable solution of (3.41) when V is independent of time. We write $\Psi = u(r)T(t)$, and find that

$$i\hbar\frac{T'}{T} = -\frac{\hbar^2}{2\mu u}\nabla^2 u + V(r) = E,$$

where E is the separation constant. Since $i\hbar T' = ET$,

$$i\hbar\frac{\partial\Psi}{\partial t} = E\Psi,$$

and hence E is the energy of the particle. The equation for u is then the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\nabla^2 u + (V(r) - E)\, u = 0$$

or

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right)\Psi = E\Psi \tag{3.42}$$

In spherical coordinates, the Laplacian takes the form:

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] \tag{3.43}$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + V(r)\right)\Psi(r,\theta,\phi) = E\Psi(r,\theta,\phi) \tag{3.44}$$

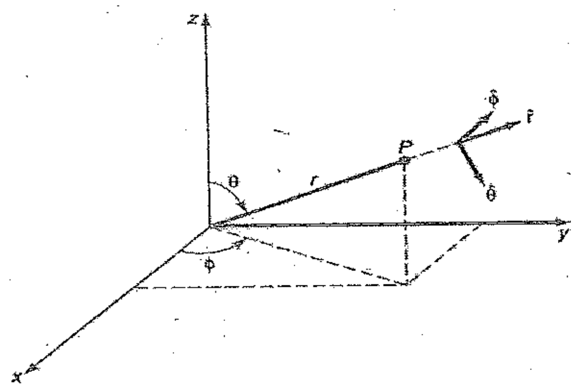

Figure 3.6 : *Spherical Coordinates*

Quantum numbers describe values of conserved quantities in the dynamics of the quantum system. The principal quantum number ($n = 1, 2, 3, 4, \ldots$) denotes the eigenvalue of Hamiltonian (H), i.e. the energy. The azimuthal quantum number ($l = 0, 1, \ldots, n-1$)

(also known as the angular quantum number or orbital quantum number) gives the orbital angular momentum through the relation $L^2 = \hbar^2 l(l+1)$. The magnetic quantum number ($m = -l, -l+1, \ldots, 0, \ldots, l-1, l$) yields the projection of the orbital angular momentum along a specified axis. (Altın, 2006)

If the potential energy and the boundary conditions are spherically symmetric, it is useful to transform H into spherical coordinates and seek solutions to Schrodinger's equation which can be written as the product of a radial portion and an angular portion: $\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$, or even $R(r)\Theta(\theta)\Phi(\phi)$. This type of solution is known as 'separation of variables'. After separating the variables of (3.44), we then multiply by $r^2/(R\Theta\Phi)$ which gives

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d}{dr}\left(\frac{r^2}{R}\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\left(\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{d^2\Phi}{d\phi^2}\right)\right)-E+V(r)\right]=0 \quad (3.45)$$

After some manipulation into (3.45), the equations for the factors become:

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \quad (3.46)$$

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\sin^2\theta\Theta = m^2\Theta \quad (3.47)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}\left[V(r)-E\right]R = l(l+1)R \quad (3.48)$$

where m^2 and $l(l+1)$ are constants on separation. After some operations, the solution of (3.46) is

$$\Phi(\phi) = (2\pi)^{-1/2}e^{im\phi} \quad (3.49)$$

Equation (3.47) is the Associated Legendre equation, so the solution is

$$\Theta(\theta) = P_{lm}(\cos\theta) \tag{3.50}$$

and normalization constant becomes

$$P_{lm}(\cos\theta) = \left[1 - \cos^2\theta\right]^{m/2} \frac{\partial^m P_l(\cos\theta)}{\partial\theta^m} \tag{3.51}$$

If we substitute $\cos\theta = \xi$, we get

$$P_l(\xi) = \frac{1}{2^l} l! \frac{\partial^l}{\partial\xi^l} \left(\xi^2 - 1\right)^l$$

where the $P_{lm}(\cos\theta)$ are Associated Legendre Polynomials. The functions Θ and Φ are often combined into a spherical harmonic $Y_{lm}(\theta, \phi)$, where

$$Y_{lm}(\theta, \phi) = CP_{lm}(\cos\theta)e^{im\phi}$$

If we rearranging equation (3.48) , we get

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) R_{nl}(r) - \frac{2\mu}{\hbar^2}\left[V(r) + \frac{l(l+1)}{2\mu r^2}\right] R_{nl}(r) + \frac{2\mu E}{\hbar^2} R_{nl}(r) = 0 \tag{3.52}$$

and also $R_{nl}(r)$ becomes

$$R_{nl}(r) = \frac{u_{nl}(r)}{r} \tag{3.53}$$

and

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) \frac{u_{nl}(r)}{r} = \frac{1}{r}\frac{d}{dr^2} u_{nl}(r) \tag{3.54}$$

By using equations (3.53) and (3.54), we get equation (3.52) of the form

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right] u_{nl}(r) = 0 \tag{3.55}$$

Following is example where exact solution is known. For no potential, $V = 0$, so the particle is free. Now we will examine this special case:

3.4.1 Free particle solution

In classical mechanics, a free particle of mass μ moves along a uniform linear trajectory. Its momentum p, its energy $E = p^2/2\mu$ and its angular momentum $L = r\ x\ p$ relative to the origin of coordinate system are constants of motion. In quantum physics, the observables p and $L = r\ x\ p$ do not commute. Hence, they represent incompatible quantities: It is not possible to measure the momentum and the angular momentum of a particle simultaneously.

Conceptually, the simplest scattering state is the free particle where potential is zero everywhere. We now look for solutions of the free particle radial Schrödinger equation (3.54). The radial Schrödinger equation for a free particle is not under any influence of potential $V(r)$ and freely travels from $-\infty$ to $+\infty$. The radial Schrödinger equation:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) R_{nl}(r) + \left[k^2 - \frac{l(l+1)\hbar^2}{r^2}\right] R_{nl}(r) = 0 \tag{3.56}$$

where $k^2 = \frac{2\mu E}{\hbar^2}$. The energy can only be positive in the case of free motion. If we change variables in equation 3.56 to $\rho = kr$ and write $R_{nl} = R_l(\rho)$, we obtain for $R_l(\rho)$ the equation:

$$\left(\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho}\right) R_l(\rho) + \left[k^2 - \frac{l(l+1)\hbar^2}{r^2}\right] R_l(\rho) = 0 \tag{3.57}$$

Which is called *spherical Bessel differential equation* whose particular solutions are $j_{l+1/2}(\rho)$ and $n_{l+1/2}(\rho)$. It is possible to write them in terms of the spherical Bessel functions:

$$j_l(\rho) = \left(\frac{\pi}{2\rho}\right)^{1/2} j_{l+1/2}(\rho) \quad . \tag{3.58}$$

and spherical Neumann functions

$$n_l(\rho) = (-1)^{l+1}\left(\frac{\pi}{2\rho}\right)^{1/2} j_{-l-1/2}(\rho) \tag{3.59}$$

where J_v is an ordinary Bessel function of order v.

The general form of the functions $j_l(\rho)$ and $n_l(\rho)$ are given by

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho} \tag{3.60}$$

$$n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\cos\rho}{\rho} \tag{3.61}$$

The asymptotic values of the spherical Bessel function for small and large ρ have the following forms

$$j_l(\rho) = \left\{ \begin{array}{ll} \frac{\rho^l}{1.3.5...(2l+1)} & \text{for } \rho \ll l \\ \frac{1}{\rho}\cos\left[\rho - \frac{\pi}{2}(l+1)\right] & \text{for } \rho \gg l \end{array} \right\} \tag{3.62}$$

The asymptotic values of the spherical Neumann function for small and large ρ are

$$n_l(\rho) = \left\{ \begin{array}{ll} \frac{1.3.5...(2l-1)}{\rho^{l+1}} & \text{for } \rho \ll l \\ \frac{1}{\rho}\sin\left[\rho - \frac{\pi}{2}(l+1)\right] & \text{for } \rho \gg l \end{array} \right\} \tag{3.63}$$

The general solution of equation (3.57) corresponding to a well-defined energy ($E = \hbar^2 k^2/2\mu$) and a well-defined orbital angular momentum l is of the form

$$R_{nl}(r) = Aj_l(kr) + Bn_l(kr) \tag{3.64}$$

Here the constant B must be zero because of the finiteness of the wave function in the origin since the spherical Neumann function $n_l(\rho)$ has a pole of order $l+1$ at origin and is therefore an irregular solution of (3.57). On the other hand, the spherical Bessel function $j_l(\rho)$ is finite at the origin and is thus a regular solution. Therefore, the radial and total wave functions of the Schrödinger equation (3.57) for a free particle are

$$R_{nl}(r) = Aj_l(kr) \quad (3.65)$$

$$\Psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \quad (3.66)$$

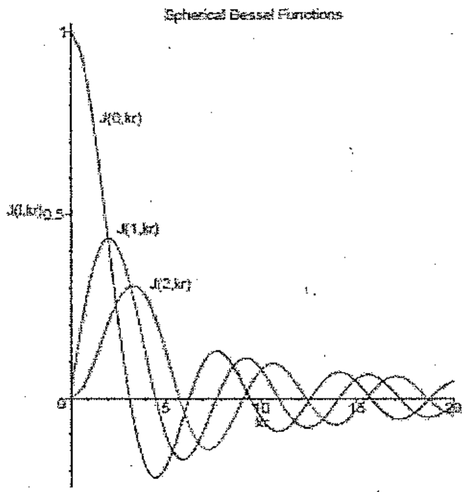

Figure 3.7 : Spherical Bessel function for different values of l.

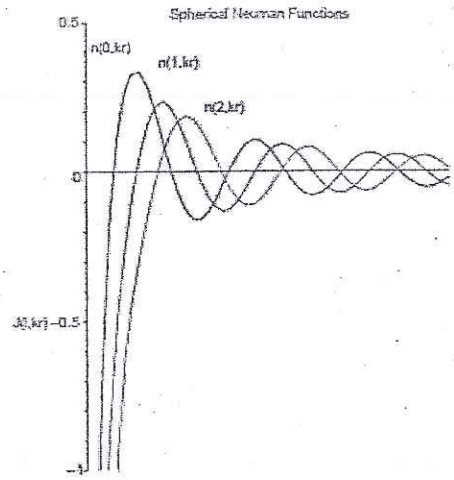

Figure 3.8 : *Spherical Neumann function for different values of l.*

Remarks:

1. The eigenvalues k^2 can take on any value in the interval of $(0, \infty)$ so that the energy $E = \frac{\hbar^2 k^2}{2\mu}$ can assume any value in this interval and the spectrum is continuous.

2. Every free particle eigenfunction can thus be labelled by the two discrete indeces l and m and by continuous index E (or k). So each energy eigenvalue is infinitely degenerate, since for a fixed value of E, the eigenfunctions are labelled by the two quantum numbers l and m such that $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l$.

Chapter 4

CONCLUSIONS

In this thesis, we obtained Bessel differential equation of second order differential equations with variable coefficients by using cylindrical coordinates of Laplace equation.

Bessel differential equation soluble by using Frobenious method and cylindrical function of the first kind and second kind, of index v $J_v(x)$ and $N_v(x)$ were obtained; respectively. If v is not an integer, the solution of this equation becomes linear combination of $J_v(x)$ and $J_{-v}(x)$. If v is an integer, the solution of this equation becomes linear combination of $J_v(x)$ and $N_v(x)$. So, modified bessel equation was discussed which can be obtained by changing variable by the formula $x = ix$. Many useful formulas for Bessel equation were obtained. Also, cylindrical functions with a half-integer index which are $J_{n+1/2}(x)$ and $J_{n-1/2}(x)$ have been shown. Moreover, we analyzed wronskian determinant which has given us linear combination of $J_v(x)$ and $J_{-v}(x)$ is linearly independent where v is noninteger and linear combination of $J_v(x)$ and $N_v(x)$ is also linearly independent where v is integer. Bessel integral and Jacobi expansion were observed in this thesis and significant funtions and integrals which are Bessel integral, Anguer function, Parseval integral, Jacobi expansion, generating function were obtained in this part. Also, many differential equations like modified bessel equation and schrödinger equation were shown that can be reduced to the Bessel equation using transformations of dependent variables.

This thesis has also given us applications of Bessel functions which are vibrations of circular membrane and Schrödinger equation. Bessel's differential equation is often encountered when solving boundary value problems, such as separable solutions to Laplace's equation or the Helmholtz equation, especially when working in cylindrical or spherical coordinates. The solution of the two dimensional wave equation of the circular membrane were discussed which involves Helmholtz equation and examined modes of vibration of circular membrane. Also, we discussed the famous problem, "Can you hear the shape of a drum?". Bessel functions are found to be the solution to the Schrödinger equation in a situation with spherical coordinates. Finally, solutions to the radial Schrödinger equation in spherical coordinates for a free particle were considered. The Bessel functions appear in many diverse scenarios, particularly situations involving cylindrical symmetry. The most difficult aspect of working with the Bessel function is determining that they can be applied through reduction of the system equation to Bessel's differential or modified equation, and then manipulating boundary conditions with appropriate application of zeroes, and the coefficient values on the argument of the Bessel function.

Conjectures and further comments

- The function z$\rightarrow J_p(\sqrt{z})$ is convex in domain if and only if

$p \neq -1$ and $p \geq -1.875$.

- The function $z \rightarrow zJ_p(\sqrt{z})$ is starlike in domain if and only if

$p \geq -0.875$.

- The function $z \rightarrow zJ_p(\sqrt{z})$ is starlike of order 1/2 in domain if and only if

$p \geq -0.875$.

- Question: What happens if p is complex?

- Radius of convexity of $z \rightarrow zJ_p(\sqrt{z})$?

- Radius of starlikeness of $z \rightarrow zJ_p(\sqrt{z})$?

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APPENDICES

APPENDICES

Appendix 1: Gamma Function

Gamma function $\Gamma(v)$ is defined as follows:

$$\Gamma(v) = \int_{0}^{\infty} e^{-t} t^{v-t} dt \tag{4.1}$$

As far as:

$$t^{v-1} = \frac{1}{v} \frac{\partial}{\partial t} t^{v} \tag{4.2}$$

By plugging (4.2) into (4.1), we get

$$v\Gamma(v) = \int_{0}^{\infty} e^{-t} \frac{d}{dt} t^{v} dt = e^{-t} t^{v} |_{\infty}^{0} + \int_{0}^{\infty} e^{-t} t^{v} dt \tag{4.3}$$

or

$$v\Gamma(v) = \Gamma(v+1) \tag{4.4}$$

Then $\Gamma(1) = 1$ and $\Gamma(2) = 1$. By induction if n is a natural number, we obtain:

$$\Gamma(n+1) = n! \tag{4.5}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{n}} (2n-1)! \tag{4.6}$$

where $(2n-1)! = 1.3.5...(2n-1)$. Then

$$\begin{aligned}
\Gamma(v) &= \frac{1}{v}\Gamma(v+1) \\
\Gamma(v) &\rightarrow \frac{1}{v}\; if \;\; v \rightarrow 0 \\
\Gamma(v-1) &= \frac{1}{v(v+1)}\Gamma(v+1) \\
\Gamma(v-n) &= \frac{1}{(v-n)(v-n+1)...v}\Gamma(v+1)
\end{aligned} \tag{4.7}$$

$\Gamma(v)$ has holes at all negative integral values of v. To find asymptotic behavior of Gamma-function as $v \rightarrow \theta$, we use so called *"Laplace Method."*

$$\begin{aligned}
\Gamma(v+1) &= \int\limits_0^\infty e^{-t}t^v dt = \int\limits_0^\infty e^{-\phi(t,v)} dt \\
\phi(t,v) &= t - v\ln(t)
\end{aligned} \tag{4.8}$$

Function $\phi(t,v)$ has a minimum at $t = v$. Indeed:

$$\frac{d\phi}{dt} = 1 - \frac{v}{t} = 0 \; if \; t = v \tag{4.9}$$

Near this minimum

$$\begin{aligned}
\phi &= \phi_0(v) + \frac{1}{2}\phi''(v)\tau^2 + ..., \qquad \tau = t - v \\
\phi_0(v) &= v - v\ln(v) \\
\phi''(v) &= \frac{1}{v}
\end{aligned}$$

Now we will replace in (4.2) $\phi\,(t,v)$ to its approximate value (4.9) and go from integration by t to intrgration τ. Without loss of accuracy we can consider that $-\infty < \tau < \infty$. Then

$$\Gamma(v+1) \approx e^{\phi_0(v)} \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2v}} d\tau \tag{4.10}$$

$$e^{-\phi_0(v)} = \left(\frac{v}{l}\right)^v$$

Now we replace $\tau = \sqrt{2v}y$ and remember that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$. We end up with the following answer:

$$\Gamma(v+1) \approx \sqrt{2\pi v}\left(\frac{v}{l}\right)^n \tag{4.11}$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{l}\right)^n$$

This is the Stirling approximation for $n = 5$, where $n! = 120$. The Stirling approximation gives $5! = 118.045$. The accuracy of the Stirling approximation is reasonable. We accept without proof:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{4.12}$$

where $\Gamma^2(1/2) = \pi$ so $\Gamma(1/2) = \sqrt{\pi}$.

For the logarithmic derivative of the gamma function

$$\psi(v) = \frac{d}{dv} \ln \Gamma(v)\Gamma = \frac{\Gamma'(v)}{\Gamma(v)} \tag{4.13}$$

the following expansion holds

$$\psi(v) = -C - \frac{1}{v} - \sum_{k=1}^{\infty}\left[\frac{1}{v+k} - \frac{1}{k}\right], \tag{4.14}$$

where C is the Euler constant; its approximate value is equal to 0.57722157.

If n is a positive integer, then

$$\psi(n+1) = -C + 1 + 1/2 + ... + 1/n \tag{4.15}$$

Appendix 2

Euler's Equation

The equation in the form

$$x^2y'' + pxy' + qy = 0 \tag{4.16}$$

with a regular singular point at x_0 is when both functions p and q are constants. Without loss of generality $x_0 = 0$.This equation is called *Euler's equation.* In thinking about possible solutions for Euler's equation ,we are led to consider power functions, since if $y(x) = x^s$, then differentiating y reduces the power by one while multiplying by x restores the power. More precisely, we have

$$xy'(x) = sx^s \; and \quad x^2y''(x) = s(s-1)x^s \, , \tag{4.17}$$

so all of the terms in Euler's equation are multiples of x^s .Making the substitutions, we get

$$\begin{aligned} x^2y'' + pxy' + qy &= [s(s-1) + ps + q]\, x^s \\ &= \left[s^2 + (p+1)s + q\right] x^s \end{aligned} \tag{4.18}$$

Thus $y(x) = x^s$ is a solution provided that

$$I(s) = s^2 + (p-1)s + q = 0 \tag{4.19}$$

The polynomial $I(s)$ is called the indicial polynomial, and equation (4.19) is called *the indicial equation.* Since it is a quadratic equation, it generally has two roots, s_1 and s_2, and $I(s)$ factors into $I(s) = (s - s_1)(s - s_2)$. The roots could be complex, but here we will only treat the case when s_1 and s_2 are real. Each real root leads to a solution of Euler's equation. Thus

$$y_1(x) = x^{s_1} \quad \textit{and} \quad y_2(x) = x^{s_2} \tag{4.20}$$

are both solutions for $x > 0$. Notice that x^s is not defined for $x < 0$ unless s is integer or $s = p/q$ where q is odd. Therefore, we limit the domain of the solution to $x > 0$. For $x < 0$, the solution is $|x^s|$. So, if the roots are real and different, then we have two solutions which are linearly independent since they are not constant multiples of each other. Therefore, the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{s_1} + c_2 x^{s_2} \tag{4.21}$$

where s_1 and s_2 are real ; c_1 and c_2 are constants. When the indicial equation has a double root, solutions $y_1(x)$ and $y_2(x)$ becomes

$$y_1(x) = x^{s_1} \quad \textit{and}\; y_2(x) = \frac{\partial y}{\partial s}(s_1, x) = \frac{\partial x^s}{\partial s}|_{s=s_1} = x^{s_1} \ln x \tag{4.22}$$

which are linearly independent.

The general result

We are ready to summarize our results about Euler's equation. However, there is one point to clear up. In the preceding discussion we have tacitly assumed that $x > 0$ and have found the solutions accordingly. But what about negative values of x? Our theorem will include the results for negative x.

Theorem 10 (1.1) *Consider the Euler equation*

$$(x-x_0)^2 y'' + p(x-x_0)y' + qy = 0 \tag{4.23}$$

and the associated indicial equation $I(s) = s(s-s_1) + ps + q = s^2 + (p-1)s + q = 0$.

1.If the indicial equation has two distinct real roots s_1 *and* s_2*, then a fundamental set of solutions defined for* $x \neq x_0$ *is*

$$y_1(x) = |x-x_0|^{s_1} \quad \text{and} \quad y_2(x) = |x-x_0|^{s_2}.$$

2. If the indicial equation has one root s_1 *of multiplicity 2, then*
$y_1(x) = |x-x_0|^{s_1}$ *and* $y_2(x) = |x-x_0|^{s_1} \ln|x-x_0|$
is a fundamental set of solutions defined for $x \neq x_0$.

Appendix 3

The Method of Frobenius

We will find solutions of Bessel equation using the method of Frobenious. It is motivated by Euler equation and power series. We will be solving equation

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{4.24}$$

Notice the coefficients $p(x)$ and $q(x)$ are analytic at o,they have power series expansions at that point. Instead of looking for a solution which is just a power series, as we do at an ordinary point, or just a power function, as we did for Euler's equation. We look for a solution which is the product of a power series and power function. A solution type

$$y(s,x) = \sum_{k=0}^{\infty} a_k x^{k+s} \tag{4.25}$$

will be called *a Frobenius solution.* The frobenius method tells us that we can seek a power series solution of the form equation (4.25). By differentiating equation (4.25), we get:

$$y'(s,x) = \sum_{k=0}^{\infty}(k+s)a_k x^{k+s-1} \tag{4.26}$$

$$y''(s,x) = \sum_{k=0}^{\infty}(k+s-1)(k+s)a_k x^{k+s-2} \tag{4.27}$$

and where the coefficients $p(x)$ and $q(x)$ are analytic at 0 and have power series expansion

$$p(x) = \sum_{k=0}^{\infty} p_k x^k \quad and \quad q(x) = \sum_{k=0}^{\infty} q_k x^k \tag{4.28}$$

which converge in an interval containing $x = 0$, We will use operator notation L and write equation C1 as

$$Ly = x^2 y'' + xp(x)y' + q(x)y = 0 \tag{4.29}$$

Substituting equations (4.26), (4.27) and (4.28) into equation (4.24), we get

$$Ly(s,x) = x^2 y'' + xp(x)y' + q(x)y$$

$$= x^s . \sum_{k=0}^{\infty} A_k(s) x^{k+s} \tag{4.30}$$

where the coefficients are

$$A_k(s) = (s+k)(s+k-1)a_k(s) + \sum_{n=0}^{k}\left[p_{k-n}(s+n) + q_{n-k}\right]a_k(s) \tag{4.31}$$

We get a solution to equation (4.24) provided that $A_k(s) = 0$ for $n \geq 0$. For $n = 0$, equation (4.31) become

$$\begin{aligned} A_0(s) &= [s(s-1)+p_0 s+q_0]\, a_0 \\ &= \left[s^2+(p_0-1)s+q_0\right] a_0 \\ &= I(s)a_0 \end{aligned} \tag{4.32}$$

where

$$I(s) = s^2 + (p_0 - 1)s + q_0 \tag{4.33}$$

is the indicial polynomial. Since we are assuming $a_0 \neq 0$, the coefficient $A_0(s) = 0$ only is $I(s) = 0$. This is the indicial equation, ant its roots are the only powers s for which there can be solutions. We will only consider the case when the indical equation has two real roots s_1 and s_2, and we will assume that they are ordered by $s_2 \leq s_1$. Then

$$A_0(s) = a_0 I(s) = a_0(s - s_1)(s - s_2) \tag{4.34}$$

Since $A_k(s) = 0$, equation (4.30) becomes

$$Ly(s,x) = A_0(s)x^s = a_0 I(s)x^s = a_0(s - s_1)(s - s_2) \tag{4.35}$$

The right-hand side of equation (4.35) vanishes for $s = s_1$ and $s = s_2$, so $y(s_1, x)$ and $y(s_2, x)$ are solutions. The solution for the larger root $s = s_1$ becomes

$$y_1(x) = y(s_1, x) = x^{s_1} \sum_{k=0}^{\infty} a_n(s_1)x^k = \sum_{k=0}^{\infty} a_k(s_1)x^{s_1+k}, \tag{4.36}$$

which is a Frobenius solution corresponding to the root s_1. The second solution when $s_1 - s_2$ is not a nonnegative integer becomes

$$y_2(x) = y(s_2, x) = x^{s_2} \sum_{k=0}^{\infty} a_k(s_2)x^k = \sum_{k=0}^{\infty} a_k(s_2)x^{s_2+k} \tag{4.37}$$

If the roots are equal, the second solution becomes

$$
\begin{aligned}
y_2(x) &= \frac{\partial y}{\partial s}(s_1, x) \\
&= x^{s_1} \ln x \sum_{k=0}^{\infty} a_k(s_1) x^k + x^{s_1} \sum_{k=0}^{\infty} a_k'(s_1) x^k && (4.38) \\
&= y_1(x) \ln x + x^{s_1} \sum_{k=1}^{\infty} a_k'(s_1) x^k && (4.39)
\end{aligned}
$$

Notice that the sum in the last expression starts at $n = 1$ instead of at $n = 0$. This is because a_0 is a constant, so $a_0' = 0$.

Appendix 4

Laplace Transform

The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

The (unilateral) Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$ is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt \tag{4.40}$$

where the parameter s is a complex number where $s = \alpha + i\beta$ with real number α and β (Abramowitz and Stegun 1972).

The unilateral Laplace transform is almost always what is meant by "the" Laplace transform, although a bilateral Laplace transform is sometimes also defined as

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0}^{\infty} f(t)e^{-st}dt \tag{4.41}$$

The inverse Laplace transform is given by the following complex integral, which is known by various names (the Bromwich integral, the Fourier-Mellin integral, and Mellin's inverse formula):

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds \tag{4.42}$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$. An alternative formula for the inverse Laplace transform is given by Post's inversion formula.

Proof of the Laplace transform of a function's derivative

It is often convenient to use the differentiation property of the Laplace transform to find the transform of a function's derivative. This can be derived from the basic expression for a Laplace transform as follows:

$$\begin{aligned}
\mathcal{L}\left\{f(t)\right\} &= \int_{0^-}^{\infty} f(t)e^{-st}dt && (4.43)\\
&= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty}\frac{e^{-st}}{-s}f'(t)dt \quad \text{(by parts)} && (4.44)\\
&= \left[-\frac{f(0)}{-s} + \frac{1}{s}\mathcal{L}\left\{f'(t)\right\}\right] && (4.45)
\end{aligned}$$

yielding

$$\mathcal{L}\left\{f'(t)\right\} = s.\mathcal{L}\left\{f'(t)\right\} - f(0) \tag{4.46}$$

and in the bilateral case,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= s\int_{-\infty}^{\infty} e^{-st}f(t)dt \\ &= s\mathcal{L}\{f(t)\}\end{aligned} \tag{4.47}$$

The general result

$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$$

where f^n is the n-th derivative of f, can then be established with an inductive argument.

The Laplace transform is a powerful integral transform used to switch a function from the time domain to the S-domain. The use of Laplace transform makes it much easier to solve linear differential equations with given initial conditions

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n\mathcal{L}\{f\} - \sum_{i=1}^{n} s^{(n-i)}f^{(i-1)}(0) \tag{4.48}$$

Consider the following differential equation:

$$\sum_{i=0}^{n} a_i f^{(i)}(t) = \phi(t) \tag{4.49}$$

$$\sum_{i=0}^{n} a_i\mathcal{L}\left\{f^{(i)}(t)\right\} = \mathcal{L}\{\phi(t)\} \tag{4.50}$$

which is equivalent to

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{\phi(t)\} + \sum_{i=1}^{n} a_i \sum_{j=0}^{i} s^{i-j}f^{(j-1)}(0)}{\sum_{i=0}^{n} a_i s^i} \tag{4.51}$$

Note that $f^{(k)}(0)$ are initial conditions. The solution for $f(t)$ will be given by applying the inverse Laplace transform to $\mathcal{L}\{f(t)\}$.

Laplace transform of Differential Equations with Polynomial Coefficients

The equation for Laplace transform is

$$\frac{d^n}{ds^n}F(s) = (-1)^n \mathcal{L}\{t^n f(t)\} \qquad (s > \alpha) \tag{4.52}$$

for $f(t)$ piecewise continuous on $[0, \infty]$ and $F(s) = \mathcal{L}\{f(t)\}$. Hence for $n = 1$,

$$\mathcal{L}\{tf(t)\} = -F'(s) \tag{4.53}$$

Then

$$\begin{aligned} \mathcal{L}\{tf'(t)\} &= -\frac{d}{ds}\mathcal{L}\{f'(t)\} \\ &= -\frac{d}{ds}\{sF(s) - f(0)\} \\ &= -sF'(s) - F(s) \end{aligned} \tag{4.54}$$

Similarly for $f''(t)$

$$\begin{aligned} \mathcal{L}\{tf''(t)\} &= -\frac{d}{ds}\mathcal{L}\{f''(t)\} \\ &= -\frac{d}{ds}\left(s^2F(s) - sf(0) - f'(0)\right) \\ &= -s^2F'(s) - 2sF(s) + f(0) \end{aligned} \tag{4.55}$$