CONVERGENCE IN THE VARIATION SEMINORM OF BERNSTEIN AND BERNSTEIN CHLODOVSKY POLYNOMIALS

by

Rifat DEMIRAY

THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE NEAR EAST UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

JUNE 2013

Rifat Demiray: CONVERGENCE IN THE VARIATION SEMINORM OF BERNSTEIN AND BERNSTEIN CHLODOVSKY POLYNOMIALS

Approval of Director of Graduate School of Applied Sciences

Prof. Dr. İlkay SALİHOĞLU

We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics

Examining Committee in Charge:

Prof. Dr. İ. Kaya Özkın,	Committee Chairman, Department of Mathematics, Near East University
Assoc. Prof. Dr. Mehmat Ali Özarslan,	Department of Mathematics, Eastern Mediterranean University
Assoc. Prof. Dr. Harun Karslı,	Supervisor, Department of Mathematics, Bolu Abant İzzet Baysal University
Assoc. Prof. Dr. Evren Hıncal,	Department of Mathematics, Near East University
Assist. Prof. Dr. Burak Şekeroğlu,	Department of Mathematics, Near East University

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name:

Signature:

Date:

ABSTRACT

Convergence in the variation seminorm of Bernstein and Bernstein Chlodovsky Polynomials

June 2013, 79 pages

This thesis is devoted to a study of the variation detracting property, convergence in variation and rates of approximation of Bernstein and Bernstein-Cholodovsky polynomials in the space of functions of bounded variation with respect to the variation seminorm. For instance, the variation detracting property $V_{[0,1]}[B_n f] \leq V_{[0,1]}[f]$ holds for all function f of bounded variation. Nevertheless, the expression $\lim_{n\to\infty} V_{[0,1]}[B_n f - f] = 0$, which represents the convergence of the polynomial $B_n f$ to the function f in the variation seminorm, is valid if and only if f is absolutely continuous. Additionally, the variation detracting property is related to the Voronovskaya-type theorems for the derivative of the polynomials. On this occasion, the Voronovskaya-type theorems having a significant place in the convergence in the variation seminorm and the relationships between these theorems and the convergence in the variation seminorm are mentioned in this thesis.

Keywords : Linear positive operators, Bernstein polynomials, Bernstein-Chlodovsky operators, Korovkin Theorem, bounded variation, variation seminorm, convergence and rate of convergence in the variation seminorm.

ÖZET

Bernstein ve Bernstein-Chlodovsky Polinomlarının varyasyon yarınormunda yakınsaklıkları

Haziran 2013, 79 sayfa

Bu tez, varyasyon yarınormuna göre sınırlı salınımlı fonsiyon uzayında Bernstein ve Bernstein-Cholodovsky polinomlarının salınım azaltma özelliği, varyasyonda yakınsaklık ve yakınsaklık hızları konusunda bir çalışmaya adanır. Örneğin, tüm sınırlı salınımlı f fonksiyonları için salınım azaltma özelliği $V_{[0,1]}[B_n f] \leq V_{[0,1]}[f]$ sağlanır. Fakat, varyasyon yarınormunda $(B_n f)$ polinomunun f fonksiyonuna yakınsamasını temsil eden $\lim_{n\to\infty} V_{[0,1]}[B_n f - f] = 0$ ifadesi ancak ve ancak f mutlak yakınsak ise vardır. Ek olarak, salınım azaltma özelliği polinomların türevleri için olan Voronovskaya tipi teoremler ile ilişkilidir. Bu vesile ile, bu tezde varyasyon yarınormundaki yakınsaklıkta önemli bir yere sahip olan Voronovskaya tipi teoremlerden ve bu teoremler ve varyasyon yarınormundaki yakınsaklık arasındaki ilişkilerinden bahsedilmiştir.

Anahtar Kelimeler : Lineer pozitif operatorler, Bernstein polinomları, Bernstein-Chlodovsky operatörleri, Bohman-Korovkin Teoremi, sınırlı salınım, varyasyon yarınormu, varyasyon yarınormunda yakınsaklık ve yakınsaklık hızı.

ACKNOWLEDGEMENTS

I would initially like to thank to my supervisor Assoc. Prof. Dr. Harun KARSLI for his invaluable feedbacks and being with me with his sympathy and support. He relied on me all the time during my study.

I would like to utter my heartfelt gratitude to Assoc. Prof. Dr. Evren HINCAL who guided me with his knowledge and comforted me during my study when I felt despondent. I am also grateful to Prof. Dr. I. Kaya OZKIN, Assist. Prof. Dr. Burak SEKEROGLU and Assist. Prof. Dr. Abdulrahman Mousa OTHMAN for believing in me. Intercalarily, I am obliged to Assist. Prof. Dr. Ismail TIRYAKI for his favor in organizing this thesis.

I really would like to express my gratitudes to Rifat REIS, Sevgi SERAT, Selen REIS, Mine REIS, Selin REIS and Turgut DEMIRAY who are my grandfather, grandmother, mother, aunt, sister and twin, respectively. They always provided everything which I needed to feel delighted.

I would like to put into words my sincere appreciation to my fiancee, Şüküfe KOCABAS who I feel like being so providential to have a place in her life. She has always been kind to me during the stressfull period and played an important role in writing this thesis.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This work is based on the field of approximation theory. The current studies concerning approximation theory mostly focus on the approximation of real-valued continuous functions by the class of algebraic polynomials.

A fundamental result for the functions approximation theory development is known as first Weierstrass approximation theorem, established by K.Weierstrass in 1885 which asserts that for each function $f \in C[a,b]$ and all $\epsilon > 0$, there is a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon$$

for any $x \in [a,b]$. This theorem was concerned with the density of the space of polynomials in $C[a,b]$. It was so arduous to comprehend the first proof of Weierstrass due to being complicated and long. Accordingly, this complexity encouraged such a lot of mathematicians to find a simpler and more apprehensible proof.

In 1912, the well-known Bernstein polynomials

$$(B_n f)(x) = B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

for any function $f(x)$ defined on $[0,1]$ were introduced by S. Bernstein (Bernstein, 1912) with the purpose of giving a simpler proof of the approximation theorem of Weierstrass. In addition to this, if $f \in C[a,b]$, then as it will be seen in Chapter 2,

$$\lim_{n\to\infty} B_n^f(x) = f(x)$$

uniformly in $[0,1]$.

In 1937, I. Cholodovsky (Cholodovsky, 1937) gave a more comprehensive proof for

Weierstrass theorem by calling into being the Bernstein-Cholodovsky operators in generalization of the Bernstein polynomials which approximate the function f defined on $[0, 1]$. These operators are given by

$$(C_n f) := \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right)$$

where f is a function defined on $[0, \infty)$ and bounded on every finite interval $[0, b] \subset [0, \infty)$ with a certain rate with $p_{k,n}$ denoting as usual

$$p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 \leq x \leq 1$$

and $(b_n)_{n=1}^{\infty}$ being a positive increasing sequence of real numbers with the properties

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{1.1}$$

As it shall be seen in Chapter 2, if

$$M(b; f) := \sup_{0 \leq x \leq b} |f(x)|$$

then if

$$\lim_{n\to\infty} \exp\left(-\alpha \frac{n}{b_n}\right) M(b_n; f) = 0 \tag{1.2}$$

for every $\alpha > 0$, it is said that $(C_n f)(x)$ converges to $f(x)$ at each point of continuity of f.

One of the simplest and most powerful proof of Weierstrass was come out by H. Bohman in 1952 and P.P. Korovkin in 1953. Bohman had the following idea: Let $L_n : C[a, b] \to C[a, b]$ be a sequence of positive linear operator. If $\left(L_n t^i\right) \rightrightarrows x^i$ $(i = 0, 1, 2)$ then

$$L_n f \rightrightarrows f \quad \text{on } [a, b].$$

Bohman proved this theorem in 1952 and a year later (in 1953) Korovkin proved the same theorem for integral type operators. On this occasion that theorem is mostly known as Bohman-Korovkin Theorem (Altomare and Campiti, 1994). The power of Bohman-Korovkin Theorem has attracted so many mathematicians and over the last sixty years, numerous research extended this theorem.

The rate of approximation by the $(B_n f)(x)$ to $f(x)$ and $(C_n f)(x)$ to $f(x)$ were formed by Voronovskaya (Voronovskaya, 1932) and J. Albrycht,J. Redecki (Albrycht and Redecki, 1960), respectively. For the former it was showed that, for bounded f on $[0, 1]$,

$$\lim_{n\to\infty} n\left[(B_n f)(x_0) - f(x_0)\right] = \frac{x_0(1-x_0)}{2} f''(x_0) \tag{1.3}$$

at each fixed point $x_0 \in [0, 1]$ for which there exists $f''(x_0) \neq 0$.

Intercalarily, for the latter, it was demonstrated that; for $\{b_n\}_{n=1}^{\infty}$ satisfying (1.1) ,

$$\lim_{n\to\infty} n\left[(C_n f)(x) - f(x)\right] = \frac{x f''(x)}{2}$$

provided (1.2) , for every $\alpha > 0$, at each point $x \geq 0$ for which $f''(x)$ exists. After 43 years of J. Albrycht and J. Radecki's proof, (1.3) was extended to first derivative of $(B_n f)(x)$ by Bardaro, Butzer, Stens, Vinti (Bardaro et.al., 2003). The theorem states for bounded f on $[0, 1]$ for which $f'''(x)$ exists at $x \in [0, 1]$,

$$\lim_{n\to\infty} n\left[(B_n f)'(x) - f'(x)\right] = \frac{1-2x}{2} f''(x) + \frac{x(1-x)}{2} f'''(x)$$

Furthermore, Butzer and Karsli (Butzer and Karsli, 2009) verified the similar theorem for first derivative of $(C_n f)$, which is given by

$$\lim_{n\to\infty} n\left[(C_n f)'(x) - f'(x)\right] = \frac{f''(x) + x f'''(x)}{2}$$

holds at each fixed point $x \geq 0$ for which $f'''(x)$ exists, provided (1.2) is satisfied for every $\alpha > 0$.

This thesis is concerned with the variation detracting property, rates of approximation of the Bernstein and Bernstein-Cholodovsky polynomials in variation seminorm. It is also investigated that the convergence in variation seminorm by $(B_n f)$ to f and $(C_n f)$ to f, such as

$$\lim_{n\to\infty} V_I\left[B_n f - f\right] = 0$$

where $V_I[f]$ is the total variation of the function f. Throughout this thesis, the class $TV(I)$ is the space of all the functions of bounded variation on I, endowed with the seminorm

$$\|f\|_{TV(I)} := V_I[f].$$

The first study about the variation detracting property and the convergence in variation of a sequence of linear positive operators was come out by Lorentz (Lorentz, 1953). He proved that B_n have

$$V_{[0,1]}\left[B_n f\right] \leq V_{[0,1]}[f]$$

and it is called the variation detracting property.

It is taken from Bardaro, Butzer, Stens, Vinti's work (Bardaro et.al., 2003) that the variation detracting property is significant to research the convergence in variation seminorm. In addition, it is known that the meaning of the total variation of a function $f \subset AC(I)$ and $L_1(I) - norm$ of f are exactly identical.

After these available studies, convergence in semi-normed space has become a new field in the theory of approximation.

CHAPTER 2

PRELIMINARIES AND AUXILIARY RESULTS

In this chapter preliminaries and auxiliary results that will be used throughout this thesis are presented. Some basic definitions and significant theorems about linear positive operators concerning approximation theory are given, as well. Addition to these, this chapter is dedicated to give some famous theorem about approximation theory such as Weierstrass, Bernstein, Cholodovsky, Bohman-Korovkin's Theorem.

Definition 2.1 *(Normed Space)*

A normed space X is a vector space with a norm defined on it. Here a norm on a (real or complex) vector space X is a real valued function on X whose value at an $x \epsilon X$ is denoted by

$$||x|| \qquad\qquad \text{(read "norm of } x\text{")}$$

and which has the properties

(N1) $||x|| \geq 0$

(N2) $||x|| = 0 \iff x = 0$

(N3) $||\alpha x|| = |\alpha| \, ||x||$

(N4) $||x + y|| \leq ||x|| + ||y||$

here x and y are arbitrary vectors in X and α is any scalar.

(E. Kreyszig, 1978)

Definition 2.2 *(Seminorm)*

A seminorm on a vector space X is a mapping $p : X \to \mathbb{R}$ satisfying (**N1**), (**N3**) and (**N4**) and a part of (**N2**) which is said that if $x = 0$ then $||x|| = 0$.

(E. Kreyszig, 1978).

By observing this definition, it can not be said that if $||x|| = 0$ then $x = 0$ in a semi-normed space. In other words the fact $||x|| = 0$ does not provide the expression $x = 0$.

Definition 2.3 *(Operator)*

Let X and Y be two linear normed function spaces. An operator $L : X \to Y$ is a rule which assigns to each function of X a function of Y. The operators are denoted by $(Lf)(x)$.

(Butzer and Nessel, 1971)

Definition 2.4 *(Linear Operator)*

Let X and Y be normed spaces and $T : D(T) \subset X \to R(T) \subset Y$. The operator T is called a linear operator if it satisfies the following conditions:

i) $T(x + y) = T(x) + T(y)$

ii) $T(\alpha x) = \alpha T(x)$ where α is a scalar.

(E. Kreyszig, 1978)

Definition 2.5 *(Positive Linear Operator)*

A linear operator L defined on a linear space of functions, V, is called positive, if

$$L(f) \geq 0, \text{ for all } f \in V, f \geq 0.$$

(Radu Paltanea, 2004)

Theorem 2.6 *Linear positive operators are monotone increasing.*

Proof. Let L be a positive linear operator. It is sufficient to show that for the functions f and g,

$$f \leq g \text{ then } L(f) \leq L(g)$$

Since $f \leq g$, it is clear that $g - f \geq 0$. Then $L(g - f) \geq 0$ because of L is positive. Besides from linearity of L, it can be written that $L(g) - L(f) \geq 0$. Therefore, $L(f) \leq L(g)$. Consequently, L is monotone increasing. □

Theorem 2.7 *If L is a linear positive operator then*

$$|L(f)| \leq L(|f|)$$

Proof. Let f be an arbitrary function. It is obvious that

$$-|f| \leq f \leq |f|$$

Since L is monotone increasing it is written that,

$$L(-|f|) \leq L(f) \leq L(|f|)$$

Because of linearity of L,

$$-L(|f|) \leq L(f) \leq L(|f|)$$

Hence,

$$|L(f)| \leq L(|f|).$$

□

Definition 2.8 *(Bounded Linear Operator)*

Let X and Y be normed spaces and $T : D(T) \rightarrow Y$ is a linear operator, where $D(T) \subset X$. The operator T is said to be bounded linear operator if there is a real number c_x such that for all $x \in D(T)$,

$$||Tx||_Y \leq c_x \, ||x||_X$$

(E. Kreyszig, 1978)

Definition 2.9 *(Uniformly continuous Function)*

A function $f : D \to \mathbb{R}$ is uniformly continuous on a set $E \subseteq D \subseteq \mathbb{R}$ if and only if for any given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(t)| < \epsilon$ for all $x, t \in E$ satisfying $|x - t| < \delta$.

(A.J.Kosmala, 2004)

Definition 2.10 *(Limit or Accumulation Point)*

Let M be a subset of a vector space X. Then a point x_0 of X (which may or may not be a point of M) is called a accumulation point of M (or limit point of M) if every neighborhood of x_0 contains at least one point $y \in M$ distinct from x_0.

(E. Kreyszig, 1978)

Definition 2.11 *(Closure)*

The set consisting of the points of M and the accumulation points of M is called the closure of M and is denoted by $\overline{M}$.

(E. Kreyszig, 1978)

Definition 2.12 *(Dense Set)*

A subset M of a metric space X is said to be dense in X if

$$\overline{M} = X.$$

(E. Kreyszig, 1978)

This definition means that every point of X is an accumulation point of M. In other words, for every point of X, a sequence in M can be found which converges to the element of X.

Definition 2.13 *(Pointwise Convergence)*

A sequence of functions $\{f_n\}$, where for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, converges (pointwise) on D to a function f if and only if for each $x_0 \in D$ the sequence of real numbers $\{f_n(x_0)\}$ converges to the real number $f(x_0)$.

(A.J.Kosmala, 2004)

Definition 2.14 *(Uniform Convergence)*

A sequence of functions $\{f_n\}$, where for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, converges uniformly to a function f if and only if for each $\epsilon > 0$ there exists $n^* \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$ and $n \geq n^*$.

(A.J.Kosmala, 2004)

Theorem 2.15 *(E. Kreyszig, 1978) Let (E, d) be a metric space. A nonempty subset M of E is closed if and only if for any sequence x_n in M, $x_n \to x_0$ implies that $x_0 \in M$.*

Proof. Let M be closed. Suppose that $x_0 \notin M$. Then $x_0 \in M^c$ where M^c is an open set. So there exists $B(x_0, r)$ such that $B(x_0, r) \subset M^c$. This means $B(x_0, r)$ does not contain any points in M distinct from x_0. This is a contradiction because of x_0 is a limit point of M.

Conversely, assume that M is not closed. Then M^c is not open. This means $\exists\, x_0 \in M^c$ such that $\forall\, \epsilon > 0$,

$$B(x_0, \epsilon) \cap M \neq \emptyset.$$

Select $\epsilon = \frac{1}{n}$ and take $x_n \in B(x_0, \epsilon) \cap M$. Then $(x_n) \subset M$ and $d(x_n, x_0) < \frac{1}{n}$. So

$$\lim_{n\to\infty} x_n = x_0 \notin \; M.$$

This is a contradiction which completes the proof. □

As it was mentioned, the significance of density of a set M in a metric space X is to find a sequence in M for every element of X such that the sequence converges to the element of X. Most of studies about approximation theory are concerned with the approximation of continuous functions by the class of polynomials. The first main study related to this was verified by K. Weierstrass in 1885. His work showed that the class of polynomials is dense in the class of continuous functions.

The following theorem is due to K. Weierstrass.

Theorem 2.16 *Each continuous real valued function f defined on* $[a, b]$ *is approximatable by algebraic polynomials.*

In other words for each $\epsilon > 0$ *there is a polynomial P with* $|f - P| < \epsilon$, $\forall\, x \in [a, b]$.

Proof. Consider the heat equation;

$$u_t = \alpha^2 u_{xx}\,, \qquad -\infty < x < \infty,\ t > 0.$$

for $u(x, t)$ a function of two variables, with initial condition

$$u(x, t) = f(x), \quad -\infty < x < \infty.$$

If the Green's Method is considered for solving the heat equation, then the solution is given by;

$$(w_n f)(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-n^2(t-x)^2} dt.$$

Let us consider the partial sum of $e^{-n^2(t-x)^2}$.

$$S_m = \sum_{k=0}^{m} \frac{[-n^2(t-x)^2]^k}{k!}.$$

Now, the following integral, which is written by using the above partial sum, is considered;

$$(P_m f)(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_m(t - x) f(t) dt.$$

Since $f(x)$ is continuous then $f(x)$ is bounded in any bounded and closed interval. In other words on any bounded and closed interval $|f(x)| \leq M$, where $M \in \mathbb{R}^+$.

The solution can be considered on $[a, b]$. So;

$$\begin{aligned} |(w_n f)(x) - (P_m f)(x)| &= \left| \frac{n}{\sqrt{2\pi}} \int_a^b f(t) e^{-n^2(t-x)^2} dt - \frac{n}{\sqrt{2\pi}} \int_a^b S_m(t-x) f(t) dt \right| \\ &\leq \frac{Mn}{\sqrt{2\pi}} \int_a^b \left| e^{-n^2(t-x)^2} - S_m(t-x) \right| dt < \frac{Mn\epsilon}{\sqrt{2\pi}}(b-a) \\ &= A\epsilon \end{aligned}$$

as $m \to \infty$, where $A = \frac{Mn}{\sqrt{2\pi}}(b-a)$.

If it is shown that $(P_m f)(x)$ is an algebraic polynomial and $(\omega_n f)(x) \to f(x)$, then the proof will be completed.

$$\begin{aligned} (P_m f)(x) &= \frac{n}{\sqrt{2\pi}} \int_a^b \sum_{k=0}^{m} \frac{[-n^2(t-x)^2]^k}{k!} f(t) dt \\ &= \frac{n}{\sqrt{2\pi}} \sum_{k=0}^{m} \frac{(-1)^k n^{2k}}{k!} \sum_{p=0}^{2k} c_k^p (-1)^p x^p \int_a^b t^{2k-p} f(t) dt. \end{aligned}$$

where $c_k^p = \frac{k!}{(k-p)!p!}$.

Hence, $(P_m f)(x) = \sum_{v=0}^{2m} A_v x^v = A_0 + A_1 x + A_2 x^2 + ... + A_{2m} x^{2m}$. So $(P_m f)(x)$ is an algebraic polynomial. In order to complete the proof, it must be shown that $(w_n f)(x) \to f(x)$ or $|(w_n f)(x) - f(x)| < \epsilon$ as $n \to \infty$.

In order to show $|(w_n f)(x) - f(x)| < \epsilon$, some properties of $(w_n f)(x)$ should be given. It is known that $(w_n f)(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-n^2(t-x)^2} dt$ where $\frac{n}{\sqrt{2\pi}} e^{-n^2(t-x)^2}$ is called the Gauss-Weierstrass kernel which is denoted by $GW_n(u) = \frac{n}{\sqrt{2\pi}} e^{-n^2u^2}$. Then,

1- $GW_n(u) > 0$.

2- $GW_n(-u) = GW_n(u)$. This implies $GW_n(u)$ is an even function.

3- $\lim_{n\to\infty} GW_n(u) = \lim_{n\to\infty} \frac{n}{\sqrt{2\pi}} e^{-n^2u^2} = \begin{cases} +\infty & , u = 0 \\ 0, & u \neq 0 \end{cases}$.

4- $\lim_{n\to\infty} \sup_{|u|\geq\delta} GW_n(u) = \lim_{n\to\infty} \frac{n}{\sqrt{2\pi}} e^{-n^2\delta^2} = 0$ and $\int_{-\infty}^{\infty} GW_n(u) du = 1$ (Gauss probability).

For all $\epsilon > 0$, there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ it is deduced that,

$$\begin{aligned}
|(w_n f)(x) - f(x)| &= \left| \int_{-\infty}^{\infty} [f(t) - f(x)] GW_n(t-x) dt \right| \\
&\leq \int_{-\infty}^{\infty} |f(t) - f(x)| \, GW_n(t-x) dt \\
&= \int_{x-\delta}^{x+\delta} |f(t) - f(x)| \, GW_n(t-x) dt + \int_{-\infty}^{x-\delta} |f(t) - f(x)| \, GW_n(t-x) dt \\
&\quad + \int_{x+\delta}^{\infty} |f(t) - f(x)| \, GW_n(t-x) dt
\end{aligned}$$

$$\begin{aligned}
&< \epsilon \int_{x-\delta}^{x+\delta} GW_n(t-x) dt + \epsilon + \epsilon \\
&< 3\epsilon.
\end{aligned}$$

Thus,

$$|f(x) - (P_m f)(x)| \leq |(w_n f)(x) - f(x)| + |(w_n f)(x) - (P_m f)(x)| < [3 + A]\epsilon.$$

In conclusion;

Every continuous function can be written as a limit of a sequence of polynomials

□

The first Weierstrass theorem was complicated and long. For this reason it was so hard to deduce. So, this situation made so many mathematicians study to find a simpler proof for Weierstrass approximation theorem. In the first instance, S.Bernstein (Bernstein, 1912) established a simpler proof by presenting Bernstein polynomials.

Definition 2.17 *(Bernstein polynomials)*

For a function $f(x)$ defined on the closed interval $[0, 1]$, the expression

$$(B_n f)(x) = B_n{}^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{2.1}$$

is called the Bernstein polynomial of order n of the function $f(x)$.

(Bernstein, 1912 see also G.G. Lorentz, 1986).

It follows from (2.1) that,

$$(B_n f)(0) = f(0) \qquad \text{and} \qquad (B_n f)(1) = f(1)$$

This means that a Bernstein polynomial for f interpolates f at both $x = 0$ and $x = 1$.

The Bernstein operator is linear, which follows from (2.1) that,

$$\begin{aligned}(B_n(\alpha f + \beta g))(x) &= \sum_{k=0}^{n} (\alpha f + \beta g)\left(\frac{k}{n}\right) p_{k,n}(x) \\ &= \sum_{k=0}^{n} (\alpha f)\left(\frac{k}{n}\right) p_{k,n}(x) + \sum_{k=0}^{n} (\beta g)\left(\frac{k}{n}\right) p_{k,n}(x) \\ &= \alpha (B_n f)(x) + \beta (B_n g)(x)\end{aligned}$$

for all f, g defined on $[0, 1]$ and all real α, β.

It can be seen readily that B_n is a positive operator. So it can be said that B_n is monotone increasing. Therefore it is concluded that,

$$m \le f(x) \le M \quad \Longrightarrow \quad m \le (B_n f)(x) \le M,\ x \in [0, 1]\,.$$

Let us prove the famous theorem of Weierstrass by the polynomials $(B_n f)(x)$. It can be made inferences that the theorem of Weierstrass is a corollary of the following theorem:

Theorem 2.18 *(Bernstein, 1912) For a function $f(x)$ bounded on $[0, 1]$, the relation*

$$\lim_{n\to\infty} B_n(x) = f(x)$$

holds at each point of continuity x of f; and the relation holds uniformly on $[0,1]$ *if* $f(x)$ *is continuous on this interval.*

Proof. It is obvious that $\sum_{k=0}^{n} p_{k,n}(x) = (x+1-x)^n = 1$. Moreover, the sums $\sum_{k=0}^{n} kp_{k,n}(x)$ and $\sum_{v=0}^{n} k^2 p_{k,n}(x)$ are found in a following way.

$$\begin{aligned}
\sum_{k=0}^{n} kp_{k,n}(x) &= \sum_{k=1}^{n} kp_{k,n}(x) = \sum_{k=1}^{n} k\frac{n!}{(n-k)!k!}x^k(1-x)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!}x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n-1} \frac{n!}{(n-1-k)!k!}x^{k+1}(1-x)^{n-1-k} \\
&= nx\sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!}x^k(1-x)^{n-1-k} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} k^2 p_{k,n}(x) &= nx\sum_{k=0}^{n-1}(k+1)\frac{(n-1)!}{(n-1-k)!k!}x^k(1-x)^{n-1-k} \\
&= nx + nx\sum_{k=0}^{n-1} k\frac{(n-1)!}{(n-1-k)!k!}x^k(1-x)^{n-1-k} \\
&= nx + nx(n-1)x \\
&= n^2x^2 - nx^2 + nx.
\end{aligned}$$

Since $x(1-x) \leq \frac{1}{4}$ on the closed interval $[0,1]$, then the following inequality can be obtained;

$$\begin{aligned}
\sum_{\left|\frac{k}{n}-x\right|\geq\delta} p_{k,n} &\leq \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \frac{(\frac{k}{n}-x)^2}{\delta^2}p_{k,n} \\
&= \frac{1}{n^2\delta^2}\sum_{\left|\frac{k}{n}-x\right|\geq\delta}(k^2-2nxk+n^2x^2)p_{k,n} \\
&\leq \frac{1}{n^2\delta^2}(n^2x^2-nx^2+nx-2n^2x^2+n^2x^2) \\
&= \frac{1}{n^2\delta^2}nx(1-x) \leq \frac{1}{4n\delta^2}
\end{aligned}$$

Because of boundness of $f(x)$, there exists $M \in \mathbb{R}^+$ such that $|f(x)| \leq M$ in $0 \leq x \leq 1$.

If x is a point of continuity of f, for a given $\epsilon > 0$, $\delta > 0$ can be found such that $|f(t) - f(x)| < \epsilon$ whenever $|x - t| < \delta$.

Hence it is written,

$$\begin{aligned}
|(B_n f)(x) - f(x)| &= \left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n} - f(x) \right| = \left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n} - \sum_{k=0}^{n} f(x) p_{k,n} \right| \\
&= \left| \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] p_{k,n} \right| \leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_{k,n} \\
&= \sum_{\left|\frac{k}{n} - x\right| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_{k,n} + \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| p_{k,n}
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon + \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left[\left| f\left(\frac{k}{n}\right) \right| + |f(x)| \right] p_{k,n} \\
&\leq \epsilon + 2M \sum_{\left|\frac{k}{n} - x\right| \geq \delta} p_{k,n} \leq \epsilon + \frac{M}{2n\delta^2}
\end{aligned}$$

Consequently,

$$|(B_n f)(x) - f(x)| \leq \epsilon + \frac{M}{2n\delta^2} \quad \text{as } n \to \infty \tag{2.2}$$

which implies that

$$(B_n f)(x) \to f(x)$$

Finally, if $f(x)$ is continuous on $[0, 1]$ then (2.2) holds with a δ independent of x so that

$$(B_n f)(x) \rightrightarrows f(x)$$

□

Lemma 2.19 *For $t \in [0,1]$ the inequality*

$$0 \leq z \leq \frac{3}{2}\sqrt{nt\,(1-t)}$$

implies

$$\sum_{|k-nt|\geq 2z\sqrt{nt(1-t)}} p_{k,n}(t) \leq 2\exp\left(-z^2\right).$$

(Albrycht and Redecki, 1960).

Bernstein polynomials was defined on bounded interval $[0,1]$. By linear substitution, the interval $[a,b]$ can be transformed into $[0,1]$. Bernstein polynomials were not consisting any problem on bounded interval for the proof of Weierstrass approximation theorem. At this stage, the major question making mathematicians think was concerned with Bernstein polynomials on an unbounded interval. In 1937, Chlodovsky solved that question by following this way:

Let the function $f(x)$ be defined on the interval $[0,b)$, $b > 0$. In order to obtain the Bernstein polynomials $B_n^f(x)$ for the interval $(0,b)$, let us define the Bernstein polynomial of $Q(y)$, $0 \leq y \leq 1$,

$$B_n^Q(y) = \sum_{k=0}^{n} Q\left(\frac{k}{n}\right)\binom{n}{k} y^k (1-y)^{n-k}$$

Let us make the subsitution $y = \frac{x}{b}$ in the polynomial $B_n^Q(y)$. So it can be seen easily that $Q(y) = f(by)$, $0 \leq y \leq 1$. Therefore

$$B_n^f(x) = \sum_{k=0}^{n} f\left(b\frac{k}{n}\right)\binom{n}{k}\left(\frac{x}{b}\right)^k\left(1-\frac{x}{b}\right)^{n-k}$$

for a constant b. It is assumed here that $b = b_n$ is a function of n.

Let's suppose that $f(x)$ is defined in $0 \leq x < \infty$. In order to obtain the relation

$$B_n^f(x) \to f(x)$$

for this interval, it must be accepted that the distance between two adjacent points $\frac{b_n}{n} \to 0$ as $n \to \infty$. This means that $b_n = o(n)$.

As it can be seen above, Chlodovsky modified Bernstein polynomials by extending the interval $[0, 1]$ into unbounded interval $[0, \infty)$. Herewith, the polynomials that Chlodovsky introduced are cited as Bernstein-Chlodovsky polynomials. Therefore Bernstein-Chlodovsky polynomials can be given as below:

Definition 2.20 *(Bernstein-Chlodovsky polynomials)*

Bernstein-Chlodovsky polynomials are given by

$$(C_n f)(x) = \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right)$$

where f is a function defined on $[0, \infty)$ and bounded on every finite interval $[0, b] \subset [0, \infty)$ with a certain rate, with $p_{k,n}$ denoting as usual

$$p_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k} \quad , \quad 0 \leq x \leq 1$$

and $(b_n)_{n=1}^{\infty}$ being a positive increasing sequence of real numbers with the properties

$$\lim_{n\to\infty} b_n = \infty \quad , \quad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

(Chlodovsky, 1937 see also Karsli, 2011).

After Chlodovsky introduced Bernstein-Chlodovsky polynomials, he proved Weierstrass approximation theorem by utilizing them.

The following theorem is due to Cholodovsky.

Theorem 2.21 *(Cholodovsky, 1937) If $b_n = o(n)$ and*

$$\lim_{n\to\infty} \exp\left(-\alpha \frac{n}{b_n}\right) M(b_n; f) = 0 \quad \text{for each } \alpha > 0,$$

then

$$\lim_{n\to\infty} (C_n f) = f(x)$$

at any point of continuity of the function f.

Proof.

$$\begin{aligned}
|(C_n f)(x) - f(x)| &= \left|\sum_{k=0}^{n}\left[f\left(\frac{kb_n}{n}\right) - f(x)\right] p_{k,n}\left(\frac{x}{b_n}\right)\right| \\
&\leq \sum_{k=0}^{n}\left|f\left(\frac{kb_n}{n}\right) - f(x)\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
&= \sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\left|f\left(\frac{kb_n}{n}\right) - f(x)\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
&\quad + \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|f\left(\frac{kb_n}{n}\right) - f(x)\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
&=: \sum_{1*} + \sum_{2*}
\end{aligned}$$

It is expected to prove that

$$\lim_{n\to\infty}\sum_{1*} = 0 \quad \text{and} \quad \lim_{n\to\infty}\sum_{2*} = 0$$

If x is a point of continuity of f, for a given $\epsilon > 0$, $\delta > 0$ is found such that $|f(t) - f(x)| < \epsilon$ whenever $|x - t| < \delta$. Therefore,

$$\begin{aligned}
\sum_{1*} &= \sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\left|f\left(\frac{kb_n}{n}\right) - f(x)\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
&< \epsilon \sum_{\left|\frac{kb_n}{n}-x\right|<\delta} p_{k,n}\left(\frac{x}{b_n}\right) \\
&\leq \epsilon
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\sum_{1*} = 0$$

Since $M(b;f) := \sup_{0\leq x\leq b}|f(x)|$,

$$\sum_{2*} = \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left| f\left(\frac{kb_n}{n}\right) - f(x) \right| p_{k,n}\left(\frac{x}{b_n}\right)$$
$$\leq 2M(b_n;f) \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right)$$

For $\left|\frac{k}{n}b_n - x\right| \geq \delta$ it is obtained that,

$$\left|k - n\frac{x}{b_n}\right| \geq \frac{n}{b_n}\delta = 2\left(\frac{\sqrt{n}\delta}{2\sqrt{x(b_n - x)}}\right)\sqrt{n\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)}$$
$$\geq 2\left(\frac{\sqrt{n}\delta}{2\sqrt{xb_n}}\right)\sqrt{n\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)}$$

Therefore according to Lemma (2.19) ,

$$\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right) \leq 2\exp\left(-\frac{\delta^2 n}{4xb_n}\right) \tag{2.3}$$

Thus,

$$\sum_{2*} \leq 4M(b_n;f)\exp\left(-\frac{\delta^2 n}{4xb_n}\right)$$

which implies

$$\lim_{n\to\infty}\sum_{2*} \leq \lim_{n\to\infty} 4M(b_n;f)\exp\left(-\frac{\delta^2 n}{4xb_n}\right) = 0$$

Consequently,

$$\lim_{n\to\infty} C_n f = f.$$

□

After Bernstein and Chlodovsky, H. Bohman gave a more general idea to prove the density and verified Weierstrass approximation theorem in 1952. One year later, P.P. Korovkin attested the same theorem for integral type operators. For this reason this theorem is known as Bohman-Korovkin Theorem.

The following theorem was given by Bohman and Korovkin and is called Bohman-Korovkin Theorem.

Theorem 2.22 *(Altomare and Campiti, 1994) Let L_n be a sequence of positive linear operators from $C[a,b]$ into itself. Assume that*

$$(L_n t^i)(x) \rightrightarrows x^i \quad (i = 0, 1, 2).$$

Then for every $f \in C[a,b]$,

$$(L_n f)(x) \rightrightarrows f(x) \quad \text{on } [a,b].$$

Proof. Let $f \in C[a,b]$. Then f is uniformly continuous and bounded on $[a,b]$. By the definition of uniformly continuous it is said that $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $|f(t) - f(x)| < \epsilon$ for all x, t in $[a,b]$ satisfying $|t - x| < \delta$.

Since $f(x)$ is bounded on $[a,b]$, there exists $M > 0$ such that $|f(x)| \leq M$ in $a \leq x \leq b$. According to the triangle inequality it is deduced that,

$$|f(t) - f(x)| \leq |f(t)| + |f(x)| \leq 2M$$

If $|t - x| \geq \delta$ then $\frac{(t-x)^2}{\delta^2} \geq 1$. So, it can be written that

$$|f(t) - f(x)| \leq 2M \frac{(t-x)^2}{\delta^2}$$

Thus,

$$\begin{aligned} \text{for } |t - x| &< \delta, \quad |f(t) - f(x)| < \epsilon \\ \text{for } |t - x| &\geq \delta, \quad |f(t) - f(x)| < 2M \frac{(t-x)^2}{\delta^2} \end{aligned}$$

it is concluded that

$$|f(t) - f(x)| < \epsilon + 2M \frac{(t-x)^2}{\delta^2} \text{ for all } t, x \in [a,b]$$

Now let us show that $(L_n f)(x) \rightrightarrows f(x)$.

$$\begin{aligned}
\left|(L_n f(t))(x) - f(x)\right| &= \left|(L_n f(t))(x) - (L_n f(x))(x) + (L_n f(x))(x) - f(x)\right| \\
&= \left|(L_n(f(t) - f(x)))(x) + f(x)(L_n 1)(x) - f(x)\right| \\
&= \left|(L_n(f(t) - f(x)))(x) + f(x)((L_n 1)(x) - 1)\right| \\
&\leq \left|(L_n(f(t) - f(x)))(x)\right| + |f(x)|\,|(L_n 1)(x) - 1|
\end{aligned}$$

By Theorem (2.7) ,

$$\left|(L_n f(t))(x) - f(x)\right| \leq (L_n |f(t) - f(x)|)(x) + |f(x)|\,|(L_n 1)(x) - 1|$$

Since L_n monotone increasing and $|f(x)| \leq M$ it is written that,

$$\left|(L_n f(t))(x) - f(x)\right| \leq \left(L_n \left(\epsilon + 2M\frac{(t-x)^2}{\delta^2}\right)\right)(x) + M\,|(L_n 1)(x) - 1)| \tag{2.4}$$

From the linearity of L_n,

$$\begin{aligned}
\left(L_n \left(\epsilon + 2M\frac{(t-x)^2}{\delta^2}\right)\right)(x) &= \epsilon(L_n 1)(x) + \frac{2M}{\delta^2} L_n((t-x)^2)(x) \\
&= \epsilon(L_n 1)(x) + \frac{2M}{\delta^2}(L_n(t^2 - 2tx + x^2))(x) \\
&= \epsilon(L_n 1)(x) + \frac{2M}{\delta^2}\{(L_n t^2)(x) - 2x(L_n t)(x) + x^2(L_n 1)(x) + 2x^2 - 2x^2\} \\
&= \epsilon(L_n 1)(x) + \frac{2M}{\delta^2}\left\{((L_n t^2)(x) - x^2) + 2x(x - (L_n t)(x)) + x^2((L_n 1)(x) - 1)\right.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(L_n \left(\epsilon + 2M\frac{(t-x)^2}{\delta^2}\right)\right)(x) &= \epsilon(L_n 1)(x) + \frac{2M}{\delta^2}\{((L_n t^2)(x) - x^2) \\
&\quad + 2x(x - (L_n t)(x)) + x^2((L_n 1)(x) - 1)\}
\end{aligned} \tag{2.5}$$

If the equality (2.5) is put in (2.4),

$$\begin{aligned}|(L_n f(t))(x) - f(x)| &\leq \epsilon(L_n 1)(x) + \frac{2M}{\delta^2}\{(L_n t^2)(x) - x^2) \\ &\quad +2x(x - (L_n t)(x)) + x^2((L_n 1)(x) - 1)\} + M\,|(L_n 1)(x) - 1)|\end{aligned}$$

Since $(L_n t^i)(x) \rightrightarrows x^i$ for $i = 0, 1, 2$ it can be seen easily from the above inequality,

$$\lim_{n\to\infty}\{\max_{a\leq x\leq b}|(L_n f(t))(x) - f(x)|\} = 0.$$

That is,

$$\lim_{n\to\infty}\|L_n(f) - f\| = 0.$$

Consequently,

$$(L_n f)(x) \rightrightarrows f(x).$$

□

The next theorem can be given for instance of Bohman-Korovkin Theorem.

Theorem 2.23 *Let* $f \in C[0, 1]$. *Then*

$$(B_n f)(x) \rightrightarrows f(x).$$

holds true.

Proof. Since $(B_n f)$ is a positive operator from $C[0, 1]$ into $C[0, 1]$, in order to prove the above theorem Bohman-Korovkin Theorem will be used . It is known that

$$\sum_{k=0}^{n} p_{k,n}(x) = 1, \quad \sum_{k=0}^{n} k p_{k,n}(x) = nx, \quad \sum_{k=0}^{n} k^2 p_{k,n}(x) = n^2x^2 - nx^2 + nx$$

Then,

$$
\begin{aligned}
(B_n1)(x) &= \sum_{k=0}^{n} 1.p_{k,n}(x) = 1 \\
(B_nt)(x) &= \sum_{k=0}^{n} \frac{k}{n} p_{k,n}(x) = \frac{1}{n} \sum_{k=0}^{n} k p_{k,n}(x) = x \\
B_nt^2)(x) &= \sum_{k=0}^{n} \frac{k^2}{n^2} p_{k,n}(x) = \frac{1}{n^2} \sum_{k=0}^{n} k^2 p_{k,n}(x) = x^2 - \frac{x^2}{n} + \frac{x}{n}
\end{aligned}
$$

Since x in $[0, 1]$, it is inferred that

$$
\begin{aligned}
|(B_n1)(x) - 1| &< \frac{1}{n} \\
|(B_nt)(x) - x| &< \frac{1}{n} \\
\left|(B_nt^2)(x) - x^2\right| &\leq \frac{2}{n}
\end{aligned}
$$

Since $\frac{1}{n}$ approaches 0 as $n \to \infty$, it can be said that

$$
\begin{aligned}
(B_n1)(x) &\rightrightarrows 1 \\
(B_nt)(x) &\rightrightarrows x \\
(B_nt^2)(x) &\rightrightarrows x^2
\end{aligned}
$$

Terefore by Bohman-Korovkin Theorem,

$$(B_nf)(x) \rightrightarrows f(x).$$

□

CHAPTER 3

FUNCTIONS OF BOUNDED VARIATION AND RELATED TOPICS

The focus of this chapter is to give some definition and theorems concerning total variation that it will be used in the topic of convergence and rate of convergence in the variation seminorm and set the relationships, which play an important role in approximation in the variation seminorm, among the spaces BV, AC, and TV. Further, it is referred to the Stieltjes integral and its relevance between Riemann integral and total variation.

3.1 Function of Bounded Variation

Let a function $f(x)$ be defined and finite on the

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

and form the sum

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

Definition 3.1 *(Total variation)*

The least upper bound of the set of all possible sums V is called the total variation of the function $f(x)$ on $[a, b]$ and is designated by $\bigvee_a^b(f)$ or $V_{[a,b]}[f]$

(I.P. Natanson,1964)

Definition 3.2 *(Finite variation)*

If $\bigvee_a^b(f) < +\infty$, then $f(x)$ is said to be a function of finite (or bounded) variation on $[a, b]$. It is also said that $f(x)$ has finite (or bounded) variation on $[a, b]$.

(I.P. Natanson, 1964)

Definition 3.3 *(BV space)*

The class of all functions of bounded variation on I is called BV space and denoted by $BV(I)$. This space can be endowed both with seminorm $|.|_{BV(I)}$ and with a norm, $||.||_{BV(I)}$, where

$$|f|_{BV(I)} := V_I[f], \qquad ||f||_{BV(I)} := V_I[f] + |f(a)|$$

$f \in BV(I)$, a being any fixed point of I.

(Octavian Agratini, 2006)

Definition 3.4 *(TV space)*

Let $I \subseteq \mathbb{R}$ be a fixed interval, and $V_I[f]$ the total variation of the function $f : I \to \mathbb{R}$. The class of all bounded functions of bounded variation on I endowed with the seminorm

$$||f||_{TV(I)} := V_I[f].$$

is called TV space and is denoted by $TV(I)$.

(Bardaro et.al., 2003)

Theorem 3.5 *A monotonic function on* $[a, b]$ *has finite variation on* $[a, b]$.

Proof. If f is a monotonically increasing function on $[a, b]$, then for any partition $\{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$,

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k) = f(b) - f(a)$$

Hence, $\bigvee_a^b(f) = f(b) - f(a) < +\infty$. This implies f is of finite variation on $[a, b]$.

It can be proven for a decreasing function in a similar way. □

Definition 3.6 *(Lipschitz condition)*

A finite function $f(x)$ defined on $[a, b]$ is said to satisfy a Lipschitz condition if there exist a constant K such that for any two points x,y in $[a, b]$,

$$|f(x) - f(y)| \leq K\,|x - y|\,.$$

(I.P. Natanson, 1964)

Theorem 3.7 *Every function of finite variation in* $[a, b]$ *is bounded in* $[a, b]$.

Proof. Let $a < x < b$ and $\{a, x, b\}$ be a partition of $[a, b]$ where $x_0 = a$, $x_1 = x$, $x_2 = b$. Then,

$\sum\limits_{k=0}^{1} |f(x_{k+1}) - f(x_k)| = |f(x) - f(a)| + |f(b) - f(x)| \leq \bigvee\limits_a^b(f)$. Since f is bounded variation on $[a, b]$, then $|f(x) - f(a)| \leq K$ where K is a nonnegative real number. It is concluded that

$$f(a) - K \leq f(x) \leq f(a) + K.$$

Since x is an arbitrary number in $[a, b]$,it is said that f is bounded on $[a, b]$. □

Theorem 3.8 *The sum, difference and product of two functions of finite variation are functions of finite variation.*

Proof. Let $f(x)$ and $g(x) \in BV\,[a, b]$. It is set that $s(x) = f(x) + g(x)$. Then,

$$|s(x_{k+1}) - s(x_k)| = |f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \leq |f(x_{k+1}) - f(x_k)| + |g(x_{k+1}) - g(x_k)|.$$

Follows from the observation, the inequality $\bigvee\limits_a^b(s) \leq \bigvee\limits_a^b(f) + \bigvee\limits_a^b(g)$ can be obtained. It is known that f and g are of bounded variation on $[a, b]$. So it can be written $\bigvee\limits_a^b(s) \leq M$, where M is a positive real number. This means that $s(x)$ is in $BV[a, b]$.

Similarly, it is shown that $f - g$ is of bounded variation on $[a, b]$.

In order to prove that $fg \in BV\,[a, b]$, let us consider a new function $t(x) = f(x)g(x)$. Then it can be written,

$$|t(x_{k+1}) - t(x_k)| = |f(x_{k+1})g(x_{k+1}) - f(x_k)g(x_k) - f(x_k)g(x_{k+1}) + f(x_k)g(x_{k+1})|$$

And from triangle inequality it is written,

$$|t(x_{k+1}) - t(x_k)| \leq |g(x_{k+1})|\,|f(x_{k+1}) - f(x_k)| + |f(x_k)|\,|g(x_{k+1}) - g(x_k)| \tag{3.1}$$

From Theorem (3.7), it is known that f and g are bounded on $[a, b]$. Therefore the inequality (3.1) implies that $\bigvee_a^b(t) = \bigvee_a^b(fg) \leq K$, where K is a positive real number. Hence $fg \in BV\,[a, b]$. □

Theorem 3.9 *Let a finite function $f(x)$ be defined on $[a, b]$ and let $a < c < b$. Then*

$$\bigvee_a^b(f) = \bigvee_a^c(f) + \bigvee_c^b(f).$$

Proof. Subdivide each of the intervals $[a, c]$ and $[c, b]$ by means of the points

$$a = y_0 < y_1 < ... < y_m = c \quad , \quad c = z_0 < z_1 < ... < z_n = b$$

Let $V_1 = \sum_{k=0}^{m-1} |f(y_{k+1}) - f(y_k)|$ and $V_2 = \sum_{k=0}^{n-1} f(z_{k+1}) - f(z_k)$. Then it is concluded that $V_1 + V_2 \leq \bigvee_a^b(f)$. Since the point sets $\{y_0, y_1, ..., y_m\}$ and $\{z_{0,}z_1, ..., z_m\}$ are arbitrary, it is created that $\bigvee_a^c(f) + \bigvee_c^b(f) \leq \bigvee_a^b(f)$.

Now subdivide the interval $[a, b]$ by means of the points

$$a = x_0 < x_1 < ... < x_n = b$$

Since $a < c < b$, suppose that $c = x_m$ where $m < n$. Then

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=m}^{n-1} |f(x_{k+1}) - f(x_k)| \leq \bigvee_a^c(f) + \bigvee_c^b(f)$$

Therefore, $\bigvee_a^b(f) \leq \bigvee_a^c(f) + \bigvee_c^b(f)$. In conclusion,

$$\bigvee_a^b(f) = \bigvee_a^c(f) + \bigvee_c^b(f)$$

□

Theorem 3.10 *A function $f(x)$ defined and finite on $[a, b]$ is a function of finite variation if and only if it is representable as the difference of two increasing functions.*

Proof. Let f_1 and f_2 be two increasing functions. By Theorem (3.5) f_1,f_2 are in $BV[a, b]$. So $f = f_1 - f_2 \in BV[a, b]$.

Conversely, setting $\pi(x) = \bigvee_a^x(f)$, where $a < x \leq b$, and $\pi(a) = 0$. It can be seen easily that π is an increasing function. Now let us consider a new function; $\nu(x) = \pi(x) - f(x)$. Firstly, it must be shown that $\nu(x)$ is an increasing function.

If $x < y$ then $\nu(y) - \nu(x) = \pi(y) - f(y) - \pi(x) + f(x) = \bigvee_a^y(f) - f(y) - \bigvee_a^x(f) + f(x)$. Follows from Theorem (3.9) it is written $\nu(y) - \nu(x) = \bigvee_a^x(f) + \bigvee_x^y(f) - f(y) - \bigvee_a^x(f) + f(x)$.Therefore it is clear that $\nu(y) - \nu(x) = \bigvee_x^y(f) - [f(y) - f(x)] \geq 0$. This implies that $\nu(x) \leq \nu(y)$. Hence $\nu(x)$ is an increasing function.

In conclusion,

$f(x) = \pi(x) - \nu(x)$. This completes the proof. □

Theorem 3.11 *Let a function $f(x)$ of finite variation be defined on the closed interval $[a, b]$. If $f(x)$ is continuous at the point x_0, then the function*

$$\pi(x) = \bigvee_a^x(f)$$

is also continuous at x_0.

Proof. Suppose that $x_0 < b$. To show that the continuity of $\pi(x)$, $\epsilon > 0$ is choosen, and the segment $[x_0, b]$ is subdivided as follows;

$$x_0 < x_1 < x_2 < ... < x_n = b$$

So

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| > \bigvee_{x_0}^{b}(f) - \epsilon$$

Since the sum V only increases when new points are added, it might be supposed that $|f(x_1) - f(x_0)| < \epsilon$. So,

$$\bigvee_{x_0}^{b}(f) < \epsilon + \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| < 2\epsilon + \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| \leq 2\epsilon + \bigvee_{x_1}^{b}(f)$$

Therefore, $\bigvee_{x_0}^{b}(f) - \bigvee_{x_1}^{b}(f) \leq 2\epsilon$. Then $\bigvee_{x_0}^{x_1}(f) \leq 2\epsilon$. Since $\pi(x) - \pi(x_0) = \bigvee_{a}^{x}(f) - \bigvee_{a}^{x_0}(f) = \bigvee_{x_0}^{x}(f)$, it is concluded that $\pi(x)$ is continuous from the right at x_0.

The other part can be proven in a similar way. Thus, $\pi(x)$ is continuous at x_0. □

Corollary 3.12 *A continuous function of finite variation on* $[a, b]$ *can be written as the difference of two continuous increasing functions.*

Theorem 3.13 *Let* f *be a function defined on* $[a, b]$*. If* f' *exists, bounded and Riemann integrable on* $[a, b]$ *then* $f \in BV[a, b]$ *and* $\bigvee_{a}^{b}(f) = \int_{a}^{b} |f'(x)|\, dx$.

Proof. Subdivide the interval $[a, b]$ by means of the points

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

It is known that f' is bounded on $[a, b]$. Then it is said that $\exists\ M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$.

Since f' exists, according to the Mean Value Theorem there exists $c_k \in \mathbb{R}$,where $x_k \leq c_k \leq x_{k+1}$, such that

$$\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = f'(c_k) \quad \text{for all } k = 0, ..., n-1$$

Therefore it can be obtained that

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} |f'(c_k)|\,|x_{k+1} - x_k| \leq M(b-a)$$

So $f \in BV[a,b]$. In addition, by using the definition of Riemann integral the following quantity is obtained:
$\bigvee_a^b(f) = \sup \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sup \sum_{k=0}^{n-1} |f'(c_k)|\,|x_{k+1} - x_k| = \lim_{n\to\infty} \sum_{k=0}^{n-1} |f'(c_k)|\,|x_{k+1} - x_k| =$
$\int_a^b |f'(x)|\,dx$. This completes the proof. □

Definition 3.14 *(Absolutely continuous Function)*

Let $f(x)$ be a finite function defined on the closed interval $[a,b]$. Suppose that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\sum_{k=1}^{n} \{f(b_k) - f(a_k)\}\right| < \epsilon$$

for all numbers $a_1, b_1, ..., a_n, b_n$ such that $a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_n < b_n$ and

$$\sum_{k=1}^{n} (b_k - a_k) < \delta$$

Then the function $f(x)$ is said to be absolutely continuous. The class of all absolutely continuous function on $[a,b]$ is denoted by $AC\,[a,b]$.

(I.P. Natanson, 1964)

Theorem 3.15 *An absolutely continuous function is uniformly continuous.*

Proof. It is obvious if n is picked as 1 in definition (3.14). □

Theorem 3.16 *If $f : [a,b] \to \mathbb{R}$ is a Lipschitz function with Lipchitz constant $M > 0$ then f is absolutely continuous on* $[a,b]$.

Proof. Let $\epsilon > 0$ and (a_k, b_k) be non-overlapping intervals in $[a, b]$ such that

$$\sum_{k=1}^{n}(b_k - a_k) < \delta.$$

According to Lipschitz condition it is concluded that $|f(b_k) - f(a_k)| \leq M_k |a_k - b_k|$, where $M_k \in \mathbb{R}^+$ for every $k = 1, ..., n$. Therefore it is obtained,

$$\sum_{k=1}^{n} |f(b_k - f(a_k)| \leq \sum_{k=1}^{n} M_k |a_k - b_k| < M.\delta \text{ where } M = \max_{1\leq k\leq n} M_k$$

picking $\delta \leq \frac{\epsilon}{M}$, the proof will be completed. Consequently f is absolutely continuous. □

Theorem 3.17 *Every absolutely continuous function on the interval* $[a, b]$ *has finite variation on* $[a, b]$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. Then for any given $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$ whenever $\sum_{i=1}^{n} |b_i - a_i| < \delta$ and (a_i, b_i) are non-overlapping intervals in $[a, b]$.

Let us consider a partition of $[a, b]$ by means $\{x_i = a + i\frac{(b-a)}{k} : 0 \leq i \leq k\}$. Hence $\bigvee_{x_{i-1}}^{x_i}(f) \leq \epsilon$ by the absolute continuity condition. There are at most the number of k of these subintervals. Therefore by Theorem (3.9) it can be obtained that $\bigvee_a^b(f) \leq \epsilon k$. Hereby, f is of bounded variation on $[a, b]$. □

Corollary 3.18 *If a function f satisfies the Lipschitz condition on* $[a, b]$*, then it is of finite variation on* $[a, b]$

Theorem 3.19 $AC[a, b]$ *is a closed subspace of* $TV[a, b]$ *in the variation seminorm.*

Proof. By Theorem (3.17), it is clear that AC is a subset of TV.

Let $(f_n)_{n\epsilon\mathbb{N}}$ be a sequence of functions in $AC[a, b]$ that converges in the variation seminorm to f. Then, given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$V_{[a,b]}[f_n - f] = \|f_n - f\| < \frac{\epsilon}{2}$$

Whereby $f_n \in AC[a,b]$, there exists a $\delta > 0$ such that, for every finite set $\{[a,b] : i = 1, 2, ..., k\}$ of nonoverlapping intervals for which $\sum_{i=1}^{k}(b_i - a_i) < \delta$ it can be written,

$$\sum_{i=1}^{k} |f_n(b_i) - f_n(a_i)| < \frac{\epsilon}{2}$$

In this manner it is deduced that,

$$\begin{aligned}
\sum_{i=1}^{k} |f(b_i) - f(a_i)| &\leq \sum_{i=1}^{k} |(f - f_n)(b_i) - (f - f_n)(a_i)| + \sum_{i=1}^{k} |f_n(b_i) - f_n(a_i)| \\
&\leq V_{[a,b]}[f - f_n] + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

.

There follows $f \in AC[a,b]$ and so $AC[a,b]$ is closed. □

Remark 3.20 *If f is absolutely continuous, then f has a derivative almost everywhere and f' is Lebesgue integrable.*

3.2 The Stieltjes Integral

Definition 3.21 *(The Stieltjes Integral)*

Let $f(x)$ and $g(x)$ be finite functions defined on the closed interval $[a,b]$. Subdivide $[a,b]$ into parts by means of the points

$$a = x_0 < x_1 < x_2 < ... < x_n = b,$$

choose a point ξ_k in $[x_k, x_{k+1}]$ for $k = 0, ..., n-1$, and form the sum

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)[g(x_{k+1}) - g(x_k)].$$

If the sum σ tends to a finite limit I as

$$\lambda = \max(x_{k+1} - x_k) \to 0$$

independently of both the method of subdivision and the choice of the points ξ_k, this limit is called the Stieltjes Integral of the function $f(x)$ with respect to $g(x)$ and it is denoted by

$$\int_a^b f(x)dg(x) \quad \text{or} \quad (S)\int_a^b f(x)dg(x).$$

(I.P. Natanson, 1964)

Theorem 3.22 *The integral*

$$\int_a^b f(x)dg(x)$$

exists if the function $f(x)$ is continuous on $[a,b]$ and $g(x)$ is of finite variation on $[a,b]$.

Proof. It can be said that g is increasing because of every function of finite variation is the difference of two increasing functions. Subdivide $[a,b]$ by means of the points

$$x_0 = a < x_1 < ... < x_n = b$$

Let m_k and M_k be the least and greatest values of $f(x)$ on $[x_{k+1}, x_k]$ and

$$s = \sum_{k=0}^{n-1} m_k[g(x_{k+1}) - g(x_k)], \quad S = \sum_{k=0}^{n-1} M_k[g(x_{k+1}) - g(x_k)]$$

Then it is reached that,

$$s \leq \sigma \leq S.$$

s does not decrease when new points of subdivision are added.To show that this, let us add the point x_c into $[x_m, x_{m+1}]$ and let's say

$$s_1 = m_0[g(x_1)-g(x_0)]+...+m_i[g(x_c)-g(x_m)]+m_j[g(x_{m+1})-g(x_c)]+...+m_{n-1}[g(x_n)-g(x_{n-1})].$$

where m_i,m_j are the least values of f on $[x_m, x_c]$ and $[x_c, x_{m+1}]$, respectively. Since,

$$s = m_0[g(x_1) - g(x_0)] + ... + m_m[g(x_{m+1}) - g(x_m)] + ... + m_{n-1}[g(x_n) - g(x_{n-1})]$$

and m_m is the least value of f on $[x_m, x_{m+1}]$ it is deduced,

$$s_1 \geq s.$$

In the similar way it can be shown that S does not increase. These mean that none of the sum s is greather than any of the sum S. That is, for two method of subdividing the segment $[a, b]$ with corresponding s_1, S_1 and s_2, S_2 we have $s_1 \leq S_2$.

Let

$$I = \sup\{s\}$$

Therefore for every subdivision of $[a, b]$,

$$s \leq I \leq S$$

and consequently,

$$|\sigma - I| < S - s$$

Since f is continuous on $[a, b]$, then f is uniformly continuous. So for any $\epsilon > 0$, a $\delta > 0$ can be found such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ for all x, y in $[a, b]$. And so,

$$M_K - m_k < \varepsilon, \quad k = 0, 1, ..., n - 1.$$

Therefore for $\lambda < \delta$,

$$\begin{aligned} S - s &= \sum_{k=0}^{n-1} M_k[g(x_{k+1}) - g(x_k)] - \sum_{k=0}^{n-1} m_k\,[g(x_{k+1}) - g(x_k)] \\ &= \sum_{k=0}^{n-1}(M_k - m_k)[g(x_{k+1}) - g(x_k)] \\ &\leq \varepsilon \sum_{k=0}^{n-1}[g(x_{k+1}) - g(x_k)] \\ &= \epsilon[g(b) - g(a)] \end{aligned}$$

Since $|\sigma - I| < S - s$ it is obtained that,

$$|\sigma - I| < \epsilon[g(b) - g(a)]$$

Consequently,

$$\lim_{\lambda \to 0} \sigma = I$$

That is,

$$I = \int_a^b f(x)dg(x)$$

□

Theorem 3.23 *For all $f_1, f_2, f \in C\,[a, b]$ and $g_1, g_2, g \in BV\,[a, b]$,*

1) $\int_a^b [f_1(x) + f_2(x)]dg(x) = \int_a^b f_1(x)dg(x) + \int_a^b f_2(x)dg(x)$.

2) $\int_a^b f(x)d[g_1(x) + g_2(x)] = \int_a^b f(x)dg_1(x) + \int_a^b f(x)dg_2(x)$.

3) If m and n are constants, then

$$\int_a^b mf(x)dng(x) = mn \int_a^b f(x)dg(x).$$

4) If $a < c < b$ then,

$$\int_a^b f(x)dg(x) = \int_a^c f(x)dg(x) + \int_c^b f(x)dg(x).$$

Theorem 3.24 *If the function $f(x)$ is continuous on $[a,b]$ and $g(x)$ has finite variation on $[a,b]$ then,*

$$\left|\int_a^b f(x)dg(x)\right| \leq M(f)\bigvee_a^b(g).$$

Proof. For an arbitrary subdivision of $[a,b]$,

$$\begin{aligned}
\left|\int_a^b f(x)dg(x)\right| &= \left|\lim_{n\to\infty}\sum_{k=0}^{n-1} f(\xi_k)[g(x_{k+1})-g(x_k)]\right| \\
&\leq \left|\lim_{n\to\infty}\sum_{k=0}^{n-1} |f(\xi_k)|\,[g(x_{k+1})-g(x_k)]\right| \\
&\leq \left|\lim_{n\to\infty}\sum_{k=0}^{n-1} \max_{x\in[a,b]}|f(x)|\,[g(x_{k+1})-g(x_k)]\right| \\
&= M(f)\left|\lim_{n\to\infty}\sum_{k=0}^{n-1}[g(x_{k+1})-g(x_k)]\right| \\
&= M(f)\,|g(b)-g(a)| \\
&\leq M(f)\bigvee_a^b(g)
\end{aligned}$$

Therefore,

$$\left|\int_a^b f(x)dg(x)\right| \leq M(f)\bigvee_a^b(g).$$

□

.

Theorem 3.25 *If the function $f(x)$ is continuous on $[a,b]$ and if the function $g(x)$ has a Riemann integrable derivative $g'(x)$ at every point of $[a,b]$, then*

$$(S)\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

Proof. Since $g(x)$ has derivative, $g(x)$ satisfies the Lipschitz condition and so $g(x)$ is of finite variation. Thus the integral on the left side exists.

Subdivide $[a, b]$ by means of the points

$$x_0 = a < x_1 < x_2 < ... < x_n = b.$$

According to the mean value theorem it is deduced,

$$g(x_{k+1}) - g(x_k) = g'(c_k)(x_{k+1} - x_k) \quad (x_k < c_k < x_{k+1})$$

The point c_k might be taken for the point ξ_k. Therefore,

$$\begin{aligned}(S)\int_a^b f(x)dg(x) &= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(c_k)[g(x_{k+1}) - g(x_k)] \\ &= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(c_k)g'(c_k)(x_{k+1} - x_k) \\ &= \lim_{n\to\infty}\sum_{k=0}^{n-1}(fg')(c_k)(x_{k+1} - x_k) \\ &= \int_a^b f(x)g'(x)dx.\end{aligned}$$

□

CHAPTER 4

VORONOVSKAYA-TYPE THEOREMS AND CONVERGENCE IN THE VARIATION SEMINORM

4.1 Voronovskaya-Type Theorem

After veryfying the convergence of $(B_nf)(x)$ and $(C_nf)(x)$, the most significant question coming to mind was rate of approximation by the $(B_nf)(x)$ to $f(x)$ and $(C_nf)(x)$ to $f(x)$. In 1932 Voronovskaya answered this question for Bernstein and in 1960 J. Albrycht and J. Redecki did for Bernstein-Cholodovsky polynomials. Sebsequent to these, Bardaro, Butzer, Stens, Vinti (Bardaro et.al., 2003) and Butzer, Karsli (Butzer and Karsli, 2009) found the solution of that question for $(B_nf)'(x)$ and $(C_nf)'(x)$, respectively.

In this section, the certain results needed to prove Voronovskaya type theorems are presented and Voronovskaya type theorems are established.

4.1.1 Bernstein polynomials case

Recall that, Bernstein polynomials, for any function $f(x)$ defined on $[0, 1]$,are defined as

$$(B_nf)(x) = B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x) \tag{4.1}$$

where

$$p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Differentiating the formula (4.1) , it can be obtained that

$$(B_nf)'(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\left\{kx^{k-1}(1-x)^{n-k} - (n-k)\,x^k\,(1-x)^{n-k-1}\right\}$$

$$(B_n f)'(x) = n \sum_{k=0}^{n-1} \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] p_{k,n-1}(x) \tag{4.2}$$

Since $\frac{d}{dx}\left(p_{k,n}(x)\right) = \frac{(k-nx)p_{k,n}(x)}{x(1-x)}$, it is also written for $(B_n f)'(x)$,

$$(B_n f)'(x) = \frac{1}{x(1-x)} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)(k - nx)p_{k,n}(x). \tag{4.3}$$

Lemma 4.1 *For* $(B_n t^s)(x)$, $s = 0, 1, 2, 3, 4, 5$ *one has for* $0 \le x \le 1$

$(B_n 1)(x) = 1$

$(B_n t)(x) = x$

$(B_n t^2)(x) = x^2 + \frac{x(1-x)}{n}$

$(B_n t^3)(x) = x^3 \left[\frac{n^2-3n+2}{n^2}\right] + x^2 \left[\frac{3(n-1)}{n^2}\right] + \frac{x}{n^2}$

$(B_n t^4)(x) = x^4 \left[\frac{n^3-6n^2+11n-6}{n^3}\right] + x^3 \left[\frac{6\left(n^2-3n+2\right)}{n^3}\right] + x^2 \left[\frac{7(n-1)}{n^3}\right] + \frac{x}{n^3}$

$(B_n t^5)(x) = x^5 \left[\frac{n^4-10n^3+35n^2-50n+24}{n^4}\right] + x^4 \left[\frac{10\left(n^3-6n^2+11n-6\right)}{n^4}\right] + x^3 \left[\frac{25(n-3n+2)}{n^4}\right] + x^2 \left[\frac{15(n-1)}{n^4}\right] +$
$x\frac{1}{n^4}$

Proof.

$$\begin{aligned}
(B_n 1)(x) &= \sum_{k=0}^{n} p_{k,n}(x) = 1 \\
(B_n t)(x) &= \sum_{k=0}^{n} \frac{k}{n} p_{k,n}\left(\frac{x}{b_n}\right) = \frac{1}{n} \sum_{k=1}^{n} k p_{k,n}(x) \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \frac{n!}{(n-1-k)!k!} x^{k+1} (1-x)^{n-1-k} \\
&= x \sum_{k=0}^{n-1} p_{k,n-1}(x) = x
\end{aligned}$$

$$\begin{aligned}
\left(B_n t^2\right)(x) &= \sum_{k=0}^{n} \frac{k^2}{n^2} p_{k,n}(x) \\
&= \frac{1}{n^2} \sum_{k=0}^{n-1} (k+1) \frac{n!}{(n-1-k)!k!} x^{k+1} (1-x)^{n-1-k} \\
&= \frac{x}{n} \left(\sum_{k=0}^{n-1} k p_{k,n-1}(x) + \sum_{k=0}^{n-1} p_{k,n-1}(x) \right) \\
&= \frac{x}{n} \left((n-1)x + 1\right) = x^2 + \frac{x(1-x)}{n}
\end{aligned}$$

$$
\begin{aligned}
(B_n t^3)(x) &= \sum_{k=0}^{n} \frac{k^3}{n^3} p_{k,n}(x) \\
&= \frac{1}{n^3} \sum_{k=0}^{n-1} (k+1)^2 \frac{n!}{(n-1-k)!k!} x^{k+1} (1-x)^{n-1-k} \\
&= \frac{x}{n^2} \left(\sum_{k=0}^{n-1} k^2 p_{k,n-1}(x) + 2 \sum_{k=0}^{n-1} k p_{k,n-1}(x) + 1 \right) \\
&= \frac{x}{n^2} \left[(n-1)\, x \left[(n-2)\, x + 1 \right] + 2\,(n-1)\, x + 1 \right] \\
&= x^3 \left(\frac{n^3 - 3n + 2}{n^2} \right) + x^2 \left(\frac{3(n-1)}{n^2} \right) + x \frac{1}{n^2}
\end{aligned}
$$

$(B_n t^4)(x)$ and $(B_n t^5)(x)$ can be found in the similar way. □

The following lemma is concerned with the moments for $(B_n f)(x)$ which will be used to established the theorems of rate of convergence of $(B_n f)(x)$ and $(B_n f)'(x)$.

Lemma 4.2 *Considering the moments*

$$W_{n,m} := \sum_{k=0}^{n} (k - nx)^m p_{k,n}(x) \qquad (m \in \mathbb{N}_0) \tag{4.4}$$

Then for $X := x(1-x)$ *there hold for all* $x \in [0,1]$ *the following identities*

$W_{n,0}(x) = 1$

$W_{n,1}(x) = 0$

$W_{n,2}(x) = nX$

$W_{n,3}(x) = nX(1-2x)$

$W_{n,4}(x) = 3(nX)^2 + (1-6X)nX$

$W_{n,5}(x) = \left(10\,(nX)^2 + (1-12X)\,nX\right)(1-2x)$

$W_{n,6}(x) = 15\,(nX)^3 + (25-130X)\,(nX)^2 + (1-30X+120X^2)nX.$

Proof. In order to complete the proof it is needed to use previous lemma. By using the previous lemma the identities can be obtained as follows;

$$
\begin{aligned}
W_{n,0}(x) &= \sum_{k=0}^{n} p_{k,n}(x) = 1 \\
W_{n,1}(x) &= \sum_{k=0}^{n} (k - nx) p_{k,n}(x) \\
&= \sum_{k=0}^{n} k p_{k,n}(x) - nx = 0.
\end{aligned}
$$

$$
\begin{aligned}
W_{n,2}(x) &= \sum_{k=0}^{n} (k - nx)^2 p_{k,n}(x) \\
&= \sum_{k=0}^{n} k^2 p_{k,n}(x) - 2xn \sum_{k=0}^{n} k p_{k,n}(x) + n^2 x^2 \\
&= b_n^2 \frac{n^2}{b_n^2} \left[x^2 + \frac{x(b_n - x)}{n} \right] - 2xnb_n \frac{nx}{b_n} + n^2 x^2 \\
&= nx(1 - x) = nX.
\end{aligned}
$$

$$
\begin{aligned}
W_{n,3}(x) &= \sum_{k=0}^{n} (k - nx)^3 p_{k,n}(x) \\
&= \sum_{k=0}^{n} k^3 p_{k,n}(x) - 3nx \sum_{k=0}^{n} k^2 p_{k,n}(x) \\
&\quad + 3n^2 x^2 \sum_{k=0}^{n} k p_{k,n}(x) - n^3 x^3 \\
&= n^3 \left[x^3 \left(\frac{n^3 - 3n + 2}{n^2} \right) + x^2 \left(\frac{3(n-1)}{n^2} \right) + x \frac{1}{n^2} \right] \\
&\quad - 3nxn^2 \left[x^2 + \frac{x(1-x)}{n} \right] + 3n^2 x^2 nx - n^3 x^3 \\
&= nx(1 - 2x)(1 - x) \\
&= nX(1 - 2x).
\end{aligned}
$$

In the similar method $W_{n,4}(x)$, $W_{n,5}(x)$ and $W_{n,6}(x)$ are calculated. □

Let us consider a Voronovskaya-type theorem for $(B_n f)(x)$ which is due to Voronovskaya (Voronovskaya, 1932). This result was first come out in the instance of Kantrovich polynomials (Voronovskaya, 1952)..

Theorem 4.3 *Let f be a bounded function on* $[0,1]$. *Then*

$$\lim_{n\to\infty} n\left[(B_nf)(x) - f(x)\right] = \frac{x(1-x)}{2} f''(x)$$

at each fixed point $x \in [0,1]$ *for which there exists* $f''(x) \neq 0$.

Proof. Since $f''(x)$ exists, from Taylor it is obtained

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left(\frac{1}{2} f''(x) + h\left(\frac{k}{n} - x\right)\right)$$

where $h(y)$ is converges to zero with y.

Since $f(x)$ is bounded it can be said that $h(y)$ is bounded for all y, saying $|h(y)| \leq M$. From Bernstein's expression and Lemma (4.2)

$$\begin{aligned}
(B_nf)(x) &= f(x) + \frac{f'(x)}{n} \sum_{k=0}^{n} (k - nx) p_{k,n}(x) + \frac{f''(x)}{2n^2} \sum_{k=0}^{n} (k - nx)^2 p_{k,n}(x) \\
&\quad + \frac{f''(x)}{2n^2} \sum_{k=0}^{n} (k - nx)^2 h\left(\frac{k}{n} - x\right) p_{k,n}(x) \\
&= f(x) + \frac{X}{2n} f''(x) + R_n(x)
\end{aligned}$$

where the remainder is given by

$$R_n(x) = \frac{f''(x)}{2n^2} \sum_{k=0}^{n} (k - nx)^2 h\left(\frac{k}{n} - x\right) p_{k,n}(x)$$

Therefore,

$$\lim_{n\to\infty} n\left[(B_nf)(x) - f(x)\right] = \lim_{n\to\infty} \left[\frac{X}{2} f''(x) + nR_n(x)\right]$$

Then in order to complete the proof it is sufficient to prove that

$$\lim_{n\to\infty} nR_n(x) = 0$$

Let's divide $R_n(x)$ into two parts as follows

$$\begin{aligned} R_n(x) &= \frac{f''(x)}{2n^2} \sum_{\left|\frac{k}{n}-x\right|<\delta} (k-nx)^2 h\left(\frac{k}{n}-x\right) p_{k,n}(x) \\ &\quad + \frac{f''(x)}{2n^2} \sum_{\left|\frac{k}{n}-x\right|\geq\delta} (k-nx)^2 h\left(\frac{k}{n}-x\right) p_{k,n}(x) \end{aligned}$$

Since $h(y)$ is converges to 0, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|h(y)| < \epsilon$ whenever $0 \leq y < \delta$. Thus,

$$\left| \frac{f''(x)}{2n^2} \sum_{\left|\frac{k}{n}-x\right|<\delta} (k-nx)^2 h\left(\frac{k}{n}-x\right) p_{k,n}(x) \right| \leq \frac{|f''(x)|}{8n}\epsilon$$

Since $|1-6X| \leq 1$, for $\left|\frac{k}{n}-x\right| \geq \delta$ from Lemma (4.2) it is obtained

$$\left| \frac{f''(x)}{2n^2} \sum_{\left|\frac{k}{n}-x\right|\geq\delta} (k-nx)^2 h\left(\frac{k}{n}-x\right) p_{k,n}(x) \right| \leq \frac{3\,|f''(x)|\,M}{32\delta^2 n^2} + \frac{|f''(x)|\,M}{8\delta^2 n^3}$$

Consequently,

$$|nR_n(x)| \leq \frac{|f''(x)|}{8}\epsilon + \frac{3\,|f''(x)|\,M}{32\delta^2 n} + \frac{|f''(x)|\,M}{8\delta^2 n^2}$$

which implies that

$$\lim_{n\to\infty} nR_n(x) = 0$$

□

Now, Voronovskaya type theorem will be presented for $(B_n f)'(x)$. Replacing f by f' and $(B_n f)(x)$ and $(B_n f)'(x)$ yields theorem (4.4) below. The first quantitative version of the following theorem is due to Bardaro, Butzer, Stens, Vinti (Bardaro et.al., 2003)..

Theorem 4.4 *If f is bounded on $[0,1]$ and $f'''(x)$ exists in a certain point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n[(B_n f)'(x) - f'(x)] = \frac{(1-2x)}{2} f''(x) + \frac{x(1-x)}{2} f'''(x).$$

Proof. By Taylor's formula it can be written,

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \frac{f''(x)}{2!} + \left(\frac{k}{n} - x\right)^3 \left[\frac{1}{6} f'''(x) + h\left(\frac{k}{n} - x\right)\right]$$

where $h(y)$ is bounded for all y, it is said that $|h(y)| \leq M$, and converges to zero. Therefore from the representation (4.3) and Lemma (4.2) the following part is obtained.

$$\begin{aligned}
(B_n f)'(x) &= \frac{1}{X} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) (k - nx) p_{k,n}(x) \\
&= \frac{f(x)}{X} \sum_{k=0}^{n} (k - nx) p_{k,n}(x) + \frac{f'(x)}{X} \sum_{k=0}^{n} (k - nx) \left(\frac{k}{n} - x\right) p_{k,n}(x) \\
&\quad + \frac{f''(x)}{2X} \sum_{k=0}^{n} (k - nx) \left(\frac{k}{n} - x\right)^2 p_{k,n}(x) + \frac{f'''(x)}{6X} \sum_{k=0}^{n} (k - nx) \left(\frac{k}{n} - x\right)^3 p_{k,n}(x) \\
&\quad + \frac{1}{X} \sum_{k=0}^{n} (k - nx) \left(\frac{k}{n} - x\right)^3 h\left(\frac{k}{n} - x\right) p_{k,n}(x) \\
&= f'(x) + \frac{(1 - 2x) f''(x)}{2n} + \frac{X f'''(x)}{2n} + \frac{(1 - 6X) f'''(x)}{6n^2} + \frac{1}{n^3} \sum_{k=0}^{n} \frac{(k - nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x)
\end{aligned}$$

The last term $\frac{1}{n^3} \sum_{k=0}^{n} \frac{(k-nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x)$ is written as;

$$\begin{aligned}
\frac{1}{n^3} \sum_{k=0}^{n} \frac{(k - nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x) &= \frac{1}{n^3} \sum_{\left|\frac{k}{n} - x\right| < \delta} \frac{(k - nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x) \\
&\quad + \frac{1}{n^3} \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \frac{(k - nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x)
\end{aligned}$$

Since $h(y)$ converges to zero, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|h(y)| < \epsilon$ for $0 < y < \delta$. It is known that $0 \leq X \leq \frac{1}{4}$. It follows from Lemma (4.2),

$$\begin{aligned}
&\frac{1}{n^3} \sum_{\left|\frac{k}{n} - x\right| < \delta} \frac{(k - nx)^4}{X} h\left(\frac{k}{n} - x\right) p_{k,n}(x) \\
\leq\ &\frac{\epsilon}{X n^3} \sum_{\left|\frac{k}{n} - x\right| < \delta} (k - nx)^4 p_{k,n}(x) \\
\leq\ &\frac{3\epsilon X}{n} + \frac{\epsilon}{n^2}(1 - 6X) \leq \frac{2\epsilon}{n}
\end{aligned}$$

For $\left|\frac{k}{n}-x\right| \geq \delta$ it is composed that $\frac{(k-nx)^2}{n^2\delta^2} \geq 1$. Therefore,

$$\begin{aligned}
& \frac{1}{n^3}\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \frac{(k-nx)^4}{X} h\left(\frac{k}{n}-x\right) p_{k,n}(x) \\
\leq\ & \frac{M}{n^3X}\sum_{\left|\frac{k}{n}-x\right|\geq\delta} (k-nx)^4 p_{k,n}(x) \\
\leq\ & \frac{M}{n^5X\delta^2}\sum_{\left|\frac{k}{n}-x\right|\geq\delta} (k-nx)^6 p_{k,n}(x) \\
\leq\ & \frac{M}{n^5X\delta^2}\left[15(nX)^3 + (25-130X)(nX)^2 + (1-30X+120X^2)(nX)\right] \\
=\ & \frac{15X^2}{n^2\delta^2}M + \frac{(25-130X)X}{n^3\delta^2}M + \frac{(1-30X+120X^2)}{n^4\delta^2}M
\end{aligned}$$

Since $|25-130X| \leq 25$ and $\left|1-30X+120X^2\right| \leq 16$, it is written that

$$\begin{aligned}
& \left|\frac{1}{n^3}\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \frac{(k-nx)^4}{X} h\left(\frac{k}{n}-x\right) p_{k,n}(x)\right| \\
\leq\ & \frac{15X^2}{n^2\delta^2}M + \frac{|25-130X|\,X}{n^3\delta^2}M + \frac{\left|1-30X+120X^2\right|}{n^4\delta^2}M \\
\leq\ & \frac{56M}{\delta^2 n^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left|\frac{1}{n^3}\sum_{k=0}^{n} \frac{(k-nx)^4}{X} h\left(\frac{k}{n}-x\right) p_{k,n}(x)\right| \\
\leq\ & \frac{2\epsilon}{n} + \frac{56M}{\delta^2 n^2}
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \left|n[(B_nf)'(x) - f'(x)] - \frac{(1-2x)}{2}f''(x) - \frac{x(1-x)}{2}f'''(x)\right| \\
\leq\ & \frac{|1-6X|}{6n}|f'''(x)| + \frac{2\epsilon}{n} + \frac{56M}{\delta^2 n} \\
\leq\ & 2\epsilon + \frac{1}{n}\left[\frac{|f'''(x)|}{6} + \frac{56M}{\delta^2}\right] \to 0 \quad \text{as } n\to\infty
\end{aligned}$$

So, the proof was completed. □

4.1.2 Chlodovsky polynomials case

It had better to reveal the definition of Bernstein-Chlodovsky polynomials, over again. These polynomials are given by

$$(C_n f)(x) = \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right)$$

where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b] \subset [0,\infty)$ with a certain rate, with $p_{k,n}$ denoting as usual

$$p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 \le x \le 1$$

and $(b_n)_{n=1}^{\infty}$ being a positive increasing sequence of real numbers wih the properties

$$\lim_{n\to\infty} b_n = \infty \quad , \quad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

Since

$$\frac{d}{dx} p_{k,n}\left(\frac{x}{b_n}\right) = \frac{kb_n - nx}{x\,(b_n - x)} p_{k,n}\left(\frac{x}{b_n}\right)$$

there follow by differentation the two fundemental representations for $(C_n f)'(x)$

$$(C_n f)'(x) = \frac{n}{b_n} \sum_{k=0}^{n-1} \left[f\left(\frac{k+1}{n} b_n\right) - f\left(\frac{k}{n} b_n\right) \right] p_{k,n-1}\left(\frac{x}{b_n}\right) \tag{4.5}$$

$$= \frac{1}{x\,(b_n - x)} \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) (kb_n - nx)\, p_{k,n}\left(\frac{x}{b_n}\right) \tag{4.6}$$

Lemma 4.5 *For $(C_n t^s)(x)$, $s = 0,1,2,3,4,5$ one has for $0 \le x \le b_n$*

$(C_n 1)(x) = 1$

$(C_n t)(x) = x$

$(C_n t^2)(x) = x^2 + \frac{x(b_n - x)}{n}$

$(C_n t^3)(x) = x^3 \left[\frac{n^2-3n+2}{n^2}\right] + x^2 \left[\frac{3b_n(n-1)}{n^2}\right] + \frac{xb_n^2}{n^2}$

$(C_n t^4)(x) = x^4 \left[\frac{n^3-6n^2+11n-6}{n^3}\right] + x^3 \left[\frac{6b_n\left(n^2-3n+2\right)}{n^3}\right] + x^2 \left[\frac{7b_n^2(n-1)}{n^3}\right] + x\frac{b_n^3}{n^3}$

$(C_n t^5)(x) = x^5 \left[\frac{n^4-10n^3+35n^2-50n+24}{n^4}\right] + x^4 \left[\frac{10b_n\left(n^3-6n^2+11n-6\right)}{n^4}\right] + x^3 \left[\frac{25b_n^2(n-3n+2)}{n^4}\right] + x^2 \left[\frac{15b_n^3(n-1)}{n^4}\right] + x\frac{b_n^4}{n^4}$

Proof.

$$
\begin{aligned}
(C_n 1)(x) &= \sum_{k=0}^{n} p_{k,n}\left(\frac{x}{b_n}\right) = 1 \\
(C_n t)(x) &= \sum_{k=0}^{n} \frac{kb_n}{n} p_{k,n}\left(\frac{x}{b_n}\right) = \frac{b_n}{n}\sum_{k=1}^{n} kp_{k,n}\left(\frac{x}{b_n}\right) \\
&= \frac{b_n}{n}\sum_{k=0}^{n-1} \frac{n!}{(n-1-k)!k!}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)^{n-1-k} \\
&= x\sum_{k=0}^{n-1} p_{k,n-1}\left(\frac{x}{b_n}\right) \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
\left(C_n t^2\right)(x) &= \sum_{k=0}^{n} \frac{k^2 b_n^2}{n^2} p_{k,n}\left(\frac{x}{b_n}\right) \\
&= \frac{b_n^2}{n^2}\sum_{k=0}^{n-1} (k+1)\,\frac{n!}{(n-1-k)!k!}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)^{n-1-k} \\
&= \frac{b_n x}{n}\left(\sum_{k=0}^{n-1} kp_{k,n-1}\left(\frac{x}{b_n}\right) + \sum_{k=0}^{n-1} p_{k,n-1}\left(\frac{x}{b_n}\right)\right) \\
&= \frac{b_n x}{n}\left(\frac{n-1}{b_n}x + 1\right) \\
&= x^2 + \frac{x(b_n - x)}{n}
\end{aligned}
$$

$$
\begin{aligned}
(C_n t^3)(x) &= \sum_{k=0}^{n} \frac{k^3 b_n^3}{n^3} p_{k,n}\left(\frac{x}{b_n}\right) \\
&= \frac{b_n^3}{n^3}\sum_{k=0}^{n-1} (k+1)^2\,\frac{n!}{(n-1-k)!k!}\left(\frac{x}{b_n}\right)^{k+1}\left(1-\frac{x}{b_n}\right)^{n-1-k} \\
&= \frac{xb_n^2}{n^2}\left(\sum_{k=0}^{n-1} k^2 p_{k,n-1}\left(\frac{x}{b_n}\right) + 2\sum_{k=0}^{n-1} kp_{k,n-1}\left(\frac{x}{b_n}\right) + 1\right) \\
&= \frac{xb_n^2}{n^2}\left[\frac{(n-1)x}{b_n}\left(\frac{(n-2)x}{b_n} + 1\right) + 2\frac{(n-1)x}{b_n} + 1\right] \\
&= x^3\left(\frac{n^3 - 3n + 2}{n^2}\right) + x^2\left(\frac{3b_n(n-1)}{n^2}\right) + x\frac{b_n^2}{n^2}
\end{aligned}
$$

$(C_n t^4)(x)$ and $(C_n t^5)(x)$ can be determined by using the proven identities. □

The moments concerning $(C_n f)$ is needed so as to verify the rate of convergence of $(C_n f)(x)$ and $(C_n f)'(x)$. For this reason the moments for $(C_n f)(x)$ will be identified in the following lemma.

Lemma 4.6 *Defining the moments*

$$T_{n,m} := \sum_{k=0}^{n} (kb_n - nx)^m p_{k,n}\left(\frac{x}{b_n}\right) \tag{4.7}$$

where $m \in \mathbb{N}_0$. *Then there hold the following identities*

$T_{n,0}(x) = 1$

$T_{n,1}(x) = 0$

$T_{n,2}(x) = nx(b_n - x)$

$T_{n,3}(x) = nx(b_n - x)(b_n - 2x)$

$T_{n,4}(x) = nx(b_n - x)\left[(b_n - x)(b_n - 2x) + x(4x - 3b_n) + 3nx(b_n - x)\right]$

and the following recurrence relation:

$$T_{n,m+1}(x) = x(b_n - x)\left[T'_{n,m}(x) + mnT_{n,m-1}(x)\right], \quad m \geq 1. \tag{4.8}$$

Proof. The proof is finalized by using the foregoing Lemma as follows:

$$\begin{aligned}
T_{n,0}(x) &= \sum_{k=0}^{n} p_{k,n}\left(\frac{x}{b_n}\right) = 1 \\
T_{n,1}(x) &= \sum_{k=0}^{n} (kb_n - nx) p_{k,n}\left(\frac{x}{b_n}\right) = b_n \sum_{k=0}^{n} k p_{k,n}\left(\frac{x}{b_n}\right) - nx = 0
\end{aligned}$$

$$\begin{aligned}
T_{n,2}(x) &= \sum_{k=0}^{n} (kb_n - nx)^2 p_{k,n}\left(\frac{x}{b_n}\right) \\
&= b_n^2 \sum_{k=0}^{n} k^2 p_{k,n}\left(\frac{x}{b_n}\right) - 2xnb_n \sum_{k=0}^{n} k p_{k,n}\left(\frac{x}{b_n}\right) + n^2x^2 \\
&= b_n^2 \frac{n^2}{b_n^2}\left[x^2 + \frac{x(b_n - x)}{n}\right] - 2xnb_n \frac{nx}{b_n} + n^2x^2 \\
&= nx(b_n - x)
\end{aligned}$$

$$
\begin{aligned}
T_{n,3}(x) &= \sum_{k=0}^{n}(kb_n - nx)^3 p_{n,k}\left(\frac{x}{b_n}\right)\\
&= b_n^3\sum_{k=0}^{n}k^3 p_{k,n}\left(\frac{x}{b_n}\right) - 3b_n^2 nx\sum_{k=0}^{n}k^2 p_{k,n}\left(\frac{x}{b_n}\right)\\
&\quad +3b_n n^2x^2\sum_{k=0}^{n}kp_{k,n}\left(\frac{x}{b_n}\right) - n^3x^3\\
&\quad -3b_n^2 nx\frac{n^2}{b_n^2}\left[x^2 + \frac{x(b_n - x)}{n}\right] + 3b_n n^2x^2\frac{n}{b_n}x - n^3x^3\\
&= nx(b_n - 2x)(b_n - x)
\end{aligned}
$$

$T_{n,4}(x)$ is obtained by following the similar technique.

Now, Let us prove the recurrence relation (4.8). Differentiating both sides of the expression (4.7) with respect to x, it is deduced

$$
\begin{aligned}
T'_{n,m}(x) &= \sum_{k=0}^{n}\left[(kb_n - nx)^m\frac{d}{dx}p_{k,n}\left(\frac{x}{b_n}\right) - mn\,(kb_n - nx)^{m-1}\,p_{k,n}\left(\frac{x}{b_n}\right)\right]\\
&= \sum_{k=0}^{n}(kb_n - nx)^m\frac{d}{dx}p_{k,n}\left(\frac{x}{b_n}\right) - mn\sum_{k=0}^{n}(kb_n - nx)^{m-1}\,p_{k,n}\left(\frac{x}{b_n}\right)\\
&= \sum_{k=0}^{n}(kb_n - nx)^m\frac{(kb_n - nx)}{x\,(b_n - x)}p_{k,n}\left(\frac{x}{b_n}\right) - mn\sum_{k=0}^{n}(kb_n - nx)^{m-1}\,p_{k,n}\left(\frac{x}{b_n}\right)
\end{aligned}
$$

and therefore

$$
T'_{n,m}(x) = \frac{1}{x\,(b_n - x)}T_{n,m+1}(x) - mnT_{n,m-1}(x)
$$

there follows the recurrence relation (4.8). □

The following lemma plays an important role in the proof of the theorems of the rate of approximation of $(C_n f)(x)$ and $(C_n f)'(x)$. On this occasion the following lemma about central moment for $(C_n f)$ and a significant result concerning the central moment will be presented.

Lemma 4.7 *For the central moments of order $m \in \mathbb{N}_0$,*

$$
T^*_{n,m}(x) := \sum_{k=0}^{n}\left(\frac{kb_n}{n} - x\right)^m p_{k,n}\left(\frac{x}{b_n}\right)
$$

one has

$T^*_{n,0}(x) = 1$

$T^*_{n,1}(x) = 0$

$T^*_{n,2}(x) = \frac{x(b_n - x)}{n}$

$T^*_{n,3}(x) = \frac{x(b_n - x)(b_n - 2x)}{n^2}$

$T^*_{n,4}(x) = \frac{x(b_n - x)[(b_n - x)(b_n - 2x) + x(4x - 3b_n) + 3nx(b_n - x)]}{n^3}$

and for any fixed $x \in [0, \infty)$*,*

$$\left|T^*_{n,m}(x)\right| \leq A_m(x)\frac{x(b_n - x)}{b_n}\left(\frac{b_n}{n}\right)^{[(m+1)/2]} \qquad (n \in \mathbb{N}, n > b_n), \tag{4.9}$$

where $A_m(x)$ *denotes a polynomial in x, of degree* $[m/2] - 1$*, with non-negative coefficients independent of n, and* $[a]$ *denotes the integral part of a.*

Proof. The identities $T^*_{n,0}(x)$, $T^*_{n,1}(x)$, $T^*_{n,2}(x)$, $T^*_{n,3}(x)$ and $T^*_{n,4}(x)$ are obtained readily by following the previous lemma. Exclusively, the inequality (4.9) will be proven.

Differentiating both sides of the expression (4.4) with respect to x, it is obtained

$$\begin{aligned} W'_{n,m}(x) &= \sum_{k=0}^{n}\left[(k - nx)^m \frac{d}{dx}p_{k,n}(x) - mn(k - nx)^{m-1} p_{k,n}(x)\right] \\ &= \sum_{k=0}^{n}(k - nx)^m \frac{d}{dx}p_{k,n}(x) - mn\sum_{k=0}^{n}(k - nx)^{m-1} p_{k,n}(x) \\ &= \sum_{k=0}^{n}(k - nx)^m \frac{(k - nx)}{x(1 - x)}p_{k,n}(x) - mn\sum_{k=0}^{n}(k - nx)^{m-1} p_{k,n}(x) \end{aligned}$$

and so

$$W_{n,m+1}(x) = x(1 - x)\left[W'_{n,m}(x) + mnW_{n,m-1}(x)\right], \qquad m \geq 1$$

The last expression can rewrite by using t variable as follows:

$$W_{n,m+1}(t) = t(1 - t)\left[W'_{n,m}(t) + mnW_{n,m-1}(t)\right], \qquad m \geq 1 \tag{4.10}$$

From this formula by using mathematical induction the following representation can be obtained:

$$W_{n,2s} = \sum_{j=1}^{s} \alpha_{j,s,n} \left(t\left(1-t\right)\right)^{j} n^{j}$$

$$W_{n,2s+1} = \left(1-2t\right) \sum_{j=1}^{s} \beta_{j,s,n} \left(t\left(1-t\right)\right)^{j} n^{j}$$

where $s \in \mathbb{N}$ and $\alpha_{j,s,n}$, $\beta_{j,s,n}$ denote real numbers independent of t and bounded uniformly in n. (see the proof in [**?**]).

If t is equal to $\frac{x}{b_n}$, $0 \le x \le b_n$, in the expression (4.10), it is concluded that,

$$T^{*}_{n,m}(x) = \frac{b_n^m}{n^m} W_{n,m}\left(\frac{x}{b_n}\right)$$

So it is deduced the following representations:

$$T^{*}_{n,2s} = \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} \alpha_{j,s,n} \left(\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\right)^{j} n^{j}$$

$$T^{*}_{n,2s+1} = \left(\frac{b_n}{n}\right)^{2s+1} \left(1-2\frac{x}{b_n}\right) \sum_{j=1}^{s} \beta_{j,s,n} \left(\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\right)^{j} n^{j}$$

for $s \in \mathbb{N}$, where $\alpha_{j,s,n}, \beta_{j,s,n}$ are independent of x and bounded uniformly in n.

Therefore for sufficiently large n (such that $n > b_n$), it is made inferences that

$$\begin{aligned}
\left|T^{*}_{n,2s}\right| &\le \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} \left|\alpha_{j,s,n}\right| \left(\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\right)^{j} n^{j} \\
&= \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} \left|\alpha_{j,s,n}\right| x^{j} \left(1-\frac{x}{b_n}\right)^{j} \left(\frac{n}{b_n}\right)^{j} \\
&\le \left(\frac{b_n}{n}\right)^{2s} \left(1-\frac{x}{b_n}\right) \sum_{j=1}^{s} \left|\alpha_{j,s,n}\right| x^{j} \left(\frac{n}{b_n}\right)^{j} \\
&= \left(\frac{b_n}{n}\right)^{2s} \frac{x}{b_n}\left(1-\frac{x}{b_n}\right) n \sum_{j=1}^{s} \left|\alpha_{j,s,n}\right| x^{j-1} \left(\frac{n}{b_n}\right)^{j-1}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{b_n}{n}\right)^{2s} \frac{x}{b_n}\left(1-\frac{x}{b_n}\right) n \sum_{j=0}^{s-1} \left|\alpha_{j+1,s,n}\right| x^j \left(\frac{n}{b_n}\right)^j \\
&\leq \left(\frac{b_n}{n}\right)^{2s} \frac{x}{b_n}\left(1-\frac{x}{b_n}\right) n \sum_{j=0}^{s-1} \left|\alpha_{j+1,s,n}\right| x^j \left(\frac{n}{b_n}\right)^{s-1} \\
&= \left(\frac{b_n}{n}\right)^{s+1} \frac{x}{b_n}\left(1-\frac{x}{b_n}\right) n \sum_{j=0}^{s-1} \left|\alpha_{j+1,s,n}\right| x^j \\
&= \frac{x\,(b_n-x)}{b_n}\left(\frac{b_n}{n}\right)^{s} \sum_{j=0}^{s-1} \gamma_{j,s} x^j
\end{aligned}
$$

where$\gamma_{j,s}$ denote non-negative independent of n and x. Since $\left|1-2\frac{x}{b_n}\right| \leq 1$, in the same way it is obtained that

$$
\left|T^*_{n,2s+1}\right| \leq \frac{x\,(b_n-x)}{b_n}\left(\frac{b_n}{n}\right)^{s+1} \sum_{j=0}^{s-1} \eta_{j,s} x^j
$$

where$\eta_{j,s}$ denote non-negative independent of n and x.

The finite sums$\sum_{j=0}^{s-1} \gamma_{j,s} x^j$ and $\sum_{j=0}^{s-1} \eta_{j,s} x^j$ represent polynomials. There provides (4.9), since $A_m(x)$ denotes a polynomial in x, of degree $[m/2]-1$, with non-negative coefficients independent of n. □

Throughout the following two theorems, Voronovskaya type theorems for $(C_n f)(x)$ and $(C_n f)'(x)$ will be given and verified. These two theorems are due to J. Albrycht, J. Redecki (Albrycht and Redecki, 1960) and Butzer, Karsli (Butzer and Karsli, 2009), respectively.

Theorem 4.8 *Let a function* f*,defined on* $[0,\infty)$*, satisfy*

$$
\lim_{n\to\infty} \frac{n}{b_n} \exp(-\alpha \frac{n}{b_n}) M(b_n; f) = 0 \quad \text{for each } \alpha > 0, \tag{4.11}
$$

and $\{b_n\}$ *being a positive sequence satisfy* (1.1) *Then there holds,*

$$
\lim_{n\to\infty} \frac{n}{b_n}\left[(C_n f)(x) - f(x)\right] = \frac{x}{2} f''(x).
$$

at each point $x \geq 0$ *at which* $f''(x)$ *exists*

Proof. Since $f''(x)$ exists, according to Taylor,

$$f\left(\frac{kb_n}{n}\right) = f(x) + \left(\frac{kb_n}{n} - x\right) f'(x) + \left(\frac{kb_n}{n} - x\right)^2 \left[\frac{f''(x)}{2} + h\left(\frac{kb_n}{n} - x\right)\right] \qquad (4.12)$$

where $h(y)$ converges to 0 with y. So,

$$\begin{aligned}(C_n f)(x) &= f(x) + f'(x) \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right) p_{k,n}\left(\frac{x}{b_n}\right) \\ &+ \frac{f''(x)}{2} \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right) \\ &+ \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^2 h\left(\frac{kb_n}{n} - x\right) p_{k,n}\left(\frac{x}{b_n}\right)\end{aligned}$$

By Lemma(4.7)

$$(C_n f)(x) = f(x) + \frac{x(b_n - x)}{2n} f''(x) + R_n(x)$$

where

$$R_n(x) = \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^2 h\left(\frac{kb_n}{n} - x\right) p_{k,n}\left(\frac{x}{b_n}\right)$$

Therefore,

$$\lim_{n\to\infty} \frac{n}{b_n} [(C_n f)(x) - f(x)] = \lim_{n\to\infty} \left[\frac{x}{2} f''(x) - \frac{x^2}{2b_n} f''(x) + \frac{n}{b_n} R_n(x)\right]$$

Since $b_n \to \infty$ as $n \to \infty$, it will be sufficient to prove

$$\lim_{n\to\infty} \frac{n}{b_n} R_n(x) = 0$$

Let us write $R_n(x)$ as follow;

$$\begin{aligned} R_n(x) &= \sum_{\left|\frac{kb_n}{n}-x\right|<\delta} \left(\frac{kb_n}{n}-x\right)^2 h\left(\frac{kb_n}{n}-x\right) p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{kb_n}{n}-x\right)^2 h\left(\frac{kb_n}{n}-x\right) p_{k,n}\left(\frac{x}{b_n}\right) \\ &= : R_{n,1}(x) + R_{n,2}(x) \end{aligned}$$

Because of convergence of $h(y)$ one has

$$\begin{aligned} \left|R_{n,1}(x)\right| &= \left|\sum_{\left|\frac{kb_n}{n}-x\right|<\delta} \left(\frac{kb_n}{n}-x\right)^2 h\left(\frac{kb_n}{n}-x\right) p_{k,n}\left(\frac{x}{b_n}\right)\right| \\ &\leq \epsilon \sum_{\left|\frac{kb_n}{n}-x\right|<\delta} \left(\frac{kb_n}{n}-x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right) \end{aligned}$$

and according to Lemma (4.7) ,

$$\left|R_{n,1}(x)\right| \leq \epsilon A_2(x) \frac{x\,(b_n - x)}{b_n}\left(\frac{b_n}{n}\right)$$

Then

$$\lim_{n\to\infty} \frac{n}{b_n} R_{n,1}(x) \leq \lim_{n\to\infty} \epsilon A_2(x)\left(x - \frac{x^2}{b_n}\right) = 0$$

The representatiaon (4.12) readily yields,

$$\left(\frac{kb_n}{n}-x\right)^2 h\left(\frac{kb_n}{n}-x\right) = f\left(\frac{kb_n}{n}\right) - f(x) - \left(\frac{kb_n}{n}-x\right) f'(x) - \left(\frac{kb_n}{n}-x\right)^2 \frac{f''(x)}{2}$$

Therefore,

$$\begin{aligned}\left|R_{n,2}(x)\right| &\leq \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|f\left(\frac{kb_n}{n}\right)\right| p_{k,n}\left(\frac{x}{b_n}\right)+|f(x)|\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right)\\ &\quad+|f'(x)|\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right| p_{k,n}\left(\frac{x}{b_n}\right)+\frac{|f''(x)|}{2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left(\frac{kb_n}{n}-x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right)\\ &=: \sum_1^*(n)+\sum_2^*(n)+\sum_3^*(n)+\sum_4^*(n)\end{aligned}$$

Since $\sup_{0\leq x\leq\alpha}|f(x)|=M(\alpha;f)$,

$$\sum_1^*(n)\leq M(b_n;f)\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right)$$

From the inequailty (2.3) ,

$$\sum_1^*(n)\leq 2M(b_n;f)\exp\left(-\frac{\delta^2 n}{4xb_n}\right)$$

So

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_1^*(n)\leq\lim_{n\to\infty}2\frac{n}{b_n}M(b_n;f)\exp\left(-\frac{\delta^2 n}{4xb_n}\right)=0$$

Now it remains to show that

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_i^*(n)=0$$

is valid for $i=2,3,4$.

Firstly for $\sum_2^*(n)$,

$$\sum_2^*(n)\leq\frac{|f(x)|}{\delta^4}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left(\frac{kb_n}{n}-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)$$

By Lemma (4.7) ,

$$\sum_2^*(n)\leq\frac{|f(x)|}{\delta^4}A_4(x)\frac{x\,(b_n-x)}{b_n}\left(\frac{b_n}{n}\right)^2$$

which implies that

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_{2}^{*}(n) \leq \lim_{n\to\infty} \frac{|f(x)|}{\delta^4} A_4(x).x \left[\frac{b_n}{n} - \frac{x}{n}\right] = 0$$

Now to the next term,

$$\begin{aligned}\sum_{3}^{*}(n) &\leq \frac{|f'(x)|}{\delta^3} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{kb_n}{n} - x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\leq \frac{|f'(x)|}{\delta^3} A_4(x) \frac{x\,(b_n - x)}{b_n} \left(\frac{b_n}{n}\right)^2\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_{3}^{*}(n) \leq \lim_{n\to\infty} \frac{|f'(x)|}{\delta^3} A_4(x).x \left[\frac{b_n}{n} - \frac{x}{n}\right] = 0$$

In the similar way for $\sum\limits_{4}^{*}(n)$,

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_{4}^{*}(n) \leq \lim_{n\to\infty} \frac{|f''(x)|}{2\delta^2} A_4(x).x \left[\frac{b_n}{n} - \frac{x}{n}\right] = 0$$

This completes the proof. □

Theorem 4.9 *Let a function f, defined on $[0,\infty)$, satisfy the growth condition* (4.11) *for every $\alpha > 0$, $\{b_n\}$ being a positive sequence satisfy* (1.1) *Then there holds,*

$$\lim_{n\to\infty} \frac{n}{b_n}[(C_n f)'(x) - f'(x)] = \frac{f''(x) + xf'''(x)}{2}. \tag{4.13}$$

at each point $x \geq 0$ at which $f'''(x)$ exists.

Proof. Firstly, it will be shown that the theorem is valid for $x = 0$.

From the representation (4.5) it can be written that

$$(C_n f)'(0) = \frac{n}{b_n}\left[f\left(\frac{b_n}{n}\right) - f(0)\right]$$

It is needed to show that

$$\lim_{n\to\infty}\frac{n}{b_n}\left\{\frac{n}{b_n}\left[f\left(\frac{b_n}{n}\right)-f(0)\right]-f'(0)\right\}=\frac{f''(0)}{2}$$

If $f'''(x)$ exists, from Taylor's formula

$$\frac{n}{b_n}\left\{\frac{n}{b_n}\left[f\left(\frac{b_n}{n}\right)-f(0)\right]-f'(0)\right\}=\frac{f''(0)}{2}+\frac{b_n}{n}\left[\frac{f'''(0)}{6}+h\left(\frac{b_n}{n}\right)\right]$$

where $h\left(\frac{b_n}{n}\right)$ converges to 0 as $n\to\infty$. Since $\frac{b_n}{n}\to 0$ as $n\to\infty$, the representation (4.13) holds. Thus the theorem is valid for $x=0$.

So let $b_n > x > 0$. By Taylor it is easily written,

$$\begin{aligned} f\left(\frac{k}{n}b_n\right) &= f(x)+\left(\frac{k}{n}b_n-x\right)f'(x)+\left(\frac{k}{n}b_n-x\right)^2\frac{f''(x)}{2} \\ &\quad +\left(\frac{k}{n}b_n-x\right)^3\left[\frac{f'''(x)}{6}+h\left(\frac{k}{n}b_n-x\right)\right] \end{aligned} \tag{4.14}$$

where $h(y)$ converges to zero with y. Then,

$$\begin{aligned} (C_nf)'(x) &= \frac{nf(x)}{x(b_n-x)}\sum_{k-0}^{n}\left(\frac{k}{n}b_n-x\right)p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad +\frac{nf'(x)}{x(b_n-x)}\sum_{k-0}^{n}\left(\frac{k}{n}b_n-x\right)^2p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad +\frac{nf''(x)}{2x(b_n-x)}\sum_{k-0}^{n}\left(\frac{k}{n}b_n-x\right)^3p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad +\frac{nf'''(x)}{6x(b_n-x)}\sum_{k-0}^{n}\left(\frac{k}{n}b_n-x\right)^4p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad +\frac{n}{x(b_n-x)}\sum_{k-0}^{n}h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4p_{k,n}\left(\frac{x}{b_n}\right) \end{aligned}$$

From Lemma (4.7) ,

$$\begin{aligned} \frac{n}{b_n}\left[(C_nf)'(x)-f'(x)\right] &= \frac{f''(x)}{2}\left(1-\frac{2x}{b_n}\right)+\frac{f''(x)}{2}x\left(1-\frac{x}{b_n}\right) \\ &\quad +\frac{f'''(x)}{6n}\left(b_n-6x+\frac{6x^2}{b_n}\right)+\frac{n}{b_n}R_n(x) \end{aligned}$$

where

$$R_n(x) := \frac{n}{x(b_n - x)} \sum_{k-0}^{n} h\left(\frac{k}{n}b_n - x\right)\left(\frac{k}{n}b_n - x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)$$

It is known that $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$, so to complete the proof, it must be proven that

$$\lim_{n\to\infty} \frac{n}{b_n} R_n(x) = 0$$

The sum $R_n(x)$ can be written as the addition of two sums as follows;

$$\begin{aligned} R_n(x) \quad : \quad &= \frac{n}{x(b_n - x)} \left\{ \sum_{\left|\frac{k}{n}b_n - x\right| < \delta} + \sum_{\left|\frac{k}{n}b_n - x\right| \geq \delta} \right\} h\left(\frac{k}{n}b_n - x\right)\left(\frac{k}{n}b_n - x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right) \\ = \quad &: \sum_1 + \sum_2 \end{aligned}$$

Since $h(y)$ converges to zero, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|h(y)| < \epsilon$ whenever $|y| < \delta$. According to Lemma(4.7) ,

$$\begin{aligned} \left|\sum_1\right| \quad &< \quad \frac{\epsilon n}{x(b_n - x)} \left|\sum_{k-0}^{n} \left(\frac{k}{n}b_n - x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right)\right| \\ &< \quad \frac{\epsilon n}{x(b_n - x)} \frac{x(b_n - x)}{b_n} A_4(x) \left(\frac{b_n}{n}\right)^2 \\ &= \quad \frac{b_n}{n} \epsilon A_4(x) \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_1 = 0$$

By the representation (4.14) it can be written,

$$\begin{aligned} \left(\frac{k}{n}b_n - x\right)^3 h\left(\frac{k}{n}b_n - x\right) \quad &= \quad f\left(\frac{k}{n}b_n\right) - f(x) - \left(\frac{k}{n}b_n - x\right) f'(x) \\ &\quad - \left(\frac{k}{n}b_n - x\right)^2 \frac{f''(x)}{2} - \left(\frac{k}{n}b_n - x\right)^3 \frac{f'''(x)}{6} \end{aligned}$$

So,

$$\begin{aligned}
\left|\sum_2\right| &= \left|\frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^3\left(\frac{k}{n}b_n-x\right)p_{k,n}\left(\frac{x}{b_n}\right)\right| \\
&\leq \frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|f\left(\frac{k}{n}b_n\right)\right|\left|\frac{k}{n}b_n-x\right|p_{k,n}\left(\frac{x}{b_n}\right) \\
&\quad +|f(x)|\frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|\frac{k}{n}b_n-x\right|p_{k,n}\left(\frac{x}{b_n}\right) \\
&\quad +|f'(x)|\frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|\frac{k}{n}b_n-x\right|^2p_{k,n}\left(\frac{x}{b_n}\right)
\end{aligned}$$

$$\begin{aligned}
&\quad +\frac{|f''(x)|}{2}\frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|\frac{k}{n}b_n-x\right|^3p_{k,n}\left(\frac{x}{b_n}\right) \\
&\quad +\frac{|f'''(x)|}{6}\frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|\frac{k}{n}b_n-x\right|^4p_{k,n}\left(\frac{x}{b_n}\right) \\
&=: \sum_3^*(n)+\sum_4^*(n)+\sum_5^*(n)+\sum_6^*(n)+\sum_7^*(n)
\end{aligned}$$

From the Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_3^*(n) &= \frac{n}{x(b_n-x)}\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left\{\left|f\left(\frac{kb_n}{n}\right)\right|\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\left|\frac{kb_n}{n}-x\right|\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\right\} \\
&\leq \frac{n}{x(b_n-x)}\left\{\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|f\left(\frac{kb_n}{n}\right)\right|^2p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \\
&\quad \left\{\sum_{\left|\frac{k}{n}b_n-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^2p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \\
&=: \frac{n}{x(b_n-x)}\sum_{3,1}^*(n)\sum_{3,2}^*(n)
\end{aligned}$$

Since $\sqrt{\sup_{0\le x\le\alpha}|f(x)|^2} = M(\alpha; f)$,

$$\sum_{3,1}^{*}(n) \le M(b_n; f)\left\{\sum_{\left|\frac{k}{n}b_n - x\right|\ge\delta} p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}}$$

It is known that from(2.3) ,

$$\sum_{\left|\frac{k}{n}b_n - x\right|\ge\delta} p_{k,n}\left(\frac{x}{b_n}\right) \le 2\exp\left(-\frac{\delta^2 n}{4xb_n}\right)$$

So it is written that

$$\sum_{3,1}^{*}(n) \le M(b_n; f)\left(2\exp\left(-\frac{\delta^2 n}{4xb_n}\right)\right)^{\frac{1}{2}}$$

$$= \sqrt{2}M(b_n; f)\exp\left(-\frac{\delta^2 n}{8xb_n}\right)$$

By Lemma (4.7),

$$\begin{aligned}
\frac{n}{x(b_n - x)}\sum_{3,2}^{*}(n) &\le \frac{n}{x\,(b_n - x)}\left\{\frac{1}{\delta^2}\sum_{\left|\frac{k}{n}b_n - x\right|\ge\delta}\left|\frac{kb_n}{n} - x\right|^4 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \\
&\le \frac{n}{x\,(b_n - x)}\frac{1}{\delta}\sqrt{A_4(x)}\sqrt{\frac{x(b_n - x)}{n}}\left(\frac{b_n}{n}\right) \\
&= \frac{1}{\delta}\frac{\sqrt{A_4(x)}}{\sqrt{x\left(1 - \frac{x}{b_n}\right)}}
\end{aligned}$$

Therefore it is deduced the following:

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_{3}^{*}(n) \le \lim_{n\to\infty}\frac{\sqrt{2A_4(x)}}{\delta\sqrt{x\left(1 - \frac{x}{b_n}\right)}}\frac{n}{b_n}\exp\left(-\frac{\delta^2 n}{8xb_n}\right)M(b_n; f) = 0$$

Now let us show that the same thing for $\sum\limits_4^*(n)$.

$$\begin{aligned}
\sum_4^*(n) \quad : & = |f(x)| \frac{n}{x(b_n - x)} \sum_{\left|\frac{k}{n}b_n - x\right| \geq \delta} \left|\frac{k}{n}b_n - x\right| p_{k,n}\left(\frac{x}{b_n}\right) \\
& \leq |f(x)| \frac{n}{x(b_n - x)} \frac{1}{\delta^5} \sum_{\left|\frac{k}{n}b_n - x\right| \geq \delta} \left|\frac{k}{n}b_n - x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right) \\
& \leq \frac{|f(x)|}{x(b_n - x)} \frac{n}{\delta^5} \frac{x(b_n - x)}{b_n} \left(\frac{b_n}{n}\right)^3 A_6(x) \\
& = \frac{|f(x)|}{\delta^5} \left(\frac{b_n}{n}\right)^2 A_6(x)
\end{aligned}$$

Then

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_4^*(n) \leq \lim_{n\to\infty} \frac{|f(x)|}{\delta^5} A_6(x) \frac{b_n}{n} = 0$$

Now to the next term,

$$\begin{aligned}
\sum_5^*(n) \quad : & = |f'(x)| \frac{n}{x(b_n - x)} \sum_{\left|\frac{k}{n}b_n - x\right| \geq \delta} \left|\frac{k}{n}b_n - x\right|^2 p_{k,n}\left(\frac{x}{b_n}\right) \\
& \leq |f'(x)| \frac{n}{x(b_n - x)} \frac{1}{\delta^4} \sum_{\left|\frac{k}{n}b_n - x\right| \geq \delta} \left|\frac{k}{n}b_n - x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right) \\
& \leq \frac{|f'(x)|}{x(b_n - x)} \frac{n}{\delta^4} \frac{x(b_n - x)}{b_n} \left(\frac{b_n}{n}\right)^3 A_6(x) \\
& = \frac{|f'(x)|}{\delta^4} \left(\frac{b_n}{n}\right)^2 A_6(x)
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_4^*(n) \leq \lim_{n\to\infty} \frac{|f'(x)|}{\delta^4} A_6(x) \frac{b_n}{n} = 0$$

For $\sum\limits_5^*(n)$ and $\sum\limits_6^*(n)$, in the same way it is composed that

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_5^*(n) \leq \lim_{n\to\infty} \frac{|f''(x)|}{2\delta^3} A_6(x) \frac{b_n}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_{6}^{*}(n) \leq \lim_{n\to\infty} \frac{|f'''(x)|}{6\delta^2} A_6(x) \frac{b_n}{n} = 0$$

Eventually, for all that the proof have been finalizied. □

4.2 Convergence in the Variation Seminorm

One of the most engrossing issue concerning approximaion theory is convergence in a semi-normed space. The first study about this subject matter was occured by Lorentz (Lorentz, 1953). Following, Bardaro, Butzer, Stens, Vinti (Bardaro et.al., 2003) specified the variation detracting property which is given by for a linear operator L,

$$V_I[Lf] \leq V_I[f]$$

is such a serious issue in order to obtain a convergence result in the variation seminorm. Throughout this section, the class $TV(I)$ is space of all the function of bounded variation on I, reported with the seminorm

$$\|f\|_{TV(I)} := V_I[f].$$

For a given $f \in TV(I)$, the sequence (L_n) converges in variation to f, if

$$\lim_{n\to\infty} V_I[L_n f - f] = 0$$

holds. This represents the $TV - approximation$ of a function f by the sequence (L_n).

The main purpose of this section is to confirm the variation detracting property and convergence in the variation seminorm for $(B_n f)$ and $(C_n f)$.

4.2.1 Bernstein polynomials case

Theorem 4.10 *If* $f \in TV[0,1]$*, then*

$$\|B_n f\|_{TV[0,1]} \leq \|f\|_{TV[0,1]}.$$

Proof. By Theorem (3.13) and the representation (4.2) it is deduced,

$$\begin{aligned}
\|B_n f\|_{TV[0,1]} &= V_{[0,1]}[B_n f] = \int_0^1 |(B_n f)'(x)|\, dx \\
&= \int_0^1 \left| n \sum_{k=0}^{n-1} \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] p_{k,n-1}(x) \right| dx \\
&\leq \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right| n \binom{n-1}{k} \int_0^1 x^k (1-x)^{n-k-1} dx
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right| n \binom{n-1}{k} B(k+1, n-k) \\
&= \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right| n \binom{n-1}{k} \frac{\Gamma(k+1)\Gamma(n-k)}{\Gamma(n+1)} \\
&= \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right| \leq V_{[0,1]}[f] = \|f\|_{TV[0,1]}
\end{aligned}$$

□

Theorem 4.11 *Let* $f \in TV[0,1]$. *There holds* $f \in AC[0,1]$ *if and only if*

$$\lim_{n\to\infty} V_{[0,1]}[B_n f - f] = 0.$$

Proof. Since f and $B_n f \in AC[0,1]$, then $B_n f - f \in AC\,[0,1]$. By following Theorem (3.13) and Remark (3.20) it is written,

$$\lim_{n\to\infty} V_{[0,1]}[B_n f - f] = \lim_{n\to\infty} \int_0^1 |(B_n f)'(x) - f'(x)|\, dx$$

From Theorem(4.4) it can be seen easily that $(B_nf)'(x) \to f'(x)$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} V_{[0,1]}[B_nf - f] = 0.$$

Conversely, since $\lim_{n\to\infty} V_{[0,1]}[B_nf - f] = 0$ it is written that $\lim_{n\to\infty} \|B_nf - f\|_{TV[0,1]} = 0$. This means that $B_nf \to f$ in TV space. Therefore f is in AC because of AC is closed. □

4.2.2 Chlodovsky polynomials case

Theorem 4.12 *If $f \in TV[0, b_n]$, then*

$$V_{[0,b_n]}[C_nf] \leq V_{[0,b_n]}[f].$$

Proof. If the representation (4.5) and Theorem (3.13) are followed, it is obtained

$$\begin{aligned} V_{[0,b_n]}[C_nf] &= \int_0^{b_n} |(C_nf)'(x)|\,dx \\ &= \int_0^{b_n} \left| \frac{n}{b_n} \sum_{k=0}^{n-1} \left[f\left(\frac{k+1}{n}b_n\right) - f\left(\frac{k}{n}b_n\right)\right] p_{k,n-1}\left(\frac{x}{b_n}\right) \right| \\ &\leq \frac{n}{b_n} \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}b_n\right) - f\left(\frac{k}{n}b_n\right)\right| \binom{n-1}{k} \int_0^{b_n} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-1-k} dx \end{aligned}$$

If it is defined that

$$t = \frac{x}{b_n}$$

It can be written

$$\begin{aligned} V_{[0,b_n]}[C_nf] &\leq \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}b_n\right) - f\left(\frac{k}{n}b_n\right)\right| n\binom{n-1}{k} \int_0^1 t^k (1-t)^{n-1-k}\,dt \\ &= \sum_{k=0}^{n-1} \left| f\left(\frac{k+1}{n}b_n\right) - f\left(\frac{k}{n}b_n\right)\right| \leq V_{[0,b_n]}[f]. \end{aligned}$$

It is known that $(C_n f)(0) = f(0)$. Since $\|f\|_{BV(I)} := V_I[f] + |f(0)|$,

$$\begin{aligned} \|(C_n f)\|_{BV[0,b_n]} &= V_{[0,b_n]}[C_n f] + |(C_n f)(0)| \\ &\leq V_{[0,b_n]}[f] + |f(0)| = \|f\|_{BV[0,b_n]} \end{aligned}$$

□

Theorem 4.13 *Let* $f \in TV[0, b_n]$, $\{b_n\}_{n=0}^{\infty}$ *satisfying* (1.1) *There holds*

$$\lim_{n\to\infty} \|C_n f - f\|_{TV[0,\infty)} = 0 \quad \Longleftrightarrow \quad f \in AC[0, b_n].$$

The proof is done in the same way with the proof of Theorem (4.11) by using Theorem (4.9) instead of Theorem (4.4) .

CHAPTER 5

RATE OF CONVERGENCE IN THE VARIATION SEMINORM

This chapter is devoted to indicate the rate of approximation by $(B_n f)$ to f and $(C_n f)$ to f in the variation seminorm.

5.1 Bernstein Polynomials Case

Theorem 5.1 *Let $g'' \in AC[0,1]$. Then*

$$V_{[0,1]}\left[B_n g - g\right] \leq \frac{C}{n}\left\{V_{[0,1]}[g] + V_{[0,1]}[g'']\right\} \qquad (n \geq 3),$$

Proof. At this time it will be used of Taylor's formula with integral remainder term,

$$\begin{aligned} g\left(\frac{k}{n}\right) &= g(x) + \left(\frac{k}{n} - x\right)g'(x) + \left(\frac{k}{n} - x\right)^2 \frac{g''(x)}{2} \\ &+ \frac{1}{2}\int_x^{\frac{k}{n}} \left(\frac{k}{n} - t\right)^2 g'''(t)dt \end{aligned}$$

Substituting $t - x = u$, it is easily reached that

$$\begin{aligned} g\left(\frac{k}{n}\right) &= g(x) + \left(\frac{k}{n} - x\right)g'(x) + \left(\frac{k}{n} - x\right)^2 \frac{g''(x)}{2} \\ &+ \frac{1}{2}\int_0^{\frac{k}{n}-x} \left(\frac{k}{n} - x - u\right)^2 g'''(x+u)du \end{aligned}$$

As it can be seen the proof of Theorem (4.4), one has

$$(B_n g)'(x) = g'(x) + \frac{(1-2x)}{2n}g''(x) + (R_n g)(x)$$

here the remainder is given by

$$(R_n g)(x) = \frac{1}{2x(1-x)} \sum_{k=0}^{n} \left[\int_0^{\frac{k}{n}-x} \left(\frac{k}{n} - x - u \right)^2 g'''(x+u)\,(k-nx)\, p_{k,n}\left(\frac{x}{b_n} \right) du \right]$$

It is known that,

$$V_{[I]} = \int_I |g'(t)|\,dt = \|g\|_{L_1(I)}$$

So, in the cause of proving the theorem it is needed to use the weighted $L_1 - norm$.

$$\begin{aligned}
\|(B_n g)' - g'\|_{L_1[0,1]} &= \int_0^1 |(B_n g)'(x) - g'(x)|\,dx \\
&= \int_0^1 \left| \frac{g''(x)}{2n}(1-2x) + (R_n g)(x) \right| dx \\
&\leq \frac{1}{2n} \int_0^1 |g''(x)|\,|1-2x|\,dx + \int_0^1 |(R_n g)(x)|\,dx \\
&= \frac{1}{2n} \int_0^1 |g''(x)|\,|1-2x|\,dx + \|(R_n g)(x)\|_{L_1[0,1]}
\end{aligned}$$

Since $|1-2x| \leq 1$, for $x \in [0,1]$, it is deduced

$$\begin{aligned}
\|(B_n g)' - g'\|_{L_1[0,1]} &\leq \frac{1}{2n} \int_0^1 |g''(x)|\,dx + \|(R_n g)(x)\|_{L_1[0,1]} \\
&= \frac{1}{2n} \|g''\|_{L_1[0,1]} + \|(R_n g)(x)\|_{L_1[0,1]}
\end{aligned}$$

Noting that

$$\left| \frac{k}{n} - v \right|^2 \leq \left| \frac{k}{n} - x \right|^2$$

holds for $\frac{k}{n} \leq v \leq x$ or $x \leq v \leq \frac{k}{n}$. Applying the substitution $x + u = v$ to $(R_n g)(x)$,

$$\begin{aligned} |(R_n g)(x)| &= \left| \frac{1}{2x(1-x)} \sum_{k=0}^{n} \left[\int_x^{\frac{k}{n}} \left(\frac{k}{n} - v \right)^2 g'''(v)\,(k - nx)\, p_{k,n}(x)\, dv \right] \right| \\ &\leq \frac{1}{2n^2 x(1-x)} \sum_{k=0}^{n} \int_x^{\frac{k}{n}} (k - vn)^2\, |k - nx|\, |g'''(v)|\, dv . p_{k,n}(x) \end{aligned}$$

Observing that $(k - nx)^3$ and $\int_x^{\frac{k}{n}} |g'''(v)|\, dv$ have the same sign. There follows that,

$$\begin{aligned} \|(R_n g)\|_{L_1[0,1]} &= \int_0^1 \left| \frac{1}{2x(1-x)} \sum_{k=0}^{n} \left[\int_0^{\frac{k}{n}-x} \left(\frac{k}{n} - x - u \right)^2 g'''(x+u)\,(k - nx)\, p_{k,n}(x)\, du \right] \right| dx \\ &\leq \frac{1}{2n^2} \int_0^1 \sum_{k=0}^{n} \frac{(k - nx)^3}{x(1-x)} \int_x^{\frac{k}{n}} |g'''(v)|\, dv . p_{k,n}(x)\, dx \\ &= \frac{1}{2n^2} \int_0^1 \sum_{k=0}^{n} \frac{(k - nx)^3}{x(1-x)} \int_0^{\frac{k}{n}} |g'''(v)|\, dv . p_{k,n}(x)\, dx + A_n g \end{aligned}$$

Here the term $A_n g$ was added and substracted, where

$$A_n g = -\frac{1}{2n^2} \int_0^1 \sum_{k=0}^{n} \frac{(k - nx)^3}{x(1-x)} \int_0^{x} |g'''(v)|\, dv . p_{k,n}(x)\, dx$$

Thus it can be written that

$$\|(R_n g)\|_{L_1[0,1]} \leq \frac{1}{2n^2} \left| \int_0^1 \sum_{k=0}^{n} \frac{(k - nx)^3}{x(1-x)} \int_0^{\frac{k}{n}} |g'''(v)|\, dv . p_{k,n}(x)\, dx \right| + |A_n g|$$

Applying Fubini's theorem,

$$\begin{aligned}\|(R_n g)\|_{L_1[0,1]} &\leq \frac{1}{2n^2}\sum_{k=0}^{n}\left|\int_0^{\frac{k}{n}}|g'''(v)|\left[\int_0^1 \frac{(k-nx)^3}{x(1-x)}.p_{k,n}(x)\,dx\right]dv\right| \\ &+\left|\frac{1}{2n^2}\int_0^1\sum_{k=0}^{n}\frac{(k-nx)^3}{x(1-x)}p_{k,n}(x)\,dx\int_0^x|g'''(v)|\,dv\right|\end{aligned}$$

From Lemma (4.2),

$$\begin{aligned}\sum_{k=0}^{n}\frac{(k-nx)^3}{x(1-x)}p_{k,n}(x) &= \frac{1}{x(1-x)}nx(1-x)(1-2x) \\ &= n(1-2x)\end{aligned}$$

Therefore,

$$\begin{aligned}A_n g &= \frac{1}{2n}\int_0^1(1-2x)dx\int_0^x|g'''(v)|\,dv \\ &\leq \frac{1}{2n}\int_0^x|g'''(v)|\,dv \leq \frac{1}{2n}\int_0^1|g'''(v)|\,dv\end{aligned}$$

The inner integral can be evaluated and estimated by

$$\begin{aligned}&\left|\int_0^1\frac{(k-nx)^3}{x(1-x)}p_{k,n}(x)\,dx\right| \\ &= \binom{n}{k}\left|\int_0^1\frac{(k-nx)^3}{x(1-x)}x^k(1-x)^{n-k}\,dx\right| \\ &= \binom{n}{k}\left|\int_0^1(k-nx)^3\,x^{k-1}(1-x)^{n-k-1}\,du\right|\end{aligned}$$

$$= \binom{n}{k} \left| \begin{array}{c} k^3 B(k, n-k) - 3k^2 n B(k+1, n-k) \\ +3kn^2 B(k+2, n-k) - n^3 B(k+3, n-k) \end{array} \right|$$

$$= \binom{n}{k} \left| \begin{array}{c} k^3 \frac{(k-1)!(n-k-1)!}{(n-1)!} - 3k^2 n \frac{k!(n-k-1)!}{n!} \\ +3kn^2 \frac{(k+1)!(n-k-1)!}{(n-1)!} - n^3 \frac{(k+2)!(n-k-1)!}{(n+2)!} \end{array} \right|$$

$$= \left| \frac{2n(n-2k)}{(n+1)(n+2)} \right|$$

Hence,

$$\left| \int_0^1 \frac{(k-nx)^3}{x(1-x)} p_{k,n}(x)\, dx \right| \le \frac{2n^2}{(n+1)(n+2)}$$

Consequently

$$\begin{aligned} \|(R_n g)(x)\|_{L_1[0,1]} &\le \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \int_0^{\frac{k}{n}} |g'''(v)|\, dv + \frac{1}{2n} \int_0^1 |g'''(v)|\, dv \\ &\le \frac{(n+1)}{(n+1)(n+2)} \int_0^1 |g'''(v)|\, dv + \frac{1}{2n} \int_0^1 |g'''(v)|\, dv \\ &= \left(\frac{1}{n+2} + \frac{1}{2n} \right) \int_0^1 |g'''(v)|\, dv \\ &\le \frac{3}{2n} \|g'''\|_{L_1[0,1]} \end{aligned}$$

So this yields

$$\left\|(B_n g)' - g'\right\|_{L_1[0,1]} \le \frac{1}{2n} \|g''\|_{L_1[0,1]} + \frac{3}{2n} \|g'''\|_{L_1[0,1]}$$

According to Stein's inequality,

$$\|g''\|_{L_1[0,1]} \le B_1 \left(\|g'\|_{L_1[0,1]} + \|g'''\|_{L_1[0,1]} \right), \text{ where } B_1 > 1$$

Therefore, it is obtained that

$$\begin{aligned}\left\|(B_ng)'-g'\right\|_{L_1[0,1]} &\leq \frac{1}{2n}B_1\left(\|g'\|_{L_1[0,1]}+\|g'''\|_{L_1[0,1]}\right)+\frac{3}{2n}\|g'''\|_{L_1[0,1]}\\ &= \frac{1}{2n}B_1\|g'\|_{L_1[0,1]}+\left(\frac{1}{2n}B_1+\frac{3}{2n}\right)\|g'''\|_{L_1[0,1]}\\ &\leq \frac{1}{n}\left(\frac{B_1}{2}+\frac{3}{2}\right)\left(\|g'\|_{L_1[0,1]}+\|g'''\|_{L_1[0,1]}\right)\\ &= \frac{C}{n}\left(\|g'\|_{L_1[0,1]}+\|g'''\|_{L_1[0,1]}\right),\text{ where } C=\frac{B_1}{2}+\frac{3}{2}\end{aligned}$$

which completes the proof. □

5.2 Chlodovsky polynomials case

Theorem 5.2 *Let $g''\in AC\,[0,b_n]$. Then*

$$V_{[0,b_n]}[C_ng-g]\leq B\frac{b_n}{n}\left\{V_{[0,b_n]}[g]+V_{[0,b_n]}\left[g''\right]\right\},$$

where $B>1$ is a constant and b_n being the same as appearing in introduction.

Proof. By Taylor formula with integral reminder term,

$$\begin{aligned}g\left(\frac{k}{n}b_n\right) &= g(x)+\left(\frac{k}{n}b_n-x\right)g'(x)+\left(\frac{k}{n}b_n-x\right)^2\frac{g''(x)}{2}\\ &\quad+\frac{1}{2}\int\limits_x^{\frac{k}{n}b_n}\left(\frac{k}{n}b_n-t\right)^2g'''(t)dt\end{aligned}$$

If the variable t is changed into the variable u by using the substitution $t-x=u$, it is written that

$$\begin{aligned}g\left(\frac{k}{n}b_n\right) &= g(x)+\left(\frac{k}{n}b_n-x\right)g'(x)+\left(\frac{k}{n}b_n-x\right)^2\frac{g''(x)}{2}\\ &\quad+\frac{1}{2}\int\limits_0^{\frac{k}{n}b_n-x}\left(\frac{k}{n}b_n-x-u\right)^2g'''(x+u)du\end{aligned}$$

So from the representation (4.6), one has

$$\begin{aligned}
(C_n g)'(x) &= \frac{1}{x(b_n-x)}\sum_{k=0}^{n} g\left(\frac{k}{n}b_n\right)(kb_n-nx)\,p_{k,n}\left(\frac{x}{b_n}\right)\\
&= \frac{g(x)}{x(b_n-x)}\sum_{k=0}^{n}(kb_n-nx)\,p_{k,n}\left(\frac{x}{b_n}\right)\\
&\quad+\frac{g'(x)}{x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)(kb_n-nx)\,p_{k,n}\left(\frac{x}{b_n}\right)\\
&\quad+\frac{g''(x)}{2x(b_n-x)}\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^2(kb_n-nx)\,p_{k,n}\left(\frac{x}{b_n}\right)\\
&\quad+(R_n g)(x)
\end{aligned}$$

where the remainder is written by

$$(R_n g)(x) = \frac{1}{2x(b_n-x)}\sum_{k=0}^{n}\left[\int_0^{\frac{k}{n}b_n-x}\left(\frac{k}{n}b_n-x-u\right)^2 g'''(x+u)\,(kb_n-nx)\,p_{k,n}\left(\frac{x}{b_n}\right)du\right]$$

By using Lemaa(4.7), it is obtained with ease,

$$(C_n g)'(x) = g'(x)+\frac{(b_n-2x)}{2n}g''(x)+(R_n g)(x)$$

Thus,

$$\begin{aligned}
\|(C_n g)'-g'\|_{L_1[0,b_n]} &= \int_0^{b_n}|(C_n g)'(x)-g'(x)|\,dx\\
&= \int_0^{b_n}\left|\frac{g''(x)}{2n}(b_n-2x)+(R_n g)(x)\right|dx\\
&\le \frac{1}{2n}\int_0^{b_n}|g''(x)|\,|b_n-2x|\,dx+\int_0^{b_n}|(R_n g)(x)|\,dx\\
&= \frac{1}{2n}\int_0^{b_n}|g''(x)|\,|b_n-2x|\,dx+\|(R_n g)(x)\|_{L_1[0,b_n]}
\end{aligned}$$

Since $|b_n - 2x| \le b_n$, for $x \in [0, b_n]$, it is deduced

$$\begin{aligned} \|(C_n g)' - g'\|_{L_1[0,b_n]} &\le \frac{b_n}{2n} \int_0^{b_n} |g''(x)|\, dx + \|(R_n g)(x)\|_{L_1[0,b_n]} \\ &= \frac{b_n}{2n} \|g''\|_{L_1[0,b_n]} + \|(R_n g)(x)\|_{L_1[0,b_n]} \end{aligned}$$

Note that

$$\left|\frac{kb_n}{n} - v\right|^2 \le \left|\frac{kb_n}{n} - x\right|^2$$

holds for $\frac{kb_n}{n} \le v \le x$ or $x \le v \le \frac{kb_n}{n}$. Applying the substitution $x + u = v$ to $(R_n g)(x)$,

$$\begin{aligned} |(R_n g)(x)| &= \left| \frac{1}{2x(b_n - x)} \sum_{k=0}^{n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\,(kb_n - nx)\, p_{k,n}\left(\frac{x}{b_n}\right) dv \right] \right| \\ &\le \frac{1}{2n^2 x(b_n - x)} \sum_{k=0}^{n} \int_x^{\frac{k}{n}b_n} (kb_n - vn)^2\, |kb_n - nx|\, |g'''(v)|\, dv . p_{k,n}\left(\frac{x}{b_n}\right) \end{aligned}$$

It is seen that $(kb_n - nx)^3$ and $\int_x^{\frac{k}{n}b_n} |g'''(v)|\, dv$ have the same sign and known that $|kb_n - nv| \le |kb_n - nx|$. From the previous inequality there follows that,

$$\begin{aligned} \|(R_n g)\|_{L_1[0,b_n]} &= \int_0^{b_n} \left| \frac{1}{2x(b_n - x)} \sum_{k=0}^{n} \left[\int_0^{\frac{k}{n}b_n - x} \left(\frac{k}{n}b_n - x - u\right)^2 g'''(x+u)\,(kb_n - nx)\, p_{k,n}\left(\frac{x}{b_n}\right) du \right] \right| dx \\ &\le \frac{1}{2n^2} \int_0^{b_n} \sum_{k=0}^{n} \frac{(kb_n - nx)^3}{x(b_n - x)} \int_x^{\frac{k}{n}b_n} |g'''(v)|\, dv . p_{k,n}\left(\frac{x}{b_n}\right) dx \\ &= \frac{1}{2n^2} \int_0^{b_n} \sum_{k=0}^{n} \frac{(kb_n - nx)^3}{x(b_n - x)} \int_0^{\frac{k}{n}b_n} |g'''(v)|\, dv . p_{k,n}\left(\frac{x}{b_n}\right) dx \end{aligned}$$

$$-\frac{1}{2n^2}\int_0^{b_n}\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}\int_0^{x}|g'''(v)|\,dv.p_{k,n}\left(\frac{x}{b_n}\right)dx$$

$$\leq \frac{1}{2n^2}\left|\int_0^{b_n}\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}\int_0^{\frac{k}{n}b_n}|g'''(v)|\,dv.p_{k,n}\left(\frac{x}{b_n}\right)dx\right|$$

$$+\left|\frac{1}{2n^2}\int_0^{b_n}\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}\int_0^{x}|g'''(v)|\,dv.p_{k,n}\left(\frac{x}{b_n}\right)dx\right|$$

Applying Fubini's theorem,

$$\|(R_ng)\|_{L_1[0,b_n]} \leq \frac{1}{2n^2}\sum_{k=0}^{n}\left|\int_0^{\frac{k}{n}b_n}|g'''(v)|\left[\int_0^{b_n}\frac{(kb_n-nx)^3}{x(b_n-x)}.p_{k,n}\left(\frac{x}{b_n}\right)dx\right]dv\right|$$

$$+\left|\frac{1}{2n^2}\int_0^{b_n}\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}p_{k,n}\left(\frac{x}{b_n}\right)dx\int_0^{x}|g'''(v)|\,dv\right| \qquad (5.1)$$

From Lemma (4.7) ,

$$\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}p_{k,n}\left(\frac{x}{b_n}\right) = \frac{1}{x(b_n-x)}nx(b_n-x)(b_n-2x)$$

$$= n(b_n-2x)$$

Therefore the latter integral of the last inequality can be written as

$$\frac{1}{2n^2}\int_0^{b_n}\sum_{k=0}^{n}\frac{(kb_n-nx)^3}{x(b_n-x)}p_{k,n}\left(\frac{x}{b_n}\right)dx\int_0^{x}|g'''(v)|\,dv$$

$$= \frac{1}{2n}\int_0^{b_n}(b_n-2x)dx\int_0^{x}|g'''(v)|\,dv$$

$$\leq \frac{b_n}{2n}\int_0^{x}|g'''(v)|\,dv \leq \frac{b_n}{2n}\int_0^{b_n}|g'''(v)|\,dv$$

Let $\frac{x}{b_n} = u$ for the former integral of the inequality (5.1). Therefore,

$$\begin{aligned}
&\left| \int_0^{b_n} \frac{(kb_n - nx)^3}{x\,(b_n - x)} p_{k,n}\left(\frac{x}{b_n}\right) dx \right| \\
&= b_n^2 \binom{n}{k} \left| \int_0^1 \frac{(k - nu)^3}{u(1-u)} u^k (1-u)^{n-k}\, du \right| \\
&= b_n^2 \binom{n}{k} \left| \int_0^1 (k - nu)^3\, u^{k-1} (1-u)^{n-k-1}\, du \right| \\
&= b_n^2 \left| \frac{2n(n-2k)}{(n+1)(n+2)} \right|
\end{aligned}$$

See in the proof of Theorem(5.1) . Hence,

$$\left| \int_0^{b_n} \frac{(kb_n - nx)^3}{x\,(b_n - x)} p_{k,n}\left(\frac{x}{b_n}\right) dx \right| \le \frac{2b_n^2 n^2}{(n+1)(n+2)}$$

Thus,

$$\begin{aligned}
\|(R_n g)(x)\|_{L_1[0,b_n]} &\le \frac{b_n^2}{(n+1)(n+2)} \sum_{k=0}^{n} \int_0^{\frac{k}{n}b_n} |g'''(v)|\, dv + \frac{b_n}{2n} \int_0^{b_n} |g'''(v)|\, dv \\
&\le \frac{b_n^2(n+1)}{(n+1)(n+2)} \int_0^{b_n} |g'''(v)|\, dv + \frac{b_n}{2n} \int_0^{b_n} |g'''(v)|\, dv \\
&= b_n \left(\frac{1}{n+2} + \frac{1}{2n} \right) \int_0^{b_n} |g'''(v)|\, dv \\
&\le \frac{3b_n}{2n} \|g'''\|_{L_1[0,b_n]}
\end{aligned}$$

So, this provides

$$\left\| \left(C_{ng}\right)' - g' \right\|_{L_1[0,b_n]} \le \frac{b_n}{2n} \|g''\|_{L_1[0,b_n]} + \frac{3b_n}{2n} \|g'''\|_{L_1[0,b_n]}$$

Then, it is deduced that

$$\begin{aligned}
\left\|\left(C_{ng}\right)' - g'\right\| &\leq \frac{b_n}{2n} B_1 \left(\|g'\|_{L_1[0,b_n]} + \|g'''\|_{L_1[0,b_n]}\right) + \frac{3b_n}{2n} \|g'''\|_{L_1[0,b_n]} \\
&= \frac{b_n}{2n} B_1 \|g'\|_{L_1[0,b_n]} + \left(\frac{b_n}{2n} B_1 + \frac{3b_n}{2n}\right) \|g'''\|_{L_1[0,b_n]} \\
&\leq \frac{b_n}{n} \left(\frac{B_1}{2} + \frac{3}{2}\right) \left(\|g'\|_{L_1[0,b_n]} + \|g'''\|_{L_1[0,b_n]}\right) \\
&= \frac{b_n}{n} B \left(\|g'\|_{L_1[0,b_n]} + \|g'''\|_{L_1[0,b_n]}\right), \text{ where } B = \frac{B_1}{2} + \frac{3}{2}
\end{aligned}$$

Consequently, the proof is completed. □

CHAPTER 6

CONCLUSION

One of the main issue in this thesis is the variation detracting property. As far as it is known, Lorentz's work (Lorentz, 1953) is the first study about this topic. Afterward, Bardaro, Butzer, Stens, Vinti's study (Bardaro et.al., 2003) showed that so as to reach a convergence resut in the variation seminorm, setting the variation detracting property is so significant. By following these, the study presented this thesis offers a new path and can be utilized to create the convergence results for other operators in the variation seminorm. One of the interesting result which is taken from this study is to obtain the convergence event in the semi-normed space which a distance can not be described. The significant thing concerning this event is to obtain the convergence in that space by setting a relationship between $L_1 - norm$ and the seminorm.

REFERENCES

[1] Agratini, O. 2006. On the variation detracting property of a class of operators. 19. 1261-1264. Appl. Math. Lett.

[2] Albrycht, J. and Radecki, J. 1960. On a generalization of the theorem of Voronovskaya. Zeszyt 2. Poznań 1-7. Zeszyty. Naukowe UAM.

[3] Altomare, F. and Campiti, M. 1994. Korovkin-type approximation theory and its applications. vol. 17. Berlin. Walter de Gruyter Studies in Math. de Gruyter&Co.

[4] Bardaro, C. et.al. 2003. Convergence in variation and rates of approximation for Berstein-type polynomials and singular convolution integral. 23 (4) . 299-346. Munich. Analysis.

[5] Bernstein, S., N. 1912/1913. Demonstration du Thĕoreme de Weierstrass fondĕe sur le calcul des probabilitĕs.13. 1-2. Comm. Soc. Math. Kharkow.

[6] Butzer, P., L. 1955. Summability of generalized Bernstein polynomials. I. 22. 617-623. Duke Math. J.

[7] Butzer, P., L. and Nessel, R., J. 1971. Fourier Analysis and Approximation. New York. London. Academic Press

[8] Butzer, P., L. and Karsli, H. 2009. Voronovskaya-type theorems for derivatives of the Berstein-Cholodovsy polynomials and the Szasz-Mirakyan operator. 49(1). 33-57. Comment. Math.

[9] Chlodovsky, I. 1937. Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein. 4. 380-393. Compositio Math.

[10] Heilmann, M. 1989. Direct and converse results for operators of Baskakov-Durrmeyer type Approx. Theory. Appl. 5. no. 1, 105-127.

[11] Karsli, H. 2012. A Voronovskaya-type theorem for the second derivative of the Bernstein–Chlodovsky polynomials. 61. 1. 9–19. Proc. Est. Acad. Sci.

[12] Karsli, H. 2013. On Convergence of Cholodovsky and Cholodovsky-Kantorovich Polynomails in the Variation Seminorm. 10. 41–56. Mediterr. J. Math.

[13] Kosmala, A., J. 2004. A Friendly Introduction to Analysis: Single and Multivariable. Second Edition.

[14] Kreyszig, E. 1978. Introductory Functional Analysis with Applications. New York. Wiley.

[15] Lorentz, G., G. 1953. Bernstein Polynomials. Toronto.University of Toronto Press.

[16] Natanson, I., P. 1955-1964. Theory of Functions of a Real Variable. Vol I. New York. Ungar.

[17] Natanson, I., P. 1955-1964. Theory of Functions of a Real Variable. Vol II. New York. Ungar.

[18] Paltanea, R. 2004. Approximation Theory Using Positive Linear Operators. Boston. Birkhauser.

[19] Voronovskaya, E. 1932. D‘etermination de la forme asymptotique dapproximation des functions par polynomes de M. Bernstein. 79. 79-85. C. R. Acad. Sci. URSS.

Further Reading

Kivinukk, Andi, Metsmägi, Tarmo. 2011. Approximation in variation by Kantorovich operators. 60(4). 201 - 209. Proceedings of Estonian Academy of Sciences.

Kivinukk, Andi, Metsmägi, Tarmo. 2011. Approximation in Variation by Meyer-König and Zeller operators. 60(2). 88 - 97. Proceedings of Estonian Academy of Sciences.