HYPERBOLIC GEOMETRY AND COMPLEX ANALYSIS

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ABSTRACT

Hyperbolic geometry, which has a large place in mathematics, consists dynamical systems, chaos theory, number theory and many more mathematics and physics area beside geometry. This geometry dating back to 19.CC, was found during the studies of understanding the axiom which is known as parallel axiom and the fifth one of the axioms published in 300 BC by Euclid. In hyperbolic geometry althought the first four postulate hold, the fifth postulate of Euclid is changed as a, a hyperbolic line and a point not on the given line, there are at least two lines parallel to the given line.The convenient meta- definition for geometry is given by Felix Clain (1849-1929) in his Erlangen programme which is published in 1872. It is 'given a set with some structure and a group of transformations that preserve that structure, geometry is the study of objects that are invariant under these transformations.' For two dimensional Euclidean geometry, the set is the plane R^2 equipped with the Euclidean distance function together with a group of transformations that preserves the distance between points. In this thesis, the defination of hyperbolic geometry given like Euclidean geometry; it will be defined a notation of distance on a set and study the transformations which preserve this distance. During this study will be studied with the upper half plane model and Poancaré madel with the complex valuable functions.

Keyword: Euclidean geometry, Hyperbolic geometry, Möbius transformations, upper half plane and Poancaré model.

ÖZET

Hiperbolik geometri, matematikte geniş yer kaplayan konulardan biri olup, geometrinin yanı sıra Dinamik Sistemler, Kaos Teorisi, Sayılar Teorisi ve daha bir çok matematik ve fizik alanlarını kapsamaktadır. 19. yüzyılda ortaya çıkan bu geometri, Öklid in Milattan önce 300 yılında yayınladığı aksiyomların beşincisi olan ve günümüzde paralellik aksiomu olarak bilinen bu aksiomu anlamaya yönelik yapılan çalışmalarda bulunmuştur. Hiperbolik geometride, Öklid geometrisindeki ilk dört aksiom sağlanmasına rağmen, beşinci aksiom; verilen bir hiperbolik doğru ve bu hiperbolik doğru üzerinde olmayan bir nokta, bu noktadan geçen ve verilen hiperbolik doğruya paralel olan en az iki doğru vardır, olarak değişir.

Geometriye, uygun bir tanımı 1892 yılında Felix Klein, yayınladığı Erlange programında şöyle tanımlamıştır;

'Geometri, bazı yapılarla verilen bir kümede, yapıları koruyan dönüşüm grupları, ve bu dönüşümler altında değişmeyen nesnelerlein incelenmesidir.' iki boyutlu Öklid geometrisi, Öklid uzaklık fonksiyonu ile birlikte noktalar arasındaki mesafeleri koruyan grup dönüşümü ile donatılmış R^2 düzlemidir. Bu tezde, hiperbolik geometri de ayni yolla tanımlanarak bir küme üzerinde uzklık kavramı tanımlayarak, bu uzaklıkları koruyan dönüşümlerle çalışılacaktır. Bu çalışma sırasında hiperbolik geometrinin modellerinden kompleks değerli fonksiyonlar ve üzerinden üst yarı düzlem modeli ve Poancaré modeli ile çalışılacaktır.

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CHAPTER 1

INTRODUCTION TO THE STUDY

1.1 Overview

In this chapter the background of the Hyperbolic Geometry will be presented.
Hyperbolic Geometry has been put forward by different mathematicians. Hyperbolic Geometry originates from Euclidean Geometry. For every line and for every point that does not lie on, there exists a unique line through the point that is parallel to the line. (Greenberg, 1993). All studies and reserches focuses on the same postulate and proves the same result. However this postulate is not true on Hyperbolic geometry. In this study this will be the main focus area.

1.2 Background of the Hyperbolic Geometry

Hyperbolic Geometry was discovered in the first half of the nineteenth century in the midst of attempt to understand the fifth postulate of Euclidean geometry. Hyperbolic geometry is the one of the non-Euclidean geometry that is the prototype of the study of geometry on space of constant negative curvature.
Hyperbolic geometry is connected the many other parts of mathematics. Like, complex analysis, topology, differential geometry, dynamical systems, number theory, geometric group theory, Riemann surface, etc.
In this thesis it will be studied in the upper half plane (also known as the Lobachevskiĭ plane) $\mathbb{H} = \{z \in \mathbb{C} : Imz > 0\}$ with the metric $ds = \frac{|dz|}{Imz}$ and the unit disc (Poincare disk) $D = \{z \in \mathbb{C} : |z| < 1\}$ with the metric $ds = \frac{2|dz|}{(1-|z|^2}$. However, to construct the hyperbolic geometry there are some other models, like, Jemisphere, Klein and Loid (hyperboloid). In order to define and give the background of Hyperbolic Geometry, It is better to start with given Euclidean postulates.

1.2.1 Five postulate of Euclidean geometry

Euclidean geometry is study of geometry in $\mathbb{R}^2$ or more generally $\mathbb{R}^n$. There are many ways to constructing Euclidean geometry, one of them Klein's Erlanges program (Klein, 1878); its give the definition of Euclidean geometry in term of Euclidean plane, equipped with the Euclidean distance function and the set of isometries that preserve the Euclidean distance. The alternatively defination can be given with Greek mathematician Euclid (c.325BC- c.265BC). In the first of his thirteen volume set ' The Elements', Euclid systematically developed Euclidean geometry. In his first book contains twenty three definitions (point, line etc.), five common notions, some proposition and following five postulates (Euclid, 1926);

(i) a straight line may be drown from any point to any other point,

(ii) a finite straight line may be extended continously in a straight line,

(iii) a circle may be drown with any center and any radius,

(iv) all right angles are equal,

(v) if a straight line falling on two straight lines makes the interior angles on the same side less than two right- angles, then the two straight lines, if extended indefinitely, meet on the side on which the angles are less than two right-angles.

It is easy to understand the first four postulates but the fifth postulate is more complex and not natural. Therefore the equivalent explanation for the fifth postulate which is given below and it is known as the *parallel postulate;*

Given any infinite straight line and a point not on that line, there exists a unique infinite straight line through that point and parallel to the given line.

For over two thousand years there are most of studies the fifth postulate. During this period most of plane geometry can be devoloped without using fifth postulate and it is not used until proposition 29 in Euclid's first book and so it is suggested that the parallel postulate is not necessary. In the same period most of mathematcians attempt to prove that the fifth postulate is not independent from the first four postulates and it can be obtained from the others first four simplier postulates. In fact,all studies turned out to be equivalent to the fifth postulate:

Proclus (412- 485), Suppose L is a line and P is a point not on L. Then there exist a unique line L' through P and parallel to (i.e. not meeting) L.

The Englishman John Wallis (1616,1703), he thought he had deduce the fifth postulate but he actually showed, there exist similar triangles of different sizes.

Girolamo Saccheri (1667-1733) who was italian mathematician, considered quadrilaterals with two base angles equal to right angle and with vertical sides having equal length and deduced concequences from the (non euclidean) possibility that the remaining two angles were not right angles.

Johann Heinrich Lambert (1728-1777) proceeded in a similar fashion and wrote an extensive work on the subject.

Kästener (1719-1800) studied with his student Klügel (1739-1812), they considered approximately thirty proof attempts for the parallel postulate.

In the nineteenth century, the decisive progress came when mathematicians leave the studies to find the contradiction the fifth postulate, and they found that,

"Given a line and a point on it, there is more then one line going through the given point that is parallel to the given line."

This postulate constucts the hyperbolic geometry.

Carl Friedrich Gauss (1777-1855), Nikolaĭ Ivanovich Lobachevskiĭ (1792- 1856) and Janos Bolyai (1802- 1860) independently developed a consistent geometry which the parallel postulate fails but the rest of Euclidean's postulates remain true.

Gauss prove that the fifth postulate is independent from the other four postulates. Indeed, he discovery that geometry for which the first four postulates hold but the parallel postulate fail. And this geometry is different from the Euclidean geometry. For instance as a known fact the sum of the three sides of triangle is 180 degrees in the Euclidean geometry but in 1824 Gauss wrote an assumtion that the sum of the sides is less then 180 degrees. And this geometry called Non-Euclidean geometry. Gauss did not publish any of his findings.

Later hyperbolic geometry rediscoverd independently by Bolyai who interest come from his father and he published his discovery as a 26 pages Appendix in his father's book in 1831 (Bolyai and Bolyai, 1913) . The third one is Lobachevskiĭ. He did more extensive studies. He developed a non- Euclidean trigonometry that paralleled the trigonometric formulas of Euclidean geometry and publish his findings in 1829. In 1837 he suggested that curved surface of constant negative curvature represent non-Euclidean geometry.

In 1868 Eugenio Beltrami, established one can constract the hyperbolic plane using standart mathematics and Euclidean geometry.

Klein (1872), study of properties of a set invariant under a transformation group.

Poincare (1854- 1912), in 1881 put forward the isometries or distance preserving bijections of upper half plane model and unit disc model are just the linear fractional transformations or Möbius maps which preserve D or $\mathbb{H}$ respectively. This makes it particularly easy to study and compute with such maps. It turns out that they are also the set of all conformal automorphisms of D or $\mathbb{H}$.

1.3 Comparison Between Euclidean And Hyperbolic Geometry

	Euclidean	**Hyperbolic**
Basic givens	points, lines, planes	points, lines, planes
Model	Euclidean plane	D or $\mathbb{H}$
Lines	Euclidean lines	arcs orthogonal to boundary
Axiom 1	any two distace point lie on a unique line	any two distance point lie on a unique line
Axiom 2	throught any point $P \notin L$ there is a unique parallel line to L	throught any point $p \notin L$ there are infinitely many parallel to L
Angle sum of triangle	π	$< \pi$
Circumferance of circle	$2\pi r$	$2\pi \sinh r$
Area of circle	πr^2	$4\pi \sinh^2 \frac{r}{2}$

1.4 Definitions of Terms

$\mathbb{H}$	:	Upper half plane
D	:	Unit disc
$\overline{D}$	:	Closure of unit disc .
$\mathbb{C}$	:	Complex plane
$\overline{\mathbb{C}}$	:	Extended complex plane.
$Aut(\overline{\mathbb{C}})$	:	Conformal Möbius group
$Aut(\mathbb{H})$	:	Group of conformal Möbius transformation which transformation $\mathbb{H}$ to $\mathbb{H}$.

CHAPTER 2

COMPLEX NUMBERS

The aim of this chaper is give the some backround of complex analysis. There will be given some basic properties and described the polar form of complex numbers. And the end of this chapter there will be defined the Stereographic projection.

Complex numbers can be defined as a ordered pairs (x, y) of real numbers, and there is a one to one correspondence between complex numbers and ordered pairs in the Euclidean plane $\mathbb{R}^2$. The real numbers correspond to the *x-axis* and the pure imaginary numbers which of form iy are corresponds to the *y-axis* in the Euclidean plane. The *y-axis* is reffered to as the imaginart axis.

The sum and product of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows;

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{2.1}$$

$$z_1.z_2 = (x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - x_2y_1) \tag{2.2}$$

Any complex number $z = (x, y)$ can be written as

$$z = (x, 0) + (0, y)$$

acording to (2.2) , it is easy to see that

$$(0, 1)(y, 0) = (0, y)$$

and if defined $i := (0, 1)$ then it will be obtained,

$$z = (x, y) = x + iy.$$

Also the square of i is;

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

In conclusion it can be said that, the complex numbers can be expression of the form,

$$z = x + iy \tag{2.3}$$

where x and y are real numbers.

Now, some basic of algebraic properties for addition and multiplication can be given,

1) $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$ (commutative laws)

2) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1z_2)z_3 = z_1(z_2z_3)$ (associative laws)

3) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (distributive law)

Complex numbers behave as same with the real numbers with respect to algebraic operations. The additive identity $0 = (0, 0)$ and multiplicative identity $1 = (1, 0)$ carry over the entire complex plane from the real numbers. That is,

$$z + 0 = (x, y) + (0, 0) = (x, y) = z$$

and

$$z.1 = (x, y)(1, 0) = (x, y) = z,$$

are satisfied for every complex number $z = (x, y) = x + iy$.

In the complex number system there is a unique additive inverse $-z = (-x, -y) = -x - iy$ for any complex number z. And additive inverse $-z$ satisfy the equation $z + (-z) = 0$. Similarly, there is a unique multiplicative invrese z^{-1} which is satisfy $z.z^{-1} = 1$.

Definition 2.1 *(Modulus)*

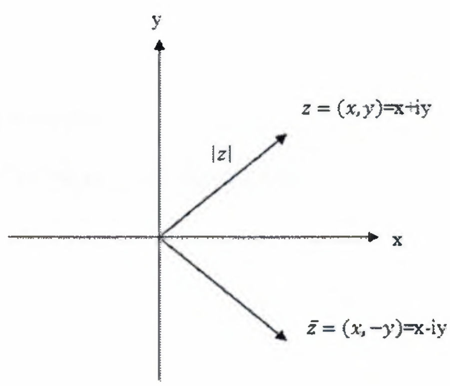
The modulus or absolute value of the complex number $z = x + iy$ is a nonegative real number denoted by the relation

$$|z| = \sqrt{x^2 + y^2}.$$

The number $|z|$ is the distance between the orgin and the point $z = (x, y)$.
From (2.3), $z^2 = (Rez)^2 + (Imz)^2$, then $Rez \leq |Rez| \leq |z|$ and $Imz \leq |Imz| \leq |z|$ are easily obtained.

Definition 2.2 *(Complex conjugate)*

The complex conjugate of a complex number $z = x + iy$ is defined to be $\overline{z} = x - iy$. Geometrically is the reflection of z in the x-axis.



Let $z = x + iy$ and the conjugate of z is $\overline{z} = x - iy$, then their addition,

$$z + \overline{z} = 2x$$

can be defined the real part of z as a,

$$x = \frac{z + \overline{z}}{2}. \tag{2.4}$$

If thinking the subtruction of z and $\bar{z}$, it will be obtained the imaginary part of z,

$$z - \bar{z} = 2iy$$

then

$$y = \frac{z - \bar{z}}{2i}. \tag{2.5}$$

Some properties of compex conjugation are;

$\bar{\bar{z}} = z$

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

$|z| = |\bar{z}|$

$|z|^2 = z.\bar{z}.$

After these properties it can be given the triangle inequalitiy which is the important appication of these properties.

Theorem 2.3 *(Triangle inequalitiy)*

If z_1 and z_2 are arbitrary complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|\,.$$

Proof.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2Re\left(z_1\overline{z_2}\right) + |z_2|^2 \\
&\leq |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

hence,

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

in conclusion;

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

is hold. □

And the other usefull identity is obtained by means of the trianle inequality. Taking any point $z \in \mathbb{C}$, it can be written by

$$z = z - w + w$$

if taking modulus both sides, and apply the triangle inequality

$$\begin{aligned} |z| &= |z - w + w| \\ &\leq |z - w| + |z| \end{aligned}$$

subtructing $|w|$ from both sides of inequality then,

$$|z| - |w| \leq |z - w|, \tag{2.6}$$

with the same idea, if z and w change their place,

$$|w| - |z| \leq |w - z| \tag{2.7}$$

since,

$$|w - z| = |-(z - w)| = |z - w|$$

then (2.7),

$$|z| - |w| \geq -|z - w| \tag{2.8}$$

and from (2.6) and (2.8),

$$||z| - |w|| \leq |z - w|.$$

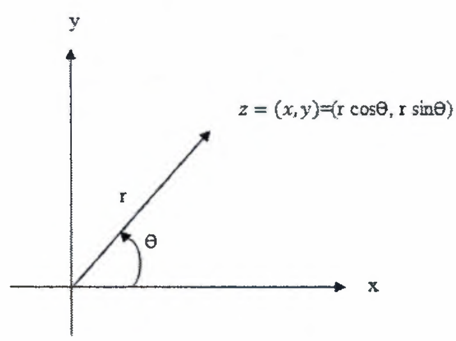
2.1 Polar Representation

Any point $z = (x, y)$ in the complex plane can be described by polar coordinates r and θ, where r is the modulus of z and θ is the angle between the vector from orgin to the point z and x-axis And the cartesian coordinates x and y can be recovered from the polar coordinates r, θ by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then the number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta).$$



Definition 2.4 *(Argument)*

The argument of z is defined by angle θ and it is written

$$\arg z = \theta.$$

Thus $\arg z$ is multivalud function, defined for $z \neq 0$.

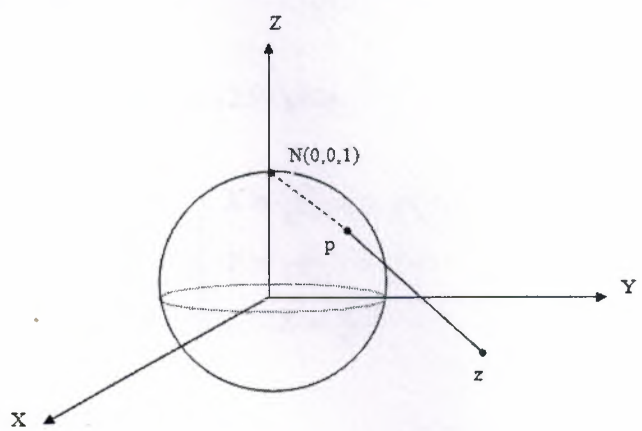
The principal vale of $\arg z$, denoted by $Argz$ is that unique θ such that $-\pi < \theta \leq \pi$. The vales of $\arg z$ are obtained from $Argz$ by adding integral multiples of 2π, such that

$$\arg z = \{Argz + 2\pi n : n = 0, \mp 1, \pm 2, \pm 3, ...\}.$$

2.2 Stereographic Projection

In order to understand the relationship among the hyperbolic models it will be used stereographic projection. In this section,it will be developed some important properties of stereographic projection.

To construct the streographic projection, let denote S with the unit sphere $x^2+y^2+z^2 = 1$ in $\mathbb{R}^3$ and let $N = (0, 0, 1)$ denote the "*North Pole*" of S.Take a point $P(X, Y, Z) \in S$, other then N, then the line connecting N and P intersects the XY- plane (which is identify the complex plane $\mathbb{C}$) at a point $z(x, y, 0)$. (as it can see following figure)



The line which is pass through N, P and z can be considered with,

$$P - z = t(N - z), \text{ where } t \in \mathbb{R},$$

and so there exist,

$$(X, Y, Z) = t(0, 0, 1) + (1 - t)(x, y, 0)$$

then,

$$\begin{cases} X = (1-t)\,x \\ Y = (1-t)\,y \\ Z = t \end{cases} . \tag{2.9}$$

Since (X, Y, Z) on the unit sphere S, there fore,

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$$

so that,

$$(1-t)^2 |z|^2 = 1 - t^2$$

from the assumtion, P is diferent point from the $N(0, 0, 1)$,so $t \neq 1$, then,

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

using this and (2.4) , the equation (2.9) yiels,

$$\begin{cases} X = \frac{2}{|z|^2+1}x = \frac{z+\bar{z}}{|z|^2+1} \\ Y = \frac{2}{|z|^2+1}y = \frac{z-\bar{z}}{|z|^2+1} \\ Z = \frac{|z|^2-1}{|z|^2+1} \end{cases} \tag{2.10}$$

conversly, it is easily obtained that,

$$\begin{cases} x = \frac{X}{1-Z} \\ y = \frac{Y}{1-Z} \end{cases} . \tag{2.11}$$

Now, it can be given the defination of streographic projection.

(Stereographic projection)

Let S denote the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ and let $N = (0, 0, 1)$ denote the "*North Pole*" of S. Given a point $P \in S$, other then N, then the line connecting N and P intersects the XY- plane (which is considered as the z- plane) at a point z. The *stereographic projection* is the map

$$\pi : \mathbb{C} \to S - \{N\} : z \to P$$

conversely,

$$\pi^{-1} : S - \{N\} \to \mathbb{C} : P \to z.$$

Note that, under stereographic projection, points on the unit circle $|z| = 1$ (in the z-plane) remain fixed (that is z=Z), forming the equator. Point outside the unit circle $|z| > 1$ project to points in the northern hemisphere, while those inside the unit circle $|z| < 1$ project to southern hemisphere. In particular the orgin of the z-plane projects to the south pole of the Reimann Sphere.

Theorem 2.5 *Suppose* $T \subset \mathbb{C}_\infty$. *Then the corresponding image of* T *on the Rimann sphere* S *is*

(a) a circle in S *not containing image* $(0, 0, 1)$, *if* T *is a circle;*

(b) a circle in S *passes through* $(0, 0, 1)$ *if* T *is a line.*

Proof. First start the proof with considering the general equation of a circle in the plane,

$$T = \left\{(x, y) : A\left(x^2 + y^2\right) + Bx + Cy + D = 0\right\} \tag{2.12}$$

the image of T under stereographic projection with using equation (2.11),

$$A\frac{1+Z}{1-Z} + B\frac{X}{1-Z} + C\frac{Y}{1-Z} + D = 0$$

then,

$$(A - D)Z + BX + XY + (A + D) = 0 \tag{2.13}$$

which is the equation of a plane in the space. Since the intersection of any plane and sphere is a circle. And the point $N = (0, 0, 1)$ is not satisfiy the equation (2.13).

Thus, the image of any circle in $\mathbb{C}_\infty$ under streographic projection is a circle not contain the point $(0,0,1)$.

Assume that, $A = 0$ in the equation (2.12) then T will be a line $\mathbb{C}_\infty$. And the image of line will be

$$BX + XY - DZ + D = 0$$

the intersection of this equation with sphere S is a circle wihich is passes through the $N = (0,0,1)$.

Hence, if T is a line then the image of T is a circle in S passes through $(0,0,1)$. □

The converse of above theorem takes the above form;

Theorem 2.6 *If T_s is acircle on the Riemann sphere S and T is its stereographic projection on $\mathbb{C}_{\infty,}$ then*

(a) T is a circle if $(0,0,1) \notin T_s$

(b) T is a line if $(0,0,1) \in T_s$.

Proof. If T_s is acircle on the sphere S, then T_s can be defined by the intersection of a plane with sphere S,

$$T_s = \{AX + BY + CZ + D = 0\} \cap \left\{AX^2 + BY^2 + CZ^2 = 1\right\}. \tag{2.14}$$

And according the (2.10),

$$A\frac{2x}{|z|^2+1} + B\frac{2y}{|z|^2+1} + C\frac{|z|^2-1}{|z|^2+1} + D = 0$$

if rewrite the above equation,

$$(C + D)\left(x^2 + y^2\right) + 2Ax + 2By - C + D = 0 \tag{2.15}$$

is obtained.

Now, from (2.14), T_s passes through $(0,0,1)$ if $C + D = 0$. Then the eqn (2.15) represents the line when $C + D = 0$, and if $C + D \neq 0$, its represent the equation of circle on

$\mathbb{C}_\infty$.

In conclution, T is a circle if $(0,0,1) \notin T_s$ and it is a line if $(0,0,1) \in T_s$. □

The length of the segment joining $Z = (X, Y, Z)$ and $W = (U, V, W)$, known as chordal distance of z from w, is defined as $\chi(z, w) = d(Z, W)$. therefore,

$$\chi(z, w) = \sqrt{(X-U)^2 + (Y-V)^2 + (Z-W)^2}$$

since, $X + Y + Z = 1$ and $U + V + W = 1$, then

$$\chi(z, w) = \sqrt{2 - 2\,(XU + YV + ZW)}$$

according to (2.10) , it is botained,

$$\chi(z, w) = \frac{2\,|z-w|}{\sqrt{1+|z|^2}\,\sqrt{1+|w|^2}}.$$

In particular, χ is defined with chordal metric that is satisfies the properties of a metric,

a) $\chi(z_1, z_2) \geq 0$

b) $\chi(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$

c) $\chi(z_1, z_2) = \chi(z_2, z_1)$

d) $\chi(z_1, z_3) \leq \chi(z_1, z_2) + \chi(z_2, z_3)$

CHAPTER 3

CONFORMAL MAPPING AND MOBIUS TRANSFORMATION

The aim of this chapter, given the definition of conforml mappings, and defined the Möbius transformations which are analytic conformal mapping. And the end of this chapter there will be given important transformations which transforms D to $D, \mathbb{H}$ to $\mathbb{H}$ and D to $\mathbb{H}$ and the following next chapter they help to construct the isometries of hyperboic geometry.

3.1 Conformal Mapping

Before given the theorem of conformality, it is needed to given the some definitions.

Definition 3.1 *(Continuity of a function)*

Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some neighborhood of z_0. $f(z)$ is continious at z_0 if the following three conditions are satisfied:

$\lim_{z \to z_0} f(z)$ exists;

$f(z_0)$ exists;

$\lim_{z \to z_0} f(z) = f(z_0)$.

A continuos function is a function that is continuous at each point of its domain.

(Analytic function)

A function $f(z)$ is analytic on the open set U if $f(z)$ is (complex) differentiable at each point of U and complex derivative $f'(z)$ is continuous on U.

Definition 3.2 *(homeomorphism)*

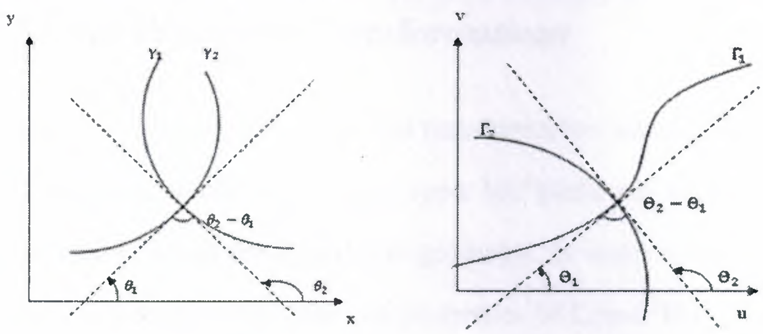
A function $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is a homeomorphism if f is a bijection and if both f and f^{-1} are continuous.

Definition 3.3 *(Automorphism)*

The set of all conformal bijections $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are defined by $Aut(\overline{\mathbb{C}})$.

Theorem 3.4 *Suppose $f(z)$ is analytic at $z_0$with $f'(z_0) \neq 0$. Let $\gamma_1 : \gamma_1(t)$ and $\gamma_2 : \gamma_2(t)$ be smooth curves in the z- plane that intersect at $z_0 =: \gamma_1\ (t_0) =: \gamma_2\ (t_0)$, with $\Gamma_1 : w_1\,(t)$ and $\Gamma_2 : w_2\,(t)$ the images of γ_1 and γ_2 respectively. Then the angle between γ_1 and γ_2 measured from γ_1 to γ_2 equal to the angle between Γ_1 and Γ_2 measured from Γ_1 to Γ_2.*

Proof. *consider that, the tangents of γ_1 and γ_2 makes angle with θ_1 and θ_2 and the argument of the tangent vectors Γ_1 and Γ_2 are Θ_1 and Θ_2 respectively and so as it can be seen below figure the angle between $\gamma_1$and γ_2 which is the angle between their tangent curves is $\theta_2 - \theta_1$, and the angle between the image curves Γ_1 and Γ_2 is $\Theta_1 - \Theta_2$.*



Firstly, For any point z_1 on the curve γ_1 other than z_0,

$$w_1 - w_0 = \frac{f(z_1) - f(z_0)}{z_1 - z_0}(z_1 - z_0).$$

Thus,

$$\arg(w_1 - w_0) = \arg\left(\frac{f(z_1) - f(z_0)}{z_1 - z_0}\right) + \arg(z_1 - z_0). \tag{3.1}$$

Note that, when z_0 approaches z_0 along the curve γ_1, $\arg(z_1 - z_0)$ approaches a value θ_1, likewise, $\arg(w_1 - w_0)$ is approaches a value Θ_1. And sice $f(z)$ is analytic at z_0, $f'(z_0) \neq 0$, therefore $f'(z_0)$ has meaning.

Now, if consider the limit of both sides of equation (3.1),

$$\Theta_1 = \arg f'(z_1) + \theta_1$$

is obtained. With the same way if take any point on γ_2, then it can be easily obtained that,

$$\Theta_2 = \arg f'(z_1) + \theta_2.$$

Then, the angle between Γ_1 and Γ_2 which is the angle between their tangent lines is,

$$\Theta_1 - \Theta_2 = \left(\arg\left(f'(z_0)\right) + \theta_2\right) - \left(\arg\left(f'(z_0)\right) + \theta_1\right) = \theta_2 - \theta_1$$

Consequently, the angle between γ_1 and γ_2 is equal to the angle between Γ_1 and Γ_2. □

If f is analytic at all points in complaex plane then the theorem (3.4) provides at all points. It gives that, if f is analytic function than it is *conformal.*

3.2 Linear Fractional Transformations

The aim of this section, give a special transformation like,maps the upper half plane on to the unit disk, upper half plane to upper half plane and unit disc to unit disc which they give isometries of the hyperbolic geometry. In order to define this special map, it will be given some definitions and properties of Linear Fractinal Transformations. Moreover, threre will be given cross ratio which is used later to define the formula for distance in the Hyperbolic geometry.

Definition 3.5 *(Linear Fractional Transformation)*

A linear fractinal transformation is a function of the form

$$w = f(z) = \frac{az + b}{cz + d}, \tag{3.2}$$

where a, b, c and d are complex constants satisfying $ad - bc \neq 0$.

Linear fractional transformations ara also called **Möbius transformations.**

If $c = 0$ then the transformatiom $f(z)$ given by (3.2) reduces to

$$f(z) = \frac{az}{d} + \frac{b}{d}$$

and it can be written

$$f(z) = Az + B,$$

where $A = \frac{a}{d}$ and $B = \frac{b}{d}$, and $ad \neq 0$, i.e. $A \neq 0$. A function of this form is called linear transformation.

It is obvious that Möbius transformations are analytic on $\mathbb{C} \setminus \{-d/c\}$. Consider the derivative,

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \qquad (z \neq -d/c),$$

since, $ad - bc \neq 0$ so that $f'(z) \neq 0$, it means that $f(z)$ is not constant.

There are special types of transformations which Möbius transformations can be written the composition of these transformations:

(i) $f(z) = z + A$, where $A \in \mathbb{C}$. *(translation),*

(ii) $f(z) = rz$, wherer $r \in \mathbb{R} \setminus \{0\}$, *(magnification),*

(iii) $f(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$ *(rotations),*

(iv) $f(z) = \frac{1}{z}$, *(inverse).*

Then, if $c \neq 0$, then the transformation (3.2) can be decomposed as,

$$\begin{aligned} f(z) &= \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz + d} \\ &= \frac{a}{c} - \left(\frac{ad - bc}{c^2}\right)\frac{1}{z + \frac{d}{c}} \end{aligned} \tag{3.3}$$

so, the Möbius transformations can be obtaind with the compsitions of the following transformations;

$f_1(z) = w_1 = z + \frac{d}{c}$ *(translatiun),*

$f_2(z) = w_2 = \frac{1}{w_1} = \frac{1}{z+\frac{d}{c}}$ *(inversion),*

$f_3(z) = w_3 = \left(-\frac{ad-bc}{c^2}\right)w_2$ (*magnification* and *rotation*),

$f_4(z) = w_4 = \frac{a}{c} + w_3$ *(translation),*

hence, $f(z) = f_1(z) \circ f_2(z) \circ f_3(z) \circ f_4(z)$.

Proposition 3.6 *Composition of two Möbius transformation is a Möbius transformation.*

Proof. *Let* $f = \frac{az+b}{cz+d}, (ad - b \neq 0)$ *and* $g = \frac{a'z+b'}{c'z+d'}, \left(a'd' - b'c' \neq 0\right)$ *are Möbius transformations. The composite function of* f *and* g *is defined by*

$$\begin{aligned}
(f \circ g)(z) &= \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} \\
&= \frac{\left(aa' + bc'\right)z + \left(ab' + bd'\right)}{(ca' + dc')z + (cb' + dd')} \\
&= \frac{Az + B}{Cz + D} \qquad (3.4)
\end{aligned}$$

where,

$$A = \left(aa' + bc'\right),\ B = \left(ab' + bd'\right),\ C = \left(ca' + dc'\right), D = \left(cb' + dd'\right). \qquad (3.5)$$

In order to say that equation(3.4) *is a Möbius transformation it must be shown*

$$AD - BC \neq 0$$

if all equations (3.5) *are put in equation*

$$\begin{aligned}
AD - BC &= \left(aa' + bc'\right).\left(cb' + dd'\right) - \left(ab' + bd'\right).\left(ca' + dc'\right) \\
&= \left(aa'cb' + aa'dd' + bc'cb' + bc'dd'\right) - \left(ab'ca' + ab'dc' + bd'ca' + bd'dc'\right) \\
&= \left(aa'dd' + bc'cb'\right) - \left(ab'dc' + bd'ca'\right) \\
&= a'd'(ad - bc) - b'c'(ad - bc) \\
&= (ad - bc)\left(a'd' - b'c'\right)
\end{aligned}$$

since $ad - b \neq 0$ *and* $a'd' - b'c' \neq 0$. *Hence,*

$$AD - BC \neq 0$$

In conclusion,

Composition of two Möbius transformation $(f \circ g)$ *is a Möbius transformation.* □

Theorem 3.7 *Möbius transformations are one- to- one mapping.*

Proof. Let f be a Möbius transformation with $f = \frac{az+b}{cz+d}, (ad - bc \neq 0)$. In order to say that f is one- to one, it must be shown thet, $z_1 = z_2$ when $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{C} \setminus \{-d/c\}$.

Since,

$$f(z_1) = f(z_2)$$

then,

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d}$$

$$\Longrightarrow \quad (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$\Longrightarrow \quad acz_1z_2 + adz_1 + bcz_2 + bd = acz_1z_2 + adz_2 + bcz_1 + bd$$

$$\Longrightarrow \quad (ad - bc) z_1 = (ad - bc) z_2$$

since, $ad - bc \neq 0$

$$z_1 = z_2.$$

□

As f is one to one, its inverse always exists. Inverse of Möbius transformation is obtained by solving the equation,

$$\begin{aligned} w &= f(z) = \frac{az + b}{cz + d} \\ &\Longrightarrow w(cz + d) = az + b \\ &\Longrightarrow z(cw - a) = b - dw \\ &\Longrightarrow z = f^{-1}(w) = \frac{dw - b}{-cw + a} \end{aligned}$$

where, $ad - (-b)(-c) = ad - bc \neq 0$, where $w \neq a/c$. It can be given the following proposition.

Proposition 3.8 *The inverse of a Möbius transformation is also Möbius transformation.*

It can be easily checked that,

$$(f \circ I)(z) = (I \circ f)(z) = f(z),$$

where $I(z) = z$ is a identity function. So, the set of all Möbius transformations is a group with respect to the composition.

Since, Möbius transformations defind on the complex plane except the points $z = -d/c$ and ∞. But it can be enlarge the definition of Möbius transformation $f(z)$ to the whole extended complex plane by including these points. Indeed as,

$$\lim_{z \to -d/c} \frac{1}{f(z)} = \lim_{z \to -d/c} \frac{cz+d}{az+b} = \frac{0}{a\frac{-d}{c}+b} = 0$$

then,

$$\lim_{z \to -d/c} f(z) = \infty.$$

Morever,

$$\lim_{z \to \infty} f(z) = \lim_{z \to 0} \frac{\frac{a}{z}+b}{\frac{c}{z}+d} = \lim_{z \to 0} \frac{a+bz}{c+dz} = \frac{a}{c}.$$

Then, it can be define, for $c \neq 0$,

$$f(z) = \begin{Bmatrix} \frac{az+b}{cz+d} \text{ if } z \neq -d/c,\ z \neq \infty \\ \infty \quad \text{if } z = -d/c \\ \frac{a}{c} \quad \text{if } z = \infty \end{Bmatrix}$$

And similarly, defined for inverse Möbius transformations;

$$f^{-1}(z) = \begin{cases} \frac{dw-b}{-cw+a}, & \text{if } z \neq a/c,\ z \neq \infty \\ \infty & \text{if } z = a/c \\ -\frac{d}{c} & \text{if } z = \infty \end{cases}.$$

It shows that both f and f^{-1} are onto surjection functions.

And as it mention before Möbius transformations are group with respect to composition in the whole extended complex plane $\overline{\mathbb{C}}$. And the Möbius groups will be shown with $M\ddot{o}b(\overline{\mathbb{C}})$.

Theorem 3.9 *Möbius maps are conformal(or angle preserving).*

Proof. Since all analytic functions are conformal.And as it was mention, all Möbius transformations are analytic functions, and so all Möbius maps are conformal. □

Theorem 3.10 *Möbius maps are Automorphism.*

Proof. Since all Möbius transformations are bijection and from previous theorem, they are conformal. Then, it can be say that they Möbius maps are Automorphism. □

3.3 Matrix Representation

The coefficients of a Möbius map can be represented by the matrix as a,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

for any Möbius transformation $M = \frac{az+b}{cz+d}$, wherer $GL(2, \mathbb{C})$ is denoted by the general linear group such that

$$GL(2, \mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \det A \neq 0 \right\}.$$

The compose of Möbius maps (as it can be seen proposition (3.6)) can be obtained by multiplying the matrices.

Proposition 3.11 *The map $F : GL(2, \mathbb{C}) \to Aut(\overline{\mathbb{C}})$ defined by*

$$F : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \left(f(z) = \frac{az+b}{cz+d} \right)$$

is a homomorphism

Proof. Consider that, two matrices with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d} \text{ and } \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \frac{a'z+b'}{c'z+d'}$$

then, the multiplication of two matrices is,

$$\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \to \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + dd')},$$

by popositon(3.6), the right side is equivalent to the composition of two Möbius map. And its shows that the homoemorphism of F. □

Corollary 3.12 *Any* $T \in Aut(\overline{\mathbb{C}})$ *can be represented by a matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

where $SL(2, \mathbb{C})$ *is defined by for any matrix* A,

$$SL(2, \mathbb{C}) = \{A \in GL(2, \mathbb{C}) : \det A = 1\}.$$

Proof. By previous proposition, T can be represented by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

with $\det A = ad - bc$.

Since the metrices $\frac{1}{\det A}A$ and A have same image under F, and the determinant of $\frac{1}{\det A}A$ is equal 1.

Hence, $A \in SL(2, \mathbb{C})$. □

3.4 Cross- Ratio

Theorem 3.13 *Given three distance points z_1, z_2, and z_3 in the extended z- plane amd three distance points w_1, w_2, and w_3 in the extendend w- plane, there exist a unique bilinear transformation $w = T(z) \in Aut(\overline{\mathbb{C}})$ such that $T(z_k) = w_k$, for $k = 1, 2, 3$.*

Proof. First assume that non of the six points is ∞. Let

$$w = T(z) = \frac{az+b}{cz+d},$$

for $k = 1, 2, 3$, it can be written,

$$w_k = \frac{az_k+b}{cz_k+d}$$

then,

$$w - w_k = \frac{az+b}{cz+d} - \frac{az_k+b}{cz_k+d}$$

with making the elementary operations, above equation can be written,

$$w - w_k = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)} \tag{3.6}$$

for $k = 1$ and $k = 3$, it can be obtained,

$$w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)} \tag{3.7}$$

and

$$w - w_3 = \frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}, \tag{3.8}$$

dividing (3.7)by (3.8),

$$\frac{w-w_1}{w-w_3} = \frac{(cz_3+d)(z-z_1)}{(cz_1+d)(z-z_3)}. \tag{3.9}$$

in above equation, if put z_2 instead of z and w_2 instead of w, then,

$$\frac{w_2 - w_1}{w_2 - w_3} = \frac{(cz_3 + d)}{(cz_1 + d)}\frac{(z_2 - z_1)}{(z_2 - z_3)}. \tag{3.10}$$

multiplying (3.9) by (3.10) , it will be obtained,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \tag{3.11}$$

If one of the points were ∞, assume $z_3 = \infty$, by taking the limit of (3.11) as z_3 approached ∞,

$$\begin{aligned}\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} &= \lim_{z_3\to\infty} \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \\ &= \frac{(z - z_1)}{(z_2 - z_1)}.\end{aligned}$$

To show that the uniqueness of $T(z_k)$, assume its not. And let $S(z_k)$ and $T(z_k)$ are both two nonlinear transformations that

$$w_k = S(z_k) = T(z_k), \; for\; k = 1, 2, 3.$$

then,

$$\left(S^{-1} \circ T\right)(z_k) = S^{-1}(T(z_k)) = S^{-1}(w_k) = z_k$$

and so

$$S^{-1} \circ T = I$$

then, obtained

$$S = T$$

which proves the uniqueness part of the theorem. □

Corollary 3.14 *Given three distance points z_1,z_2,and z_3 in the extended z-plane there exists a unique bilinear transformation $w = T(z)$ such that $T(z_1) = 0$, $T(z_2) =$*

1, $T(z_3) = \infty$. *And it is given by*

$$w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Proof. From the previous theorem, it is known that for any z_1, z_2, and z_3, there exist a unique transformation,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

if put 0,1 and ∞ instead of w_1, w_2, w_3 respectively,

$$w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)},$$

the proof is completed. □

Definition 3.15 *(Cross- Ratio)*

Let, z_1, z_2, z_3, z be distinct points in $\overline{\mathbb{C}}$, the *cross- ratio* of these points is defined by

$$[z_1, z_2; z_3, z] = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

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Theorem 3.16 *Let A be either a circle or a line in* $\mathbb{C}$. *Then A has the equation*

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0,$$

where $\alpha, \gamma \in \mathbb{R}$ *and* $\beta \in \mathbb{C}$.

Proof. Since the equation,

$$a(x^2 + y^2) + bx + cy + d = 0, \text{ where } a, b, c, d \in \mathbb{R}, \tag{3.12}$$

is an equation of circle when $a \neq 0$, or an equation of a line when $a = 0$. Recalling that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and $x^2 + y^2 = |z|^2$, substituting these expression into (3.12) ,then,

$$a|z|^2 + b\frac{z+\bar{z}}{2} + c\frac{z-\bar{z}}{2i} + d = 0$$

and recombine gives,

$$a|z|^2 + \frac{b-ci}{2}z + \frac{b+ci}{2}\bar{z} + d = 0$$

consider, $a = \alpha, d = \gamma \in \mathbb{R}$ and $\frac{b-c\,i}{2} = \beta \in \mathbb{C}$. Then if $a = \alpha$ equal to zero, then above equation gives a line, unless it gives a circle in $\mathbb{C}$. Hence any circle or line in the complex plane can be written as a form,

$$\alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \gamma = 0.$$

□

Proposition 3.17 *Let A be either a circle or a line in $\mathbb{C}$ with satisfying the equation $\alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \gamma = 0$. Suppose $\beta \in \mathbb{R}$. Then A is either circle with centre on real axis or a vertical straight line.*

Proof. From the proof of the previous theorem, $\frac{b-c\,i}{2} = \beta$ if $\beta \in \mathbb{C}$. And assume $a \neq 0$ in the equation (3.12) , then it can be recomposed as a

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{c}{2a}\right)^2 = \frac{b^2 + c^2 - 4ad}{4a^2},$$

then the center of the circle is,

$$\left(-\frac{b}{2a}, -\frac{c}{2a}\right)$$

since $\frac{b}{2} = Re\beta$ and $\frac{-c}{2} = Im\beta$, and so the center can be expressed,

$$\left(\frac{-Re\beta}{a}, \frac{Im\beta}{a}\right).$$

Hence, if $\beta \in \mathbb{R}$, then $Im\beta = 0$. Therefore if $\beta \in \mathbb{R}$, the center of the equation $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, is on the real axis.

Now, assume $a = 0$, then the equation (3.12) will be,

$$bx + cy + d = 0,$$

and the slope of above line is $\frac{-b}{c}$,and it can be expressed as a $\frac{Re\beta}{Im\beta}$. If $\beta \in \mathbb{R}$, $Im\beta = 0$ where it gives the vertical line. □

Lemma 3.18 *Every Möbius transformation maps circles and lines into circles and lines in* $\overline{\mathbb{C}}$.

Proof. From theorem (3.16), the equation of a circle or a line can be regarded as,

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0. \tag{3.13}$$

Let,

$$w = \frac{az+b}{cz+d},$$

then

$$z = \frac{dw-b}{-cw+a}.$$

Substituting this into (3.13),

$$\alpha\left(\frac{dw-b}{-cw+a}\right)\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right)+\beta\left(\frac{dw-b}{-cw+a}\right)+\bar{\beta}\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right)+\gamma=0,$$

it can be recomposed as a,

$$\left(\alpha d^2-\beta cd-\bar{\beta}cd+\gamma c^2\right)w\bar{w}+\left(-\alpha bd+\beta ad+\bar{\beta}bc-\gamma ac\right)w+\left(-\alpha bd+\beta bc+\bar{\beta}ad-\gamma ac\right)\bar{w}+\left(\alpha b^2-\beta al\right. \tag{3.14}$$

Since, a, b, c, d,α and γ are real number,β is a complex number, and it is known that $2Re\beta = \beta + \overline{\beta}$. And so, if it is considered

$$\alpha d^2 - 2cdRe\beta + \gamma c^2 = \alpha_1,$$

$$-\alpha bd + \beta ad + \overline{\beta} bc - \gamma ac = \beta_1, \text{ whenever } -\alpha bd + \beta bc + \overline{\beta} ad - \gamma ac = \overline{\beta_1}$$

and

$$\alpha b^2 - \beta ab - \overline{\beta} ab + \gamma c^2 = \gamma_1$$

then the equation (3.14) can be regarded as,

$$\alpha_1 w\overline{w} + \beta_1 w + \overline{\beta_1}\overline{w} + \gamma_1 = 0$$

with $\alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}$. Clearly, the last equation represent the equation of a circle when $\alpha_1 \neq 0$, and a line when $\alpha_1 = 0$. This complete the proof. □

3.5 Automurphism of the Unit Disc And Upper Half Plane

In this section, it will be introduce some particular Möbius maps which transform upper half plane to the unit disc, upper half plane to itself and unit disk to unit disk. And it will be start in this section with Cayley transformation which is transform upper half plane to the unit circle it will be introduce the Möbius map;

$$C : z \longrightarrow \frac{z - i}{z + i} = w$$

which is called *Cayley transformation.*

Lemma 3.19 *Cayley transformation induces a conformal automorphism from $\mathbb{H}$ to D*

Proof. Take any three poins $0, 1$ and ∞ on the $\mathbb{R} \cup \{\infty\}$, from the theorem (3.13), there exist a unique transformation which is transform the thre points to another three poins on the circle.with $C(0) = -1$, $C(\infty) = 1$ and $C(1) = -i$, which is give the cyley transformation

$$w = \frac{z-i}{z+i}.$$

So the cyley transformation transform the boundary of $\mathbb{H}$ onto boundary of unit disk. Let take any point in the upper half plane it can be easily seen that the cyley transformation transforms the point in the unit disk. Therefore it is an Möbius map from $\mathbb{H}$ to D. □

3.5.1 Automorphism of the unit disc D

Automorphism of unit disk is defind by

$$Aut(D) = \left\{T \in Aut(\overline{\mathbb{C}}) : T(D) = D\right\}.$$

Theorem 3.20 *The set $Aut(D)$ is the subgroup of $Aut(\overline{\mathbb{C}})$ of Möbius maps of the form*

$$T(z) = \frac{az+\overline{c}}{cz+\overline{a}}, \text{ with } |a|^2 + |b|^2 = 1$$

Proof. Since the unit circle is defined by $|z|^2 = 1$, and from lemma (3.18), every Möbius transformation maps circles into circles in $\overline{\mathbb{C}}$. Take any Möbius transformation

$$T(z) = w = \frac{az+b}{cz+d} \text{ with } ad - bc = 1$$

then ,

$$z = \frac{dw-b}{-cw+a} \text{ and } \overline{z} = \frac{\overline{dw}-\overline{b}}{-\overline{cw}+\overline{a}}$$

is obtained. Put them into unit circle,

$$\left(\frac{dw-b}{-cw+a}\right)\left(\frac{\overline{dw}-\overline{b}}{-\overline{cw}+\overline{a}}\right) = 1$$

$$|d|^2|w|^2 - \bar{b}dw - b\bar{d}\bar{w} + |b|^2 = |c|^2|w|^2 - c\bar{a}w - a\bar{c}\bar{w} + |a|^2$$

$$\left(|d|^2 - |c|^2\right)|w|^2 + \left(-\bar{b}d + c\bar{a}\right)w + \left(-b\bar{d} + a\bar{c}\right)\bar{w} + |b|^2 - |a|^2 = 0$$

is obtained. In order to obtained unit circle

$$-\bar{b}d + c\bar{a} \tag{3.15}$$

and

$$-b\bar{d} + a\bar{c} \tag{3.16}$$

must be equal to zero and

$$\frac{|a|^2 - |b|^2}{|d|^2 - |c|^2} = 1$$

and so

$$|a|^2 - |b|^2 = |d|^2 - |c|^2 \neq 0 \tag{3.17}$$

From equation (3.16)

$$-b\bar{d} + a\bar{c} = 0 \Rightarrow a\bar{c} = b\bar{d}$$

from here, let

$$\frac{a}{\bar{d}} = \frac{b}{\bar{c}} = \lambda$$

then,

$$a = \lambda\bar{d} \text{ and } b = \lambda\bar{c} \tag{3.18}$$

put them on equation (3.17)

$$|d|^2 - |c|^2 = |a|^2 - |b|^2 = |\lambda|^2\left(|d|^2 - |c|^2\right)$$

$$\Rightarrow |d|^2 - |c|^2 = |\lambda|^2\left(|d|^2 - |c|^2\right)$$

and from $ad - bc = 1$,

$$\lambda\left(d\bar{d} - c\bar{c}\right) = 1.$$

because of $d\bar{d} = |d|^2$ and $c\bar{c} = |c|^2$ are real, λ must be real, therefore $\lambda = \pm 1$, the sign of λ is depend on $|d|^2 - |c|^2$.

The transform of inside of the unit circle is inside of the unit circle nad as it mentiond before the point $\frac{-d}{c}$ is tranform ∞ under the Möbius transformation so that point must be keep out the inside of the unit circle. Therefore

$$\left|\frac{-d}{c}\right| > 1 \Rightarrow |d| > |c|$$

than,

$$|d| - |c| > 0.$$

It is implise that λ must be equal to 1. If put 1 instead of λ in the equation (3.18), then,

$$a = \bar{d},\ b = \bar{c}$$

and so,

$$\bar{a} = d,\ \bar{b} = c$$

are obtained. Finally, the Möbius transformation $T(z)$ will be equal,

$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \text{ with } |a|^2 - |c|^2 = 1.$$

In conclusion,

the Möbius transformation which is translate the unit circle to unit circle is given by

$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \text{ with } |a|^2 - |c|^2 = 1.$$

□

3.5.2 *Automorphism of The upper half plane* $\mathbb{H}$

Before given the Automprphism of the upper half plae, it will be defined Automorphism of upper half plane,

$$Aut(\mathbb{H}) = \left\{T \in Aut(\overline{\mathbb{C}}) : T(\mathbb{H}) =, \mathbb{H}\right\}.$$

Theorem 3.21 *The set $Aut(\mathbb{H})$ of analytic bijections $\mathbb{H} \to \mathbb{H}$ is the subgroup of $Aut(\overline{\mathbb{C}})$ of Möbius maps of the form;*

$$f(z) = \frac{az - b}{cz - d}, \text{ with } a, b, c, d \in \mathbb{R}, \ ad - bc > 0. \tag{3.19}$$

Proof. It is needed to show that the Möbius transformation which transform the upper half plane to apper half plane is in the form of (3.19). So the $Imf(z)$ under the transformation must be equal to zero whenever $Imz > 0$. Take a Möbius transformation,

$$f(z) = \frac{az + b}{cz + d},$$

first it must be computed $Imf(z)$, since $Imz = \frac{1}{2i}(z - \bar{z})$ and if apply this for $f(z)$, then,

$$\begin{aligned} Imf(z) &= \frac{1}{2i}\left(\frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d}\right) \\ &= \frac{1}{2i}\frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{(cz + d)(c\bar{z} + d)} \end{aligned}$$

after making some elimination, it will be obtained,

$$Imf(z) = \frac{1}{2i}\frac{ad - bc}{|cz + d|^2}(z - \bar{z})$$

which is equivalent,

$$Imf(z) = \frac{ad - bc}{|cz + d|^2} Imz.$$

Now, Since Imz and $|cz + d|^2$ greater then zero and so for making, $Imf(z) > 0$, $ad - bc$ must be greater then zero. And this completes the proof. □

To move from D to $\mathbb{H}$, it will be intoduce the Möbius map;

$$C : z \to \frac{z - i}{z + i} = w$$

which is called *Cayley transformation.*

Lemma 3.22 *Cayley transformation induces a conformal automorphism from $\mathbb{H}$ to D*

Proof. Take any three poins $0, 1$ and ∞ on the $\mathbb{R} \cup \{\infty\}$, from the theorem (3.13), there exist a unique transformation which is transform the three points to another three points on the circle.with $C(0) = -1$, $C(\infty) = 1$ and $C(1) = -i$, which is give the cyley transformation

$$w = \frac{z - i}{z + i}.$$

So the cyley transformation transform the boundary of $\mathbb{H}$ onto boundary of unit disk. Let take any point in the upper half plane it can be easily seen that the cyley transformation transforms the point in the unit disk. Therefore it is an Möbius map from $\mathbb{H}$ to D. □

CHAPTER 4

HYPERBOLIC GEOMETRY

4.1 Upper Half Plane Of Hyperbolic Geometry

In this section a review of background knowlage on upper half plane of hyperbolic will be presented. In the framework of upper half plane method, length and distance in hyperboic geometry will be presented. Then aspects, definitions will be listed. Morever some formulas will be illustrated. Follwing this namely path integrals will be highlighted. Finally relevant aspects of the upper half plane will be mention with the empirical studies contacted on this area.

Definition 4.1 *(upper half plane model)*

The upper hyperbolic plane $\mathbb{H}$ model is the upper half plane in the complex plane and its defined by,

$$\mathbb{H} = \{z \in \mathbb{C} : Imz > 0\}$$

with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Definition 4.2 *(boundary of* $\mathbb{H}$*)*

The boundary of $\mathbb{H}$ is defined to be the set $\partial H = \{z \in \mathbb{C} : Im(z) = 0\} \cup \{\infty\}$. That is ∂H is the real axis together with the point ∞.

Definition 4.3 *(Hyperbolic Line)*

There are two seeningly types of Hyperbolic line,both defined in terms of Euclidean objects in $\mathbb{C}$. One is the intersection of $\mathbb{H}$ with an Euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$. The other is the intersection of $\mathbb{H}$ with an Euclidean circle centred on the real axis in $\mathbb{R}$.

Proposition 4.4 *For each pair p and q of distinct points in $\mathbb{H}$, there exist a unique hyperbolic line ℓ in $\mathbb{H}$ passing through p and q.*

Proof. Firstly, it will be shown that the existence of the hyperbolic line passing through p and q. There are two cases to consider the line;

First, assume $Rep = Req$, it means, if p and q are defined by $p = p_1 + ip_2$ and $q = q_1 + iq$, then $p_1 = q_1$.From Euclidean geometry, it is known that, there is a line passes through p and q with equation,

$$y - p_2 = \frac{q_2 - p_2}{q_1 - p_1}(x - p_1), \tag{4.1}$$

and from the assumption $p_1 = q_1$, the line which passes through p and q is perpendicular to the $\mathbb{R}$ axis in the Euclidean plane. According to definition (4.3), the intersection of $\mathbb{H}$ with line ℓ is a hyperbolic line which passing through p and q.

Now, assume $Rep \neq Req$, again the line joining p and q is defined with equation (4.1). Let perpendicular bisector of this line is K, and since K and $\mathbb{R}$- axis are not parallel each other so they intersect at any point, let says c. The euclidean circle that passes through p and q has its center on K. Assume a circle A with the center C and radius will be $|c - p|$. Since the center c lies on the perpendicular bisector K so $|c - p| = |c - q|$, it means the circle A passes through both p and q. According to (4.1) it can be said that the intersection of $\mathbb{H}$ with a circle A is a Hyperbolic line.

In order to complete the proof, it must be shown that the uniqueness of this hyperbolic lines. As it was mentioned before, for the circle passing through p and q, the center must be on the perpendicular bisector. Moreover the perpendicular bisector and R-axis intersect only one point c, and so there exist a unique Euclidean circle centered on R and passes through p and q. And there exists unique Euclidean line passes through p and q. This completes the proof. □

Definition 4.5

Two hyperbolic lines in $\mathbb{H}$ are parallel if they are disjoint.

As it was mentioned in the introduction part Hyperbolic geometry is a one of the non

Euclidean geometry which is not satisfied the fifth postulate of Euclidean geometry. The following theorem gives the importance of the between Euclidean geometry and hyperbolic geometry.

Theorem 4.6 *Let ℓ be a hyperbolic line in $\mathbb{H}$, and let p be a point in $\mathbb{H}$ not on ℓ. Then, there exist infinitely many distinct lines through that are parallel to ℓ.*

Proof. *Let the Hyperbolic line ℓ contained from any Euclidean line L. Since p is not on L then from the Euclidean fifth postulate there exist a K Euclidean line wich is passes through p and parallel to L. Becouse of ℓ is hyperbolic line, L is perpendicular to $\mathbb{R}$, then K must be perpendicular to $\mathbb{R}$ as well. And so according to Definition*(4.3) *one hyperbolic line in $\mathbb{H}$ through p and parallel to L is the intersection of $\mathbb{H}$ and K.*

In order to show that there exist another line passes through p and parallel to L; Let take any point x on $\mathbb{R}$-axis between L and K. Therefore Rep $\neq$ Rex, there exist an Euclidean circle A which passes through p and x, and has a center on $\mathbb{R}$- axis. By this construction, A and L are disjoint. And so the Hyperbolic line $\mathbb{H} \cap A$ and ℓ are disjoint, that is reached that $\mathbb{H} \cap A$ is another Hyperbolic parallel line to ℓ.

Because of,there are infinetly many points on $\mathbb{R}$ between K and L, its gives that there are infinitely many distance Hyperbolic lines through p and parallel to ℓ.

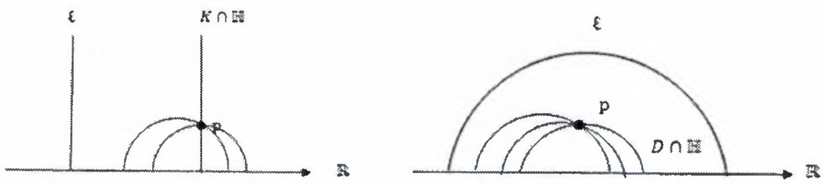
To complete the proof, It needs to show that, If ℓ is contained any circle, there exist infinitely many parallel Hyperbolic lines parallel to ℓ.

Now, assume that ℓ contained from an Euclidean circle A, and p is a point in $\mathbb{H}$ not on A. Let D be the Euclidean circle which has a same center with A and passes through p. Since, circles with have same center are disjont, then from Definition (4.5) *one Hyperbolic line passes through p and parallel to ℓ is $\mathbb{H} \cap D$.*

In order to construct second hyperbolic line, take any point x on $\mathbb{R}$ between D and A. There exist an Euclidean circle E, which passes through x and p. From construction of E, A and E are disjoint and so $\mathbb{H} \cap A$ that is hyperbolic line ℓ and $\mathbb{H} \cap E$ are disjoint then the second hyperbolic line passes through p and parallel to ℓ is $\mathbb{H} \cap E$.

Finally, therefore infinitely many points on $\mathbb{R}$ between A and D, there are infinitely many distinct hyperbolic lines through p and parallel to ℓ.

Consequently, for any line l in $\mathbb{H}$, there exist infinitely many distinct hyperbolic lines through any point p not on l. □



4.2 Length And Distance In Hyperbolic Geometry

In this section, it wil be given definitions and theorems for hyperbolic distance, in addition this there will be given most usefull distance formulas. Before define the length in hyperbolic geometry, it is needed to recall from calculus the definition of element of arc-length.

4.2.1 Path integrals

A path σ in the plane $\mathbb{R}^2$ is a function $\sigma : [a,b] \longrightarrow \mathbb{R}^2$ that is continuous on $[a,b]$ and differentiable on (a,b) with continuous derivative. The image of the interval under σ is a *curve* in $\mathbb{R}^2$. The Euclidean length of σ is given by the integral

$$length(\sigma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$$

where σ can be written with $\sigma = (x(t), y(t))$, $t \in [a,b]$, and $\sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the *arc-length element* in $\mathbb{R}^2$.

If it is assumed σ as a path into the complex plane, then σ can be written with $\sigma = x(t) + iy(t)$, the length of σ,

$$length(\sigma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b |\sigma'(t)|\, dt$$

then, it can be rewritting as a

$$length(\sigma) = \int_{\sigma} |dz|,$$

where $|dz| = |\sigma'(t)|\, dt$ is the standart Euclidean element of arc-lenght in $\mathbb{C}$.

Then the path integral of any continuous function can be obtained with using above notations, that is let f be a continuous function $f : \mathbb{C} \longrightarrow \mathbb{R}$. The path integral of f along the a path $\sigma : [a, b] \longrightarrow \mathbb{C}$ is the integral

$$\int_{\sigma} f(z)\, |dz| = \int_{a}^{b} f(\sigma(t)) \quad |\sigma'(t)|\, dt,$$

it can be thought $f(z)\,|dz|$ as a new element of arc-length and it rise the following defination.

Definition 4.7

For a path $\sigma : [a, b] \longrightarrow \mathbb{C}$, the lenght of σ with respect to the element of arc-length $f(z)\,|dz|$ is defined to be the integral

$$length_f(\sigma) = \int_{\sigma} f(z)\, |dz| = \int_{a}^{b} f(\sigma(t)) \quad |\sigma'(t)|\, dt.$$

4.2.2 Hyperbolic length and distance

This section will be start with the defination of hyperbolic length and hyperbolic distance in $\mathbb{H}$ .After that there will be given theorems which gives the distance between two hyperbolic points on the hyperbolic vertical line and then the preserving of distace under the $Aut(\mathbb{H})$.

To define them it is will be used a suitable arc- lenght element on $\mathbb{H}$ invariant under the action of $Aut(\mathbb{H})$.

Definition 4.8 *(Length in $\mathbb{H}$)*

For a piecewise path $\sigma : [a, b] \longrightarrow \mathbb{H}$, the hyperbolic length of σ defined to be

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Im z} |dz| = \int_{a}^{b} \frac{1}{Im\,(\sigma(t))} |\sigma'(t)|\, dt.$$

Definition 4.9 *(Distance)*

Let $z, z' \in \mathbb{H}$. The *hyperbolic distance* between z and z' is defined by

$$d_{\mathbb{H}}\,(z, z') = \inf \{length_{\mathbb{H}}\,(\sigma) : \ \sigma \text{ is a piecewise differentiable path with end points } z \text{ and } z'\}.$$

Definition 4.10 *(Geodesic)*

The paths of shortest hyperbolic length between points are called hyperbolic geodesics.

Theorem 4.11 *Vertical lines are geodesics in $\mathbb{H}$. Moreover if $b > a$, then*

$$d_{\mathbb{H}}\,(ai, bi) = \ln\left(\frac{b}{a}\right).$$

Proof. By defination (4.8) , the lenght of any path σ is defined by,

$$length_{\mathbb{H}}(\sigma) = \int_{a}^{b} \frac{1}{Im\,(\sigma(t))} |\sigma'(t)|\, dt.$$

Let take the path from ai to bi as a $\sigma(t) = it$ with $a \leq t \leq b$, which is the vertical line in $\mathbb{H}$, then

$$\begin{aligned} length_{\mathbb{H}}(\sigma) &= \int_{a}^{b} \frac{1}{t} i dt \\ &= \ln b - \ln a \\ &= \ln \frac{b}{a}. \end{aligned}$$

Now, consider any path $\sigma = x(t) + iy(t) : [0, 1] \longrightarrow \mathbb{H}$ joining ai and bi. Again from definition (4.8),

$$\begin{aligned} lenght_{\mathbb{H}}(\sigma) &= \int_0^1 \frac{1}{Im(\sigma(t))} |\sigma'(t)|\, dt \\ &= \int_0^1 \frac{1}{y(t)} \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &\geq \int_0^1 \frac{1}{y(t)} \sqrt{(y'(t))^2} dt \\ &= \int_0^1 \frac{1}{y(t)} |y'(t)|\, dt \\ &\geq \int_0^1 \frac{1}{y(t)} y'(t) dt \\ &= \ln \frac{b}{a}. \end{aligned}$$

Hence, the infimum length of paths joining ai and bi is $\ln \frac{b}{a}$. The equality is hold only $x'(t) = 0$, so $x(t)$ is constant, it means that the length which is the joining ai and bi is the vertical line in $\mathbb{H}$. This complete the proof. □

Proposition 4.12 *Let γ be a Möbius transformation of $\mathbb{H}$ and let z and $z' \in \mathbb{H}$. Then*

$$d_{\mathbb{H}}(\gamma(z), \gamma(z')) = d_{\mathbb{H}}(z, z').$$

Proof. Let σ be a path from z to z'. Then the path from $\gamma(z)$ to $\gamma(z')$ is $\gamma(\sigma)$. In order to show that the distace of two points is equal to the distance under the Möbius transformation of that points, it is sufficient to show that

$$length_{\mathbb{H}}(\gamma \circ \sigma) = length(\sigma).$$

Let take a Möbius transformation $\gamma(z) = \frac{az+b}{cz+d}$,with a, b, c and c are real, then,

$$|\gamma'(z)| = \frac{ad - bc}{|cz + d|^2}$$

and

$$Im(\gamma(z)) = \frac{ad - bc}{|cz + d|^2} Im(z),$$

they can be proved easily. Take $z = \sigma(t)$,

$$\begin{aligned}
length_{\mathbb{H}}(\gamma \circ \sigma) &= \int \frac{1}{Im\left[(\gamma \circ \sigma)(t)\right]} \left|(\gamma \circ \sigma)'(t)\right| dt \\
&= \int \frac{1}{Im\left(\gamma(\sigma(t))\right)} |\gamma'(\sigma(t))|\,|\sigma'(t)|\,dt \\
&= \int \frac{1}{\frac{ad-bc}{|c\sigma(t)+d|^2} Im\sigma(t)} \frac{ad - bc}{|c\sigma(t) + d|^2} |\sigma'(t)|\,dt \\
&= \int \frac{1}{Im(\sigma(t))} |\sigma'(t)|\,dt. \\
&= length_{\mathbb{H}}(\sigma).
\end{aligned}$$

□

Since, with the aid of cross ratio, it can be found a $Aut(\mathbb{H})$ that is transform any line in $\mathbb{H}$ to the vertical line (namely y-axis). The importance of the above theorem, it proves that all $Aut(\mathbb{H})$ preserve the hyperbolic distances. And so the previous two theorem construct significant structure in the following sections to determine the formula of distance of any two point.

4.2.3 Metric spaces

Definition 4.13 *(A Metric on a set X)*

A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying three conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y \in X$ (triangle inequality).

If d is a metric on X, it is referred to a *metric space* (X, d).

Theorem 4.14 $(\mathbb{H}, d_{\mathbb{H}})$ *is a path metric space.*

Proof. To show that $d_{\mathbb{H}}$ does define a metric. $d_{\mathbb{H}}$ must be satisfies the three conditions in the definition (4.13).

Let $\Gamma[x, y]$ denoted by the set of all piecewise σ paths $\sigma : [a, b] \to \mathbb{H}$ with $\sigma(a) = x$ and $\sigma(b) = y$. Now, consider the path $\sigma : [a, b] \to \mathbb{H}$ in $\Gamma[x, y]$, from the definition of $length_{\mathbb{H}}(\sigma)$,

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Imz} |dz| = \int_a^b \frac{1}{Im(\sigma(t))} |\sigma'(t)| \, dt,$$

since above integral is always nonegative, therefore $length_{\mathbb{H}}(\sigma)$ is nonnegative for every path in $\Gamma[x, y]$ and so the infimum $d_{\mathbb{H}}(x, y)$ of these integrals are nonnegative which is complete the first condition of defination (4.13).

Now to show that $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$, for all x, y in $\mathbb{H}$, it is needed to compare the length of paths in $\Gamma[x, y]$ and $\Gamma[y, x]$. Let $\sigma(a, b) \to \mathbb{H}$ be a path in $\Gamma[x, y]$ and define $\gamma : [a, b] \to [a, b]$ with $\gamma = a + b - t$ and the composition of σ and γ, $\sigma \circ \gamma : [a, b] \to \mathbb{H}$ lies in $\Gamma(y, x)$, in fact $\sigma \circ \gamma(a) = \sigma(b) = y$ and $\sigma \circ \gamma(b) = \sigma(a) = x$. And the length of $\sigma \circ \gamma$,

$$\begin{aligned} length_{\mathbb{H}}(\sigma \circ \gamma) &= \int_{\sigma \circ \gamma} \frac{1}{Imz} |dz| \\ &= \int_a^b \frac{1}{Im((\sigma \circ \gamma)(t))} |(\sigma \circ \gamma)'(t)| \, dt \\ &= \int_a^b \frac{1}{Im((\sigma(\gamma(t)))} |(\sigma'(\gamma(t))| \, |\gamma'(t)| \, dt, \end{aligned}$$

if make a subsitution with $s = \gamma(t)$, with, $s = \gamma(a) = b$, $s = \gamma(b) = a$ and $s' = \gamma'(t) = -dt$, then,

$$\begin{aligned} length_{\mathbb{H}}(\sigma \circ \gamma) &= -\int_b^a \frac{1}{Im(\sigma(s))} |\sigma'(s)| \, ds \\ &= \int_a^b \frac{1}{Im(\sigma(s))} |\sigma'(s)| \, ds \\ &= length_{\mathbb{H}}(\sigma). \end{aligned}$$

Hence, every path in $\Gamma[y,x]$ has the same length with the path in $\Gamma[x,y]$, by composing with the appropriate γ. Using the same argument, every path in $\Gamma[y,x]$ and in $\Gamma[x,y]$ has equal length.

In particular, two sets of hyperbolic length,

$$\{length_{\mathbb{H}}(\sigma),\ \sigma \in \Gamma[x,y]\} \text{ and } \{length_{\mathbb{H}}(\rho),\ \rho \in \Gamma[y,x]\}$$

are equal. Thus they have the same infimum, and it shows that $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(y,x)$.
To complete the proof, it must be satisfied the last condition of definition (4.13). Suppose it is not satisfied for $d_{\mathbb{H}}$, then there exist three distance points x, y, and z such that,

$$d_{\mathbb{H}}(x,z) > d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z),$$

and so,

$$d_{\mathbb{H}}(x,z) - (d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)) > 0,$$

let,

$$d_{\mathbb{H}}(x,z) - (d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)) = \epsilon.$$

Since, $d_{\mathbb{H}}(x,y) = \inf\{length_{\mathbb{H}}(f) : f \in \Gamma[x,y]\}$, there exists a path $f : [a,b] \to \mathbb{H}$ in $\Gamma[x,y]$, such that,

$$length_{\mathbb{H}}(f) < d_{\mathbb{H}}(x,y) + \frac{\epsilon}{2}.$$

Similarly, $d_{\mathbb{H}}(y,z) = \inf\{length_{\mathbb{H}}(g) : g \in \Gamma[y,z]\}$, and there exists a path $g : [b,c] \to \mathbb{H}$ in $\Gamma[y,z]$, such that,

$$length_{\mathbb{H}}(g) < d_{\mathbb{H}}(y,z) + \frac{\epsilon}{2}.$$

Assume, $h : [a,c] \to \mathbb{H}$ in $\Gamma[x,z]$ be a concatenation of f and g, and so it is lies in $\Gamma[x,z]$, then

$$length_{\mathbb{H}}(h) = length_{\mathbb{H}}(f) + length_{\mathbb{H}}(g).$$

Therefore,

$$\begin{aligned} d_{\mathbb{H}}(x,z) &\leq length_{\mathbb{H}}(h) \\ &= length_{\mathbb{H}}(f) + length_{\mathbb{H}}(g) \\ &< d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) + \epsilon, \end{aligned}$$

then,

$$d_{\mathbb{H}}(x,z) - (d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)) < \epsilon$$

which is a contradiction. This complete the proof. □

4.2.4 *Isometries of* $\mathbb{H}$

An isomerty of a metric space (X, d) is an homeomorphism f of X that preserves distance. That is, an isometry of (X, d) is a homeomorphism f of X for which

$$d(x,y) = d(f(x), f(y))$$

for every pair x and y of points of X.

Proposition 4.15 *Let x, y, and z be distinct point in $\mathbb{H}$.Then,*

$$d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z)$$

if and only if y is contained in the hyperbolic line segment joining x to y.

Proof. There exist a Möbius transformation m in $M\ddot{o}b(\mathbb{H})$ such that $m(x) = i$ and $m(z) = \alpha i$. From Proposition (4.12) and Theorem (4.11) ,

$$\ln \alpha = d_{\mathbb{H}}(i, \alpha i) = d_{\mathbb{H}}(x,z),$$

and let, $m(y) = a + ib$. There are several cases;

As a first case, let y lies on the hyperbolic line segment joining x to y. Then, $m(y)$ lies on the line segment joining $m(x)$ to $m(z)$, In particular $(y) = bi$ and $1 \leq b \leq \alpha$, therefore

$$d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(i,bi) = \ln b,$$

and

$$d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(bi,\alpha i) = \ln\frac{\alpha}{b} = \ln\alpha - \ln b,$$

and so,

$$d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z)\,. \tag{4.2}$$

Now, assume that y does not lie on the hyperbolic line segment joining x to z. Then, $m(y)$ can be lie on the positive imaginary axis such that $a = 0$ or not on the positive imaginary axis so that $a \neq 0$ and so there are two cases for this step,

Let $m(y)$ lies on the positive imaginary axis with not on the hyperbolic line segment joining x to z and so either $0 < b < 1$ or $b > \alpha$, and $a = 0$.

If $0 < b < 1$, as $\ln b < 0$,

$$d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(i,bi) = -\ln b$$

and

$$d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(bi,\alpha i) = \ln\frac{\alpha}{b} = \ln\alpha - \ln b,$$

and then,

$$d_{\mathbb{H}}(y,z) = \ln\alpha - \ln b > \ln\alpha + \ln b = d_{\mathbb{H}}(x,z) - d_{\mathbb{H}}(x,y)$$

and so,

$$d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) \tag{4.3}$$

is obtained.

And If $b > \alpha$, then $\ln b > \ln\alpha$, and

$$d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(i,bi) = \ln b,$$

and

$$d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(bi, \alpha i) = \ln \frac{b}{\alpha} = \ln b - \ln \alpha,$$

then,

$$d_{\mathbb{H}}(y,z) = \ln b - \ln \alpha$$

with added both sides $\ln b$, then,

$$\ln b + d_{\mathbb{H}}(y,z) = 2\ln b - \ln \alpha > \ln \alpha$$

it is obtained that,

$$d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) > d_{\mathbb{H}}(x,z). \tag{4.4}$$

Let $m(y)$ does not lie on the positive imaginary axis with not on the hyperbolic line segment joining x to z, and so $a \neq 0$. It can be observation that, whenever $a \neq 0$,

$$d_{\mathbb{H}}(i, bi) < d_{\mathbb{H}}(i, a+bi) = d_{\mathbb{H}}(x,y), \tag{4.5}$$

likewise,

$$d_{\mathbb{H}}(bi, \alpha i) < d_{\mathbb{H}}(a+bi, \alpha i) = d_{\mathbb{H}}(y,z). \tag{4.6}$$

If $1 < b < \alpha$, then according to equation (4.2),

$$d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(i,bi) + d_{\mathbb{H}}(bi,\alpha i)$$

if apply the equations (4.5) and (4.6) then,

$$d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z).$$

And similarly, If $b > \alpha$ or $0 < b < 1$, then according from equations (4.3) and (4.4), both case have,

$$d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(i,bi) + d_{\mathbb{H}}(bi,\alpha i) < d_{\mathbb{H}}(i,a+bi) + d_{\mathbb{H}}(a+bi,\alpha i) = d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)$$

is obtained.

In conclusion,

the only case in which $d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)$, is y lies on the hyperbolic line joining x to z. This complete the proof. □

The previous proposition observation that hyperbolic line segments can be characterized purely in terms of hyperbolic distance.

Lemma 4.16 *Every hyperbolic isometry of* $\mathbb{H}$ *takes hyperbolic lines to hyperbolic lines.*

Proof. From previous proposition, assume y lies on the line segmet ℓ_{xz} joining x to z, then

$$d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z).$$

Taking f as a hyperbolic isometry, so it preserves hyperbolic distance, therefore,

$$d_{\mathbb{H}}(f(x), f(z)) = d_{\mathbb{H}}(f(x), f(y)) + d_{\mathbb{H}}(f(y), f(z))$$

is hold. From above equation, it can be say that $f(y)$ lies on the hyperbolic line segmet $\ell_{f(x)f(z)}$ joining $f(x)$ to $f(z)$. And so

$$f(\ell_{xz}) = \ell_{f(x)f(z)}.$$

Hence, the hyperbolic isometry f takes hyperbolic lines to hyperbolic lines. □

Theorem 4.17 $Isom(\mathbb{H}, d_{\mathbb{H}}) = Aut(\mathbb{H})$

Proof. Since all Möbius transformations are bijection and invertable, and from proposition (4.12) every element of $Aut(\mathbb{H})$ is a hyperbolic isometry, and so $Aut(\mathbb{H}) \subset Isom(\mathbb{H}, d_{\mathbb{H}})$. In order the complete the proof it must be shown that $Isom(\mathbb{H}, d_{\mathbb{H}}) \subset Aut(\mathbb{H})$.

Let $f(z)$ be a hyperbolic isometry function and for any two points p and $q \in \mathbb{H}$, let ℓ_{pq} be a hyperbolic line segment joining p to q . From definition of isometry

$$\ell_{pq} = f(\ell_{pq}).$$

Assume, ℓ be a perpendicular bisector of hyperbolic line ℓ_{pq} which is a hyperbolic line

$$\ell = \{z \in \mathbb{H} : d_{\mathbb{H}}(p, z) = d_{\mathbb{H}}(q, z)\}.$$

And so $f(\ell)$ is a perpendicular bisector of $f(\ell_{pq}) = \ell_{f(p)f(q)}$.

Now, normalize the hyperbolic isometry f. Let take any two points x and y on the positive imaginary axis I in $\mathbb{H}$. Since there exists an element γ of $Aut(\mathbb{H})$ that satisfies $\gamma(f(x)) = x$ and $\gamma(f(y)) = y$ because $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(f(x), f(y))$. In particular, $\gamma \circ f$ fixes both x and y it means that $\gamma \circ f$ takes I to I.

If take another point on I then it can be preserved by determined by the two hyperbolic distances $d_{\mathbb{H}}(x, z)$ and $d_{\mathbb{H}}(y, z)$ and as both hyperbolic distances are preserved by $\gamma \circ f$.

Now, let w be any point in $\mathbb{H}$ that does not lie on I, let ℓ be a hyperbolic line throught w that is perpendicular to I. ℓ can be describe as a hyperbolic line contained in the Euclidean circle with Euclidean center 0 and radius $|w|$ .Since ℓ is a perpendicular bisector of any line segment in I. And since $\gamma \circ f$ fixes every point of I then $\gamma \circ f(\ell) = \ell$.Assume ℓ and I intersect at a point z, therefore, $d_{\mathbb{H}}(z, w) = d_{\mathbb{H}}(\gamma \circ f(z), \gamma \circ f(w))$ and since $\gamma \circ f$ preserves I then it must be $\gamma \circ f$ fixes w. It means that $\gamma \circ f$ fixes every point of $\mathbb{H}$, It shows $\gamma \circ f$ is a identity function. And so $f = \gamma^{-1}$ therefore f is an element of $Aut(\mathbb{H})$. This complete the proof. □

4.2.5 More formula for distance

There are several useful ways of writing down the distance between two points without having to integrate along arcs. One of this method is a cross ratio;

Proposition 4.18

$$d_{\mathbb{H}}(z,w) = \ln[z,z';w,w'] \tag{4.7}$$

where z' and w' are the endpoints on $\partial\mathbb{H}$ of the geodesic that joins z to w.

Proof. In order the proof of this proposition, it is enough to check that the left hand side of equality (4.7) equal to the right had side.

Since there exists an isometric function f such that the points z, z', w and w' maps to $f(z) = i,\ f(z') = 0,\ f(w) = \alpha i$ and $f(w') = \infty$. And then from theorems (4.12)and (4.11),

$$\begin{aligned} d_{\mathbb{H}}(z,w) &= d_{\mathbb{H}}(f(z), f(w)) \\ &= d_{\mathbb{H}}(i, \alpha i) = \ln \alpha. \end{aligned}$$

And the cross ratio of these four points,

$$\begin{aligned} [z,z';w,w'] &= \frac{(w'-z)(z'-w)}{(w'-w)(z'-z)} \\ &= \frac{(f(w')-f(z))(f(z')-f(w))}{(f(w')-f(w))(f(z')-f(z))} \\ &= \frac{(f(w')-i)(0-\alpha i)}{(f(w')-\alpha i)(0-i)} \\ &= \frac{\alpha i}{i} \lim_{f(w')\to\infty} \frac{f(w')-i}{f(w')-\alpha i} \\ &= \alpha. \end{aligned}$$

Then,

$$\ln[z,z';w,w'] = \ln \alpha.$$

Hence, the left hand side is equal to right hand side and so, the equality (4.7) is hold.

□

And the another useful way to calculate the Hyperbolic distance is given the following proposition;

Proposition 4.19 *Given two points* $z,\ w \in \mathbb{H}$;

$$\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2ImzImw}. \tag{4.8}$$

Proof. To prove this proposition, the same proof technique will be used with above proposition.

As a first let find the right hand side of the equality (4.8), let f be a isometric function, with $f(z) = i$ and $f(w) = \alpha i$, then

$$\begin{aligned} 1 + \frac{|z-w|^2}{2ImzImw} &= 1 + \frac{|f(z) - f(w)|^2}{2Imf(z)\,Imf(w)} \\ &= 1 + \frac{|i - \alpha i|^2}{2.1.\alpha} \\ &= \frac{2\alpha + (1-\alpha)^2}{2\alpha} \\ &= \frac{1+\alpha^2}{2\alpha}. \end{aligned}$$

And to find the left hand side, according from (4.11), the distace from z to w is,

$$d_{\mathbb{H}}(z,w) = \ln \alpha.$$

And so,

$$\begin{aligned} \cosh d_{\mathbb{H}}(z,w) &= \cosh \ln \alpha \\ &= \frac{e^{\ln\alpha} + e^{-\ln\alpha}}{2} \\ &= \frac{\alpha + \alpha^{-1}}{2} \\ &= \frac{\alpha^2+1}{2\alpha}. \end{aligned}$$

conclude that, left hand side equal to right hand side for all $z, w \in \mathbb{H}$. □

CHAPTER 5

THE POINCARÉ DISC MODEL

Up to now it is studied the upper half plane model of hyperbolic geomerty but there are several other useful models to study hyperbolic geomerty and one of them which will be described in this chapter is Poincaré disc model. There are different ways to construct the Poincaré Disc Model, and in this thesis it will construct with using the upper half plane.

Definition 5.1 *The disc* $D = \{z \in \mathbb{C} : |z| < 1\}$ *is called Poincaré disc. The circle* $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ *is called the circle at* ∞ *or boundary of* D*.*

Definition 5.2 *(Geodesics in D)*

The geodesics in the Poincaré Disc Model of hyperbolic geometry are the arcs of circles and diameters in D that meet ∂D orthogonally.

Recall from in chapter 1, the Möbius transformation from $\mathbb{H}$ to D is given with the Cayley mapping,

$$h(z) = \frac{z - i}{z + i}.$$

The inverse function of Cayley mappind is defined by D to $\mathbb{H}$,

$$g(z) = h^{-1}(z) = \frac{iz + i}{-z + 1}.$$

For any piecewise σ path $\sigma : [a, b] \to D$, the composition $g \circ \sigma : [a, b] \to \mathbb{H}$ is a path in $\mathbb{H}$, and the length of $g \circ \sigma$ in $\mathbb{H}$ is given by,

$$\begin{aligned} length_{\mathbb{H}}(g \circ \sigma) &= \int_{g \circ \sigma} \frac{1}{Imz} |dz| \\ &= \int_a^b \frac{1}{Im((g \circ \sigma)(t))} \left|(g \circ \sigma)'(t)\right| dt \\ &= \int_a^b \frac{1}{Im(g(\sigma(t))} |g'(\sigma(t))| \, |\sigma'(t)| \, dt \end{aligned}$$

and it is easy to calculate that,

$$Im(g(\sigma(t))) = Im\frac{\sigma(t) - i}{\sigma(t) + i} = \frac{1 - |\sigma(t)|^2}{|-\sigma(t) + 1|^2}$$

and

$$|g'(\sigma(t))| = \frac{2}{|-\sigma(t) + 1|^2},$$

and so

$$\frac{1}{Im(g(\sigma(t))} |g'(\sigma(t))| = \frac{2}{1 - |\sigma(t)|^2}$$

Hence,

$$length_{\mathbb{H}} (g \circ \sigma) = \int_a^b \frac{2}{1 - |\sigma(t)|^2} |\sigma'(t)|\, dt.$$

Then the length of the path σ in D is defined by,

$$length_D (\sigma) = \int_a^b \frac{2}{1 - |\sigma(t)|^2} |\sigma'(t)|\, dt.$$

The distance between two points $z,w \in D$ is defined by taking the length of the shortest path between them:

$$d_D(z, w) = \inf \{length_D (\sigma) : \ \sigma \text{ is a piecewise differentiable path from } z \text{ to } w\}.$$

Proposition 5.3 *Let* $\sigma : [0, r] \to D$, *then*

$$d_D(0, r) = \ln\left(\frac{1 + r}{1 - r}\right).$$

Morever, the real axis is the unique geodesic joining to 0 *to* r.

Proof. Let define a path $\sigma [0, r] \to D$ with $\sigma(t) = t$. Then,

$$\begin{aligned}
length_D(\sigma) &= \int_\sigma \frac{2}{1-|z|^2}|dz| \\
&= \int_0^r \frac{2}{1-t^2}dt \\
&= \int \left(\frac{1}{1+r} + \frac{1}{1-r}\right)dt \\
&= \ln\left[\frac{1+r}{1-r}\right].
\end{aligned}$$

To show that this length is minimum, take polar coordinates (r, θ) in D and recall that $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

Now, consider any path $\sigma = r(t) + i\theta(t) : [0, 1] \longrightarrow D$ joining 0 and r. Again from definition (4.8),

$$\begin{aligned}
lenght_{\mathbb{H}}(\sigma) &= \int_0^1 \frac{2}{1-|\sigma(t)|^2}|\sigma'(t)|\, dt \\
&= \int_0^1 \frac{1}{1-r^2}\sqrt{(r'(t))^2 + r^2\,(\theta'(t))^2 dt} \\
&\geq \int_0^1 \frac{1}{1-r^2(t)}\sqrt{(r'(t))^2 dt} \\
&= \int_0^1 \left[\frac{1}{1+r(t)} + \frac{1}{1-r(t)}\right]|r'(t)|\, dt \\
&\geq \int_0^1 \left[\frac{1}{1+r(t)} + \frac{1}{1-r(t)}\right] r'(t)dt \\
&= \ln\left[\frac{1+r(t)}{1-r(t)}\right] \\
&= \ln\left[\frac{1+r}{1-r}\right].
\end{aligned}$$

Hence, the infimum length of paths joining 0 and r is $\ln\left[\frac{1+r}{1-r}\right]$. The equality is hold only $\theta'(t) = 0$, it means that the length which is the joining 0 and r is the real axis. This complete the proof. □

From previous proposition,

$$d_D(0,r) = \ln\left(\frac{1+r}{1-r}\right),$$

if it solve for r,

$$r = \frac{e^{d_D(0,r)} - 1}{e^{d_D(0,r)} - 1},$$

since $\tanh z = \frac{e^{2z}+1}{e^{2z}-1}$, then,

$$r = \tanh\left(\frac{1}{2}d_D(0,r)\right).$$

Above formula is related with the radius of hyperbolic circle and Euclidean circle. In euclidean geometry, a circle is defined as the locus of all points a fixed distance from a fixed point and the definition of circle in the hyperbolic plane is given following definition.

Definition 5.4 *(Hyperbolic Circle)*

A *hyperbolic circle* in D is a set in D of the form

$$C = \{y \in D : d_D(x,y) = \rho\}$$

where $x \in D$ and $\rho > 0$ are fixed. x is refer to the *hyperbolic centre* of C and ρ is refer the *Hyperbolic radius of C.*

Lemma 5.5 *The circumference of a hyperbolic circle of radius ρ is* $2\pi\sinh(\rho)$.

Proof. Let start the proof with take any hyperbolic circle with hyperbolic center 0 and hyperbolic radius ρ, then from the definition (5.4),

$$\rho = d_D(0,r) = \ln\left(\frac{1+r}{1-r}\right).$$

Then, $r = \tanh(\frac{1}{2}\rho)$, and so the hyperbolic circle is also Euclidean circle with center 0 and radius r. Since the Euclidean circle can be parametrized by,

$$\gamma : [0, 2\pi] \to D,\ \gamma(t) = re^{it},\ \text{where } 0 \le t \le 2\pi,$$

with

$$|\gamma(t)| = r \text{ and } |d\gamma(t)| = rdt.$$

Then, the hyperbolic length of $\gamma(t)$,

$$d_D = \int_\gamma \frac{2\,|dz|}{1-|z|^2} = \int_0^{2\pi} \frac{2r}{1-r^2}dt = \frac{4\pi r}{1-r^2}.$$

And since, $r = \tanh\left(\frac{1}{2}\rho\right)$, then,

$$\begin{aligned} d_D &= \frac{4\pi\ \tanh\frac{1}{2}\rho}{1-(\tanh\frac{1}{2}\rho)^2} \\ &= \frac{4\pi\ \tanh\frac{1}{2}\rho}{\operatorname{sech}^2\frac{1}{2}\rho} \\ &= 4\pi\sinh\left(\frac{1}{2}\rho\right)\cosh\left(\frac{1}{2}\rho\right). \end{aligned}$$

Since $\sinh(2t) = 2\sinh(t)\cosh(t)$, and so the length of hyperbolic circle will be,

$$d_D = 2\pi\sinh(\rho).$$

□

Lemma 5.6 *The area of a hyperbolic circle of radius ρ is* $4\pi\sinh(\frac{1}{2}\rho)$.

Proof. The integral from the zero to radius of the hyperbolic circumference of the circle gives that the area of the circle. And so,

$$Area = \int_0^\rho 2\pi\sinh\rho\ d\rho = 2\pi\cosh\rho\ |_0^\rho = 2\pi\,(\cos h\rho - 1)$$

since $\sinh\rho = \frac{\cosh 2\rho - 1}{2}$, then,

$$Area = 4\pi \sinh^2\left(\frac{1}{2}\rho\right).$$

□

CHAPTER 6

CONCLUSION

In this thesis studied with hyperbolic geometry which is the one of the non Euclidean geometry. It is construct with using upper half plane model and Poancaré and complex variable functions. The Möbius transformations played most imprtant role in the hyperbolic geometry. Here with using the Möbius transformations obtained the some formulas to find length and distace in hyperbolic geometry. And stereographic projection and hyperbolic geomtery make easiest to study with spherical geometry.

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