## COMPLEX ANALYSIS AND RIEMANN MAPPING THEOREM

## A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By AYÇA GÜLFİDAN

In Partial Fulfillment of the Requirements for The Degree of Master of Science In The Department of Mathematics

NICOSIA, 2015

# Ayça Gülfidan: COMPLEX ANALYSIS AND RIEMANN MAPPING THEOREM

#### Approval of Director of Graduate School of Applied Sciences

## Prof. Dr. İlkay SALİHOĞLU

#### We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics

**Examining Committee in Charge:** 

Prof. Dr. F. Kaya Özkın

Assoc. Prof. Dr. Mehmetali Özarslan,

Assoc. Prof. Dr. Evren Hincal,

Committee Chairman And Supervisor Department of Mathematics Near East University

Department of Mathematics, Eastern Mediterranean University

Department of Mathematics, Near East University

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Signature: Date:

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#### ACKNOWLEDGEMENTS

First of all, I feel very responsible to thank my supervisor Prof.Dr. Kaya I. OZKIN for working hard with me.

Also, I would like to thank Assoc.Prof.Dr Evren HINCAL, Assist.Prof.Dr. Abdurrahman Mousa OTHMAN, Assist.Prof.Dr. Burak SEKEROGLU, for guiding me while I was writing my dissertation. I appreciate Assoc.Prof.Dr. Harun KARSLI for helping me organise my dissertation.

I shouldn't forget to thank my mother, Cemile GULFIDAN, my father Halil GULFIDAN, and my brother Ömer GULFIDAN for supporting me from their heart.

I am very thankful to my best friend Selen VAIZ for encouraging me in this hard journay. I also appreciate Meral İpek AZIMLI for spending effort for me and believing me from her heart.

#### ABSTRACT

Eventhough, real differential functions are one of the main subject of real analysis, they are also take place in the centre of complex functions theorem.

Functions which are both holomorphic and bijective, are called biholomorphic functions. If two spaces have biholomorphism between them, then these two spaces are biholomorphically equivalent.

Biholomorphically equivalence is very important in complex analysis, because instead of working on complicated spaces, we can work on simpler work on known spaces.

For example, suppose U and V are both open subsets of  $\mathbb{C}$ . If  $f: U \to V$  and mapping is biholomorphic, then  $gof: U \to \mathbb{C}$  must be holomorphic in order to make  $g: V \to \mathbb{C}$  function holomorphic.

Basically, in comlpex functions theorem, spaces which are holomorphically equivalence are identical.

Riemann Mapping Theorem is a big result for sufficient of conditions of biholomorphic equivalence in complex function theory. Riemann Mapping Theorem provides an easy way for building biholomorphically equivalence. It quarantees the presence of biholomorphic functions and it also shows that, building biholomorphic transformations between spaces is unnecessary.

**Keywords:** Biholomorphically equivalent, Open Mapping Theorem, Montel's Theorem, Hurwitz's Theorem, Mobius Transformations, Conformal Mapping, Riemann Mapping Theorem.

#### ÖZET

Her nekadar reel diferensiyellenebilir fonksiyonlar reel analizin temel konularından biri ise de, holomorfik fonksiyonlar, karmaşık fonksiyonlar teorisinin merkezinde yer alırlar. Burada holomorfik ve birebir ve üzerine olan fonksiyonlara biholomorfik fonksiyonlar denilecektir. Aralarında biholomorfizm olan iki uzay biholomorfik eşdeğerdirler.

Karmasık analizde biholomorfik eşdeğerlilik önemlidir. Çünkü bu sayede yapısı hayli karısık olan bir uzayla calısmak yerine, yapısı daha yakından bilinen bir uzayl alarak calısmak mümkün olabilmektedir. Örneğin, U ve V, C nin açık alt kümeleri oldugunu varsayalım. Eger  $f: U \to V$  donüsümü biholomorfik ise bu durumda herhangi bir  $g: U \to \mathbb{C}$  fonksiyonun holomorfik olması icin gerek ve yeter kosul,  $go f: U \to \mathbb{C}$  bileske foksiyonunun holomorfik olmasıdır.

Temelde karmaşık fonksiyonlar teorisinde holomorfik olarak eşdeğer uzaylar aslında özdeştirler.

Karmaşık fonksiyonlar teorisinde Riemann dönüşüm teoremi, biholomorfik eşdeğerlik için hangi koşulların yeterli olacağını belirtmesi bakımından çok önemli büyük bir sonuçtur. Uzaylar arasında biholomorfik dönüşümler inşaa etmek genellikle zordur.

Riemann dönüşüm teoreminin sagladığı kolaylık belirli tipten uzaylar arasında biholomorfik fonksiyonun varlığını garanti etmesi ile artık uzaylar arasında biholomorfik dönüşümler insaa etmenin gereksiz olacağıdır.

Anahtar Sözcükler: Biholomirfik Eşdeğerlik, Açık Dönüşüm Teorisi, Montel Teorisi, Hurwitz's Teorisi, Mobyüs Dönüşümleri, Konform Dönüşümler, Riemann Dönüşüm Teorisi.

#### **CHAPTER 1**

#### **INTRODUCTION TO COMPLEX NUMBERS**

The goal of this chapter is to understand complex numbers, complex functions and their properties. The chapter is written in order to explain the main topic of my dissertation. The difference between complex functions and real functions are mentioned and suitable examples are given. Later on this chapter, topologic properties of complex planes are discussed. The most importantly, we will see derivatives of complex functions and we will also talk about what are the conditions on these subjects.

#### **1.1 Definition of Complex Numbers**

We can represented complex number in cartesian form, polar form and spherical form.

#### **1.1.1** Cartesian form

Let

$$\mathbb{C} = \{(x, y) : x \text{ and } y \text{ are in } \mathbb{R}\}.$$

We call  $\mathbb{C}$  the set of all complex numbers. This means that z = (x, y) = x + iy is complex number, where  $x, y \in \mathbb{R}$  and  $i \in \mathbb{C}$ , and x is called the real part of the given complex number, which is

$$Rez = x$$

and similarly y is called the imaginary part of the given complex number, which is

$$Imz = y$$
.

Addition to these  $\bar{z}$  is called conjugate of z, which is,

$$\overline{z} = x - iy.$$

and the modulus of a complex number z , that is

$$|z| = \sqrt{x^2 + y^2}.$$

and also this is positive real number.

We will talk about some basic properties of complex numbers in cartesian form. Assume that  $z_1, z_2, z_3 \in \mathbb{C}$ , which are,  $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_3 = x_3 + iy_3$  where all components of all z are in  $\mathbb{R}$ .

If  $z_1 = z_2$  then we have

$$x_1 + iy_1 = x_2 + iy_2 \iff x_1 = x_2$$
 and  $y_1 = y_2$ .

• If  $z_1 = z_2$  and  $z_2 = z_3$  then  $z_1 = z_3$ .

For z = x + iy where  $x, y \in \mathbb{R}$  and  $i \in \mathbb{C}$  then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

• The conjugate of  $\overline{z_1}$  is equal to  $z_1$ . That is

$$\overline{\overline{z_1}} = \overline{x_1 - \iota y_1} = z_1.$$

•  $|z_1|^2 = z_1. \overline{z_1}.$ 

From assertion, we have  $z_1 = x_1 + iy_1$  and clearly  $\overline{z_1} = x_1 - iy_1$ .

$$z_{1}.\overline{z_{1}} = (x_{1} + iy_{1}).(x_{1} - iy_{1})$$
  
=  $x_{1}^{2} - ix_{1}.y_{1} + ix_{1}.y_{1} + y_{1}^{2}$   
=  $x_{1}^{2} + y_{1}^{2}$  (1.1)

We know that

$$|z_1| = \sqrt{x_1^2 + y_1^2} \tag{1.2}$$

If we take square 1.2 side by side, then we have

$$|z_1|^2 = x_1^2 + y_1^2 \tag{1.3}$$

We can see that 1.1 and 1.3 are equal. Finally, we have

 $|z_1|^2 = z_1.\overline{z_1}.$ 

- $|\text{Rez}_1| < |z_1|$  and  $|\text{Imz}_1| < |z_1|$
- $|\overline{z_1}| = |z_1|$
- $|z_1, z_2| = |z_1|, |z_2|$
- $|z_1 + z_2| < |z_1| + |z_2|$ . This is called triangle inequality.

Since Rez = x and Imz = y then we have

$$x = \frac{z + \overline{z}}{2}$$
 and  $y = \frac{z - \overline{z}}{2i}$ .

- |z| = 0 if and only if z = 0.
- $Re(z_1 \pm z_2) = Rez_1 \pm Rez_2$ .
- $Im(z_1 \pm z_2) = Imz_1 \pm Imz_2$ .
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$  where  $|z_2| \neq 0$ .

#### 1.1.2 Polar form

Suppose that r = |z| and  $\theta$  is argument of any complex number z = x + iy, then,

$$x = rcos\theta$$
 and  $y = rsin\theta$ .

and  $z = r(\cos\theta + i\sin\theta)$  is called polar form of z.

Clearly,  $r = |z| = \sqrt{x^2 + y^2}$  and  $\theta = \tan\left(\frac{y}{x}\right) = \text{Argz}$ . Argz is called principle argument, where  $0 \le \theta \le 2\pi$ . Then

$$\arg z = \operatorname{Arg} z + 2k\pi$$
 where  $k \in \mathbb{Z}$ .

For example, let z = 2 + 2i and we have  $r = |z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$  and also  $\theta = \frac{\pi}{4}$ .

$$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

is polar form of z = 2 + 2i.

Now, we talk about some basic properties of complex numbers in polar form. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
 and  $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ .

- $z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))$ .
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left( \cos(\theta_1 \theta_2) + i\sin(\theta_1 \theta_2) \right)$ where  $z_2 \neq 0$ .
- $\arg(z_1, z_2) = \arg z_1 + \arg z_2$ .
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 \arg z_2$  where  $z_2 \neq 0$ .

**Example 1.1** *Find* Argz *and* arg *z where*  $z = \sqrt{3} - i$ .

**Solution.** We have z lies in fourth quadrant of complex plane. The modulus of z is

$$|z| = \sqrt{3+1} = 2.$$

Since z lies in fourth part in  $\mathbb{C}$ , the principal argument of z is equal to  $\operatorname{Arg} z = \frac{11\pi}{6}$ . We know that,  $\operatorname{arg} z = \operatorname{Arg} z + 2k\pi$  where  $k \in \mathbb{Z}$ . And finally we have,

$$\arg z = \frac{11\pi}{6} + 2k\pi, k \in \mathbb{Z}.$$

#### 1.1.3 Stereographic Projection



Let Z(x, y, 0) is any point in  $\mathbb{C}$ . Through the points N and Z we draw the straight line NZ intersecting the sphere S at a point  $A(x_1, x_2, x_3)$ . Then A is called the stereographic projection. Consider the unit sphere S in  $\mathbb{R}^3$ , that is

$$S = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 1\}$$

Equation of the line M passing through Z and N

$$M = \{(x_1, x_2, x_3): ((1 - t)x, (1 - t)y, t), t \in \mathbb{R}\}$$
$$AZ = t(N - Z)$$
$$A - Z = Nt - Zt$$
$$A = Nt + (1 - t).z$$
$$A = (0,0,1)t + (1 - t).Z$$
$$A = \{(1 - t)x, (1 - t)y, t): t \in \mathbb{R}\}.$$

Then we must have

$$(1-t)^{2}x^{2} + (1-t)^{2}y^{2} + t^{2} = 1$$

$$(1-t)^{2}(x^{2} + y^{2}) + t^{2} = 1$$

$$(1-t)^{2}|z|^{2} = 1 - t^{2}$$

$$|z|^{2} - 1 = t(|z|^{2} + 1)$$

$$t = \frac{|z|^{2} - 1}{|z|^{2} + 1}$$
(1.4)

We have

$$x_1 = (1 - t)x \tag{1.5}$$

If we put 1.4 in 1.5 then we have

$$x_{1} = \left[1 - \frac{|z|^{2} - 1}{|z|^{2} + 1}\right] \cdot x$$

$$x_{1} = \left[\frac{|z|^{2} + 1 - |z|^{2} + 1}{|z|^{2} + 1}\right] x$$

$$x_{1} = \frac{2x}{|z|^{2} + 1} = \frac{2Rez}{|z|^{2} + 1}$$
(1.6)

Similarly, we have

$$x_2 = (1 - t)y \tag{1.7}$$

If we put 1.4 in 1.7, we have,

$$x_2 = \frac{2y}{|z|^2 + 1} = \frac{2Imz}{|z|^2 + 1}$$
(1.8)

And so we have clearly

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{1.9}$$

From 1.6, 1.8, and 1.9 we have

$$A(x_1, x_2, x_3) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

**Example 1.2** Write the given complex number  $z = \frac{i}{1+i}$  in three form of complex numbers.

**Solution.** Now, we multiply numerator and denumerator of the given number with 1 - i then we have

$$z = \frac{i}{1+i} = \frac{i}{1+i} \cdot \frac{1-i}{1-i}$$
$$z = \frac{i-i^2}{1^2+1^2} = \frac{1+i}{2} = \frac{1}{2} + \frac{i}{2}$$

This is the cartesian form of the given z. Now, we compute  $\theta$  and r, which are;

$$r = |z| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{2}$$

and

$$\theta = \operatorname{Arg}\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{\pi}{4}$$

Therefore

$$z = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

this is polar form of the given z. And finally,

$$x_{1} = \frac{2x}{|z|^{2} + 1} = \frac{2\frac{1}{2}}{(\frac{\sqrt{2}}{2})^{2} + 1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$
$$x_{2} = \frac{2y}{|z|^{2} + 1} = \frac{2\frac{1}{2}}{(\frac{\sqrt{2}}{2})^{2} + 1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$
$$x_{3} = \frac{|z|^{2} - 1}{|z|^{2} + 1} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}$$

Therefore  $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$  is the spherical form of the given z.

**Definition 1.3**  $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$  is called the extended complex plane.

In general if for  $z \in \mathbb{C}, z \to A(x_1, x_2, x_3)$  on  $S \setminus N$  then

$$|z| > K \implies x_3 > \frac{K^2}{1 + K^2}$$
  
 $|z| < K \implies x_3 < \frac{K^2}{1 + K^2}$ 

Under stereographic projection, we have

$$|z| < 1 \iff x_3 < \frac{1}{2} \Rightarrow x_1^2 + x_2^2 < \frac{3}{4}$$
$$|z| > 1 \iff x_3 > \frac{1}{2} \Rightarrow x_1^2 + x_2^2 > \frac{3}{4}$$

 $|z| = 1 \Leftrightarrow$  this is the equator z = Z.

Now we assume that  $(x_n, y_n, z_n)$  is a sequences of S which converge (0,0,1) and let  $\{z_n^*\}$  be the corresponding sequence of points in  $\mathbb{C}$ . Now, we show that if  $|z_n^*| \to \infty$  then  $(x_n, y_n, z_n) \to (0,0,1)$ .

Let  $z_n^* = x_n^* + i y_n^*$ 

$$x_n = \frac{2x_n^*}{|z_n^*|^2 + 1}$$
$$y_n = \frac{2y_n^*}{|z_n^*|^2 + 1}$$
$$z_n = \frac{|z_n^*|^2 - 1}{|z_n^*|^2 + 1}.$$

If  $|z_n^*| \to \infty$  we have

$$\lim_{|z_n^*| \to \infty} \frac{2x_n^*}{|z_n^*|^2 + 1} = 0$$
$$\lim_{|z_n^*| \to \infty} \frac{2y_n^*}{|z_n^*|^2 + 1} = 0$$
$$\lim_{|z_n^*| \to \infty} \frac{|z_n^*|^2 - 1}{|z_n^*|^2 + 1} = 1.$$

Now, if we show  $\Theta(\infty) = N$ , then we say that  $\Theta$  is a continuous function.

$$\Theta(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)$$

at  $|z| = \infty$  we have

$$x_{1} = \frac{2x}{|z|^{2} + 1} = \frac{2x}{\omega + 1} = 0$$

$$x_{2} = \frac{2y}{|z|^{2} + 1} = \frac{2y}{\omega + 1} = 0$$

$$x_{3} = \frac{|z|^{2} - 1}{|z|^{2} + 1} = \frac{|z|^{2}(1 - \frac{1}{|z|^{2}})}{|z|^{2}(1 + \frac{1}{|z|^{2}})} = \frac{1 - \frac{1}{\omega}}{1 + \frac{1}{\omega}} = 1.$$

Hence  $\Theta(\infty) = N$ , which show that  $\Theta$  is a continuous function. Now we can define a function  $\Theta: \mathbb{C} \to S$ . We can see that  $\Theta$  is one-to-one and onto. Hence  $\Theta$  has inverse function, which is  $\Theta^{-1}: S \to \mathbb{C}$ . We know that  $x_1 = (1 - t)x$ ,  $x_2 = (1 - t)y$  and  $t = x_3$ . If we put  $t = x_3$  in  $x_1$  and  $x_2$  we have

$$x_1 = (1 - x_3)x$$
 and  $x_2 = (1 - x_3)y$ 

$$x = \frac{x_1}{1 - x_3} \text{ and } y = \frac{x_2}{1 - x_3}$$
(1.10)  
$$\Theta^{-1}(A) = z = \frac{x_1}{1 - x_3} + i \frac{x_2}{1 - x_3}.$$

Now we define  $f(x_1, x_2) = x_1 + ix_2$  and  $g(x_3) = 1 - x_3$  where f and g are continuous on  $\mathbb{C}$ . We know that if f and g on  $\mathbb{C}$  then  $\frac{f}{g}$  continuous where  $x_3 \neq 1$ . Therefore  $\Theta^{-1}$  is continuous where  $x_3 \neq 1$ . Now, we show that  $\Theta^{-1}$  is continuous at infinity.

Let us take any sequence  $\{z_n\}$ . If  $z \to (0,0,1)$  then

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} x_n + i \lim_{n \to \infty} y_n$$

from 1.10 we have

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{x_1}{1 - x_3} = \infty$$
$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_2}{1 - x_3} = \infty$$
$$\implies z_n \to \infty.$$

Hence  $\Theta^{-1}$  is continuous function.

Finally we can say that  $\theta$  is one-to-one, onto and continuous and  $\theta^{-1}$  is continuous, then  $\theta$  is called homeomorphism.

Now we can define

$$\Psi(z_1, z_2) = d(Z_1, Z_2)$$

is Euclidean distance between  $Z_1$  and  $Z_2$ , which are given  $Z_1(a_1, b_1, c_1)$  and  $Z_2(a_2, b_2, c_2)$  as well. Addition to these,  $Z_1$  and  $Z_2$  are image of  $z_1 = x + iy$  and  $z_2 = x' + iy'$ .

Since  $Z_1$  and  $Z_2$  are on the sphere *S*, then we have

$$a_1^2 + b_1^2 + c_1^2 = 1 \text{ and } a_2^2 + b_2^2 + c_2^2 = 1$$
 (1.11)

We know that the distance between  $z_1$  and  $z_2$ , is defined as

$$\Psi(z_1, z_2) = d(Z_1, Z_2).$$

Hence we have

$$\Psi(z_1, z_2) = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}$$
$$= \sqrt{a_1^2 - 2a_1a_2 + a_2^2 + b_1^2 - 2b_1b_2 + c_1^2 - 2c_1c_2 + c_2^2}.$$

from 1.11 we have

$$\Psi(z_1, z_2) = \sqrt{2 - 2(a_1a_2 + b_1b_2 + c_1c_2)}$$
$$= \sqrt{2} \cdot \sqrt{1 - a_1a_2 - b_1b_2 - c_1c_2}.$$
(1.12)

since  $Z_1$  and  $Z_2$  are on S, clearly we have

$$a_{1} = \frac{2x}{|z_{1}|^{2} + 1}, b_{1} = \frac{2y}{|z_{1}|^{2} + 1}, c_{1} = \frac{|z_{1}|^{2} - 1}{|z_{1}|^{2} + 1}$$
$$a_{2} = \frac{2x'}{|z_{2}|^{2} + 1}, b_{2} = \frac{2y'}{|z_{2}|^{2} + 1}, c_{2} = \frac{|z_{2}|^{2} - 1}{|z_{2}|^{2} + 1}.$$
(1.13)

If we put 1.13 in 1.12, we have

$$\Psi(z_1, z_2) = \frac{2 \cdot |z_1 - z_2|}{\sqrt{|z_1|^2 + 1} \cdot |z_2|^2 + 1}.$$

If  $z_2$  is the point at infinity then

$$\Psi(z_1, \infty) = \lim_{z_2 \to \infty} \frac{2 \cdot |z_1 - z_2|}{\sqrt{|z_1|^2 + 1} \cdot |z_2|^2 + 1}$$
$$= \lim_{|z_2| \to \infty} \frac{2|z_2| \cdot \left| \frac{z_1}{|z_2|} - 1 \right|}{\sqrt{|z_1|^2 + 1} \cdot |z_2| \cdot \sqrt{1 + \frac{1}{|z_2|^2}}}$$
$$= \lim_{|z_2| \to \infty} \frac{2}{|z_1| + 1} = \frac{2}{|z_1| + 1}.$$

We have some important theorems about stereographic projection.

**Theorem 1.4** *Streographic projection take circle to circle and lines.* 

**Proof.** We assume that K is a circle on Riemann sphere.

$$K = \{(x_1, x_2, x_3): Ax_1 + Bx_2 + Cx_3 + D = 0 \text{ and } x_1^2 + x_2^2 + x_3^2 = 1\}.$$

If K passes through (0,0,1) then we have

$$A. 0 + B. 0 + C. 1 + D = 0$$
  
 $C + D = 0.$ 

from definition of stereographic projection we have

$$x_1 = \frac{2x}{|z|^2 + 1}, x_2 = \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}$$
(1.14)

If we put 1.14 in

$$Ax_1 + Bx_2 + Cx_3 + D = 0$$

then we have

$$\frac{2Ax}{1+x^2+y^2} + \frac{2By}{1+x^2+y^2} + \frac{C(x^2+y^2-1)}{1+x^2+y^2} + D = 0$$
  
$$2Ax + 2By + C(x^2+y^2) - C + D + D(x^2+y^2) = 0$$
  
$$(x^2+y^2)(C+D) + 2Ax + 2By + D - C = 0$$
(1.15)

If C + D = 0, then 1.15 becomes

$$2Ax + 2By + D - C = 0$$

which is equation of a line. If  $C + D \neq 0$  then 1.15 is

$$(C+D)(x^{2}+y^{2}) + 2Ax + 2By + D + C = 0$$

is the equation of a circle.  $\square$ 

**Theorem 1.5** Let  $K \in \mathbb{C}_{\infty}$ . Then the corresponding image of K on S is

- *a is a circle in S not containing* (0,0,1) *if K is circle*
- *a line in S containing* (0,0,1) *if K is a line.*

**Proof.** Now, we consider the general equation of a circle in  $\mathbb{C}$ .

$$K = \{(x, y): A(x^{2} + y^{2}) + Bx + Cy + D = 0\}.$$
(1.16)

Now we put 1.10 in 1.16 then we have,

$$A\left[\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2}\right] + B\frac{x_1}{1-x_3} + C\frac{x_2}{1-x_3} + D = 0$$
$$A\frac{(x_1^2 + x_2^2)}{(1-x_3)^2} + \frac{Bx_1 + Cx_2}{1-x_3} + D = 0$$
$$(A-D)x_3 + Bx_1 + Cx_2 + D = 0.$$
(1.17)

Let A = 0. Clearly, from 1.16 we have

$$Bx + Cy + D = 0$$

this is equation of the line. Since A = 0 from 1.17

$$-Dx_3 + Bx_1 + Cx_2 + D = 0$$
$$Bx_1 + Cx_2 + D(1 - x_3) = 0.$$

this equation containing (0,0,1). If K is a line then  $(0,0,1) \in S$ . Now we assume that  $A \neq 0$  and D = 0. Then from 1.16 we have

$$A(x^2 + y^2) + Bx + Cy = 0$$

this is equation of circle and

$$Bx_1 + Cx_2 + Ax_3 = 0 \tag{1.18}$$

we can see 1.18 not containing (0,0,1).  $\Box$ 

#### **1.2 Topology of the Complex Plane**

In this section we will talk about the topology of the complex plane and we will see difference between them.

#### **Definition 1.6** (open set)

A set is open in C if it contains none of its boundary points. (Churchill and Brown, 1990).

#### **Definition 1.7** (closed set)

A set is closed in C if it contains all of its boundary points. (Churchill and Brown, 1996).

#### **Definition 1.8** (smooth curve)

"Suppose that a curve C in the plane is parametrized by x = x(t) and y = y(t) where  $a \le t \le b$ . If x' and y' are continuous on [a, b] and not simultaneously zero on (a, b)." (Zill and Shanahan, 1940).

#### **Definition 1.9** (piece-wise smooth curve)

"C is a piece-wise smooth curve in  $\mathbb{C}$  if it consists of a finite number of smooth curves  $C_1, C_2, ..., C_n$  joined end to end, that is, the terminal point of on curve  $C_k$  coinciding with the initial point of the next curve  $C_{k+1}$ ." (Zill and Shanahan, 1940).

#### **Definition 1.10** (connected)

"An open set S is connected in  $\mathbb{C}$  if every pair of points  $z_1$  and  $z_2$  contained in S can be joined by a curve that lies entirely in S." (Mathews and Howell , 2006).

#### **Definition 1.11** (simply connected and multiply connected)

"A domain D in  $\mathbb{C}$  is simply connected if its complement with respect to  $\mathbb{C}_{\infty}$ . A domain that is not simply connected is called multiply connected domain, that is, it has "*holes*" in it." (Zill and Shanahan , 1940).

#### **Definition 1.12** (domain)

"A domain is a nonempty open connected set in C." (Ponnusamy, 2005).

#### **Definition 1.13** (region)

"A domain together with some, none or all of its boundary points is reffered to as a region in  $\mathbb{C}$ ." (Ponnusamy , 2005).

#### Definition 1.14 (bounded and unbounded)

"A set S is bounded in C if  $\exists M > 0$  such that |z| < M whenever  $z \in S$  else it is unbounded." (Stein and Shakarchi, 2003).

#### **Definition 1.15** (compact set)

" If a set S is closed and bounded then it is called compact in C." (Zill and Shanahan , 1940).

#### **1.3 Function of a Complex Variable**

#### **Definition 1.16** (complex function)

Let A and B be two nonempty subset of C. A function from A to B is a rule, f, which assigns each  $z_0 = x_0 + iy_0 \in A$  a unique element  $w_0 = u_0 + iv_0 \in B$ . The number  $w_0$  is called the values of f at  $z_0$  and we write  $w_0 = f(z_0)$ . If z varies in A then w = f(z) varies in B. We also write  $f: A \to B, z \to w = f(z)$ . We have two real-valued functions  $u: A \to \mathbb{R}, v: A \to \mathbb{R}$ , then by defining  $f(z) = u(x, y) + iv(x, y), (x, y) \in A$ . We obtain  $f: A \to C$ , where A is subset of C.

" In this section introduction, we defined a real-valued function of a real variable to be a function whose whose domain and range are subsets of the set  $\mathbb{R}$  of real numbers. Because  $\mathbb{R}$  is a subset of the set  $\mathbb{C}$  of the complex numbers, every real-valued function of a real variable is also a complex function. We will soon see that real-valued functions of two real variables x and y are also special types of complex analysis. This functions will play an important role in the study of complex numbers." (Zill and Shanahan [3], 1940).

If w = f(z) is a complex function, then the image of z = x + iy under f is w = u + iv.

For example; let z = x + iy and  $w = 2z^2$  is complex function. Then the image of z under  $w = 2z^2$ .

$$w = 2z^{2} = 2(x + iy)^{2} = 2x^{2} - 2y^{2} + 4xyi$$
$$u(x, y) = 2x^{2} - 2y^{2} \text{ and } v(x, y) = 4xy.$$

Where u is real part, v is imaginary part of the given complex function.

"A useful tol for the study of real functions in elementary calculus is the graph of the function. Recall that if y = f(x) is a real-valued function of a real variable x, then the graph of f is defined to be the set of all points (x, f(x)) in the two-dimensional Cartesian plane. An analogous definition can be made for a complex function. However, if w = f(z) is a complex function, the both z and w lie in complex plane. It follows that the set of all points (z, f(z)) lies in four-dimensional space. Ofcourse, a subset of four-dimensional space cannot be easily illustrated. Therefore; we cannot draw the graph a complex function." (Zill and Shanahan, 1940).

Consider the real function  $f(x) = x \ge 3$ . We can draw the given real function in Cartesian plane. But if w = f(z) = iz is a complex function then the image of  $x \ge 3$  under w = iz, we have

$$w = f(3 + iy) = i(3 + iy) = 3i - y$$

w = 3i - y where  $y \in \mathbb{R}$ , is the line v = 3 in w-plane. Finally we have x = 3 in z-plane mapped onto v = 3 in w-plane under w = iz. Since  $Rez = x \ge 3$ ,  $y \in \mathbb{R}$  this is mapped onto  $Imw = v \ge 3$ ,  $u \in \mathbb{R}$  under w = iz.

#### 1.4 Continuity, Differentiable and Analyticity

#### **Definition 1.17** (continuity)

"Let  $A \subset \mathbb{C}$  be an open set and let  $f: A \to \mathbb{C}$  be a function. We say f is continuous at  $z_0 \in A$  if and only if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

and that is continuous on A if f is continuous at each point  $z_0$  in A." (Marsen and Hoffman, 1987).

#### **Definition 1.18** (differentiability)

" A complex valued function f(z) is differentiable at  $z_0$  if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. The function f is said to be differentiable on D if it is differentiable at every points of D." (Gamelin, 2001).

#### **Definition 1.19** (analyticity)

"Let D be an open set in  $\mathbb{C}$  and f is a complex valued function on D. The function f is analytic or holomorphic at the point  $z_0 \in D$  if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

converge to a limit. The function f is said to be holomorphic or analytic on D if f is holomorphic or analytic at every point of D." (Stein and Shakarchi, 2003).

#### **Definition 1.20** (*Cauchy-Riemann equations*)

Properties of real and imaginary parts of the differentiable function f(z) = u(x, y) + iv(x, y) will be deduced by specializing the mode of approach. Firstly, we assume that *h* approaches to zero along the real axis

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h}$$
$$= \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h}.$$

Since *f* is differentiable at z = x + iy, then

$$\lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} \text{ and } \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h}$$

must be exists. And also we know that

$$\lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} = \frac{\partial u}{\partial x}$$

and

$$\lim_{h\to 0}\frac{v(x+h,y)-v(x,y)}{h}=\frac{\partial v}{\partial x}.$$

Thus we have

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \coloneqq \frac{\partial f}{\partial x}$$
(1.19)

Now we assume that h approach 0 along the imaginary axis. Then for  $h = ih_2$ ,  $h_2$  is real we have

$$\lim_{h_2 \to 0} \frac{f(z+ih_2) - f(z)}{ih_2} = \lim_{h_2 \to 0} \frac{u(x, y+h_2) - u(x, y)}{ih_2} + i \lim_{h_2 \to 0} \frac{v(x, y+h_2) - v(x, y)}{ih_2}$$
$$\lim_{h_2 \to 0} \frac{f(z+ih_2) - f(z)}{ih_2} = \frac{1}{i} \lim_{h_2 \to 0} \frac{u(x, y+h_2) - u(x, y)}{h_2} + \lim_{h_2 \to 0} \frac{v(x, y+h_2) - v(x, y)}{h_2}$$

$$=\frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \coloneqq -i\frac{\partial f}{\partial y}.$$
 (1.20)

**Example 1.21** Show that the given complex function  $f(z) = z^2$  satisfies Cauchy Riemann equations.

Solution.

$$f(z) = z^{2} = (x + iy)^{2} = x^{2} - y^{2} + 2xyi$$
$$u(x, y) = x^{2} - y^{2} \text{ and } v(x, y) = 2xy$$
$$u_{x} = 2x, v_{y} = 2x \implies u_{x} = v_{y}$$
$$u_{y} = -2y, v_{x} = 2y \implies u_{y} = -v_{x}$$
$$\implies \text{satisfies all Cauchy-Riemann equations.}$$

Note that since we talked about complex functions in cartesian form and polar form, now we will talk in terms of z and  $\overline{z}$ . We know that, z = x + iy gives that

$$x = \frac{z + \overline{z}}{2}$$
 and  $y = \frac{z - \overline{z}}{2i}$ 

If we write complex function

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Also we can write

$$f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left[ \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right] = \frac{1}{2} \left[ (u_x - v_y) + i (v_x + u_y) \right]$$

and  $f_{\bar{z}} = 0$  is equivalent to the system

$$f_x = -if_y \text{ or } u_x = v_y, \ u_y = -v_x.$$

Thus we have following theorem.

**Theorem 1.22** "A necessary condition for a function f to be differentiable at a point a is that satisfies the equation  $f_{\bar{z}} = 0$  at a." (Ponnusamy, 2005).

Now, we will see, we can write CR-equations in polar form. Let f be differentiable at a point z. Since we can write any complex function in polar form, we have

$$f(r,\theta) = u(r,\theta) + iv(r,\theta)$$
  
$$z = re^{i\theta} \text{ and also } x = r\cos\theta, y = r\sin\theta.$$
(1.22)

We have from Chain rule;

$$\frac{du}{dr} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \,. \tag{1.23}$$

Now in 1.22 we take the derivatives x and y with respect to r and put in 1.23 and we have

$$\frac{du}{dr} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \sin\theta.$$
(1.24)

Similarly, we have from Chain rule;

$$\frac{du}{d\theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}.$$
(1.25)

Now in 1.22 we take derivative x and y with respect to  $\theta$  and put in 1.25 and we have;

$$\frac{du}{d\theta} = \frac{\partial u}{\partial x} \cdot (-r\sin\theta) + \frac{\partial u}{\partial y} \cdot (r\cos\theta).$$
(1.26)

Similarly, again we have from chain rule;

$$\frac{dv}{dr} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}.$$
(1.27)

Now, we already have derivative of x and y with respect to  $\theta$ . We can put in 1.27 and we have;

$$\frac{dv}{dr} = \frac{\partial v}{\partial x} \cdot \cos\theta + \frac{\partial v}{\partial y} \cdot \sin\theta.$$
(1.28)

Finally we have

$$\frac{dv}{d\theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta}.$$
(1.29)

Now we already have derivative of x and y with respect to  $\theta$ . We can put in 1.29

$$\frac{dv}{d\theta} = \frac{\partial v}{\partial x}.\left(-r\sin\theta\right) + \frac{\partial v}{\partial y}.\left(r\cos\theta\right).$$
(1.30)

Now, we know that CR-equations in cartesian form, that is,

$$u_x = v_y, u_y = -v_x \tag{1.31}$$

If we put 1.31 in 1.24, 1.26, 1.28 and 1.30 we have,

$$\frac{dv}{d\theta} = r\left(-\frac{\partial v}{\partial x}.(\sin\theta) + \frac{\partial v}{\partial y}.(r\cos\theta)\right) = r\left(\frac{\partial u}{\partial y}(\sin\theta) + \frac{\partial u}{\partial x}(\cos\theta)\right) = r\frac{\partial u}{\partial r}$$
(1.32)

$$\frac{du}{d\theta} = r\left(-\frac{\partial v}{\partial y}.(\sin\theta) - \frac{\partial u}{\partial x}.(\cos\theta)\right) = -r\left(\frac{\partial v}{\partial y}(\sin\theta) + \frac{\partial v}{\partial x}(\cos\theta)\right) = -r\frac{\partial v}{\partial r} \quad (1.33)$$

and in 1.32 and 1.33 are called CR-equations in polar form.

**Example 1.23** Show that the given complex function  $f(r, \theta) = 2r\cos\theta + i2r\sin\theta$  satisfies *Cauchy-Riemann equations in polar coordinates.* 

Solution.

$$u(r,\theta) = 2r\cos\theta$$
 and  $v(r,\theta) = 2r\sin\theta$   
 $u_r = 2\cos\theta, v_{\theta} = 2r\cos\theta \implies ru_r = v_{\theta}$   
 $u_{\theta} = -2r\sin\theta, v_r = 2\sin\theta \implies -rv_r = u_{\theta}$   
 $\implies$  satisfies Cauchy-Riemann equations in polar coordinates.

CR-equations are necessary but not sufficient for derivative of a complex function. If any complex function differentiable then f must be satisfy CR-equations but it is conversely not true. We have following useful example.

**Example 1.24** Determine  $f(z) = \begin{cases} \frac{xy}{x^2+y^2}; & \text{if } z \neq 0 \\ 0 & \text{; if } z = 0 \end{cases}$  is differentiable or not at z = 0.

**Solution.** We can see that f(z) = 0 at z = 0.

$$u_x = v_y = 0$$
 and  $u_y = -v_x = 0$ 

satisfies CR-equations but on the line y = mx ( $m \neq 0$ )

$$f'(0) = \lim_{h \to 0} \frac{f(h + imh) - f(0)}{h + imh} = \lim_{h \to 0} \frac{\frac{h \cdot mh}{h^2 + m^2 h^2}}{h + imh} = \infty$$

.

doesn't exists. Hence f is not differentiable at z = 0.

**Theorem 1.25** "Let f(z) = u(x, y) + iv(x, y) be defined in a domain D, and let u and v have continuous partial derivatives that satisfy the Cauchy Riemann equations  $u_x = v_y = 0$  and  $u_y = -v_x = 0$  for all points in D. Then f(z) is analytic in a domain D." (Ponnusamy, 2005).

#### **Definition 1.26** (singular point)

" If a function f fails to be analytic at a point  $z_0$  but is analytic at some point in every neighborhood of  $z_0$ , then  $z_0$  is called a singular point of f." (Churchill and Brown , 1996).

#### **Definition 1.27** (*residue*)

" If a complex function f has an isolated singularity at a point  $z_0$ , then f has a Laurent series representation

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

which converges for all z near  $z_0$ . More precisely, the representation is valid in some deleted neighborhood of  $z_0$  or punctured open disk  $0 < |z - z_0| < R$ . The coefficient  $a_{-1}$  of  $\frac{1}{z - z_0}$  in

the Laurent series given above is called the residue of the function." (Zill and Shanahan, 1940).

#### **Definition 1.28** (discrete or isolated)

" A singular point  $z_0$  is said to be isolated if , in addition, there is a deleted neighborhood  $0 < |z - z_0| < \varepsilon$  of  $z_0$  throughout which f is analytic." (Churchill and Brown, 1996).

#### **Definition 1.29** (Taylor expansion)

" Let f is analytic function throughout an open disc  $|z - z_0| < R_0$ , centered at  $z_0$  and with radius  $R_0$ , then at each point z in that disc, f(z) has a series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0) \cdot (z - z_0)^n}{n!}$$

is called a Taylor expansion of f." (Churchill and Brown, 1990).

#### **Definition 1.30** (zero and pole of holomorphic function)

"Let f is analytic in an open domain D. If for  $z \in D$ , f(z) = 0 then z is called zero of holomorphic function f. We say that  $z_0$  is a pole of f and the smallest  $n \in \mathbb{N}$  such that

$$(z-z_0)^n f(z)$$

is bounded near  $z_0$  is called the order of the pole at  $z_0$ ." (*Zill and Shanahan*, 1940).

#### **Definition 1.31** (meromorphic function)

" A function f is said to be meromorphic in a domain D if it is analytic throughout D except for poles." (Churchill and Brown, 2009).

#### **Definition 1.32** (sequences and subsequences)

A mapping  $\mathbb{N} \to \mathbb{C}$ ,  $n \to z_n$  is called a sequence. Suppose  $\{z_n\}$  is a sequence of points in  $\mathbb{C}$  and that  $\{n_k\}$  is a strictly increasing sequence of natural numbers. Then the sequence  $\{z_{n_k}\}$  is called a subsequence of  $\{z_n\}$ .

#### **Definition 1.33** (converge uniformly)

Consider an open set  $U \subset \mathbb{C}$  and a sequence of functions  $\{f_j\}$  converge uniformly to  $f_0$  if  $\forall \mathcal{E} > 0$  there exist  $J \in \mathbb{N} \forall j > J$  such that  $\forall z \in U |f_j(z) - f_0(z)| < \mathcal{E}$ . Note that this J must work for all  $z \in U$ , it depends only on  $\mathcal{E}$ .

#### **Definition 1.34** (converge normally)

We say that  $\{f_j\}$  converge normally to  $f_0$  if for each compact  $K \subset U$  and  $\forall \mathcal{E} > 0$  there exist  $J \in \mathbb{N} \ \forall j > J$  such that  $\forall z \in K |f_j(z) - f_0(z)| < \mathcal{E}$ .

#### **CHAPTER 2**

#### **CONFORMAL MAPPING AND MOBIUS TRANSFORMATIONS**

The goal of this chapter is, given some basic definition, theorem and properties about conformal mapping and Mobius transformations. We are going to see how the last chapter is important.

#### 2.1 Conformal Mapping

#### **Definition 2.1** (homeomorphism)

A function  $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$  is a homeomorphism if f is a bijection and if both f and  $f^{-1}$  are continuous.

#### **Definition 2.2** (automorphism)

The set of all conformal bijections  $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$  are defined by  $Aut(\overline{\mathbb{C}})$ .

**Theorem 2.3** If f is an analytic function in a domain D containing  $z_0$ , and if  $f'(z_0) \neq 0$ , then w = f(z) is a conformal mapping at  $z_0$ .

**Proof.** Assume that f is analytic function in a domain D containing  $z_0$  and  $f'(z_0) \neq 0$ . Let  $C_1$  and  $C_2$  be two smooth curves in D and  $C_1$  parametrized by  $z_1(t)$  and  $C_2$  parametrized by  $z_2(t)$ , respectively, with  $z_1(t_0) = z_2(t_0) = z_0$ . Let w = f(z) maps the curves  $C_1$  and  $C_2$  onto the curves  $\psi_1$  and  $\psi_2$ . Let  $\theta$  is angle between  $C_1$  and  $C_2$  and  $\phi$  is angle between image of  $C_1$  and  $C_2$ , respectively,  $\psi_1$  and  $\psi_2$ . Now we have to show that  $\theta = \phi$ .  $\psi_1$  and  $\psi_2$  are parametrized by  $w_1(t) = f(z_1(t))$  and  $w_2(t) = f(z_2(t))$ . Now, we compute the tangent

vectors  $w_1'$  and  $w_2'$  to  $\psi_1$  and  $\psi_2$  at  $f(z_0) = f(z_1(t_0)) = f(z_2(t_0))$ . We can use the chain rule;

$$w_{1}' = w_{1}'(t_{0}) = f'(z_{1}(t_{0})) \cdot z_{1}'(t_{0}) = f'(z_{0}) \cdot z_{1}'$$
$$w_{2}' = w_{2}'(t_{0}) = f'(z_{2}(t_{0})) \cdot z_{2}'(t_{0}) = f'(z_{0}) \cdot z_{2}'$$

We have already  $C_1$  and  $C_2$  are smooth and both  $z'_1$  and  $z'_2$  are nonzero. By hypothesis we have  $f'(z_0) \neq 0$ . Hence,

$$\phi = \arg(w_{2}') - \arg(w_{1}') = \arg(f'(z_{0}), z_{2}') - \arg(f'(z_{0}), z_{1}')$$
  

$$\phi = \arg(f'(z_{0})) + \arg(z_{2}') - \arg(f'(z_{0})) - \arg(z_{1}')$$
  

$$\phi = \arg(z_{2}') - \arg(z_{1}')$$
  

$$\phi = \theta.$$

Therefore we completed this proof.  $\Box$ 

**Example 2.4** Find all points where the mapping  $f(z) = \cos z$  is conformal.

**Solution.** Firstly we have f(z) = cosz. Now, if we take the first derivative of the given function, we have,

f'(z) = -sinz

 $-sinz = 0 \iff z = n\pi$  where  $n = 0 \pm 1, \pm 2, ...$ 

f(z) is conformal mapping at z for all  $z \neq n\pi$  where  $n = 0 \pm 1, \pm 2, ...$ 

**Example 2.5** Determine where the complex mapping f(z) = z.  $e^{\frac{z^3}{3}+5}$  is conformal.

**Solution.** Firstly we have  $f(z) = z \cdot e^{\frac{z^3}{3} + 5}$ . Now we take the first derivative of the given function with respect to *z*, we have,

$$f'(z) = e^{\frac{z^3}{3} + 5} + e^{\frac{z^3}{3} + 5} \cdot z \cdot z^2$$
  
=  $e^{\frac{z^3}{3} + 5} (1 + z^3)$   
=  $e^{\frac{z^3}{3} + 5} (1 + z^3) = 0 \iff (1 + z^3) = 0$   
=  $z^3 = -1 \implies z = (-1)^{\frac{1}{3}}$ 

Now, we have three roots of *z*, that is,

$$z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$
$$z_1 = \cos\pi + i\sin\pi$$
$$z_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

Then we have,

$$z_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_1 = -1$$
$$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

f is conformal mapping at z for all  $z \neq \frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i.$ 

Example 2.6 Find the image

$$H = \{z \in \mathbb{C} : Imz > 0\}$$

under

$$f(z) = \frac{i-z}{i+z}.$$

Solution. Let we define

$$w = \frac{i-z}{i+z}$$

This means that

$$iw + wz = i - z$$
$$wz + z = i - iw$$
$$z(1 + w) = i(1 - w)$$
$$z = i\frac{1 - w}{1 + w}$$

Let w = u + iv

$$z = \frac{i - ui + v}{1 + u + iv} \cdot \frac{1 + u - iv}{1 + u - iv}$$
$$z = \frac{i + ui + v - ui - ui^2 - uv + v + uv - iv^2}{(1 + u)^2 + v^2}$$

since 
$$Imz > 0$$

$$\frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0$$
$$1 - u^2 - v^2 > 0$$
$$1 > u^2 - v^2.$$

Hence the image of H under f is unit disc.

Example 2.7 Find the image of

$$K = \{z = x + iy : |z| < 1 \text{ and } y > 0\}$$

under

$$f(z) = \frac{1+z}{1-z}.$$

Solution. Let we define

$$w = \frac{1+z}{1-z}$$

This means that

$$w - zw = 1 + z$$
$$w - 1 = z(w + 1)$$

$$z = \frac{w-1}{w+1}.$$

Since |z| < 1 we have

$$\begin{aligned} \left|\frac{w-1}{w+1}\right| < 1 \\ |w-1| < |w| + 1 \\ (w-1). (\overline{w} - 1) < (w+1). (\overline{w} + 1) \\ |w|^2 - w - \overline{w} + 1 < |w|^2 + w + \overline{w} + 1 \\ 0 < w + \overline{w} \\ 0 < u. \end{aligned}$$

Let w = u + iv. Since y > 0, we have
$$z = \frac{w-1}{w+1} = \frac{u+iv-1}{u+iv+1} \cdot \frac{u+1-iv}{u+1-iv}$$
$$z = \frac{u^2+u-iuv+v^2-u-1+iv}{(u+1)^2+v^2}$$
$$Imz = \frac{2v}{(u+1)^2+v^2} > 0 \implies v > 0.$$

Hence, the image of K under f is first quadrant of  $\mathbb{C}$  plane.

## **2.2 Mobius Transformations**

In this section we will talk about mobius transformations and their basic properties. Mobius transformations are very important because this transformations are basic of conformal mapping.

**Definition 2.8** Let *a*, *b*, *c*, *d*  $\in \mathbb{C}$ , *ad*  $-bc \neq 0$  and  $T: \mathbb{C}_{\infty} \to \mathbb{C}_{\infty}$  then

$$T(z) = \frac{az+b}{cz+d}$$
(2.1)

and

$$T(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$
(2.2)

is called Mobius transformation.

2.1 is called automorphism and 2.2 is called anti-automorphism. We will discuss some basic properties about automorphism and anti-automorphism of  $\mathbb{C}_{\infty}$ .

- Every T anti-automorphism, it can be define composition of automorphism and complex conjugate transformation of C∞.
- The composition of two anti-automorphisms is automorphism.
- A composition of an automorphism and an anti-automorphism again antiautomorphism.

• The automorphism keep constant of the magnitude of angles but inverse the direction.

Now, in this part when we investigate mobius transformations, we will consider mobius transformation in form 2.1.

 $\Delta = ad - bc$  is called determinant of mobius transformation T. We must take  $ad - bc \neq 0$  because in definition of mobius transformation gives us

$$T(z) - T(w) = \frac{(ad - bc).(z - w)}{(cz + d).(cw + d)}$$
(2.3)

If  $ad - bc \neq 0$  then we take T(z) = T(w). This means that T will be constant function. In the same time if  $ad - bc \neq 0$  then T is one-to-one function. Mobius transformation independent to coefficient a, b, c, d. If  $\lambda \in \mathbb{C} - \{0\}$  then,  $\lambda a, \lambda b, \lambda c, \lambda d$  gives us

$$T(z) = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{\lambda (az + b)}{\lambda (cz + d)} = \frac{az + b}{cz + d}$$

in this manner again to get T. Therefore, we multiply numerator and denumerator of 2.1 with

$$\lambda = \frac{1}{\pm \sqrt{ad - bc}}$$

Then we have

$$T(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{\pm\sqrt{ad-bc}}z + \frac{b}{\pm\sqrt{ad-bc}}}{\frac{c}{\pm\sqrt{ad-bc}}z + \frac{d}{\pm\sqrt{ad-bc}}}$$
(2.4)

the determinant of 2.4 is

$$\left(\frac{a}{\pm\sqrt{ad-bc}},\frac{d}{\pm\sqrt{ad-bc}}\right) - \left(\frac{b}{\pm\sqrt{ad-bc}}\frac{c}{\pm\sqrt{ad-bc}}\right) = \frac{ad-bc}{ad-bc} = 1$$

Consequently, we can take instead of  $\Delta = ad - bc \neq 0$ ,  $\Delta = ad - bc = 1$ .

**Definition 2.9** Let  $T(z) = \frac{az+b}{cz+d}$ , a, b, c,  $d \in \mathbb{C}$ ,  $\Delta = ad - bc \neq 0$  then  $T(\infty) = \frac{a}{c}$  and  $T\left(-\frac{d}{c}\right) = \infty$ .  $\infty$ . If c = 0 then  $T(\infty) = \infty$ . **Theorem 2.10** *Every mobius transformation from*  $\mathbb{C}_{\infty}$  *to*  $\mathbb{C}_{\infty}$  *are one-to-one and onto.* 

For identity transformation I(z) = z,  $a = d \neq 0$  and b = c = 0 then we can write

$$I(z) = z = \frac{az + 0}{0z + d}$$
(2.5)

Hence I(z) = z identity transformation is any mobius transformation. Addition to these the inverse mobius transformation of T is

$$T^{-1}(z) = \frac{dz - b}{-cz + a}$$
(2.6)

and clearly we can say that  $T^{-1}$  transformation is also mobius transformation. Therefore we can give the following theorem.

Theorem 2.11 The set

$$M = \left\{ T: \mathbb{C}_{\infty} \to \mathbb{C}_{\infty}: T(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad-bc \neq 0 \right\}$$

is a group with respect to composition.

## **Definition 2.12** (*similarity transformation*)

 $U(z) = az + b, a, b \in \mathbb{C}, a \neq 0$  is called similarity transformation.

Let  $H = \{U: \mathbb{C} \to \mathbb{C}: U(z) = az + b, a, b \in \mathbb{C}, a \neq 0\}$  is the set of similarity transformations. Since  $I(z) \in H$  then  $H \neq 0$ . Addition to this we can write

$$U(z) = az + b = \frac{az + b}{0z + 1}.$$

Then  $H \subseteq M$ .

Specially H is closed with respect to composition. Then  $H \leq M$ .

**Definition 2.13** (general linear group)

 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2x2}, det A = ad - bc \neq 0 \text{ is called General Linear Group and we can show in form } GL(2, \mathbb{C}).$ 

Now, we can define relation between Mobius transformation and matrices

$$\Phi: GL(2, \mathbb{C}) \to M$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to T, T(z) = \frac{az+b}{cz+d}, ad - bc \neq 0.$$

**Theorem 2.14**  $\Phi$ :  $GL(2, \mathbb{C}) \to M$  is a homeomorphism. Separately since  $\Phi$ :  $GL(2, \mathbb{C}) \to M$  is onto,  $\Phi$  is called epimorphism and its kernel defined as

$$K = Ker\Phi = \{A \in GL(2, \mathbb{C}) : \Phi(A) = z\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{az+b}{cz+d} = z \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = d \neq 0, b = c = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right\}$$

If we take  $a = \lambda$  then

$$K = Ker\Phi = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \neq 0 \right\}$$
$$= \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \lambda \neq 0 \right\}$$
$$= \{\lambda I : \lambda \neq 0\}.$$

This is proof of the following theorem.

**Theorem 2.15** Let I is an identity matrix in form 2x2. Then kernel of homeomorphism  $\Phi$ ,

$$K = Ker\Phi = \{\lambda I: \lambda \in \mathbb{C}\}$$
(2.7)

 $A, B \in GL(2, \mathbb{C})$  two matrices defined as the same mobius transformation in  $\mathbb{C}_{\infty}$  if and only if  $A^{-1}B \in K$  that is to say for  $\lambda \neq 0$   $B = \lambda A$ .

A necessary of first isomorphism theorem, for  $\Phi$  transformation  $\Phi$ : GL(2,  $\mathbb{C}$ )  $\rightarrow M$  then

$$M \cong GL(2,\mathbb{C}) \setminus K = GL(2,\mathbb{C}) \setminus \{\lambda I \colon \lambda \neq 0\}$$

 $GL(2, \mathbb{C}) \setminus K$  is called Projective General Linear Group and we can define this  $PGL(2, \mathbb{C})$ . Because of this, it can be  $M \cong PGL(2, \mathbb{C})$ . For every  $A, B \in GL(2, \mathbb{C})$  since det(A, B) = det(A).det(B) then

det: 
$$GL(2, \mathbb{C}) \rightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

is called homeomorphism.

$$Ker(\det) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A \in GL(2, \mathbb{C}), \det A = 1 \right\} = SL(2, \mathbb{C})$$

the kernel set of function of determinant is called special linear group. Because of determinant is onto, necessary of first isomorphism theorem, we have

$$GL(2,\mathbb{C})\setminus SL(2,\mathbb{C})\cong\mathbb{C}^*$$

If  $B \in GL(2, \mathbb{C})$  then  $\lambda^2 = \det B$  and for  $A \in SL(2, \mathbb{C}), B = \lambda A$ . Since  $\Phi(B) = A$  it can be show this for every mobius transformation in  $\mathbb{C}_{\infty}$  write in form

$$T(z) = \frac{az+b}{cz+d}; ad-bc = 1.$$

Additionally, under  $\Phi$ ,  $SL(2, \mathbb{C})$  in  $PGL(2, \mathbb{C})$ . This is proof of following theorem.

**Theorem 2.16**  $M \cong PGL(2, \mathbb{C}) = PSL(2, \mathbb{C}).$ 

#### **2.2.1 Special types of mobius transformations**

## **Definition 2.17** (translation)

 $T_t(z) = z + t, t \in \mathbb{C}$  is called translation. The transformation translation keeps constant of infinity and it makes a movement on  $\mathbb{C}$  plane.

#### **Definition 2.18** (rotation)

 $R_{\theta}(z) = e^{i\theta}z$ ,  $\theta \in \mathbb{R}$  is called rotation. If  $\theta > 0$  then rotation is counterclockwise, if  $\theta < 0$  then rotation is clockwise, The transformation rotation makes 0 and infinity constant.

**Definition 2.19** (inversion)

 $J(z) = \frac{1}{z}$  is called inversion.

## **Definition 2.20** (magnification)

 $M_r(z) = rz$ ,  $r \neq 0$ ,  $r \in \mathbb{C}$  is called magnification.

The transformation magnification makes 0 and infinity constant. The transformation magnification affects as similarity on  $\mathbb{C}$  plane and longtitude is expanded or shrinked by the r multiplier.

**Theorem 2.21** *Every mobius transformation can write composition of translation, magnification, rotation and inversion.* 

Proof. Every mobius transformation knows in form

$$T(z) = \frac{az+b}{cz+d}; ad-bc = 1$$

If c = 0 then  $T(z) = \frac{az+b}{d}$ ,  $a, d \neq 0$ . Let  $\frac{a}{d} = re^{i\theta}$  and  $\frac{b}{d} = t$ . Then  $T(z) = re^{i\theta}z + t$ .

therefore, we have

$$T(z) = T_t(z)oM_r(z)oR_{\theta}(z)$$

This transformation makes infinity constant. If  $c \neq 0$  then we can use ad - bc = 1, so we can write

$$T(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c^2(z+\frac{d}{c})}$$

Because of there we can write

$$T(z) = T_{\frac{a}{c}o} J_{-c(cz+d)} = \left(T_{\frac{a}{c}o} Jo T_{-cd}o M_{-c^2}\right)(z). \square$$

#### **Definition 2.22** (Euclidean circle)

The set of points in  $\mathbb{C}$  satisfy  $|z - z_0| = r$ , r > 0 are called Euclidean circle.

The set of points in  $\mathbb{C}$  satisfies |z - a| = |z - b|,  $a \neq b$  are called Euclidean line. Euclidean circle or  $L \cup \{\infty\}$  are called circle in  $\mathbb{C}_{\infty}$ . In there *L* is any Euclidean line.

**Theorem 2.23** If a circle in  $\mathbb{C}_{\infty}$  then the image of C under mobius transformation T, T(C) is a circle in  $\mathbb{C}_{\infty}$ .

**Proof.** We know that general equation of circle is

$$Az\overline{z} + Bz + \overline{Bz} + D = 0; A, D \in \mathbb{R}.$$

the transformation in form

$$T(z) = \frac{az+b}{cz+d}; a, b, c, d \in \mathbb{C}, ad-bc \neq 0$$

We have to show that T(C) is a circle.

$$T(z) = z' = \frac{az+b}{cz+d}$$
$$z'(cz+d) = az+b$$
$$czz'+dz' = az+b$$
$$czz'-az = b-dz'$$
$$z = \frac{b-dz'}{cz'-a}.$$

since  $z = \frac{b - dz'}{cz' - a}$ , clearly we have

$$\bar{z} = \frac{-\overline{dz'} + \bar{b}}{\overline{cz'} - a}$$

If we put z and  $\bar{z}$  in general equation of a circle, we have

$$A\left(\frac{b-dz'}{cz'-a}\right) \cdot \left(\frac{-\overline{dz'}+\overline{b}}{\overline{cz'}-\overline{a}}\right) + B\left(\frac{b-dz'}{cz'-a}\right) + \overline{B}\left(\frac{-\overline{dz'}+\overline{b}}{\overline{cz'}-\overline{a}}\right) + D = 0$$

Clearly we have

$$\begin{bmatrix} Ad\bar{d} - B\bar{c}d - \bar{B}c\bar{d} + Dc\bar{c} \end{bmatrix} z'\bar{z'} + \begin{bmatrix} -A\bar{b}d + B\bar{a}d + \bar{B}\bar{b}c - D\bar{a}c \end{bmatrix} z' + \begin{bmatrix} -A\bar{b}d + Bb\bar{c} + \bar{B}a\bar{d} - Da\bar{c} \end{bmatrix} \bar{z'} + \begin{bmatrix} Ab\bar{b} - B\bar{a}b + \bar{B}a\bar{b} + Da\bar{a} \end{bmatrix} = 0$$

In this section the coefficient of  $z'\overline{z'}$  is real. Because  $d\overline{d}$  and  $c\overline{c}$  are real. Similarly the coefficient of z' and  $\overline{z'}$  are real. Because of there, image of circle under T(z) again circle.

## 2.2.2 Reflection

Let C is a circle in  $\mathbb{C}_{\infty}$ , which is

$$az\overline{z} + bz + bz + c = 0 a, c \in \mathbb{R} and b \in \mathbb{C}$$

If  $a \neq 0$  then C is an Euclidean circle in  $\mathbb{C}$ .

Let the center of C is w and radius is r.  $\forall z \in \mathbb{C} - \{w\}, \exists z^*$  on the line which include w and z, such that

$$|z - w| = |z^* - w| = r$$

 $z, z^*$  and w are on the same line. There,  $z^*$  is called reflection of z with respect to C.

#### **Definition 2.24** (reflection transformation)

Let w is a center and r is radius of C.  $T_{w,r}(z) = z^*$  is called reflection transformation.

From this definition we can see the point z goes to infinity, when w approaches to  $z^*$ . Similarly, when w approaches to z,  $z^*$  goes to infinity. Therefore, we define  $T_{w,r}(w) = \infty$ and  $T_{w,r}(\infty) = w$ . And  $T_{w,r}(z)$  is continuous and magnificate to any transformation in  $\mathbb{C}_{\infty}$ in the same time. Seperately,

$$T_{w,r}^{2}(z) = (T_{w,r}oT_{w,r})(z) = T_{w,r}(z^{*}) = z = I(z)$$

If  $T_{w,r}(z) = I(z)$  then  $z \in \mathbb{C}$ . In other words, reflection of any points on C is itself. If  $z \neq w$ , then

$$|(\bar{z} - \bar{w}).(z^* - w)| = |z - w|.|z^* - w| = r^2$$

Since (z - w) and  $(z^* - w)$  on the same line

$$\arg(z-w) = \arg(z^*-w)$$

then

$$\arg(\overline{z} - \overline{w}) = -\arg(z^* - w)$$

therefore

$$T_{w,r}(z) = z^* = \frac{r^2}{\overline{z} - \overline{w}} + w.$$

**Theorem 2.25** If z and  $z^*$  are reflect points with respect to C then the image of z and  $z^*$ under  $T(z) = \frac{az+b}{cz+d}$ , which are w and w<sup>\*</sup> are reflect points with respect to C.

# CHAPTER 3 HOLOMORPHIC FUNCTIONS

This chapter is dedicated to give some definition, lemma and some theorems about holomorphic functions, Fundamental Theorem for Contour Integral, Cauchy's Integral Theorem and its applications, such as Cauchy's Theorem or Cauchy-Goursat Theorem, Bounding Theorem, Taylor's Theorem, Cauchy's Integral Formula, Cauchy's Integral Formula for Derivatives, Cauchy's Inequality, Cauchy Residue Theorem, Argument Principle Theorem, Rouche's Theorem, Open Mapping Theorem, Maximum Modulus Theorem, Maximum Modulus Principle, Montel's Theorem and Hurwitz's Theorem, as well.

#### **Definition 3.1** (*antibiholomorphic*)

If the analytic function preserve angle but not preserve its orientation then this function is called antibiholomorphic.

**Example 3.2** We have  $f(z) = re^{i\theta}$  where  $0 \le \theta \le 2\pi$ . We assume that the angle  $\theta$  has positive orientation. For conjugate of f is  $\overline{f(z)} = re^{-i\theta}$ . And its angle is  $-\theta$ . Also we can say, f preserve magnitude of angle but not preserve is orientation. Then f is antibiholomorphic function.

Now, we are going to start useful lemma about holomorphic functions.

**Lemma 3.3** Suppose that  $U \subset \mathbb{C}$  is simply connected and open that  $f: U \to \mathbb{C}$  is holomorphic and nowhere vanishing. Then there exists a holomorphic function  $g: U \to \mathbb{C} - \{0\}$  such that

$$[g(z)]^2 = f(z) \qquad \forall z \in U.$$

**Proof.** Let *U* is simply connected and open subset of  $\mathbb{C}$ . Since *f* is nonvanishing, we can say that  $\frac{f'(z)}{f(z)}$  is also holomorphic. Then there exists a holomorphic function  $h: U \to \mathbb{C}$  such that

$$h(z) = \ln[f(z)] \tag{3.1}$$

If we take derivative 3.1 with respect to z then we have

$$h'(z) = \frac{f'(z)}{f(z)}$$
(3.2)

Since  $\frac{f'(z)}{f(z)}$  holomorphic, then h'(z) must be holomorphic, as well. Now we choose  $z_1 \in U$ . If we put  $z_1$  in 3.1, we have

$$h(z_1) = \ln[f(z_1)]$$

Clearly, we have

$$e^{h(z_1)} = f(z_1) \tag{3.3}$$

Now,  $\forall z \in U$ , from 3.2

$$[f(z).e^{-h(z)}]' = f'(z).e^{-h(z)} - f(z).h'(z).e^{-h(z)}$$
$$= f'(z).e^{-h(z)} - f(z).\frac{f'(z)}{f(z)}.e^{-h(z)}$$
$$= e^{-h(z)}[f'(z) - f'(z)]$$
$$= 0.$$

 $\Rightarrow$   $f'(z).e^{-h(z)}$  is constant. Now, from 3.3 we have

$$f(z_1) \cdot e^{-h(z)} = 1$$

Hence we have

$$f(z) = e^{h(z)} \ \forall z \in U.$$

Finally, we have

$$g(z) = e^{\frac{h(z)}{2}} \tag{3.4}$$

If we take square of 3.4

$$[g(z)]^{2} = \left[e^{\frac{h(z)}{2}}\right]^{2} = e^{h(z)} = f(z)$$

## 3.1 Cauchy's Theorem and Its Applications

Most of powerful theorems proved in this section. Importance of Cauchy's Theorem lies in its applications. Addition to these, there is a such a good relationship between the different theorems.

First we will talk about some famous theorem which are Fundamental Theorem of Contour Integral and Green's Theorem. Because consequence of these theorems are very useful for Cauchy Integral Theorem and its applications.

#### **Theorem 3.4** (Fundamental Theorem for Contour Intergral)

Let f is continuous on a domain D and  $F' \equiv f$  in D. Then any C in D with initial point  $z_1$  and terminal point  $z_2$ ,

$$\oint f(z)dz = F(z_2) - F(z_1).$$

**Proof.** Let C is smooth curve and parametrized by z = z(t),  $a \le t \le b$ . Clearly,  $z(a) = z_1$  and  $z(b) = z_2$ . Since  $F'(z) = f(z) \forall z \in D$ ; we have

$$\oint f(z)dz = \int_{a}^{b} f(z(t)) \cdot z'(t)dt = \int_{a}^{b} F'(z(t)) \cdot z'(t)dt$$
(3.5)

We know that

$$\frac{d}{dt}F(z(t)) = F'(z(t)).z'(t)$$
(3.6)

If we put 3.6 in 3.5, we have

$$\oint f(z)dz = \int_a^b \frac{d}{dt} \cdot F(z(t)) \cdot dt$$

$$= F(z(b)) - F(z(a))$$
$$= F(z_2) - F(z_1). \Box$$

**Theorem 3.5** (Cauchy Integral Theorem of Cauchy-Goursat Theorem)

Suppose that a function f is analytic in a simply connected domain D and f' is continuous in

*D.* Then  $\forall C$  in *D* such that

$$\oint f(z)dz = 0.$$

**Proof.** Let f(z) = u(x, y) + iv(x, y). Since f is analytic, f(z) satisfies Cauchy Riemann equations, that is,

$$u_x = v_y \text{ and } u_y = -v_x. \tag{3.7}$$

And also we can write dz = dx + idy. Hence clearly we have

$$\oint f(z)dz = \oint (u + iv). (dx + idy)$$
$$= \oint (udx - vdy) + i \oint (vdx + udy)$$

From Green's theorem we have

$$\oint (udx - vdy) = \iint (-v_x - u_y) dxdy$$

and

$$\oint (vdx + udy) = \iint (u_x - v_y) dxdy$$

This means that

$$\oint f(z)dz = \iint (-v_x - u_y)dxdy + i \iint (u_x - v_y)dxdy$$
(3.8)

If we put 3.7 in 3.8 then we have

$$\oint f(z)dz = \iint (u_y - u_y)dxdy + i \iint (v_y - v_y)dxdy$$
$$\oint f(z)dz = 0 + 0i. \Box$$

Example 3.6 Use 3.5 and evaluate

$$\oint \frac{z-2}{z^2-z} dz$$

where C: |z - i| = 1.

**Solution.** We have two singular points, which are z = 0 and z = 1. And also they lie in C. We can write again the given function in this form, that is,

$$\frac{z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$
$$A(z-1) + Bz = z - 2$$
$$Az - A + Bz = z - 2$$
$$A + B = 1, \quad -A = -2$$
$$A = 2 \text{ and } B = -1.$$

Now, we have

$$\oint \frac{z-2}{z^2-z} dz = \oint \frac{2}{z} dz - \oint \frac{1}{z-1} dz$$
$$= 2.2\pi i - 1.2\pi i$$
$$= 4\pi i - 2\pi i$$
$$= 2\pi i.$$

Example 3.7 Use 3.5 and evaluate

$$\oint \frac{1}{z^2 + 4} dz$$

*where* C: |z - i| = 4.

**Solution.** We have two singular points, which are z = -2i and z = 2i. And also they lie in C. We can write again the given function in this form, that is,

 $\frac{1}{z^2 + 4} = \frac{A}{z - 2i} + \frac{B}{z + 2i}$ A(z + 2i) + B(z - 2i) = 1Az + 2Ai + Bz - 2Bi = 1A + B = 0 and 2A - 2B = -i $A = -\frac{i}{4}, B = \frac{i}{4}.$ 

We have

$$\oint \frac{1}{z^2 + 4} dz = -\frac{i}{4} \oint \frac{1}{z - 2i} dz + \frac{i}{4} \oint \frac{1}{z + 2i} dz$$
$$= -\frac{i}{4} \cdot 2\pi i + \frac{i}{4} \cdot 2\pi i$$
$$= 0.$$

**Remark 3.8** If  $a \in \mathbb{C}$  in simple closed C', then for  $k \in \mathbb{Z}$  we have

$$\oint \frac{dz}{(z-a)^k} = \begin{cases} 0 \ ; \quad k \neq 1 \\ 2\pi i \ ; k = 1 \end{cases}$$

Now we will apply the given remark in this following example.

**Example 3.9** Evaluate the given integral

$$I = \oint \left[\frac{5}{z+2} - \frac{1}{(z+2i)^2}\right] dz$$

where C: |z| = 5.

Solution. Firstly, we can separate the given integral

$$\oint \frac{5}{z+2} dz - \oint \frac{1}{(z-2i)^2} dz.$$

We have zeros of the given functions are z = -2 and z = 2i. They are also inside |z| = 5. From remark, we have

$$I = 5.2\pi i - 0 = 10\pi i.$$

Now, we will give some information about Bounding theorem. This is very useful for Cauchy Integral formula.

## **Theorem 3.10** (Bounding Theorem or ML-Inequality)

If h is continuous on a smooth curve C and if  $|h(z)| \le M \forall z$  on C then

$$\left|\oint h(z)dz\right| \le ML$$

where L is length of C.

**Proof.** From triangle inequality;

$$\left|\oint h(z)dz\right| = \left|\sum_{k=1}^{n} h(z_k^*) \cdot \Delta z_k\right| \le \sum_{k=1}^{n} |h(z_k^*)| \cdot |\Delta z_k| \le M \sum_{k=1}^{n} |\Delta z_k|$$
(3.9)

where  $\Delta z_k$  is distance between two points on C. It cannot be greater than length L of C. Hence we have;

$$|\Delta z_k| \le L \tag{3.10}$$

If we put 3.10 in 3.9 we have

$$\left|\oint h(z)dz\right| \leq ML.$$

## Theorem 3.11 (Cauchy Integral Formula)

Let g is a holomorphic in a simply connected domain D and C is any simple closed contour in D. Then

$$g(a) = \frac{1}{2\pi i} \oint \frac{g(z)}{z-a} dz.$$

**Proof.** Suppose that *D* be a simply connected domain, C is a simple closed contour and a interior point of C. Addition to these, let  $C_1$  be a circle at a,  $C_1$  lies in C. We can write

$$\int_C \frac{g(z)}{z-a} dz = \int_{C_1} \frac{g(z)}{z-a} dz$$

From 3.5, we have

$$\int_{C_1} \frac{g(z)}{z-a} dz = 2\pi i$$

$$\int_{C_1} \frac{g(z)}{z-a} dz = \int_{C_1} \frac{g(z) + g(a) - g(a)}{z-a} dz$$

$$= g(a) \oint \frac{dz}{z-a} + \oint \frac{g(z) - g(a)}{z-a} dz \qquad (3.11)$$

Again from 3.5, we have

$$\oint \frac{dz}{z-a} = 2\pi i \tag{3.12}$$

If we put 3.12 in 3.11, we have

$$\oint \frac{dz}{z-a} = 2\pi i. g(a) + \int_{C_1} \frac{g(z) - g(a)}{z-a} dz$$
(3.13)

Since g is continuous, we have  $\forall E > 0$  there exists  $\delta > 0$  such that |g(z) - g(a)| < Ewhenever  $|z - a| < \delta$ . Now, if we choose for  $C_1$ ,  $|z - a| < \frac{\delta}{2}$ , then from Bounding theorem; we have

$$\left| \int_{C_1} \frac{g(z) - g(a)}{z - a} dz \right| \le \frac{\varepsilon}{\frac{\delta}{2}} \cdot 2\pi \cdot \frac{\delta}{2} = 2\pi \varepsilon$$
(3.14)

This happens only bif the integral is 0. Thus

$$\oint \frac{g(z)}{z-a} dz = 2\pi i g(a) \tag{3.15}$$

Finallyi if we divided 3.15 side by side by  $2\pi i$ , then we have

$$\frac{1}{2\pi i}\oint \frac{g(z)}{z-a}dz = g(a).$$

## Example 3.12 Evaluate

$$\oint \frac{z}{z^2 + 9} dz,$$

where C: |z - i| = 3.

## Solution.

 $z^2 + 9 = 0 \implies z = \pm 3i$  are zeros of the given function. Now, we can see z = 3i inside of C. Hence from Cauchy Integral formula we have;

$$\oint \frac{z}{z^2 + 9} dz = \oint \frac{z}{(z - 3i).(z + 3i)} dz = \oint \frac{z}{z + 3i} dz$$

We can choose  $f(z) = \frac{z}{z+3i}$  is analytic in C.

$$\oint \frac{z}{z^2 + 9} dz = f(3i) \cdot 2\pi i = \frac{3i}{6i} \cdot 2\pi i = \pi i$$

## **Theorem 3.13** (Cauchy's Integral Formula for Derivatives)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for  $z_0$  in C,

$$f^{n}(z_{0}) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_{0})^{n+1}} dz.$$

**Proof.** Proof by induction. We already find for n = 0 in 3.11. Now, we have to find for n = 1. Let C is  $\sigma = \sigma(t)$ . And clearly we have  $d\sigma = \sigma'(t)dt$  for  $a \le t \le b$ . We use 3.11 and write

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\sigma)d\sigma}{\sigma - z} = \frac{1}{2\pi i} \int_{a}^{b} \frac{f(\sigma(t)) \cdot \sigma'(t)dt}{\sigma(t) - z}$$

Let we assume that

$$k(z,t) = \frac{f(\sigma(t)) \cdot \sigma'(t)}{\sigma(t) - z}$$

and derivative of k(z, t) equals to

$$\frac{\partial k}{\partial z} = k_z(z, t) = \frac{f(\sigma(t)) \cdot \sigma'(t)}{\sigma(t) - z}$$

Now we have

$$f'(z) = \frac{1}{2\pi i} \int_{a}^{b} \frac{f(\sigma(t)) \cdot \sigma'(t) d\sigma}{(\sigma(t) - z)^{2}}$$

For n = 1 is true. Now, we assume that for n = k is also true. And finally we have to show that n = k + 1 is true. If we continuous, we will see it is true.  $\Box$ 

**Example 3.14** *Evaluate the given integral* 

$$\oint \frac{e^{2z}}{z^4} dz$$

*On* C: |z| = 1.

**Solution.** We know that z = 0 is singular point of the given function and z = 0 lies in C: |z| = 1.

$$\oint \frac{f(z)}{(z-0)^{3+1}} dz = \oint \frac{e^{2z}}{(z-0)^{3+1}} dz = \frac{2\pi i}{3!} \cdot f^{\prime\prime\prime}(0)$$

We have

$$f(z) = e^{2z}$$
$$f'(z) = 2e^{2z}$$
$$f''(z) = 4e^{2z}$$
$$f'''(z) = 8e^{2z}$$

If we put 0 instead of z, then we have f'''(0) = 8. Then

$$\oint \frac{e^{2z}}{(z-0)^{3+1}} dz = \frac{2\pi i}{3!} \cdot 8 = \frac{8\pi i}{3}$$

**Example 3.15** *Evaluate the given integral* 

$$\oint \frac{z+1}{z^4+2iz^3} dz$$

*where* C: |z| = 1.

**Solution.** We know that z = 0 is pole of order 3 and z = -2i are singular point of the given function and only z = 0 lies in C: |z| = 1. Then we have

$$\oint \frac{z+1}{z^3(z+2i)} dz = \oint \frac{\frac{z+1}{z+2i}}{z^3} dz$$

$$\oint \frac{\frac{z+1}{z+2i}}{z^{2+1}} dz = \frac{2\pi i}{3} f''(0)$$

We have

$$f(z) = \frac{z+1}{z+2i}$$
$$f'(z) = \frac{2i-1}{(z+2i)^2}$$
$$f''(z) = \frac{2-4i}{(z+2i)^3}$$

If we put 0 instead of z.

$$f''(0) = \frac{2+i}{4}$$

then we have

$$\oint \frac{\frac{z+1}{z+2i}}{z^{2+1}} dz = \frac{2\pi i}{3} \cdot \left(\frac{2+i}{4}\right) = \frac{\pi(2i-1)}{6}$$

Now we will talk about Taylor's theorem. Because its application very useful for Cauchy's Inequality.

## **Theorem 3.16** (Taylor's Theorem)

Let f(z) be analytic in a domain D whose boundary is C. If  $z_0$  is a point in D, then f(z) may be expressed as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n$$

and the series converges for  $|z - z_0| < \delta$ , where  $\delta$  is distance from  $z_0$  to the nearest point on *C*.

**Proof.** Suppose that circle  $C_1$ . From Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\rho)d\rho}{\rho - z}$$
(3.16)

Let  $|z - z_0| = r$ 

$$\begin{aligned} r &= |z - z_0| < |\rho - z_0| = m \\ \\ &= \frac{1}{\rho - z} = \frac{1}{\rho - z_0} \cdot \left[ \frac{1}{1 - \frac{z - z_0}{\rho - z_0}} \right] \\ &= \frac{1}{\rho - z_0} \left[ 1 + \frac{z - z_0}{\rho - z_0} + \left( \frac{z - z_0}{\rho - z_0} \right)^2 + \left( \frac{z - z_0}{\rho - z_0} \right)^3 + \dots + \left( \frac{z - z_0}{\rho - z_0} \right)^{n-1} + \frac{\left( \frac{z - z_0}{\rho - z_0} \right)^n}{1 - \frac{z - z_0}{\rho - z_0}} \right] \\ f(z) &= \frac{1}{2\pi i} \oint \frac{f(\rho)}{\rho - z} d\rho + \frac{(z - z_0)}{2\pi i} \oint \frac{f(\rho) d\rho}{(\rho - z)^2} + \dots + \frac{(z - z_0)^{n-1}}{2\pi i} \oint \frac{f(\rho) d\rho}{(\rho - z)^n} + R_n \end{aligned}$$

where

$$R_n = \frac{1}{2\pi i} \oint \left(\frac{z - z_0}{\rho - z_0}\right)^n \frac{f(\rho)}{\rho - z} d\rho$$

But from 3.11 we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots + \frac{f^{n-1}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n$$
$$\lim_{n \to \infty} R_n = 0.$$

Let  $|f(z)| \leq M$  on  $C_1$ . Then we have

$$R_{n} \leq \frac{1}{2\pi} \oint \left| \frac{z - z_{0}}{\rho - z_{0}} \right|^{n} \left| \frac{f(\rho)}{\rho - z} \right| |\rho| \leq \frac{M}{2\pi} \left( \frac{r}{m} \right)^{n} \int \frac{|d\rho|}{|\rho - z|}$$
$$\frac{1}{|\rho - z|} = \frac{1}{|\rho - z_{0}| - |z - z_{0}|} = \frac{1}{m - r}$$
$$R_{n} \leq \frac{M}{2\pi i (m - r)} \left( \frac{r}{m} \right)^{n} \oint |d\rho| = \frac{M \cdot m}{m - r} \left( \frac{r}{m} \right)^{n}$$

Since r < m,

$$\lim_{n \to \infty} \left(\frac{r}{m}\right)^n = 0$$

Hence,

$$\sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n.$$

## **Theorem 3.17** (Cauchy's Inequality)

Let  $U \subset \mathbb{C}$  be open and let  $f: U \to \mathbb{C}$  be a holomorphic function. Suppose that  $p \in U$  and that for some r > 0 we have  $\overline{D(p,r)} \subset U$ . Set  $M = \sup_{z \in \overline{D(p,r)}} |f(z)|$ . Then for  $n \in \mathbb{Z}^+$ , we have;

$$\left|\frac{\partial^n f}{\partial z^n}(p)\right| \le \frac{M.\,n!}{r^n}.$$

**Proof.** Now, we assume that  $\psi: [0,1] \to \partial D(p,r)$  be the (counterclockwise) path around the boundary of the disc D(p,r). For  $n \in \mathbb{Z}^+$ , from Cauchy Integral Formula for derivatives, we have;

$$\frac{\partial^n f}{\partial z^n}(p) = \frac{n!}{2\pi i} \oint \frac{f(\alpha) d\alpha}{(\alpha - p)^{n+1}}$$
(3.17)

Now, we compute

$$\left|\oint \frac{f(\alpha)d\alpha}{(\alpha-p)^{n+1}}\right| = \left|\int_0^1 \frac{f(\psi(t))}{(\psi(t)-p)^{n+1}} \cdot \frac{d\psi}{dt} \cdot dt\right|$$

from triangle inequality;

$$\left| \int_{0}^{1} \frac{f(\psi(t))}{(\psi(t) - p)^{n+1}} \cdot \frac{d\psi}{dt} \cdot dt \right| \leq \int_{0}^{1} \left| \frac{f(\psi(t))}{(\psi(t) - p)^{n+1}} \right| \cdot \left| \frac{d\psi}{dt} \right| \cdot dt$$

$$\leq \sup_{t \in [0,1]} \frac{|f(\psi(t))|}{|\psi(t) - p|^{n+1}} \int_{0}^{1} \left| \frac{d\psi}{dt} \right| \cdot dt$$
(3.18)

The length of path  $\psi$  is  $2\pi r$ . If we put this length in 3.18, then we have

$$\left| \int_{0}^{1} \frac{f(\psi(t))}{(\psi(t) - p)^{n+1}} \cdot \frac{d\psi}{dt} \cdot dt \right| \le \sup_{t \in [0,1]} \frac{|f(\psi(t))|}{|\psi(t) - p|^{n+1}} 2\pi r$$
(3.19)

Now, if we take the absolute of 3.17 then

$$\left|\frac{\partial^n f}{\partial z^n}(p)\right| = \left|\frac{n!}{2\pi i} \oint \frac{f(\alpha) d\alpha}{(\alpha - p)^{n+1}}\right| \le \frac{n!}{2\pi} \frac{\sup_{t \in [0,1]} |f(\psi(t))|}{|\alpha - p|^{n+1}} 2\pi r$$
(3.20)

From assertion we know that

$$\sup_{t \in [0,1]} |f(\psi(t))| = M$$
(3.21)

Finally, if we put 3.21 in 3.20, then we have

$$\left|\frac{\partial^n f}{\partial z^n}(p)\right| \le \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{M \cdot n!}{r^n}$$

## **Theorem 3.18** (Cauchy Residue Theorem)

"Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points  $z_k$  (k = 0, 1, 2, ..., n) inside C, then

$$\oint f(z)dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z).$$

(Churchill and Brown, 2009).

**Example 3.19** Use 3.18 and evaluate the given integral

$$\oint \frac{dz}{(z-1)^2(z-3)}$$

*where* C: |z| = 2.

**Solution.** We can see there only z = 1 lies in |z| = 2. We have

$$\frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2} \cdot \frac{1}{-2+z-1}$$
$$= \frac{1}{(z-1)^2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{1-\frac{z-1}{2}}$$
$$= \frac{1}{(z-1)^2} \left(-\frac{1}{2}\right) \cdot \left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots\right]$$
$$= \frac{-\frac{1}{2}}{(z-1)^2} + \frac{-\frac{1}{4}}{z-1} - \frac{1}{8} - \cdots$$

This means that

$$Res[f(z),1] = \frac{1}{4}$$

Hence

$$\oint f(z)dz = -\frac{1}{4} \cdot 2\pi i = -\frac{\pi i}{2}.$$

Example 3.20 Use 3.18 and evaluate the given integral

$$\oint \frac{dz}{z^2 - 4}$$

where C: |z - 1| = 2.

**Solution.** Only z = 2 lies in C. Then we have

$$\frac{1}{z^2 - 4} = \frac{1}{(z - 2).(z + 2)}$$
$$= \frac{1}{z - 2} \cdot \frac{1}{4 + z - 2}$$
$$= \frac{1}{z - 2} \cdot \left(\frac{1}{4}\right) \cdot \frac{1}{1 + \frac{z - 2}{4}}$$

$$= \frac{1}{z-2} \cdot \left(\frac{1}{4}\right) \cdot \left[1 - \frac{z-2}{4} + \frac{(z-2)^2}{4^2} - \cdots \right]$$
$$= \frac{1/4}{z-2} - \frac{1}{16} + \frac{z-2}{64} - \cdots$$
$$\Rightarrow Res[f(z), 2] = \frac{1}{4}$$
$$\oint f(z)dz = \frac{1}{4} \cdot 2\pi i = \frac{\pi i}{2}.$$

## **3.2 Zeros of Holomorphic Functions**

In this section, we will demonstrate the zero of holomorphic functions which are not identically zero are isolated.

**Theorem 3.21** Let  $U \subset \mathbb{C}$  be connected and open set and let  $f: U \to \mathbb{C}$  be a holomorphic. If f is not identically zero, then the zeros of f are isolated.

**Proof.** Firstly, assume that f is not identically zero. We have two cases:

*Case 1*: Let *f* has no zeros. This is trivial solution.

<u>*Case 2*</u>: Suppose that f has zeros. Let  $z_0$  be a zeros of f. Then

$$x = \min\left\{k \in \mathbb{Z}^+ : \left(\frac{\partial}{\partial z}\right)^n f(z_0) \neq 0\right\}.$$

From assertion we know that f is holomorphic, and also Taylor's theorem, we can say that

$$f(z) = \sum_{j=x}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \cdot \frac{(z-z_0)^j}{j!} \text{ on } D(p,r)$$

Similarly,

$$g(z) = \sum_{j=x}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \cdot \frac{(z-z_0)^{j-x}}{j!}$$
(3.2)

From 3.2 we can say that g is holomorphic on D and so g is continuous on D. From  $\left(\frac{\partial}{\partial z}\right)^j f(z_0) \neq 0$ , we have  $g(z_0) \neq 0$ . Since  $g(z_0) \neq 0$ 

$$\exists \delta > 0, \delta < r \ \forall z \in D(z_0, r)$$
$$|z - z_0| < \delta \implies |g(z) - g(z_0)| < |g(z_0)|$$

We know that g has no zero in  $D(z_0, \delta)$ . Also we have

$$f(z) = g(z) \cdot (z - z_0)^x \ \forall z \in D(z_0, \delta)$$
$$f(z) = 0 \iff z = z_0$$

Hence all zeros of f must be isolated.  $\Box$ 

**Lemma 3.24** Let  $U \subset \mathbb{C}$  be open, and  $q \in U$  with  $\overline{D(q,r)} \subset U$ . Assume that f is a holomorphic function on U which a zero of order n at q and no other zeros in  $\overline{D(p,r)}$ . Then

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = n.$$

Proof. Consider

$$h(z) = \frac{f(z)}{(z-q)^n} = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j} (q) (z-q)^{j-n}$$

since  $h(z) = \frac{f(z)}{(z-q)^n}$ , clearly

$$f(z) = h(z).(z-q)^n$$

where h(z) is an analytic and nonzero at q. Now, if we take derivative of f with respect to z, we have

$$f(z) = h'(z). (z - q)^n + n. h(z). (z - q)^{n-1}$$

Clearly we have;

$$\frac{f'(z)}{f(z)} = \frac{h'(z).(z-q)^n + n.h(z).(z-q)^{n-1}}{h(z).(z-q)^n}$$

$$=\frac{h'(z)}{h(z)}+\frac{n}{z-q}$$

 $\Rightarrow \frac{h'(z)}{h(z)}$  is holomorphic and nonzero on  $\overline{D(q,r)}$ . By Cauchy Integral Theorem; we have

$$\oint \frac{f'(z)}{f(z)} dz = \oint \frac{h'(z)}{h(z)} dz + \oint \frac{n}{z-q} dz$$

clearly,

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = n.$$

#### **Theorem 3.25** (Argument Principle Theorem)

If f is analytic and nonzero at point of a simple closed positively oriented contour C and is meromorphic inside C, then

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = N_o(f) - N_p(f)$$

where  $N_o(f)$  and  $N_p(f)$  are respectively, the number of zeros and poles of f inside C.

**Proof.** Suppose f is an analytic and nonzero function. Assume that

$$G(z) = \frac{f'(z)}{f(z)}$$

Since f is an analytic, G is an also analytic and nonzero there. Consider  $z_0$  inside C that is zero of f of order m. Then we know that f can be written

$$f(z) = (z - z_0)^n . h(z)$$

where h(z) is analytic and nonzero at  $z_0$ . Now, if we take derivative of f with respect to z, we have,

$$f'(z) = m.(z - z_0)^{m-1}.h(z) + (z - z_0)^m.h'(z)$$

From this, we have

$$G(z) = \frac{f'(z)}{f(z)}$$
  
=  $\frac{m.(z - z_0)^{m-1}.h(z) + (z - z_0)^m.h'(z)}{(z - z_0)^m.h(z)}$   
=  $\frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$ 

Since  $\frac{h'(z)}{h(z)}$  is analytic at  $z_0$ , this representation shows that G has a simple pole at  $z_0$  with residue equal to m. Now we have two cases:

<u>*Case 1*</u>: If f has a pole of order k at  $z_p$ , then

$$f(z) = \frac{H(z)}{(z - z_p)^k}$$
(3.23)

where H(z) is analytic at  $z_p$  and  $H(z_p) \neq 0$ . We already have

$$G(z) = \frac{f'(z)}{f(z)}$$

Take the derivative of 3.23 with respect to z

$$f'(z) = \frac{H'(z).(z - z_p)^k - k.H(z).(z - z_p)^{k-1}}{(z - z_p)^{2k}}$$
(3.24)

If we put 3.24 in 3.23 we have,

$$G(z) = \frac{\frac{H'(z).(z - z_p)^k - k.H(z).(z - z_p)^{k-1}}{(z - z_p)^{2k}}}{\frac{H(z)}{(z - z_p)^k}}$$
$$= \frac{H'(z)}{H(z)} - \frac{k}{z - z_p}$$

Since  $\frac{H'(z)}{H(z)}$  is analytic at  $z_p$ , we find that G has a simple pole at  $z_p$  with residue equal to minus k. Finally by Residue theorem, the image of G around C must equal  $2\pi i$ .

$$\oint G(z)dz = \oint \frac{f'(z)}{f(z)}dz = 2\pi i \left[ N_o(f) - N_p(f) \right].$$

<u>*Case 2*</u>: If *f* has no poles inside C, then  $N_p(f) = 0$  and we have

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = N_o(f)$$

where  $N_o(f)$  is the number of zero of f inside C.  $\Box$ 

Now, we can say that our job will be easier because of Rouche's Theorem. Now we will see that applying to this theorem is very useful for Open Mapping Theorem.

## Theorem 3.28 (Rouche's Theorem)

Suppose f(z) and g(z) are analytic inside and on a simple closed contour C, with

$$|g(z)| \le |f(z)|$$

on C. Then f(z) + g(z) has the same number of zeros as f(z) inside C.

**Proof.** Firstly we take the absolute value of f(z) + g(z). We have

$$|f(z) + g(z)| \le |f(z)| + |g(z)|$$

Since g(z) has no zero on C we can divide both side by |g(z)|, then we have

$$\frac{|f(z) + g(z)|}{|g(z)|} < \left|\frac{f(z)}{g(z)}\right| + 1, \forall z \in \mathbb{C}.$$

for  $z \in \mathbb{C} \frac{f(z)}{g(z)}$  cannot be equal to zero. The image of *C* is  $C^*$  under the mapping  $\frac{f(z)}{g(z)}$  doesn't contain  $[0, \infty)$  and the function defined by

$$w(z) = \log \frac{f(z)}{g(z)} = \ln \left| \frac{f(z)}{g(z)} \right| + i \arg \frac{f(z)}{g(z)}$$
$$= \ln r + i\Phi.$$

where  $\frac{f(z)}{g(z)} = re^{i\theta} \neq 0$  and  $0 \le \theta \le 2\pi$  is analytic in a simply connected domain  $D^*$  in  $C^*$ .

$$w'(z) = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}$$

so  $w(z) = \log \frac{f(z)}{g(z)}$  is antiderivative of  $\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \forall z \in D^*$ .  $C^*$  is closed curve in  $D^*$ . From 3.5 we have

$$\oint \left[\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right] dz = \oint \frac{f'(z)}{f(z)} dz - \oint \frac{g'(z)}{g(z)} dz$$
$$= Z_f - Z_f = 0$$
$$\implies Z_f = Z_g. \ \Box$$

Example 3.29 Use 3.28 and determine the number of zeros of the given function

$$h(z) = z^7 + 4z - 1$$

*where* |z| = 3*.* 

Solution. Let  $f(z) = z^7$  and g(z) = 4z - 1 that is, h(z) = f(z) + g(z).

$$|f(z)| = |z|^7 = 3^7 = 2187$$
  
 $|g(z)| = |4z - 1| \le 4|z| + 1 = 4.3 + 1 = 13$   
 $|g(z)| = 13 < |f(z)| = 2187.$ 

h(z) has seven zeros on |z| = 3.

Example 3.30 Use 3.28 and determine the number of zeros of the given function

$$h(z) = z^5 - 7z + 6$$

*where* |z| = 2*.* 

**Solution.** Let  $h(z) = z^5$  and g(z) = -7z + 6. We have

$$|f(z)| = |z^5| = |z|^5 = 2^5 = 32$$
  
 $|g(z)| = |-7z + 6| \le 7|z| + 6 = 20.$ 

this means that

$$|g(z)| = 20 < 32 = |f(z)|$$

Hence from 3.28 f(z) and h(z) have five zeros inside |z| = 2.

#### **Theorem 3.31** (Open Mapping Theorem)

A nonconstant analytic function maps open sets onto open sets.

**Proof.** Let  $U \subset \mathbb{C}$  be open set. Now, we suppose that  $f: U \to \mathbb{C}$  be analytic and nonconstant at  $z_0$ . We have to show that f(U) is open. Let we choose r > 0 such that we define new function, that is  $g(z) = f(z) - w_0$  is analytic in  $\overline{D(z_0, r)}$  and it has no contain zero on  $|z - z_0| = r$ . Now, we will show that M be the minimum value of |g(z)| on  $|z - z_0| = r$ . We will show that

$$D(w_0,M) \subset f(U)$$

Let  $w_1 \in D(w_0, M)$ . Then we have

$$|w_0 - w_1| < M \le |f(z) - w_0|$$

From 3.28;

$$(f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$$

 $f(z) - w_1$  and  $f(z) - w_0$  have the same number of zeros in  $D(z_0, r)$ . Since g(z) has at least one zero, then we can say that  $f(z) - w_1$  has at least one zero, as well. Since  $w_1$  is arbitrary, we must have

$$D(w_0, M) \subset f(U).$$

Open Mapping Theorem has two results:

#### **Corollary 3.32** (Maximum Modulus Principle)

Let  $U \subset \mathbb{C}$  be open and connected and f is a holomorphic on U. Let  $\exists q \in U$  such that  $|f(z)| \leq |f(q)| \ \forall z \in U$ . Then f is constant.

**Proof.** Suppose that  $U \subset \mathbb{C}$  be open and connected and f is a holomorphic on U. Pick  $q \in U$ . Since f is bounded we have

$$|f(z)| \le |f(q)| = M$$

Since f is holomorphic function, by Cauchy's Inequality, we have

$$|f'(z_0)| \le \frac{M}{r}$$
$$\lim_{r \to \infty} \frac{M}{r} = 0.$$

 $|f'(z_0)| \le 0 \Leftrightarrow f'(z_0) = 0$ . Hence *f* is constant.  $\Box$ 

#### **Corollary 3.33** (*Maximum Modulus Theorem*)

Let  $U \subset \mathbb{C}$  be bounded, open and connected. Let f be a function which is continuous on  $\overline{U}$  and holomorphic on U. Then the maximum value of |f| must occur on  $\partial U$ .

**Proof.** Since  $\overline{U}$  is compact, |f| must occur on  $\overline{U}$ . We have two cases.

<u>*Case 1*</u>: Suppose that f is constant. If f is constant, result is obvious.

<u>*Case 2*</u>: Suppose that f is nonconstant. Since f is nonconstant from Maximum Modulus Principle f cannot have a maximum on U, so it must occur on  $\overline{U}$ .  $\Box$ 

#### **3.3 Sequences of Holomorphic Functions**

We will discuss a few result concerning sequence of holomorphic function. In this section we will talk about Hurvitz's Theorem. Because this is very useful for prove that the function is injective. Additon to these we will talk about Montel's Theorem. Because this is central key for Riemann Mapping Theorem.

Suppose that  $U \subset \mathbb{C}$  is open and that F is family of uniformly bounded holomorphic functions on U. Then for every sequence  $\{f_j\} \subset F$  there is a subsequence  $\{f_{j_k}\}$  which converges normally to a holomorphic function  $f_0$ .

**Theorem 3.37** Let  $U \subset \mathbb{C}$  be open and  $q \in U$ . Let F be a family of holomorphic functions  $f: U \to D(0,1), f(q) = 0$ . Then there is a sequence  $\{f_j\}$  in F which converges normally to a holomorphic function  $f_0: U \to D(0,1)$  such that  $|f'(p)| \leq |f'_0(p)| \forall f \in F$ .

#### Proof. Let

$$w = \sup\{|f'(p): f \in F|\}$$

there exists  $\{f_j\} \subset F$  we have  $|f'_j(p)| \to w$ . From assertion we have each function map to unit disc. Hence  $\{f_j\}$  bounded uniformly 1. From 3.36, subsequence  $\{f_{j_k}\}$  converge normally  $f_0$ . From application of 3.31 and 3.36 we know that  $\{|f'_{j_k}(p)|\}$  converge and uniformly bounded,

$$\left\{\left|f_{j_k}'(p)\right|\right\} \to \left|f_0'(p)\right|.$$

If we choose  $|f'_0(p)| = w$ , then proof is completed.  $\Box$ 

#### Theorem 3.38 (Hurwitz's Theorem)

Let  $U \subset \mathbb{C}$  be open and connected. Suppose that  $\{f_j\}$  is a sequence of nonvanishing functions which are holomorphic on U. If this sequence converge normally to a holomorphic function  $f_0$ , then  $f_0$  nonvanishing or  $f_0 \equiv 0$ .

**Proof.** We assume that  $f_0 \neq 0$  but it vanishes at  $q \in U$  with multiplicity n. Since zeros of f are isolated, we have r > 0 such that  $\overline{D(q,r)} \subset U$  and  $f_0$  nonvanishing on  $\overline{D(q,r)} - \{q\}$ . From theorem 3.25, we have

$$\frac{1}{2\pi i} \oint \frac{f_0'(z)}{f_0(z)} dz = n$$
(3.25)

Since  $\forall j$ , we have

$$\frac{1}{2\pi i} \oint \frac{f_j'(z)}{f_j(z)} dz = 0$$
(3.26)

We know that  $\{f_j\}$  and  $\{f_j'\}$  are converge uniformly to  $f_0$  and  $f_0'$  on  $\partial D(q, r)$ , 3.25 must be converge uniformly to 3.26. Its contradiction because n is nonzero positive integer. Hence we have if  $f_0 \neq 0$  then  $f_0$  is nonvanishing.  $\Box$ 

Now, we are going to talk about Schwarz Lemma and its corollary. They are very important application for Riemann Mapping Theorem.

#### Lemma 3.39 (Schwarz Lemma)

Let  $f: D(0,1) \rightarrow D(0,1)$  be analytic function which maps the unit disc D(0,1) to itself. If f(0) = 0 then

$$|f(z)| \le |z| \text{ for } 0 < |z| < 1$$
  
 $|f'(0)| \le 1.$ 

**Proof.** Let  $g(z) = \frac{f(z)}{2}$ . Then g is analytic for 0 < |z| < 1 and it has singular point at z = 0, since f(0) = 0. It becomes analytic at 0 if we define

$$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = f'(0)$$

fix 0 < r < 1, for |z| = r

$$|g(z)| = \left|\frac{f(z)}{z}\right| < \frac{1}{r}$$

from Maximum Modulus Theorem for analytic g, it follows that  $|g(z)| < \frac{1}{r}$  for  $|z| \le r$ . Fix  $z \in D(0,1)$  and let  $r \to 1^-$  to get

$$|g(z)| \le 1$$

This is true  $\forall z \in D(0,1)$  and

$$\left|\frac{f(z)}{z}\right| \le 1, \qquad 0 < |z| < 1$$
$$|f(z)| \le |z|, \quad 0 \ne z \in D(0,1).$$

If |z| < 1 then

$$|g(0)| \le 1$$
  
 $|g(0)| = |f'(0)| \le 1.$ 

**Corollary 3.40**  $\Phi: D(0,1) \to D(0,1)$  conformal. There exists  $|\lambda| = 1$  and  $a \in D(0,1)$  such that

$$\Phi(z) = \lambda \frac{z-a}{1-\bar{a}z}.$$

## CHAPTER 4

## **RIEMANN MAPPING THEOREM**

The goal of this chapter is Riemann Mapping Theorem. In this chapter we are going to talk about this theorem and its importance. This means that, we will show that U and D(0,1) are biholomorphically equivalent.

**Theorem 4.1** Let U be a simply connected domain and  $U \subset \mathbb{C}$  but never  $U = \mathbb{C}$ . Then  $U \equiv D(0,1)$ .

**Proof.** Assume that *U* be a simply connected domain and  $U \subset \mathbb{C}$ . We have to show that *U* and D(0,1) are biholomorphically equivalent. Let fix  $p \in U$ . We consider  $f: U \to D(0,1)$  injective holomorphic function such that f(p) = 0. Now, we assume that *F* be a family of *f*. Firstly, we have to show that *F* is nonempty. We choose any  $k \in U$  and we define new function  $\theta(k) = z - k$ . We already shown in lemma 3.3, we can find a holomorphic function  $\Phi$  such that  $\Phi^2 = \theta$ .

Pick some  $x \in \Phi(U)$ . From lemma 3.3, since  $\Phi$  is nonconstant and holomorphic we can apply theorem 3.31.

 $\exists r > 0$  such that  $D(x, r) \subset \Phi(U)$ .

From assertion, since  $\theta$  is one-to-one, if  $\Phi(z_1) = \Phi(z_2)$  or  $\Phi(z_1) = -\Phi(z_2)$  then  $z_1 = z_2$ . This implies

$$D(-x,r) \cap \Phi(U) = \emptyset.$$

then we can write

$$f(z) = \frac{r}{2[\Phi(z) + x]}$$

We have

$$|\Phi(z) + x| \ge r \qquad \forall z \in U$$

Hence  $f: U \rightarrow D(0,1)$ .

We know that  $\Phi$  is one-to-one. Clearly, f is also one-to-one. Hence  $f \in F \neq \emptyset$ . Secondly, since  $F \neq \emptyset$ , in order to find any functions which are biholomorphic between U and D(0,1). We can use theorem 3.37 we have  $\{f_j\}$  a sequence in F converge normally to  $f_0: U \to D(0,1)$  such that  $f_0(p) = 0$ . So

$$|f_0'(p)| = \sup_{f \in F} |f'(p)|.$$

Now, we want to prove injectivity. Therefore we will apply theorem 3.38. If we show  $f_0(z) - f_0(z_0)$  is not identically zero then that is enough for injectivity. For every  $j \in \mathbb{N}$ , we define

$$h_j(z) = f_j(z) - f_j(z_0)$$
 on  $U - \{z_0\}$ 

From assertion we have for every *j*, all  $f_j$  are one-to-one. And also every  $h_j$  nonvanishing on  $U - \{z_0\}$ .

This means that  $f_0$  is one-to-one. In order to prove that  $U \equiv D(0,1)$ , we must show that  $f_0$  is onto. Suppose that  $f_0$  is not onto then there exists  $a \in D(0,1)$  which is not contained in image of  $f_0$ . Now we will see that Mobius transformations are very important to this. Now, we consider new function on U, that is

$$\Psi(z) = \frac{f_0(z) - a}{1 - \bar{a}f_0(z)} \tag{4.1}$$

Clearly,  $\Psi$  is one-to-one and nonvanishing and maps to unit disc. Since U is simply connected domain from lemma 3.3 we can find a holomorphic function

$$m^2 = \Psi \tag{4.2}$$

Now, we use Mobius transformations and we can define

$$T(z) = \frac{m(z) - m(p)}{1 - \overline{m(p)}m(z)}$$
(4.1)

Clearly,  $T: U \to D(0,1)$  holomorphic and T(p) = 0. Hence  $T \in F$ . Now we are going to show that

$$|\mathsf{T}'(\mathsf{p})| > |f_0'(\mathsf{p})|$$

Now we take derivative of 4.3 with respect to z, that is,

$$T'(z) = \frac{m'(z) - \overline{m(p)} \cdot m'(z) \cdot m(z) + \overline{m(p)} \cdot m(z) - \overline{m(p)} \cdot m(p)}{\left[1 - \overline{m(p)} \cdot m(z)\right]^2}$$

$$T'(z) = \frac{m'(z) \left[1 - m(z) \cdot \overline{m(p)} \cdot \right] + \overline{m(p)} \cdot \left[m(z) - m(p)\right]}{\left[1 - \overline{m(p)} \cdot m(z)\right]^2}$$
(4.4)

Now if we put p instead of z in 4.4, then we have

$$T'(p) = \frac{m'(p) \cdot [1 - m(p) \cdot \overline{m(p)}]}{[1 - \overline{m(p)} \cdot m(p)]^2}$$
$$T'(p) = \frac{m'(p)}{[1 - \overline{m(p)} \cdot m(p)]}$$
$$T'(p) = \frac{m'(p)}{1 - |m(p)|^2}$$
(4.5)

Now we take absolute value of 4.5, we have

$$|T'(p)| = \left|\frac{m'(p)}{1 - |m(p)|^2}\right|$$
(4.6)

From assertion, if we take derivative of 4.2 with respect to z, then we have

$$\Psi'(z) = 2.\,m(z).\,m'(z) \tag{4.7}$$

If we put p instead of z in 4.7 then we have

$$\Psi'(p) = 2.m(p).m'(p)$$

Clearly we have

$$m'(p) = \frac{\Psi'(p)}{2.m(p)}$$
(4.8)

Again from assertion we have  $f_0(p) = 0$ . If we put p instead of z in 4.1, we have

$$\Psi(p) = \frac{f_0(p) - a}{1 - \overline{a}.f_0(p)}$$

Clearly we have

$$\Psi(p) = -a \tag{4.9}$$

Now we put p instead of z in 4.2 then

$$m^2(p) = \Psi(p) \tag{4.10}$$

If we put 4.9 in 4.10, then we have

$$m^2(p) = -a$$
 (4.11)

We take absolute value both side of 4.11, we have

$$|m^2(p)| = |a| (4.12)$$

If we take square root of 4.12 then we have

$$|m(p)| = \sqrt{|a|} \tag{4.13}$$

Now, we take absolute value of 4.2 then we have

$$|\Psi(p)| = |m(p)|^2 \tag{4.14}$$

Note that from assertion

$$f_0(p) = 0 (4.15)$$

Now, we put 4.2, 4.5, 4.11, 4.13, 4.14 and 4.15 in 4.6 we have

$$|T'(p)| = \left|\frac{m'(p)}{1 - |m(p)|^2}\right| = \left|\frac{\frac{\Psi(p)}{2.m(p)}}{1 - |\Psi(p)|}\right|$$

$$|T'(p)| = \frac{\Psi'(p)}{(1 - |\Psi(p)|) \cdot 2m(p)}$$
$$|T'(p)| = \left| \frac{f_0'(p) \cdot [1 - \bar{a} \cdot f_0'(p)] + \bar{a} \cdot f_0'(p) \cdot [f_0'(p - a)]}{[1 - \bar{a} \cdot f_0(p)]^2 \cdot 2 \cdot m(p) \cdot (1 - |\Psi(p)|)} \right|$$

Since  $f_0(p) = 0$ 

$$|T'(p)| = \left| \frac{f'_0(p) \cdot (1-0) + f'_0(p) \cdot \bar{a} \cdot (-a)}{(1-0)^2 \cdot 2m(p) \cdot (1-|\Psi(p)|)} \right|$$
$$|T'(p)| = \left| \frac{f'_0(p) \cdot (1-|a|^2)}{2 \cdot m(p) \cdot (1-|a|)} \right|$$
$$|T'(p)| = \left| \frac{f'_0(p) \cdot (1-|a|) \cdot (1+|a|)}{2 \cdot m(p) \cdot (1-|a|)} \right|$$
$$|T'(p)| = \left| \frac{f'_0(p) \cdot (1+|a|)}{2 \cdot \sqrt{|a|}} \right|$$

We can see there

$$\frac{1+|a|}{2\sqrt{|a|}} \ge 1$$

So we have

$$|T'(p)| = |f'_0(p)|$$

Its contradiction.  $f_0$  is onto. Since one-to-one and onto then  $f^{-1}$  exists and holomorphic too. So we have  $f_0$  is biholomorphic functions which is  $f_0: U \to D(0,1)$ . Proof is completed.  $\Box$ 

**Example 4.2** Let  $f: \mathbb{C} \to D(0,1)$  be biholomorphic. For  $a \in \mathbb{C}, b \in D(0,1) - \{0\}$  such that f(a) = b. We want a biholomorphic function which maps a to a zero.

**Solution.** We know that from 3.40  $\Phi$ : D(0,1)  $\rightarrow$  D(0,1) conformal. Then there exists  $|\lambda| = 1$  and  $b \in D(0,1)$  such that

$$\Phi(z) = \lambda \frac{z - b}{1 - \overline{b}z}$$

for this question we assume  $\lambda = 1$ . And we have Mobius transformation;

$$\Phi(z) = \frac{z-b}{1-\overline{b}z}$$

We know that every Mobius transformations are biholomorphic and composite of mobius transformations again biholomorphic. Clearly,  $\Phi of: \mathbb{C} \to D(0,1)$ . We can see there this is one-to-one. Now, we have to show that this function takes a to zero.

$$\Phi of(z) = \frac{f(z) - b}{1 - \overline{b}f(z)}$$

Since f(a) = b, we have

$$\Phi of(a) = \frac{f(a) - b}{1 - \overline{b}f(a)} = \frac{b - b}{1 - \overline{b}b} = 0.$$

**Example 4.3** Find a conformal bijection mapping  $U = \{z \in \mathbb{C} : |z| < 1 \text{ and } Imz > 0\}$  to D(0,1) such that  $\frac{i}{2}$  to 0.

**Solution.** Let  $f: U \to \mathbb{C}$  is analytic such that

$$f(z) = \frac{z-1}{z+1}, \qquad f\left(\frac{i}{2}\right) = \frac{-3+4i}{5}$$

we can see there the image of that  $\frac{i}{2}$  in second quadrant of  $\mathbb{C}$ . Let

$$w = \frac{z - 1}{z + 1}$$
$$z = \frac{1 + w}{1 - w}.$$

Let w = u + iv. Since |z| < 1,

$$|1+w| < |1-w|$$
  
 $(1+w).(1+\overline{w}) < (1-w).(1-\overline{w})$ 

$$w + \overline{w} < 0$$
$$u < 0.$$

since Imz > 0,

$$Im\left[\frac{1+u+iv}{1-u-iv},\frac{1-u+iv}{1-u+iv}\right] > 0$$
$$\left[\frac{v(1+u+1-u)}{(1-u)^2+v^2}\right] > 0$$
$$v > 0.$$

Secondly we can define a new function  $g(z^*) = (z^*)^2$  conformal except 0. We can see there the image of second quadrant of  $\mathbb{C}$  is

$$g(z^*) = (z^*)^2 = (x^* + iy^*)^2$$
$$g(z^*) = (x^*)^2 - (y^*)^2 + 2x^*y^*i.$$

Since  $Rez^* < 0$  and  $Imz^* > 0$  then

$$v^* = 2x^*y^* < 0$$
$$v^* < 0.$$

Finally, we can define new analytic function

$$h(z^{**}) = \frac{z^{**} + i}{z^{**} - i}.$$

we will find image of  $Imz^{**} < 0$  under this function. That is;

$$w^{**} = \frac{z^{**} + i}{z^{**} - i}$$
$$z^{**} = i\frac{w^{**} + 1}{w^{**} - 1}.$$

Since  $Imz^{**} < 0$  we have;

$$Im\left[i\frac{w^{**}+1}{w^{**}-1}\right] < 0$$

$$Im\left[\frac{(1+u^{**})i-v^{**}}{u^{**}-1+iv^{**}}\cdot\frac{u^{**}-1-iv^{**}}{u^{**}-1-iv^{**}}\right] < 0$$
$$\frac{(1+u^{**})\cdot(u^{**}-1)\cdot i+i(v^{**})^2}{(u^{**}-1)^2+(v^{**})^2} < 0$$
$$(u^{**})^2 - 1 + (v^{**})^2 < 0$$
$$(u^{**})^2 + (v^{**})^2 < 1.$$

This is D(0,1). Now, we take compositon *f*, *g* and *h*;

$$(hogof)(z) = \frac{\left[\frac{z-1}{z+1}\right]^2 + i}{\left[\frac{z-1}{z+1}\right]^2 - i}$$
$$(hogof)(z) = \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2}$$
$$(hogof)(z) = i\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}.$$

We can see that

$$(hogof)\left(\frac{i}{2}\right) = -\frac{i}{7}$$

But we want to make 0. From 3.40 we have for any a in lower half plane then the map

$$\frac{z-a}{z-\bar{a}}$$

Will suffice as a replacement for *h*. We can see there

$$(gof)\left(\frac{i}{2}\right) = \frac{-7 - 24i}{25}.$$

If we change h(z)

$$\tilde{h} = \frac{z - (\frac{-7 - 24i}{25})}{z - (\frac{-7 + 24i}{25})}$$

Finally if we compose f, g and  $\tilde{h}$  then we have

$$(\tilde{h}ogof)(z) = \frac{25\left[\frac{z-1}{z+1}\right]^2 + 7 + 24i}{25\left[\frac{z-1}{z+1}\right]^2 + 7 - 24i}$$

This is conformal bijection and  $(\tilde{h}ogof)(\frac{i}{2}) = 0.$ 

## CHAPTER 5 CONCLUSION

Finally, instead of working in complicated spaces, Mobius transformations and Riemann Mapping Theorem provides easier ways to find holomorphic functions between known spaces. In my future life I would like work on this subject.

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