CHAPTER 1

INTRODUCTION AND BASIC DEFINITIONS

1.1 Introduction

In applied mathematics and physics, orthogonal polynomials have an important place. Moreover, geometrically, orthogonal polynomials are the basic of vector spaces and so any member of this vector space can be expanding a series of orthogonal polynomials.

Almost four decades ago, Konhauser (1965-1967) found a pair of orthogonal polynomials which satisfy an additional condition, which is a generalization of orthogonality condition. These polynomials are called biorthogonal polynomials. After Konhauser's study, several properties of these polynomials and another biorthogonal polynomial pairs was found.

In 2007, Şekeroğlu, Srivastava and Taşdelen gave a general definition of q-biorthogonal polynomials and obtained main properties of them.

After these study, some special properties was obtained for q-biorthogonal polynomials.

In 2008, Srivastava , Taşdelen and Şekeroğlu studied several generating functions for q-biorthogonal Konhauser polynomials.

In this work, general and basic properties of biorthogonal polynomials are given and two types of biorthogonal polynomials which are namely Konhauser polynomials and Jacobi type biorthogonal polynomials are investigated.

In the first chapter, several basic definitions and theorems about q-analysis theory are given.

In the second chapter, definition and main theorems of about orthogonal polynomials theory are given and some special orthogonal polynomial families are given.

In the third chapter, definition and main theorems of about biorthogonal polynomials are obtained and some special biorthogonal polynomial families are given.

In the fourth chapter, definition and main theorems of about q-orthogonal polynomials theory and some special q-orthogonal polynomial families are given.

In the fifth chapter, definition and main theorems of about q-biorthogonal polynomials theory and some q-biorthogonal polynomial families are given.

In the sixth chapter, are given conclusions.

1.2 Gamma Function

The definition of a special function which is defined by using an improper integral is given below. This function is called Gamma Function and has several applications in Mathematics and Mathematical Physics.

Definition 1.1(Rainville, 1965)

The improper integral

$$\int_0^\infty t^{x-1}e^{-t}dt$$

converges for any $x > 0$ is called "Gamma Function" and is denoted by Γ.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Some basic properties of Gamma function are given without their proofs (Rainville, 1965).

$$\int_0^\infty t^n e^{-t} dt = n! = \Gamma(n+1),$$

where n is a positive integer.

$$n.\Gamma(n) = \Gamma(n+1),$$

and

$$\Gamma(2b)\sqrt{\pi} = 2^{1-2b}\Gamma(b)\Gamma\left(b+\frac{1}{2}\right),$$

where Re(b) > 0.

$$\Gamma(2b+n)\sqrt{\pi}21-2b-n = \Gamma\left(b+\frac{1}{2}n\right)\Gamma\left(b+\frac{1}{2},n+\frac{1}{2}\right),$$

where Re(b) > 0 and n is non-negative.

$$\Gamma(a) = (a)^n \frac{(n-1)!}{(a)_n},$$

where Re(a) > 0 and n is non-negative integer.

Definition 1.2(Askey, 1999)

Let x be a real or complex number and n is non-negative integer,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\ldots(x+n-1),$$

$$(x)_0 = 1\,,$$

$$(x)_1 = x\,,$$

$$(x)_2 = x^2 + x,$$

is known "Pochammer Symbol".

There are some properties of Pochammer symbol.

1.

$$(c+n)_k = \frac{(c)_{n+k}}{(c)_n}\,,$$

where c is real or complex number and n and k are natural numbers.

2.

$$\frac{n!}{(n-k)!} = \frac{(-n)_k}{(-1)^k}\,,$$

where n and k are natural numbers.

3.

$$\frac{(c)_{2k}}{2^{2k}} = \left(\frac{c}{2}\right)_k \cdot \left(\frac{c+1}{2}\right)_k,$$

where c is a complex number and k is a natural number.

4.

$$\frac{(2k)!}{2^{2k}k!} = \left(\frac{1}{2}\right)_k,$$

where k is a natural number.

There is a useful lemma for Pochammer symbol. Proof of this lemma can be obtained by directly and elementarily (Raiville, 1965).

Lemma 1.1

Let α be real or complex number and n is non-negative integer,

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n \cdot \left(\frac{\alpha+1}{2}\right)_n, \qquad (1.1)$$

Proof

$$(\alpha)_{2n} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+2n-1)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha}{2}+1\right)\ldots\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}+1\right)\ldots\left(\frac{\alpha}{2}+n-1\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}+1\right)$$

$$\ldots\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

1.3 q- Analysis

After the systematic development of calculus by Leibniz and Newton in the after half of the seventeenth century, Mathematicians attempted to improve new techniques. One of them is q-Analysis, which is an important generalization of standard techniques.

Definition 1.3

Let q$\in \mathbb{R} \setminus \{1\}$. Then the q-analogue of a number a is given by

$$[a]_q = \frac{1-q^a}{1-q}$$

Definition 1.4

For a real or complex number q ($|q| < 1$), $(a;q)_n$ is given by

$$(a;q)_n = \begin{cases} 1 & ; \quad (n=0) \\ \prod_{j=0}^{n-1} (1-aq^j) & ; \quad (n \in \mathbb{N} = \{1,2,\dots\}) \end{cases}$$

and

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j)$$

Definition 1.5

Let q$\in \mathbb{R} \setminus \{1\}$. Then the q-Pochhammer symbol is defined by

$$[a]_{n;q} = \prod_{m=0}^{n-1} [a+m]_q$$

for a reel parameter a.

Definition 1.6

Let q∈ $\mathbb{R}\setminus\{1\}$. Then the q-analogue of n! is given by

$$[n]_q! = \prod_{m=1}^{n}[m]_q \ , \quad [0]_q! = 1$$

for a natural parameter n.

Definition 1.7

Let q∈ $\mathbb{R}/\{1\}$. The q-derivative operator D_q is defined by

$$D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}\ , \qquad (1.2)$$

Notation

It is significant to mention about when $q \to 1^-$, it is said that $[a]_q \to a$ and $[a]_{n,q} \to (a)_n$, $[n]_q! \to n!$ and $D_q f(x) \to \frac{d}{dx}\big(f(x)\big)$ where $(a)_n$ is called Pochhammer symbol for a natural number n and a real number a.

Example 1

Let a is a reel number then

$$D_q(x^a) = [a]_q x^{a-1}.$$

Solution

If the representation (4.1) is applied to the expression x^a, it is deduced that

$$D_q(x^a) = \frac{(qx)^a - x^a}{(q-1)x} = \frac{1-q^a}{1-q}x^{a-1} = [a]_q x^{a-1}.$$

Example 2

The q-exponential function is

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(x)^k}{[k]_q!} \;. \qquad (1.3)$$

a is a real number then

$$D_q\big(e_q(ax)\big) = ae_q(x).$$

Solution

With the help of the representation (1.3) it can be written that,

$$e_q(x) = D_q\left(\sum_{k=0}^{\infty} \frac{(ax)^k}{[k]_q!}\right) = \sum_{k=0}^{\infty} \frac{(a)^k}{[k]_q!}\; D_q(x^k)\;.$$

If example 1 is used in above the equation , it is readily obtained,

$$e_q(x) = \sum_{k=1}^{\infty} \frac{(a)^k}{[k]_q!}\;[k]_q x^{k-1} = a\sum_{k=1}^{\infty} \frac{(ax)^{k-1}}{[k-1]_q!} = ae_q(ax)\,.$$

Lemma 1.2

Let f(x) and g(x) be two piecewise continuous function in (a,b). Then we have

$$D_q\{f(x)g(x)\} = f(x)D_q\{g(x)\} + g(x)D_q\{f(x)\}$$

$$+(q-1)xD_q\{f(x)\}D_q\{g(x)\}. \qquad (1.4)$$

Proof

If the representation (1.2) is applied to the left side of the equation (1.4), it is written that,

$$D_q\big(f(x)g(x)\big) = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}$$

if the quantity $f(x)g(x)$ add to and substract from the equation above, the following relation is found easily.

$$D_q\big(f(x)g(x)\big) = \frac{f(qx)g(qx) - f(x)g(x) + f(x)g(x) - f(x)g(x)}{(q-1)x}$$

$$= f(x)\frac{\big(g(qx) - g(x)\big)}{(q-1)x} + g(x)\frac{\big(f(qx) - f(x)\big)}{(q-1)x}$$

$$= f(x)D_q g(x) + g(qx)D_q f(x)$$

if the equation

g(qx)=g(x)+(q-1)x$D_q g(x)$

that was obtained with the help of (1.2) is used, the proof will be completed.

Definition 1.8

The q-integral of a piecewise continuous function f(x) in (a,b) is defined as follows:

$$\int_a^b f(x)d_q x = \sum_{n=0}^{\infty}(bq^n - bq^{n+1})f(bq^n) - \sum_{n=0}^{\infty}(aq^n - aq^{n+1})f(aq^n)$$

and

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j f\left(q^j\right). \tag{1.5}$$

If f is a continuous function, the meaning of q-integral exactly equals to Riemann integral when q approaches to 1^-. In order to show this, let us take f(x) as x^α. assume that F(x) is an antiderivative of f(x). There follows F(x)= $\frac{x^{a+1}}{a+1}$.

$$\sum_{n=0}^{\infty} f(bq^n)\,(bq^n - bq^{n+1})$$

$$= \sum_{n=0}^{\infty} (bq^n)^\alpha (bq^n - bq^{n+1}) = b^{\alpha+1}(1-q)\sum_{n=0}^{\infty} q^{(\alpha+1)n}$$

$$= b^{\alpha+1}\frac{1-q}{1-q^{\alpha+1}}$$

then

$$\lim_{q\to 1^-} b^{\alpha+1}\frac{1-q}{1-q^{\alpha+1}} = \frac{b^{\alpha+1}}{\alpha+1} = F(b)$$

in the similar way,

$$\lim_{q\to 1^-} \alpha^{\alpha+1}\frac{1-q}{1-q^{\alpha+1}} = F(\alpha).$$

Consequently,

$$\lim_{q\to 1^-}\int_a^b f(x)d_qx = F(b) - F(a) = \int_a^b f(x)dx.$$

Theorem 1.1

The q-integral of a piecewise continuous function f(x) in (a,b) is defined as follows:

$$\int_a^b f(x)d_qx = \sum_{n=0}^{\infty}(bq^n - bq^{n+1})f(bq^n) - \sum_{n=0}^{\infty}(aq^n - aq^{n+1})f(aq^n)$$

and

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j f(q^j).$$

Lemma 1.3 (q-partial integration)

The q-partial integration is defined by

$$\int_0^{\infty} f(x)D_q\{g(x)\}d_qx = \lim_{n\to\infty}\{f(q^{-n})g(q^{-n}) - f(q^{n+1})g(q^{n+1})\}$$

$$-\int_0^{\infty} g(x)D_q\{f(x)\}d_qx - (q-1)\int_0^{\infty} xD_q\,\{f(x)\}D_q\{g(x)\}d_qx.$$

for two piecewise continuous f(x) and g(x).

Proof

If q-integral is applied to both sides on the expression (1.4) on [0,∞), by using the definition (1.5), it is found that

$$\int_0^{\infty} f(x)D_qg(x)d_qx = \lim_{n\to\infty}\left((1-q)\sum_{k=-n}^{n} q^kD_q(fg)(q^k)\right)$$

$$-\int_0^{\infty} g(x)D_qf(x)d_qx - (q-1)\int_0^{\infty} x\left(D_qf(x)\right)\left(D_qg(x)\right)d_qx. \quad (1.6)$$

If the definition of q-derivative operator is applied to the first term of the right side of (1.6), it is reached that

$$\lim_{n\to\infty}\left((1-q)\sum_{k=-n}^{n} q^k D_q(fg)(q^k)\right)$$

$$=\lim_{n\to\infty}\left((1-q)\sum_{k=-n}^{n} q^k \times \sum_{k=-n}^{n} q^k \frac{(fg)(q^{k+1})-(fg)(q^k)}{(q-1)q^k}\right)$$

$$=\lim_{n\to\infty}\sum_{k=-n}^{n}\left((fg)(q^{k+1})-(fg)(q^k)\right)$$

$$=\lim_{n\to\infty}(f(q^{-n})g(q^{-n})-f(q^{n+1})g(q^{n+1})).$$

Definition 1.9

Jackson defined a q-analogue of the gamma function as

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \qquad 0<q<1.$$

Note that Γ_q satisfies the functional equation

$$\Gamma_q(x+1)=\frac{q^x-1}{q-1}\Gamma_q(x).$$

He also showed that $lim_{q\to 1^-}\ \Gamma_q(x)=\Gamma_q$. Askey proved the integral formula

$$\int_{\mathbf{0}}^{\infty}\frac{x^\alpha dx}{(-(1-q)x;q)_\infty}=\frac{\Gamma(-\alpha)\Gamma(\alpha+1)}{\Gamma_q(-\alpha)}\ , \quad 0<q<1, Re(\alpha)>0. \quad (1.7)$$

CHAPTER 2

ORTHOGONAL POLYNOMIALS

In this section, definitions and main properties of orthogonal polynomials which are a special case of the biorthogonal polynomials are given. (Askey, 1999)

2.1 ORTHOGONAL POLYNOMIALS

Definition 2.1

A polynomial is a function p whose value at x is

P(x) = $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

where $a_n, a_{n-1}, \dots, a_2, a_1$ *and* a_0, called the coefficients of the polynomial, are constants and , if n >0, then $a_n \neq 0$. The number n, the degree of the highest power of x in the polynomial, is called the degree of the polynomial. (The degree of the zero polynomial is not defined.)

Definition 2.2

Let w(x) is a weight function and p_n(x) polynomials are defined over the interval [a,b], if

$$\int_a^b w(x) p_n(x) p_m(x)\, dx = 0\,, \qquad m \neq n\,, \qquad (2.1)$$

is satisfied, then the polynomials $p_n(x)$ are called orthogonal with respect to the weight function w(x) are the interval (a,b), m and n are degrees of polynomials.

There is an additional condition for the orthogonal polynomials which makes them orthonormal.

Definition 2.3

If the polynomials p_n(x) are orthogonal with respect to the weight function w(x), over the interval (a,b) and

$$\|p_n(x)\|^2 = \int_a^b \int_a^b w(x)p_n^2(x)dx = 1 \quad , \qquad m = n \, ,$$

is satisfied , then the polynomials $p_n(x)$ are called orthonormal.

There is an equivalent condition for the orthogonality relation (2.1) which is given below.

Theorem 2.1 (Askey, 1999)

It is sufficient for the orthogonality of the polynomials on the interval [a,b] with respect to the weight function w(x) to satisfy the condition

$$\int_a^b w(x)\phi_n(x)x^i dx = 0 \, , \qquad i = 0,1,2, \dots, n-1 \qquad (2.2)$$

here, $\phi_n(x)$ is polynomial of degree n.

Proof

If the polynomials $\phi_n(x)$ and $\phi_m(x)$ are orthogonal on the interval [a,b] with respect to w(x) then

$$\int_a^b w(x)\phi_n(x)x^i dx = 0 \quad , \quad m \neq n$$

x^i, can be written as linear combinations,

$$x^i = a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + \cdots + a_i\phi_i = \sum_{m=0}^{i} a_m\phi_m(x),$$

substituting this in (2.2).

$$\int_a^b w(x)\phi_n(x)x^i dx = \int_a^b w(x)\phi_n(x)\left\{\sum_{m=0}^{i} a_m\phi_m(x)\right\}dx$$

$$= \sum_{m=0}^{i} a_m \int_a^b w(x)\,\phi_n(x)\phi_m(x)dx = 0$$

for 0≤ $m \leq i$, $\phi_n(x)$ and $\phi_m(x)$ where 0≤ $m < n$. Hence,

$$\int_a^b w(x)\,\phi_n(x)x^i dx = 0\,, \qquad i = 0,1,2,\ldots,n-1.$$

Orthogonal polynomials have several important properties. In this section, general definitions of these properties are given and then obtained special form of them for well-known orthogonal polynomial families.

Definition 2.4 (Askey, 1999)

Any polynomial family $\phi_n(x)$, which is orthogonal on the interval [a,b] with respect to the weight function w(x), satisfies the recurrence formula

$\phi_{n+1}(x) - (xA_n + B_n)\phi_n(x) + C_n\phi_{n-1}(x) = 0$ here A_n,B_n and C_n are constants which depend on n.

Definition 2.5 (Askey , 1999)

Rodrigues Formula for orthogonal polynomials are written as

$$\phi_n(x) = A_n \frac{1}{w(x)} \frac{d^n}{dx^n}[w(x)u^n(x)], \qquad n = 0,1,2,\ldots \qquad (2.3)$$

here, $\phi_n(x)$ polinomials are orthogonal with respect to the weight function w(x) and u^n is a polynomial of x.

Definition 2.6 (Askey, 1999)

If the two variable function F(x,t) has a Taylor series as in the form of

$$F(x,t) = \sum_{n=0}^{i} a_n \, \phi_n(x) t^n \,, \qquad (2.4)$$

with respect to one of its variables t, then the function F(x,t) is called the generating function for the polynomials $\{\phi_n(x)\}$.

Definition 2.7 (Bilateral Generating Funtion)

If the three variable function H(x,y,t) has a Taylor series in the form of

$$H(x,y,t) = \sum_{n=0}^{\infty} c_n f_n(x) g_n(x) t^n$$

with respect to one of its variables, t, then the function H(x,y,t) is the bilateral generating function for the families $f_n \; and \; g_n$.

Definition 2.8 (Bilinear Generating Function)

If the three variable function G(x,y,t) has a Taylor series in the form of

$$G(x,y,t) = \sum_{n=0}^{\infty} c_n f_n(x) g_n(y) t^n$$

with respect to one of its variables , t , then the function G(x,y,t) is the bilinear generating function for the families function f_n and g_n .

2.2 Some Special Orthogonal Polynomial Families

Some well-know orthogonal polynomials families which have several applications in applied mathematics are given at this section. These polynomial families have several properties which are common and obtainable for any orthogonal polynomial family.

2.2.1 Laguerre Polynomials (Rainville, 1965)

For $\alpha > -1$, the $L_n^{(\alpha)}$(x) polynomials, which are orthogonal on 0≤ $x < \infty$ with respect to the weight function w(x)=$x^{\alpha}e^{-x}$ and which are known as Laguerre polynomials are given by,

$$\phi_n(x) = L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \quad , \qquad n = 0,1,2,....$$

The special case $\alpha = 0$ is $L_n^{(\alpha)}$(x) = L_n(x). Let us give the first five Laguerre polynomials,

$L_0(x) = 1$

$L_1(x) = -x + 1$

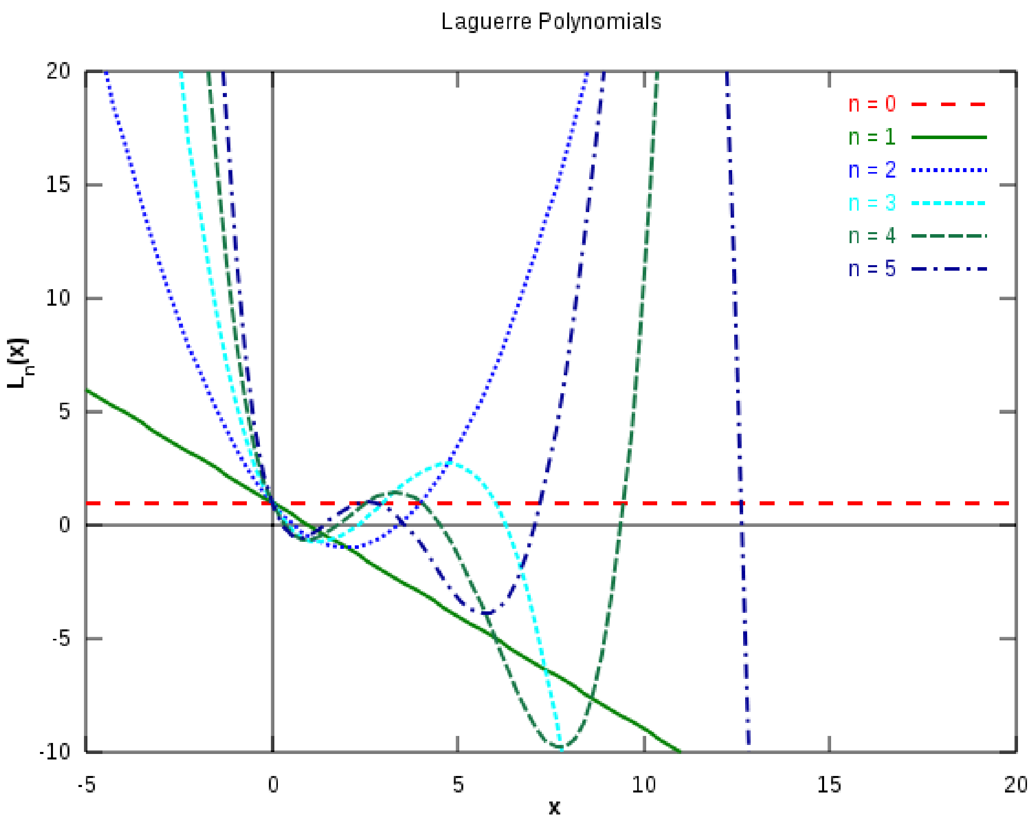
$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$

$L_3(x) = \frac{1}{6}(-x^3 - 16x^2 - 18x + 6)$

$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$

The graphs of first six Laguerre polynomials $L_0(x), L_1(x), L_2(x), L_3(x),\ L_4(x)$ and $L_5(x)$ are shown in the figure:



Several properties of Laguerre polynomials similar to orthogonal polynomials can be obtained. One of these properties is that it satisfies second order differential equations. Starting from $\frac{d}{dx}[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)]$, we obtain Leguerre differential equation,

$$xy'' + (\alpha + 1 - x)y' + ny = 0,$$

where the solution of this differential equation are Laguerre polynomials can be obtained.

Let us start with the equation below:

$$\frac{d}{dx}[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)]= x^{\alpha}e^{-x}[x\frac{d^2}{dx^2}L_n(x) \ +(\alpha+1-x)\frac{d}{dx}L_n(x)]$$

it can be written as linear combinations,

$$x\frac{d^2}{dx^2}L_n(x) \ +(\alpha+1-x)\frac{d}{dx}L_n(x) = \sum_{i=1}^{n}\alpha_i L_i(x).$$

Therefore,

$$\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right] = x^{\alpha}e^{-x}\sum_{i=1}^{n}\alpha_i L_i(x)$$

by integrate over the interval (0,∞), it is deduced that

$$\int_0^{\infty} L_j(x)\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]dx = \int_0^{\infty} L_j(x)x^{\alpha}e^{-x}\sum_{i=1}^{n}\alpha_i L_i\,(x)$$

$$= \alpha_j\int_0^{\infty} x^{\alpha}e^{-x}L^2{}_j(x)dx + \sum_{\substack{i=1\\ j=1\\ i\neq j}}^{n}\alpha_i\int_0^{\infty} e^{-x}x^{\alpha}L_j(x)L_i(x)dx$$

it is known that the Laguerre polynomial are orthogonal, then

$$\int_0^{\infty} e^{-x}x^{\alpha}L_j(x)L_i(x)dx \ = 0\,, \qquad i\neq j$$

Consequently,

$$\int_0^{\infty} L_j(x)\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]dx = \alpha_j\int_0^{\infty} x^{\alpha}e^{-x}L^2{}_j(x)dx + 0$$

$$\alpha_j = \frac{\int_0^\infty L_j(x) \frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{d}{dx} L_n(x)\right] dx}{\int_0^\infty x^\alpha e^{-x} L^2{}_j(x) dx}$$

and $\frac{d}{dx} L_n(x) = y'$, $\frac{d^2}{dx^2} L_n(x) = y''$ so

xy'' +$(\alpha + 1 - x)y' + \lambda y$ = 0

$\lambda_n =$ -n($\frac{n-1}{2}$. $(x)'' + (\alpha + 1 - x)'$) = n.

Then the following differential equation is obtained.

xy'' +$(\alpha + 1 - x)y' + ny$ = 0.

The generating function for the Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = \frac{1}{(1-t)} exp\left(\frac{-xt}{1-t}\right), \tag{2.5}$$

can be written. For obtaining the $\left\| L_n^{(\alpha)}(x) \right\|$ norm of Laguerre polynomials, the generating function (2.5) is rewritten as in the form of

$$\sum_{m=0}^{\infty} e^{-x} L_m^{\alpha}(x) t^m = e^{-x} \frac{1}{(1-t)} exp\left(\frac{-xt}{1-t}\right) , \tag{2.6}$$

by multiplying both sides of (2.5) by w(x) = e^{-x} where m $\neq n$. If (2.5) and (2.6) are multiplied side by side and integrate over the interval (0,∞)

$$\sum_{n,m=0}^{\infty} \left[\int_0^\infty e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx\right] t^{n+m} = \frac{1}{(1-t)^2} \int_0^\infty exp\left(\frac{x(1+t)}{t-1}\right)$$

is obtained. If left hand side of the last equation is separated for m = n and m ≠ n, and take the integral at right hand side,

$$\sum_{n,m=0}^{\infty}\left[\int_{0}^{\infty} e^{-x}L_n^2(x)dx\right]t^{2n} + \sum_{n,m=0}^{\infty}\left[\int_{0}^{\infty} e^{-x}\,L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx\right]\,t^{n+m}$$

$$=\frac{1}{(1-t)^2}\cdot\frac{1-t}{(1+t)}=\frac{1}{1-t^2}$$

is obtained. By using the orthoganality of Laguerre polynomials, for n=m , second integral at the left hand side is equal to zero.

If the Taylor series ,

$$\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n\,.$$

is used on the right hand side of the last equality, then

$$\sum_{n,m=0}^{\infty}\left[\int_{0}^{\infty} e^{-x}L_n^2(x)dx\right]t^{2n} = \sum_{n=0}^{\infty}t^{2n} \quad ,$$

is obtained. Thus , equality of the coefficient of t^{2n} in both sides give the norm of Laguerre polynomials as

$$\left\|L_n^{(\alpha)}(x)\right\|^2 = \int_{0}^{\infty} e^{-x}L_n^2(x)dx = 1.$$

Finally the recurrence relation for Laguerre polynomial $L_n^{\alpha}(x)$ is given as,

$(n+1)L_{n+1}^{(\alpha)}(x) + (x-2n-1-\alpha)L_n^{(\alpha)}(x) + (n+\alpha)\,L_{n-1}^{(\alpha)}(x) = 0$

2.2.2 Jacobi Polynomials (Askey, 1999)

For $\alpha > -1\,, \beta > -1,$ the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which is orthogonal on the interval -1≤ $x \leq 1$ with respect to the weight function w(x)=$(1-x)^{\alpha}(1+x)^{\beta}$, are given by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x+1)^k(x-1)^{n-k}\ ,$$

n=0,1,2,… .

If $\alpha = \beta$, the polynomials $P_n^{(\alpha,\beta)}(x),$ are called “Ultraspherical Polynomials”.

Some special cases of Jacobi polynomials which depend on the values of α and β are given below:

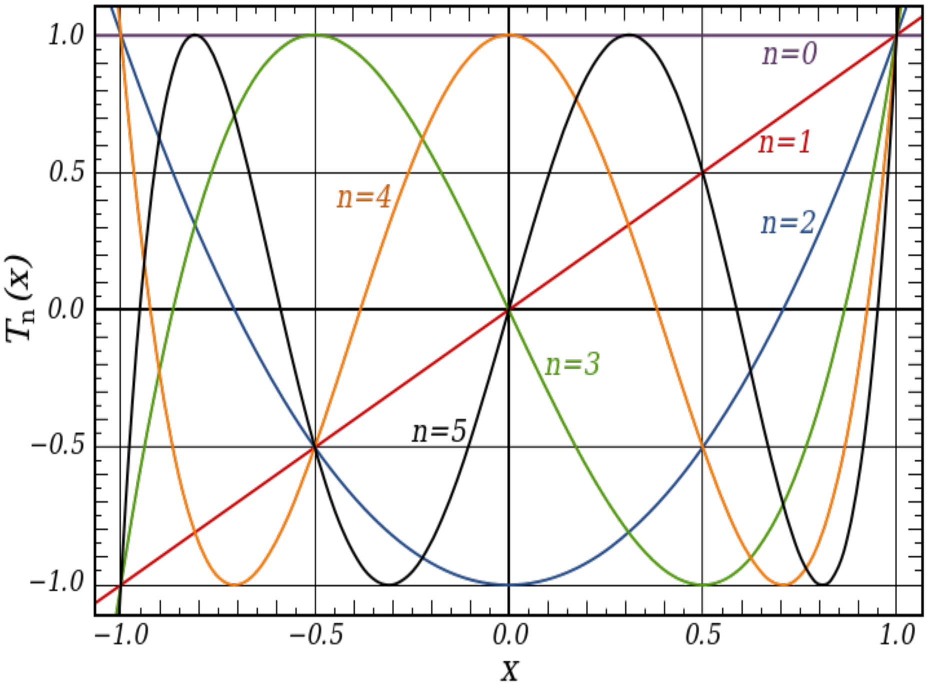
1. For $\alpha = \beta = -\frac{1}{2}$, the polynomials

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \sum_{k=0}^{[n/2]}\frac{n!\,x^{n-2k}(x^2-1)^k}{(2k)!\,(n-2k)!} = T_n(x)\ ,$$

are called “I. Type Chebyshev Polynomials”.
Some of the polynomials $T_n(x)$ are

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$

The graphs of first six I. Type Chebyshev Polynomials $T_0(x), T_1(x), T_2(x),$ $T_3(x), T_4(x)$ and $T_5(x)$ are shown in the figure:



2. For $\alpha = \beta = 0$, the polynomials

$$P_n^{(0,0)}(x) = 2^{-n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} = P_n(x),$$

are called " Legendre Polynomials". Let us give the first five Legendre polynomials;

$P_0(x) = 1$

$P_1(x) = \mathrm{x}$

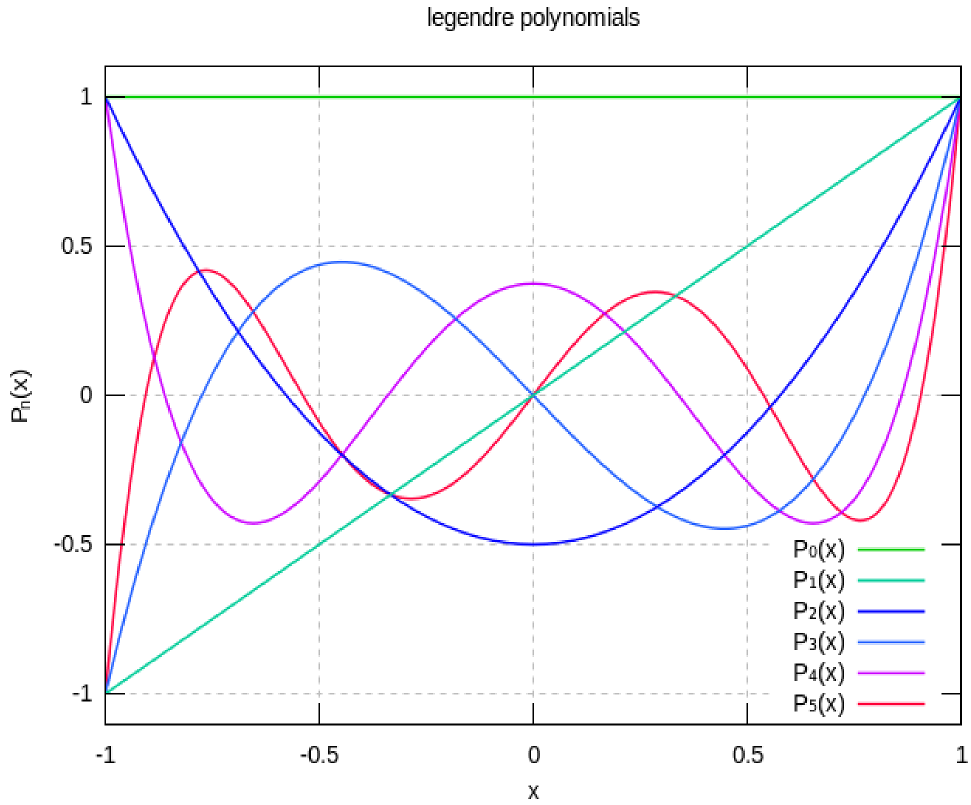
$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(3x^2 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$

The graphs of first six Legendre polynomials $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$ and $P_5(x)$ are shown in the figure:



Here

$$\left[\frac{n}{2}\right] = \begin{cases} \dfrac{n}{2} & if\ n\ is\ even, \\ \dfrac{n-1}{2} & if\ n\ is\ odd. \end{cases}$$

If $\frac{d}{dx}\left[(1-x^2)(1-x)^{\alpha}(1+x)^{\beta}\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\right]$, is used to start, the Jabobi differential equation can be obtain as

(1-x^2)$y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \beta + \alpha + 1)y = 0$,

which has the solutions as Jacobi polynomials.

Generating function for the Jacobi polynomials are given as

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n$$
$$= \frac{2^{\alpha+\beta}}{\sqrt{1-2tx+t^2}\,[1-t+\sqrt{1-2xt+t^2}][1+t+\sqrt{1-2xt+t^2}]}.$$

Finally, the recurrence relation for Jacobi polynomials are given as

2(n+1)(n+$\alpha+\beta-1$)(2n+$\beta+\alpha$)$P_{n+1}^{(\alpha,\beta)}(x)$ - $[(2n+\alpha+\beta+1)(\alpha^2-\beta^2)(2n+\alpha\beta+\beta)x]P_n^{(\alpha,\beta)}(x)+2(n+\alpha)(\alpha+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x)=0$.

2.2.3 Hermite Polynomials (Askey, 1999)

The $H_n(x)$ Hermit polynomials, which are orthogonal on the interval -∞ $< x <$ ∞ with respect to the weight function w(x)=e^{-x^2} given by,

$$\phi_n(x) = H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!\,(n-2k)!}(2x)^{n-2k} ,$$

n=0,1,2, … .

Some of the polynomials $H_n(x)$ are,

$H_0(x) = 1$

$H_1(x) = 2x$

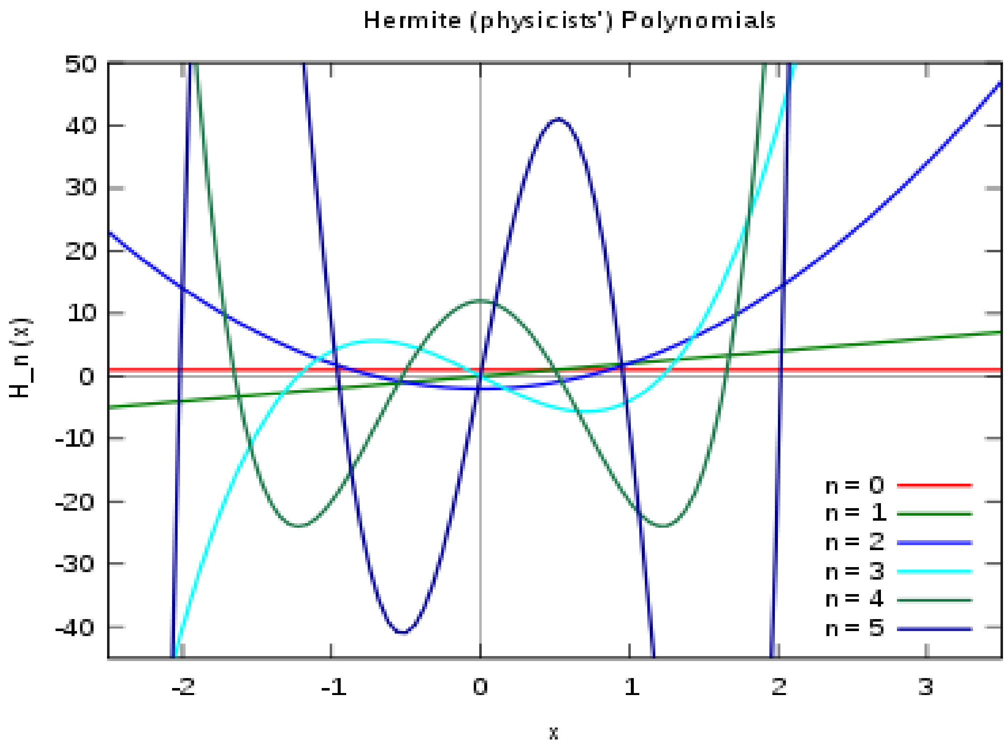
$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

The graphs of first six Hermit polynomials $H_0(x), H_1(x), H_2(x), H_3(x), H_4(x)$ and $H_5(x)$ are shown in the figure:



Rodrigues formula for Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2}).$$

The generating function for the Hermite polynomials

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{2.7}$$

Norm of the Hermite polynomials

$$\|H_n(x)\|^2 = \int_{-\infty}^{\infty} e^{-x^2} H_n(x)dx = 2^n\sqrt{\pi}\, n!$$

Form the equation

$$\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}H_n(x)\right]$$

The Hermite differential equation can be obtained as

$$y'' - 2xy' + 2ny = 0,$$

Which has the solution as Hermite polynomials.

Finally, the recurrence relation for the Hermite polynomials given as

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \quad (2.8)$$

By using generating function, (2.7), we can obtain the recurrence relation above by following steps.

Take the derivative of both side in (2.7) with respect to t.

$$(2x-2t)e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}nt^{n-1}$$

$$(2x-2t)\sum_{n=0}^{\infty}\frac{H_n(x)}{n!} = \sum_{n=1}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1}$$

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n - \sum_{\substack{n=0\\ n\to n-1}}^{\infty}\frac{2H_n(x)}{n!}t^{n+1} = \sum_{n=0}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1}$$

if the indices are manipulated to make all powers of t as t^n,

$$\sum_{n=0}^{\infty}\frac{2xH_n(x)}{n!}t^n-\sum_{n=0}^{\infty}\frac{2H_{n-1}(x)}{(n-1)!}t^n=\sum_{n=0}^{\infty}\frac{H_{n+1}(x)}{n!}t^n$$

and open some terms to start the summations from 1,

$$2xH_0(x)-\sum_{n=1}^{\infty}(2xH_n(x)-2nH_{n-1}(x))\frac{t^n}{n!}=H_0(x)+\sum_{n=1}^{\infty}H_{n+1}(x)\frac{t^n}{n!}$$

is obtained. By the equality of the coefficients of the term $\frac{t^n}{n!}$,

$2xH_n(x)-2nH_{n-1}(x)=H_{n+1}(x),$

can be written, which gives the recurrence relation (2.8).

CHAPTER 3

BIORTHOGONAL POLYNOMIALS

In 1951, L.Spencer and U.Fano introduced a particular pair of biorthogonal polynomial sets in carrying out calculations involving the penetration of gamma rays through matter Spencer and Fano did not establish any general properties of biorthogonal polynomial sets, but essentially utilized the biorthogonality of polynomials in x and polynomials in x^2 with respect to the weight function $x^{\alpha}e^{-x}$, where α is a nonnegative integer, over the interval (0,∞).

3.1 Biorthogonal Polynomials

Definition 3.1

Let r(x) and s(x) be real polynomials in x of degree h > 0 and k > 0 , respectively. Let $R_m(x) and\ S_n(x)$ denote polynomials of degree m and n in r(x) and s(x) , respectively. Then $R_m(x) and\ S_n(x)$ are polynomials of degree mh and nk in x. Here, the polynomials r(x) and s(x) are called basic polynomial.

Notation 3.1

Let $[R_m(x)]$ denote the set of polynomials R_0, R_1, R_2, . . . of degree 0,1,2,… in r(x). Let $[S_m(x)]$ denoted the set of polynomials S_0, S_1, S_2, ... of degree 0,1,2, . . . in s(x).

Definition 3.2 (Konhouser, 1965)

The real-valued function p(x) of the real variable x is an admissible weight function on the finite or infinite interval (a,b) if all the moments

$$I_{i,j} = \int_a^b p(x)[r(x)]^i[s(x)]^j dx, \qquad i,j = 0,1,2,\ldots$$

exist, with

$$I_{0,0} = \int_a^b p(x)dx \neq 0.$$

For orthogonal polynomials, it is customary to require p(x) be non-negative on the interval (a,b). This requirement is necessary fot the establishment of certain properties for biorthogonal polynomials, this is found necessarily to require that p(x) be either nonnegative or non-positive , with $I_{0,0} \neq 0$, on the interval (a,b).

Definition 3.3 (Konhouser, 1965)

The polynomial sets $R_m(x) and\ S_n(x)$ are biorthogonal over the interval (a,b) with respect to the admissible weight function p(x) and the basic polynomials r(x) and s(x) provided the orthogonality conditions

$$J_{m,n} = \int_a^b p(x)R_m(x)S_n(x)dx = \begin{cases} 0\ , & m \neq n \\ \neq 0\ , & m = n \end{cases}, \quad m,n = 0,1,2,\ldots \quad (3.1)$$

are satisfied.

The orthogonality conditions (3.1) are analogous to the requirements (1.1) for the orthogonality of a single set of polynomials. Following (1.9), it was pointed out that the requirement that the different from m=n was redundant. The requirement in (1.1) that $J_{m,n}$ be different from zero is not redundant. Polynomial sets $[R_m(x)]$ and $[S_n$ (x)] exist such that

$$J_{m,n} = \begin{cases} 0\ , & m \neq n \\ \neq 0, & m = n \end{cases}, \ \text{m,n=0,1,2,…} \ , \quad \text{and}\ J_{k,k} = 0.$$

Definition 3.4

If the leading coefficient of polynomial is unity. The polynomial is called monic.

Now, let give the alternative definition for biorthogonality condition the following theorem is the analogue of the Theorem (2.3) which gives an alternative definition for orthogonality condition.

Theorem 3.1 (Konhouser, 1965)

If p(x) is an admissible weight function over the interval (a,b) and if the basic polynomials r(x) and s(x) are such that for n=0,1,2,…,

$$\int_a^b p(x)[r(x)]^j S_n(x)dx = \begin{cases} 0, & j = 0,1,2,\dots,n-1 \\ \neq 0, & j = n \end{cases} \tag{3.2}$$

and

$$\int_a^b p(x)[s(x)]^j R_m(x)dx = \begin{cases} 0, & j = 0,1,2,\dots,m-1, \\ \neq 0, & j = m, \end{cases} \tag{3.3}$$

are satisfied, then

$$\int_a^b p(x)R_m(x)S_n(x)dx = \begin{cases} 0, & m \neq n \\ \neq 0, & n = m \end{cases}, \qquad m,n = 0,1,2,\dots \tag{3.4}$$

holds. Conversely, when (3.4) holds then both (3.2) and (3.3) hold.

Proof

If (2.3) and (2.4) hold, then constants, $c_{m,j}, j$ =0,1,…,m, ($c_{m,m} \neq 0$), exist such that

$$R_m(x) = \sum_{j=0}^{m} c_{m,j}[r(x)]^j.$$

If m$\leq n$, then

$$\int_a^b p(x)R_m(x)S_n(x)dx = \int_a^b p(x)\sum_{j=0}^{m} c_{m,j}[r(x)]^j S_n(x)dx$$

$$= \sum_{j=0}^{m} c_{m,i}\int_a^b p(x)[r(x)]^j S_n(x)dx.$$

in virtue of (3.2),

$$\int_a^b p(x)[r(x)]^j S_n(x)dx$$

vanishes except where j=m=n.

If m > n, then constants $d_{n,j}$, j=0,1,…,n ($d_{n,n} \neq 0$), exist such that

$$S_n(x) = \sum_{j=0}^{m} d_{n,j}[s(x)]^j,$$

and the argument is completed as in the case m $\leq n$.

Now, assume that (3.4) holds. Then constants $e_{m,i}$ *and* $f_{n,i}$ exist such that

$$[r(x)]^j = \sum_{i=0}^{j} e_{m,i}\, R_i(x)\ ,$$

and

$$[s(x)]^j = \sum_{i=0}^{j} f_{n,i}\, S_i(x)\ .$$

If $0 \leq j \leq n,$ then

$$\int_a^b p(x)[r(x)]^j S_n(x)dx = \int_a^b p(x) \sum_{i=0}^{j} e_{m,i} R_i(x) S_n(x)dx$$

$$= \sum_{i=0}^{j} e_{m,i} \int_a^b p(x) R_i(x) S_n(x)dx.$$

If i=1,2,3,…,j , j < n, each interval on the right side is zero since (3.4) holds. If j = n , the interval on the right side is different from zero. Therefore (3.2) holds. In like manner (3.3) can be established.

3.2 Investigation Of Sufficient Conditions Which Ensure The Existence Of Biorthogonal Polynomials

The determinant Δ_n depends upon the moments $I_{i,j}, whith, in\ turn, depend\ upon\ the\ basic\ polynomials\ r(x) and\ s(x)$, the weight function p(x) and the interval (a,b). It is natural to attempt to attempt to determine sufficient conditions which ensure that $\Delta_n \neq 0$, n=1,2,3,… .In this direction, partial results will be obtained.

Notation

The determinant Δ_n is given by

$$\Delta_n = \begin{vmatrix} I_{0,0} & I_{0,1} & \cdots & I_{0,n-1} \\ I_{1,0} & I_{1,1} & \cdots & I_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n-1} \end{vmatrix}$$

If p(x) is an admissible weight function then $\Delta_1 = I_{0,0} \neq 0$.

Theorem

Given the basic polynomials r(x) and s(x) and an arbitrary admissible weight function p(x) on the interval (a,b), polynomial sets $[R_m(x)]$ and $[S_m(x)]$ satisfying the biorthogonality requirement (3.4) exist if and only if the determinant Δ_n is different from zero for n=1,2,3,… . Moreover. The polynomials are unique, each to within a multiplicative constant.

Proof

It is convenient to use the equivalent conditions (3.2) and (3.3) in place of (3.4). Coefficients $c_{n,0}, c_{n,1}, \dots, c_{n,n}$ and $d_{n,0}, d_{n,1}, \dots, d_{n,n}$, with $c_{n,n}, d_{n,n} \neq 0$, are required such that for n=0,1,2,…

$$\int_a^b p(x)\sum_{i=0}^{n} c_{n,i}[r(x)]^i[s(x)]^j dx = \begin{cases} 0 & , \ j = 0,1,\dots,n-1 \\ \neq 0 & , \quad j = n \end{cases}, \qquad (3.5)$$

and such that

$$\int_a^b p(x)\sum_{i=0}^{n} d_{n,i}[r(x)]^i[s(x)]^j dx = \begin{cases} 0 & , \ j = 0,1,\dots,n-1 \\ \neq 0 & , \quad j = n \end{cases}. \qquad (3.6)$$

In term of the moments $I_{i,j}$, requirements (3.5) and (3.6) may be written

$$\sum_{i=0}^{n} c_{n,i} I_{i,j} = \begin{cases} 0 & , \quad j = 0,1,\dots,n-1 \\ \neq 0 & , \quad j \neq n \end{cases} \qquad (3.7)$$

and

$$\sum_{i=0}^{n} d_{n,i} I_{i,j} = \begin{cases} 0 & , \quad j = 0,1,\dots,n-1 \\ \neq 0 & , \quad j \neq n \end{cases}. \qquad (3.8)$$

The first n requirements of (3.7) constitute a system of linear equations in the n unknowns $c_{n,o}/c_{n,n}$, $c_{n,1}/c_{n,n}$,. . . , $c_{n,n-1}/c_{n,n}$. The system will have a unique solution if and only if the coefficient determinant, which is precisely Δ_n , is different from zero.

The first n requirements of (3.8) constitute a system of n linear equations in the n unknowns $d_{n,o}/d_{n,n}$, $d_{n,1}/d_{n,n}$,. . . , $d_{n,n-1}/d_{n,n}$. The determinant of this system is the transpose Δ_n^T of Δ_n and is nonzero if and only if Δ_n is nonzero. Therefore, if $\Delta_n \neq 0$, n=1,2,3,… , then polynomials can be found which satisfy the first n requirements of both (3.7) and (3.8). Moreover, the polynomials are unique, each to within a multiplicative constant.

The (n+1)st requirements of (3.7) and (3.8) must snow be examined. The (n+1)st requirement of (3.7) is that

$$\sum_{i=0}^{n} c_{n,i} I_{i,n} \neq 0. \tag{3.9}$$

If we replace $c_{n,i}$ by its value as determined from the solution of the system of n equations the left side of (3.9) is $c_{n,n}/\Delta_n$ times the determinant Δ_{n+1} must be different from zero for n=1,2,3,… . ın life manner, the (n+1)st requirement for (3.8) leads to the requirement that

$$\sum_{i=0}^{n} d_{n,i} I_{n,i} = \frac{d_{n,n}\Delta_{n+1}}{\Delta_n} \neq 0.$$

Multiplication of the polynomials $R_n(x)$ and $S_n(x)$

By constants, which are not necessarily the same for every value of n, does not affect the satisfaction of requirements (3.7) and (3.8). In conclusion, a necessary and sufficient condition for the existence of biorthogonal polynomials are that the determinants Δ_n be different from zero for n=1,2,3,… .

In regard to the basic polynomials r(x) and s(x), it is clear that the biorthogonal polynomials determined by the basic polynomials r(x)+μ and s(x)+v, where μ and v are constants, are identical because of the orthogonality of $R_n(x)$ and $S_n(x)$ with respect to the constants $S_0(x)$ and $R_0(x)$, respectively.

3.3 Some Special Biorthogonal Polynomial Families

After Konhoauser found the general property of biorthogonal polynomials, there was a rise in interest of this topic. Then, in 1967 he defined a pair of biorthogonal polynomial family. This situation had the interest of this topic reached to the top point. The polynomial family that is called biorthogonal polynomial defined by Laguerre polynomial is also called Konhouser polynomial. After that year, the mathematicians were established biorthogonal polynomial family defined by the property of classic orthogonal polynomial.

3.3.1 Biorthogonal Polynomials Defined By Laguerre Polynomials

In this section, it will be mentioned about the pair of biorthogonal polynomial defined by Konhauser in 1967.

$Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ polynomials are given by, $\alpha > -1$,

$$Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)} \qquad (3.10)$$

and

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n \qquad (3.11)$$

the expressions (3.10) and (3.11) yield,

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k) x^i dx = \begin{cases} 0 \quad , & i = 0,1,\dots,n-1 \\ \neq 0 \quad , & i = n \end{cases} \tag{3.12}$$

and

$$\int_0^\infty x^\alpha e^{-x} Y_n^\alpha(x;k) x^{ki} dx = \begin{cases} 0 \quad , & i = 0,1,\dots,n-1 \\ \neq 0 \quad , & i = n \end{cases} \tag{3.13}$$

respectively. If (3.10) is put in (3.12), there follows,

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k) x^i dx$$

$$= \int_0^\infty x^\alpha e^{-x} \frac{\Gamma(kn+\alpha+1)}{n!} \times \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} x^i dx$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{\Gamma(kj+\alpha+1)} \int_0^\infty e^{-x} x^{kj+\alpha+1} dx$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\Gamma(kj+\alpha+i+1)}{\Gamma(kj+\alpha+1)} \,. \tag{3.14}$$

It can be obtained easily that,

$$D^i x^{kj+\alpha+i}\Big|_{x=1} = (kj+\alpha+i)(kj+\alpha+i-1)\dots(kj+\alpha+1) x^{kj+\alpha}\Big|_{x=1}$$

$$= (kj+\alpha+i)(kj+\alpha+i-1)\dots\frac{\Gamma(kj+\alpha+1)}{\Gamma(kj+\alpha+1)}$$

$$= \frac{\Gamma(kj+\alpha+i+1)}{\Gamma(kj+\alpha+1)}.$$

By following the quantity that is was obtained above, the expression (3.14) can be written that,

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k) x^i dx = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j \binom{n}{j}\left(D^i x^{kj+\alpha+i}\Big|_{x=1}\right)$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} D^i x^{\alpha+i}\sum_{j=0}^{n}(-1)^j \binom{n}{j} x^{kj}\Big|_{x=1}$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} D^i x^{\alpha+1}(1-x^k)^n\Big|_{x=1}.$$

This expression is equal to zero for i=0,1,…,n-1 however it is different from zero for i=n. This shows that the expression (3.12) is satisfied. In s similar way, it shows that (3.13) is satisfied as well.

Theorem

If (3.12) and (3.13) are satisfied, $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ is biorthogonal with respect to $x^\alpha e^{-x}$ over the interval (0,∞) by Theorem 3.2.1.

That is,

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k) Y_n^\alpha(x;k) dx = \begin{cases} 0 \quad , & m \neq n \\ \neq 0 \quad , & m = n \end{cases} \tag{3.15}$$

is satisfied. In fact for m$\leq n$, it is written

$$J_{m,n} = \int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k) Y_n^\alpha(x;k) dx$$

$$= \int_0^\infty x^\alpha e^{-x}\left\{\frac{1}{m!}\sum_{i=0}^{m}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_m\right\} Z_n^\alpha(x;k) dx$$

$$= \frac{1}{m!}\sum_{i=0}^{m}\frac{1}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_m\left\{\int_0^\infty x^\alpha e^{-x}Z_n^\alpha(x;k)x^i dx\right\}$$

$$= \begin{cases} 0 & , \quad i = 0,1,\dots,n-1, n \neq m \\ \neq 0 & , \quad i = n = m \end{cases}.$$

For $m \geq n$, it is obtained

$$J_{m,n} = \int_0^\infty x^\alpha e^{-x}\, Z_n^\alpha(x;k)Y_n^\alpha(x;k)dx$$

$$= \int_0^\infty x^\alpha e^{-x}\left\{\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}\right\}Y_n^\alpha(x;k)dx$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{\Gamma(kj+\alpha+1)}\left\{\int_0^\infty x^\alpha e^{-x}Y_n^\alpha(x;k)x^i dx\right\}$$

$$= \begin{cases} 0 & , \quad j = 0,1,\dots,n-1, n \neq m \\ \neq 0 & , \quad j = n = m \end{cases}.$$

These expressions show that the equation (3.15) is satisfied.

In the definition of (3.10) and (3.11), it is seen that the polynomials $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ can be reduced to Laguerre polynomials defined by (3.15) for k=1. In a similar way the orthogonality relations (3.12) and (3.13) can be reduced to the orthogonality relation of Laguerre polynomials defined by (3.14) for k=1. So, the polynomials $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ are called biorthogonal polynomial defined by Laguerre polynomial and Konhauser polynomials.

3.3.2 Biorthogonal Polynomials Defined By Jacobi Polynomials

At this time, the pair of polynomials that were defined by Madhekar and Thakare and can be reduced to Jacobi Polynomials for k=1 will be defined.

$J_n(\alpha, b, k; x)$ and $K_n(\alpha, b, k; x)$ polynomials are given by

$$J_n(\alpha, b, k; x) = \frac{(1+\alpha)_{kn}}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}} \tag{3.16}$$

and

$$K_n(\alpha, b, k; x) = \sum_{r=0}^{n} \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} \frac{(1+\beta)_n}{n!\,r!\,(1+\beta)_{n-r}} \left(\frac{s+\alpha+1}{k}\right)_n \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^n \tag{3.17}$$

respectively.

The polynomial family $J_n(\alpha, b, k; x)$ and $K_n(\alpha, b, k; x)$ that were given in (3.16) and (3.17) are biorthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ over the interval (-1,1).

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} J_n(\alpha, b, k; x)\, K_m(\alpha, b, k; x) dx = \begin{cases} 0 \;, & m \neq n \\ \neq 0 \;, & m = n \end{cases} ; m, n = 0,1, \ldots$$

For k=1, both of $J_n(\alpha, b, k; x)$ and $K_n(\alpha, b, k; x)$ polynomials are reduced to Jacobi polynomials. $P_n^{(\alpha,\beta)}(x)$ and is called biorthogonal polynomials defined by Jacobi polynomials.

CHAPTER 4

q-ORTHOGONAL POLYNOMIALS

4.1 q-Orthogonal Polynomials

q-orthogonal polynomial family is a generalization of classic orthogonal polynomials family. These generalization has may common properties with orthogonal polynomials.

Definition 4.1(q-orthogonal polynomial)

For $|q| < 1$, let w(x;q) be a positive weight function which is defined on the set $\{\alpha q^n, bq^n; n \in \mathbb{N}_0\}$. If the polynomials $\{P_n(x;q)\}_{n\in\mathbb{N}_0}$ satisfy the following property:

$$\int_a^b P_m(x;q)P_n(x;q)w(x;q)d_qx = \begin{cases} 0 & (m \neq n) \\ \neq 0 & (m = n) \end{cases}, (m,n \in \mathbb{N}_0), \qquad (4.1)$$

then the polynomial $P_n(x;q)$ are called q-orhogonal polynomials with respect to the weight function w(x;q) over the interval (a,b).

For q→ 1^- , (4.1) q-orthogonality condition gives the orthogonality condition (2.1).

4.2 Some Special q-Orthogonal Polynomials Families

4.2.1 q-Laguerre Polynomial

The ordinary Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{(\alpha+1)_k k!},$$

where $\alpha_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)$. These polynomials satisfy the orthogonality relation

$$\int_0^\infty L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x) x^\alpha e^{-x} dx = \begin{cases} \dfrac{\Gamma(\alpha+n+1)}{n!} & , m = n \\ 0 \quad , & m \neq n \end{cases}$$

There is a q-analogue of these polynomials which is defined as

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (1-q)^k (q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k (q;q)_k}. \qquad (4.2)$$

Note that $L_n^{(\alpha)}(x;q) \to L_n^{(\alpha)}(x)$ as q$\to 1^-$.

One orthogonal relation is:

Theorem 1.

For $\alpha > -1$, the following integral is given by

$$\int_0^\infty L_n^{(\alpha)}(x;q)\, L_m^{(\alpha)}(x;q) \frac{x^\alpha dx}{(-(1-q)x;q)_\infty}$$

$$= \begin{cases} \dfrac{\Gamma(\alpha+1)\Gamma(-\alpha)(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)(q;q)_n q^n} & , \quad m = n, \\ 0 \qquad , & m \neq n. \end{cases} \qquad (4.3)$$

Proof

Firstly, it should be shown that

$$\int_0^\infty L_n^{(\alpha)}(x;q) x^m \frac{x^\alpha dx}{(-(1-q)x;q)_\infty} = 0\, , m < n. \quad (4.4)$$

In fact by (1.7),

$$\int_0^\infty L_n^{(\alpha)}(x;q)\frac{x^{\alpha+m}dx}{(-(1-q)x;q)_\infty}$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k(1-q)^k}{(q^{\alpha+1};q)_k(q;q)_k}\times\int_0^\infty\frac{x^{k+\alpha+m}}{(-(1-q)x;q)_\infty}dx$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\times\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k\Gamma(-k-\alpha-m)\Gamma(1+k+\alpha+m)}{(1-q)^{-k}(q^{\alpha+1};q)_k(q;q)_k\Gamma(-k-\alpha-m)}.$$

By the reflection formula for the gamma function and the functional equation for the q-gamma function, it is reached that,

$$=\frac{\pi(q^{\alpha+1};q)_n\,csc(-\alpha\pi-m\pi)}{(q;q)_n\Gamma_q(-\alpha-m)}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k(-)^k(q^{-\alpha-k-m};q)_k}{(q^{\alpha+1};q)_k(q;q)_k}$$

$$=\frac{\pi(q^{\alpha+1};q)_n\,csc(-\alpha\pi-m\pi)}{(q;q)_n\Gamma_q(-\alpha-m)}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(q^{n+\alpha+1})^k q^{-\alpha k-mk-\binom{k+1}{2}}(q^{\alpha+m+1};q)_k}{(q^{\alpha+1};q)_k(q;q)_k}$$

$$=\frac{(q^{\alpha+1};q)_n\Gamma(-\alpha)\Gamma(\alpha+1)(q^{-\alpha-m};q)_m(-)^m}{(q;q)_n(1-q)^m\Gamma_q(-\alpha)}$$

$$\times\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{\alpha+m+1};q)_k(q^{n-m})^k}{(q^{\alpha+1};q)_k(q;q)_k}.$$

There is a sum due to Heine [see 3.p.68],

$$\sum_{k=0}^{\infty}\frac{(\alpha;q)_k(b;q)_k}{(c;q)_k(q;q)_k}\left(\frac{c}{ab}\right)^k=\frac{(\frac{c}{\alpha};q)_\infty(\frac{c}{b};q)_\infty}{(c;q)_\infty(\frac{c}{ab};q)_\infty}$$

in particular when $\alpha=q^{-n}$,

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k (b;q)_k}{(c;q)_k (q;q)_k} \left(\frac{cq^n}{b}\right)^k = \frac{(c/b\,;q)_n}{(c;q)_n}$$

hence the integral (4.4) can be written as the following way.

$$\int_0^\infty L_n^{(\alpha)}(x;q) \frac{x^{m+\alpha}dx}{(-(1-q)x;q)_\infty}$$

$$= \frac{(q^{\alpha+1};q)_n \Gamma(-\alpha)\Gamma(\alpha+1)(q^{\alpha+1};q)_m (q^{-m};q)_n}{(q;q)_n \Gamma_q(-\alpha)(1-q)^m q^{\alpha m+\binom{m+1}{2}} (q^{\alpha+1};q)_n}$$

$$= \begin{cases} 0 & ,if \quad m<n \\ \dfrac{\Gamma(-\alpha)\Gamma(\alpha+1)(q^{\alpha+1};q)_n(-)^n}{\Gamma_q(-\alpha)q^{\alpha n+n^2+n}(1-q)^n} & , \quad m=n \end{cases}$$

since $L_n^{(\alpha)}(x;q) = \left(\frac{(-)^n q^{n^2+n\alpha}(1-q)^n}{(q;q)_n}\right) x^n + \cdots$, it is obtained,

$$\int_0^\infty (L_n^{(\alpha)}(x;q))^2 \frac{x^\alpha dx}{(-(1-q)x;q)_\infty} = \frac{\Gamma(-\alpha)\Gamma(\alpha+1)(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)(q;q)_n q^n},$$

which proves the theorem.

The measure can be normalized so that the total mass in one, and it is deduced that,

$$\int_0^\infty L_n^{(\alpha)}(x;q)\, L_m^{(\alpha)}(x;q) \frac{x^\alpha \Gamma_q(-\alpha)dx}{(-(1-q)x;q)_\infty \Gamma(-\alpha)\Gamma(\alpha+1)} = \begin{cases} \dfrac{(q^{\alpha+1},q)_n}{(q;q)_n q^n}, & m=n \\ 0\,, & m\neq n. \end{cases}$$

There is another orthogonality relation using Ramanujan’s sum.

4.2.2 q- Hermite Polynomial

The continuous q-Hermit polynomials $\{H_n(x|q)\}$ are generated by the recursion relation

$$2xH_n(x|q) = H_{n+1}(x|q) + (1-q^n)H_{n-1}(x|q), \tag{4.5}$$

and the initial conditions

$$H_0(x|q) = 1\,, \quad H_1(x|q) = 2x. \tag{4.6}$$

Our fist take is to derive generating function for $\{H_n(x|q)\}$. Let

$$H(x,t) = \sum_{n=0}^{\infty} H_n(x|q)\frac{t^n}{(q;q)_n}.$$

Multiply (4.5) by $\frac{t^n}{(q;q)_n}$, add for n=1,2,3,… , and take into account the initial condition (4.6). we obtain the functional equation

$$H(x,t) - H(x,qt) = 2xtH(x,t) - t^2H(x,t).$$

Therefore

$$H(x,t) = \frac{H(x,qt)}{1-2x+t^2} = \frac{H(x,qt)}{(1-te^{i\theta})(1-te^{-i\theta})}\,, \tag{4.7}$$

x=cosθ.

This suggests iterating the functional equation (4.7) to get

$$H(cos\theta,t) = \frac{H(cos\theta,q^nt)}{(te^{i\theta},te^{-i\theta};q)_n}.$$

As n→ ∞ , H(x,q^nt) → H(x,0) = 1.

CHAPTER 5

q-BIORTHOGONAL POLYNOMIALS

Firstly in 1983, q-biorthogonal polynomial that is q-analogue of biorthogonal polynomials were come out by Al-Salam and Verma (1983) These polynomials defined by (5.1) and (5.2) are seen as the definition of q-Konhouser polynomials that is q-analogue of Konhouser polynomial. In 1992, Jain and Srivastava obtained the alternative definitions of these polynomials.

Basic definitions and general conditions of q-biorthogonality are given by Şekeroğlu , Srivastava and Taşdelen (2007).

5.1 q-Biorthogonal Polynomials

Definition 5.1

For $|q| < 1$, let r(x;q) and s(x;q) be polynomials in x of degrees h and k, respectively (h,k∈ ℕ). Also let $R_m(x;q) and\ S_n(x;q)$ denote polynomials of degrees m and n in r(x;q) and s(x;q), respectively. Then $R_m(x;q)$ and $S_n(x;q)$ are polynomials of degrees mh and nk in x . The polynomials r(x;q) and s(x;q) are called the q-basic polynomials.

For $|q| < 1$, let $\{R_n(x;q)\}_{n=0}^{\infty}$ denote the set of polynomials

$$R_0(x;q), R_1(x;q), \dots, R_n(x;q), \dots$$

of degrees

$$0,1,\dots,n,\dots \qquad in\ r(x;q).$$

Similarly , let $\{S_n(x;q)\}_{n=0}^{\infty}$ denote the set of polynomials

$$S_0(x;q), S_1(x;q), \dots, S_n(x;q), \dots$$

of degrees

$$0,1,\dots,n,\dots \qquad in\ s(x;q).$$

Definition 5.2

For $|q| < 1$, let w(x;q) be an admissible weight function which is defined on the set

$$\{aq^n, bq^n; n \in \mathbb{N}_0\}.$$

If the polynomial sets

$\{R_n(x;q)\}_{n=0}^{\infty}$ and $\{S_n(x;q)\}_{n=0}^{\infty}$

satisfy the following condition:

$$\int_a^b R_m(x;q)S_n(x;q)w(x;q)d_qx = \begin{cases} 0\ . & (m \neq n) \\ \neq 0 & (m = n) \end{cases} \quad (m, n \in \mathbb{N}_0) \qquad (5.1)$$

then the polynomial sets

$\{R_n(x;q)\}_{n=0}^{\infty}$ and $\{S_n(x;q)\}_{n=0}^{\infty}$

are said to be q-biorthogonal over the integral (a,b) with respect to the weight function w(x;q) and the q-basic polynomials r(x;q) and s(x;q).

The q-biothogonality condition (5.1) is analogous to the q-orthogonality condition (4.1). We also note that, when q$\to 1-$, the q-biorthogonality condition (5.1) gives us the usual biorthogonality condition (3.1).

5.2 Some Special q-Biorthogonal Polynomial Families

5.2.1 q-Konhauset Polynomials

Remark 1. If we take the weight function

$$w(x;q) = x^{\alpha}e_q(-x)$$

over the interval (0,∞), we obtain the following q-Konhauser polynomials:

$$Z_n^{(\alpha)}(x,k;q)=\frac{(q^{1+\alpha};q)_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j q^{\frac{1}{2}kj(kj-1)+kj(n+\alpha+1)}}{(q^k;q^k)_j(q^{1+\alpha};q)_{jk}}x^{kj} \quad (5.2)$$

and

$$Y_n^{(\alpha)}(x,k;q)=\frac{1}{(q;q)_n}\sum_{r=0}^{n}\frac{x^r q^{\frac{1}{2}r(r-1)}}{(q;q)_r}\sum_{j=0}^{r}\frac{(q^{-r};q)_j(q^{1+\alpha+j};q^k)_n}{(q;q)_j}q^j\,, \quad (5.3)$$

which were considered by Al-Salam and Verma [1], who proved that

$$\int_0^\infty Z_n^{(\alpha)}(x,k;q)Y_m^{(\alpha)}(x,k;q)x^\alpha e_q(-x)d_qx=\begin{cases}0 \ (n\neq m),\\ \neq 0 \ (n=m).\end{cases}$$

Equation (5.4) does indeed exhibit the fact that the polynomials $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$ are q-biorthogonal polynomials with respect to the weight function $x^\alpha e_q(-x)$ over the interval (0,∞).

Remark 2. For k=1, the q-Konhauser polynomials in (5.2) and (1.4) reduce to the q-Laguerre polynomials given by (4.2)

Remark 3. Just as we indicated in the preceding section, Jain and Srivastava gave another pair of q-Konhauser polynomials which are defined by

$$Z_n^{(\alpha)}(x,k/q)=\frac{(\alpha q;q)_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j}{(\alpha q;q)_{kj}}\frac{(xq)^{kj}}{(q^k;q^k)_j}$$

and

$$Y_n^{(\alpha)}(x,k/q) = \frac{1}{(q;q)_n}\sum_{j=0}^{n}\frac{(xq)^j}{(q;q)_j}\sum_{i=0}^{j}\frac{(q^{-j};q)_i(\alpha q^{i+1};q^k)_n}{(q;q)_i}q^{(j-n)i}.$$

CHAPTER 6

CONCLUSIONS

In this thesis, definitions and basic properties of q-biorthogonal polynomials are given and q-Konhauser polynomials $Y_n(x, k; q)$ and $Z_n(x, k; q)$ are defined.

For k=1, q-Konhauser polynomials give the q-Laguerre polynomials. So, several properties of q-Laguerre polynomials can be generalized for the q-Konhauser polynomials.

Moreover , new q-biorthogonal families can be investigated and by using q-biorthogonality process, they can be obtained.

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