# COMPARISON BETWEEN NUMERICAL AND EXACT SOLUTIONS OF RICCATI DIFFERENTIAL EQUATIONS USING EXCEL

**A THESIS SUBMITTED TO**

**THE GRADUATE SCHOOL OF APPLIED SCIENCES**

**OF**

**NEAR EAST UNIVERSITY**

**By**


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**In Numerical Analysis for Ordinary Differential Equations**

**The Degree of Master of Science**

**InMathematics**


**NICOSIA 2015**

Ali Wahid Nwry: Comparison between Numerical and Exact Solutions of Riccati Differential Equations Using Excel

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# DECLARATION

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# ABSTRACT

The thesis deals with the numerical solutionsof various forms ofnonlinear Riccati Differential Equation. In doing that, several different numerical methods are used and for each numerical method a nonlinear Riccati Differential Equation was used as an illustrative example.The work provides an opportunity to judge and compare the adequacy of the numerical methods compared with the available close form solutions. The use of excel worksheet provides an easy way for implementing the numerical algorithms and also an easy and interactive way to see the effect of the step size graphically and immediately.

In each case a graphical representation for both exact and numerical solutions are presented and the results compare very well for majority of the cases without any need for finer step size .

**Keywords:** Riccati differential equation; Runge-Kutta method; absolute and % Relative errors; graphical figure representations.

## ÖZET

Bu tez doğrusal olmayan Riccati türev denkleminin çeşitli yönlerinin sayısal çözümleriyle ilgilenmektedir. Bunu yaparken birçok farklı sayısal metodlar kullanılmıştır ve her bir sayısal metod için doğrusal olmayan Riccati türev denklemi, tanımlayıcı bir örnek olarak kullanılmıştır. Çalışma en yakın çözümleriyle karşılaştırılmış yeterli sayısal metodları yargılamak ve karşılaştırmak için olanak sağlar. Excel sayfasının kullanımı sayısal algoritmaları uygulamaya koyarak kolay bir yön sağlamıştır ve ayrıca grafiksel ve hızlı bir şekilde basamak değerinin etkisini görmek için kolay ve interktif bir yön olmuştur.

Her bir bulguda doğru ve sayısal çözümler için grafiksel gösterimler uygulanmıştır ve sonuçlar bulguları detaylı bir basamak değeri kullanmaya gerek kalmadan karşılaştırmıştır.



**Anahtar sözcükler**:Riccati türev denklemi, Runge-Kutta metodu, mutlak ve yüzdelik oransal hatalar, grafiksel gösterimler.

## ACKNOWLEDGEMENTS

First and most importantly I would like to thank God for providing me the means and the strength enabling me to complete this work, because it is my belief that ALLA is always supporting us to make a best things in our life.

My special thanks and appreciation goes to my supervisor Dr. Abdulrahman Othman; for all his advices, assistances, instructions because he supported and guided me to always learn more he was a wonderful instructor throughout my study of MSc and also I extend my gratitude to his family.

My special thanks goes to our Director Prof. Dr. İ. Kaya Özkin, Assist. Prof. Dr. Evren Hinçal and my grateful to all others Teachers in our department.

My special thanks goes to my wonderful father because he continuously motivated me to finish this hard work. And also thanks for my family and my wife because she always encouraged me to gain more things in the real life. General thanks to all my friends.

This thesis was generally supported by the Department of mathematics of Near East University. I have great thanks for all supporter.

*The most important things that should be known, for achieving every things and operating all works, we are being require forhelping, supporting and motivating from our family, friends, and all peoples, in so far as we must have an appreciation for all of them.*

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## LIST OF SYMBOLS USED

| | |
|---|---|
| $y' = dy/dt$ | First derivative of dependent variable $y$ with respect to the independent variable t. |
| $y$ | Dependent variable |
| $t$ | Independent variable |
| $p$ | Continuous Function |
| $Q$ | Real constant |
| $r$ | Real constant |
| $t_o$ | Independent initial variable |
| $y_0$ | Dependent initial variable |
| $f$ | Function |
| $f(t, y)$ | Ordinary Differential Equations |
| □ | Step size (Increment variable |
| $\xi$ | Some Numbers between $t_i$ and $t_{i+1}$ |
| $y(t_i)$ | Exact solution |
| $w_i$ | Approximate solution |
| $k_j$ | Coefficients |
| $k_j$ | Coefficients |
| $y^{(n)} = d^n y/dt^n$ | Higher order derivatives |
| $du/dt$ | First derivative dependent variable $u$ to the independent variable $t$ |
| $\alpha$ | Value of initial condition |
| $i$ | Some positive integer number |
| $j$ | Some positive integer number |
| $M$ | Constant number satisfies $\|y''(t)\| \leq M$, for all $t \in (a, b)$. |
| L | The Lipschitz constant |
| $a$ | Lower bound of open interval $(a, b)$ |
| $b$ | Upper bound of open interval $(a, b)$ |
| $N$ | Positive integer number |

## ABBREVIATIONS USED

| | |
|---|---|
| ODEs | Ordinary Differential Equations |
| RDE | Riccati Differential Equation |
| RIVPs | Riccati Initial ValueProblems |
| IVPs | Initial Value Problems |
| TE | Truncation Error |
| RK-Method | Runge-Kutta Method |
| RKF-Method | Runge-Kutta-Fehlberg Method |
| RKV-Method | Runge-Kutta-Verner Method |
| MOE | Microsoft Office Excel |
| ABEM | Adams Bashforth Explicit Method |

# CHPTER 1

# INTRODUCTION

This thesis investigates an interesting type of Ordinary Differential Equations (ODEs) known as Riccati Differential Equations (RDE). The general form of the RDE together with its initial condition make a Riccati Initial value problem (RIVP) which is represented by

$$y' = p\ t\ y^2 + q\ t\ y + r\ t \quad , \quad y\ t_0 \ = y_0 \ , \qquad t_i \leq t \leq t_{i+1} \tag{1.1}$$

, to refers to the initial value of t, $t_i = t_0 + i$ and $t_{i+1} = t_i +$ , is the step size.

Provided that $p\ t\ \not\equiv 0$ , where $p, q, r$ are continuous functions [1], [4], [11], [14]. This equation is nonlinear first order differential equation because it contains $y^2$ and $dy/dt$. In fact that, to solve the Riccati differential equation by using someknown numerical methods that are used for solving Initial Value Problems (IVP) to identify the approximate solution [14], after that I will compare its solution to the exact solution so that we will judge the performance of these methods and judge them accordingly. This way we may gain the experience of judging which method is more suitable for any particular IVP.

It is known that most of the RDEs do not have the exact solution [21], that is, general solution cannot be obtained by the classical methods of solving ODEs or by direct integration using the elementary calculus or functionmanipulations [22]. However, general solutions may be obtained for some forms of RDE by knowing/guessing a particular solution first and hoping that this leads to a general solution [1], [2], [6], [11], and [12].

With the recent advancement in computer hardware and software developments, it is more efficient to seek numerical approximations to the solutions of IVP than going through hard working and tedious manipulations [28]. In fact this piece of work demonstrates exactly that. This is particularly so if Excel Spread Sheet is utilized as we can see later in the following chapters. One important point about Excel Sheet is that it is available almost in every desk/laptop and even in many smart mobile phones. I hope that the reader will appreciate what I mean they finish reading this thesis. The actual work starts in chapter two and onwards, with using the well-known classical numerical methods to bring the greatest and most appropriate methods to the reader in a simple and interesting way.

In the chapters that follow, numerical methods used here include. Euler's Method [3], [5], [7], [8], [10], [15], [16], [18], [19], Taylor's Method [3], [17], Runge-Kutta-Method [17], Runge-Kutta-Fehlberg Method [3], Runge-Kutta-Verner Method [3], and Adams-Bashforth Explicit Method [3]. In all the methods mentioned above, the absolute and the relative error between the numerical and the close form solutions are presented for the sake of comparison.

## 1.1 Literature Review

The Riccati Differential Equation (RDE), named after the Italian mathematician and nobleman Jacopo Francesco Riccati (1676-1754) [26]. One way to solve a RDE, need to have a particular solution. If we don't have at least one particular solution, then it cannot be possible to determine the general solution or no chance to solve such differential equation exactly [27]. Anequation (1.1) may be possible to take the integration by using some methods like the linear or separable variables, or it may have coefficients that are homogenous and of the same degree. If the RDE is not considered in the case of elementary function then we cannot find the integration by using the mentioned methods, may be solve by power series.Sometimes possible to determine a solution of a RDE by trail, such as $y = at^b$ ,or $y = ae^{bt}$ , with unspecified constants $a$ and $b$ . If the integral can be achieved then the general solution is available from it [1].Another work that we have seen over RDE is solved by a succession for two substitutions provided that we have the particular solution $y_1$of the given equation, normally by obtaining $y = y_1 + u$ then we can reduce Riccati's Equation into a Bernoulli equation and then reduce to the linear equation by using $w = u^{-1}$, the reduction can be fulfilled provided that the particular solution is known [2]. Assume that some of the particular solution of the equation (1.1) is known then the general solution can be found by using substitution $y = y_1 \; t \; + 1/u(t)$, after that we must verify that $u \; t$ satisfies the linear equation$u' = - \; q + 2py_1 \; u - p$ . Observe that $u \; t$ will include a unique arbitrary constant [11]. Another way for solving RDE is to assume that $u \; t$ is a solution of equation (1.1) then we can use substitution $y = u + 1/v$ in order to reduce the (1.1) into the linear equation in $v$ [12].In actually we have seen some works that a number of mathematicians have studied RDE, involving several of the Bernoulli, Riccati himself, and his son Vincenzo. At the end of 1723, it was known that equation (1.1) cannot be solved in the case of elementary functions, after that, Euler stated that if the particular solution of (1.1) is known then by using the substitution $y = u + 1/v$, converts the RDE in to the linear Differential Equation in $v$, and then we can get the general solution, he also said that, if the two particular solutions are available then the general solution is considerable in case of simple quadrature [6]. There are some works in the literature on the RDE that we have seen throughout investigation of this research, they are tried to determine the approximate solution to the fractional Riccati differential equation [21]. Another work for solving RDE is generalized Chebyshev wavelet operational matrix, in fact that used operational matrix with collocation points converts the fractional order RDE into a system of algebraic equations, also presented the accuracy and efficiency of the method computed by numerical examples [4]. Also in another paper thatthe author studied the general RDE by using the iterative decomposition method, the given equation includes one with variable coefficient and one in matrix form, constructed the consideration of comparison between the decomposition method and some existing methods, observe that the solution by computing some numerical examples [20]. Finally we can state that probably there are more works on the numerical solution of the Riccati Differential Equations but maybe, it is impossible to consult all the available research and that would be beyond the scope of this dissertation.

## 1.2 purpose

Our purpose here is to present a variety of numerical solutions to the RDE using Excel spread sheet and compare the result in each case with available close form solutions. A graphical representation is used to illustrate the comparison of each numerical solution with the exact solution, in fact, in most of the cases, the numerical and close form solutions compare very well.

# CHAPTER 2

# NUMERICAL METHODS

Numerical methods have been developed to provide an approximate solutions to ODEs that have no analytical solutions which is the case in most real life applications or as an easy alternative method for problems with complicated algebraic and/or functional manipulations. Often, even in these situations one still needs to seek some computer soft wares that are capable of algebraic and functional manipulations. This is particularly truein the caseof RiccatiDifferential Equation [23]. So if we cannot solve RIVP or any other differential equation analytically, then we must rise to seek another method for solving them approximately [7]. If the RDE has no solution exactly then we are going to use the numerical methods to find approximate solution. It is clear that if it has the actual solution then it is not necessary to evaluate the numerical solutions for ODEs [25]. Also in actually if we have the differential equation together with an initial condition then we can constitute the initial value problem or sometimes it can be written briefly as IVP, in our case because we are interested in the RDE then we can consider as an Riccati InitialValue Problem RIVP.

## 2.1 Euler’s Method

Euler’s method is a common and basic method for solvingIVPs but it is not very precise and also it is not very accurate numerical method, but it is mostly used forpresenting many ideas that are contained in the numerical methods for solving the initial value problems [19]. In this case,we will show how to apply Euler’s Methodto a form of RDE that represents Riccati Initial Value Problem (RIVP). Next we apply Euler’s Method to RIVPas an example and use Excel to implement the method and use its graphical facility to present the results.

### 2.1.1 Algorithm and Truncation Error of Euler's Method

Since the general form of the ordinary differential equation with initial condition has the form;

$$y' = f(t,y), \qquad t_0 \le t \le t_1, \qquad y(t_0) = y_0 \qquad (2.1.1.1)$$

And also the general form of the RDE given as;

$$y' = p(t)y^2 + q(t)y + r(t), \qquad t_0 \le t \le t_1, \qquad y(t_0) = y_0 \qquad (2.1.1.2)$$

We can see that the left hand side of the Riccati Differential Equation (2.1.1.2) with the left hand side of the equation (2.1.1.1) are equal, this means that the right hand sides are equal. Since in the point $(t_i, y(t_i))$ and let $t_{i+1} = t_i + h$ then the general form of the Euler's method was given as follow;

$$y_{i+1} = y_i + hf(t_i, y_i), \qquad (2.1.1.3)$$

This formula can be obtained from Taylor's series when $n = 1$, the benefit of this formula is need not to take the differentiation for the function $f$ [30].

Then we can substitute the right hand side of the RDE in the equation (2.1.1.3), this yields;

$$y_{i+1} = y_i + h\left(p(t_i)y_i^2 + q(t_i)y_i + r(t_i)\right), \qquad (2.1.1.4)$$

Since $t_{i+1} = t_i + h$, then put $i = 0,1,2,\dots,N$ to get

$t_{0+1} = t_0 + h = t_1 \Rightarrow t_2 = t_1 + h \Rightarrow t_3 = t_2 + h$, and so on,

Now we can consider the algorithm of Euler's formula over the RDE for 10 steps as follows;

$y(t_0 + h) = y(t_0) + h\{p(t_0)(y(t_0))^2 + q(t_0)y(t_0) + r(t_0)\}$,

$y(t_1 + h) = y(t_1) + h\{p(t_1)(y(t_1))^2 + q(t_1)y(t_1) + r(t_1)\}$,

$y(t_2 + h) = y(t_2) + h\{p(t_2)(y(t_2))^2 + q(t_2)y(t_2) + r(t_2)\}$,

.

.

$y(t_9 + h) = y(t_9) + h\{p(t_9)(y(t_9))^2 + q(t_9)y(t_9) + r(t_9)\}$

$$\text{Truncation Error} = \frac{h^2}{2!}y''(\xi), \qquad t_i < \xi < t_i + h \qquad (2.1.1.5)$$

The above relation is obtained by taking the first and second terms from the Taylor's series and truncating the third term involving $\frac{h^2}{2!}y''(\xi)$ [5], [7], and [19]. This term is known as the truncation error.

### 2.1.2 Calculation of Absolute and Relative Percentage of Errors

The absolute error is computed by the form $|y(t_i) - w_i|$, where the $y(t_i)$ value denote the exact solution and $w_i$ value denote to the approximation solution at $t = t_i$, as we can see in the formulation of absolute error, It is the absolute value of the difference between the approximate solution and the exact solution [3], [9].

Another formula for measuring the error is the percentage relative error which is calculated by this formula

$$\% \, RelativeError = \left|\frac{y(t_i) - w_i}{y(t_i)}\right| \times 100 \qquad (2.1.2)$$

Hopefully, if the error is small it indicate that the method used to determine the approximation solution is very good the results are near to the actual solution [9].

### 2.1.3 Error Bounds for Euler's Method.

A detailed analysis of the error bound for Euler's method is given in [1], however, only an outline of the analysis will be given here to help the reader to compute the error bound when they wish doing so. The actual error bound is given by:

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left(e^{L(t_i-a)} - 1\right) \quad (2.1.3)$$

Where: $|y(t_i) - w_i|$ represents the error between Numerical and exact solution. $M$ is a constant satisfies $|y''(t)| \leq M$, for all $t \in (a, b)$.

L is the Lipschitz constant and h is the step size.

It is generally accepted that, the smaller the step size h, the better the accuracy. But we have to be careful with this. There is a limit for, how small h should be to obtain the best solution. If we take a step size smaller than this limit we may get less accurate solution [1]. The other problem with small step size is more computation and more round of errors.

### 2.1.4 Application of Euler's Formula by Using Microsoft Office Excel (MOE) over the RDEs

If the step size $h = 0.1$ and let $t_0 = 0$, then we can consider the algorithm of Euler's Method by using Excelas the following constraint steps:

i. Devote the first column to count the iteration numbers starting from $i = 0, 1, 2, \dots, 10$.
ii. Dedicate the second column to put the value of step size $h$.
iii. Assign the third column to adding the initial variable $t_0$ with the step size $h$ in order to make the new variables $t_1, t_2, \dots, t_{10}$.
iv. Devoted the forth column to the exact solution.
v. Fifth column assigned to obtaining the Euler's formula beside the column of the exact solution to ease the comparison.
vi. Compute the $f(t_i, w_i)$ in the possible column of excel sheet.

The following table illustrates all the steps above and to see more detail of the implementations see (Appendix A).

Table 2.1: Illustration of the exact solution and Euler's Method by Excel.

| Iteration | $h$ | $t_i$ | Exact Solution | Euler's Method $w_i$ | $f(t_i, w_i)$ |
|---|---|---|---|---|---|
| 0 | 0.1 | $t_0$ | Inter the exact solution in the first cell and then drag to down with respect to the desired interval to compare with the numerical solution. | $w_0 = y_0$ | $f(t_0, w_0)$ |
| 1 | 0.1 | $t_1 = t_0 + h$ | | $w_1 = w_0 + h\{f(t_0, w_0)\}$ | $f(t_1, w_1)$ |
| 2 | 0.1 | $t_2 = t_1 + h$ | | $w_2 = w_1 + h\{f(t_1, w_1)\}$ | $f(t_2, w_2)$ |
| 3 | 0.1 | $t_3 = t_2 + h$ | | $w_3 = w_2 + h\{f(t_2, w_2)\}$ | $f(t_3, w_3)$ |
| 4 | 0.1 | $t_4 = t_3 + h$ | | $w_4 = w_3 + h\{f(t_3, w_3)\}$ | $f(t_4, w_4)$ |
| 5 | 0.1 | $t_5 = t_4 + h$ | | $w_5 = w_4 + h\{f(t_4, w_4)\}$ | $f(t_5, w_5)$ |
| 6 | 0.1 | $t_6 = t_5 + h$ | | $w_6 = w_5 + h\{f(t_5, w_5)\}$ | $f(t_6, w_6)$ |
| 7 | 0.1 | $t_7 = t_6 + h$ | | $w_7 = w_6 + h\{f(t_6, w_6)\}$ | $f(t_7, w_7)$ |
| 8 | 0.1 | $t_8 = t_7 + h$ | | $w_8 = w_7 + h\{f(t_7, w_7)\}$ | $f(t_8, w_8)$ |
| 9 | 0.1 | $t_9 = t_8 + h$ | | $w_9 = w_8 + h\{f(t_8, w_8)\}$ | $f(t_9, w_9)$ |
| 10 | 0.1 | $t_{10} = t_9 + h$ | | $w_{10} = w_9 + h\{f(t_9, w_9)\}$ | $f(t_{10}, w_{10})$ |

Example [1]: (Form 1 of RIVP)

Use Euler's Method to determine the approximate solutions of the following RIVP

$$y' = \quad 1 + \frac{2}{t} \quad - \quad 2 + \frac{2}{t} \quad y + y^2 \quad , \qquad y \; 1 \; = \frac{5}{2} \; , \quad 1 \leq t \leq 2$$

And compare it with the actual solution given as $y = (3 + 3t - t^2)/(3t - t^2)$.

Solution: we recognized the given differential equation is the Riccati differential equation. Now we are ready to allying the equation (2.1.1.4) and then see the table (2.2) and figure (2.1) to realize how to create the tabulations and figures using Microsoft Excel.

Table 2.2: Illustration of the exact solution and Euler's Method.

| ▯ | $t_i$ | Exact Solution | Euler's Method | $f(t_i, y\ t_i\ )$ | Absolute Error | % Relative Error |
|---|---|---|---|---|---|---|
| 0.1 | 1 | 2.500000 | 2.500000 | -0.750000 | 0.000000 | 0.00 |
| 0.1 | 1.1 | 2.435407 | 2.425000 | -0.560284 | 0.010407 | 0.43 |
| 0.1 | 1.2 | 2.388889 | 2.368972 | -0.407536 | 0.019917 | 0.83 |
| 0.1 | 1.3 | 2.357466 | 2.328218 | -0.279249 | 0.029248 | 1.24 |
| 0.1 | 1.4 | 2.339286 | 2.300293 | -0.166799 | 0.038993 | 1.67 |
| 0.1 | 1.5 | 2.333333 | 2.283613 | -0.063822 | 0.049720 | 2.13 |
| 0.1 | 1.6 | 2.339286 | 2.277231 | 0.034780 | 0.062055 | 2.65 |
| 0.1 | 1.7 | 2.357466 | 2.280709 | 0.133499 | 0.076757 | 3.26 |
| 0.1 | 1.8 | 2.388889 | 2.294059 | 0.236745 | 0.094830 | 3.97 |
| 0.1 | 1.9 | 2.435407 | 2.317733 | 0.349334 | 0.117673 | 4.83 |
| 0.1 | 2 | 2.500000 | 2.352667 | 0.477041 | 0.147333 | 5.89 |

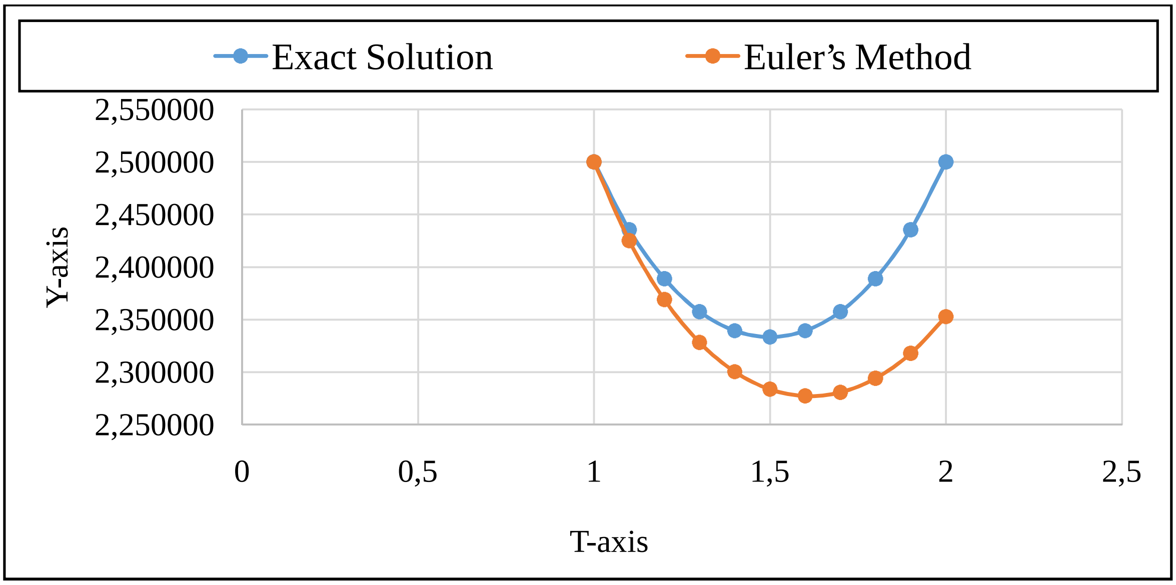


Figure 2.1:Euler's Methodand exact solution when ▯ = 0.1.

## Calculation for Truncation Error in Euler's Method

In this case computed TE of Euler's method by using equation (2.1.1.5), also in the table (2.2.1) illustrated the computation which is presented in the below;

Table 2.2.1: Illustration of Truncation Error of Euler's Method.

| [illegible] | $t_i$ | Euler's Method ($w_i$) | y' | y" | Truncation Error |
|---|---|---|---|---|---|
| 0.1 | 1 | 2.50000 | -0.75000 | 2.25000 | 0.01125 |
| 0.1 | 1.1 | 2.42500 | -0.56028 | 1.77726 | 0.00889 |
| 0.1 | 1.2 | 2.36897 | -0.40754 | 1.46477 | 0.00732 |
| 0.1 | 1.3 | 2.32822 | -0.27925 | 1.25966 | 0.00630 |
| 0.1 | 1.4 | 2.30029 | -0.16680 | 1.13134 | 0.00566 |
| 0.1 | 1.5 | 2.28361 | -0.06382 | 1.06224 | 0.00531 |
| 0.1 | 1.6 | 2.27723 | 0.03478 | 1.04321 | 0.00522 |
| 0.1 | 1.7 | 2.28071 | 0.13350 | 1.07119 | 0.00536 |
| 0.1 | 1.8 | 2.29406 | 0.23675 | 1.14848 | 0.00574 |
| 0.1 | 1.9 | 2.31773 | 0.34933 | 1.28298 | 0.00641 |
| 0.1 | 2 | 2.35267 | 0.47704 | 1.48985 | 0.00745 |

## 2.2 Taylor's Method

In the previous section, we introducedan illustration of Euler's method applied to a form of RDE.As it is expected the Euler's method produced a poor approximation compared with the exact solution Figure 2.1.Next, Taylor's method is presented to solve an RDE. This method is based on Taylor's series truncated at n$^{th}$ term leaving the $(n+1)^{th}$term and onwards as the error terms by which we judge the accuracy of the method. Since the first of these terms namely $(n+1)^{th}$is the largest of the error terms it is used to judge the size of the error in the approximation and it is called the truncation error. The decision of how many terms should be included in the approximation and where to truncate the Taylor Series is a matter of striking a balance between the accuracy required and the availability of real and computer time [23]. In the following sections we present two approximations based on Taylor's series [25], they are Taylor's approximation of order two and of order four.

A Riccati differential equation$y'(t) = p(t)y^2 + q(t)y + r(t)$ with the initial condition$y(t_0) = y_0$, can be formulated and used to demonstrate the Taylor's method for approximating the solution of IVP.WE will attempt to establish a good procedure for the implementation in Excel program. Depending on the order of the Taylor's method we select, it is required to compute$y'(t), y''(t), y'''(t)$ .... as necessary.Great accuracy can be achieved by using higher order Taylor methods but this may be on the account of some laborious and tedious algebraic manipulations involving higher derivatives of $f(t_i, w_i)$ [24]. Therefore, one should strike balance between the need for the accuracy and the demand for real and computational time when choosing a numerical approximation.

Next we look in to the Taylor's approximation of order two;

The general Taylor approximation of order n[3], [8] is given by

$w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ , For each $i = 0,1,\dots,N-1$.

Where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{3!}f''(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{n-1}(t_i, w_i)$$

In fact that if the initial condition is known and the derivatives of $f(t_i, w_i)$ at a point $t = t_0$ is defined we can compute the solution to the RDE at any selection of the value $t$ [7], a detailed procedure on how to implement the Taylors method in an easy to follow bullet point steps and tabulations followed by graphical presentation of the approximations compared with the available exact solutions.

### 2.2.1 Algorithm and Truncation Error of Taylor's Method of Order-Two

The Riccati deferential equation in the form of the initial value problem is

$$y' = f(t, y) = p(t)y^2 + q(t)y + r(t) \quad , \quad y(t_0) = y_0$$

In the interval $[t_0, b]$ ;

Select the step size $h = (b - t_0) / N$ . $t_{i+1} = t_i + h$ , $i = 0,1,2, \dots , N - 1$

Use the following second –order Taylor series method formulas;

$$\left.\begin{aligned} w_{i+1} &= w_i + hT^{(2)}(t_i, w_i) \\ T^{(2)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \end{aligned}\right\} \qquad (2.2.1.1)$$

$$\text{Truncation Error} = \frac{h^3}{3!} y'''(\xi) \quad , \quad t_i < \xi < t_i + h \qquad (2.2.1.2)$$

Which is taken from the general Taylor series expansion [3], [7]. In this formula contain an initial point or initial condition which is consist of $y(t_i)$ , also we have $y'(t_i)$ which is take to the Riccati differential equation which is denoted by RDE, it clear that each of the initial condition with the Riccati differential equation are make the initial value problem which is denoted by IVP, and also contain $y''(t_i)$ we are necessary to take a derivative for RDE.

For explain this type of approximation, by bring the following example we can give the more illustration about it.

Since it is known that taking the derivatives to general RDE may be seldom or it tis the hard work because the Riccati differential equation is nonlinear. Consequently, we should be make the derivatives and construct the algorithm over the specific example. (See Appendix B.1) to understand the desired method.

### 2.2.1.1 Application of an Order Two Taylor's MethodtoRDE Using Microsoft Office Excel

Assuming a step size $h = 0.1$, we summarize the procedure of the algorithm by the following steps.

i. Repeat the first and second steps of section (2.1.3).

ii. Assign the first column to adding the initial variable $t_0$with the step size $h$ in order to make the new independent variable$t_1, t_2, \dots, t_{10}$.

iii. In the second column calculate the numerical approximation $w_i$ to $y_i$ according to Taylor's Method of order two.In the next column calculate the exact solution $y_i$ for comparison purpose.

iv. Third and fourth columns devoted to the $f(t_i, y(t_i))$, $f'(t_i, y(t_i))$ respectively. Last column is used to evaluate $T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$.

Table 2.3: A tabulated illustration of Taylor's Method for order two by using Excel.

| $t_i$ | Tylor's Method (order 2) | $f(t_i, w_i)$ | $f'(t_i, w_i)$ | $T^{(2)}(t_i, w_i)$ |
|---|---|---|---|---|
| $t_0$ | $w_0$ | $f(t_0, w_0)$ | $f'(t_0, w_0)$ | $T^{(2)}(t_0, w_0)$ |
| $t_1 = t_0 + h$ | $w_1 = w_0 + hT^{(2)}(t_0, w_0)$ | $f(t_1, w_1)$ | $f'(t_1, w_1)$ | $T^{(2)}(t_1, w_1)$ |
| $t_2 = t_1 + h$ | $w_2 = w_1 + hT^{(2)}(t_1, w_1)$ | $f(t_2, w_2)$ | $f'(t_2, w_2)$ | $T^{(2)}(t_2, w_2)$ |
| $t_3 = t_2 + h$ | $w_3 = w_2 + hT^{(2)}(t_2, w_2)$ | $f(t_3, w_3)$ | $f'(t_3, w_3)$ | $T^{(2)}(t_3, w_3)$ |
| $t_4 = t_3 + h$ | $w_4 = w_3 + hT^{(2)}(t_3, w_3)$ | $f(t_4, w_4)$ | $f'(t_4, w_4)$ | $T^{(2)}(t_4, w_4)$ |
| $t_5 = t_4 + h$ | $w_5 = w_4 + hT^{(2)}(t_4, w_4)$ | $f(t_5, w_5)$ | $f'(t_5, w_5)$ | $T^{(2)}(t_5, w_5)$ |
| $t_6 = t_5 + h$ | $w_6 = w_5 + hT^{(2)}(t_5, w_5)$ | $f(t_6, w_6)$ | $f'(t_6, w_6)$ | $T^{(2)}(t_6, w_6)$ |
| $t_7 = t_6 + h$ | $w_7 = w_6 + hT^{(2)}(t_6, w_6)$ | $f(t_7, w_7)$ | $f'(t_7, w_7)$ | $T^{(2)}(t_7, w_7)$ |
| $t_8 = t_7 + h$ | $w_8 = w_7 + hT^{(2)}(t_7, w_7)$ | $f(t_8, w_8)$ | $f'(t_8, w_8)$ | $T^{(2)}(t_8, w_8)$ |
| $t_9 = t_8 + h$ | $w_9 = w_8 + hT^{(2)}(t_8, w_8)$ | $f(t_9, w_9)$ | $f'(t_9, w_9)$ | $T^{(2)}(t_9, w_9)$ |
| $t_{10} = t_9 + h$ | $w_{10} = w_9 + hT^{(2)}(t_9, w_9)$ | $f(t_{10}, w_{10})$ | $f'(t_{10}, w_{10})$ | $T^{(2)}(t_{10}, w_{10})$ |

**An example [11]**:

Determine the approximate solution of the given IVRP using Taylor's Method of order two

$$y' = y^2 - 2ty + t^2 + 1 \quad , \qquad y\ 0\ = \frac{1}{2} \ , \quad 0 \le t \le 1$$

When the actual solution is given by: $y = t + 1/(2 - t)$.

**Solution:**

We recognize that the given differential equation is the Initial Value Riccati Problem.

Now we are ready to apply the equation (2.2.1.1) and then see the table (2.4) and figure (2.2) to see how to construct the tabulations and figures by Microsoft office excel.

Table 2.4: Illustration the tabulation of Taylor's Method for order two.

| ? | $t_i$ | Exact solution ($y_i$) | Tylor's Method ($w_i$) (order2) | $f(t_i, y(t_i))$ | $f'(t_i, y\ t_i\ )$ | $T^{2}(t_i, y\ t_i\ )$ |
|---|---|---|---|---|---|---|
| 0.1 | 0 | 0.500000 | 0.500000 | 1.250000 | 0.250000 | 1.262500 |
| 0.1 | 0.1 | 0.626316 | 0.626250 | 1.276939 | 0.293478 | 1.291613 |
| 0.1 | 0.2 | 0.755556 | 0.755411 | 1.308482 | 0.358668 | 1.326415 |
| 0.1 | 0.3 | 0.888235 | 0.888053 | 1.345806 | 0.460705 | 1.368841 |
| 0.1 | 0.4 | 1.025000 | 1.024937 | 1.390546 | 0.616133 | 1.421353 |
| 0.1 | 0.5 | 1.166667 | 1.167072 | 1.444985 | 0.843675 | 1.487169 |
| 0.1 | 0.6 | 1.314286 | 1.315789 | 1.512354 | 1.165475 | 1.570628 |
| 0.1 | 0.7 | 1.469231 | 1.472852 | 1.597300 | 1.609249 | 1.677763 |
| 0.1 | 0.8 | 1.633333 | 1.640628 | 1.706656 | 2.212069 | 1.817259 |
| 0.1 | 0.9 | 1.809091 | 1.822354 | 1.850737 | 3.027362 | 2.002105 |
| 0.1 | 1 | 2.000000 | 2.022565 | 2.045638 | 4.138466 | 2.252562 |

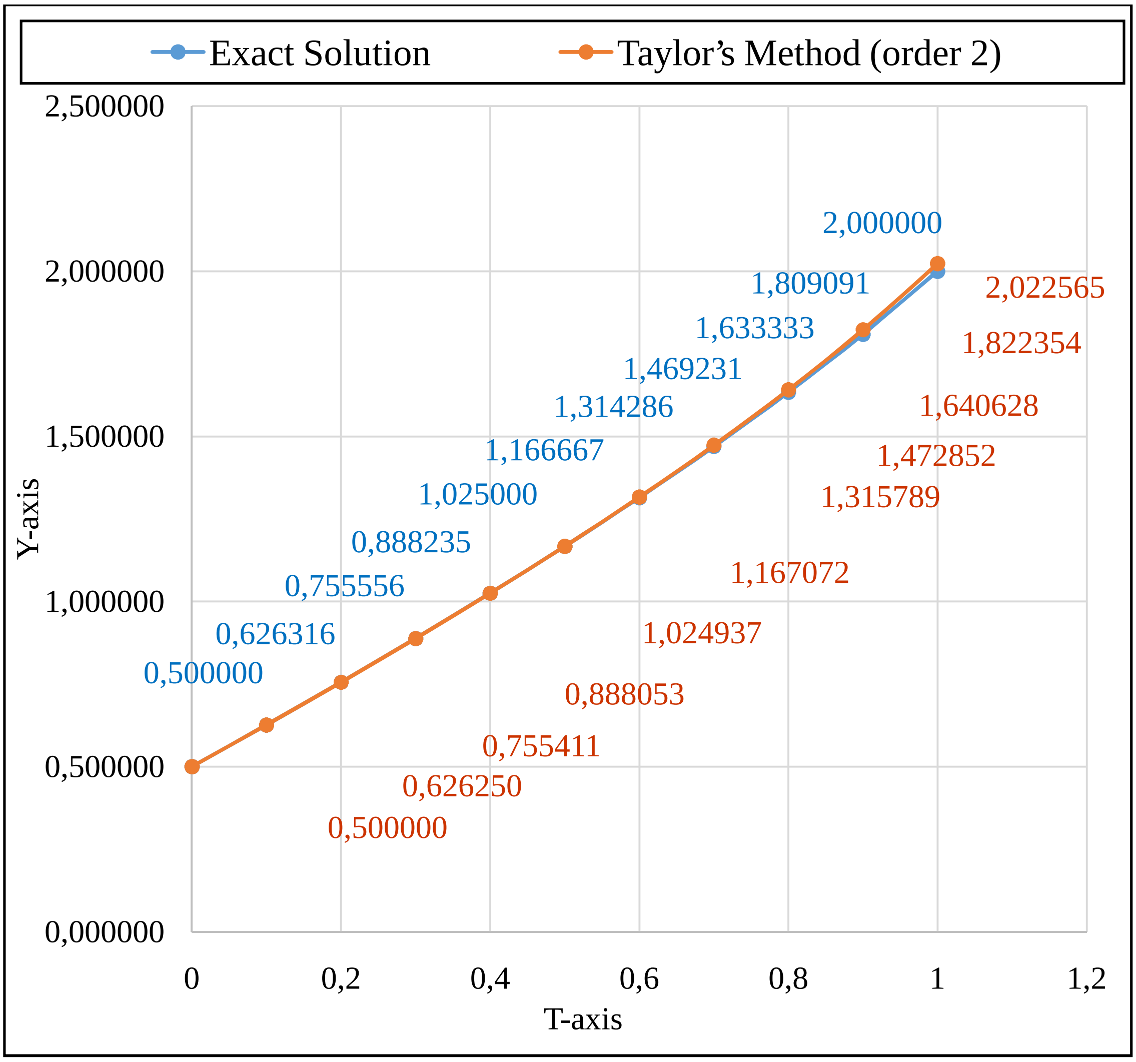


Figure 2.2:Taylor's Method of order twoand exact solution when ⍰ = 0.1.

It is clear from the figure that there is a very good agreement between the numerical and the exact solutions. This is better than our expectations for the following reasons. Firstly the method is not of higher order, hence we do not expect an accurate solution, and secondly the step size is not small. Therefore, for any reason, if we seek to improve the approximation further, we can either choose a higher order method or use a finer step size ⍰.

## 2.2.2 Taylor's Method of Order Four

One of the ways to improve the accuracy of the approximation of the numerical method is to go for higher order Taylor approximation. In this case we chose a fourth order Taylor's series expansion,Of course this demands more effort, real and computingtime. A Taylor method of order four [23], [25] will have the form:

$$w_{i+1} = w_i + hT^{(4)}(t_i, w_i) \qquad (2.2.2.1)$$

$$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2!}f'(t_i, w_i) + \frac{h^2}{3!}f''(t_i, w_i) + + \frac{h^3}{4!}f'''(t_i, w_i)$$

With the truncation error

$$\text{Error} = \frac{y^{(5)}(\xi_i)}{5!}h^5, \quad when\, t_i \leq \xi_i \leq t_{i+1} \qquad (2.2.2.2)$$

The truncation error can be used as an indication to the accuracy of the method prior to the implementation of the method [15], [30]. A detailed procedure of implementing the method in a tabulated form is presented and a graphical representation of the numerical and the close form results are presented for comparison.

### 2.2.2.1 Application of an Order Four Taylor's method to RIVP Using Microsoft Office Excel

Again assuming $h = 0.1$ and $t_0 = 0$ and in a procedure similar to that of section (2.2.1.1) we can summarize the algorithm of Taylor's Method of order four, noting that, here we need additional derivatives$f''$and$f'''$to compute a fourth order Taylor approximation (See Appendix B.2).

Table 2.5: Comparison of the exact and 4th order Taylor's Solution

| Iteration | □ | $t_i$ | Exact Solution | Tylor's Method (order 4) |
|---|---|---|---|---|
| 0 | 0.1 | $t_0$ | Inter the exact solution in the first cell and then drag to down with respect to the desired interval to compare with the numerical solution. | $w_0$ |
| 1 | 0.1 | $t_1 = t_0 +$ □ | | $w_1 = w_0 +$ □$T^{4}\ \ t_0, w_0$ |
| 2 | 0.1 | $t_2 = t_1 +$ □ | | $w_2 = w_1 +$ □$T^{4}\ \ t_1, w_1$ |
| 3 | 0.1 | $t_3 = t_2 +$ □ | | $w_3 = w_2 +$ □$T^{4}\ \ t_2, w_2$ |
| 4 | 0.1 | $t_4 = t_3 +$ □ | | $w_4 = w_3 +$ □$T^{4}\ \ t_3, w_3$ |
| 5 | 0.1 | $t_5 = t_4 +$ □ | | $w_5 = w_4 +$ □$T^{4}\ \ t_4, w_4$ |
| 6 | 0.1 | $t_6 = t_5 +$ □ | | $w_6 = w_5 +$ □$T^{4}\ \ t_5, w_5$ |
| 7 | 0.1 | $t_7 = t_6 +$ □ | | $w_7 = w_6 +$ □$T^{4}\ \ t_6, w_6$ |
| 8 | 0.1 | $t_8 = t_7 +$ □ | | $w_8 = w_7 +$ □$T^{4}\ \ t_7, w_7$ |
| 9 | 0.1 | $t_9 = t_8 +$ □ | | $w_9 = w_8 +$ □$T^{4}\ \ t_8, w_8$ |
| 10 | 0.1 | $t_{10} = t_9 +$ □ | | $w_{10} = w_9 +$ □$T^{4}\ \ t_9, w_9$ |

Table 2.5.1: Illustration of coefficients $f, f', f''$, and $f'''$ related to the table 2.5.

| $f(t_i, w_i)$ | $f'(t_i, w_i)$ | $f''(t_i, w_i)$ | $f'''(t_i, w_i)$ | $T^{4}\ (t_i, y\ \ t_i\ )$ |
|---|---|---|---|---|
| $f(t_0, w_0)$ | $f'(t_0, w_0)$ | $f''(t_0, w_0)$ | $f'''(t_0, w_0)$ | $T^{(4)}\ \ t_0, w_0$ |
| $f(t_1, w_1)$ | $f'(t_1, w_1)$ | $f''(t_1, w_1)$ | $f'''(t_1, w_1)$ | $T^{4}\ \ t_1, w_1$ |
| $f(t_2, w_2)$ | $f'(t_2, w_2)$ | $f''(t_2, w_2)$ | $f'''(t_2, w_2)$ | $T^{4}\ \ t_2, w_2$ |
| $f(t_3, w_3)$ | $f'(t_3, w_3)$ | $f''(t_3, w_3)$ | $f'''(t_3, w_3)$ | $T^{4}\ \ t_3, w_3$ |
| $f(t_4, w_4)$ | $f'(t_4, w_4)$ | $f''(t_4, w_4)$ | $f'''(t_4, w_4)$ | $T^{4}\ \ t_4, w_4$ |
| $f(t_5, w_5)$ | $f'(t_5, w_5)$ | $f''(t_5, w_5)$ | $f'''(t_5, w_5)$ | $T^{4}\ \ t_5, w_5$ |
| $f(t_6, w_6)$ | $f'(t_6, w_6)$ | $f''(t_6, w_6)$ | $f'''(t_6, w_6)$ | $T^{4}\ \ t_6, w_6$ |
| $f(t_7, w_7)$ | $f'(t_7, w_7)$ | $f''(t_7, w_7)$ | $f'''(t_7, w_7)$ | $T^{4}\ \ t_7, w_7$ |
| $f(t_8, w_8)$ | $f'(t_8, w_8)$ | $f''(t_8, w_8)$ | $f'''(t_8, w_8)$ | $T^{4}\ \ t_8, w_8$ |
| $f(t_9, w_9)$ | $f'(t_9, w_9)$ | $f''(t_9, w_9)$ | $f'''(t_9, w_9)$ | $T^{4}\ \ t_9, w_9$ |
| $f(t_{10}, w_{10})$ | $f'(t_{10}, w_{10})$ | $f''(t_{10}, w_{10})$ | $f'''(t_{10}, w_{10})$ | $T^{4}\ \ t_{10}, w_{10}$ |

**An example [11]:**

Determine the approximate solutions by Taylor's Method of order four for the following initial value problem

$$y' = y^2 - \frac{y}{t} - \frac{1}{t^2} \quad , y(0.5) = 2.36364 \text{ for } \quad 0.5 \leq t \leq 1.5$$

When the actual solution is given as

$$y(t) = \frac{2t}{3 - t^2} + \frac{1}{t}$$

**Solution:** we recognize the given differential equation is the Riccati differential equation. Now we are ready to apply the equation (2.2.2.1) to the problem and then see the results in table (2.6) and figure (2.3) to judge the approximation against the exact solution.

Table 2.6: illustration of Tylor's Method of order four and Exact solution when $h = 0.1$

| $t_i$ | $y_i$ (exact) | $w_i$ (Taylor) | $f(t_i, w_i)$ | $f'(t_i, w_i)$ | $f''(t_i, w_i)$ | $f'''(t_i, w_i)$ | $T^{(4)}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 2.36364 | 2.36364 | -3.14050 | 16.88956 | -93.15429 | 776.02091 | -2.41894 |
| 0.6 | 2.12121 | 2.12174 | -1.81222 | 10.48321 | -42.42832 | 321.25278 | -1.34539 |
| 0.7 | 1.98634 | 1.98720 | -0.93070 | 7.51701 | -19.50609 | 163.12418 | -0.58056 |
| 0.8 | 1.92797 | 1.92915 | -0.25233 | 6.26240 | -6.51118 | 107.34904 | 0.05441 |
| 0.9 | 1.93303 | 1.93459 | 0.35852 | 6.12069 | 3.57110 | 101.07003 | 0.67472 |
| 1 | 2.00000 | 2.00206 | 1.00618 | 7.02476 | 15.13626 | 138.92983 | 1.38844 |
| 1.1 | 2.13814 | 2.14090 | 1.81075 | 9.37910 | 33.86855 | 254.89504 | 2.34677 |
| 1.2 | 2.37179 | 2.37558 | 2.96929 | 14.44027 | 72.68880 | 578.33880 | 3.83655 |
| 1.3 | 2.75396 | 2.75924 | 4.89917 | 25.81036 | 171.76864 | 1619.16498 | 6.54344 |
| 1.4 | 3.40659 | 3.41358 | 8.70405 | 55.67720 | 496.70020 | 6019.94789 | 12.5665 |
| 1.5 | 4.66667 | 4.67024 | 18.25317 | 160.9926 | 2075.0475 | 35821.2171 | 31.2537 |

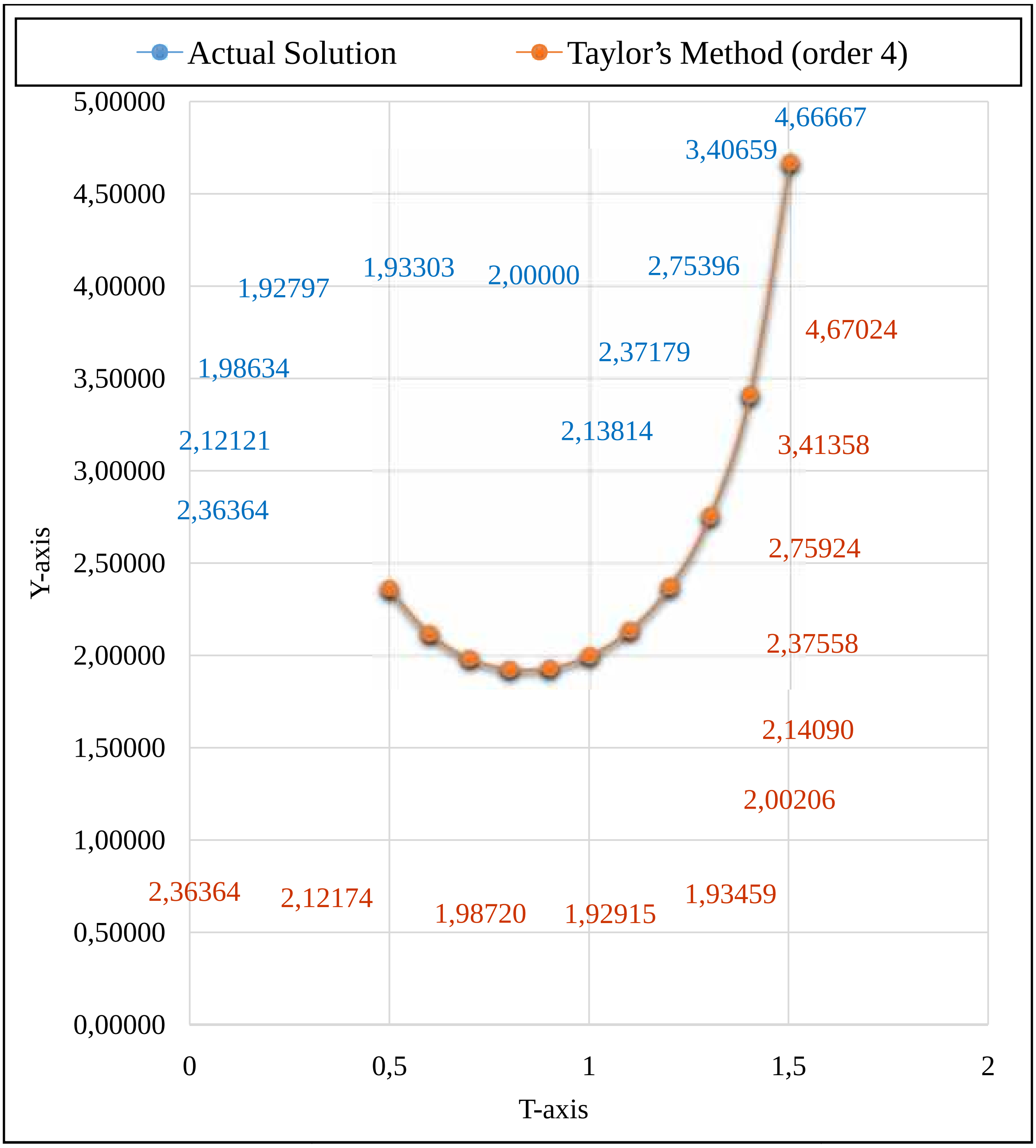


Figure 2.3: Taylor's Method of order four and exact solution when $h = 0.1$.

Again as the figure shows the exact and the numerical solutions compare very well with an absolute error around 7%.

## 2.3 Runge-Kutta Method

In the earlier sections, we have given some illustrations about the Euler’s method, the improved Euler method and the Taylor’s method and used all these methods to find the approximate solution of the RDE in the form of an initial value problem

$$y' = p\ t\ y^2 + q\ t\ y + r\ t\ , \qquad y\ t_0\ = y_0 \qquad (2.3)$$

In using the above methods we have come across some difficulties that we highlight here and because of these difficulties we begin to think of exploring better and more efficient methods. To summarize the difficulties;

1. The Euler’s method is not a good method because it is not accuracy method when we compare it to the other methods and the Euler’s formula is cheap on computer timewhen applied to RDE; so if you need to achieve more accurate solution then you will need to take the smaller step size.However, it may be impossible to achieve that every time [7].
2. The Taylor’s method involvessome tedious differentiation, which can be difficult to implement. Depending on the form of RDE this can be hard work and may be impossible. [7], [13].

Because of the above reasons, we will explore another class of methods for solving IVPsknown as Runge-Kutta (R.K)methods [25]. These methods produce very good results almost all the time and they are reasonably easy to implement, furthermore, they do not require very small step size □ [17], therefore they are regarded to be the most popular methods for solving IVPs.

### 2.3.1 Runge-Kutta Method of Order Two or Improved Euler Method

For solving the initial value problem by using the Runge-Kutte method of order two, we can use the following formula

$$k_1 = □f(t_i, y_i)$$

$$k_2 = □f(t_i + □, y_i + □k_1)$$

$$y_{i+1} = y_i + \frac{1}{2}\ k_1 + k_2 \qquad (2.3.1)$$

Also the above formulas are known by motivated Euler's method. This is the simplest of (R.K) class of methods. Before we begin to explore the higher order of these methods [5], [7], [8], [17], [18], weusean R.K of order two and apply the method to approximate the solution of RDE.

### 2.3.1.1 Application of RK-Method ofOrder Two to RDEs Using Excel

Let $h = 0.1$ then to apply this method we must perform the following steps.

i. Generate the time sequence, $t_0$, $t_1, t_2, \ldots, t_{10}$. By adding the increment value $h$ to the initial variable$t_0$.
ii. Evaluate the coefficient $k_1 = hf(t_i, w_i)$ in one of the Excel columns.
iii. Evaluate the coefficient $k_2 = hf(t_i + h, w_i + hk_1)$ in another Excel column.
iv. Evaluate the numerical values $w_i$ of RK-Method of order two $w_{i+1} = w_i + \frac{1}{2}\ k_1 + k_2$ .
v. Calculate the exact solution for comparison.

See Appendix C.1 in order to understand the illustrations.

Table 2.7: illustration of RK- Method for order two.

| $t_i$ | Runge-Kutte method (order 2). $w_{i+1} = w_i + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_i, w_i)$ | $k_2 = □f(t_i + □, w_i + □k_1)$ |
|---|---|---|---|
| $t_0$ | $w_0 = y_0$ | $k_1 = □f(t_0, w_0)$ | $k_2 = □f(t_0 + □, w_0 + □k_1)$ |
| $t_1$ | $w_1 = w_0 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_1, w_1)$ | $k_2 = □f(t_1 + □, w_1 + □k_1)$ |
| $t_2$ | $w_2 = w_1 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_2, w_2)$ | $k_2 = □f(t_2 + □, w_2 + □k_1)$ |
| $t_3$ | $w_3 = w_2 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_3, w_3)$ | $k_2 = □f(t_3 + □, w_3 + □k_1)$ |
| $t_4$ | $w_4 = w_3 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_4, w_4)$ | $k_2 = □f(t_4 + □, w_4 + □k_1)$ |
| $t_5$ | $w_5 = w_4 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_5, w_5)$ | $k_2 = □f(t_5 + □, w_5 + □k_1)$ |
| $t_6$ | $w_6 = w_5 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_6, w_6)$ | $k_2 = □f(t_6 + □, w_6 + □k_1)$ |
| $t_7$ | $w_7 = w_6 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_7, w_7)$ | $k_2 = □f(t_7 + □, w_7 + □k_1)$ |
| $t_8$ | $w_8 = w_7 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_8, w_8)$ | $k_2 = □f(t_8 + □, w_8 + □k_1)$ |
| $t_9$ | $w_9 = w_8 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_9, w_9)$ | $k_2 = □f(t_9 + □, w_9 + □k_1)$ |
| $t_{10}$ | $w_{10} = w_9 + \frac{1}{2}\ k_1 + k_2$ | $k_1 = □f(t_{10}, w_{10})$ | $k_2 = □f(t_{10} + □, w_{10} + □k_1)$ |

**An example [7]:**

Approximate the solution of the given IVP by using RK-Method of order two

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \quad , \qquad y(1) = -1, \quad 1 \leq t \leq 2$$

When the actual solution given as $y = -1/t$.

**Solution:** we recognized the given differential equation is the Riccati differential equation. Now we are ready to apply the equation (2.3.1) as a result we can observe the table (2.8) to compare the numerical and exact solutions. These results are also presented graphically in figure (2.4) to indicate that how to make the tabulations and figures by Microsoft office excel.

Table 2.8: illustration of tabulation of RK- Method for order two and exact solution when  = 0.1.

|  | $t_i$ | Exact solution | RK-Method (order 2) | $k_1$ | $k_2$ | Absolute Error | % Relative Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 1 | -1.000000 | -1.000000 | 0.100000 | 0.091645 | 0.000000 | 0.00 |
| 0.1 | 1.1 | -0.909091 | -0.904178 | 0.083089 | 0.079430 | 0.004913 | 0.54 |
| 0.1 | 1.2 | -0.833333 | -0.822918 | 0.070302 | 0.069671 | 0.010415 | 1.25 |
| 0.1 | 1.3 | -0.769231 | -0.752932 | 0.060399 | 0.061738 | 0.016299 | 2.12 |
| 0.1 | 1.4 | -0.714286 | -0.691864 | 0.052572 | 0.055188 | 0.022422 | 3.14 |
| 0.1 | 1.5 | -0.666667 | -0.637984 | 0.046274 | 0.049705 | 0.028683 | 4.30 |
| 0.1 | 1.6 | -0.625000 | -0.589994 | 0.041128 | 0.045056 | 0.035006 | 5.60 |
| 0.1 | 1.7 | -0.588235 | -0.546902 | 0.036863 | 0.041071 | 0.041333 | 7.03 |
| 0.1 | 1.8 | -0.555556 | -0.507936 | 0.033283 | 0.037618 | 0.047620 | 8.57 |
| 0.1 | 1.9 | -0.526316 | -0.472485 | 0.030244 | 0.034599 | 0.053831 | 10.23 |
| 0.1 | 2 | -0.500000 | -0.440064 | 0.027638 | 0.031935 | 0.059936 | 11.99 |

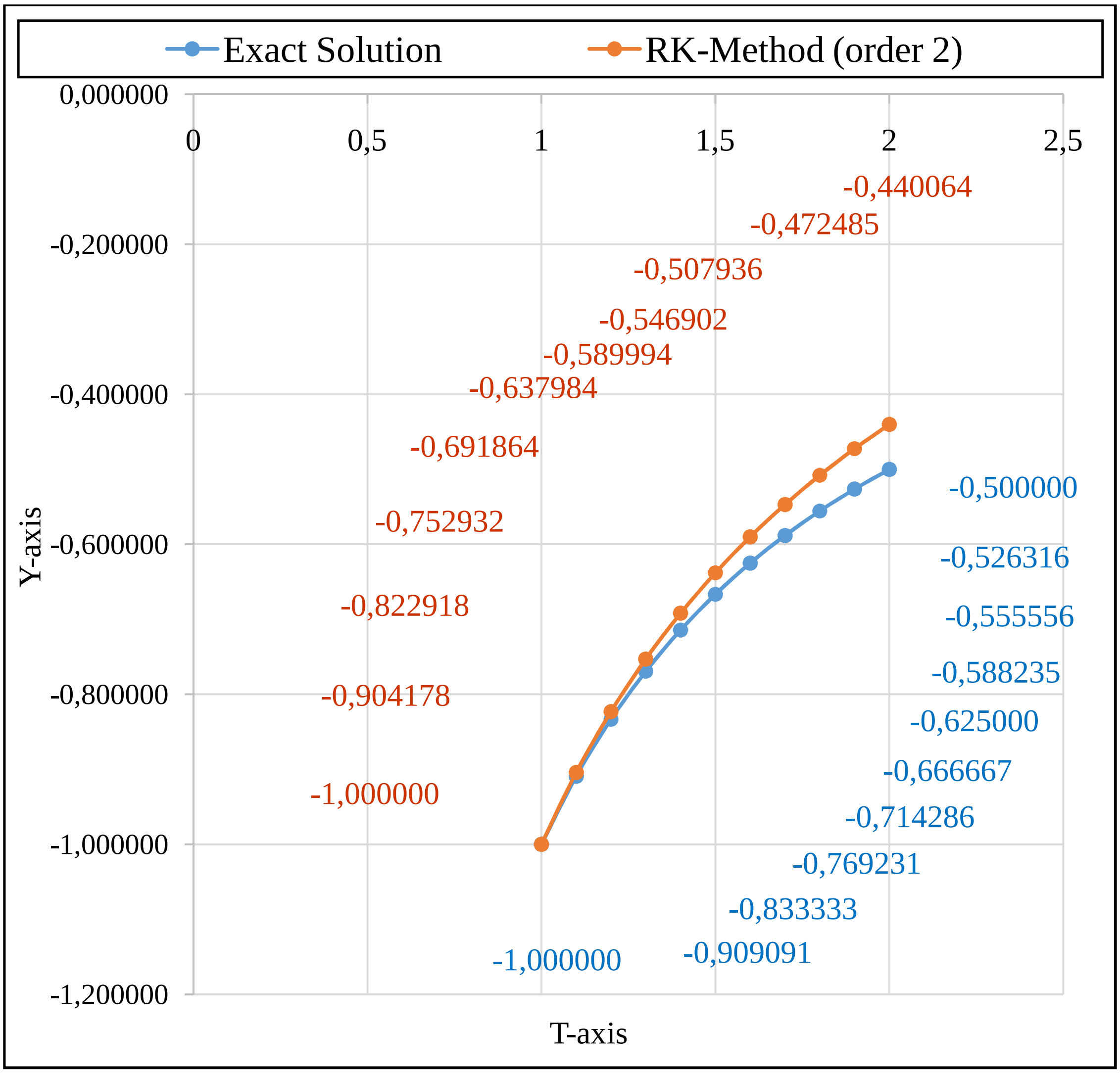


Figure 2.4: RK-Method of order two and exact solution when ℎ = 0.1.

The figure shows a comparison between the numerical and the exact solutions. There is an error of 12% at the end of the time interval. In many situations this approximation is no acceptable. Therefore, one should either use a smaller step size h or choose a higher order than two say R.K of order three or order four that will be discussed later.

### 2.3.2 Runge-Kutta Method of Order Four:

The initial value problem has the form

$$y' = f(t, y) \quad , \quad y(t_0) = y_0 \tag{2.3.2.1}$$

And the general form of IVRP has the form

$$y' = p(x)y^2 + q(x)y + r(x) \quad , \quad y(t_0) = y_0 \, ,$$

$$t_{i+1} = t_i + h \quad , \quad i \geq 0 \tag{2.3.2.2}$$

Then one can apply an R.K method of order four directly. The order four R.K method we use here has the form [3], [7], [8], [15], [16], [17], [18], [19], [25], [30].

$$\left\{ \begin{array}{c} k_1 = hf(t_i, y_i) \\ k_2 = hf\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right) \\ k_3 = hf\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right) \\ k_4 = hf(t_i + h, y_i + k_3) \end{array} \right\} \tag{2.3.2.3}$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.3.2.4}$$

#### 2.3.2.1 Application of Order Four RK-Method to RDE Using Excel

Let $h = 0.1$, then to apply this method we must perform the following steps;

i. Repeat the first and second steps of section (2.3.1.1).
ii. Evaluate the coefficient $k_1$, $k_2$, $k_3$ and $k_4$ as considered in the table (2.3.2.1).
iii. At the end, compute the numerical values for the given formula of RK-Method of order four $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

For further details of RK-Method of order four see (Appendix C.2).

Table 2.9: Illustration of RK- Method ofOrder Four.

| $t_i$ | Runge-Kutte method (order 4). $w_{i+1} = w_i + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_i, w_i)$ |
|---|---|---|
| $t_0$ | $w_0 = y_0$ | $k_1 = \square f(t_0, w_0)$ |
| $t_1$ | $w_1 = w_0 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_1, w_1)$ |
| $t_2$ | $w_2 = w_1 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_2, w_2)$ |
| $t_3$ | $w_3 = w_2 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_3, w_3)$ |
| $t_4$ | $w_4 = w_3 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_4, w_4)$ |
| $t_5$ | $w_5 = w_4 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_5, w_5)$ |
| $t_6$ | $w_6 = w_5 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_6, w_6)$ |
| $t_7$ | $w_7 = w_6 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_7, w_7)$ |
| $t_8$ | $w_8 = w_7 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_8, w_8)$ |
| $t_9$ | $w_9 = w_8 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_9, w_9)$ |
| $t_{10}$ | $w_{10} = w_9 + \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $k_1 = \square f(t_{10}, w_{10})$ |

Table 2.9.1: Illustration of coefficients $k_2$, $k_3$, $k_4$ related to the table 2.9.

| | | |
|---|---|---|
| $k_2 = \square f \quad t_i + \frac{1}{2}\square, w_i + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_i + \frac{1}{2}\square, w_i + \frac{1}{2}k_2$ | $k_4 = \square f(t_i + \square, w_i + k_3)$ |

| | | |
|---|---|---|
| $k_2 = \square f \quad t_0 + \frac{1}{2}\square, w_0 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_0 + \frac{1}{2}\square, w_0 + \frac{1}{2}k_2$ | $k_4 = \square f(t_0 + \square, w_0 + k_3)$ |
| $k_2 = \square f \quad t_1 + \frac{1}{2}\square, w_1 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_1 + \frac{1}{2}\square, w_1 + \frac{1}{2}k_2$ | $k_4 = \square f(t_1 + \square, w_1 + k_3)$ |
| $k_2 = \square f \quad t_2 + \frac{1}{2}\square, w_2 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_2 + \frac{1}{2}\square, w_2 + \frac{1}{2}k_2$ | $k_4 = \square f(t_2 + \square, w_2 + k_3)$ |
| $k_2 = \square f \quad t_3 + \frac{1}{2}\square, w_3 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_3 + \frac{1}{2}\square, w_3 + \frac{1}{2}k_2$ | $k_4 = \square f(t_3 + \square, w_3 + k_3)$ |
| $k_2 = \square f \quad t_4 + \frac{1}{2}\square, w_4 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_4 + \frac{1}{2}\square, w_4 + \frac{1}{2}k_2$ | $k_4 = \square f(t_4 + \square, w_4 + k_3)$ |
| $k_2 = \square f \quad t_5 + \frac{1}{2}\square, w_5 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_5 + \frac{1}{2}\square, w_5 + \frac{1}{2}k_2$ | $k_4 = \square f(t_5 + \square, w_5 + k_3)$ |
| $k_2 = \square f \quad t_6 + \frac{1}{2}\square, w_6 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_6 + \frac{1}{2}\square, w_6 + \frac{1}{2}k_2$ | $k_4 = \square f(t_6 + \square, w_6 + k_3)$ |
| $k_2 = \square f \quad t_7 + \frac{1}{2}\square, w_7 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_7 + \frac{1}{2}\square, w_7 + \frac{1}{2}k_2$ | $k_4 = \square f(t_7 + \square, w_7 + k_3)$ |
| $k_2 = \square f \quad t_8 + \frac{1}{2}\square, w_8 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_8 + \frac{1}{2}\square, w_8 + \frac{1}{2}k_2$ | $k_4 = \square f(t_8 + \square, w_8 + k_3)$ |
| $k_2 = \square f \quad t_9 + \frac{1}{2}\square, w_9 + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_9 + \frac{1}{2}\square, w_9 + \frac{1}{2}k_2$ | $k_4 = \square f(t_9 + \square, w_9 + k_3)$ |
| $k_2 = \square f \quad t_{10} + \frac{1}{2}\square, w_{10} + \frac{1}{2}k_1$ | $k_3 = \square f \quad t_{10} + \frac{1}{2}\square, w_{10} + \frac{1}{2}k_2$ | $k_4 = \square f(t_{10} + \square, w_{10} + k_3)$ |

**An example [8]**:

Here we use the same equation that we have used in an order two R.K method and also we use the same step size $h = 0.1$, this way we can directly see the accuracy between an order two an order four R.K method, the RDE equation is

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \quad , \qquad y(1) = -1 , \quad 1 \le t \le 2$$

With the actual solution given as $y = -1/t$.

**Solution:** we know that the given differential equation is the Riccati differential equation. Now we are ready to apply the equation (2.3.2.4) and then see the numerical and the exact solutions table (2.10) a graphical representation of the results are also presented in figure (2.5) for the purpose of comparison.

Table 2.10: Illustration of RK- Method of order four and exact solution when $h = 0.1$.

| h | $t_i$ | Exact solution | RK-Method (order 4) | $k_1$ | $k_2$ | $k_3$ | $k_4$ |
|---|---|---|---|---|---|---|---|
| 0.1 | 1 | -1.000000 | -1.000000 | 0.100000 | 0.090929 | 0.090497 | 0.082607 |
| 0.1 | 1.1 | -0.909091 | -0.909090 | 0.082645 | 0.075770 | 0.075472 | 0.069421 |
| 0.1 | 1.2 | -0.833333 | -0.833332 | 0.069445 | 0.064111 | 0.063898 | 0.059156 |
| 0.1 | 1.3 | -0.769231 | -0.769229 | 0.059172 | 0.054951 | 0.054795 | 0.051010 |
| 0.1 | 1.4 | -0.714286 | -0.714283 | 0.051021 | 0.047623 | 0.047506 | 0.044437 |
| 0.1 | 1.5 | -0.666667 | -0.666664 | 0.044445 | 0.041670 | 0.041580 | 0.039057 |
| 0.1 | 1.6 | -0.625000 | -0.624997 | 0.039063 | 0.036767 | 0.036697 | 0.034598 |
| 0.1 | 1.7 | -0.588235 | -0.588232 | 0.034602 | 0.032681 | 0.032627 | 0.030861 |
| 0.1 | 1.8 | -0.555556 | -0.555552 | 0.030864 | 0.029241 | 0.029197 | 0.027699 |
| 0.1 | 1.9 | -0.526316 | -0.526312 | 0.027701 | 0.026317 | 0.026281 | 0.024998 |
| 0.1 | 2 | -0.500000 | -0.499996 | 0.025000 | 0.023810 | 0.023781 | 0.022675 |

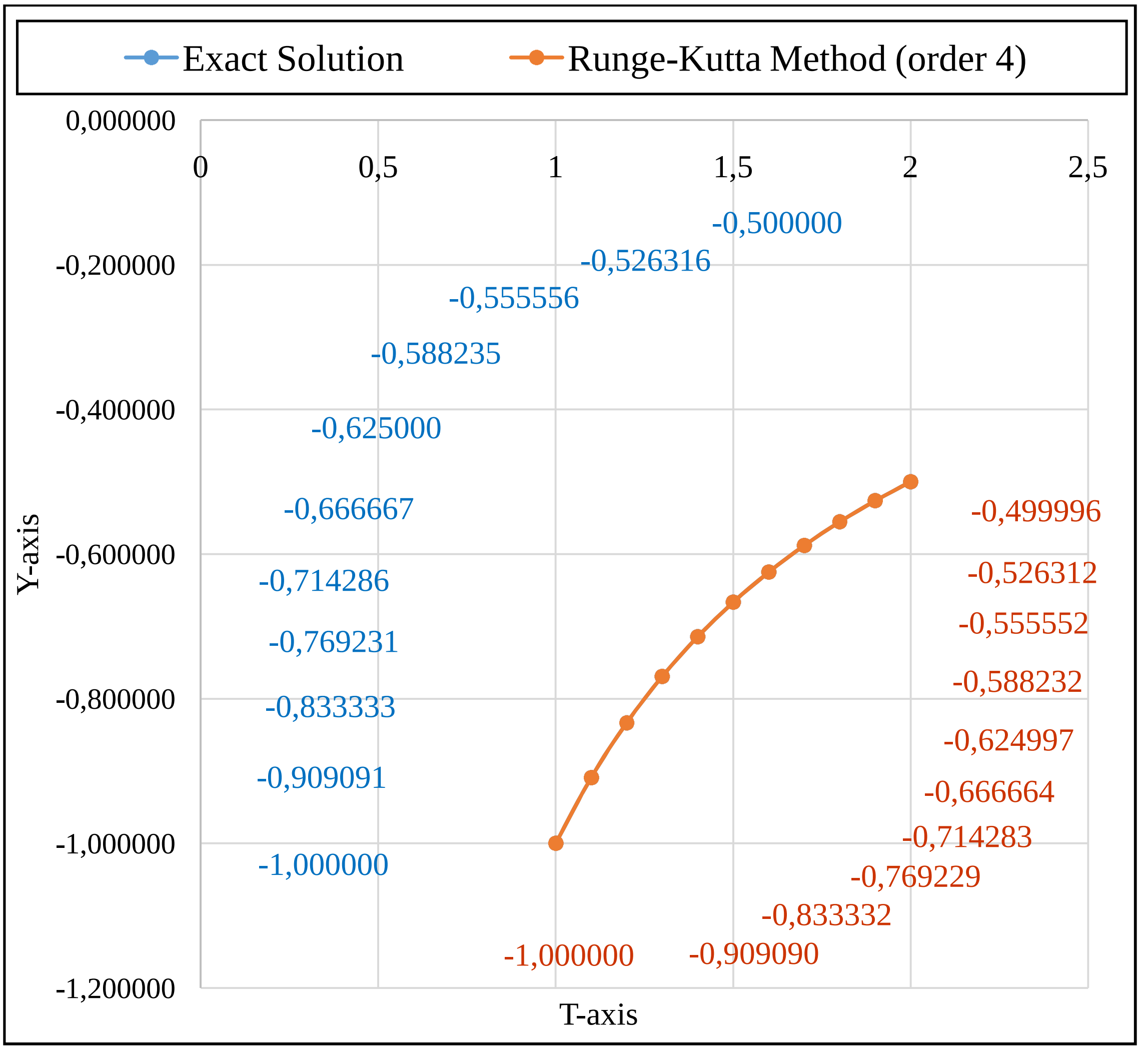


Figure 2.5: RK-Method of order four and exact solution when ℎ = 0.1.

## 2.4 Runge-Kutta-Fehlberg method

Another method that is used for solving the initial value problem is the RKF-Method. The Runge-Kutta method of order five with local truncation error given as

$$\tilde{w}_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \qquad (2.4.1)$$

, use to compute the error in a Runge-Kutta method of order four given as

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5 \qquad (2.4.2)$$

Where

$$\left.\begin{aligned}
k_1 &= hf\left(t_i, w_i\right), \\
k_2 &= hf\left(t_i + \frac{1}{4}h, w_i + \frac{1}{4}k_1\right), \\
k_3 &= hf\left(t_i + \frac{3}{8}h, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right), \\
k_4 &= hf\left(t_i + \frac{12}{13}h, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right), \\
k_5 &= hf\left(t_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right), \\
k_6 &= hf\left(t_i + \frac{1}{2}h, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)
\end{aligned}\right\} \qquad (2.4.3)$$

The above formula is known as RKF-Methods [3].

Now, to see the application of the Runge-Kutta-Fehlberg method to the Riccati differential equation visit the (Appendix D).

### 2.4.1 Application of R.K Fehlberg to RDE Using Excel

Let $h = 0.1$ then to apply this method we must be perform the following steps;

i. Repeat the first and second steps of section (2.3.1.1).
ii. Compute $k_1$, $k_2$, $k_3$, $k_4$, $k_5$and $k_6$and organize their columns in a sequential order because the computation of any ki depends on the values of the previous $k_{i-1}, .. k_0$.
iii. Finally, compute the approximate $w_i$ for RKF-Method of order five using (2.4.1) and also to find RKF-Method of order four use equation (2.4.2).

**An example [1]:**

Determine the approximate solutions by RKF-Method of order four and five over the following initial value problem

$$y' = 1 + \frac{2}{t} - 2 + \frac{2}{t} \ y + y^2 \quad , \qquad y \ 1 = \frac{5}{2} \ , \quad 1 \le t \le 2$$

When the actual solution given as $y = (3 + 3t - t^2)/(3t - t^2)$.

**Solution**: we recognized the given differential equation is the Riccati differential equation. Now we are ready to apply the equation (2.4.1), (2.4.2) and (2.4.3) and then see the table (2.11) and figure (2.6), (2.7) to understand that how to make the tabulations and figures by Microsoft office excel.

Table 2.11: Illustration of RKF- Method for order four and five and exact solution when ⍰ = 0.1.

| $i$ | ⍰ | $t_i$ | Exact solution | RKF (Order 4) | RKF (Order5) |
|---|---|---|---|---|---|
| 0 | 0.1 | 1 | 2.5 | 2.5 | 2.5 |
| 1 | 0.1 | 1.1 | 2.435407 | 2.435407 | 2.435407 |
| 2 | 0.1 | 1.2 | 2.388889 | 2.388889 | 2.388889 |
| 3 | 0.1 | 1.3 | 2.357466 | 2.357466 | 2.357466 |
| 4 | 0.1 | 1.4 | 2.339286 | 2.339286 | 2.339286 |
| 5 | 0.1 | 1.5 | 2.333333 | 2.333333 | 2.333333 |
| 6 | 0.1 | 1.6 | 2.339286 | 2.339286 | 2.339286 |
| 7 | 0.1 | 1.7 | 2.357466 | 2.357466 | 2.357466 |
| 8 | 0.1 | 1.8 | 2.388889 | 2.388889 | 2.388889 |
| 9 | 0.1 | 1.9 | 2.435407 | 2.435407 | 2.435407 |
| 10 | 0.1 | 2 | 2.5 | 2.500001 | 2.5 |

Table 2.11.1: Illustration of Coefficients related to the table (2.11).

| k1 | k2 | k3 | k4 | k5 | k6 |
|---|---|---|---|---|---|
| -0.075 | -0.06961 | -0.06694 | -0.05631 | -0.05489 | -0.06442 |
| -0.05494 | -0.05063 | -0.04846 | -0.03972 | -0.03853 | -0.04641 |
| -0.03858 | -0.03495 | -0.03311 | -0.02557 | -0.02453 | -0.03136 |
| -0.02457 | -0.02137 | -0.01972 | -0.01288 | -0.01191 | -0.01815 |
| -0.01196 | -0.00898 | -0.00743 | -0.00089 | 4.68E-05 | -0.00595 |
| -1.2E-08 | 0.002914 | 0.004448 | 0.011045 | 0.01201 | 0.005925 |
| 0.011958 | 0.014967 | 0.016571 | 0.023586 | 0.024632 | 0.018122 |
| 0.02457 | 0.027843 | 0.029611 | 0.037465 | 0.038659 | 0.031326 |
| 0.03858 | 0.042326 | 0.044376 | 0.053618 | 0.055049 | 0.046368 |
| 0.054944 | 0.059441 | 0.061937 | 0.073347 | 0.075149 | 0.064364 |
| 0.075 | 0.08066 | 0.083847 | 0.098624 | 0.101007 | 0.086947 |

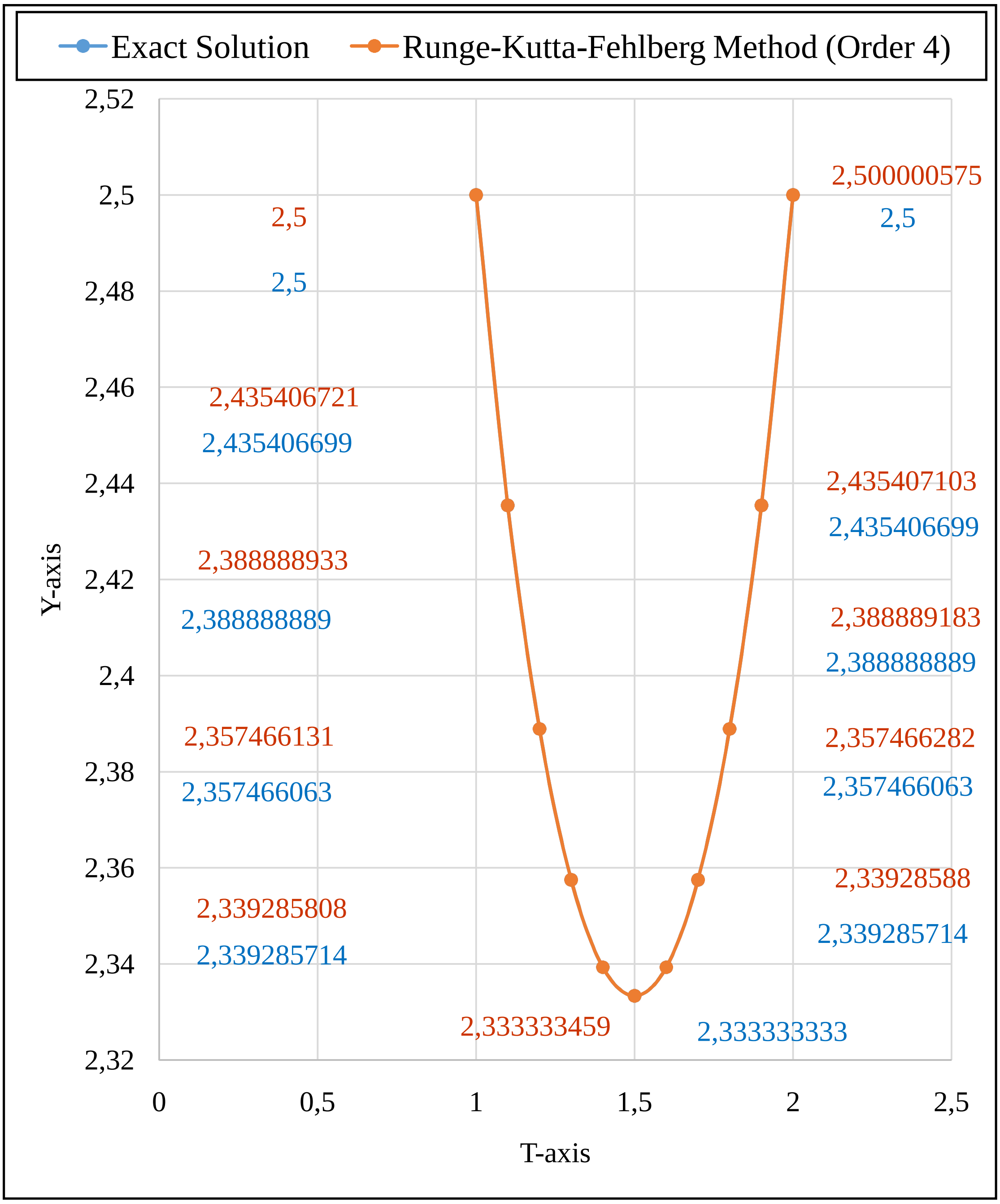


Figure 2.6: RKF-Method of order fourand exact solution when ℎ = 0.1.

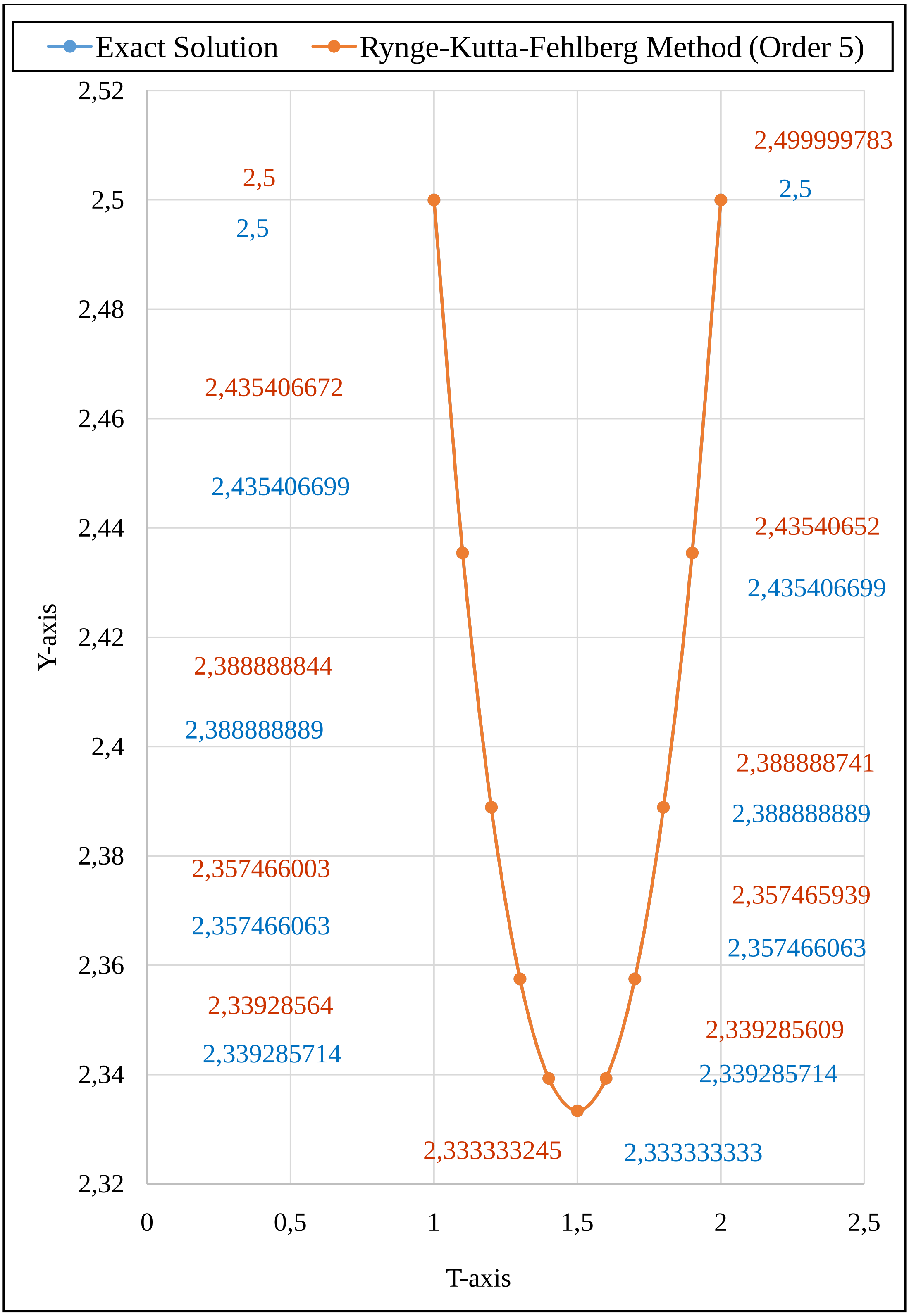


Figure 2.7: RKF-Method of order five and exact solution when ⍰ = 0.1.

## 2.5 Runge-Kutta-Verner Method

The following formula known as Runge-Kutta-Verner method.

The Runge-Kutta -Verner method for fifth-order is given by

$$w_{i+1} = w_i + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6, \qquad (2.5.1)$$

The Runge-Kutta -Verner method for sixth-order is given by

$$\widetilde{w}_{i+1} = w_i + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7 + \frac{43}{616}k_8, \qquad (2.5.2)$$

Where the coefficients consist of

$$\left.\begin{aligned}
k_1 &= hf\left(t_i, w_i\right), \\
k_2 &= hf\left(t_i + \frac{1}{6}h, w_i + \frac{1}{6}k_1\right), \\
k_3 &= hf\left(t_i + \frac{4}{15}h, w_i + \frac{4}{75}k_1 + \frac{16}{75}k_2\right), \\
k_4 &= hf\left(t_i + \frac{2}{3}h, w_i + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right), \\
k_5 &= hf\left(t_i + \frac{5}{6}h, w_i - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4\right), \\
k_6 &= hf\left(t_i + h, w_i + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5\right), \\
k_7 &= hf\left(t_i + \frac{1}{15}h, w_i - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5\right), \\
k_8 &= hf\left(t_i + h, w_i + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5 \right. \\
&\qquad \left. + \frac{3850}{26703}k_7\right),
\end{aligned}\right\}$$

The above formulas can be used to determine the approximation solutionof the Riccati Differential Equation [3]. In fact that six order Runge-Kutta -Verner method use for calculating the error in a five order Runge-Kutta -Verner method. See (Appendix E).

### 2.5.1 Application of RK-Verner Method TO RDE Using Excel

Let ⍰ = 0.1 (or can be choose any value of ⍰) then to apply this method we must be perform the following constraints;

i. Generate the step size by adding the increment value ⍰ to the initial variable $t_0$in order to make $t_1, t_2, \ldots, t_{10}$.
ii. Compute $k_1$, $k_2$, $k_3$, $k_4$, $k_5$, $k_6$, $k_7$and $k_8$respectively provided that for any coefficients $k_i$must be assign the identified column in the excel sheet.
iii. Apply the equation (2.5.1) to achieve the RKV-Method of order five and also Apply the equation (2.5.2) to achieve the RKV-Method of order six provided that for any desired formula must specified the one column in the excel sheet.
iv. Finally, sometimes putted the one column to the exact solution in order to indicate the comparison between the actual and the numerical solutions.

Notice that because the formulas and the coefficients are too long, we tried to apply the method over the examples to generation the tabulations by using MOE, keep your mind to get further more the expositions about the computations (see Appendix E).

An example [1]:
Determine the approximate solutions by RKV-Method of order five and six over the following initial value problem

$$y' = -\frac{2+t}{t(1+t)^2} - \frac{2+t-t^2}{t(1+t)}y + (1+t)y^2 \quad , \qquad y(1) = -\frac{1}{2}, \quad 1 \le t \le 3$$

When the actual solution given as $y = -1/(1+t)$.

Solution: we recognized the given differential equation is the Riccati differential equation. Now we are ready to applying the equation (2.5.1), (2.5.2) and (2.5.3) and then see the table (2.12) and figure (2.8) and (2.9) to understand that how to create the tabulations and figures by Microsoft office excel.

Table 2.12: Illustration of Runge-Kutte –Verner method of order five and six and Exact solution when $\varepsilon$ = 0.05.

| $\varepsilon$ | $t_i$ | Exact Solution | R K V- Method (Order 5) | R K V-Method (Order 6) |
|---|---|---|---|---|
| 0.05 | 1 | -0.50000 | -0.50000 | -0.50000 |
| 0.05 | 1.05 | -0.48780 | -0.48486 | -0.48510 |
| 0.05 | 1.1 | -0.47619 | -0.47109 | -0.47151 |
| 0.05 | 1.15 | -0.46512 | -0.45847 | -0.45902 |
| 0.05 | 1.2 | -0.45455 | -0.44682 | -0.44746 |
| 0.05 | 1.25 | -0.44444 | -0.43601 | -0.43671 |
| 0.05 | 1.3 | -0.43478 | -0.42591 | -0.42664 |
| 0.05 | 1.35 | -0.42553 | -0.41643 | -0.41718 |
| 0.05 | 1.4 | -0.41667 | -0.40750 | -0.40826 |
| 0.05 | 1.45 | -0.40816 | -0.39906 | -0.39981 |
| 0.05 | 1.5 | -0.40000 | -0.39104 | -0.39178 |
| 0.05 | 1.55 | -0.39216 | -0.38342 | -0.38414 |
| 0.05 | 1.6 | -0.38462 | -0.37614 | -0.37684 |
| 0.05 | 1.65 | -0.37736 | -0.36919 | -0.36986 |
| 0.05 | 1.7 | -0.37037 | -0.36252 | -0.36317 |
| 0.05 | 1.75 | -0.36364 | -0.35612 | -0.35674 |
| 0.05 | 1.8 | -0.35714 | -0.34997 | -0.35056 |
| 0.05 | 1.85 | -0.35088 | -0.34405 | -0.34461 |
| 0.05 | 1.9 | -0.34483 | -0.33834 | -0.33887 |
| 0.05 | 1.95 | -0.33898 | -0.33283 | -0.33333 |
| 0.05 | 2 | -0.33333 | -0.32750 | -0.32798 |

Table 2.12.1: Illustration of Coefficients related to the table (2.12).

| k1 | k2 | k3 | k4 | k5 | k6 | k7 | k8 |
|---|---|---|---|---|---|---|---|
| 0.01250 | -0.02459 | 0.01364 | -0.00180 | 0.07778 | -0.02497 | 0.01900 | -0.02135 |
| 0.01151 | -0.02263 | 0.01252 | -0.00120 | 0.06919 | -0.02196 | 0.01738 | -0.01868 |
| 0.01068 | -0.02077 | 0.01159 | -0.00065 | 0.06183 | -0.01933 | 0.01600 | -0.01635 |
| 0.00999 | -0.01903 | 0.01080 | -0.00016 | 0.05548 | -0.01703 | 0.01481 | -0.01432 |
| 0.00939 | -0.01740 | 0.01012 | 0.00027 | 0.04998 | -0.01502 | 0.01377 | -0.01254 |
| 0.00888 | -0.01590 | 0.00953 | 0.00065 | 0.04519 | -0.01325 | 0.01286 | -0.01097 |
| 0.00843 | -0.01451 | 0.00901 | 0.00099 | 0.04101 | -0.01168 | 0.01205 | -0.00960 |
| 0.00803 | -0.01324 | 0.00855 | 0.00128 | 0.03735 | -0.01030 | 0.01134 | -0.00838 |
| 0.00767 | -0.01207 | 0.00814 | 0.00153 | 0.03412 | -0.00908 | 0.01070 | -0.00731 |
| 0.00734 | -0.01099 | 0.00777 | 0.00174 | 0.03127 | -0.00800 | 0.01012 | -0.00636 |
| 0.00705 | -0.01001 | 0.00744 | 0.00193 | 0.02874 | -0.00703 | 0.00960 | -0.00552 |
| 0.00678 | -0.00911 | 0.00713 | 0.00209 | 0.02649 | -0.00617 | 0.00912 | -0.00477 |
| 0.00653 | -0.00829 | 0.00685 | 0.00222 | 0.02448 | -0.00541 | 0.00869 | -0.00411 |
| 0.00630 | -0.00753 | 0.00659 | 0.00233 | 0.02269 | -0.00473 | 0.00829 | -0.00351 |
| 0.00608 | -0.00685 | 0.00634 | 0.00242 | 0.02107 | -0.00412 | 0.00792 | -0.00299 |
| 0.00588 | -0.00622 | 0.00612 | 0.00250 | 0.01962 | -0.00357 | 0.00758 | -0.00252 |
| 0.00569 | -0.00564 | 0.00591 | 0.00256 | 0.01831 | -0.00308 | 0.00726 | -0.00210 |
| 0.00551 | -0.00512 | 0.00571 | 0.00261 | 0.01712 | -0.00264 | 0.00697 | -0.00172 |
| 0.00534 | -0.00464 | 0.00552 | 0.00265 | 0.01604 | -0.00225 | 0.00670 | -0.00139 |
| 0.00518 | -0.00420 | 0.00534 | 0.00267 | 0.01506 | -0.00189 | 0.00644 | -0.00109 |
| 0.00502 | -0.00380 | 0.00518 | 0.00269 | 0.01417 | -0.00158 | 0.00620 | -0.00082 |

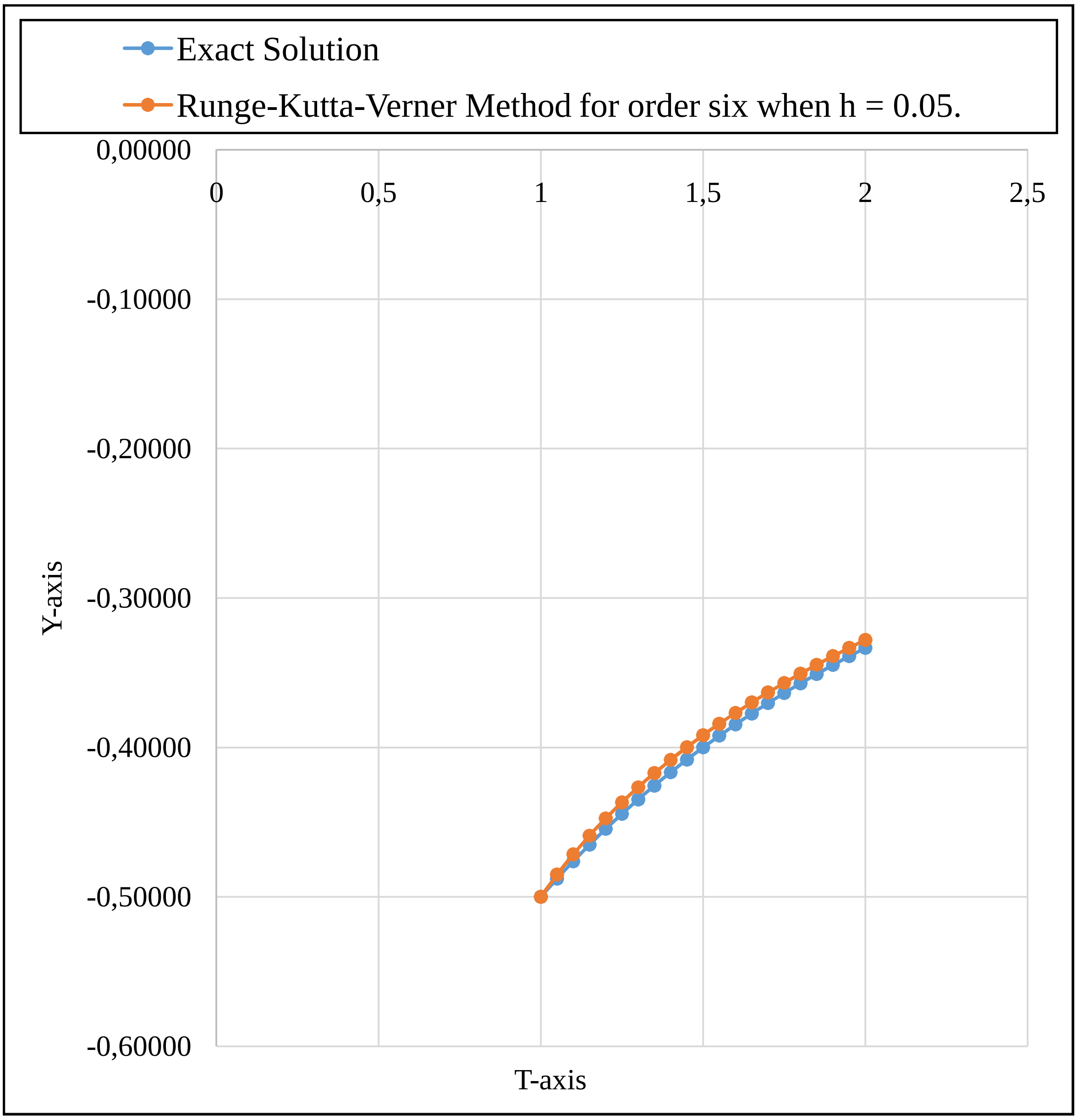


Figure 2.8: RKV-Method of order fiveand exact solution when ? = 0.05.

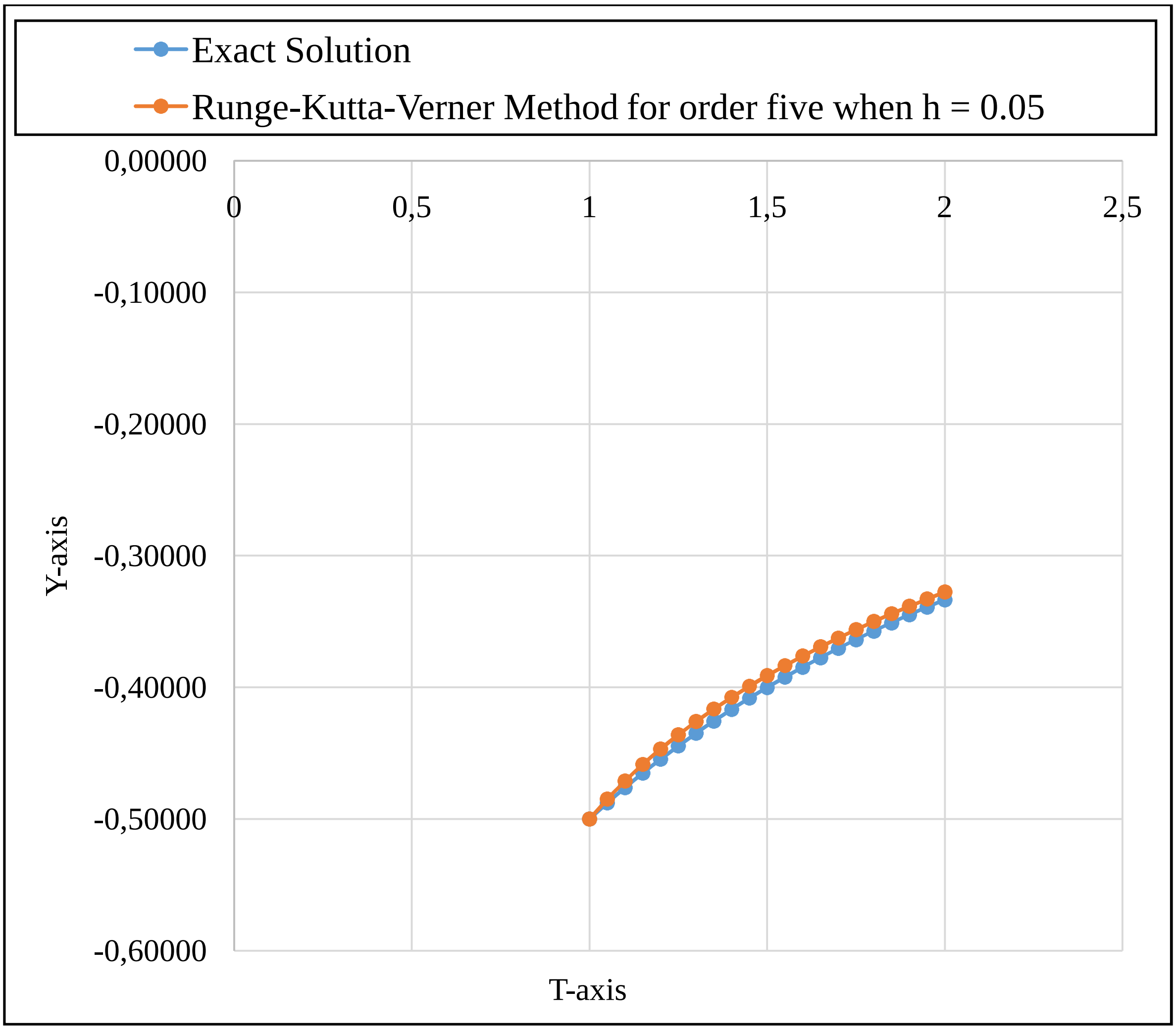


Figure 2.9: RKV-Method of order six and exact solution when ℎ = 0.05.

## 2.6 System of Differential Equations

This part of our thesis is devoted to discuss the numerical solution of higher-order initial value problems of ordinary differential equation. So, to deal with this type of problems,we transform the higher order equation into the system of first-order differential equation.

### 2.6.1 Transform the Higher Order Differential Equation to The first Order System OFDifferential Equation

Given the higher-order differential equation of the form

$$y^{(N)} = f\left(t, y(t), y'(t), y''(t), \ . \ . \ . \ . \ , y^{(N-1)}(t)\right) \qquad (2.6.1)$$

It can be transformed into the system of $N$ first-order by using the following substitutions.

Let$u_1(t) = y(t)$ , $u_2(t) = y'(t)$ , $u_3(t) = y''(t)$ , ……… . , $u_{N-1}(t) = y^{(N-1)}(t)$ , then

$$\left\{ \begin{array}{l} \dfrac{du_1}{dt} = f_1(t, u_1, u_2, \dots, u_N) \\ \dfrac{du_2}{dt} = f_2(t, u_1, u_2, \dots, u_N) \\ \dfrac{du_3}{dt} = f_3(t, u_1, u_2, \dots, u_N) \\ \quad . \\ \quad . \\ \dfrac{du_N}{dt} = f_2(t, u_1, u_2, \dots, u_N) \end{array} \right\}$$

, where $a \leq t \leq b$ with the following initial conditions [9];

$$u_1(a) = \alpha_1 , u_2(t) = \alpha_2 , \ . \ . \ . \ , u_n(t) = \alpha_n$$

Indeed, all the numerical methods that mentioned in this thesis can be applied to approximate the solutions of this system.However, here we choose to use a fourth order Runge-Kutta method approximate the solution to this system.Because this method has been proven to be one of the most accurate methods for systems.

The following formulas used for solving the system of first order initial value problems;

$$\left\{ \begin{aligned} w_{1,i+1} &= w_{1,i} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ w_{2,i+1} &= w_{2,i} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \end{aligned} \right\}$$

Where the coefficients given as the following form; Where $i = 0, 1, 2, \ldots, N-1$.

$$\left\{ \begin{aligned} k_1 &= hf_1\ t_i, u_{1,i}, u_{2,i} \\ l_1 &= hf_2\ t_i, u_{1,i}, u_{2,i} \\ k_2 &= hf_1\ t_i + \frac{h}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2} \\ l_2 &= hf_2\ t_i + \frac{h}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2} \\ k_3 &= hf_1\ t_i + \frac{h}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2} \\ l_3 &= hf_2\ t_i + \frac{h}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2} \\ k_4 &= hf_1\ t_i + h, u_{1,i} + k_3, u_{2,i} + l_3 \\ l_4 &= hf_2\ t_i + h, u_{1,i} + k_3, u_{2,i} + l_3 \end{aligned} \right\}$$

By using these formulas we can generate the numerical solutions to the system of the differential equations [3], [7], [29]. Note that equations (2.6.3), (2.6.4) are used for solving a second order DE.

## 2.6.2 Application ofRK-Method of Order Four to a System of DEs Using Excel

Let $h = 0.1$ then to apply this method we must be perform the following steps;

i. Select the suitable value of $h$ here we use $h = 0.1$.
ii. Generate the sequence $t_0, t_1, t_2, \ldots, t_{10}$ by adding h to $t_0$ and so on.

iii. Evaluate all the required coefficients in a sequential order, so that the computation of each of $k_1$, $l_1$, $k_2$, $l_2$, $k_3$, $l_3$, $k_4$and $l_4$ are possible as they depend on the previously calculated ones.

iv. Compute the numerical solutions using equation (2.6.3).

v. Note that we are explained the method by giving the example (F.1) and Also observe that we used RKV-Method to find the numerical solution in the example (F.2) of Appendix F.

Table 2.13Illustration Of RK-Method Of Order Four Applied to System Of Differential Equations Using Excel.

| $t_i$ | $w_{1,i+1} = w_{1,i} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ RK (order 4) | $w_{2,i+1} = w_{2,i} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ RK (order 4) |
|---|---|---|
| $t_0$ | $w_{1,0} = y_0$ | $w_{2,0} = y'_0$ |
| $t_1$ | $w_{1,1} = w_{1,0} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,1} = w_{2,0} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_2$ | $w_{1,2} = w_{1,1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,2} = w_{2,1} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_3$ | $w_{1,3} = w_{1,2} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,3} = w_{2,2} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_4$ | $w_{1,4} = w_{1,3} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,4} = w_{2,3} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_5$ | $w_{1,5} = w_{1,4} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,5} = w_{2,4} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_6$ | $w_{1,6} = w_{1,5} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,6} = w_{2,5} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_7$ | $w_{1,7} = w_{1,6} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,7} = w_{2,6} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_8$ | $w_{1,8} = w_{1,7} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,8} = w_{2,7} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_9$ | $w_{1,9} = w_{1,8} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,9} = w_{2,8} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |
| $t_{10}$ | $w_{1,10} = w_{1,9} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $w_{2,10} = w_{2,9} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ |

Table 2.13.1: Illustration of coefficients $k_1$, $l_1$ related to the table 2.13.

| $K_1 =$ ⍰$f_1\ t_i, u_{1,i}, u_{2,i}$ | $L_1 =$ ⍰$f_2\ t_i, u_{1,i}, u_{2,i}$ |
|---|---|
| $k_1 =$ ⍰$f_1\ t_0, u_{1,0}, u_{2.0}$ | $l_1 =$ ⍰$f_2\ t_0, u_{1,0}, u_{2.0}$ |
| $k_1 =$ ⍰$f_1\ t_1, u_{1,1}, u_{2,1}$ | $l_1 =$ ⍰$f_2\ t_1, u_{1,1}, u_{2,1}$ |
| $k_1 =$ ⍰$f_1\ t_2, u_{1,2}, u_{2,2}$ | $l_1 =$ ⍰$f_2\ t_2, u_{1,2}, u_{2,2}$ |
| $k_1 =$ ⍰$f_1\ t_3, u_{1,3}, u_{2,3}$ | $l_1 =$ ⍰$f_2\ t_3, u_{1,3}, u_{2,3}$ |
| $k_1 =$ ⍰$f_1\ t_4, u_{1,4}, u_{2,4}$ | $l_1 =$ ⍰$f_2\ t_4, u_{1,4}, u_{2,4}$ |
| $k_1 =$ ⍰$f_1\ t_5, u_{1,5}, u_{2,5}$ | $l_1 =$ ⍰$f_2\ t_5, u_{1,5}, u_{2,5}$ |
| $k_1 =$ ⍰$f_1\ t_6, u_{1,6}, u_{2,6}$ | $l_1 =$ ⍰$f_2\ t_6, u_{1,6}, u_{2,6}$ |
| $k_1 =$ ⍰$f_1\ t_7, u_{1,7}, u_{2,7}$ | $l_1 =$ ⍰$f_2\ t_7, u_{1,7}, u_{2,7}$ |
| $k_1 =$ ⍰$f_1\ t_8, u_{1,8}, u_{2,8}$ | $l_1 =$ ⍰$f_2\ t_8, u_{1,8}, u_{2,8}$ |
| $k_1 =$ ⍰$f_1\ t_9, u_{1,9}, u_{2,9}$ | $l_1 =$ ⍰$f_2\ t_9, u_{1,9}, u_{2,9}$ |
| $k_1 =$ ⍰$f_1\ t_{10}, u_{1,10}, u_{2,10}$ | $l_1 =$ ⍰$f_2\ t_{10}, u_{1,10}, u_{2,10}$ |

Table 2.13.2: Illustration of coefficients $k_2$, $l_2$related to the table 2.13.

| $K_2 = \square f_1 \left( t_i + \frac{\square}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2} \right)$ | $L_2 = \square f_2 \left( t_i + \frac{\square}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2} \right)$ |
|---|---|
| $k_2 = \square f_1 \left( t_0 + \frac{\square}{2}, u_{1,0} + \frac{k_1}{2}, u_{2,0} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_0 + \frac{\square}{2}, u_{1,0} + \frac{k_1}{2}, u_{2,0} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_1 + \frac{\square}{2}, u_{1,1} + \frac{k_1}{2}, u_{2,1} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_1 + \frac{\square}{2}, u_{1,1} + \frac{k_1}{2}, u_{2,1} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_2 + \frac{\square}{2}, u_{1,2} + \frac{k_1}{2}, u_{2,2} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_2 + \frac{\square}{2}, u_{1,2} + \frac{k_1}{2}, u_{2,2} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_3 + \frac{\square}{2}, u_{1,3} + \frac{k_1}{2}, u_{2,3} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_3 + \frac{\square}{2}, u_{1,3} + \frac{k_1}{2}, u_{2,3} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_4 + \frac{\square}{2}, u_{1,4} + \frac{k_1}{2}, u_{2,4} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_4 + \frac{\square}{2}, u_{1,4} + \frac{k_1}{2}, u_{2,4} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_5 + \frac{\square}{2}, u_{1,5} + \frac{k_1}{2}, u_{2,5} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_5 + \frac{\square}{2}, u_{1,5} + \frac{k_1}{2}, u_{2,5} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_6 + \frac{\square}{2}, u_{1,6} + \frac{k_1}{2}, u_{2,6} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_6 + \frac{\square}{2}, u_{1,6} + \frac{k_1}{2}, u_{2,6} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_7 + \frac{\square}{2}, u_{1,7} + \frac{k_1}{2}, u_{2,7} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_7 + \frac{\square}{2}, u_{1,7} + \frac{k_1}{2}, u_{2,7} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_8 + \frac{\square}{2}, u_{1,8} + \frac{k_1}{2}, u_{2,8} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_8 + \frac{\square}{2}, u_{1,8} + \frac{k_1}{2}, u_{2,8} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_9 + \frac{\square}{2}, u_{1,9} + \frac{k_1}{2}, u_{2,9} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_9 + \frac{\square}{2}, u_{1,9} + \frac{k_1}{2}, u_{2,9} + \frac{l_1}{2} \right)$ |
| $k_2 = \square f_1 \left( t_{10} + \frac{\square}{2}, u_{1,10} + \frac{k_1}{2}, u_{2,10} + \frac{l_1}{2} \right)$ | $l_2 = \square f_2 \left( t_{10} + \frac{\square}{2}, u_{1,10} + \frac{k_1}{2}, u_{2,10} + \frac{l_1}{2} \right)$ |

Table 2.13.3:Illustration of coefficients $k_3$, $l_3$related to the table 2.13.

| $K_3 = \Box f_1 \ \ t_i + \frac{\Box}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2}$ | $L_3 = \Box f_2 \ \ t_i + \frac{\Box}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2}$ |
|---|---|
| $k_3 = \Box f_1 \ \ t_0 + \frac{\Box}{2}, u_{1,0} + \frac{k_2}{2}, u_{2,0} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_0 + \frac{\Box}{2}, u_{1,0} + \frac{k_2}{2}, u_{2,0} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_1 + \frac{\Box}{2}, u_{1,1} + \frac{k_2}{2}, u_{2,1} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_1 + \frac{\Box}{2}, u_{1,1} + \frac{k_2}{2}, u_{2,1} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_2 + \frac{\Box}{2}, u_{1,2} + \frac{k_2}{2}, u_{2,2} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_2 + \frac{\Box}{2}, u_{1,2} + \frac{k_2}{2}, u_{2,2} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_3 + \frac{\Box}{2}, u_{1,3} + \frac{k_2}{2}, u_{2,3} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_3 + \frac{\Box}{2}, u_{1,3} + \frac{k_2}{2}, u_{2,3} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_4 + \frac{\Box}{2}, u_{1,4} + \frac{k_2}{2}, u_{2,4} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_4 + \frac{\Box}{2}, u_{1,4} + \frac{k_2}{2}, u_{2,4} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_5 + \frac{\Box}{2}, u_{1,5} + \frac{k_2}{2}, u_{2,5} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_5 + \frac{\Box}{2}, u_{1,5} + \frac{k_2}{2}, u_{2,5} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_6 + \frac{\Box}{2}, u_{1,6} + \frac{k_2}{2}, u_{2,6} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_6 + \frac{\Box}{2}, u_{1,6} + \frac{k_2}{2}, u_{2,6} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_7 + \frac{\Box}{2}, u_{1,7} + \frac{k_2}{2}, u_{2,7} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_7 + \frac{\Box}{2}, u_{1,7} + \frac{k_2}{2}, u_{2,7} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_8 + \frac{\Box}{2}, u_{1,8} + \frac{k_2}{2}, u_{2,8} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_8 + \frac{\Box}{2}, u_{1,8} + \frac{k_2}{2}, u_{2,8} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_9 + \frac{\Box}{2}, u_{1,9} + \frac{k_2}{2}, u_{2,9} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_9 + \frac{\Box}{2}, u_{1,9} + \frac{k_2}{2}, u_{2,9} + \frac{l_2}{2}$ |
| $k_3 = \Box f_1 \ \ t_{10} + \frac{\Box}{2}, u_{1,10} + \frac{k_2}{2}, u_{2,10} + \frac{l_2}{2}$ | $l_3 = \Box f_2 \ \ t_{10} + \frac{\Box}{2}, u_{1,10} + \frac{k_2}{2}, u_{2,10} + \frac{l_2}{2}$ |

Table 2.13.4:Illustration of coefficients $k_4$, $l_4$related to the table 2.13.

| $K_4 = □f_1\ t_i + □, u_1 + k_3, u_{2,i} + l_3$ | $L_4 = □f_2\ t_i + □, u_{1,i} + k_3, u_{2,i} + l_3$ |
|---|---|
| $k_4 = □f_1\ t_0 + □, u_{1,0} + k_3, u_{2,0} + l_3$ | $l_4 = □f_2\ t_0 + □, u_{1,0} + k_3, u_{2,0} + l_3$ |
| $k_4 = □f_1\ t_1 + □, u_{1,1} + k_3, u_{2,1} + l_3$ | $l_4 = □f_2\ t_1 + □, u_{1,1} + k_3, u_{2,1} + l_3$ |
| $k_4 = □f_1\ t_2 + □, u_{1,2} + k_3, u_{2,2} + l_3$ | $l_4 = □f_2\ t_2 + □, u_{1,2} + k_3, u_{2,2} + l_3$ |
| $k_4 = □f_1\ t_3 + □, u_{1,3} + k_3, u_{2,3} + l_3$ | $l_4 = □f_2\ t_3 + □, u_{1,3} + k_3, u_{2,3} + l_3$ |
| $k_4 = □f_1\ t_4 + □, u_{1,4} + k_3, u_{2,4} + l_3$ | $l_4 = □f_2\ t_4 + □, u_{1,4} + k_3, u_{2,4} + l_3$ |
| $k_4 = □f_1\ t_5 + □, u_{1,5} + k_3, u_{2,5} + l_3$ | $l_4 = □f_2\ t_5 + □, u_{1,5} + k_3, u_{2,5} + l_3$ |
| $k_4 = □f_1\ t_6 + □, u_{1,6} + k_3, u_{2,6} + l_3$ | $l_4 = □f_2\ t_6 + □, u_{1,6} + k_3, u_{2,6} + l_3$ |
| $k_4 = □f_1\ t_7 + □, u_{1,7} + k_3, u_{2,7} + l_3$ | $l_4 = □f_2\ t_7 + □, u_{1,7} + k_3, u_{2,7} + l_3$ |
| $k_4 = □f_1\ t_8 + □, u_{1,8} + k_3, u_{2,8} + l_3$ | $l_4 = □f_2\ t_8 + □, u_{1,8} + k_3, u_{2,8} + l_3$ |
| $k_4 = □f_1\ t_9 + □, u_{1,9} + k_3, u_{2,9} + l_3$ | $l_4 = □f_2\ t_9 + □, u_{1,9} + k_3, u_{2,9} + l_3$ |
| $k_4 = □f_1\ t_{10} + □, u_{1,10} + k_3, u_{2,10} + l_3$ | $l_4 = □f_2\ t_{10} + □, u_{1,10} + k_3, u_{2,10} + l_3$ |

**An example [13]:**

Solve the second-order initial value problem

$$ty'' - y' - t^3 y = 0 \quad , \quad for\ \ 1 \le t \le 2 \quad , y\ 1 = 2.776347 \quad , y'\ 1 = 2.169817$$

By transforming to the system of the first-order initial value problems and use a fourth order R.K method to approximate the solution with □ = 0.1 , Tolerance =$10^{-6}$ , given that the actual solutions are given by

$$y(t) = \frac{3}{2}\exp\ \frac{t^2}{2} + \frac{1}{2}\exp\ -\frac{t^2}{2}$$

$$y'\ t = \frac{3}{2}\,t\exp\ \frac{t^2}{2} - \frac{1}{2}\,t\exp\ -\frac{t^2}{2}\ .$$

**Solution:**Applying the equation (2.6.2), (2.6.3) and (2.6.4) and organizing the numerical and the exact in a table (2.14) this show an easy way for comparison. A graphical representation is also presented in figure (2.10).

Table 2.14: Illustration ofRK Method for Order Four andExact Solution When $h = 0.1$.

| $i$ | $h$ | $t_i$ | $y(t) = u_{1,i}$ exact | $w_{1,i}$ RK (order 4) | $y'(t) = u_{2,i}$ exact | $w_{2,i}$ RK (order 4) |
|---|---|---|---|---|---|---|
| 0 | 0.1 | 1 | 2.776347 | 2.776347 | 2.169817 | 2.169817 |
| 1 | 0.1 | 1.1 | 3.019916 | 3.019913 | 2.721225 | 2.721223 |
| 2 | 0.1 | 1.2 | 3.325026 | 3.325020 | 3.405928 | 3.405923 |
| 3 | 0.1 | 1.3 | 3.706745 | 3.706734 | 4.260344 | 4.260334 |
| 4 | 0.1 | 1.4 | 4.184340 | 4.184321 | 5.332640 | 5.332622 |
| 5 | 0.1 | 1.5 | 4.782652 | 4.782621 | 6.686999 | 6.686967 |
| 6 | 0.1 | 1.6 | 5.533978 | 5.533931 | 8.409506 | 8.409454 |
| 7 | 0.1 | 1.7 | 6.480651 | 6.480580 | 10.616339 | 10.616255 |
| 8 | 0.1 | 1.8 | 7.678585 | 7.678478 | 13.465235 | 13.465100 |
| 9 | 0.1 | 1.9 | 9.202194 | 9.202034 | 17.171668 | 17.171450 |
| 10 | 0.1 | 2 | 11.151252 | 11.151011 | 22.031833 | 22.031481 |

Table 2.14.1: Illustration of Coefficients related to the table (2.14).

| $K_1$ | $L_1$ | $K_2$ | $L_2$ | $K_3$ | $L_3$ | $K_4$ | $L_4$ |
|---|---|---|---|---|---|---|---|
| 0.216982 | 0.494616 | 0.241712 | 0.548256 | 0.244394 | 0.552173 | 0.272199 | 0.612963 |
| 0.272122 | 0.612793 | 0.302762 | 0.680649 | 0.306155 | 0.685625 | 0.340685 | 0.762858 |
| 0.340592 | 0.762630 | 0.378724 | 0.849122 | 0.383048 | 0.855561 | 0.426148 | 0.954470 |
| 0.426033 | 0.954156 | 0.473741 | 1.065294 | 0.479298 | 1.073758 | 0.533409 | 1.201469 |
| 0.533262 | 1.201028 | 0.593314 | 1.344994 | 0.600512 | 1.356272 | 0.668889 | 1.522514 |
| 0.668697 | 1.521888 | 0.744791 | 1.709862 | 0.754190 | 1.725067 | 0.841203 | 1.943176 |
| 0.840945 | 1.942277 | 0.938059 | 2.189607 | 0.950426 | 2.210322 | 1.061978 | 2.498672 |
| 1.061626 | 2.497373 | 1.186494 | 2.825236 | 1.202887 | 2.853724 | 1.346998 | 3.237776 |
| 1.346510 | 3.235888 | 1.508304 | 3.673680 | 1.530194 | 3.713200 | 1.717830 | 4.228452 |
| 1.717145 | 4.225695 | 1.928430 | 4.814484 | 1.957869 | 4.869752 | 2.204120 | 5.566021 |
| 2.203148 | 5.561979 | 2.481247 | 6.359513 | 2.521124 | 6.437401 | 2.846888 | 7.385073 |

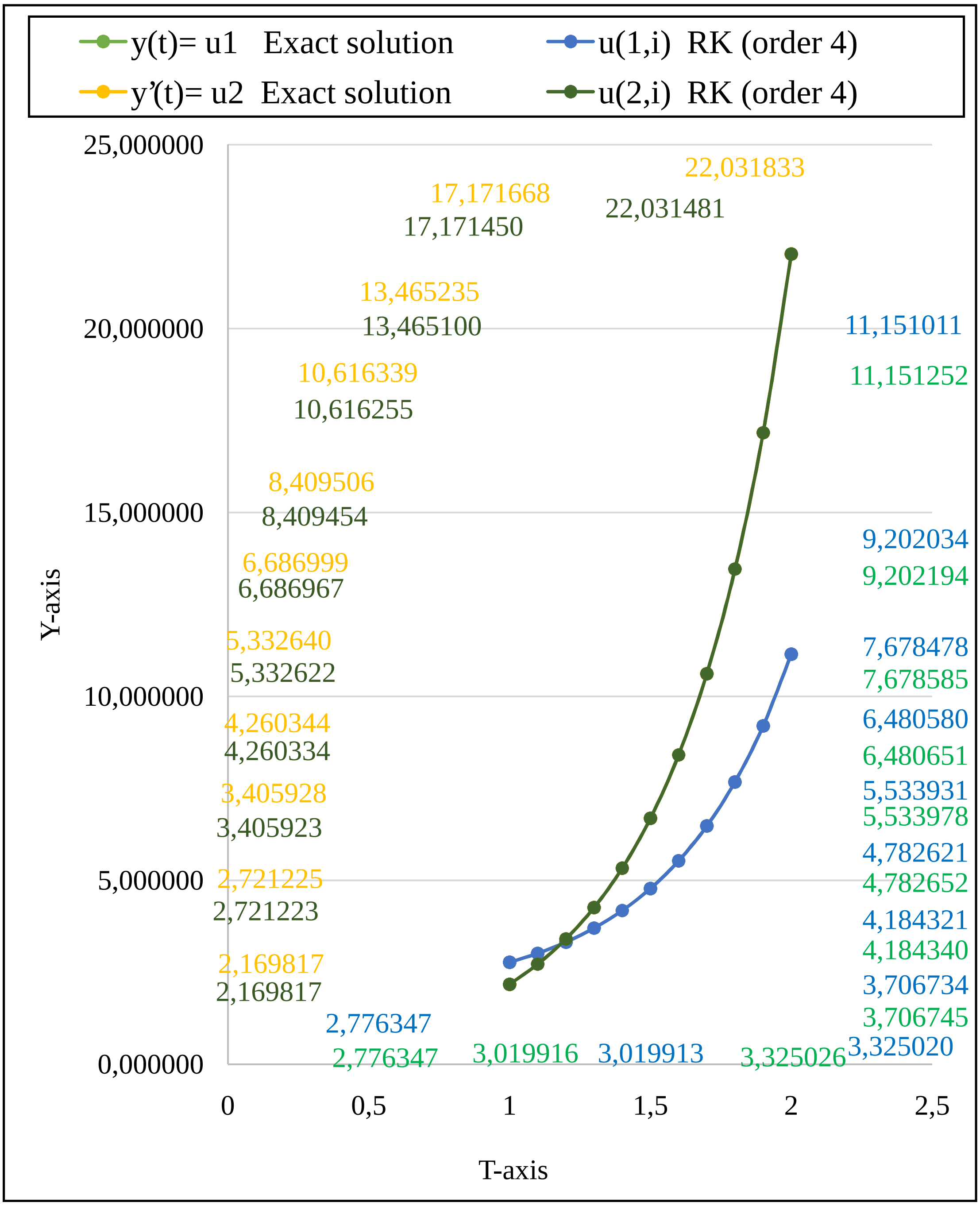


Figure 2.10: RK-Method of Order Four for System of Differential Equationand Exact Solution When ℎ = 0.1.

## 2.7 Adams-Bashforth Explicit Methods

Another class of methods for numerically approximating the solution of IVP are known as Multistep Methods. Among these, few of these methods are very well known. They are:

Adams-Bashforth 2-Step explicit method:

$$w_{i+1} = w_i + \frac{1}{2}\left[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})\right], \qquad where\, i = 1,2,3, \ldots\ldots, N-1$$

Adams-Bashforth 3-Step explicit method:

$$w_{i+1} = w_i + \frac{1}{12}\left[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})\right],$$
$$where\, i = 2,3,4, \ldots\ldots, N-1$$

Adams-Bashforth 4-Step explicit method:

$$w_{i+1} = w_i + \frac{1}{24}\left[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})\right],$$
$$where\, i = 3,4,5, \ldots\ldots, N-1$$

Adams-Bashforth 5-Step explicit method:

$$w_{i+1} = w_i + \frac{1}{720}\left[1901f(t_i, w_i) - 2774f(t_{i-1}, w_{i-1}) + 2616f(t_{i-2}, w_{i-2}) - 1274f(t_{i-3}, w_{i-3}) + 251f(t_{i-4}, w_{i-4})\right],$$
$$where\, i = 4,5,6, \ldots\ldots, N-1$$

Any of the above Schemes can be considered for determining the approximate solutions of the Riccati Differential Equation [9]. See (Appendix G).

### 2.7.1 Application of Adams Bashforth Explicit Method to RDE Using Excel

Let $h = 0.1$ then to apply this method we must be perform the following steps;

i. Generate the time sequence $t_0,\ t_1, t_2, \dots, t_{10}$, by adding the increment value $h$ to the initial variable $t_0$ in order to calculate $t_1, t_2, \dots, t_{10}$.

ii. Use the numerical methods to determine the approximate solution. A drawback of the multistep method is that they require extra initial values compared to the single initial value in the case of R.K methods. For instance an Adam-Bashforth explicit method there is the need for values of $w_0, w_1, w_2$ before we can compute $w_3$. For that one can use a single step method such as Euler's method or R.K method to evaluate the required initial values.

iii. Calculate the exact solution to the given Riccati differential equation in one of the columns of excel sheet next to the column of Runge-Kutta Method, for easy comparison..

iv. Finally, compute the numerical solution using the preferred formula of ABE-Method. In this case the selected methods is the three step ABE given by:

$$w_{i+1} = w_i + \frac{1}{12} h\left[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})\right],$$

$$\text{where } i = 2,3,4, \dots\dots, N - 1.$$

The procedures are presented in table (2.15) and for further details see the Appendix G.

Table 2.15: Illustration of Three Step Adams-Bashforth Explicit Method.

| $t_i$ | RK-Method (order 4) | Adams-Bashforth (3-Step) | $f(t_i, w_i)$ (3-Step) |
|---|---|---|---|
| $t_0$ | $w_0 = y_0$ | Taken from the RK-Method $w_0$ | $f(t_0, w_0)$ |
| $t_1$ | $w_1$ | Taken from the RK-Method $w_1$ | $f(t_1, w_1)$ |
| $t_2$ | $w_2$ | Taken from the RK-Method $w_2$ | $f(t_2, w_2)$ |
| $t_3$ | $w_3$ | $w_3 = w_2 + \frac{1}{12}[23f(t_2, w_2) - 16f(t_1, w_1) + 5f(t_0, w_0)]$ | $f(t_3, w_3)$ |
| $t_4$ | $w_4$ | $w_4 = w_3 + \frac{1}{12}[23f(t_3, w_3) - 16f(t_2, w_2) + 5f(t_1, w_1)]$ | $f(t_4, w_4)$ |
| $t_5$ | $w_5$ | $w_5 = w_4 + \frac{1}{12}[23f(t_4, w_4) - 16f(t_3, w_3) + 5f(t_2, w_2)]$ | $f(t_5, w_5)$ |
| $t_6$ | $w_6$ | $w_6 = w_5 + \frac{1}{12}[23f(t_5, w_5) - 16f(t_4, w_4) + 5f(t_3, w_3)]$ | $f(t_6, w_6)$ |
| $t_7$ | $w_7$ | $w_7 = w_6 + \frac{1}{12}[23f(t_6, w_6) - 16f(t_5, w_5) + 5f(t_4, w_4)]$ | $f(t_7, w_7)$ |
| $t_8$ | $w_8$ | $w_8 = w_7 + \frac{1}{12}[23f(t_7, w_7) - 16f(t_6, w_6) + 5f(t_5, w_5)]$ | $f(t_8, w_8)$ |
| $t_9$ | $w_9$ | $w_9 = w_8 + \frac{1}{12}[23f(t_8, w_8) - 16f(t_7, w_7) + 5f(t_6, w_6)]$ | $f(t_9, w_9)$ |
| $t_{10}$ | $w_{10}$ | $w_{10} = w_9 + \frac{1}{12}[23f(t_9, w_9) - 16f(t_8, w_8) + 5f(t_7, w_7)]$ | $f(t_{10}, w_{10})$ |

**An example:**

Determine the approximate solutions by ABEM of the given initial value problem

$$y' = \frac{t}{1+t^2} + \frac{y}{t} + \frac{y^2}{t(1+t^2)}, \qquad y(1) = 0, \quad 1 \le t \le 2$$

When the actual solution given by $y = (t^2 - t)/(t+1)$.

**Solution:** we recognized the given differential equation is the Riccati differential equation. Now we are ready to apply the method of three step ABEM to the given IVP The table (2.16) shows this approximation, an order 2 R.K method and the exact solution. The figure (2.11) gives a graphical representation of the ABEM and the exact solution. As we can see the results compare extremely well.

Table 2.16: Illustration of Three Step Adams-Bashforth Explicit Method, RK-Method and The Exact solution.

| ⍰ | $t_i$ | Exact Solution | RK-Method (order 2) | Adams-Bashforth (3-Step) | $f(t_i, w_i)$ |
|---|---|---|---|---|---|
| 0.1 | 1 | 0.000000 | 0.000000 | 0.000000 | 0.500000 |
| 0.1 | 1.1 | 0.052381 | 0.052211 | 0.052381 | 0.546485 |
| 0.1 | 1.2 | 0.109091 | 0.108764 | 0.109091 | 0.586777 |
| 0.1 | 1.3 | 0.169565 | 0.169091 | 0.169525 | 0.621893 |
| 0.1 | 1.4 | 0.233333 | 0.232720 | 0.233255 | 0.652713 |
| 0.1 | 1.5 | 0.300000 | 0.299252 | 0.299888 | 0.679911 |
| 0.1 | 1.6 | 0.369231 | 0.368353 | 0.369088 | 0.704034 |
| 0.1 | 1.7 | 0.440741 | 0.439736 | 0.440569 | 0.725528 |
| 0.1 | 1.8 | 0.514286 | 0.513156 | 0.514087 | 0.744761 |
| 0.1 | 1.9 | 0.589655 | 0.588403 | 0.589431 | 0.762040 |
| 0.1 | 2 | 0.666667 | 0.665292 | 0.666417 | 0.777620 |

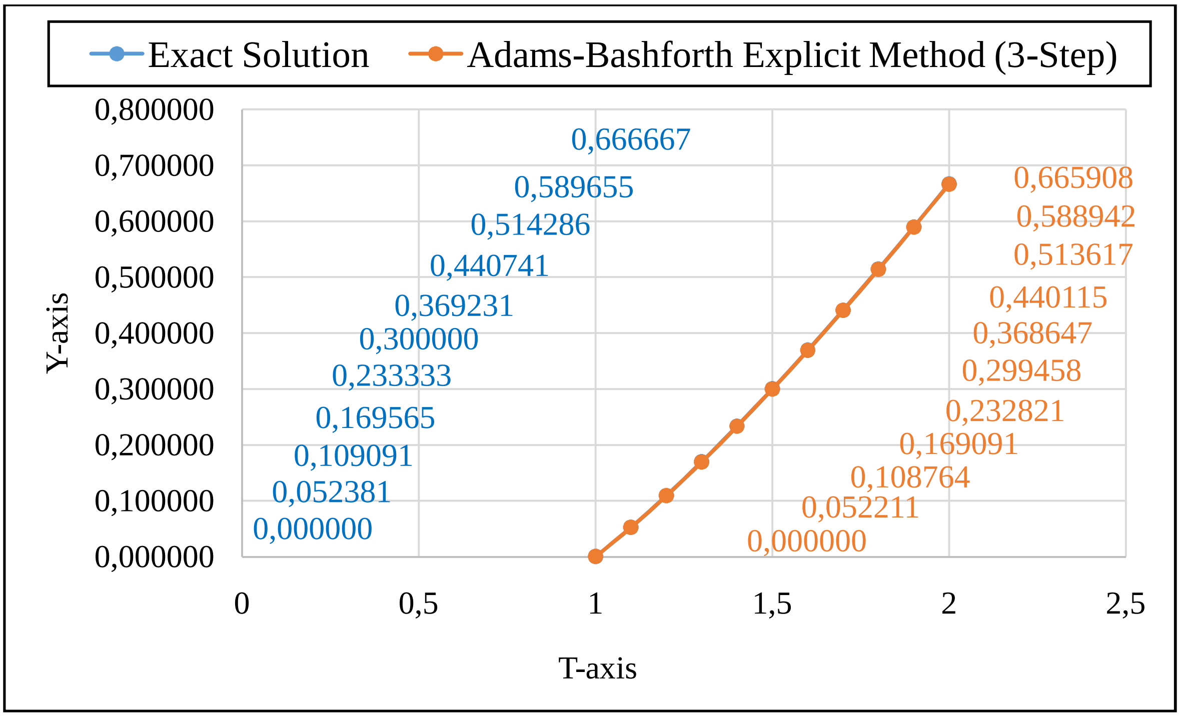


Figure 2.11: Adams Bashforth Explicit Method for three step and exact solutions when ⍰ = 0.1.

# CHAPTER 3

# RESULTS AND DISCUSSIONS

In this section we will attempt to present an overall two way comparison for a range of known numerical methods and their performance in solving RIVPs. On the other hand we will demonstrate the effect of varying the step size on their performance. The overall results are summarized and presented in a tabulated format and also graphically for ease of comparison.

The one-step numerical methods that are used for this comparison study include; Euler's Method, Taylor's Method of order four, Runge-Kutta Method of order four, Runge-Kutta-Fehlberg Method of order four and Runge-Kutta-Verner Method of order five. Among the multi-step methods, we chose to use the Adams-Bashforth Explicit Method.

## 3.1 Numerical Method's Capability with the Various Step Size $h$

Case 1: Step size $h = 0.5$

Taking the step size $h = 0.5$ and the interval $0 \le t \le 1$ then$t_0 = 0$, $t_1 = 0.5$ and$t_2 = 1$. The approximate solutions shownin table (3.1). The last row of the table gives the numerical solutions at $t = t_2$and these are;Euler's Method $= 1.820313$, Taylor's Method of order four $= 1.839480$, Runge-Kutta Method of order four $= 1.999419$, Runge-Kutta-Fehlberg Method of order four $= 2.000069$, Runge-Kutta-Verner Method for order five $= 2.000019$ and Adams-Bashforth Explicit Methods of order five $= 1.937500$. The exact solution at $t = t_2$ is 2.0.

Notice that each of the Euler's Method and Taylor's Method give poor performance and this is in line with our expectations of these methods. On the other hand theRunge-Kutta Method, Runge-Kutta-Fehlberg, and Runge-Kutta-Verner Method produced solutions that compare extremely well with exact solution. While Adams-Bashforth Explicit Methods gave solutions that lie somewhere between the two groups of the methods. These results are also presented graphically for the ease of comparison figure (3.1) to figure (3.6).

Table 3.1: Illustration Of Exact Solution, Euler's Method, (Taylor, RK andRKF) Methods for Order Four and (RKV andABE) Methods for Order Five when □ = 0.5.

| Exact | Euler | Taylor (order 4) | Runge-Kutta (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (order 5) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 1.166667 | 1.125000 | 1.133789 | 1.166610 | 1.166674 | 1.166669 | 1.166667 |
| 2.000000 | 1.820313 | 1.839480 | 1.999419 | 2.000069 | 2.000019 | 1.937500 |

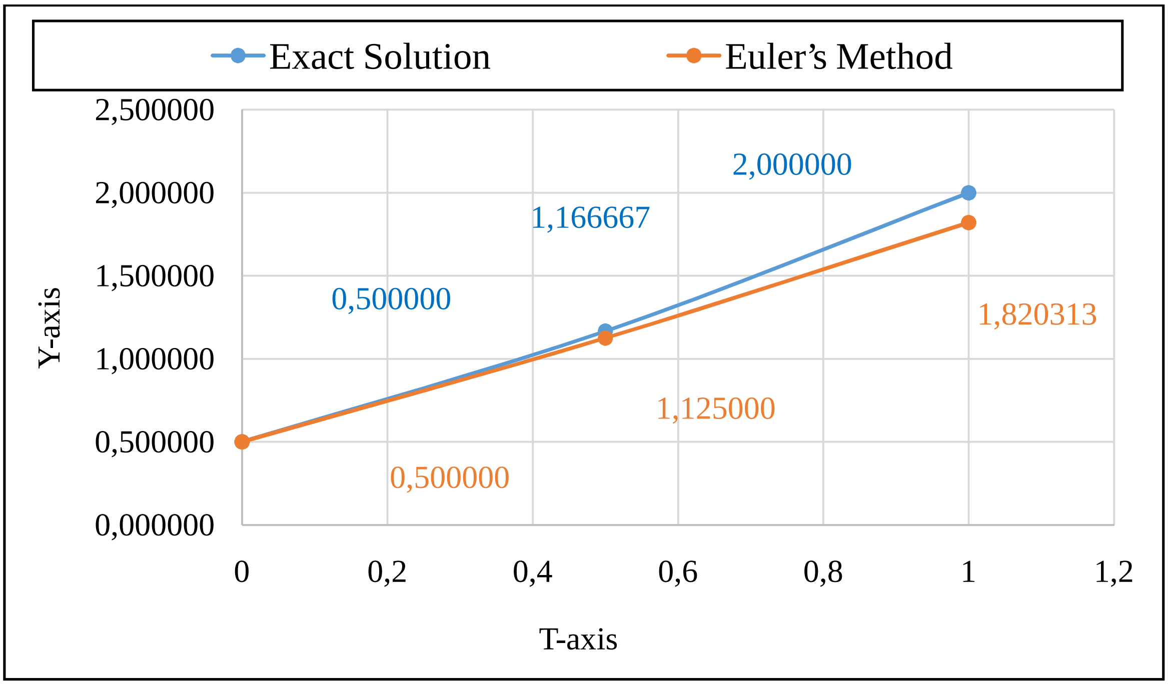


Figure 3.1: Euler's Method and exact solution when □ = 0.5.

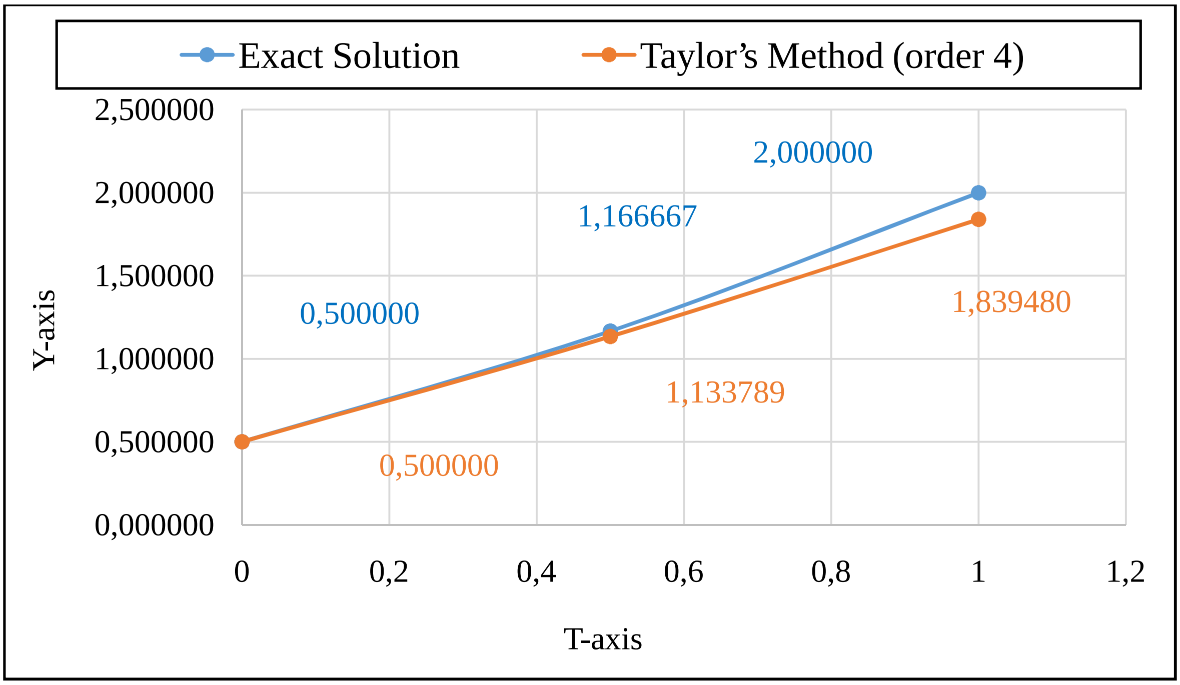


Figure 3.2: Taylor's Method of order four and exact solution when ℎ = 0.5.

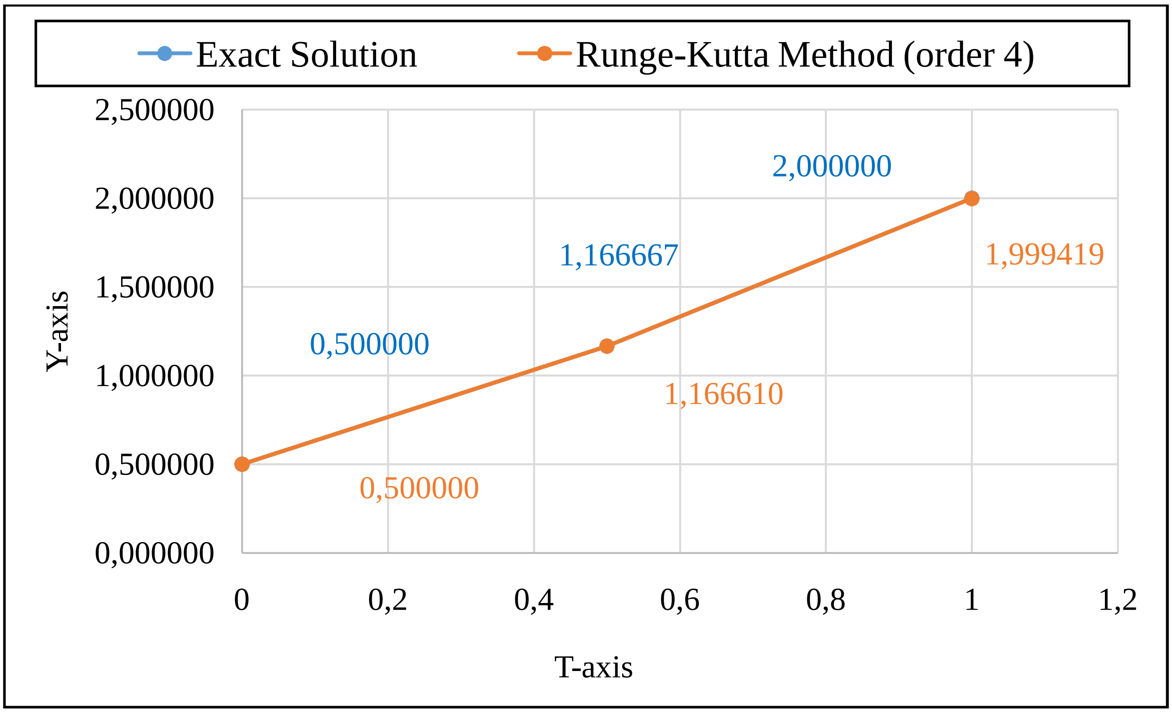


Figure 3.3: RK-Method of order four and exact solutions when ℎ = 0.5.

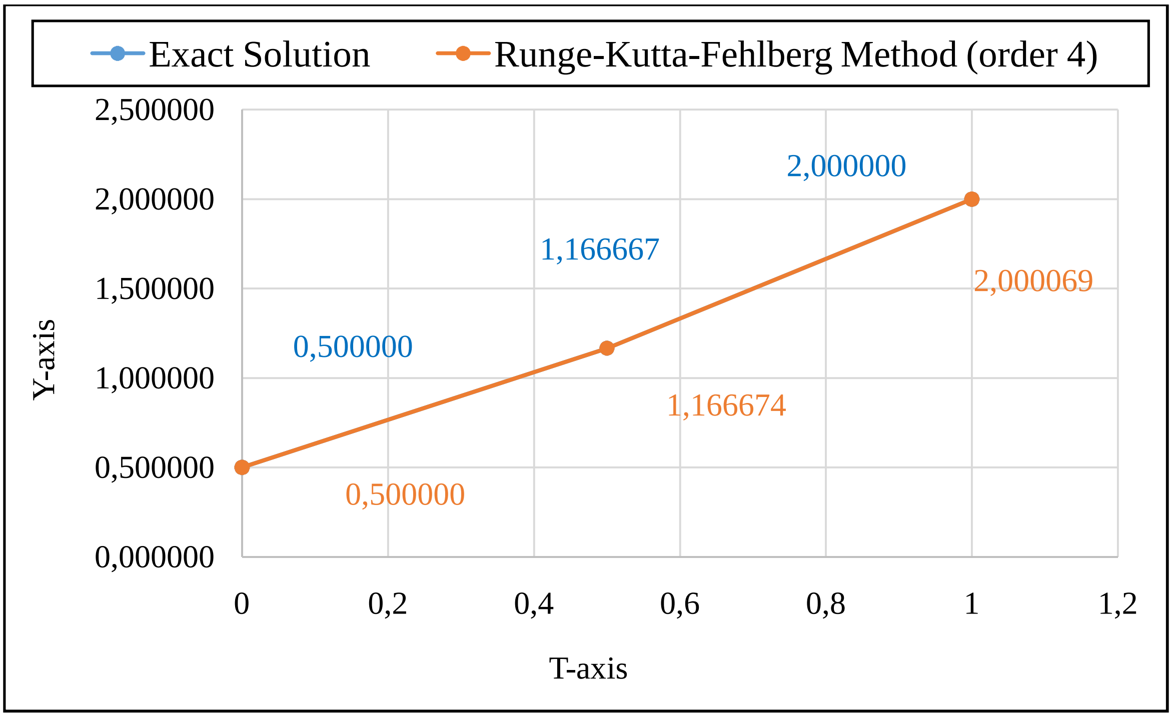


Figure 3.4: RKF-Method of order four and exact solutions when ℎ = 0.5.

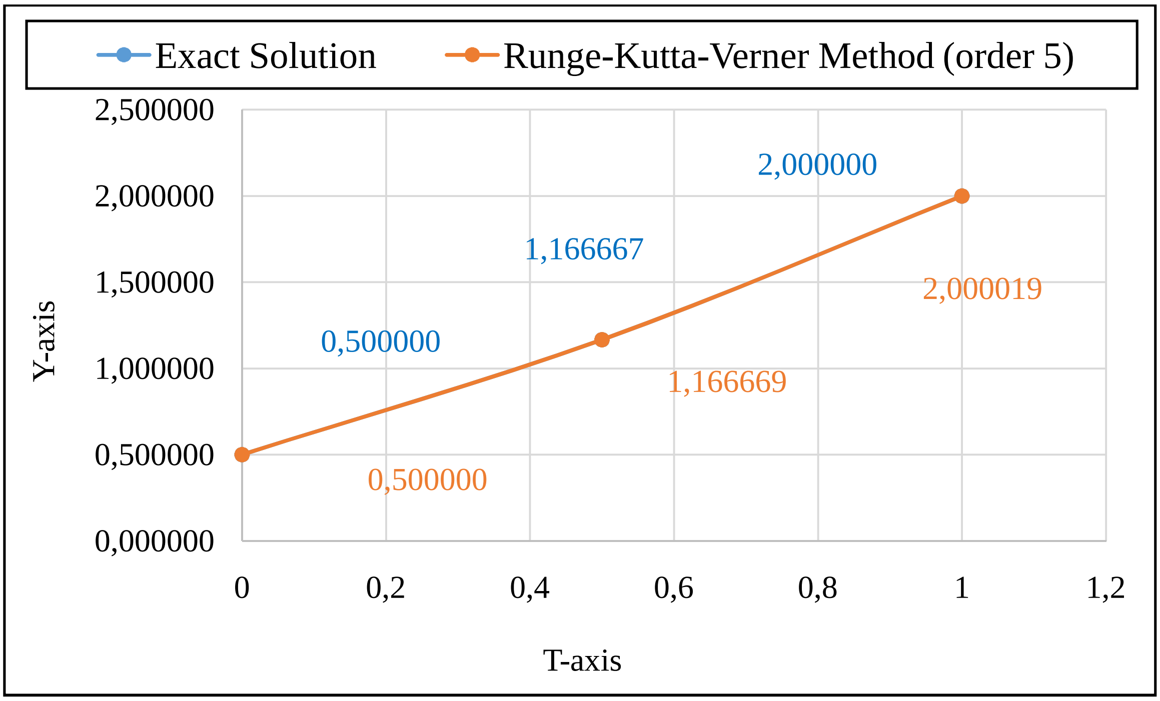


Figure 3.5: RKV-Method of order five and exact solutions when ℎ = 0.5.

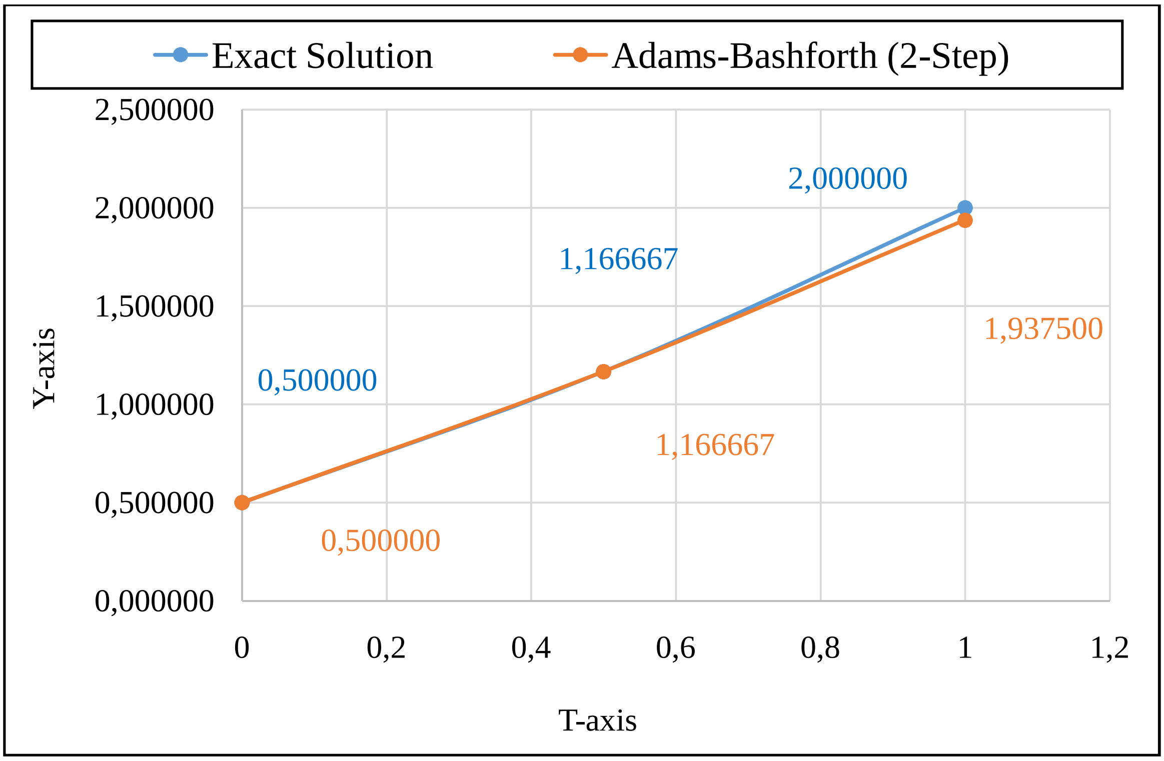


Figure 3.6: Adams Bashforth Explicit Method for two step and exact solutions when ℎ = 0.5.

## Case 2: Step size $h = 0.25$

In this case after reducing the value of $h$ from the h = 0.5 to h = 0.25 then we would expect that the accuracy of the solutions are increased and the size of error also decreased. Fortunately, all the methods above except Euler's method showed better results.This is also in line with our understanding of the methods because Euler's method is the most elementary among the others.

Table (3.2) summarizes these results with $h = 0.25$. The rows give the solutions at the times $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ $and$ $t_4 = 1$.

At the last row of the table shows the solutions at $t = 1$ and the values are;

Euler's Method $= 1.883090$, Taylor's Method of order four $= 1.968497$, Runge-Kutta Method of order four $= 1.999956$, Runge-Kutta-Fehlberg Method of order four $= 2.000007$, Runge-Kutta-Verner Method of order five $= 2.000001$ and Adams-Bashforth Explicit Methods for order five $= 1.995931$.See figures (3.7) to figure (3.12).

Table 3.2: Illustration Of Exact Solution, Euler's Method, (Taylor, RK andRKF) Methods for Order Four and (RKV andABE) Methods for Order Five when $h = 0.25$.

| Exact Solution | Euler's Method | Taylor's Method (order 4) | Runge-Kutta Method (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (order 5) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 0.821429 | 0.812500 | 0.818176 | 0.821428 | 0.821429 | 0.821429 | 0.821429 |
| 1.166667 | 1.141602 | 1.155969 | 1.166663 | 1.166667 | 1.166667 | 1.166667 |
| 1.550000 | 1.494515 | 1.528939 | 1.549987 | 1.550002 | 1.550000 | 1.550000 |
| 2.000000 | 1.883090 | 1.968497 | 1.999956 | 2.000007 | 2.000001 | 1.995931 |

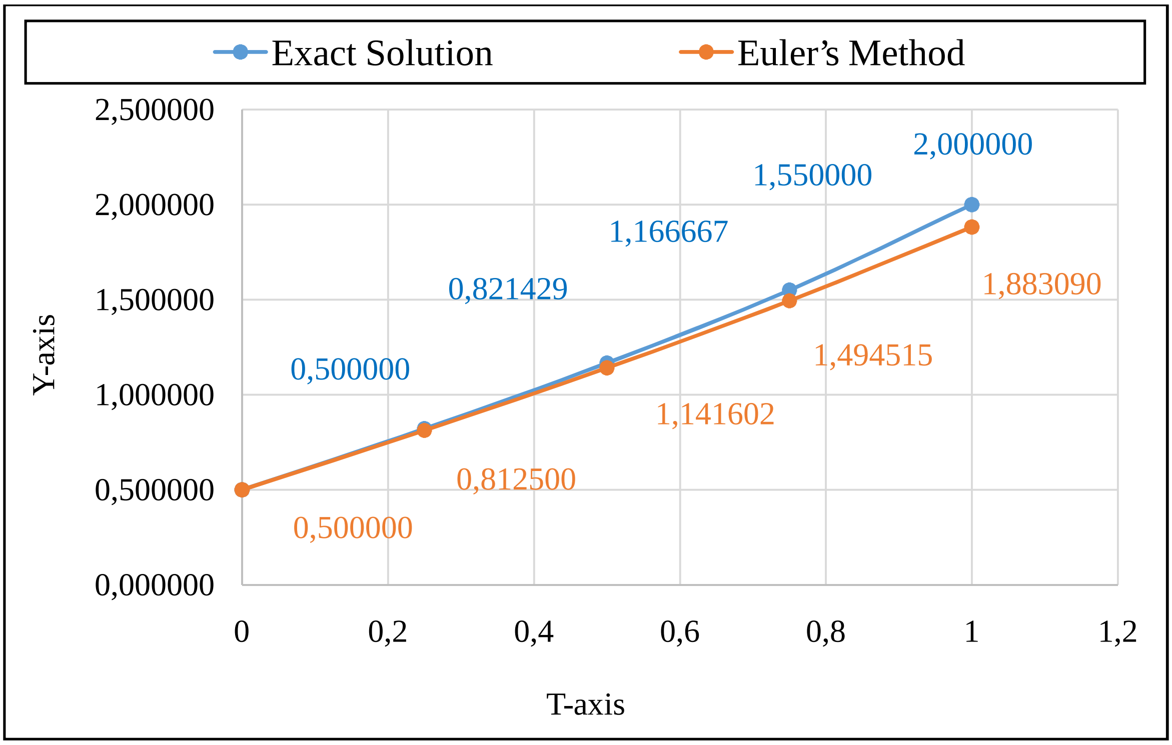


Figure 3.7: Euler's Methodand Exact Solutions When ℎ = 0.25.

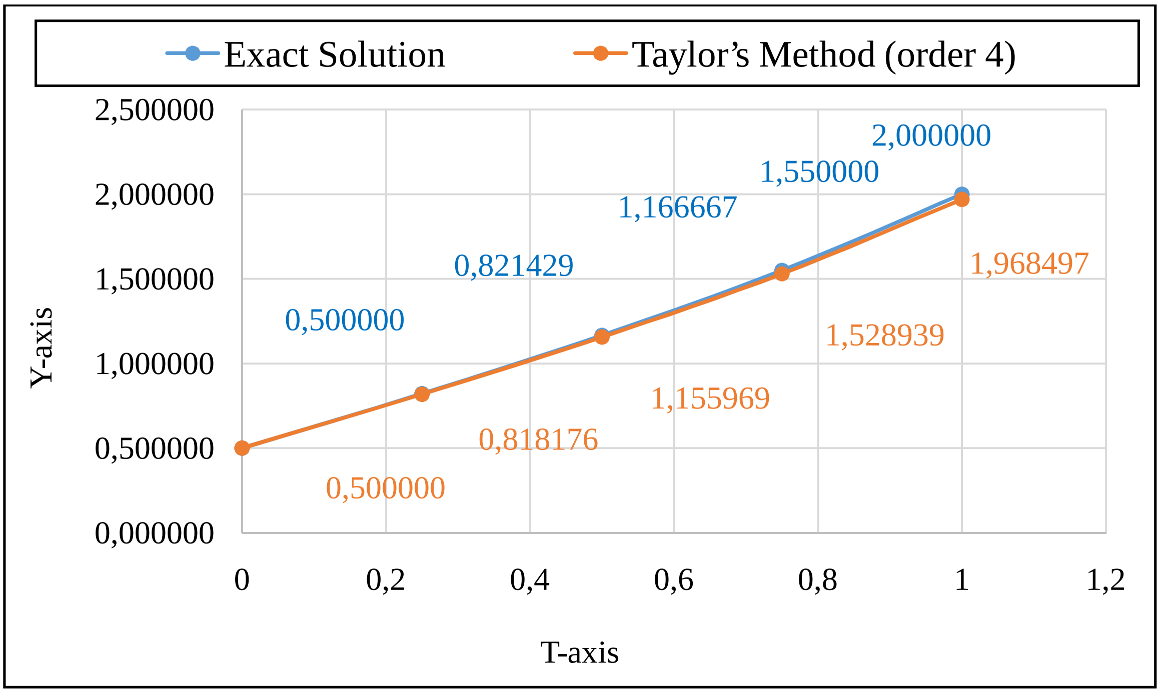


Figure 3.8: Taylor's Method of Order Fourand Exact Solutions, ℎ = 0.25.

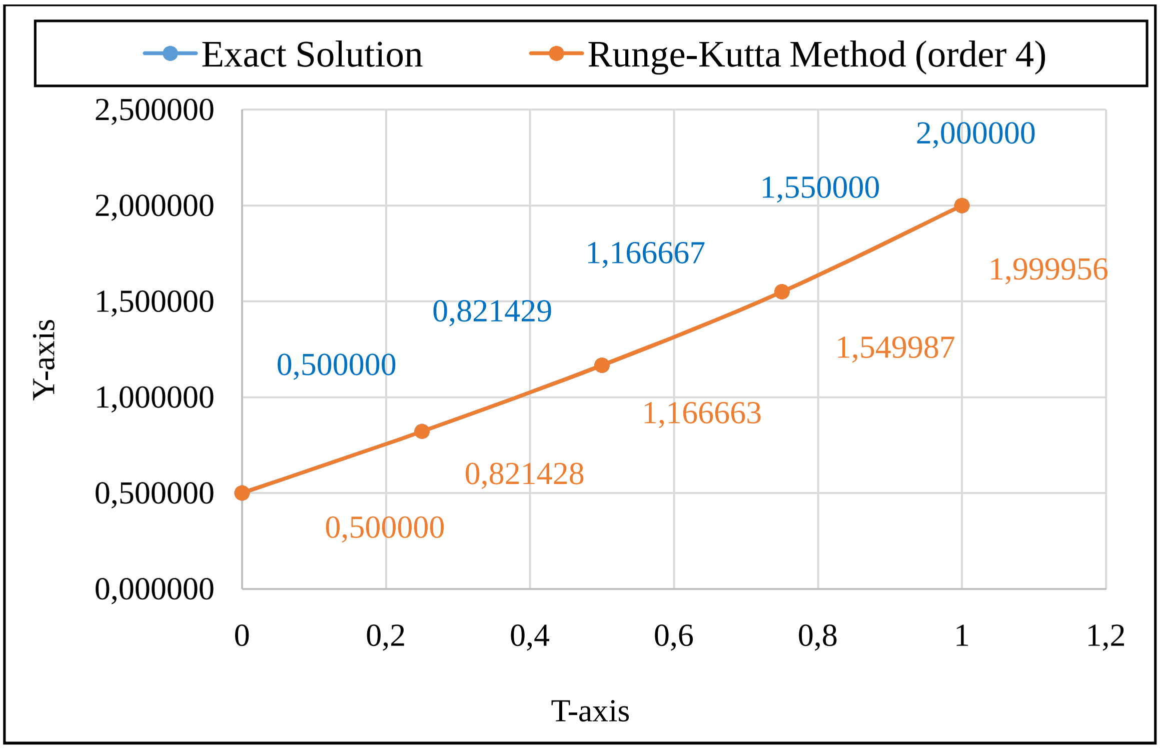


Figure 3.9: RK-Method of Order Four and Exact Solutions, ⍰ = 0.25.

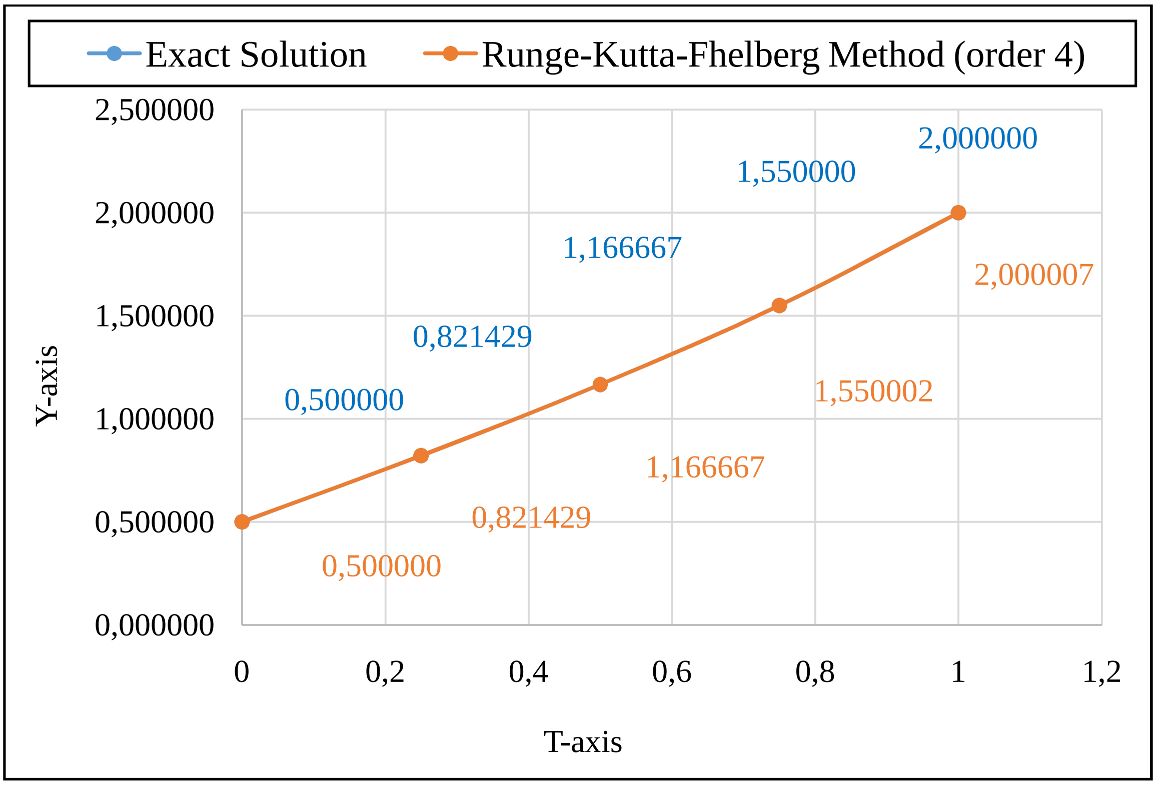


Figure 3.10: RKF-Method of Order Fourand Exact Solutions, ⍰ = 0.25.

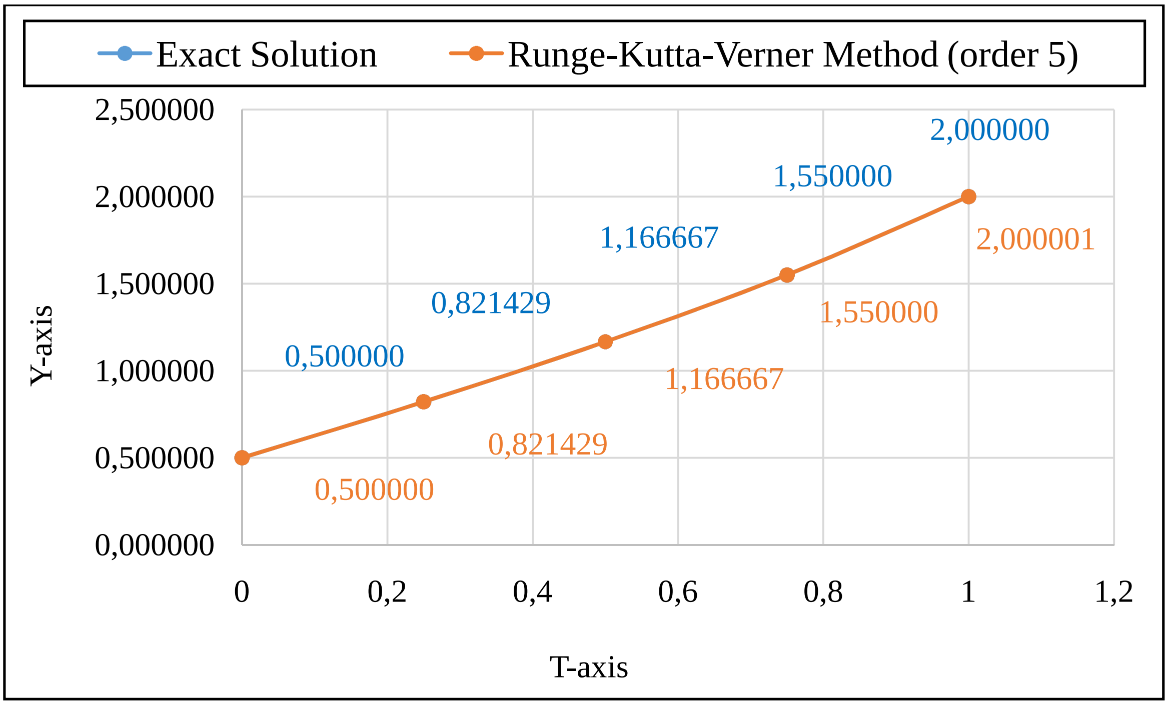


Figure 3.11: RKV-Method of Order Five and Exact Solutions, □ = 0.25.

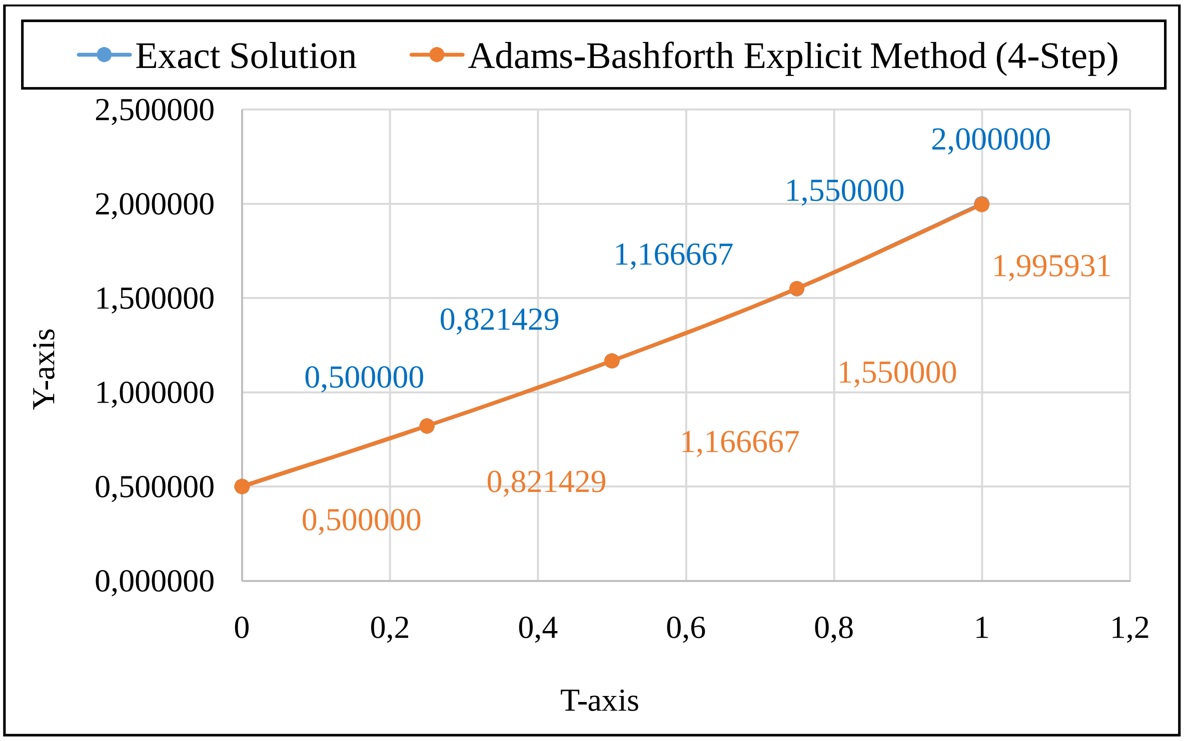


Figure 3.12: Four Step Adams Bashforth Explicit Method and Exact Solutions, □ = 0.25.

## Case 3: Step size $h = 0.1$

We further reduced the step size to $h = 0.1$ and carry out the computations as in the case of $h = 0.5$ and $h = 0.25$ and as we expect there are improvements in results as we can see from the table (3.3). The results at the last row of the table are at $t = 1$ and the values obtained are; Euler's Method $= 1.942205$, Taylor's Method for order four $= 2.012006$, Runge-Kutta Method for order four $= 1.999999$, Runge-Kutta-Fehlberg Method for order four $= 2.000000$, Runge-Kutta-Verner Method for order five $= 2.000000$ and Adams-Bashforth Explicit Methods for order five $= 1.999829$. The graphical representations aregiven in figure (3.13) to figure (3.18).

Table 3.3: Illustration of Exact Solution, Euler's Method, (Taylor, RK and RKF) Methods of Order Four and (RKV and ABE) Methods ofOrder Five when $h = 0.1$.

| Exact Solution | Euler's Method | Taylor's Method (order 4) | Runge-Kutta (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (order 5) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 0.626316 | 0.625000 | 0.626139 | 0.626316 | 0.626316 | 0.626316 | 0.626316 |
| 0.755556 | 0.752563 | 0.755105 | 0.755556 | 0.755556 | 0.755556 | 0.755556 |
| 0.888235 | 0.883095 | 0.887448 | 0.888235 | 0.888235 | 0.888235 | 0.888235 |
| 1.025000 | 1.017095 | 1.023904 | 1.025000 | 1.025000 | 1.025000 | 1.025000 |
| 1.166667 | 1.155176 | 1.165441 | 1.166667 | 1.166667 | 1.166667 | 1.166662 |
| 1.314286 | 1.298101 | 1.313335 | 1.314286 | 1.314286 | 1.314286 | 1.314272 |
| 1.469231 | 1.446836 | 1.469263 | 1.469231 | 1.469231 | 1.469231 | 1.469203 |
| 1.633333 | 1.602612 | 1.635465 | 1.633333 | 1.633333 | 1.633333 | 1.633281 |
| 1.809091 | 1.767031 | 1.814975 | 1.809090 | 1.809091 | 1.809091 | 1.808996 |
| 2.000000 | 1.942205 | 2.012006 | 1.999999 | 2.000000 | 2.000000 | 1.999829 |

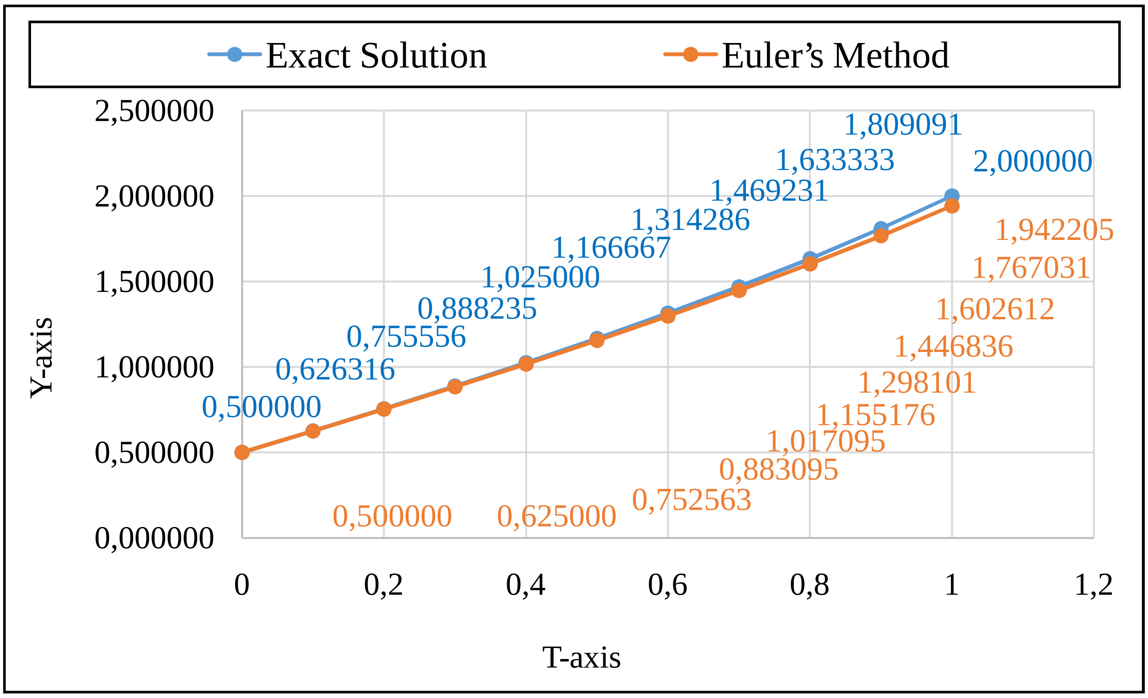


Figure 3.13: Euler's Method and exact solutions when ▯ = 0.1.

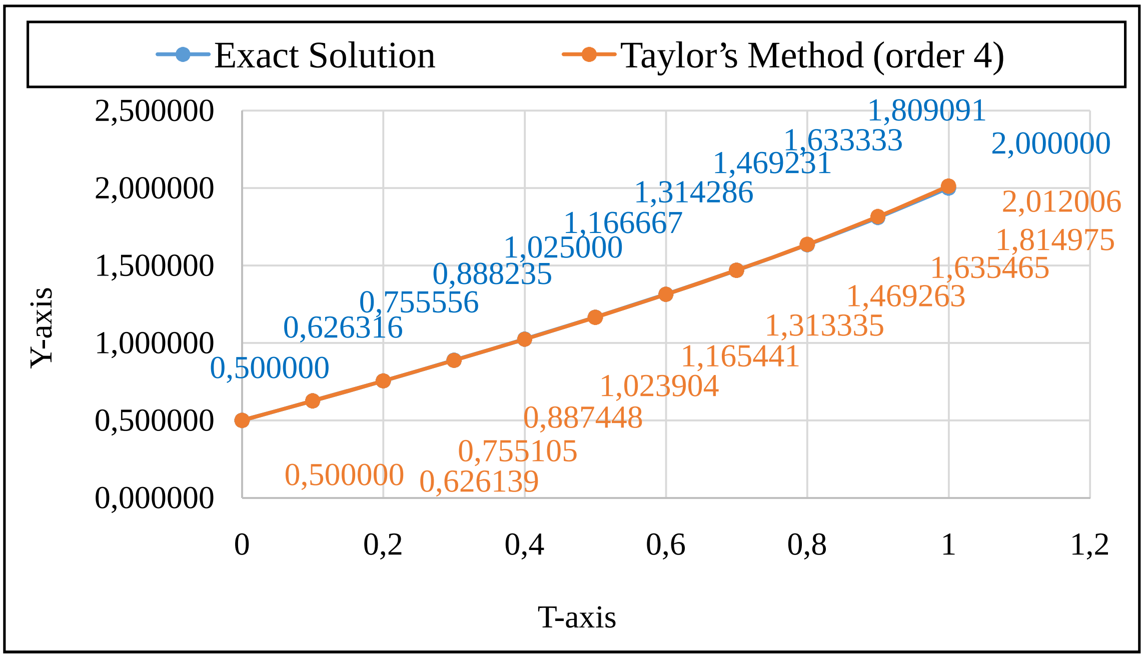


Figure 3.14: Taylor's Method of order fourand exact solutions when ▯ = 0.1.

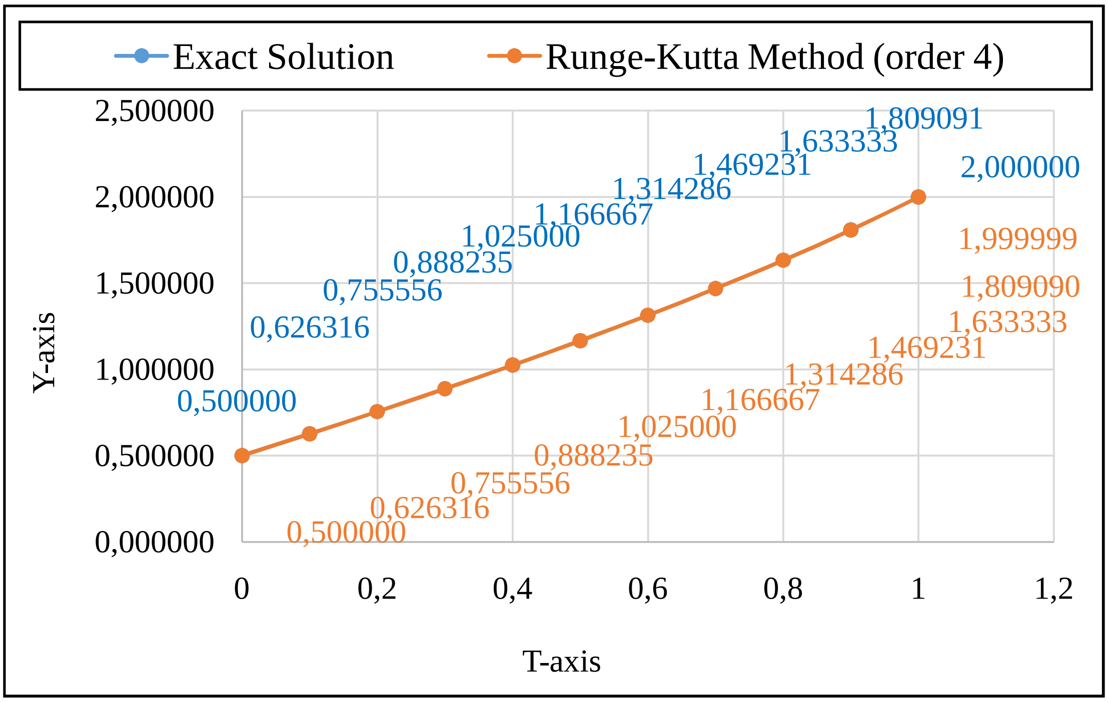


Figure 3.15: RK-Method of order four and exact solutions when ⍰ = 0.1.

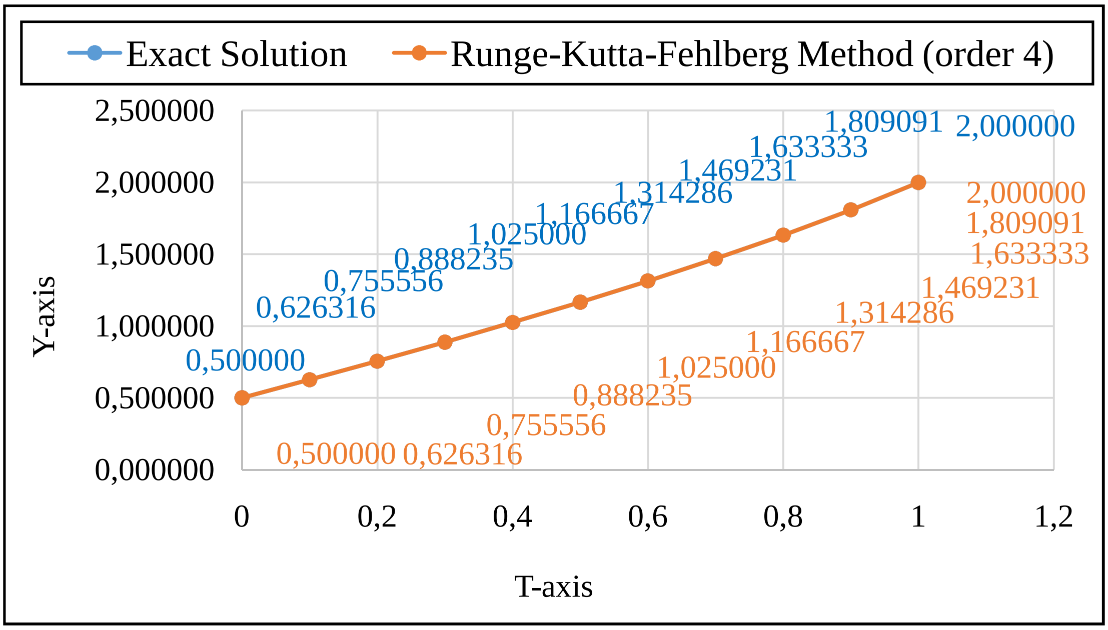


Figure 3.16: RKF-Method of Order Fourand Exact Solutions, ⍰ = 0.1.

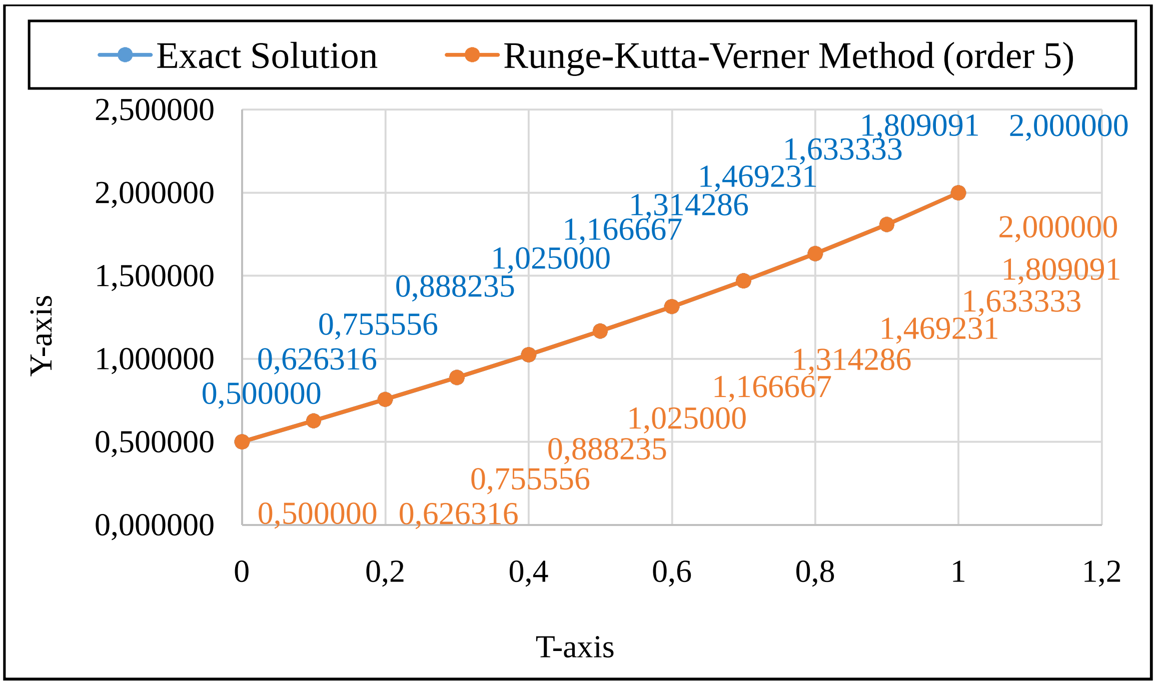


Figure 3.17: RKV-Method of Order Fiveand Exact Solutions, ⍰ = 0.1.

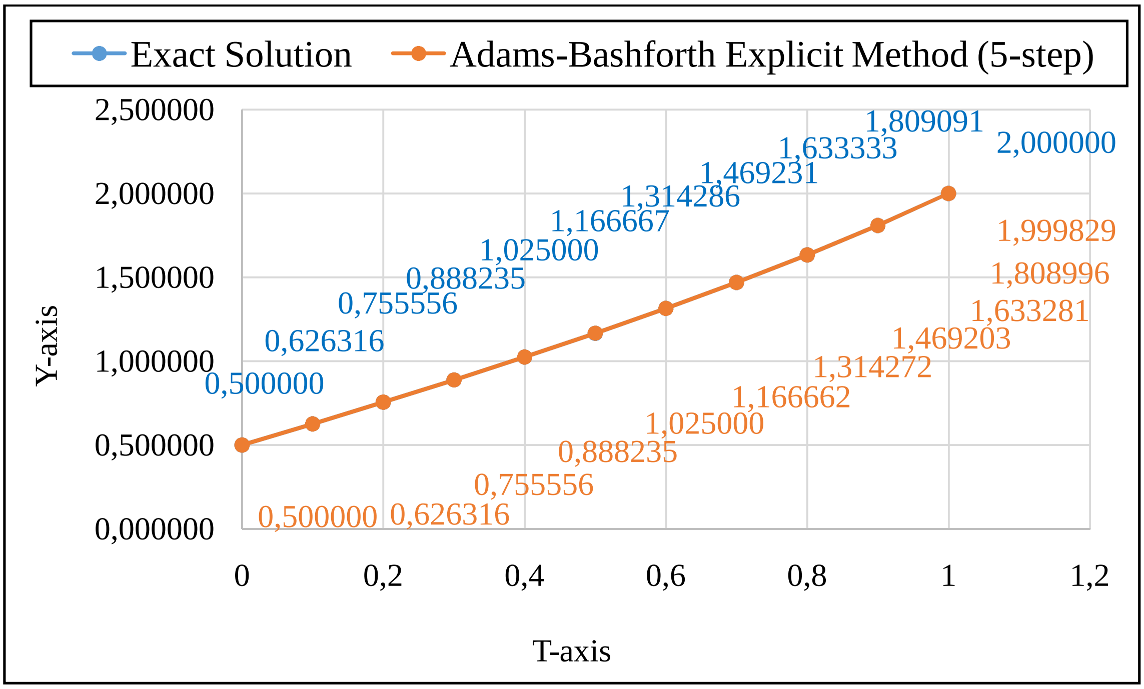


Figure 3.18: Five SepAdams Bashforth Explicit Method Exact Solution when ⍰ = 0.1.

# CHAPTER 4

# CONCLUSIONS

The lessons we have learned from this interesting piece of work can be summarized as follows; The very basic Euler's method is very simple easy to implement therefore, it is useful for learners of numerical analysis and also beginners programmers to use the method as a practice aiming to further improve their numerical knowledge and also their programming skills. Another area that we recommend the use of this method is to compute additional starting values that required when using Multistep methods for solving IVPs. Otherwise if an accurate result is required for academic or scientific purposes, certainly this method is not recommended.

Regarding the Taylor method, Reasonable results can be achieved if higher order Taylor's method is employed, but this is often very costly because the need for evaluating higher order derivatives, where some times can be very tedious and even impossible to obtain. Hence this group of methods are also not recommended for serious scientific or academic work. Coming to Linear Multistep methods it is fairly easy to implement, the drawbacks are that they require additional initial values that are not readily available, therefore by the time you program a single step method to obtain these additional values one will be tempted to continue with this method to produce the full solution of the problem in hand.

The method that stands out among all the numerical methods are the fourth order Runge-Kutte method. These methods produce very good and impressive results, they are easy to implement and do not require additional starting values. Therefore definitely they are our choice of recommendation.

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# APPENDIXES

## Appendix A. Approximation Solution from RDE by Euler's Method

**Example (A.1):**

By using Euler's method find the approximate solution for the Riccati differential equation $y' = x^2 + 1 - 2xy + y^2$ when the initial condition consist of $y(0) = \frac{1}{2}$ and its interval is $0 \leq t \leq 1$, also use step size $h = 0.1$ *and* $h = 0.01$ to indicate that which one can give us the best accurate solution, if you know that the exact solution considered as

$$y = \frac{1}{2 - t} + t$$

### A.1.1 Solution of example (A.1) when $(h = 0.1)$

Euler's method is given by $y_{i+1} = y_i + h f(t_i, y(t_i))$, when $i \geq 0$

Since $t_0 = 0$ and $h = 0.1$ then for finding next $t_i$ we can use the formula $t_i = t_{i-1} + h$ as:

$t_1 = t_0 + h \Rightarrow t_1 = 0 + 0.1 = 0.1$

$t_2 = t_1 + h \Rightarrow t_2 = 0.1 + 0.1 = 0.2$, and so on until to get the upper bound of the required interval, which is $t_{10} = t_9 + h = 0.9 + 0.1 = 1$.

To illustrate the procedure, note that the value of step size $h = 0.1$ is used to calculate $t_i$ with respect to the interval $0 \leq t \leq 1$, which is specified in this experiment the value of $t_0 = 0$ and we found $t_1, t_2$ *and* $t_{10}$, also notice that we stopped when the value of $t = 1$ which is $t_{10}$ and all these $t_i$ values are being use for identifying the approximate solution of the points which are denoted by $y_1, y_2, \ldots, y_i$. To make the procedure approximation, Euler's formula which is

$y_{i+1} = y_i + h f(t_i, y(t_i))$, And since $y_0 = 1/2$ with $h = 0.1$, we obtain

$y_1 = y_0 + h f(t_0, y(t_0)) \Rightarrow y_1 = y_0 + h(t_0^2 + 1 - 2t_0y_0 + y_0^2) \Rightarrow$

$$= 0.5 + 0.1(0 + 1 - (2 \cdot 0 \cdot 0.5) + 0.5^2) =$$

$$y_1 = 0.625,$$

$y_2 = y_1 + h f(t_1, y(t_1)) \Rightarrow y_2$

$$= 0.625 + 0.1(0.1^2 + 1 - (2 \cdot 0.1 \cdot 0.625) + 0.625^2) = 0.752563;$$

So

$y_{10} = y_9 + h f(t_9, y(t_9)) = 1.767031 + 0.1(0.9^2 + 1 - 2 \cdot 0.9 \cdot 1.767031 + 1.767031^2) =$

$$y_{10} = 1.942205.$$

From the preceding calculation, it is found the approximation solution by used each of $t_0 = 0$, $h = 0.1$ and $y_0 = 1/2$ then by used the Euler's formula $y_1 = y_0 + h\, f(t_0, y(t_0))$, could be found the value of $y_1 = 0.62500$.

For find the value of $y_2$, can be constructed the second step by take a result of the first Euler step and applying the formula of $y_2 = y_1 + h\, f(t_1, y(t_1))$, it is found that $y_2 = 0.75256$.

These computation are continued for evaluating all values of $y(t_i)$ until we found $y_{10}$, finally the formula of $y_{10} = y_9 + h\, f(t_9, y(t_9))$ give the approximation value of $y_{10} = 1.94220$.

With regarded to the calculation above and by observe that the table (A.1) that explained how can find the approximate solution of the initial value problem of the example (A.1) which is computed by (MICROSOFT OFFICE EXCEL) and compared with the actual solution, reminder that in here all these statements are computed when the step size $h = 0.1$, and clearly the approximation solution is near to the close form solution, therefore, we can say that the method that used to construct the implementation by this software is mostly succeeded but not very accurate comparably with exact solution.

Table (A.1): Illustration of Euler's Method and the exact solution

When $h = 0.1$.

| $i$ | $h$ | $t_i$ | Exact solution | Euler's method | $f(ti, y(ti))$ |
|---|---|---|---|---|---|
| 0 | 0.1 | 0 | 0.50000 | 0.50000 | 1.25000 |
| 1 | 0.1 | 0.1 | 0.62632 | 0.62500 | 1.27563 |
| 2 | 0.1 | 0.2 | 0.75556 | 0.75256 | 1.30533 |
| 3 | 0.1 | 0.3 | 0.88824 | 0.88310 | 1.34000 |
| 4 | 0.1 | 0.4 | 1.02500 | 1.01710 | 1.38081 |
| 5 | 0.1 | 0.5 | 1.16667 | 1.15518 | 1.42926 |
| 6 | 0.1 | 0.6 | 1.31429 | 1.29810 | 1.48735 |
| 7 | 0.1 | 0.7 | 1.46923 | 1.44684 | 1.55776 |
| 8 | 0.1 | 0.8 | 1.63333 | 1.60261 | 1.64419 |
| 9 | 0.1 | 0.9 | 1.80909 | 1.76703 | 1.75174 |
| 10 | 0.1 | 1 | 2.00000 | 1.94220 | 1.88775 |

So in the table (A.2) generated the part of error by subtracting the approximation solution from the exact solution, notice that the computation of absolute error at the last step when $h = 0.1$ is $2.00000 - 1.94220 = 0.05780$ and calculated the percentage relative error which is 2.89.

Table (A.2): Illustration the relative and relative percentage of errors When $h = 0.1$.

| Iterations | Absolute error | % Relative Error |
|---|---|---|
| 0 | 0.00000 | 0.00 |
| 1 | 0.00132 | 0.21 |
| 2 | 0.00299 | 0.40 |
| 3 | 0.00514 | 0.58 |
| 4 | 0.00790 | 0.77 |
| 5 | 0.01149 | 0.98 |
| 6 | 0.01618 | 1.23 |
| 7 | 0.02240 | 1.52 |
| 8 | 0.03072 | 1.88 |
| 9 | 0.04206 | 2.32 |
| 10 | 0.05780 | 2.89 |

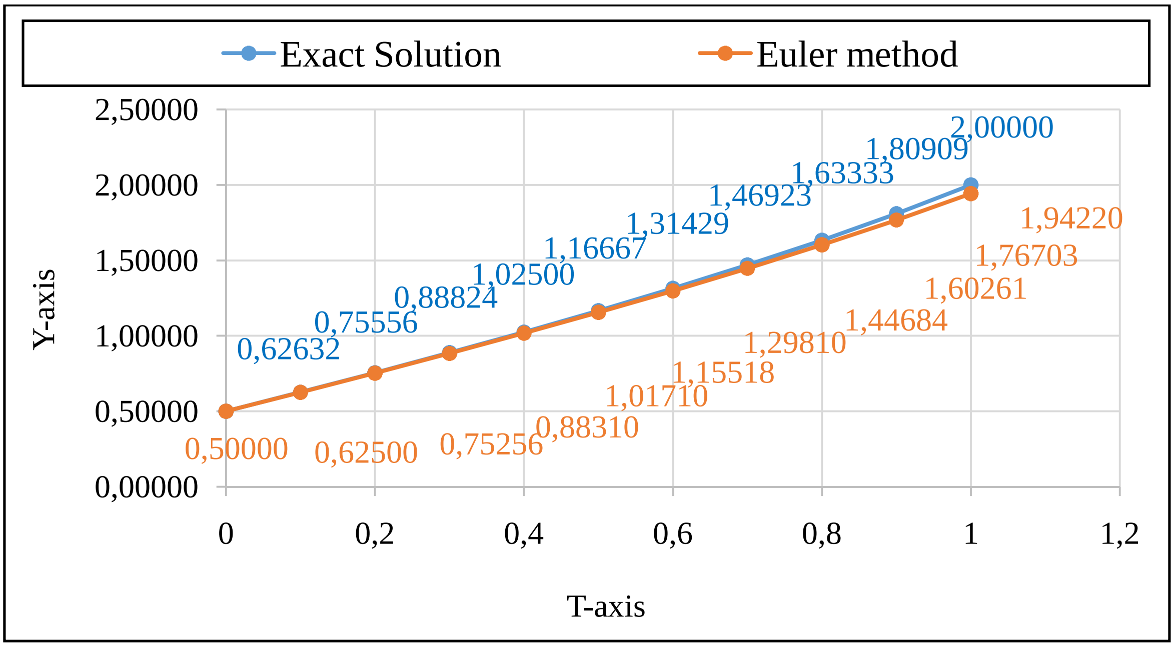


Figure (A.1): Approximate and exact solutions for example (A.1).

### A.1.2 Solution of example (A.1) when ($h = 0.01$)

Continuously in the table (A.3), we found the approximation solution by using the Euler's method when $h = 0.01$, note that all computation by this method has been repeated like the previous illustrations but in this situation we just changed the value of $h$ with respect to the interval $0 \leq t \leq 1$. Accordingly, by addition the value of step size $h = 0.01$ for the lower bound of our interval, it seems each of $t_1 = t_0 + 0.01 = 0 + 0.01 = 0.01$, and so on for finding $t_2, t_3$ until getting the upper bound with regarded the given interval as $t_{100} = t_{99} + 0.01 = 0.99 + 0.0.1 = 1$.

As we have seen that from the preceding implementations of Euler's formula and if we take the smallest step size which is $h = 0.01$ and since $y_0 = 1/2$, then the value of $y_1 = y_0 + h \cdot f(t_i, y(t_i)) = 0.5 + 0.01 \cdot (1.25000) = 0.51250$, and so on for finding each of the $y_5 = 0.56275$, $y_{10} = 0.62617$, and the last one is $y_{100} = 1.99321$.

Also with the calculations of approximation solution, evaluated the error part when $h = 0.01$ by using subtracting the approximate solution from the exact solution as

$$\text{Error part} = |\text{actual solution} - \text{approximate solution}|$$

For instance that, in the last step of table (A.3), the error part is $2.00000 - 1.98646 = 0.00679$.

From the above illustrations observe that the approximation solution is very accuracy because the value of $h$ is small and it is near of the actual solution. And also the error is a small when we have taken the value of $h$ is small. Finally throughout using Euler's method, we understand that although the Euler's method is not more accurate, the results are fortunately good and we can say that it is wonderful because the value of step size $h$ is small. Also the Microsoft office excel is indeed available to compute the solution because it can give us the mostly restriction and accurate results.

To see more information about computation of the Euler's method, visit the tabulation interpretation in the below that considered each of the exact solution with numerical solution and relative error.Note that the tolerance $= 10^{-5}$ and we are taken the some steps of the solution as you can see in the tabulation because all the rows have the same computations.

Table (A.3): Illustration of Euler's Method and the Exact solution and also represented the absolute and relative percentage errors when ? = 0.01.

| $i$ | ? | $t_i$ | Exact solution | Euler's method | $f(ti, y(ti))$ | Absolute Error | % Relative Error |
|---|---|---|---|---|---|---|---|
| 0 | 0.01 | 0 | 0.50000 | 0.50000 | 1.25000 | 0.00000 | 0.00 |
| 1 | 0.01 | 0.01 | 0.51251 | 0.51250 | 1.25251 | 0.00001 | 0.00 |
| 5 | 0.01 | 0.05 | 0.56282 | 0.56275 | 1.26292 | 0.00007 | 0.01 |
| 10 | 0.01 | 0.1 | 0.62632 | 0.62617 | 1.27686 | 0.00014 | 0.02 |
| 15 | 0.01 | 0.15 | 0.69054 | 0.69031 | 1.29194 | 0.00023 | 0.03 |
| 20 | 0.01 | 0.2 | 0.75556 | 0.75523 | 1.30828 | 0.00032 | 0.04 |
| 25 | 0.01 | 0.25 | 0.82143 | 0.82100 | 1.32604 | 0.00043 | 0.05 |
| 30 | 0.01 | 0.3 | 0.88824 | 0.88768 | 1.34537 | 0.00056 | 0.06 |
| 35 | 0.01 | 0.35 | 0.95606 | 0.95536 | 1.36646 | 0.00070 | 0.07 |
| 40 | 0.01 | 0.4 | 1.02500 | 1.02414 | 1.38955 | 0.00086 | 0.08 |
| 45 | 0.01 | 0.45 | 1.09516 | 1.09411 | 1.41488 | 0.00105 | 0.10 |
| 50 | 0.01 | 0.5 | 1.16667 | 1.16540 | 1.44276 | 0.00126 | 0.11 |
| 55 | 0.01 | 0.55 | 1.23966 | 1.23814 | 1.47354 | 0.00151 | 0.12 |
| 60 | 0.01 | 0.6 | 1.31429 | 1.31249 | 1.50764 | 0.00180 | 0.14 |
| 65 | 0.01 | 0.65 | 1.39074 | 1.38861 | 1.54555 | 0.00213 | 0.15 |
| 70 | 0.01 | 0.7 | 1.46923 | 1.46672 | 1.58786 | 0.00251 | 0.17 |
| 75 | 0.01 | 0.75 | 1.55000 | 1.54704 | 1.63527 | 0.00296 | 0.19 |
| 80 | 0.01 | 0.8 | 1.63333 | 1.62984 | 1.68864 | 0.00349 | 0.21 |
| 85 | 0.01 | 0.85 | 1.71957 | 1.71545 | 1.74900 | 0.00412 | 0.24 |
| 90 | 0.01 | 0.9 | 1.80909 | 1.80424 | 1.81764 | 0.00485 | 0.27 |
| 95 | 0.01 | 0.95 | 1.90238 | 1.89665 | 1.89614 | 0.00573 | 0.30 |
| 100 | 0.01 | 1 | 2.00000 | 1.99321 | 1.98646 | 0.00679 | 0.34 |

In the figure (A.2) note that the graph is plotted by Microsoft office excel and putted $h = 0.01$ when constructed the graph as started at $t_0 = 1$ and at the end $t_{10} = 2$ ,also in this graph represented the approximate solution which is showed by the blue color and it is mostly approach to the exact solution which is plotted by the orange color. As illustrated earlier the numerical solution normally computed by Euler's method to determine the approximate solution, in fact that, these approximate and the exact solutions are not very different when we arrive that at $y_{10} = 2.00000$ and $y_{10} = 1.99321$ respectively.

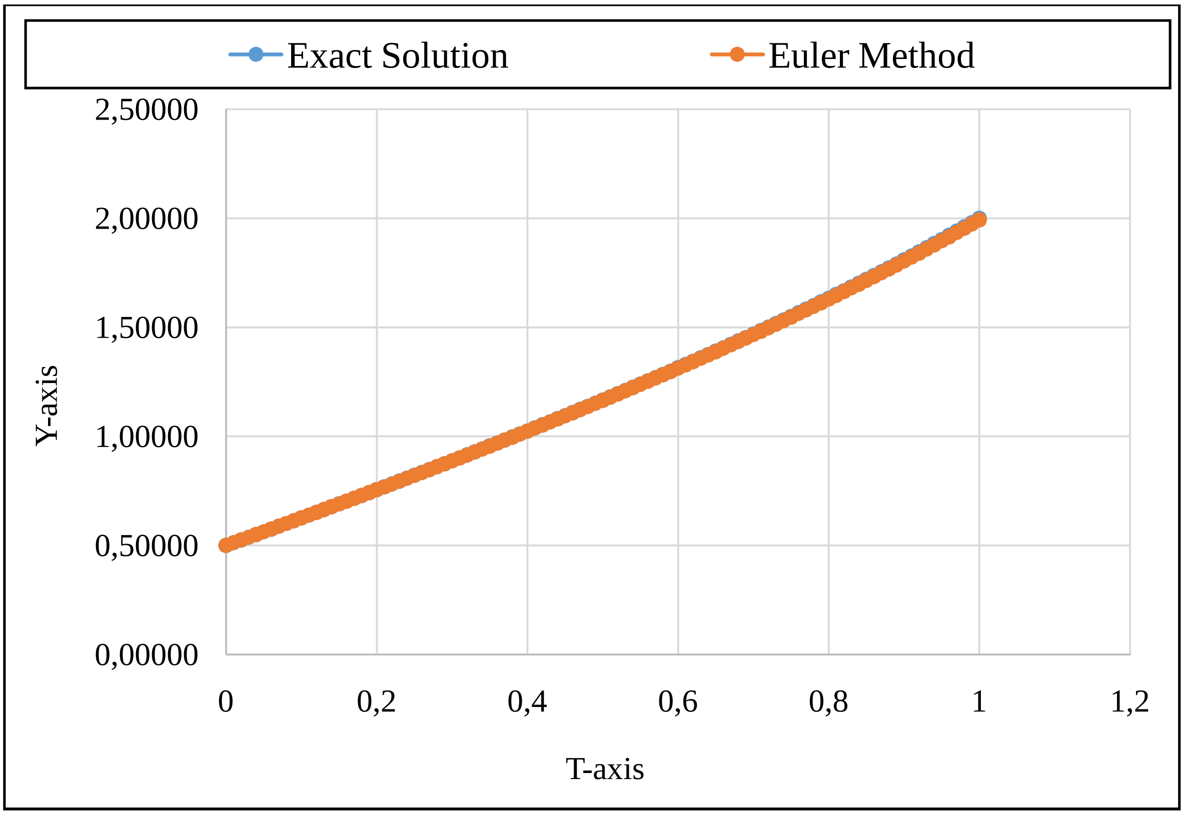


Figure (A.2): Approximate and exact solutions for example (A.1)

When $h = 0.01$.

# Appendix B. Approximation Solution from RDE by Taylor's Method

## B.1 Approximation Solution from RDE by Taylor's Method of Order-Two

**Example (B.1):** Apply Tylor's method of order two to the initial value problem

$$y' = 1 + t - (1 + 2t)y + ty^2, \qquad y(0) = 3, \quad for\ 0 \leq t \leq 1$$

Using $h = 0.1$ and $h = 0.01$ and find the accuracy of the solution between the exact and approximate solutions, when the actual solution is

$$y = 1 + \frac{1}{t + 1 - \frac{1}{2}e^t}$$

### B.1.1 Approximation solution when ($h = 0.1$)

This simple example is selected to explain the Tylor's method so that we can arise to verify the calculation algorithm. The exact solution that obtained in the table (B.1) in order to compare with our numerical approximation result and to indicate that the relativity error throughout the computations.

Since the initial condition is considered, then the first part is known from the initial condition is $y(0) = 3$. Subsequently, we can compute the second term which is the first derivative by substituting $t = 0$, $y = 3$ in the Riccati differential equation of the example (B.1):

$$f(t_0, y(t_0)) = y'(t_0) = 1 + t_0 - (1 + 2t_0)y_0 + t_0 y_0^2$$

$$= 1 + 0 - (1 + 2(0))(3) + (0)(3)^2 = -2;$$

In order to apply a Tylor's method of order two, we must be go to find the second derivative from the first derivative equation $f(t, y(t)) = y' = 1 + t - (1 + 2t)y + ty^2$ with respect to the independent variable $t$, as we obtain in the following steps:

$$y'' = 1 - ((1 + 2t)y' + 2y) + 2tyy' + y^2$$

$$= 1 - ((1 + 2t)(1 + t - (1 + 2t)y + ty^2) + 2y) + 2ty(1 + t - (1 + 2t)y + ty^2) + y^2$$

$$= 1 - (1 - 3t - 2t^2 + y + 4ty + 4t^2y - ty^2 - 2t^2y^2) - 2y + 2ty + 2t^2y - 2ty^2 - 4t^2y^2 + 2t^2y^3 + y^2$$

$$y'' = -3t - 2t^2 - y + 6ty + 6t^2y - 3ty^2 - 6t^2y^2 + 2t^2y^3 + y^2$$

$$f'(t_0, y_0) = y''(t_0)$$

$$= -3t_0 - 2t_0^2 - y_0 + 6t_0y_0 + 6t^2y_0 - 3t_0y_0^2 - 6t^2y_0^2 + 2t^2y_0^3 + y_0^2$$

$$= -3(0) - 2(0)^2 - 3 + 6(0)(3) + 6(0)^2(3) - 3(0)(3)^2 - 6(0)^2(3)^2 + 2(0)^2(3)^3 + (3)^2 = 6;$$

Then, we can compute the first three steps as the following steps;

$$T^{(2)}(t_0, w_0) = f(t_0, w_0) + \frac{h}{2}f'(t_0, w_0) = -2 + \frac{0.1}{2}(6) = -1.7;$$

Since the initial condition given as;

$$w_0 = y(t_0) = y(0) = 3,$$

So $\quad w_1 = w_0 + hT^{(2)}(t_0, w_0) = 3 + 0.1(-1.7) = 2.83;$

$$T^{(2)}(t_1, w_1) = f(t_1, w_1) + \frac{h}{2}f'(t_1, w_1) = -1.49511 + \frac{0.1}{2}(4.29679974)$$

$$= -1.280270013;$$

$$w_2 = w_1 + hT^{(2)}(t_1, w_1) = 2.83 + 0.1(-1.280270013) = 2.701973;$$

$$T^{(2)}(t_2, w_2) = f(t_2, w_2) + \frac{h}{2}f'(t_2, w_2) = -1.122631 + \frac{0.1}{2}(3.255068) = -0.959877$$

$$w_3 = w_2 + hT^{(2)}(t_2, w_2) = 2.701973 + 0.1(-0.959877) = 2.605985;$$

.

.

And the last step is;

$$T^{(2)}(t_9, w_9) = f(t_9, w_9) + \frac{h}{2}f'(t_9, w_9) = 0.528602 + \frac{0.1}{2}(3.156990) = 0.686451$$

$$w_{10} = w_9 + hT^{(2)}(t_9, w_9) = 2.502116 + 0.1(0.686451) = 2.570761.$$

All the calculation of approximation and the actual solutions are shown in the table (B.1), it also again computed by Microsoft office excel. Note that, in this kind of our tabulation obtained the exact solution in the fourth column whose results indicate the comparison with the approximate solutions. The solutions which was tabulated consist of 10 steps as presented in the below. And also a plot of the approximate and the exact solutions are explained in the figure (B.1).

Table (B.1): illustration of Tylor's Method and Exact solution when $h = 0.1$

| $i$ | $h$ | $t_i$ | Exact solution ($y_i$) | Tylor's Method ($w_i$) (order 2) | $f(t_i, y(t_i))$ | $f'(t_i, y(t_i))$ | $T^{2}(t_i, y(t_i))$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 0 | 3.000000 | 3.000000 | -2.000000 | 6.000000 | -1.700000 |
| 1 | 0.1 | 0.1 | 2.826769 | 2.830000 | -1.495110 | 4.296800 | -1.280270 |
| 2 | 0.1 | 0.2 | 2.696933 | 2.701973 | -1.122631 | 3.255068 | -0.959877 |
| 3 | 0.1 | 0.3 | 2.599819 | 2.605985 | -0.832229 | 2.609489 | -0.701754 |
| 4 | 0.1 | 0.4 | 2.528847 | 2.535810 | -0.592325 | 2.223278 | -0.481161 |
| 5 | 0.1 | 0.5 | 2.480080 | 2.487694 | -0.381077 | 2.027384 | -0.279708 |
| 6 | 0.1 | 0.6 | 2.451504 | 2.459723 | -0.181248 | 1.994553 | -0.081521 |
| 7 | 0.1 | 0.7 | 2.442744 | 2.451571 | 0.023370 | 2.131180 | 0.129929 |
| 8 | 0.1 | 0.8 | 2.455118 | 2.464564 | 0.251394 | 2.482645 | 0.375526 |
| 9 | 0.1 | 0.9 | 2.492095 | 2.502116 | 0.528602 | 3.156990 | 0.686451 |
| 10 | 0.1 | 1 | 2.560405 | 2.570761 | 0.896530 | 4.387231 | 1.115892 |

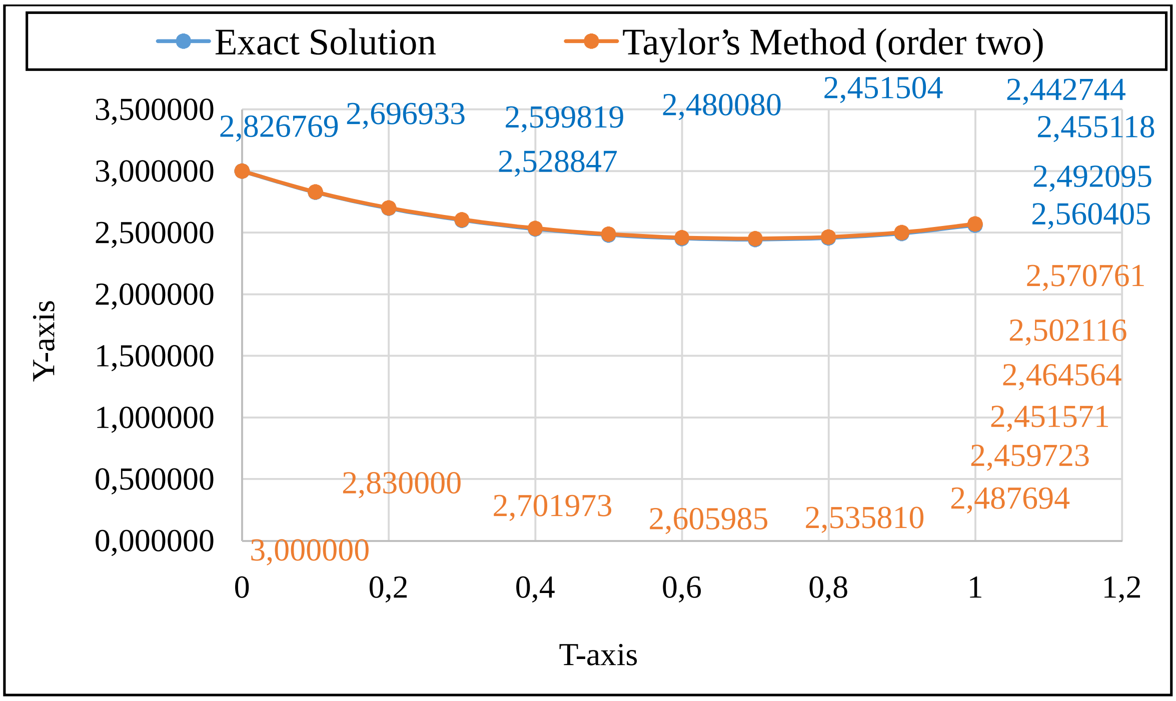


Figure (B.1): Approximate and exact solutions for example (B.1) when $h = 0.1$.

#### B.1.1.1 Calculation of absolute and relative percentage of errors:

The absolute error is computed by this form $|y(t_i) - w_i|$, when $y(t_i)$ denote the exact solution and $w_i$ value denote the approximate solution, as we can see in the formula of absolute error, we can compute it by subtracting the approximation solution from the exact solution.

Normally, if we subtract $w_{10} = 2.570761$ which is the approximate solution from $y_{10} = 2.560405$ which is the exact solution then the absolute error is 0.010356. Subsequently, we hope that any case of finding the solution by using a numerical methods, if the error is small or keep the error small, then meaning that the method that used to find the approximate solution is good and the result is more accuracy. Also the better measurement of finding the accuracy of the solution is percentage relative error which is calculated by this formula

$$\%\ Relative\ Error = \frac{y(t_i) - w_i}{y(t_i)} \ \square\ 100$$

As we showed in the table (B.2), the relative percentage error in the first step is 0.00, in the second step is 0.11, and in the third step is 0.19 and so on until we get the last step which is the 0.40.

Table (B.2): Illustration the relative and relative percentage of errors when □ = 0.1.

| Iterations | Absolute error | % Relative Error |
|---|---|---|
| 0 | 0.000000 | 0.00 |
| 1 | 0.003231 | 0.11 |
| 2 | 0.005040 | 0.19 |
| 3 | 0.006166 | 0.24 |
| 4 | 0.006963 | 0.28 |
| 5 | 0.007614 | 0.31 |
| 6 | 0.008219 | 0.34 |
| 7 | 0.008827 | 0.36 |
| 8 | 0.009446 | 0.38 |
| 9 | 0.010021 | 0.40 |
| 10 | 0.010356 | 0.40 |

### B.1.2 Approximation solution when ($h = 0.01$)

All these computations of this case are generated like the previous case, we just are changed the value of step size $h$ which is equal to 0.01. Successively, it has achieved more accuracy of the solution compare to the first case. Also we choose only some of the steps of the solutions because the table is too long and it is need to devote more places.

Table (B.3): Illustration of Tylor's Method and Exact solution $h = 0.01$.

| i | $t_i$ | Exact solution ($y_i$) | Tylor's method ($w_i$) | $f(t_i, y(t_i))$ | $f'(t_i, y(t_i))$ | $T^{2}(t_i, y(t_i))$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 3.000000 | 3.000000 | -2.000000 | 6.000000 | -1.970000 |
| 1 | 0.01 | 2.980296 | 2.980300 | -1.941084 | 5.785794 | -1.912155 |
| 5 | 0.05 | 2.907071 | 2.907087 | -1.725238 | 5.033199 | -1.700072 |
| 10 | 0.1 | 2.826769 | 2.826797 | -1.493078 | 4.284756 | -1.471655 |
| 15 | 0.15 | 2.757213 | 2.757250 | -1.294061 | 3.699793 | -1.275562 |
| 20 | 0.2 | 2.696933 | 2.696976 | -1.121031 | 3.239815 | -1.104832 |
| 25 | 0.25 | 2.644771 | 2.644821 | -0.968462 | 2.877424 | -0.954075 |
| 30 | 0.3 | 2.599819 | 2.599873 | -0.831995 | 2.592937 | -0.819030 |
| 35 | 0.35 | 2.561363 | 2.561420 | -0.708109 | 2.372183 | -0.696248 |
| 40 | 0.4 | 2.528847 | 2.528908 | -0.593884 | 2.205049 | -0.582859 |
| 45 | 0.45 | 2.501853 | 2.501917 | -0.486827 | 2.084527 | -0.476405 |
| 50 | 0.5 | 2.480080 | 2.480146 | -0.384730 | 2.006106 | -0.374699 |
| 55 | 0.55 | 2.463329 | 2.463398 | -0.285555 | 1.967420 | -0.275717 |
| 60 | 0.6 | 2.451504 | 2.451576 | -0.187333 | 1.968091 | -0.177492 |
| 65 | 0.65 | 2.444607 | 2.444681 | -0.088064 | 2.009777 | -0.078015 |
| 70 | 0.7 | 2.442744 | 2.442820 | 0.014391 | 2.096408 | 0.024873 |
| 75 | 0.75 | 2.446132 | 2.446210 | 0.122432 | 2.234684 | 0.133606 |
| 80 | 0.8 | 2.455118 | 2.455198 | 0.238882 | 2.434912 | 0.251057 |
| 85 | 0.85 | 2.470206 | 2.470287 | 0.367196 | 2.712351 | 0.380758 |
| 90 | 0.9 | 2.492095 | 2.492177 | 0.511755 | 3.089368 | 0.527202 |
| 95 | 0.95 | 2.521734 | 2.521814 | 0.678308 | 3.598900 | 0.696302 |
| 100 | 1 | 2.560405 | 2.560482 | 0.874622 | 4.290148 | 0.896073 |

In the other hand, in this situation it is need to take the smallest step size ▯because the Taylor’s method is not very stronger method to accomplish the accuracy of the solution and because its formula contained the nth order derivatives of the function probably it cannot possible to use this method  as generally. See the table and observe that recorded all the information about everything that founded by the Taylor’s method for order two which was desired in the preceding of this case.

Table (B.4): Illustration the relative and relative percentage of errors

When ▯ = 0.01.

| Iterations | Absolute error | % Relative Error |
|---|---|---|
| 0 | 0.000000 | 0.00 |
| 1 | 0.000004 | 0.00 |
| 5 | 0.000016 | 0.00 |
| 10 | 0.000028 | 0.00 |
| 15 | 0.000037 | 0.00 |
| 20 | 0.000044 | 0.00 |
| 25 | 0.000049 | 0.00 |
| 30 | 0.000054 | 0.00 |
| 35 | 0.000058 | 0.00 |
| 40 | 0.000061 | 0.00 |
| 45 | 0.000064 | 0.00 |
| 50 | 0.000067 | 0.00 |
| 55 | 0.000069 | 0.00 |
| 60 | 0.000072 | 0.00 |
| 65 | 0.000074 | 0.00 |
| 70 | 0.000076 | 0.00 |
| 75 | 0.000078 | 0.00 |
| 80 | 0.000080 | 0.00 |
| 85 | 0.000081 | 0.00 |
| 90 | 0.000081 | 0.00 |
| 95 | 0.000080 | 0.00 |
| 100 | 0.000077 | 0.00 |

A plot of the exact and numerical approximate solutions are presented in the figure (B.2) , notice that we cannot think that by the difference or distinct between the approximate and the actual solutions because the founded results are mostly accurate solutions and fortunately the numerical approximation result is too near of the exact solution. Occasionally, if we return to the first case then you think that this distinction of the results refer to the using the small step size $h$ and sometimes to the usage of methods.

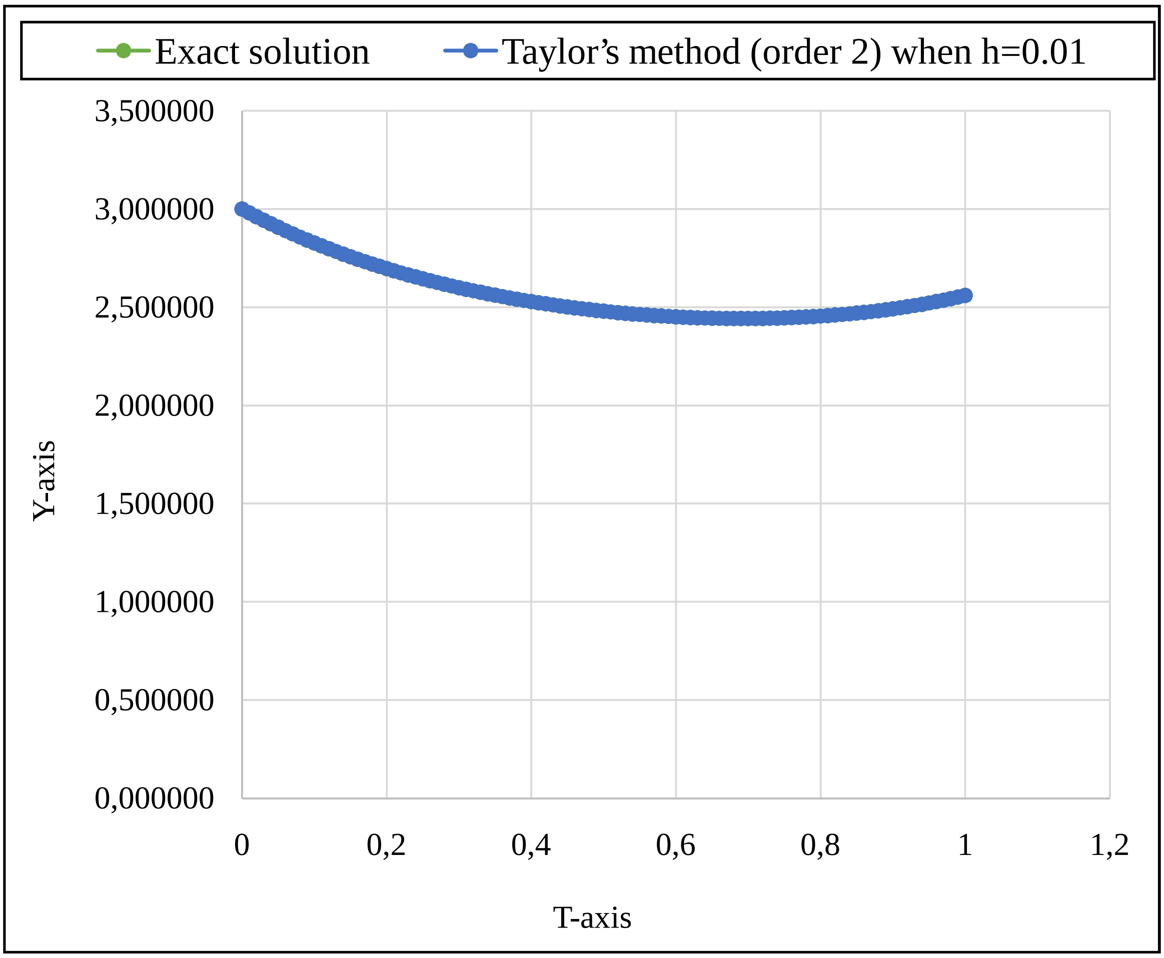


Figure (B.2): Approximate and exact solutions for example (B.1) when $h = 0.01$.

## B.2 Approximation Solution from RDE by Taylor's Method Of Order-Four

### Example (B.2):

Apply Taylor's method of order four to approximate the solution to the following initial-value problem

$$f(t, y(t)) = y' = y^2 - \frac{y}{t} - \frac{1}{t^2} \quad , \qquad for \quad 0.5 \le t \le 1.5$$

the initial condition at $t = 0.5$ is $y = 2.363636$ . Recall that the actual solution is given by

$$y(t) = \frac{2t}{3 - t^2} + \frac{1}{t}$$

### Solution:

In fact that, this normally an interest example is taken to explain the method and we recognized the example that the Riccati differential equation which is one of the important and wonderful type of the ordinary differential equations. Since our initial condition which is the first term with respect to the desired RDE is known, and it is given in the example $y(0.5) = 2.3636362.$

To compute the fourth order of Taylor's method and because we have the ordinary differential equation that it is specified in the example which is $f(t_i, y(t_i))$ then we are working to calculate the required first three derivatives by differentiating the given equation $f(t, y(t)) = y' = y^2 - \frac{y}{t} - \frac{1}{t^2}$ with respect to the independent variable $t$

Since

$$f(t, y(t)) = y' = y^2 - \frac{y}{t} - \frac{1}{t^2}$$

Take the first derivative to the above equation to obtain that;

$$f'(t, y(t)) = 2yy' - \left(\frac{ty' - y}{t^2}\right) + \frac{2}{t^3}$$

$$f'(t, y(t)) = 2y\left(y^2 - \frac{y}{t} - \frac{1}{t^2}\right) - \left(\frac{t\left(y^2 - \frac{y}{t} - \frac{1}{t^2}\right) - y}{t^2}\right) + \frac{2}{t^3}$$

$$= 2y^3 - \frac{2y^2}{t} - \frac{2y}{t^2} - \frac{ty^2 - y - \frac{1}{t} - y}{t^2} + \frac{2}{t^3}$$

$$= 2y^3 - \frac{2y^2}{t} - \frac{2y}{t^2} - \frac{y^2}{t} + \frac{y}{t^2} + \frac{1}{t^3} + \frac{y}{t^2} + \frac{2}{t^3}$$

$$f'\ t,y\ t\ \ = y'' = 2y^3 - \frac{3y^2}{t} + \frac{3}{t^3}\ ;$$

And take the second derivative to the previous equation to achieve that;

$$f''\ t,y\ t\ \ = y\ ^3\ = 6y^2y' - \frac{3\ 2tyy' - y^2}{t^2} - \frac{9}{t^4}$$

$$= 6y^2\ \ y^2 - \frac{y}{t} - \frac{1}{t^2}\ \ - \frac{3\ \ 2ty\ \ y^2 - \frac{y}{t} - \frac{1}{t^2}\ \ - y^2}{t^2} - \frac{9}{t^4}$$

$$= 6y^4 - \frac{6y^3}{t} - \frac{6y^2}{t^2} - \frac{3\ \ 2ty^3 - 2y^2 - \frac{2y}{t} - y^2}{t^2} - \frac{9}{t^4}$$

$$= 6y^4 - \frac{6y^3}{t} - \frac{6y^2}{t^2} - \frac{3\ \ 2ty^3 - 2y^2 - \frac{2y}{t} - y^2}{t^2} - \frac{9}{t^4}$$

$$= 6y^4 - \frac{6y^3}{t} - \frac{6y^2}{t^2} - \frac{6y^3}{t} + \frac{9y^2}{t^2} + \frac{6y}{t^3} - \frac{9}{t^4}$$

$$f''\ t,y\ t\ \ = y\ ^3\ = 6y^4 - \frac{12y^3}{t} + \frac{3y^2}{t^2} + \frac{6y}{t^3} - \frac{9}{t^4}\ ;$$

And also take the third derivative to the previous equation to achieve that;

$$f^{(3)}\ t,y\ t\ \ = y\ ^4$$

$$= 24y^3y' - \frac{12\ 3ty^2y' - y^3}{t^2} + \frac{3\ 2t^2yy' - 2y^2t}{t^4} + \frac{6\ t^3y' - 3yt^2}{t^6} + \frac{36}{t^5}$$

$$= 24y^3(y^2 - \frac{y}{t} - \frac{1}{t^2}) - \frac{12\ \ 3ty^2(y^2 - \frac{y}{t} - \frac{1}{t^2}) - y^3}{t^2} + \frac{3\ \ 2t^2y(y^2 - \frac{y}{t} - \frac{1}{t^2}) - 2y^2t}{t^4} + \frac{6\ \ t^3(y^2 - \frac{y}{t} - \frac{1}{t^2}) - 3yt^2}{t^6} + \frac{36}{t^5}$$

$$= 24y^5 - \frac{24y^4}{t} - \frac{24y^3}{t^2} - \frac{12\ \ 3ty^4 - 4y^3 - \frac{3y^2}{t}}{t^2} + \frac{3\ 2t^2y^3 - 4y^2t - 2y}{t^4} + \frac{6\ t^3y^2 - 4t^2y - t}{t^6} + \frac{36}{t^5}$$

$$= 24y^5 - \frac{24y^4}{t} - \frac{24y^3}{t^2} - \frac{36y^4}{t} + \frac{48y^3}{t^2} + \frac{36y^2}{t^3} + \frac{6y^3}{t^2} - \frac{12y^2}{t^3} - \frac{6y}{t^4} + \frac{6y^2}{t^3} - \frac{24y}{t^4} - \frac{6}{t^5} + \frac{36}{t^5}$$

$$f^{(3)}(t,y(t)) = y^{(4)} = 24y^5 - \frac{60y^4}{t} + \frac{30y^3}{t^2} + \frac{30y^2}{t^3} - \frac{30y}{t^4} + \frac{30}{t^5}\ .$$

Now, we substitute each of $y'$, $y''$, $y^{(3)}$, $y^{(4)}$ in the equation (3.2.1), as follows;

$$T^{(4)}(t_i,w_i) = f(t_i,w_i) + \frac{h}{2}f'(t_i,w_i) + \frac{h^2}{3!}f''(t_i,w_i) + \frac{h^3}{4!}f'''(t_i,w_i)$$

$$= \left(y^2 - \frac{y}{t} - \frac{1}{t^2}\right) + \frac{1}{2}h\left(2y^3 - \frac{3y^2}{t} + \frac{3}{t^3}\right) + \frac{1}{6}h\left(6y^4 - \frac{12y^3}{t} + \frac{3y^2}{t^2} + \frac{6y}{t^3} - \frac{9}{t^4}\right)$$
$$+ \frac{1}{24}h\left(24y^5 - \frac{60y^4}{t} + \frac{30y^3}{t^2} + \frac{30y^2}{t^3} - \frac{30y}{t^4} + \frac{30}{t^5}\right)$$

Since the initial condition $y(t_0) = y(0.5) = w_0 = 2.363636$ then again by equation (2.2.2.1) we obtain;

$$w_1 = w_0 + h[T^{(4)}(t_0,w_0)]$$

$$w_1 = w_0 + h\left[\left(y_0^2 - \frac{y_0}{t_0} - \frac{1}{t_0^2}\right) + \frac{1}{2}h\left(2y_0^3 - \frac{3y_0^2}{t_0} + \frac{3}{t_0^3}\right)\right.$$
$$+ \frac{1}{6}h\left(6y_0^4 - \frac{12y_0^3}{t_0} + \frac{3y_0^2}{t_0^2} + \frac{6y_0}{t_0^3} - \frac{9}{t_0^4}\right)$$
$$\left.+ \frac{1}{24}h\left(24y_0^5 - \frac{60y_0^4}{t_0} + \frac{30y_0^3}{t_0^2} + \frac{30y_0^2}{t_0^3} - \frac{30y_0}{t_0^4} + \frac{30}{t_0^5}\right)\right]$$

$$w_1 = 2.363636$$
$$+ (0.1)\left[(2.363636)^2 - \frac{2.363636}{0.5} - \frac{1}{(0.5)^2}\right.$$
$$+ \frac{1}{2}(0.1)\left(2(2.363636)^3 - \frac{3(2.363636)^2}{0.5} + \frac{3}{(0.5)^3}\right)$$
$$+ \frac{1}{6}(0.1)\left(6(2.363636)^4 - \frac{12(2.363636)^3}{0.5} + \frac{3(2.363636)^2}{(0.5)^2}\right.$$
$$\left.+ \frac{6(2.363636)}{(0.5)^3} - \frac{9}{(0.5)^4}\right)$$
$$+ \frac{1}{24}(0.1)\left(24(2.363636)^5 - \frac{60(2.363636)^4}{0.5} + \frac{30(2.363636)^3}{(0.5)^2}\right.$$
$$\left.\left.+ \frac{30(2.363636)^2}{(0.5)^3} - \frac{30(2.363636)}{(0.5)^4} + \frac{30}{(0.5)^5}\right)\right] = 2.121742;$$

Or we can use another technique to calculate $w_i$ as the following steps;

$$f(t_0,y_0) = y_0^2 - \frac{y_0}{t_0} - \frac{1}{t_0^2} = (2.363636)^2 - \frac{2.363636}{0.5} - \frac{1}{(0.5)^2} = -3.140496;$$

$$f'(t_0, y_0) = 2y_0^3 - \frac{3y_0^2}{t_0} + \frac{3}{t_0^3} = 2(2.363636)^3 - \frac{3(2.363636)^2}{0.5} + \frac{3}{(0.5)^3} = 16.889557;$$

$$f''(t_0, y_0) = 6y_0^4 - \frac{12y_0^3}{t_0} + \frac{3y_0^2}{t_0^2} + \frac{6y_0}{t_0^3} - \frac{9}{t_0^4}$$

$$= 6(2.363636)^4 - \frac{12(2.363636)^3}{0.5} + \frac{3(2.363636)^2}{(0.5)^2} + \frac{6(2.363636)}{(0.5)^3}$$

$$- \frac{9}{(0.5)^4} = -93.154293;$$

$$f^{(3)}(t_0, y_0) = 24y_0^5 - \frac{60y_0^4}{t_0} + \frac{30y_0^3}{t_0^2} + \frac{30y_0^2}{t_0^3} - \frac{30y_0}{t_0^4} + \frac{30}{t_0^5}$$

$$= 24(2.363636)^5 - \frac{60(2.363636)^4}{0.5} + \frac{30(2.363636)^3}{(0.5)^2}$$

$$+ \frac{30(2.363636)^2}{(0.5)^3} - \frac{30(2.363636)}{(0.5)^4} + \frac{30}{(0.5)^5} = 776.020913$$

So

$$w_1 = w_0 + hT^{(4)}(t_0, w_0)$$

$$= w_0 + h\left[f(t_0, w_0) + \frac{h}{2}f'(t_0, w_0) + \frac{h^2}{3!}f''(t_0, w_0) + \frac{h^3}{4!}f'''(t_0, w_0)\right]$$

$$= (2.363636) + (0.1)\left[(-3.140496) + \frac{0.1}{2}(16.889557) + \frac{(0.1)^2}{6}(-93.154293)\right.$$

$$\left. + \frac{(0.1)^3}{24}(776.020913)\right] = 2.121742;$$

$$w_2 = w_1 + hT^{(4)}(t_1, w_1)$$

$$= w_1 + h\left[f(t_1, w_1) + \frac{h}{2}f'(t_1, w_1) + \frac{h^2}{3!}f''(t_1, w_1) + \frac{h^3}{4!}f'''(t_1, w_1)\right]$$

$$= 2.121742 + (0.1)\left[(-1.812225) + \frac{0.1}{2}(10.483215) + \frac{(0.1)^2}{6}(-42.428321)\right.$$

$$\left. + \frac{(0.1)^3}{24}(321.252784)\right] = 1.987203;$$

$$w_3 = w_2 + hT^{(4)}(t_2, w_2)$$

$$= w_2 + h\left[f(t_2, w_2) + \frac{h}{2}f'(t_2, w_2) + \frac{h^2}{3!}f''(t_2, w_2) + \frac{h^3}{4!}f'''(t_2, w_2)\right]$$

$= 1.987203 + 0.1 \quad -0.930702 + \frac{h}{2} \; 7.517007 + \frac{h^2}{3!} \; -19.506091$

$+ \frac{h^3}{4!} \; 163.124181 \quad = 1.929147;$

And so on, until to get the last step which is $w_{10}$ with respect to the required interval.

$w_{10} = w_9 + h \, T^{4} \;\; t_9, w_9$

$= w_9 + h \;\; f \;\; t_9, w_9 \;\; + \frac{h}{2} f' \;\; t_9, w_9 \;\; + \frac{h^2}{3!} f'' \;\; t_9, w_9 \;\; + \frac{h^3}{4!} f''' \;\; t_9, w_9$

$= \; 3.413579 + 0.1 \quad 8.704045 + \frac{h}{2} \; 55.677199 + \frac{h^2}{3!} \; 496.700200$

$+ \frac{h^3}{4!} \; 6019.947886 \quad = 4.670236.$

Table (B.5) contains the numerical approximation of Taylor's method of the forth order and it is clear that it is superior. In fact that, it is interested that the method will give the successive approximation and more accurate result sufficiently when we are using the step size $h = 0.1$, also it can be observed that the errors decrease when we are using the smallest step size $h$ otherwise the errors will be increase when we will include the greatest value of $h$.

Furthermore in the table (B.5) calculated the exact solution as it is shown in second column, and computed each of the first, second and third derivatives with respect to the independent variable $t$ or in the other words, we must take the derivatives by repeated the differentiation of this function with respect to the desired derivatives as it is obtained in the algorithm. This work of differentiations are very messy and we should be able to create this work out for each equation and this approximation not often applied in practice, however we must be working strictly with this differentiation because it sometimes need to take more extra time to complete the steps of our working. In order to attain the precisely approximation result, we used the step size $h = 0.1$ which is a small positive increment in $t$ ,as indicated in the exposition of the tabulation, computed that, $t_1 = t_0 + h = 0.5 + 0.1 = 0.6$, $t_2 = t_1 + h = 0.6 + 0.1 = 0.7$ , $t_3 = t_2 + h = 0.7 + 0.1 = 0.8$, until we support the last specified node point with respect to the interval which is $t_{10} = t_1 + h = 1.4 + 0.1 = 1.5$.

All the illustrations above about the technique of this method with comparing with the exact solution interpreted in the table (B.5). Also in the figure (B.3), plotted the Taylor's method of order four with the exact solution and because the step size ⍰ is small then, the distance between the exact and the approximate solutions are not appear and also meaning that the numerical method is very superior.

Table (B.5): illustration of Tylor's Method of order four and Exact solution when ⍰ = 0.1

| $t_i$ | Exact $y_i$ | Taylor $w_i$ | $f(t_i, w_i)$ | $f'(t_i, w_i)$ | $f''(t_i, w_i)$ | $f'''(t_i, w_i)$ | $T^{(4)}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 2.36364 | 2.36364 | -3.14050 | 16.88956 | -93.15429 | 776.02091 | -2.41894 |
| 0.6 | 2.12121 | 2.12174 | -1.81222 | 10.48321 | -42.42832 | 321.25278 | -1.34539 |
| 0.7 | 1.98634 | 1.98720 | -0.93070 | 7.51701 | -19.50609 | 163.12418 | -0.58056 |
| 0.8 | 1.92797 | 1.92915 | -0.25233 | 6.26240 | -6.51118 | 107.34904 | 0.05441 |
| 0.9 | 1.93303 | 1.93459 | 0.35852 | 6.12069 | 3.57110 | 101.07003 | 0.67472 |
| 1 | 2.00000 | 2.00206 | 1.00618 | 7.02476 | 15.13626 | 138.92983 | 1.38844 |
| 1.1 | 2.13814 | 2.14090 | 1.81075 | 9.37910 | 33.86855 | 254.89504 | 2.34677 |
| 1.2 | 2.37179 | 2.37558 | 2.96929 | 14.44027 | 72.68880 | 578.33880 | 3.83655 |
| 1.3 | 2.75396 | 2.75924 | 4.89917 | 25.81036 | 171.76864 | 1619.16498 | 6.54344 |
| 1.4 | 3.40659 | 3.41358 | 8.70405 | 55.67720 | 496.70020 | 6019.94789 | 12.5665 |
| 1.5 | 4.66667 | 4.67024 | 18.25317 | 160.9926 | 2075.0475 | 35821.2171 | 31.2537 |

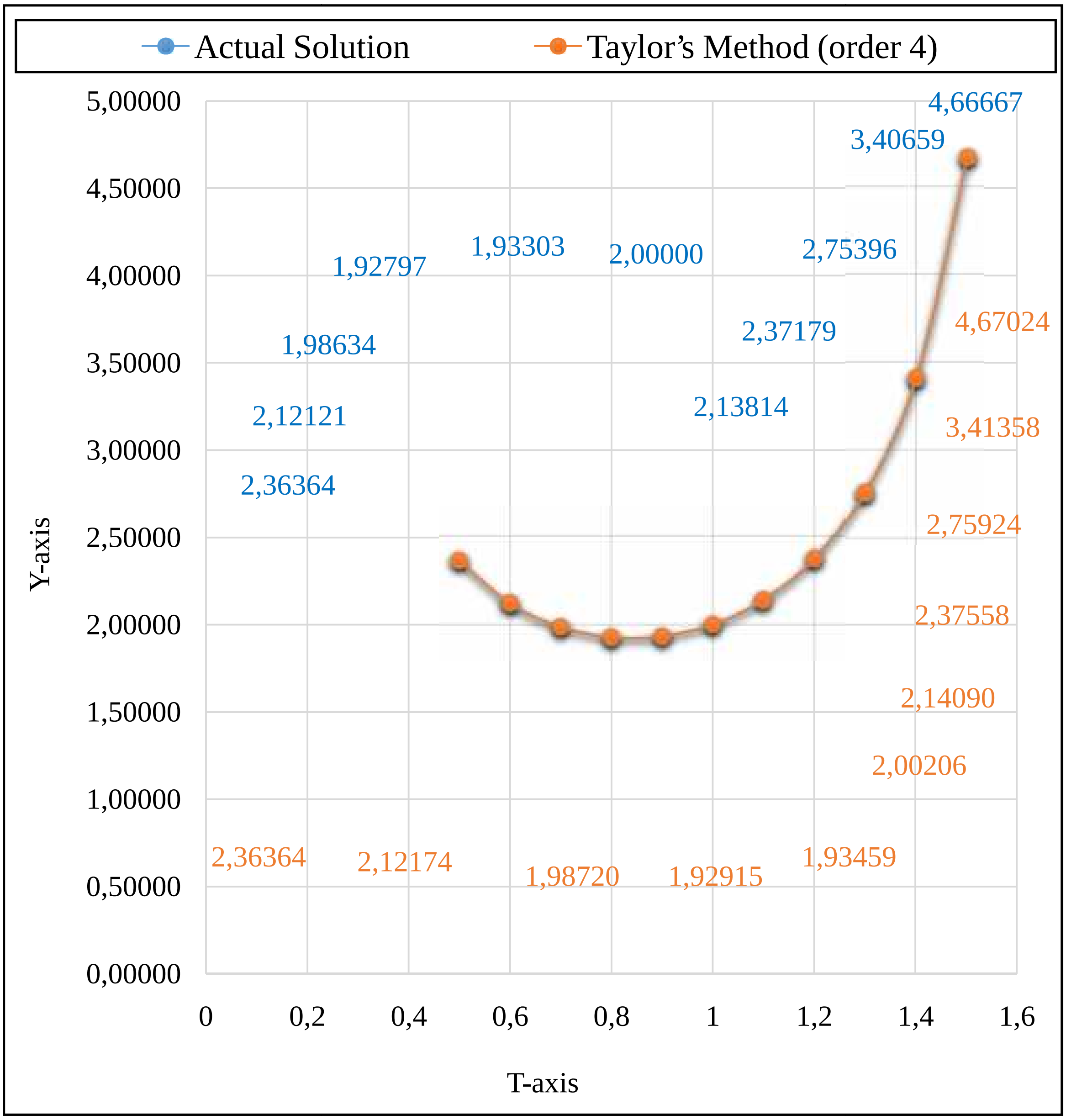


Figure (B.3): Approximate and exact solutions for example (B.2) when $h = 0.1$.

### B.2.1.1 Calculation of absolute and relative percentage of errors:

As mentioned earlier, the formula to find the absolute error is $|y(t_i) - w_i|$, since the close form solution denoted by $y(t_i)$ and the approximate solution is represented by $w_i$. Normally, we thought that any step of the approximation solution contains its error, however, with respect to the table (B.6), first step is computed by subtracting the approximate value which is $w_0 = 2.36364$ from the exact solution which is $y(t_i) = 2.36364$ then by used the above formula as $|2.36364 - 2.36364| = 0.000000$ , the absolute error is 0.00, second step calculated as $|2.12121 - 2.12174| = 0.000530$, third step computed as $|1.98634 - 1.98720| = 0.000863$, and so on, until we get the last step which is computed as $|4.66667 - 4.67024| = 0.003569$.

As presented the formula of percentage relative of error in the preceding sections, therefore it is not necessary to repeat this formula, we just indicate the results errors which founded by it. The percentagerelative error is 0.00 in the first step, it is 0.02 in the second and in the end step it is 0.08. Since it is perfect that, if the numericalresult is approach to the actual solution then the value ofpercentage Relative and absolute errors are decreased, recall that if the exact solution equal to zero then the %Relative error is undefined.

Table (B.6):Illustration the relative and relative percentage of errors.

| $i$ | Absolute Error | % Relative Error |
|---|---|---|
| 0 | 0.000000 | 0.00 |
| 1 | 0.000530 | 0.02 |
| 2 | 0.000863 | 0.04 |
| 3 | 0.001180 | 0.06 |
| 4 | 0.001559 | 0.08 |
| 5 | 0.002060 | 0.10 |
| 6 | 0.002762 | 0.13 |
| 7 | 0.003785 | 0.16 |
| 8 | 0.005271 | 0.19 |
| 9 | 0.006985 | 0.21 |
| 10 | 0.003569 | 0.08 |

# Appendix C. Approximation Solution from RDE by Runge - Kutta Method

## C.1 Approximation Solution from RDE by Runge - Kutta Method of Order-Two (Improved Euler Method)

**Example (c.1):** Find the approximate solution for the following initial value problem

$$y' = \frac{t}{1+t^2} + \frac{y}{t} + \frac{y^2}{t(1+t^2)} \quad , \quad y(1) = 0 \quad , \quad 1 \le t \le 2$$

, by using Runge-Kutte method of order two, after that compare with actual solution which is given as

$y = (t^2 - t)/(t+1)$ , and also determine the error with explain its solutions by graphical representation.

**Solution:**

We accepted this form of the ordinary differential equation is a Riccati differential equitation which is nonlinear and first order differential equation, so we apply the Runge-Kutte method for order two for computing the approximate solutions.

Since $t_0 = 1$ and $y_0 = 0$ then

$t_1 = t_0 + h = 1 + 0.1 = 1.1$,

$t_2 = t_1 + h = 1.1 + 0.1 = 1.2$,

$t_3 = t_3 + h = 1.2 + 0.1 = 1.3$,

.

.

$t_{10} = t_9 + h = 1.9 + 0.1 = 2$.

Consequently, we can continue to calculate $k_1, k_2$ and because the initial condition is known then, fortunately we can compute the formula of the Runge-Kutte method for order two as follow;

$$k_1 = hf\left(t_0, y_0\right) = h \times \left(\frac{t_0}{1+t_0^2} + \frac{y_0}{t_0} + \frac{y_0^2}{t_0\left(1+t_0^2\right)}\right),$$

$$= 0.1 \times \left(\frac{1}{1+(1)^2} + \frac{0}{1} + \frac{(0)^2}{1 \times \left(1+(1)^2\right)}\right),$$

$$= 0.05000,$$

$$k_2 = hf\left(t_0 + h, y_0 + hk_1\right) = h \times \left\{\frac{t_0 + h}{1+(t_0+h)^2} + \frac{y_0 + k_1}{(t_0 + h)} + \frac{(y_0 + k_1)^2}{(t_0 + h) \times \left(1 + (t_0 + h)^2\right)}\right\}$$

$$= 0.1 \times \left(\frac{1+0.1}{1+(1+0.1)^2} + \frac{0+0.05000}{1+0.1} + \frac{\left(0+0.05000\right)^2}{\left(1+0.1\right) \times \left(1+\left(1+0.1\right)^2\right)}\right),$$

$$= 0.05442,$$

$$y_{i+1} = y_i + \frac{1}{2}\left(k_1 + k_2\right) \Rightarrow y_1 = y_0 + \frac{1}{2} \times \left(k_1 + k_2\right),$$

$$y_1 = 0 + 05 \times \{0.05000 + 0.05442\} = 0.05221$$

And to find $y_2$ then we obtain again the above formula as;

$$k_1 = hf\left(t_1, y_1\right) = h \times \left(\frac{t_1}{1+t_1^2} + \frac{y_1}{t_1} + \frac{y_1^2}{t_1\left(1+t_1^2\right)}\right),$$

$$= 0.1 \times \left(\frac{1.1}{1+(1.1)^2} + \frac{0}{1.1} + \frac{(0)^2}{1.1 \times \left(1+(1.1)^2\right)}\right),$$

$$= 0.05463,$$

$$k_2 = hf\left(t_1 + h, y_1 + hk_1\right) = h \times \left\{\frac{t_1 + h}{1+(t_1+h)^2} + \frac{y_1 + k_1}{(t_1 + h)} + \frac{(y_1 + k_1)^2}{(t_1 + h) \times \left(1 + (t_1 + h)^2\right)}\right\}$$

$$= 0.1 \times \left(\frac{1.1+0.1}{1+(1.1+0.1)^2} + \frac{0.05221+0.05463}{1.1+0.1} + \frac{\left(0.05221+0.05463\right)^2}{\left(1.1+0.1\right) \times \left(1+\left(1.1+0.1\right)^2\right)}\right),$$

$$= 0.05847,$$

$$y_2 = y_1 + \frac{1}{2} \times \left(k_1 + k_2\right) \Rightarrow y_2 = 0.05221 + \frac{1}{2} \times \left(0.05463 + 0.05847\right)$$

$$y_2 = 0.10876,$$

And also each of the results of the $y_3, y_4, \ldots \ldots, y_9$ specified as;

$$y_3 = 0.16909,$$
$$y_4 = 0.23272,$$
$$y_5 = 0.29925,$$
$$y_6 = 0.36835,$$
$$y_7 = 0.43974,$$
$$y_8 = 0.51316,$$
$$y_9 = 0.58840,$$

So, the calculation to last step which is $y_{10}$ interpreted in the following statements as;

$$k_1 = h f(t_9, y_9) = h \times \left( \frac{t_9}{1 + t_9^2} + \frac{y_9}{t_9} + \frac{y_9^2}{t_9(1 + t_9^2)} \right),$$

$$= 0.1 \times \left( \frac{1.9}{1 + (1.9)^2} + \frac{0.58840}{1.9} + \frac{(0.58840)^2}{1.9 \times (1 + (1.9)^2)} \right),$$

$$= 0.07614,$$

$$k_2 = h f(t_9 + h, y_9 + k_1)$$

$$= h \times \left( \frac{t_9 + h}{1 + (t_9 + h)^2} + \frac{y_9 + k_1}{t_9 + h} + \frac{(y_9 + k_1)^2}{(t_9 + h) \times (1 + (t_9 + h)^2)} \right),$$

$$= 0.1 \times \left( \frac{1.9 + 0.1}{1 + (1.9 + 0.1)^2} + \frac{0.58840 + 0.07614}{1.9 + 0.1} + \frac{(0.58840 + 0.07614)^2}{(1.9 + 0.1) \times (1 + (1.9 + 0.1)^2)} \right),$$

$$= 0.07764,$$

$$y_{10} = y_9 + \frac{1}{2} \times (k_1 + k_2) \Rightarrow y_{10} = 0.58840 + \frac{1}{2} \times (0.07614 + 0.07764),$$

$$y_{10} = 0.66529.$$

From the preceding computations and with observe or look that on the table (C.1), we can understand that Runge-Kutte method is one of the essential method to achieve the best accurate result although the step size is not a smallest value. Notice that, it is not necessary to put the small step size  because this method itself can give the strictly and accurate solution through the implementation.

Fortunately, if we compare the approximate and the exact solutions then we are agreed that there is no more the difference between both of the results, also meaning that if we cannot solve the problem analytically then we can use this type of numerical method immediately to realize the solution.

Also, in the table (C.1) computed the errors between the actual and the approximate solutions, the last two column of this table, obtained the absolute and percentage relative errors respectively, for instance that, the absolute error at the end row of the solutions between both the actual and approximate solutions consist of 0.00137 which is calculated by subtracting the approximate solution from the exact solution.

Finally, at the end column, computed the percentage relative errors as we have seen in the previous sections, the amount of the percentage relative error in the last step consist of 0.21.

See all the computations through the table (C.1) in the next page with graphical representation.

Table (C.1): Illustration of Runge-Kutte method for order two and Exact

| $i$ | □ | $t_i$ | Exact solution | Runge-Kutte method (order 2) | $k_1$ | $k_2$ | Absolute Error | % Relative Error |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 1 | 0.00000 | 0.00000 | 0.05000 | 0.05442 | 0.00000 | Undefined |
| 1 | 0.1 | 1.1 | 0.05238 | 0.05221 | 0.05463 | 0.05847 | 0.00017 | 0.32 |
| 2 | 0.1 | 1.2 | 0.10909 | 0.10876 | 0.05865 | 0.06201 | 0.00033 | 0.30 |
| 3 | 0.1 | 1.3 | 0.16957 | 0.16909 | 0.06215 | 0.06511 | 0.00047 | 0.28 |
| 4 | 0.1 | 1.4 | 0.23333 | 0.23272 | 0.06523 | 0.06784 | 0.00061 | 0.26 |
| 5 | 0.1 | 1.5 | 0.30000 | 0.29925 | 0.06794 | 0.07026 | 0.00075 | 0.25 |
| 6 | 0.1 | 1.6 | 0.36923 | 0.36835 | 0.07035 | 0.07242 | 0.00088 | 0.24 |
| 7 | 0.1 | 1.7 | 0.44074 | 0.43974 | 0.07249 | 0.07435 | 0.00100 | 0.23 |
| 8 | 0.1 | 1.8 | 0.51429 | 0.51316 | 0.07441 | 0.07608 | 0.00113 | 0.22 |
| 9 | 0.1 | 1.9 | 0.58966 | 0.58840 | 0.07614 | 0.07764 | 0.00125 | 0.21 |
| 10 | 0.1 | 2 | 0.66667 | 0.66529 | 0.07769 | 0.07906 | 0.00137 | 0.21 |

solution when □ = 0.1.

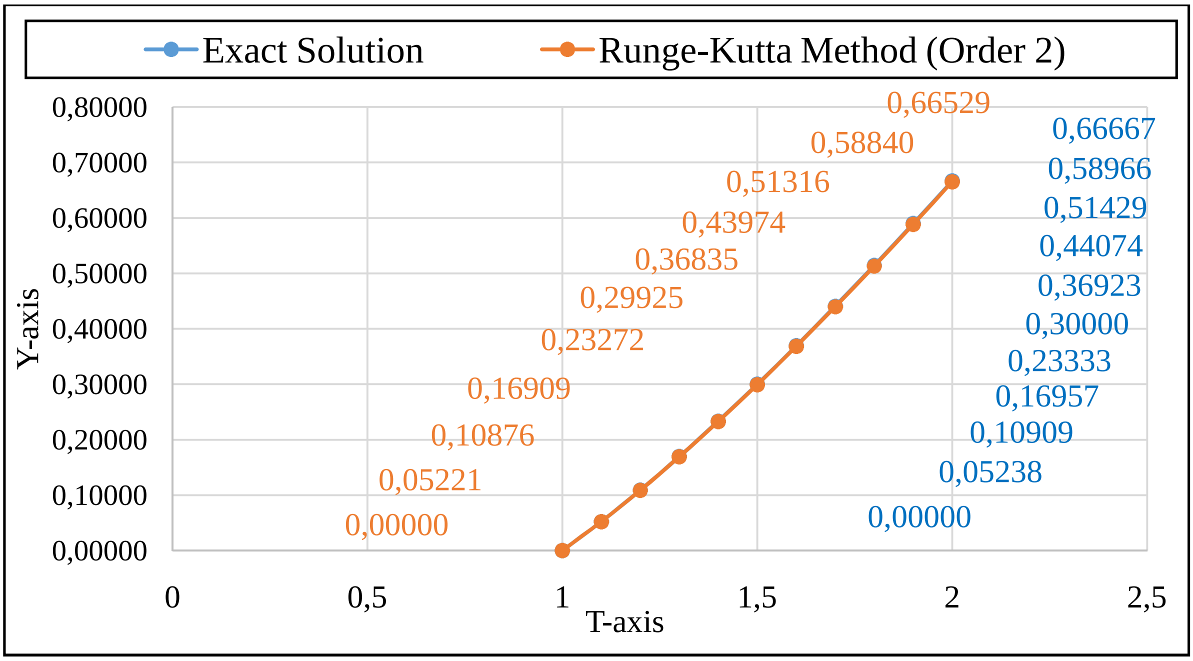


Figure (C.1): Approximate and exact solutions for example (C.1) when $h = 0.1$.

## C.2 Approximation Solution from RDE by Runge - Kutta Method of Order-Four

**Example (C.2):** Find the approximate solution for the following initial value problem

$$y' = \left(1+\frac{2}{t}\right) - \left(2+\frac{2}{t}\right)y + y^2 \quad , \quad y(1) = \frac{5}{2} \quad , \quad 1 \le t \le 2$$

, by using Runge-Kutte method of order four, after that compare with actual solution which is given as $y = (3 + 3t - t^2)/(3t + t^2)$ , and determine the error with explain by graphical representation.

### C.2.1 Approximation Solution of example (C.2) when ($h = 0.1$)

Since $t_0 = 1$ then byusing this form$t_{i+1} = t_i + h$, we obtain

$t_1 = t_0 + h = 1 + 0.1 = 1.1$,

$t_2 = t_1 + h = 1.1 + 0.1 = 1.2$,, and so on

.

.

$t_{10} = t_9 + h = 1.9 + 0.1 = 2$.

Now, we are continue to compute the approximate solution and since the initial condition given as $y(1) = 5/2$ then, we find that

$$k_1 = hf(t_0, y_0) = 0.1\left(\left(1+\frac{2}{t_0}\right) - \left(2+\frac{2}{t_0}\right)y_0 + y_0^2\right),$$

$$= 0.1\left(\left(1+\frac{2}{1}\right) - \left(2+\frac{2}{1}\right)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2\right),$$

$$= -0.07500$$

$$k_2 = hf\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right),$$

$$= h\left(\left(1+\frac{2}{t_0+\frac{1}{2}h}\right) - \left(2+\frac{2}{t_0+\frac{1}{2}h}\right)\left(y_0+\frac{1}{2}k_1\right) + \left(y_0+\frac{1}{2}k_1\right)^2\right),$$

$$= 0.1\left[\left(1+\frac{2}{1+\frac{1}{2}\,0.1}\right)-\left(2+\frac{2}{1+\frac{1}{2}\,0.1}\right)\left(\frac{5}{2}+\frac{1}{2}-0.07500\right)+\left(\frac{5}{2}+\frac{1}{2}-0.07500\right)^2\right] = -0.06468$$

$$k_3 = hf\left(t_i+\frac{1}{2}h, y_i+\frac{1}{2}k_2\right)$$

$$= h\left[\left(1+\frac{2}{t_0+\frac{1}{2}h}\right)-\left(2+\frac{2}{t_0+\frac{1}{2}h}\right)\left(y_0+\frac{1}{2}k_2\right)+\left(y_0+\frac{1}{2}k_2\right)^2\right],$$

$$= 0.1\left[\left(1+\frac{2}{1+\frac{1}{2}\,0.1}\right)-\left(2+\frac{2}{1+\frac{1}{2}\,0.1}\right)\left(\frac{5}{2}+\frac{1}{2}-0.06468\right)+\left(\frac{5}{2}+\frac{1}{2}-0.06468\right)^2\right] = -0.06415$$

$$k_4 = hf\left(t_i+h, y_i+k_3\right),$$

$$= h\left[\left(1+\frac{2}{t_0+h}\right)-\left(2+\frac{2}{t_0+h}\right)\left(y_0+k_3\right)+\left(y_0+k_3\right)^2\right],$$

$$= 0.1\left[\left(1+\frac{2}{1+0.1}\right)-\left(2+\frac{2}{1+0.1}\right)\left(\frac{5}{2}\pm 0.06415\right)+\left(\frac{5}{2}\pm 0.06415\right)^2\right]$$

$$= -0.05490$$

Thus, we can apply the equation (2.3.2.4) to achieve the next $y_i$ as follow;

$$y_{i+1} = y_i+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right),$$

$$y_1 = y_0+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right),$$

$$y_1 = \left(\frac{5}{2}\right)+\frac{1}{6}\left(-0.07500+2(-0.06468)+2(-0.06415)+-0.05490\right),$$

$$= 2.43541$$

Now, we are going to find $y_2$ by using the entire information of $y_1$ , naturally, we must be find each of the $k_1, k_2, k_3, k_4$ as we have seen in the previous calculations, after that it can be easy to compute $y_2$ in order to complete the calculations;

Since $t_1 = 1.1$ then $y_1 = 2.43541$ , subsequently we find that;

$$k_1 = hf(t_1, y_1) = 0.1\left[\left(1 + \frac{2}{t_1}\right) - \left(2 + \frac{2}{t_1}\right) y_1 + y_1^2\right],$$

$$= 0.1\left[\left(1 + \frac{2}{1.1}\right) - \left(2 + \frac{2}{1.1}\right)(2.43541) + (2.43541)^2\right] = -0.05494,$$

$$k_2 = hf\left(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right),$$

$$= h\left[\left(1 + \frac{2}{t_1 + \frac{1}{2}h}\right) - \left(2 + \frac{2}{t_1 + \frac{1}{2}h}\right)\left(y_1 + \frac{1}{2}k_1\right) + \left(y_1 + \frac{1}{2}k_1\right)^2\right],$$

$$= 0.1\left[\left(1 + \frac{2}{1.1 + (0.5)(0.1)}\right) - \left(2 + \frac{2}{1.1 + (0.5)(0.1)}\right)\left(2.43541 + \frac{1}{2}(-0.05494)\right) + \left(2.43541 + \frac{1}{2}(-0.05494)\right)^2\right]$$

$$= -0.04663,$$

$$k_3 = hf\left(t_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right),$$

$$= h\left[\left(1 + \frac{2}{t_1 + \frac{1}{2}h}\right) - \left(2 + \frac{2}{t_1 + \frac{1}{2}h}\right)\left(y_1 + \frac{1}{2}k_2\right) + \left(y_1 + \frac{1}{2}k_2\right)^2\right],$$

$$= 0.1\left[\left(1 + \frac{2}{1.1 + \frac{1}{2}(0.1)}\right) - \left(2 + \frac{2}{1.1 + \frac{1}{2}(0.1)}\right)\left(2.43541 + \frac{1}{2}(-0.04663)\right) + \left(2.43541 + \frac{1}{2}(-0.04663)\right)^2\right] = -0.04618$$

$$k_4 = hf\left(t_1 + h, y_1 + k_3\right),$$

$$= h\left[\left(1 + \frac{2}{t_1 + h}\right) - \left(2 + \frac{2}{t_1 + h}\right)\left(y_1 + k_3\right) + \left(y_1 + k_3\right)^2\right],$$

$$= 0.1\left[\left(1 + \frac{2}{1.1 + 0.1}\right) - \left(2 + \frac{2}{1.1 + 0.1}\right)\left(2.43541 + (-0.04618)\right)\right.$$

$$\left. + \left(2.43541 + (-0.04618)\right)^2\right] = -0.03854$$

Thus, we can apply the equation (5.5) to achieve the next $y_i$ as follow;

$$y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

$$y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

$$= 2.43541 + \frac{1}{6}\left[-0.05494 + 2(-0.04663) + 2(-0.04618) + (-0.03854)\right]$$

$$= 2.38889$$

Because the calculations of the next $y_i$ values same as the preceding calculations then we only refer to their values as follow;

$$y_3 = 2.35746,\ y_4 = 2.33928,\ y_5 = 2.33333,\ y_6 = 2.33928,\ y_7 = 2.35746,$$

$$y_8 = 2.38889,\ y_9 = 2.43540,\ y_{10} = 2.49999.$$

After we found the last $y_i$ which is $y_{10}$ with respect to the desired interval, fortunately we have achieved the interested result to this type of the differential equation which is Riccati differential equation. And also indicated that the Runge-Kutte method that it was used through the implementation of this method to find the approximation solution is a mostly accurate method compare to the all these method that used to find the approximation solution, notice that, it is not necessary to take the small step size $h$.

As mentioned earlier, all the computations are indicated in the table (C.2) with representation of the actual solution which is obtained in the third column. Nate that, this tabulation applied by using the (Microsoft Office Excel)which is putted by one of the most important program that it can give us a best and accurate solution.

Table (C.2): Illustration of Runge-Kutte method for order four and Exact

| $t_i$ | Exact solution | Runge-Kutte method (order 4) | $k_1$ | $k_2$ | $k_3$ | $k_4$ |
|---|---|---|---|---|---|---|
| 1 | 2.50000 | 2.50000 | -0.07500 | -0.06468 | -0.06415 | -0.05490 |
| 1.1 | 2.43541 | 2.43541 | -0.05494 | -0.04663 | -0.04618 | -0.03854 |
| 1.2 | 2.38889 | 2.38889 | -0.03858 | -0.03156 | -0.03115 | -0.02454 |
| 1.3 | 2.35747 | 2.35746 | -0.02457 | -0.01833 | -0.01796 | -0.01193 |
| 1.4 | 2.33929 | 2.33928 | -0.01196 | -0.00613 | -0.00576 | 0.00003 |
| 1.5 | 2.33333 | 2.33333 | 0.00000 | 0.00573 | 0.00613 | 0.01198 |
| 1.6 | 2.33929 | 2.33928 | 0.01196 | 0.01791 | 0.01835 | 0.02460 |
| 1.7 | 2.35747 | 2.35746 | 0.02457 | 0.03108 | 0.03160 | 0.03861 |
| 1.8 | 2.38889 | 2.38889 | 0.03858 | 0.04606 | 0.04671 | 0.05498 |
| 1.9 | 2.43541 | 2.43540 | 0.05494 | 0.06396 | 0.06482 | 0.07504 |
| 2 | 2.50000 | 2.49999 | 0.07500 | 0.08639 | 0.08759 | 0.10084 |

solution when □ = 0.1

## C.2.1.1 Absolute and Percentage Relative Errors:

In the preceding sections, we indicated that how can find the absolute and relative percentage errors and given their formulas that use to find the amount values between the actual and the numerical solutions. So, we don't like to write their rules again, only we arise to interpret the errors with respect to the desired example.

As presented in the table (C.3), the absolute error in all situations equal to zero except the last step (situation) that the amount value of its error equal to the $0.00001$. In actually, these all amounts are attained by subtracting the numerical solution from the actual solution. Note that, in this case, taken the tolerance equal to $10^{-5}$ and $□ = 0.1$.

Evidently we have expressed in the table (C.3), the percentage relative error through the all situations equal to zero because firstly: the method that used to find the numerical solution is more extremely accurate method although the step size □ is not too small. Secondly: this amount of error is percentage relativity which its tolerance equal to $10^{-2}$ and keep in your mind always, it can be measured the percentage relative error by subtracting the numerical solution from the exact solution and divide by the exact solution.

Table (C.3) illustration the relative and relative percentage of errors

| $i$ | Absolute Error | % Relative Error |
|---|---|---|
| 0 | 0.00000 | 0.00 |
| 1 | 0.00000 | 0.00 |
| 2 | 0.00000 | 0.00 |
| 3 | 0.00000 | 0.00 |
| 4 | 0.00000 | 0.00 |
| 5 | 0.00000 | 0.00 |
| 6 | 0.00000 | 0.00 |
| 7 | 0.00000 | 0.00 |
| 8 | 0.00000 | 0.00 |
| 9 | 0.00000 | 0.00 |
| 10 | 0.00001 | 0.00 |

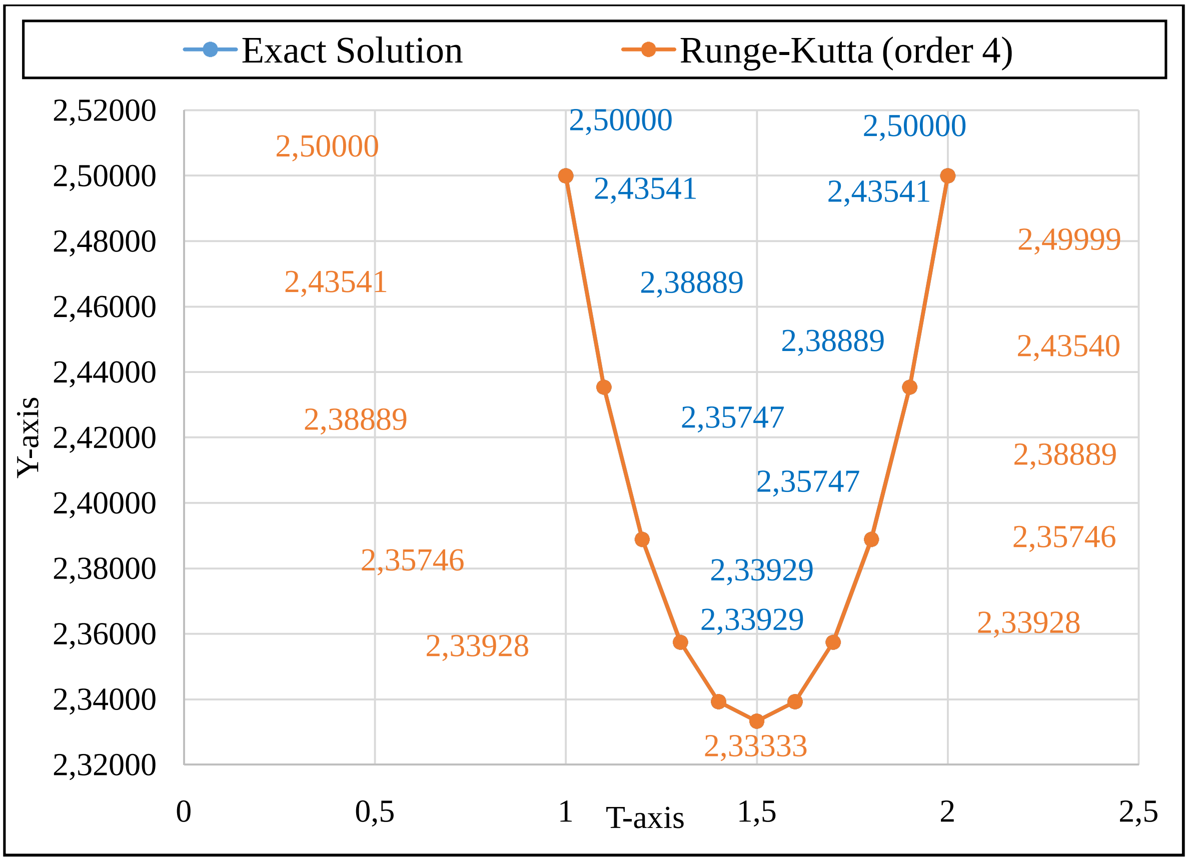


Figure (C.2): Approximate and exact solutions for example (C.2)

When $h = 0.1$.

### C.2.2 Approximation Solution of example (C.2) when ($h = 0.01$)

In actually, there is no necessary to use the smallest step size $h$ when we bring the Runge-Kutte method for solving the initial value problem for ordinary differential equation, because the method naturally can give us the accurate result although the value of step size is not very small. But in this example we take the smallest step size $h$ only for indicate the mostly accurate in the approximate solution and for give more representation about the illustrations of this method.

In fact that, in the previous case we explained the method widely then, in this case we do not write all of the steps of the solution again, because we changed only the value of step size $h$ and also we take the some steps of the tabulation because it is too long which is consist of one hundred iterations . In the next page expressed each of the tabulation and figure of this case when $h = 0.01$.

Table (C.4) illustration of Runge-Kutte method for order four and Exact

| $i$ | $t_i$ | Exact | Runge-Kutte | $k_1$ | $k_2$ | $k_3$ | $k_4$ |
|---|---|---|---|---|---|---|---|

solution when = 0.01.

| | | solution | Method (order 4) | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2.50000 | 2.50000 | -0.00750 | -0.00739 | -0.00739 | -0.00728 |
| 1 | 1.01 | 2.49261 | 2.49261 | -0.00728 | -0.00717 | -0.00717 | -0.00706 |
| 5 | 1.05 | 2.46520 | 2.46520 | -0.00644 | -0.00634 | -0.00634 | -0.00624 |
| 10 | 1.1 | 2.43541 | 2.43541 | -0.00549 | -0.00541 | -0.00540 | -0.00532 |
| 15 | 1.15 | 2.41011 | 2.41011 | -0.00464 | -0.00456 | -0.00456 | -0.00448 |
| 20 | 1.2 | 2.38889 | 2.38889 | -0.00386 | -0.00378 | -0.00378 | -0.00371 |
| 25 | 1.25 | 2.37143 | 2.37143 | -0.00313 | -0.00307 | -0.00306 | -0.00300 |
| 30 | 1.3 | 2.35747 | 2.35747 | -0.00246 | -0.00239 | -0.00239 | -0.00233 |
| 35 | 1.35 | 2.34680 | 2.34680 | -0.00181 | -0.00175 | -0.00175 | -0.00169 |
| 40 | 1.4 | 2.33929 | 2.33929 | -0.00120 | -0.00114 | -0.00113 | -0.00107 |
| 45 | 1.45 | 2.33482 | 2.33482 | -0.00059 | -0.00053 | -0.00053 | -0.00047 |
| 50 | 1.5 | 2.33333 | 2.33333 | 0.00000 | 0.00006 | 0.00006 | 0.00012 |
| 55 | 1.55 | 2.33482 | 2.33482 | 0.00059 | 0.00065 | 0.00065 | 0.00071 |
| 60 | 1.6 | 2.33929 | 2.33929 | 0.00120 | 0.00126 | 0.00126 | 0.00132 |
| 65 | 1.65 | 2.34680 | 2.34680 | 0.00181 | 0.00188 | 0.00188 | 0.00194 |
| 70 | 1.7 | 2.35747 | 2.35747 | 0.00246 | 0.00252 | 0.00252 | 0.00259 |
| 75 | 1.75 | 2.37143 | 2.37143 | 0.00313 | 0.00320 | 0.00321 | 0.00328 |
| 80 | 1.8 | 2.38889 | 2.38889 | 0.00386 | 0.00393 | 0.00393 | 0.00401 |
| 85 | 1.85 | 2.41011 | 2.41011 | 0.00464 | 0.00472 | 0.00472 | 0.00480 |
| 90 | 1.9 | 2.43541 | 2.43541 | 0.00549 | 0.00558 | 0.00558 | 0.00568 |
| 95 | 1.95 | 2.46520 | 2.46520 | 0.00644 | 0.00654 | 0.00654 | 0.00664 |
| 100 | 2 | 2.50000 | 2.50000 | 0.00750 | 0.00761 | 0.00761 | 0.00773 |

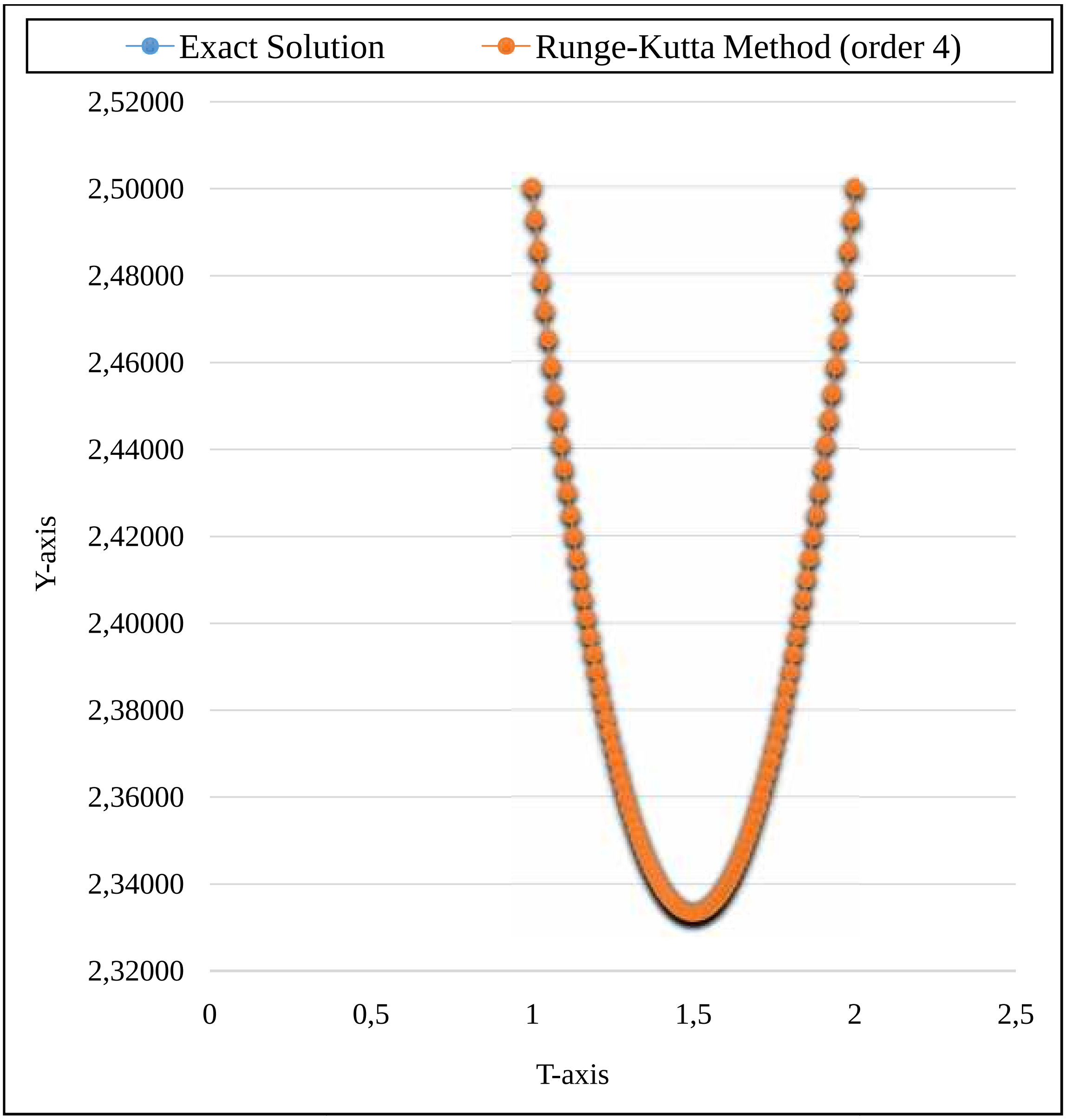


Figure (C.3): Approximate and exact solutions for example (C.2)

When $h = 0.01$

## Appendix D. Approximation Solution from RDE by Runge - Kutta-Fehlberg Method

**Example (D.1):** Use the Runge-Kutta-Fehlberg method to determine the approximate solution to the initial value problem

$$y' = 9t + 3 - 18t \; y + 9t - 3 \; y^2 \quad , \quad y \; 0 = 2 \; , \; 0 \leq t \leq 1 \; ,$$

, with $h = 0.1$ and compare to the exact solution which is given as

$$y = (3 + 3t - t^2) / (3t - t^2).$$

**Solution:**

We recognized the given example is one type of the ordinary differential equation which is Riccati differential equation, so we are ready to apply the desired method as;

Since, we have the initial condition $w_0$ then we can determine $w_1$ by using $h = 0.1$,

$$k_1 = hf \; t_0, w_0 = (0.1)(9t_0 + 3 - 18t_0 \; y_0 + 9t_0 - 3 \; y_0)^2$$

$$= (0.1) \; 9 \; 0 + 3 - 18 \; 0 \; (2) + 9(0) - 3 \; 2^2 = -0.6000000;$$

$$k_2 = hf \; t_0 + \frac{1}{4}h, w_0 + \frac{1}{4}k_1$$

$$= (0.1) \; 9 \; 0 + \frac{1}{4} \; 0.1 + 3 - 18 \; 0 + \frac{1}{4} \; 0.1 \quad 2 + \frac{1}{4} \; -0.6000000$$

$$+ \; 9 \; 0 + \frac{1}{4} \; (0.1) \; - 3 \; 2 + \frac{1}{4} \; -0.6000000^{\;2} = -0.4554938;$$

$$k_3 = hf \; t_i + \frac{3}{8}h, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2$$

$$= (0.1) \; 9 \; 0 + \frac{3}{8} \; (0.1)$$

$$+ \; 3 - 18 \; 0 + \frac{3}{8} \; (0.1) \quad 2 + \frac{3}{32} \; -0.6000000$$

$$+ \; \frac{9}{32} \; -0.4554938$$

$$+ \; 9 \; 0 + \frac{3}{8} \; (0.1)$$

$$- 3 \quad 2 + \frac{3}{32} \; -0.6000000 + \frac{9}{32} \; -0.4554938^{\;2}$$

$$= -0.4218215;$$

$$k_4 = hf\ \ t_0 + \frac{12}{13}h, w_0 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3$$

$$= (0.1)\ \ 9\ \ 0 + \ \frac{12}{13}\ (0.1)$$

$$+\ \ 3 - 18\ \ 0 + \frac{12}{13}\ (0.1)\ \ \ 2 + \frac{1932}{2197}\ \ -0.6000000$$

$$-\ \frac{7200}{2197}\ \ -0.4554938\ + \frac{7296}{2197}\ \ -0.4218215$$

$$+\ \ 9\ \ 0 + \frac{12}{13}\ (0.1)$$

$$-\ 3\ \ \ 2 + \frac{1932}{2197}\ \ -0.6000000\ - \frac{7200}{2197}\ \ -0.4554938$$

$$+\ \frac{7296}{2197}\ \ -0.4218215\ ^{2}\ \ = -0.2383604;$$

$$k_5 = hf\ \ t_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4$$

$$= (0.1)\ \ 9\ \ 0 + 0.1$$

$$+\ \ 3 - 18\ \ 0 + 0.1\ \ \ 2 + \frac{439}{216}\ \ -0.6000000\ - 8\ -0.4554938$$

$$+\ \frac{3680}{513}\ \ -0.4218215\ - \frac{845}{4104}\ \ -0.2383604$$

$$+\ \ 9\ \ 0 + 0.1$$

$$-\ 3\ \ \ 2 + \frac{439}{216}\ \ -0.6000000\ - 8\ -0.4554938$$

$$+\ \frac{3680}{513}\ \ -0.4218215\ - \frac{845}{4104}\ \ -0.2383604\ ^{2}\ \ = -0.1763148;$$

$$k_6 = □ f \quad t_i + \frac{1}{2} □, w_i - \frac{8}{27} k_1 + 2k_2 - \frac{3544}{2565} k_3 + \frac{1859}{4104} k_4 - \frac{11}{40} k_5$$

$$= 0.1 \quad 9 \quad 0 + \quad \frac{1}{2} \quad 0.1$$

$$+ \quad 3 - 18 \quad 0 + \quad \frac{1}{2} \quad 0.1 \qquad 2 - \quad \frac{8}{27} \quad -0.6000000 \quad + \; 2 \; -0.4554938$$

$$- \quad \frac{3544}{2565} \quad -0.4218215 \quad + \quad \frac{1859}{4104} \quad -0.2383604$$

$$- \quad \frac{11}{40} \quad -0.1763148$$

$$+ \quad 9 \quad 0 + \quad \frac{1}{2} \quad 0.1$$

$$- \; 3 \quad 2 - \quad \frac{8}{27} \quad -0.6000000 \quad + \; 2 \; -0.4554938$$

$$- \quad \frac{3544}{2565} \quad -0.4218215 \quad + \quad \frac{1859}{4104} \quad -0.2383604$$

$$- \quad \frac{11}{40} \quad -0.1763148 \quad {}^{2} \quad = -0.3962347$$

$$\tilde{w}_1 = w_0 + \frac{16}{135} k_1 + \frac{6656}{12825} k_3 + \frac{28561}{56430} k_4 - \frac{9}{50} k_5 + \frac{2}{55} k_6$$

$$= \quad 2 + \quad \frac{16}{135} \quad -0.6000000 \quad + \frac{6656}{12825} \quad -0.4218215 \quad + \frac{28561}{56430} \quad -0.2383604$$

$$- \frac{9}{50} \quad -0.1763148 \quad + \frac{2}{55} \quad -0.3962347 \quad = \; 1.6063597$$

And

$$w_1 = w_0 + \frac{25}{216} k_1 + \frac{1408}{2565} k_3 + \frac{2197}{4104} k_4 - \frac{1}{5} k_5$$

$$= \quad 2 + \frac{25}{216} \quad -0.6000000 \quad + \frac{1408}{2565} \quad -0.4218215 \quad + \frac{2197}{4104} \quad -0.2383604$$

$$- \frac{1}{5} \quad -0.1763148 \quad = \; 1.6066671$$

All the calculations of algorithm above are shown in the table (D.1), the accuracy has been happened from the fourth and fifth order of the RKF-Method compare to the actual solution meaning that it is appear the increasing of the accuracy to get the certain of the calculation. Also in the table (D.3) expressed each of the absolute and the percentage relative errors with respect to the Runge-Kutta-Fehlberg method for order four and order five respectively.it is clear that the percentage relative error through the Runge-Kutta-Fehlberg method of order five is less than the percentage relative error through the Runge-Kutta-Fehlberg method of order four.

Table (D.1): Illustration of Runge-Kutte –Fehlberg method for order four and five and Exact solution when  = 0.1.

| $i$ |  | $t_i$ | Exact solution | RKF (order4) | RKF (order 5) |
|---|---|---|---|---|---|
| 0 | 0.1 | 0 | 2.0000000 | 2.0000000 | 2.0000000 |
| 1 | 0.1 | 0.1 | 1.5883330 | 1.6066671 | 1.6063597 |
| 2 | 0.1 | 0.2 | 1.3781808 | 1.4133309 | 1.4128873 |
| 3 | 0.1 | 0.3 | 1.2551537 | 1.2981237 | 1.2975965 |
| 4 | 0.1 | 0.4 | 1.1772976 | 1.2217283 | 1.2211448 |
| 5 | 0.1 | 0.5 | 1.1255748 | 1.1677040 | 1.1670803 |
| 6 | 0.1 | 0.6 | 1.0900958 | 1.1279539 | 1.1273003 |
| 7 | 0.1 | 0.7 | 1.0652216 | 1.0979977 | 1.0973213 |
| 8 | 0.1 | 0.8 | 1.0475142 | 1.0751165 | 1.0744226 |
| 9 | 0.1 | 0.9 | 1.0347712 | 1.0575272 | 1.0568196 |
| 10 | 0.1 | 1 | 1.0255290 | 1.0439824 | 1.0432642 |

Table (D.2): Illustration of the Coefficients with respect the Runge-Kutte – Fehlberg method.

| $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ |
|---|---|---|---|---|---|
| -0.6000000 | -0.4554938 | -0.4218215 | -0.2383604 | -0.1763148 | -0.3962347 |
| -0.2591190 | -0.2174694 | -0.2040951 | -0.1457141 | -0.1334662 | -0.1916635 |
| -0.1443233 | -0.1268853 | -0.1205083 | -0.0938403 | -0.0894146 | -0.1143513 |
| -0.0919359 | -0.0829493 | -0.0794298 | -0.0647615 | -0.0625769 | -0.0759800 |
| -0.0634091 | -0.0581108 | -0.0559497 | -0.0468705 | -0.0455915 | -0.0538129 |
| -0.0459367 | -0.0425026 | -0.0410663 | -0.0349645 | -0.0341332 | -0.0396373 |
| -0.0343008 | -0.0319174 | -0.0309052 | -0.0265580 | -0.0259793 | -0.0298935 |
| -0.0260708 | -0.0243346 | -0.0235909 | -0.0203670 | -0.0199453 | -0.0228452 |
| -0.0200005 | -0.0186941 | -0.0181324 | -0.0156794 | -0.0153630 | -0.0175679 |
| -0.0153994 | -0.0143968 | -0.0139656 | -0.0120718 | -0.0118304 | -0.0135317 |
| -0.0118562 | -0.0110791 | -0.0107453 | -0.0092739 | -0.0090881 | -0.0104094 |

Table (D.3): Illustration the relative and relative percentage of errors.

| $i$ | Absolute Error (RKF- Order 4) | Absolute Error (RKF- Order 5) | % Relative Error (RKF- Order 4) | % Relative Error (RKF- Order 5) |
|---|---|---|---|---|
| 0 | 0.0000000 | 0.0000000 | 0.00 | 0.00 |
| 1 | 0.0183341 | 0.0180267 | 1.15 | 1.13 |
| 2 | 0.0351501 | 0.0347065 | 2.55 | 2.52 |
| 3 | 0.0429700 | 0.0424428 | 3.42 | 3.38 |
| 4 | 0.0444307 | 0.0438472 | 3.77 | 3.72 |
| 5 | 0.0421292 | 0.0415054 | 3.74 | 3.69 |
| 6 | 0.0378581 | 0.0372044 | 3.47 | 3.41 |
| 7 | 0.0327761 | 0.0320997 | 3.08 | 3.01 |
| 8 | 0.0276024 | 0.0269084 | 2.64 | 2.57 |
| 9 | 0.0227561 | 0.0220485 | 2.20 | 2.13 |
| 10 | 0.0184534 | 0.0177352 | 1.80 | 1.73 |

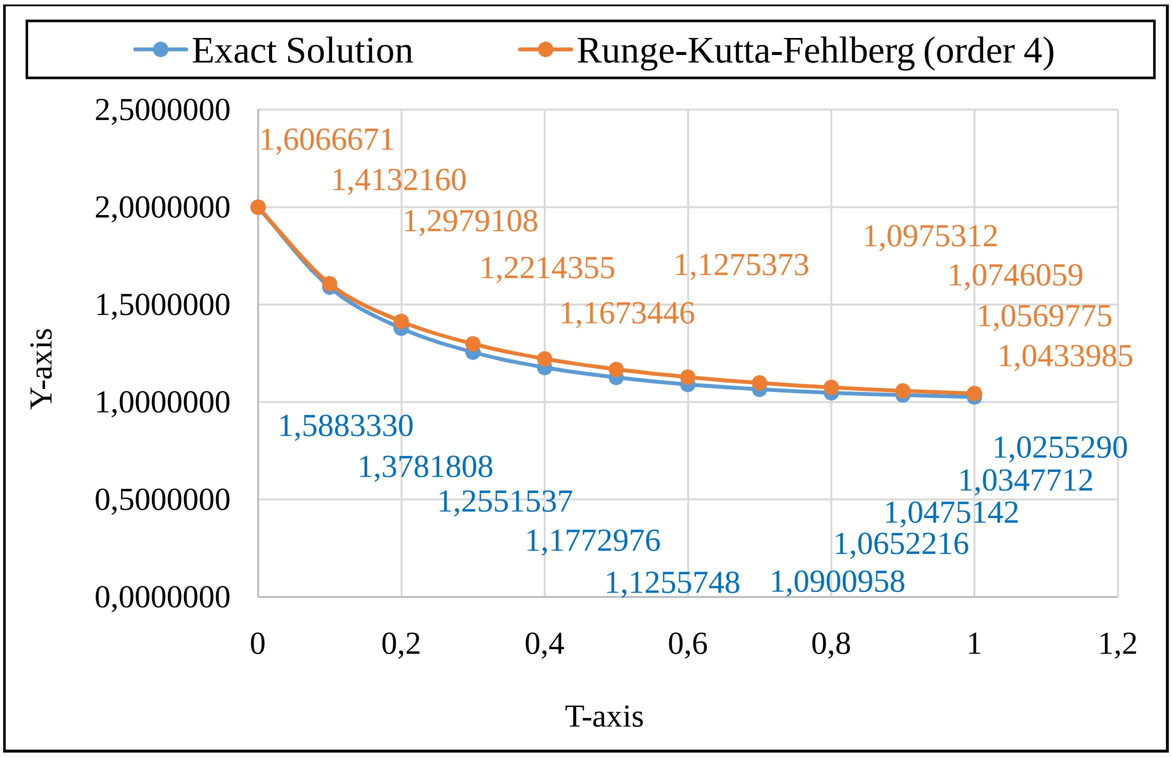


Figure (D.1): Approximate and exact solutions for example (D.1)

when ⍰ = 0.1.

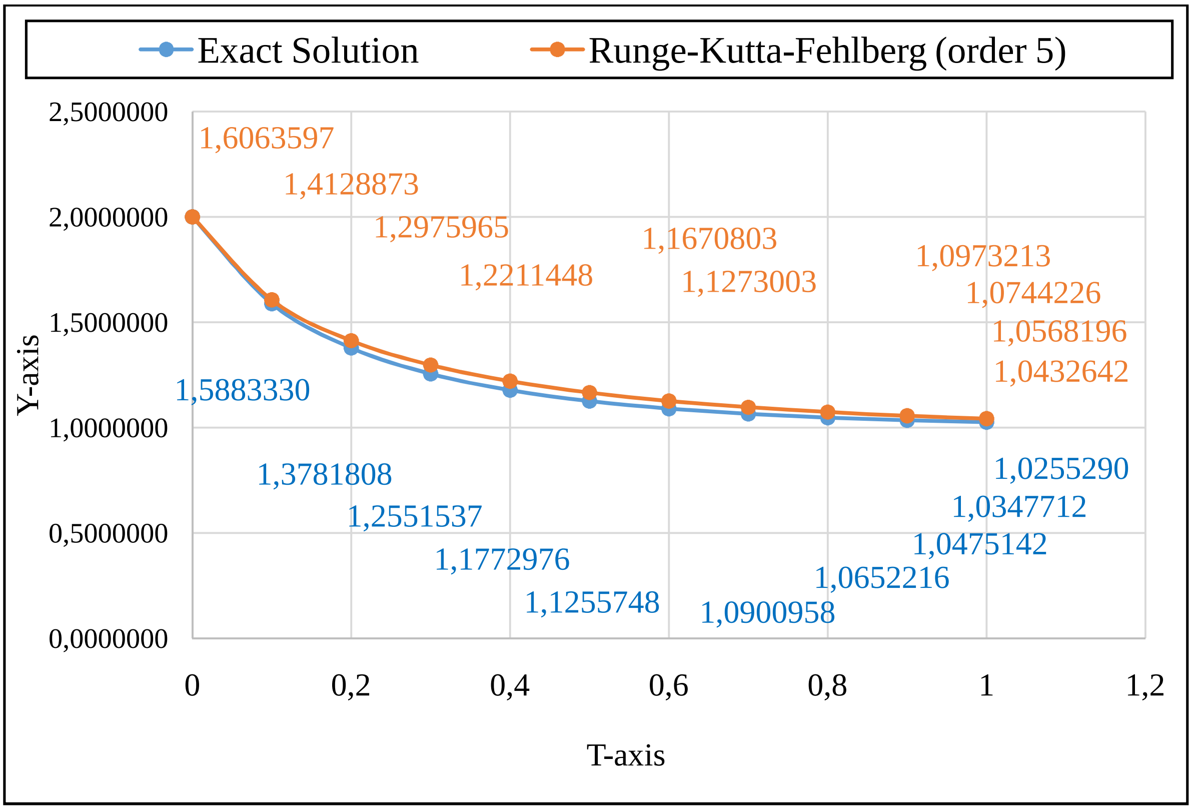


Figure (D.2): Approximate and exact solutions for example (D.1)

when ⍰ = 0.1

# Appendix E. Approximation Solution from RDE by Runge - Kutta-Verner Method

**Example (E.1):** Use the Runge-Kutta-Verner method to determine the approximate solution to the initial value problem

$$y' = 1 + 2y - y^2 \quad , \quad y(0) = 0.483649 \quad , \quad 0 \le t \le 1 \ ,$$

, and compare to the actual solution for this Riccati differential equation which is given as

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}\,t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) .$$

**Solution:** let $h = 0.1$ and, since the initial condition is given then we can compute all the desired coefficients as;

$$k_1 = hf(t_i, w_i) = 0.1\left(1 + 2(0.483649) - (0.483649)^2\right) = 0.173338125;$$

$$k_2 = hf\left(t_i + \frac{1}{6}h, w_i + \frac{1}{6}k_1\right)$$

$$= 0.1\left(1 + 2\left(0.483649 + \left(\frac{1}{6}\right)(0.173338125)\right)\right.$$

$$\left. - \left(0.483649 + \left(\frac{1}{6}\right)(0.173338125)\right)^2\right) = 0.17623811;$$

$$k_3 = hf\left(t_i + \frac{4}{15}h, w_i + \frac{4}{75}k_1 + \frac{16}{75}k_2\right)$$

$$= 0.1\left(1 + 2\left(0.483649 + \left(\frac{4}{75}\right)(0.173338125) + \left(\frac{16}{75}\right)(0.17623811)\right)\right.$$

$$\left. - \left(0.483649 + \left(\frac{4}{75}\right)(0.173338125) + \left(\frac{16}{75}\right)(0.17623811)\right)^2\right)$$

$$= 0.17795611;$$

$$k_4 = hf\left(t_i + \frac{2}{3}h, w_i + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right)$$

$$= 0.1\{1 + 2 \quad 0.483649 + \frac{5}{6} \quad 0.173338125 \; - \; \frac{8}{3} \quad 0.17623811$$

$$+ \; \frac{5}{2} \quad 0.17795611$$

$$- \quad 0.483649 + \frac{5}{6} \quad 0.173338125 \; - \; \frac{8}{3} \quad 0.17623811$$

$$+ \; \frac{5}{2} \quad 0.17795611 \quad ^{2} = 0.184241;$$

$$k_5 = \;?f \quad t_i + \frac{5}{6}?, w_i - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4$$

$$= 0.1 \quad 1 + 2 \quad 0.483649 - \frac{165}{64} \quad 0.173338125 \; + \; \frac{55}{6} \quad 0.17623811$$

$$- \; \frac{425}{64} \quad 0.17795611 \; + \; \frac{85}{96} \quad 0.184241$$

$$- \quad 0.483649 - \frac{165}{64} \quad 0.173338125 \; + \; \frac{55}{6} \quad 0.17623811$$

$$- \; \frac{425}{64} \quad 0.17795611 \; + \; \frac{85}{96} \quad 0.184241 \quad ^{2} = 0.186580;$$

$$k_6 = \;?f \quad t_i + \;?, w_i + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5$$

$$= 0.1 \quad 1 + 2 \quad 0.483649 + \frac{12}{5} \quad 0.173338125 \; - 8 \; 0.17623811$$

$$+ \; \frac{4015}{612} \quad 0.17795611 \; - \; \frac{11}{36} \quad 0.184241 \; + \; \frac{88}{255} \quad 0.186580$$

$$- \quad 0.483649 + \frac{12}{5} \quad 0.173338125 \; - 8 \; 0.17623811$$

$$+ \; \frac{4015}{612} \quad 0.17795611 \; - \; \frac{11}{36} \quad 0.184241 \; + \; \frac{88}{255} \quad 0.186580 \quad ^{2}$$

$$= 0.188799;$$

$$k_7 = \square f \quad t_i + \frac{1}{15}\square, w_i - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5$$

$$= 0.1 \quad 1 + 2 \quad 0.483649 - \frac{8263}{15000} \quad 0.173338125 \quad + \frac{124}{75} \quad 0.17623811$$

$$- \frac{643}{680} \quad 0.17795611 \quad - \frac{81}{250} \quad 0.184241 \quad + \frac{2484}{10625} \quad 0.186580$$

$$- \quad 0.483649 - \frac{8263}{15000} \quad 0.173338125 \quad + \frac{124}{75} \quad 0.17623811$$

$$- \frac{643}{680} \quad 0.17795611 \quad - \frac{81}{250} \quad 0.184241 \quad + \frac{2484}{10625} \quad 0.186580 \qquad ^{2}$$

$$= 0.174517;$$

$$k_8 = \square f \quad t_i + \square, w_i + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5$$

$$+ \frac{3850}{26703}k_7$$

$$= 0.1 \quad 1 + 2 \quad 0.483649 + \frac{3501}{1720} \quad 0.173338125 \quad - \frac{300}{43} \quad 0.17623811$$

$$+ \frac{297275}{52632} \quad 0.17795611 \quad - \frac{319}{2322} \quad 0.184241$$

$$+ \frac{24068}{84065} \quad 0.186580 \quad + \frac{3850}{26703} \quad 0.174517$$

$$- \quad 0.483649 + \frac{3501}{1720} \quad 0.173338125 \quad - \frac{300}{43} \quad 0.17623811$$

$$+ \frac{297275}{52632} \quad 0.17795611 \quad - \frac{319}{2322} \quad 0.184241$$

$$+ \frac{24068}{84065} \quad 0.186580 \quad + \frac{3850}{26703} \quad 0.174517 \qquad ^{2} = 0.188798.$$

Now, we are continue to applying the Runge-Kutta method for fifth-order as;

$$w_1 = w_0 + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6$$

$$= 0.483649 + \frac{13}{160}\ 0.173338125 + \frac{2375}{5984}\ 0.17795611 + \frac{5}{16}\ 0.184241$$
$$+ \frac{12}{85}\ 0.186580 + \frac{3}{44}\ 0.188799 = 0.665150.$$

And then we obtain the Runge-Kutta method for sixth-order as;

$$\widetilde{w}_1 = w_0 + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7 \frac{43}{616}k_8$$

$$= 0.483649 + \frac{3}{40}\ 0.173338125 + \frac{875}{2244}\ 0.17795611 + \frac{23}{72}\ 0.184241$$
$$+ \frac{264}{1955}\ 0.186580 + \frac{125}{11592}\ 0.174517\ \frac{43}{616}\ 0.188798 = 0.665150.$$

As we have seen from the previous algorithm, we solved the nonlinear Riccati differential equation by using the Runge-Kutta-Verner method until we represent that how to find the solution for this type of ordinary differential equation. Clearly, for any ODEs in the form of the initial value problem, it can be find the approximate solution by obtaining one of the technique from the numerical methods.

All the evaluations of the coefficients and the Runge-Kutta-Verner method of order five and six are considered in the table (E.1), (E.2) respectively. Observe that, if we focus on the results that is explained within the tabulation then we can understand that there is no any deferent between the actual and the approximate solutions because the method that used to solve the problem was super numerical method.

Although the value of step size  is not too small,weattained the mostly accurate and fortunateapproximate solutions because this technique is more extremely developed compared with the other numerical methods.

Table (E.1): Illustration of Runge-Kutte –Verner method for order five and six and Exact solution when □ = 0.1.

| $i$ | □ | $t_i$ | Exact Solution | R K V - (Fifth order) | R K V - (Sixth order) |
|---|---|---|---|---|---|
| 0 | 0.1 | 0 | 0.483649 | 0.483649 | 0.483649 |
| 1 | 0.1 | 0.1 | 0.665150 | 0.665150 | 0.665150 |
| 2 | 0.1 | 0.2 | 0.859142 | 0.859142 | 0.859142 |
| 3 | 0.1 | 0.3 | 1.058640 | 1.058640 | 1.058640 |
| 4 | 0.1 | 0.4 | 1.255827 | 1.255827 | 1.255827 |
| 5 | 0.1 | 0.5 | 1.443240 | 1.443240 | 1.443240 |
| 6 | 0.1 | 0.6 | 1.614845 | 1.614845 | 1.614845 |
| 7 | 0.1 | 0.7 | 1.766694 | 1.766694 | 1.766694 |
| 8 | 0.1 | 0.8 | 1.897050 | 1.897050 | 1.897050 |
| 9 | 0.1 | 0.9 | 2.006074 | 2.006074 | 2.006074 |
| 10 | 0.1 | 1 | 2.095286 | 2.095286 | 2.095286 |

Table (E.2): Illustration of the Coefficients with respect the Runge-Kutte – Verner method.

| k1 | k2 | k3 | k4 | k5 | k6 | k7 | k8 |
|---|---|---|---|---|---|---|---|
| 0.17334 | 0.17624 | 0.17796 | 0.18424 | 0.18658 | 0.18880 | 0.17452 | 0.18880 |
| 0.18879 | 0.19080 | 0.19193 | 0.19574 | 0.19698 | 0.19802 | 0.18962 | 0.19802 |
| 0.19802 | 0.19884 | 0.19923 | 0.19999 | 0.19994 | 0.19966 | 0.19837 | 0.19966 |
| 0.19966 | 0.19916 | 0.19875 | 0.19636 | 0.19500 | 0.19346 | 0.19948 | 0.19346 |
| 0.19346 | 0.19170 | 0.19057 | 0.18539 | 0.18295 | 0.18035 | 0.19278 | 0.18035 |
| 0.18035 | 0.17760 | 0.17592 | 0.16867 | 0.16550 | 0.16218 | 0.17928 | 0.16218 |
| 0.16220 | 0.15880 | 0.15679 | 0.14837 | 0.14486 | 0.14118 | 0.16086 | 0.14118 |
| 0.14122 | 0.13755 | 0.13543 | 0.12670 | 0.12319 | 0.11946 | 0.13978 | 0.11947 |
| 0.11953 | 0.11592 | 0.11385 | 0.10549 | 0.10223 | 0.09868 | 0.11811 | 0.09870 |
| 0.09878 | 0.09544 | 0.09356 | 0.08599 | 0.08312 | 0.07992 | 0.09747 | 0.07993 |
| 0.08003 | 0.07709 | 0.07545 | 0.06889 | 0.06646 | 0.06369 | 0.07888 | 0.06371 |

| $i$ | Absolute Error (RKV- Order 4) | Absolute Error (RKV- Order 5) | % Relative Error (RKV- Order 5) | % Relative Error (RKV- Order 6) |
|---|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.00 | 0.00 |
| 1 | 0.0000000048 | 0.0000000007 | 0.00 | 0.00 |
| 2 | 0.0000000072 | 0.0000000004 | 0.00 | 0.00 |
| 3 | 0.0000000079 | 0.0000000001 | 0.00 | 0.00 |
| 4 | 0.0000000075 | 0.0000000002 | 0.00 | 0.00 |
| 5 | 0.0000000052 | 0.0000000005 | 0.00 | 0.00 |
| 6 | 0.0000000015 | 0.0000000001 | 0.00 | 0.00 |
| 7 | 0.0000000157 | 0.0000000019 | 0.00 | 0.00 |
| 8 | 0.0000000392 | 0.0000000061 | 0.00 | 0.00 |
| 9 | 0.0000000704 | 0.0000000114 | 0.00 | 0.00 |
| 10 | 0.0000001058 | 0.0000000165 | 0.00 | 0.00 |

Table (E.3): Illustration the relative and relative percentage of errors.

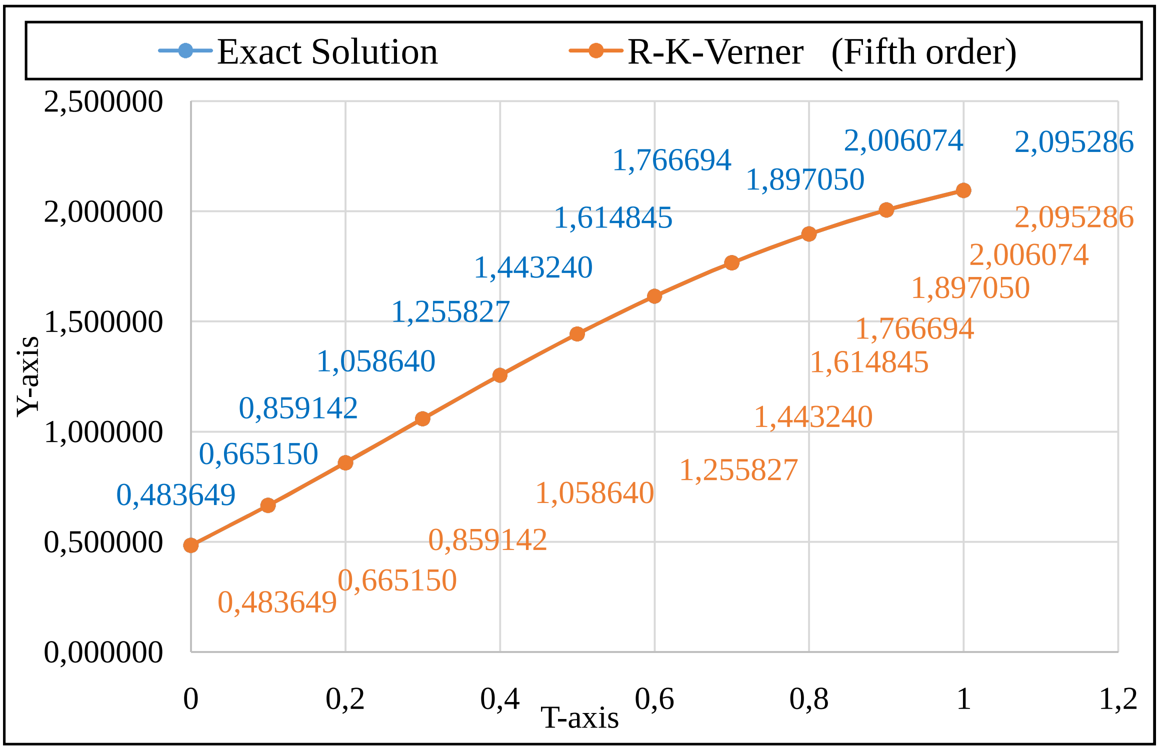

Figure (E.1): R-K Verner method of the fifth order and exact solutions for example (E.1) when ? = 0.1.

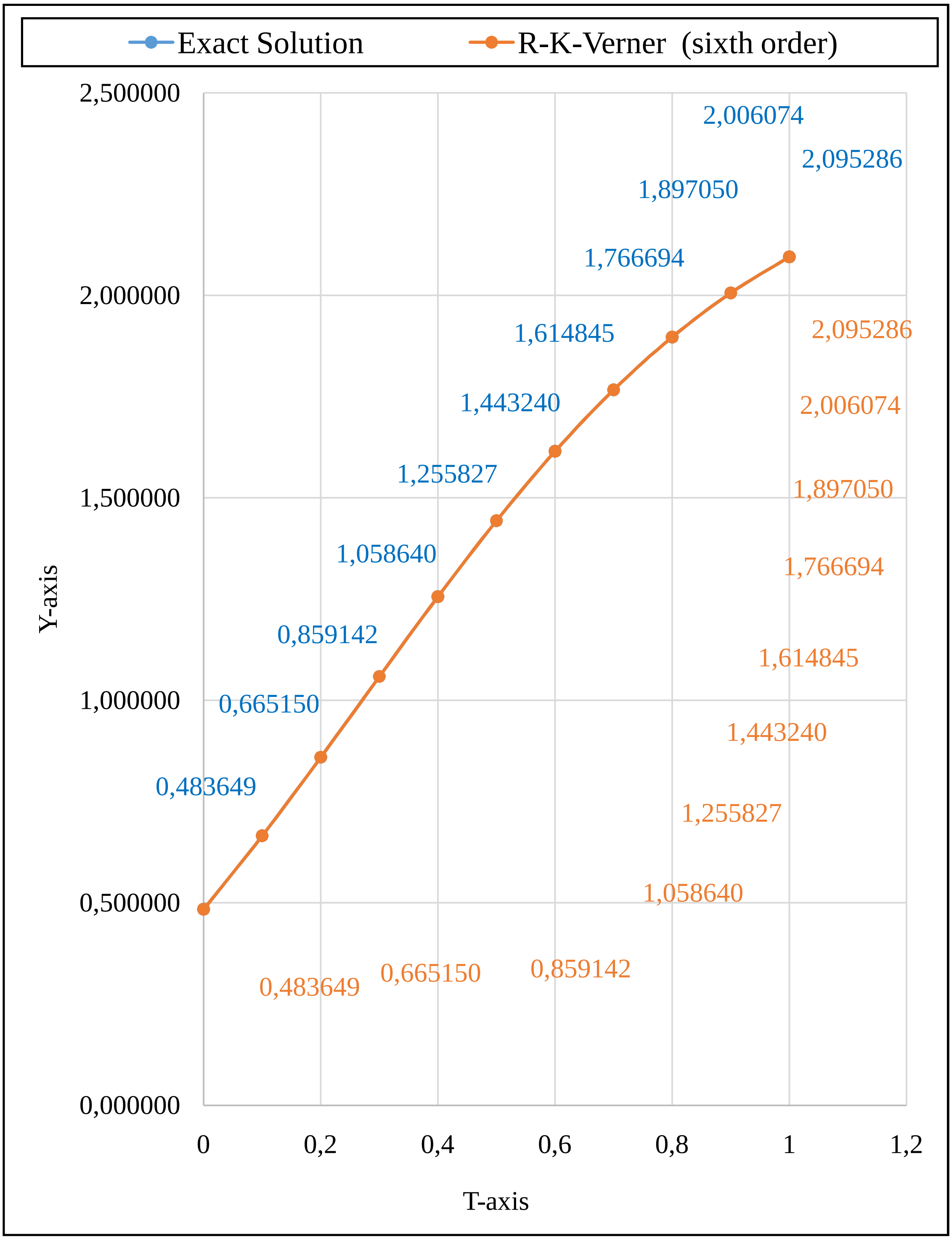

Figure (E.2): R-KVerner method of the sixth order and exact solutions for example (E.1) when $h = 0.1$.

# Appendix F. Approximation Solution from System of Differential Equations

**Example (F.1):**

By using the Runge-Kutta method of the fourth order transform the second-order initial value problem

$$ty'' - y' - t^3 y = 0 \quad , \quad for \;\; 1 \le t \le 2 \quad , y(1) = 2.776347 \quad , y'(1) = 2.169817$$

Into the system of the first-order initial value problems, to determine the approximate solution when $h = 0.1$ , Tolerance $=10^{-6}$ , and if the actual solutions given as follows

$$y(t) = \frac{3}{2}\exp\left(\frac{t^2}{2}\right) + \frac{1}{2}\exp\left(-\frac{t^2}{2}\right)$$

$$y'(t) = \frac{3}{2}\,t\exp\left(\frac{t^2}{2}\right) - \frac{1}{2}\,t\exp\left(-\frac{t^2}{2}\right) .$$

**Solution:**

Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$ , then by an assumption of the transformation as mentioned earlier, then it can be transform the second-order equation into the system of the first order as;

$$u_1'(t) = u_2(t) = f_1(t, u_1, u_2)$$

$$u_2'(t) = \frac{1}{t}u_2 + t^2 u_1 = f_2(t, u_1, u_2)$$

, with the initial condition $u_1(1) = y(1) = 2.776347$ and $u_2(1) = y'(1) = 2.169817$.

Now, we are ready to implement the Runge-Kutta method of the order four which its formulas is given in the equation (4.5.3),(4.5.4) as;

$$k_1 = hf_1(t_0, u_1, u_2) = 0.1(u_2) = 0.1(2.169817) = 0.216982;$$

$$l_1 = hf_2(t_0, u_1, u_2) = 0.1\left(\frac{1}{t_0}u_2 + t_0^2 u_1\right) = 0.1\left(\frac{1}{1}(2.169817) + (1)^2(2.776347)\right)$$

$$= 0.494616;$$

$$k_2 = \square f_1 \ \ t_0 + \frac{\square}{2}, u_1 + \frac{k_1}{2}, u_2 + \frac{l_1}{2} \ = 0.1 \ \ u_2 + \frac{1}{2} l_1$$

$$= 0.1 \ \ 2.169817 + \frac{1}{2} \ 0.494616 \quad = 0.241712;$$

$$l_2 = \square f_2 \ \ t_0 + \frac{\square}{2}, u_1 + \frac{k_1}{2}, u_2 + \frac{l_1}{2}$$

$$= \square \quad \frac{1}{t_0 + \frac{1}{2}\square} \quad u_2 + \frac{1}{2} l_1 \ + \ t_0 + \frac{1}{2}\square \ ^2 \ u_1 + \frac{1}{2} k_1$$

$$= 0.1 \quad \frac{1}{1 + \frac{1}{2} \ 0.1} \quad 2.169817$$

$$+ \frac{1}{2} \ 0.494616 \quad + \quad 1 + \frac{1}{2} \ 0.1 \ ^2 \quad 2.776347 + \frac{1}{2} \ 0.216982$$

$$= 0.548256;$$

$$k_3 = \square f_1 \ \ t_0 + \frac{\square}{2}, u_1 + \frac{k_2}{2}, u_2 + \frac{l_2}{2} \ = \square \ u_2 + \frac{1}{2} l_2 \ = 0.1 \ \ 2.169817 + \frac{1}{2} \ 0.548256$$

$$= 0.244394;$$

$$l_3 = \square f_2 \ \ t_0 + \frac{\square}{2}, u_1 + \frac{k_2}{2}, u_2 + \frac{l_2}{2}$$

$$= \square \quad \frac{1}{t_0 + \frac{1}{2}\square} \quad u_2 + \frac{1}{2} l_2 \ + \ t_0 + \frac{1}{2}\square \ ^2 \ u_1 + \frac{1}{2} k_2$$

$$= 0.1 \quad \frac{1}{1 + \frac{1}{2} \ 0.1} \quad 2.169817 + \frac{1}{2} \ 0.548256$$

$$+ \quad 1 + \frac{1}{2} \ 0.1 \ ^2 \quad 2.776347 + \frac{1}{2} \ 0.241712 \quad = 0.552173;$$

$$k_4 = \square f_1 \ t_0 + \square, u_1 + k_3, u_2 + l_3 \ = \square \ u_2 + l_3 \ = 0.1 \ 2.169817 + 0.552173$$

$$= 0.272199;$$

$$l_4 = hf_2\left(t_0 + h, u_1 + k_3, u_2 + l_3\right) = h\left[\frac{1}{t_0 + h}\left(u_2 + l_3\right) + \left(t_0 + h\right)^2\left(u_1 + k_3\right)\right]$$

$$= 0.1\left[\frac{1}{1 + 0.1}\left(2.169817 + 0.552173\right)\right.$$

$$\left. + \left(1 + 0.1\right)^2\left(2.776347 + 0.244394\right)\right] = 0.612963.$$

So by equation (4.5.3), we obtain;

$$w_{1,1} = w_{1,0} + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 2.776347 + \frac{1}{6}\left(0.216982 + 2\left(0.241712\right) + 2\left(0.244394\right) + 0.272199\right)$$

$$= 3.019913$$

And

$$w_{2,1} = w_{2,0} + \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

$$= 2.169817 + \frac{1}{6}\left(0.494616 + 2\left(0.548256\right) + 2\left(0.552173\right) + 0.612963\right)$$

$$= 2.721223.$$

From the algorithm above, Observe that to the evaluation of the coefficients we must be calculate $k_1$before$l_1$, $k_2$ before $l_2$and so on, because $l_1$, $l_2$depended on the $k_1$, $k_2$respectively, after that we were agreed to calculating each of the $w_{1,j}$ and $w_{2,j}$ and then because in this example especially the differential equation in the situate of the second order then naturally it is transformed only to the two first order differential equation.

Normally, we have done successfully the computations by using the Runge-Kutte method for the fourth order to calculate the value of $w_{1,1} = 3.019913$ since the actual solution corresponded to the amount value is $u_{1,1} = 3.019916$and $w_{2,1} = 2.721223$ since the actual solution corresponded to this amount value is $u_{2,1} = 2.721225$ and so on until preserved the last stage which is $w_{1,10} = 11.151011$ however the exact solution corresponded to this approximate solution is $u_{1,10} = 11.151252$ and also $w_{2,10} = 22.031481$ however the exact solution corresponded to this approximation solution is $u_{2,10} = 22.031833$ , it is meaning or indicate that we can evaluate each of$w_{1,j}$ and $w_{2,j}$for $j = 0,1,2,\ldots\ldots\ldots,10$, as represented the calculations from the table (F.1). Finally, considered the illustrations above by graphical representation in the figure (F.1), also measured the absolute and relative percentage of errors in the table (F.3). Thus, we are fortunately conserved the accurate numerical solution compared to the exact solution.

Table (F.1): Illustration of R-K method for order four and Exact solution when $h = 0.1$.

| $i$ | $h$ | $t_i$ | $y(t) = u_{1,j}$ exact | $w_{1,j}$ RK (order 4) | $y'(t) = u_{2,j}$ exact | $w_{2,j}$ RK (order 4) |
|---|---|---|---|---|---|---|
| 0 | 0.1 | 1 | 2.776347 | 2.776347 | 2.169817 | 2.169817 |
| 1 | 0.1 | 1.1 | 3.019916 | 3.019913 | 2.721225 | 2.721223 |
| 2 | 0.1 | 1.2 | 3.325026 | 3.325020 | 3.405928 | 3.405923 |
| 3 | 0.1 | 1.3 | 3.706745 | 3.706734 | 4.260344 | 4.260334 |
| 4 | 0.1 | 1.4 | 4.184340 | 4.184321 | 5.332640 | 5.332622 |
| 5 | 0.1 | 1.5 | 4.782652 | 4.782621 | 6.686999 | 6.686967 |
| 6 | 0.1 | 1.6 | 5.533978 | 5.533931 | 8.409506 | 8.409454 |
| 7 | 0.1 | 1.7 | 6.480651 | 6.480580 | 10.616339 | 10.616255 |
| 8 | 0.1 | 1.8 | 7.678585 | 7.678478 | 13.465235 | 13.465100 |
| 9 | 0.1 | 1.9 | 9.202194 | 9.202034 | 17.171668 | 17.171450 |
| 10 | 0.1 | 2 | 11.151252 | 11.151011 | 22.031833 | 22.031481 |

Table (F.2): Illustration of the Coefficients with respect the Runge-Kutte method.

| $K_1$ | $L_1$ | $K_2$ | $L_2$ | $K_3$ | $L_3$ | $K_4$ | $L_4$ |
|---|---|---|---|---|---|---|---|
| 0.216982 | 0.494616 | 0.241712 | 0.548256 | 0.244394 | 0.552173 | 0.272199 | 0.612963 |
| 0.272122 | 0.612793 | 0.302762 | 0.680649 | 0.306155 | 0.685625 | 0.340685 | 0.762858 |
| 0.340592 | 0.762630 | 0.378724 | 0.849122 | 0.383048 | 0.855561 | 0.426148 | 0.954470 |
| 0.426033 | 0.954156 | 0.473741 | 1.065294 | 0.479298 | 1.073758 | 0.533409 | 1.201469 |
| 0.533262 | 1.201028 | 0.593314 | 1.344994 | 0.600512 | 1.356272 | 0.668889 | 1.522514 |
| 0.668697 | 1.521888 | 0.744791 | 1.709862 | 0.754190 | 1.725067 | 0.841203 | 1.943176 |
| 0.840945 | 1.942277 | 0.938059 | 2.189607 | 0.950426 | 2.210322 | 1.061978 | 2.498672 |
| 1.061626 | 2.497373 | 1.186494 | 2.825236 | 1.202887 | 2.853724 | 1.346998 | 3.237776 |
| 1.346510 | 3.235888 | 1.508304 | 3.673680 | 1.530194 | 3.713200 | 1.717830 | 4.228452 |
| 1.717145 | 4.225695 | 1.928430 | 4.814484 | 1.957869 | 4.869752 | 2.204120 | 5.566021 |
| 2.203148 | 5.561979 | 2.481247 | 6.359513 | 2.521124 | 6.437401 | 2.846888 | 7.385073 |

Table (F.3): Illustration the relative and relative percentage of errors.

| $i$ | Absolute Error $u_{1,j}$ R-K method (order 4) | Absolute Error $u_{2,j}$ R-K method (order 4) | % Relative Error $u_{1,j}$ R-K method (order 4) | % Relative Error $u_{2,j}$ R-K method (order 4) |
|---|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 0.00 | 0.00 |
| 1 | 0.000003 | 0.000002 | 0.00 | 0.00 |
| 2 | 0.000006 | 0.000006 | 0.00 | 0.00 |
| 3 | 0.000012 | 0.000011 | 0.00 | 0.00 |
| 4 | 0.000019 | 0.000019 | 0.00 | 0.00 |
| 5 | 0.000030 | 0.000031 | 0.00 | 0.00 |
| 6 | 0.000047 | 0.000051 | 0.00 | 0.00 |
| 7 | 0.000071 | 0.000083 | 0.00 | 0.00 |
| 8 | 0.000107 | 0.000135 | 0.00 | 0.00 |
| 9 | 0.000160 | 0.000218 | 0.00 | 0.00 |
| 10 | 0.000240 | 0.000352 | 0.00 | 0.00 |

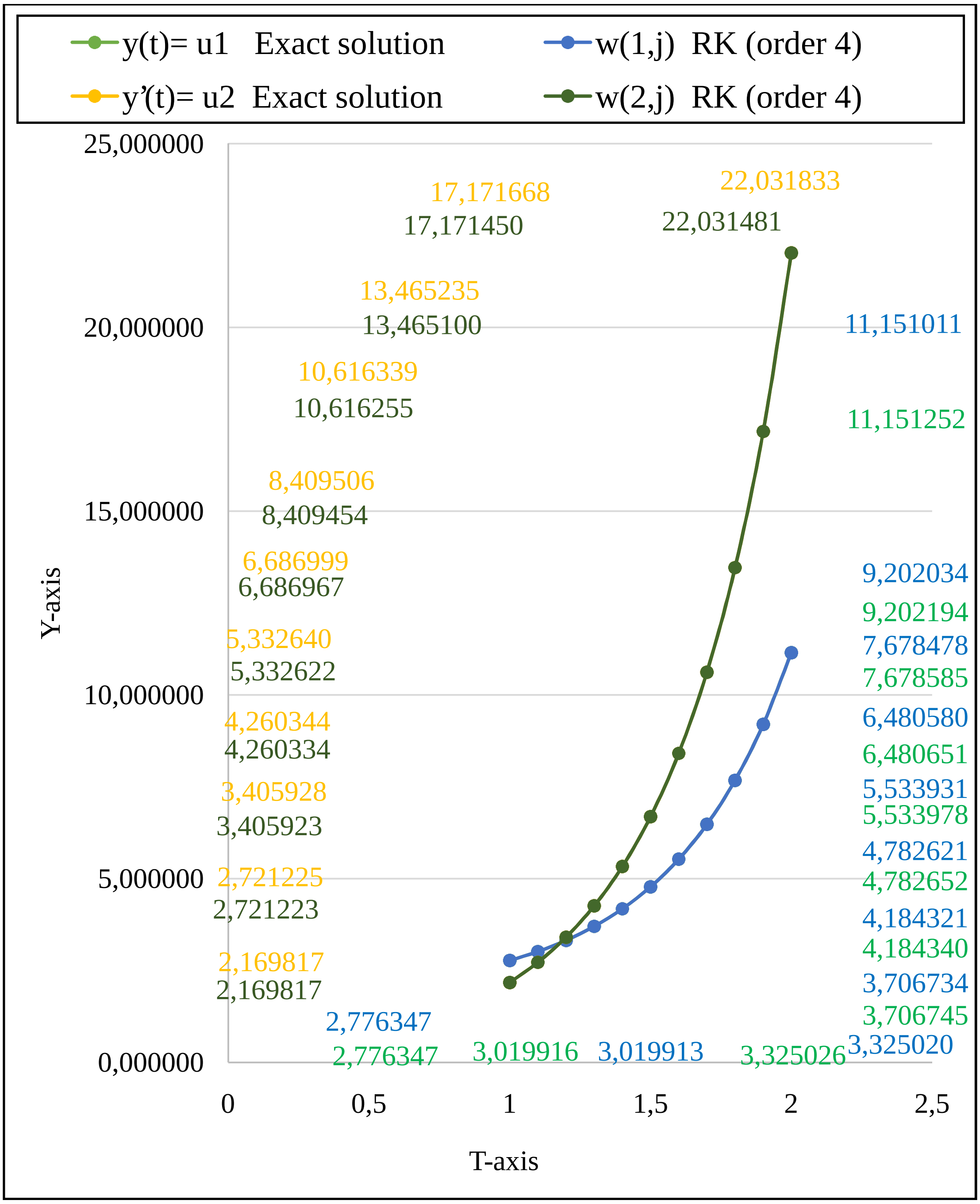


Figure (F.1): Approximate and exact solutions for example (F.1) when ℎ = 0.1.

**Special solution of the example (F.1):**

In actually we bring this example because the equation is a second order and it's linear equation therefore it can be transformed to the Riccati differential equation. In fact that, the general purpose of our working throughout of this thesis is the investigation over this type of the ordinary differential equations. So, we want show that how to convert this linear equation into the Riccati differential equation by take a some assumptions;

Since the equation is given as;

$$ty'' - y' - t^3 y = 0$$

$$y = e^{\ y_1 dt} \ , \ y' = y_1 e^{\ y_1 dt} \ , \quad y'' = {y_1}^2 e^{\ y_1 dt} + {y'}_1 e^{\ y_1 dt}$$

, then by substitute the right hand side of each of $y$ , $y'$ $and$ $y''$in the linear second order equation, we can get the Riccati differential equation as follows;

$t\ {y_1}^2 e^{\ y_1 dt} + {y'}_1 e^{\ y_1 dt} \ - \ y_1 e^{\ y_1 dt} \ - t^3 \ e^{\ y_1 dt} \ = 0$

$t{y_1}^2 e^{\ y_1 dt} + y_1' \, t \, e^{\ y_1 dt} - y_1 e^{\ y_1 dt} - t^3 e^{\ y_1 dt} = 0$

$e^{\ y_1 dt} \ \ t\, y_1^2 + y_1' \, t - y_1 - t^3 \ = 0$

$t\, y_1^2 + y_1' \, t - y_1 - t^3 = 0$ , both sides divide by ( $t$ ), and then

$y_1^2 + y_1' - \frac{1}{t} y_1 - t^2 = 0$

$y_1' = t^2 + \frac{1}{t} y_1 - y_1^2$ , and we set$y_1 = t$ to be a solution of the Riccati differential equation, to verify that substitute the value of $y_1$in the RDE and also take the differentiation of the $y_1$ and then note that;

$$y_1' = t^2 + \frac{1}{t} \, t - t^2 = 1$$

And

$y_1' = 1$

then, clearly the right hand sides are equal meaning that $y_1 = t$ when it is devoted as a solution of the Riccati differential equation has been verified [13]. Accordinglyto the illustration of the given example in the above then we can state that although the given equation in the situate of second order, we can make the Riccati differential equation from the second order DE by substitute the desired assumption in the equation of second order and then we are achieve the RDE. Note that this illustration considered only for explain that how to make the RDE from the second order differential equation.

**Example (F.2):**

By using the Runge-Kutta-Verner method of the fifth and sixth orders in order to transform the second-order initial value problem

$$t^2y'' + ty' - 4y = t^3 \quad , \quad for \;\; 1 \le t \le 2 \quad , y(1) = 2 \quad , y'(1) = 1$$

Into the system of the first-order initial value problems, to determine the approximate solution when $h = 0.1$ , Tolerance $=10^{-5}$ , and if the actual solutions given as follows

$$y(t) = t^2 + \frac{4}{5}t^{-2} + \frac{1}{5}t^3 \qquad and \qquad y'(t) = 2t - \frac{8}{5}t^{-3} + \frac{3}{5}t^2.$$

**Solution:** the given second order differential equation can be expressed as the easier form;

$$t^2y'' + ty' - 4y = t^3 \quad \Rightarrow \quad y'' + \frac{1}{t}y' - \frac{4}{t^2}y = t \quad \Rightarrow \quad y'' = -\frac{1}{t}y' + \frac{4}{t^2}y + t$$

Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$, and by take the differentiation, yields;

$$u_1'(t) = u_2(t) = f_1(t, u_1, u_2)$$

$$u_2'(t) = -\frac{1}{t}u_2 + \frac{4}{t^2}u_1 + t = f_2(t, u_1, u_2)$$

Now, the system of the first-order equations has been succeeded then we can apply the preferred Runge-Kutta-Verner method, at the beginning, we can calculate the coefficients as follows;

Since the initial conditions given as

$w_{1,0} = y(t_0) = y(1) = 2$ and $w_{2,0} = y'(t_0) = y'(1) = 1$, thus

$$k_1 = hf_1(t_0, u_1, u_2) = h(u_2) = 0.1(1) = 0.10000;$$

$$l_1 = hf_2(t_0, u_1, u_2) = h\left(-\frac{1}{t}u_2 + \frac{4}{t^2}u_1 + t\right) = 0.1\left(-\frac{1}{1}(1) + \frac{4}{(1)^2}(2) + 1\right) = 0.80000;$$

$$k_2 = hf_1\left(t_0 + \frac{1}{6}h, u_1 + \frac{1}{6}k_1, u_2 + \frac{1}{6}l_1\right) = h\left(u_2 + \frac{1}{6}l_1\right) = 0.1\left(1 + \frac{1}{6}(0.80000)\right)$$

$$= 0.11333;$$

$$l_2 = hf_2\left(t_0 + \frac{1}{6}h, u_1 + \frac{1}{6}k_1, u_2 + \frac{1}{6}l_1\right)$$

$$= h\left(-\frac{1}{\left(t + \frac{1}{6}h\right)}\left(u_2 + \frac{1}{6}l_1\right) + \frac{4}{\left(t + \frac{1}{6}h\right)^2}\left(u_1 + \frac{1}{6}k_1\right)\right.$$

$$\left. + \left(t + \frac{1}{6}h\right)\right)$$

$$= \; 0.1 \left( -\frac{1}{1+\frac{1}{6}\,0.1} \left(1+\frac{1}{6}\,0.80000\right) \right.$$

$$\left. + \frac{4}{\left(1+\frac{1}{6}\,0.1\right)^2}\left(2+\frac{1}{6}\,0.10000\right) + \left(1+\frac{1}{6}\,0.1\right) \right) = 0.78363;$$

$$k_3 = \square f_1\left(t_0 + \frac{4}{15}\square, u_1 + \frac{4}{75}k_1 + \frac{16}{75}k_2, u_2 + \frac{4}{75}l_1 + \frac{16}{75}l_2\right)$$

$$k_3 = \square\left(u_2 + \frac{4}{75}l_1 + \frac{16}{75}l_2\right) = 0.1\left(1 + \frac{4}{75}\,0.80000 + \frac{16}{75}\,0.78363\right) = 0.12098;$$

$$l_3 = \square f_2\left(t_0 + \frac{4}{15}\square, u_1 + \frac{4}{75}k_1 + \frac{16}{75}k_2, u_2 + \frac{4}{75}l_1 + \frac{16}{75}l_2\right)$$

$$l_3 = \square\left( -\frac{1}{(t_0+\frac{4}{15}\square)}\left(u_2 + \frac{4}{75}l_1 + \frac{16}{75}l_2\right) + \frac{4}{(t_0+\frac{4}{15}\square)^2}\left(u_1 + \frac{4}{75}k_1 + \frac{16}{75}k_2\right)\right.$$

$$\left. + \left(t_0 + \frac{4}{15}\square\right)\right) =$$

$$= \; 0.1\left( -\frac{1}{1+\frac{4}{15}\,0.1}\left(1 + \frac{4}{75}\,0.80000 + \frac{16}{75}\,0.78363\right)\right.$$

$$+ \frac{4}{\left(1+\frac{4}{15}\,0.1\right)^2}\left(2 + \frac{4}{75}\,0.10000 + \frac{16}{75}\,0.11333\right)$$

$$\left. + \left(1 + \frac{4}{15}\,0.1\right)\right) = 0.75501;$$

$$k_4 = \square f_1\left(t_0 + \frac{2}{3}\square, u_1 + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3, u_2 + \frac{5}{6}l_1 - \frac{8}{3}l_2 + \frac{5}{2}l_3\right)$$

$$k_4 = \square\left(u_2 + \frac{5}{6}l_1 - \frac{8}{3}l_2 + \frac{5}{2}l_3\right) = 0.1\left(1 + \frac{5}{6}\,0.80000 - \frac{8}{3}\,0.78363 + \frac{5}{2}\,0.75501\right)$$

$$= 0.14645;$$

$$l_4 = \square f_2\left(t_0 + \frac{2}{3}\square, u_1 + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3, u_2 + \frac{5}{6}l_1 - \frac{8}{3}l_2 + \frac{5}{2}l_3\right)$$

$$l_4 = h\left[-\frac{1}{(t_0 + \frac{2}{3}h)}\left(u_2 + \frac{5}{6}l_1 - \frac{8}{3}l_2 + \frac{5}{2}l_3\right)\right.$$

$$\left. + \frac{4}{(t_0 + \frac{2}{3}h)^2}\left(u_1 + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right) + \left(t_0 + \frac{2}{3}h\right)\right]$$

$$l_4 = 0.1\left[-\frac{1}{1 + \frac{2}{3}(0.1)}\left(1 + \frac{5}{6}(0.80000) - \frac{8}{3}(0.78363) + \frac{5}{2}(0.75501)\right)\right.$$

$$+ \frac{4}{\left(1 + \frac{2}{3}(0.1)\right)^2}\left(2 + \frac{5}{6}(0.10000) - \frac{8}{3}(0.11333) + \frac{5}{2}(0.12098)\right)$$

$$\left. + \left(1 + \frac{2}{3}(0.1)\right)\right] = 0.70188;$$

$$k_5 = hf_1\left(t_0 + \frac{5}{6}h, u_1 - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4, u_2 - \frac{165}{64}l_1 + \frac{55}{6}l_2 - \frac{425}{64}l_3\right.$$

$$\left. + \frac{85}{96}l_4\right)$$

$$k_5 = h\left(u_2 - \frac{165}{64}l_1 + \frac{55}{6}l_2 - \frac{425}{64}l_3 + \frac{85}{96}l_4\right)$$

$$= 0.1\left(1 - \frac{165}{64}(0.10000) + \frac{55}{6}(0.78363) - \frac{425}{64}(0.75501)\right.$$

$$\left. + \frac{85}{96}(0.70188)\right) = 0.17286;$$

$$l_5 = hf_2\left(t_0 + \frac{5}{6}h, u_1 - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4, u_2 - \frac{165}{64}l_1 + \frac{55}{6}l_2 - \frac{425}{64}l_3\right.$$

$$\left. + \frac{85}{96}l_4\right)$$

$$l_5 = h\left[-\frac{1}{t_0 + \frac{5}{6}h}\left(u_2 - \frac{165}{64}l_1 + \frac{55}{6}l_2 - \frac{425}{64}l_3 + \frac{85}{96}l_4\right)\right.$$

$$\left. + \frac{4}{\left(t_0 + \frac{5}{6}h\right)^2}\left(u_1 - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4\right) + \left(t_0 + \frac{5}{6}h\right)\right]$$

$$l_5 = \ 0.1 \quad - \frac{1}{1 + \frac{5}{6}\ 0.1} \quad 1 - \frac{165}{64}\ 0.80000 \ + \frac{55}{6}\ 0.78363 \ - \frac{425}{64}\ 0.75501$$

$$+ \frac{85}{96}\ 0.70188$$

$$+ \quad \frac{4}{1 + \frac{5}{6}\square^{\ 2}} \quad 2 - \frac{165}{64}\ 0.10000 \ + \frac{55}{6}\ 0.11333 \ - \frac{425}{64}\ 0.12098$$

$$+ \frac{85}{96}\ 0.14645 \quad + \quad 1 + \frac{5}{6}\ 0.1 \qquad = 0.66701;$$

$$k_6 = \square f_1 \ \ t_0 + \ \square, u_1 + \frac{12}{5}k_1 - \ 8\ k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5, u_2 + \frac{12}{5}l_1 - 8\ l_2$$

$$+ \frac{4015}{612}l_3 - \frac{11}{36}l_4 + \frac{88}{255}l_5$$

$$k_6 = \square \ \ u_2 + \frac{12}{5}l_1 - 8\ l_2 + \frac{4015}{612}l_3 - \frac{11}{36}l_4 + \frac{88}{255}l_5$$

$$= \ 0.1 \ \ 1 + \frac{12}{5}\ 0.80000 \ - \ 8 \ \ 0.78363 \ + \frac{4015}{612}\ 0.75501$$

$$- \frac{11}{36}\ 0.70188 \ + \frac{88}{255}\ 0.66701 \quad = \ 0.16198;$$

$$l_6 = \square f_2 \ \ t_0 + \ \square, u_1 + \frac{12}{5}k_1 - \ 8\ k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5, u_2 + \frac{12}{5}l_1 - \ 8\ l_2$$

$$+ \frac{4015}{612}l_3 - \frac{11}{36}l_4 + \frac{88}{255}l_5$$

$$l_6 = \ \square \quad - \frac{1}{t_0 + \ \square} \quad u_2 + \frac{12}{5}l_1 - 8\ l_2 + \frac{4015}{612}l_3 - \frac{11}{36}l_4 + \frac{88}{255}l_5$$

$$+ \quad \frac{4}{t_0 + \ \square^{\ 2}} \quad u_1 + \frac{12}{5}k_1 - \ 8\ k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5$$

$$+ \quad t_0 + \ \square$$

$$l_6 = \ 0.1 \quad - \frac{1}{1+\ 0.1} \quad 1 + \frac{12}{5}\ 0.80000\ - 8\ \ 0.78363\ + \frac{4015}{612}\ 0.75501$$

$$- \frac{11}{36}\ 0.70188\ + \frac{88}{255}\ 0.66701$$

$$+ \quad \frac{4}{1+\ 0.1\ ^{2}} \quad 2 + \frac{12}{5}\ 0.10000\ - 8\ \ 0.11333\ + \frac{4015}{612}\ 0.12098$$

$$- \frac{11}{36}\ 0.14645\ + \frac{88}{255}\ 0.17286 \quad + \quad 1 +\ 0.1 \quad = 0.67082;$$

$$k_7 = \square f_1\ \ t_0 + \frac{1}{15}\square, u_1 - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5, u_2$$

$$- \frac{8263}{15000}l_1 + \frac{124}{75}l_2 - \frac{643}{680}l_3 - \frac{81}{250}l_4 + \frac{2484}{10625}l_5$$

$$k_7 = \square\ \ u_2 - \frac{8263}{15000}l_1 + \frac{124}{75}l_2 - \frac{643}{680}l_3 - \frac{81}{250}l_4 + \frac{2484}{10625}l_5$$

$$k_7 = \ 0.1\ \ 1 - \frac{8263}{15000}\ 0.80000\ + \frac{124}{75}\ 0.78363\ - \frac{643}{680}\ 0.75501\ - \frac{81}{250}\ 0.70188$$

$$+ \frac{2484}{10625}\ 0.66701 \quad = \ 0.10695;$$

$$l_7 = \square f_2\ \ t_0 + \frac{1}{15}\square, u_1 - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5, u_2$$

$$- \frac{8263}{15000}l_1 + \frac{124}{75}l_2 - \frac{643}{680}l_3 - \frac{81}{250}l_4 + \frac{2484}{10625}l_5$$

$$l_7 = \square \quad - \frac{1}{t_0 + \frac{1}{15}\square} \quad u_2 - \frac{8263}{15000}l_1 + \frac{124}{75}l_2 - \frac{643}{680}l_3 - \frac{81}{250}l_4 + \frac{2484}{10625}l_5$$

$$+ \quad \frac{4}{(t_0 + \frac{1}{15}\square)^2} \quad u_1 - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4$$

$$+ \frac{2484}{10625}k_5 \quad + \quad t_0 + \frac{1}{15}\square \quad ;$$

$$l_7 = 0.1 \quad - \frac{1}{1 + \frac{1}{15}\ 0.1} \quad 1 - \frac{8263}{15000}(0.80000) + \frac{124}{75}(0.78363)$$

$$- \frac{643}{680}(0.75501) - \frac{81}{250}(0.70188) + \frac{2484}{10625}(0.66701)$$

$$+ \frac{4}{1 + \frac{1}{15}\ 0.1}^{2} \quad 2 - \frac{8263}{15000}(0.10000) + \frac{124}{75}(0.11333)$$

$$- \frac{643}{680}(0.12098) - \frac{81}{250}(0.14645) + \frac{2484}{10625}(0.17286)$$

$$+ \quad 1 + \frac{1}{15}(0.1) \quad = 0.78815;$$

$$k_8 = \square f_1 \quad t_0 + \square, u_1 + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297575}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5$$

$$+ \frac{3850}{26703}k_7, u_2 + \frac{3501}{1720}l_1 - \frac{300}{43}l_2 + \frac{297575}{52632}l_3 - \frac{319}{2322}l_4 + \frac{24068}{84065}l_5$$

$$+ \frac{3850}{26703}l_7$$

$$k_8 = \square\{u_2 + \frac{3501}{1720}l_1 - \frac{300}{43}l_2 + \frac{297575}{52632}l_3 - \frac{319}{2322}l_4 + \frac{24068}{84065}l_5 + \frac{3850}{26703}l_7\}$$

$$k_8 = \quad 0.1 \quad 1 + \frac{3501}{1720}\ 0.80000 \ - \frac{300}{43}\ 0.78363 \ + \frac{297575}{52632}\ 0.75501$$

$$- \frac{319}{2322}\ 0.70188 \ + \frac{24068}{84065}\ 0.66701 \ + \frac{3850}{26703}\ 0.78815 \quad = 0.16337;$$

$$l_8 = \square f_2 \quad t_0 + \square, u_1 + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297575}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5$$

$$+ \frac{3850}{26703}k_7, u_2 + \frac{3501}{1720}l_1 - \frac{300}{43}l_2 + \frac{297575}{52632}l_3 - \frac{319}{2322}l_4 + \frac{24068}{84065}l_5$$

$$+ \frac{3850}{26703}l_7$$

$$l_8 = \square \quad -\frac{1}{t_0 + \square} \quad u_2 + \frac{3501}{1720} l_1 - \frac{300}{43} l_2 + \frac{297575}{52632} l_3 - \frac{319}{2322} l_4 + \frac{24068}{84065} l_5$$
$$+ \frac{3850}{26703} l_7$$
$$+ \quad \frac{4}{t_0 + \square^{\;2}} \quad u_1 + \frac{3501}{1720} k_1 - \frac{300}{43} k_2 + \frac{297575}{52632} k_3 - \frac{319}{2322} k_4$$
$$+ \frac{24068}{84065} k_5 + \frac{3850}{26703} k_7 \quad + \quad t_0 + \square \quad ;$$

$$l_8 = \quad 0.1 \quad -\frac{1}{1 + 0.1} \quad 1 + \frac{3501}{1720}\ 0.80000 - \frac{300}{43}\ 0.78363 + \frac{297575}{52632}\ 0.75501$$
$$- \frac{319}{2322}\ 0.70188 + \frac{24068}{84065}\ 0.66701 + \frac{3850}{26703}\ 0.78815$$
$$+ \quad \frac{4}{1 + 0.1^{\;2}} \quad 2 + \frac{3501}{1720}\ 0.10000 - \frac{300}{43}\ 0.11333$$
$$+ \frac{297575}{52632}\ 0.12098 - \frac{319}{2322}\ 0.14645 + \frac{24068}{84065}\ 0.17286$$
$$+ \frac{3850}{26703}\ 0.10695 \quad + \quad 1 + 0.1 \quad = 0.67193.$$

So The Runge-Kutta-Verner method for fifth-order is given as follows;

$$w_{1,1} = w_{1,0} + \frac{13}{160} k_1 + \frac{2375}{5984} k_3 + \frac{5}{16} k_4 + \frac{12}{85} k_5 + \frac{3}{44} k_6$$
$$w_{1,1} = \quad 2.00000 + \frac{13}{160}\ 0.10000 + \frac{2375}{5984}\ 0.12098 + \frac{5}{16}\ 0.14645 + \frac{12}{85}\ 0.17286$$
$$+ \frac{3}{44}\ 0.16198 \quad = 2.13736.$$

And

$$w_{2,1} = w_{2,0} + \frac{13}{160} l_1 + \frac{2375}{5984} l_3 + \frac{5}{16} l_4 + \frac{12}{85} l_5 + \frac{3}{44} l_6$$
$$w_{2,1} = \quad 1.00000 + \frac{13}{160}\ 0.80000 + \frac{2375}{5984}\ 0.75501 + \frac{5}{16}\ 0.70188 + \frac{12}{85}\ 0.66701$$
$$+ \frac{3}{44}\ 0.67082 \quad = 1.72390.$$

Also the Runge-Kutta -Verner method for sixth-order is given as follows;

$$\widetilde{w}_{1,1} = w_{1,0} + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7\frac{43}{616}k_8$$

$$\widetilde{w}_{1,1} = 2.00000 + \frac{3}{40}\ 0.10000 + \frac{875}{2244}\ 0.12098 + \frac{23}{72}\ 0.14645 + \frac{264}{1955}\ 0.17286 + \frac{125}{11592}\ 0.10695 + \frac{43}{616}\ 0.16337 = 2.13736.$$

And

$$\widetilde{w}_{2,1} = w_{2,0} + \frac{3}{40}l_1 + \frac{875}{2244}l_3 + \frac{23}{72}l_4 + \frac{264}{1955}l_5 + \frac{125}{11592}l_7\frac{43}{616}l_8$$

$$\widetilde{w}_{2,1} = 1.00000 + \frac{3}{40}\ 0.80000 + \frac{875}{2244}\ 0.75501 + \frac{23}{72}\ 0.70188 + \frac{264}{1955}\ 0.66701 + \frac{125}{11592}\ 0.78815 + \frac{43}{616}\ 0.67193 = 1.72408.$$

As illustrated the computations from the algorithm above, notice that we must be calculate $k_1$before$l_1$, $k_2$before $l_2$and so on because each of the coefficients $l_1, l_2, l_3, \ldots$ , $l_8$depended on the coefficients $k_1, k_2, k_3, \ldots$ , $k_8$respectively. Also observe that for computing the Runge-Kutta-Verner method should be able to evaluating the sixteen coefficients if you were converting the second-order to the first-order differential equations, therefore if the order of differential equations changed then the procedure of the constructing the system of the differential equations is also changed.

Accordingly, we can expect that the six order Runge-Kutta-Verner method obtained to estimate the error in the order five in their method, the whole calculations registered in the table (F.4) offered the huge exposition through the numerical solutions.

So the amount value of the approximate solution is $w_{1,1} = 2.13736$ with respect to the order fifth and $\widetilde{w}_{1,1} = 2.13736$ with respect to the order sixth since normally the exact solution corresponded to those amount values is $u_{1,1} = 2.13736$ , the value of the approximate solution is $w_{1,2} = 2.34114$ with respect to the fifth order and $\widetilde{w}_{1,2} = 2.34116$ with respect to the order sixth since typically the exact solution corresponded to those amount values is $u_{1,2} = 2.34116$ and so on until to do the last stage which is the amount value of the approximate solution is $w_{1,10} = 5.79982$ with respect to the fifth order and $\widetilde{w}_{1,10} = 5.80000$ with respect to the order sixth since fortunately the exact solution corresponded to those amount values is $u_{1,10} = 5.80000$.

And also the value of the approximate solution is $w_{2,1}$ = 1.72390 with respect to the order fifth and $\widetilde{w}_{2,1}$ = 1.72408 with respect to the order sixth since normally the exact solution corresponded to those amount values is $u_{2,1}$ = 1.72390 , the value of the approximate solution is $w_{2,2}$ = 2.33807 with respect to the fifth order and $\widetilde{w}_{2,2}$ = 2.33822 with respect to the order sixth, since typically the exact solution corresponded to those amount values is $u_{2,2}$ = 2.33807 and so on until to do the last stage which is the amount value of the approximate solution is $w_{2,10}$ = 6.19994 with respect to the fifth order and $\widetilde{w}_{2,10}$ = 6.19742 with respect to the order sixth since fortunately the exact solution corresponded to those amount values is $u_{2,10}$ = 6.20000.

Finally, throughout the table (F.6), (F.7) described the relative and percentage relative of errors, recall that in the preceding sections interpreted their formulas. And also within the figure (F.2) and (F.3) plotted the graphical depictions.

Table (F.4): Illustration of Runge-Kutte-Verner method for order (5 and 6)

| $i$ | ⍰ | $t_i$ | $Y(t) = u_{1,j}$ Exact solution | $w_{1,j}$ R-K-V (5 order) | $\widetilde{w}_{1,j}$ R-K-V (6 order) | $Y'(t) = u_{2,j}$ Exact solution | $w_{2,j}$ R-K-V (5 order) | $\widetilde{w}_{2,j}$ R-K-V (6 order) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 1 | 2.00000 | 2.00000 | 2.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.1 | 1.1 | 2.13736 | 2.13736 | 2.13736 | 1.72390 | 1.72390 | 1.72408 |
| 2 | 0.1 | 1.2 | 2.34116 | 2.34114 | 2.34116 | 2.33807 | 2.33807 | 2.33822 |
| 3 | 0.1 | 1.3 | 2.60277 | 2.60275 | 2.60278 | 2.88573 | 2.88573 | 2.88571 |
| 4 | 0.1 | 1.4 | 2.91696 | 2.91692 | 2.91697 | 3.39291 | 3.39290 | 3.39263 |
| 5 | 0.1 | 1.5 | 3.28056 | 3.28050 | 3.28056 | 3.87593 | 3.87591 | 3.87534 |
| 6 | 0.1 | 1.6 | 3.69170 | 3.69162 | 3.69171 | 4.34538 | 4.34536 | 4.34444 |
| 7 | 0.1 | 1.7 | 4.14942 | 4.14931 | 4.14942 | 4.80833 | 4.80830 | 4.80701 |
| 8 | 0.1 | 1.8 | 4.65331 | 4.65319 | 4.65332 | 5.26965 | 5.26961 | 5.26793 |
| 9 | 0.1 | 1.9 | 5.20341 | 5.20326 | 5.20341 | 5.73273 | 5.73268 | 5.73059 |
| 10 | 0.1 | 2 | 5.80000 | 5.79982 | 5.80000 | 6.20000 | 6.19994 | 6.19742 |

and exact solutions of example (F.2) when ⍰ = 0.1.

Table (F.5): Illustration of the Coefficients with respect the Runge-Kutte-Verner method.

| $k_1$ | $l_1$ | $k_2$ | $l_2$ | $k_3$ | $l_3$ | $k_4$ | $l_4$ |
|---|---|---|---|---|---|---|---|
| 0.10000 | 0.80000 | 0.11333 | 0.78363 | 0.12098 | 0.75501 | 0.14645 | 0.70188 |
| 0.17239 | 0.65985 | 0.18339 | 0.72335 | 0.19134 | 0.63158 | 0.19238 | 0.61897 |
| 0.23381 | 0.57548 | 0.24340 | 0.70412 | 0.25190 | 0.55678 | 0.23320 | 0.56999 |
| 0.28857 | 0.52406 | 0.29731 | 0.71118 | 0.30654 | 0.51104 | 0.27036 | 0.54208 |
| 0.33929 | 0.49294 | 0.34751 | 0.73594 | 0.35762 | 0.48337 | 0.30496 | 0.52765 |
| 0.38759 | 0.47481 | 0.39550 | 0.77312 | 0.40662 | 0.46736 | 0.33783 | 0.52207 |
| 0.43454 | 0.46523 | 0.44229 | 0.81940 | 0.45450 | 0.45909 | 0.36957 | 0.52244 |
| 0.48083 | 0.46146 | 0.48852 | 0.87259 | 0.50191 | 0.45611 | 0.40062 | 0.52688 |
| 0.52696 | 0.46171 | 0.53466 | 0.93125 | 0.54929 | 0.45682 | 0.43131 | 0.53416 |
| 0.57327 | 0.46482 | 0.58101 | 0.99437 | 0.59696 | 0.46017 | 0.46188 | 0.54344 |
| 0.61999 | 0.46999 | 0.62783 | 1.06122 | 0.64514 | 0.46542 | 0.49252 | 0.55416 |
| $k_5$ | $l_5$ | $k_6$ | $l_6$ | $k_7$ | $l_7$ | $k_8$ | $l_8$ |
| 0.17286 | 0.66701 | 0.16198 | 0.67082 | 0.10695 | 0.78815 | 0.16337 | 0.67193 |
| 0.30074 | 0.51368 | 0.16523 | 0.64461 | 0.18787 | 0.65051 | 0.17434 | 0.63425 |
| 0.41161 | 0.41557 | 0.17083 | 0.63254 | 0.25712 | 0.56615 | 0.18641 | 0.61486 |
| 0.51402 | 0.34949 | 0.17616 | 0.63105 | 0.31957 | 0.51350 | 0.19767 | 0.60815 |
| 0.61254 | 0.30310 | 0.18030 | 0.63701 | 0.37809 | 0.48052 | 0.20755 | 0.61010 |
| 0.70974 | 0.26942 | 0.18300 | 0.64812 | 0.43445 | 0.46017 | 0.21600 | 0.61792 |
| 0.80710 | 0.24431 | 0.18433 | 0.66275 | 0.48975 | 0.44819 | 0.22314 | 0.62973 |
| 0.90551 | 0.22518 | 0.18440 | 0.67977 | 0.54474 | 0.44192 | 0.22917 | 0.64427 |
| 1.00551 | 0.21039 | 0.18340 | 0.69840 | 0.59991 | 0.43963 | 0.23427 | 0.66066 |
| 1.10747 | 0.19882 | 0.18148 | 0.71810 | 0.65559 | 0.44020 | 0.23862 | 0.67832 |
| 1.21161 | 0.18972 | 0.17877 | 0.73850 | 0.71203 | 0.44284 | 0.24235 | 0.69683 |

Table (F.6): Illustration the relative and relative percentage of errors.

| $i$ | Absolute Error $w_{1,j}$ RKV (5 order) | Absolute Error $\widetilde{w}_{1,j}$ RKV (6 order) | % Relative Error $w_{1,j}$ RKV (5 order) | % Relative Error $\widetilde{w}_{1,j}$ RKV (6 order) |
|---|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00 | 0.00 |
| 1 | 0.00000 | 0.00000 | 0.00 | 0.00 |
| 2 | 0.00001 | 0.00000 | 0.00 | 0.00 |
| 3 | 0.00002 | 0.00000 | 0.00 | 0.00 |
| 4 | 0.00004 | 0.00001 | 0.00 | 0.00 |
| 5 | 0.00006 | 0.00001 | 0.00 | 0.00 |
| 6 | 0.00008 | 0.00001 | 0.00 | 0.00 |
| 7 | 0.00010 | 0.00001 | 0.00 | 0.00 |
| 8 | 0.00013 | 0.00001 | 0.00 | 0.00 |
| 9 | 0.00015 | 0.00000 | 0.00 | 0.00 |
| 10 | 0.00018 | 0.00000 | 0.00 | 0.00 |

Table (F.7): Illustration the relative and relative percentage of errors

| $i$ | Absolute Error $w_{1,j}$ RKV (5 order) | Absolute Error $\widetilde{w}_{1,j}$ RKV (6 order) | % Relative Error $w_{1,j}$ RKV (5 order) | % Relative Error $\widetilde{w}_{1,j}$ RKV (6 order) |
|---|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00 | 0.00 |
| 1 | 0.00000 | 0.00019 | 0.00 | 0.01 |
| 2 | 0.00000 | 0.00015 | 0.00 | 0.01 |
| 3 | 0.00000 | 0.00002 | 0.00 | 0.00 |
| 4 | 0.00001 | 0.00028 | 0.00 | 0.01 |
| 5 | 0.00001 | 0.00059 | 0.00 | 0.02 |
| 6 | 0.00002 | 0.00094 | 0.00 | 0.02 |
| 7 | 0.00003 | 0.00132 | 0.00 | 0.03 |
| 8 | 0.00004 | 0.00172 | 0.00 | 0.03 |
| 9 | 0.00005 | 0.00214 | 0.00 | 0.04 |
| 10 | 0.00006 | 0.00258 | 0.00 | 0.04 |

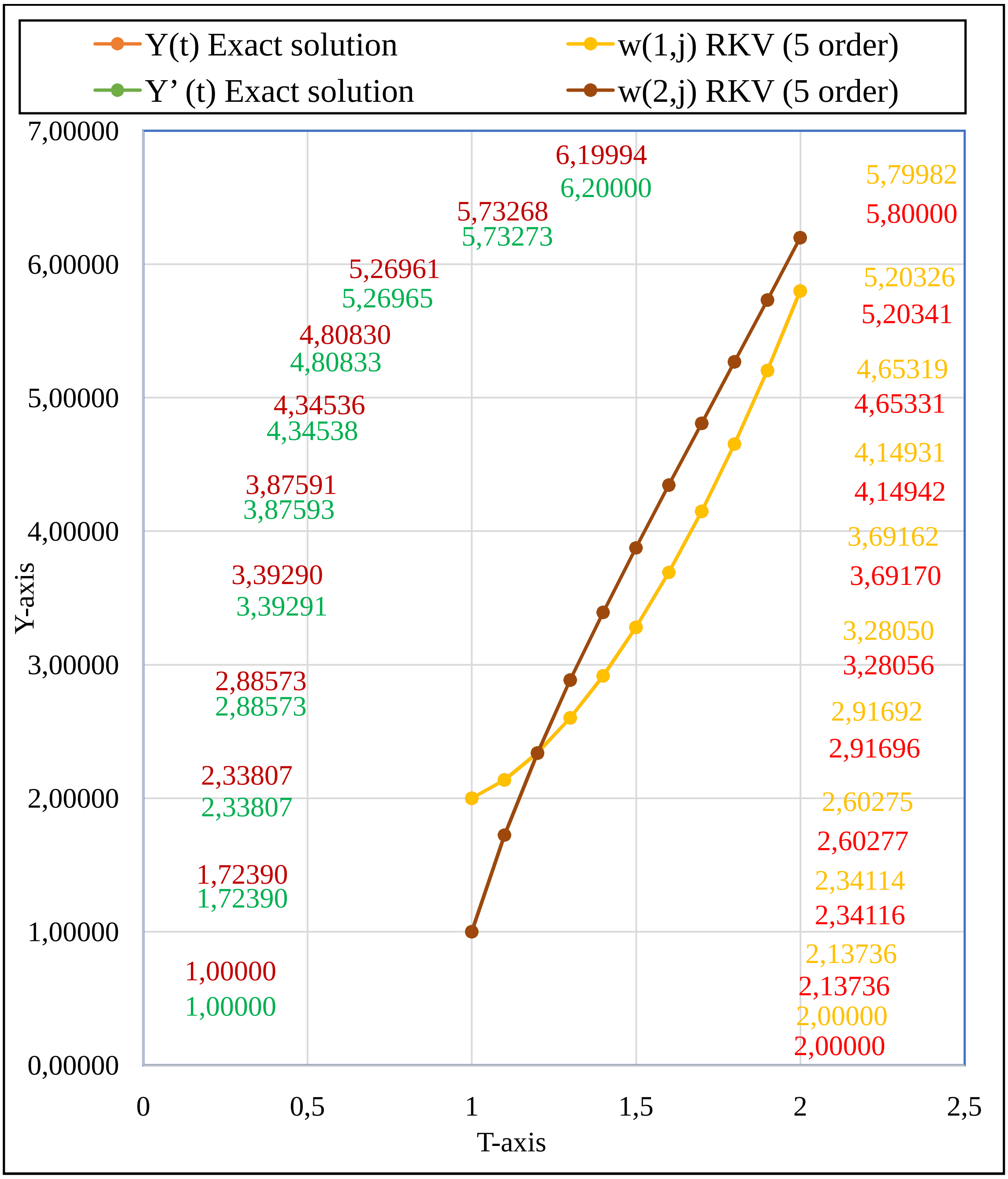


Figure (F.2): Approximate and exact solutions for example (F.2)

when ℎ = 0.1.

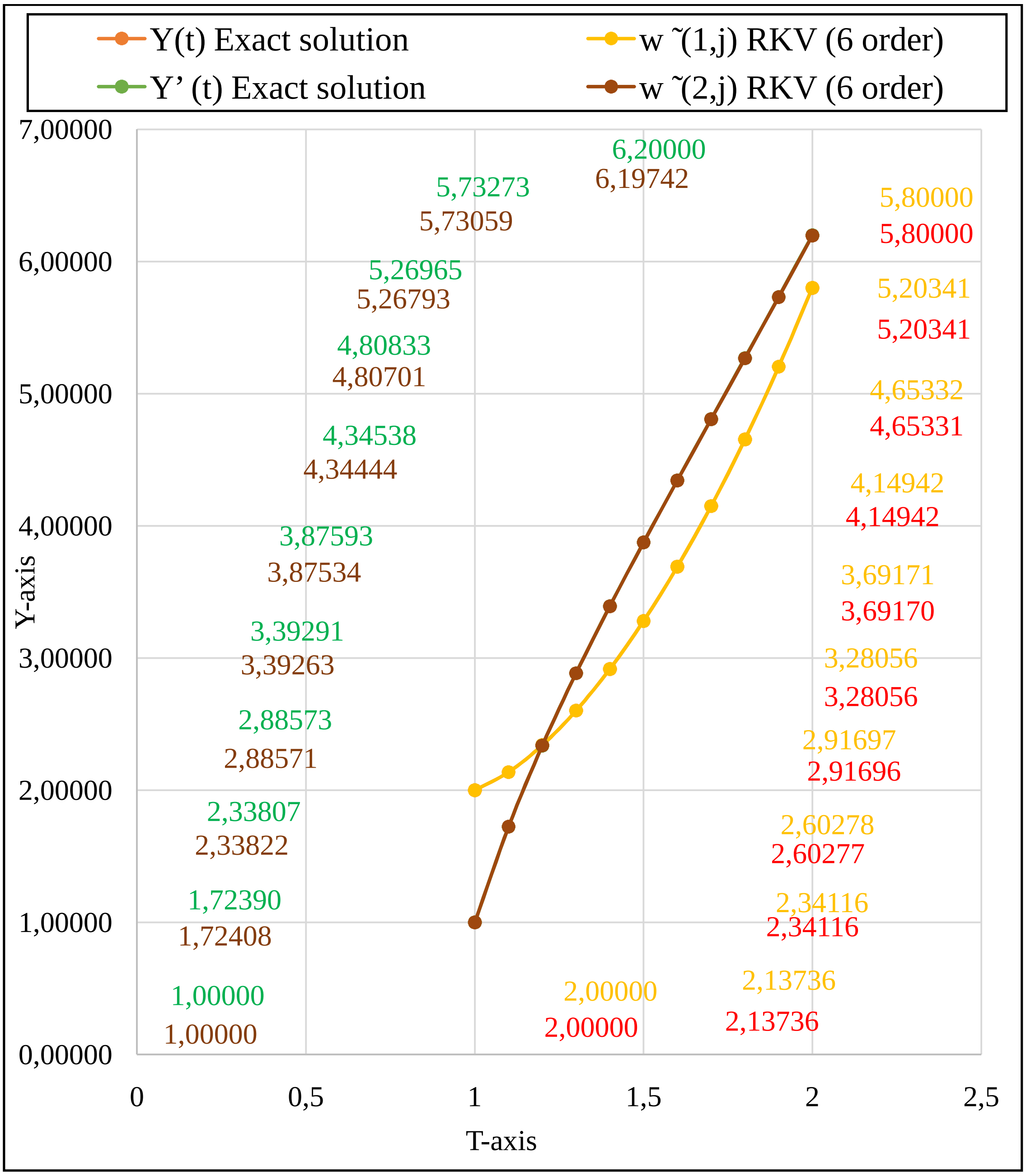


Figure (F.3): Approximate and exact solutions for example (F.2) when ℎ = 0.1.

## Appendix G. Approximation Solution from RDE by Adams-Bashforth Explicit Methods

**Example (G.1):** find the numerical solution by using Adams-Bashforth (2-Step, 3-Step, 4-Step, 5-Step) methods to the initial value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \ , \quad y\ 1\ = -1\ , \quad 1 \le t \le 2$$

, and it compare to the exact solution which is given by $y = -1/t$ .

**Solution:**

Firstly, we recognized the given equation is a Riccati differential Equation, secondly because it is specified in the form of the initial value problem then normally we can solve it by Adams-Bashforth-explicit methods as desired in the example.

The first five approximation solution where found by Runge-Kutta method of the order fourth is consist of $w_0 = y\ 1\ = -1$, $w_1 = y\ 1.1\ = -0.909090$, $w_2 = y\ 1.2\ = -0.833332$, $w_3 = y\ 1.3\ = -0.769229$, $w_4 = y\ 1.4\ = -0.714283$ then we can use those approximation solutions to determine the (One-Two-Three-Four-Five)-Steps of the Adams-Bashforth Method as follows;

**First:** computation for the Adams-Bashforth 2-Step Method;

$$w_2 = w_1 + \frac{1}{2}(0.1)\ 3f\ t_1, w_1\ - f\ t_0, w_0$$

$$y\ 1.2\ \approx w_2 = -0.909090 + \frac{0.1}{2}\ 3\ 0.826447\ - 1.000000\ = -0.835123;$$

$$w_{10} = w_9 + \frac{1}{2}(0.1)\ 3f\ t_9, w_9\ - f\ t_8, w_8$$

And

$$y\ 2\ \approx w_{10} = -0.535011 + \frac{0.1}{2}\ 3\ 0.272356\ - 0.304139\ = -0.509365$$

**Second:** computation for the Adams-Bashforth 3-Step Method;

$$w_3 = w_2 + \frac{0.1}{12}\ 23f\ t_2, w_2\ - 16f\ t_1, w_1\ + 5f\ t_0, w_0$$

$$y(1.3) \approx w_3 = -0.833332 + \frac{0.1}{12}\left[23(0.694446) - 16(0.826447) + 5(1.000000)\right]$$
$$= -0.768756;$$

And

$$w_{10} = w_9 + \frac{0.1}{12}\left[23f(t_9, w_9) - 16f(t_8, w_8) + 5f(t_7, w_7)\right]$$
$$y(2) \approx w_{10} = -0.524446 + \frac{0.1}{12}\left[23(0.277989) - 16(0.309593) + 5(0.346929)\right]$$
$$= -0.497988;$$

**Third:** computation for the Adams-Bashforth 4-Step Method;

$$w_4 = w_3 + \frac{1}{24}\left[55f(t_3, w_3) - 59f(t_2, w_2) + 37f(t_1, w_1) - 9f(t_0, w_0)\right]$$
$$y(1.4) \approx w_4 = w_3 + \frac{0.1}{24}\left[55f(t_3, w_3) - 59f(t_2, w_2) + 37f(t_1, w_1) - 9f(t_0, w_0)\right]$$
$$= -0.769229 + \frac{0.1}{24}\left[55(0.591718) - 59(0.694446) + 37(0.826447) - 9(1.000000)\right]$$
$$= -0.714434;$$

And

$$y(2) \approx w_{10} = w_9 + \frac{0.1}{24}\left[55f(t_9, w_9) - 59f(t_8, w_8) + 37f(t_7, w_7) - 9f(t_6, w_6)\right]$$
$$= -0.526809 + \frac{0.1}{24}\left[55(0.276748) - 59(0.308392) + 37(0.345786) - 9(0.390413)\right]$$
$$= -0.500532.$$

**Forth:** computation for the Adams-Bashforth 5-Step Method;

$$w_5 = w_4 + \frac{1}{720}\left[1901f(t_4, w_4) - 2774f(t_3, w_3) + 2616f(t_2, w_2) - 1274f(t_1, w_1) + 251f(t_0, w_0)\right]$$
$$y(1.5) \approx w_5 = w_4 + \frac{0.1}{720}\left[1901f(t_4, w_4) - 2774f(t_3, w_3) + 2616f(t_2, w_2) - 1274f(t_1, w_1) + 251f(t_0, w_0)\right]$$
$$= -0.714283 + \frac{0.1}{720}\left[1901(0.510206) - 2774(0.591718) + 2616(0.694446) - 1274(0.826447) + 251(1.000000)\right] = -0.666609;$$

And

$$y(2) \approx w_{10} = w_9 + \frac{1}{720}\left[1901f(t_9, w_9) - 2774f(t_8, w_8) + 2616f(t_7, w_7) - 1274f(t_6, w_6) + 251f(t_5, w_5)\right]$$

$$= -0.526159 + \frac{1}{720}\left[1901(0.277091) - 2774(0.308720) + 2616(0.346091) - 1274(0.390687) + 251(0.444483)\right] = -0.499831.$$

As presented earlier the algorithm to the Adams-Bashforth Method to the deferent steps (orders) can be achieved by applying its formulas, and then by using a one of the programing of computer that it can available to implement this method. Fortunately, as we have reviewed from the preceding evaluations then we understand that it must be calculate some steps of the approximation solutions before we are starting to compute the required step throughout implementing the Adams-Bashforth Explicit Method. In this case, we were evaluated the some steps of the approximate solution by using the R-K Method. In the table (G.2) through the next page devoted the first column to the calculation of $t_i$ which were starting from $t_0 = 1$, $t_1 = 1.1$, $t_2 = 1.2$, ……, $t_{10} = 2$ with respect to the selected interval $1 \leq t \leq 2$.

Also the second column assigned to the exact solution so that we bring it to compare with approximate solution, the third column involved the computation of the Runge-Kutta method for the four order and also remain columns obtained to the calculations of the deferent steps to the Adams-Bashforth Methods. Consequently, we have achieved the mostly accurate numerical solutions compared to the actual solution although the value of the step size $h$ is not too small. Nate that, the tolerance which was taken in here is TOL= $10^{-6}$. And also in the extension of the table (G.2) obtained the associated functions to the deferent orders (steps) of the Adams-Bashforth methods.

Within the table (G.1), interpreted each of the absolute and percentage relative errors, normally the absolute error calculated by subtracting the approximate solution from the actual solution were in the first case is $|-1.000000 - (-1.000000)| = 0.000000$ , in the second case is $|-0.909091 - (-0.909090)| = 0.000001$, and so on until the end situation is $|-0.500000 - (-0.499831)| = 0.000169$. And also considered the percentage relative error which is assigned by the important form to determine the errors.

In the first situation, the % Relative Error is 0.00, the second situation is 0.00 and the last situation is 0.03. It is indicate the method could be give us the best accurate solution. Note that these amount values had been taken with respect to the fifth order Adams-Bashforth methods.

All these illustrations are structured by the graphical interpretation in the figure (G.1) with identified all its points inside the figure.

Table (G.1): Illustration the relative and relative percentage of errors.

| Iteration | Absolute Error | % Relative Error |
|---|---|---|
| 0 | 0.000000 | 0.00 |
| 1 | 0.000001 | 0.00 |
| 2 | 0.000002 | 0.00 |
| 3 | 0.000002 | 0.00 |
| 4 | 0.000002 | 0.00 |
| 5 | 0.000057 | 0.01 |
| 6 | 0.000099 | 0.02 |
| 7 | 0.000120 | 0.02 |
| 8 | 0.000140 | 0.03 |
| 9 | 0.000157 | 0.03 |
| 10 | 0.000169 | 0.03 |

Table (G.2): Illustration of ABE-Method for order (2, 3, 4 and 5), RK-Method (order 4) and exact solutions of example (G.1) when $h = 0.1$.

| $t_i$ | Exact Solution | RK-method (order 4) | Adams-Bashforth (2-Step) | Adams-Bashforth (3-Step) | Adams-Bashforth (4-Step) | Adams-Bashforth (5-Step) |
|---|---|---|---|---|---|---|
| 1 | -1.000000 | -1.000000 | -1.000000 | -1.000000 | -1.000000 | -1.000000 |
| 1.1 | -0.909091 | -0.909090 | -0.909090 | -0.909090 | -0.909090 | -0.909090 |
| 1.2 | -0.833333 | -0.833332 | -0.835123 | -0.833332 | -0.833332 | -0.833332 |
| 1.3 | -0.769231 | -0.769229 | -0.772503 | -0.768756 | -0.769229 | -0.769229 |
| 1.4 | -0.714286 | -0.714283 | -0.718772 | -0.713431 | -0.714434 | -0.714283 |
| 1.5 | -0.666667 | -0.666664 | -0.672185 | -0.665534 | -0.666932 | -0.666609 |
| 1.6 | -0.625000 | -0.624997 | -0.631423 | -0.623643 | -0.625338 | -0.624901 |
| 1.7 | -0.588235 | -0.588232 | -0.595475 | -0.586687 | -0.588634 | -0.588116 |
| 1.8 | -0.555556 | -0.555552 | -0.563547 | -0.553838 | -0.556005 | -0.555416 |
| 1.9 | -0.526316 | -0.526312 | -0.535011 | -0.524446 | -0.526809 | -0.526159 |
| 2 | -0.500000 | -0.499996 | -0.509365 | -0.497988 | -0.500532 | -0.499831 |

Related to the table (G.2)

| $f(t_i, w_i)$ (2-Step) | $f(t_i, w_i)$ (3-Step) | $f(t_i, w_i)$ (4-Step) | $f(t_i, w_i)$ (5-Step) |
|---|---|---|---|
| 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 0.826447 | 0.826447 | 0.826447 | 0.826447 |
| 0.692950 | 0.694446 | 0.694446 | 0.694446 |
| 0.589188 | 0.592081 | 0.591718 | 0.591718 |
| 0.506979 | 0.510814 | 0.510098 | 0.510206 |
| 0.440735 | 0.445198 | 0.444268 | 0.444483 |
| 0.386569 | 0.391471 | 0.390413 | 0.390687 |
| 0.341710 | 0.346929 | 0.345786 | 0.346091 |
| 0.304139 | 0.309593 | 0.308392 | 0.308720 |
| 0.272356 | 0.277989 | 0.276748 | 0.277091 |
| 0.245230 | 0.251002 | 0.249734 | 0.250085 |

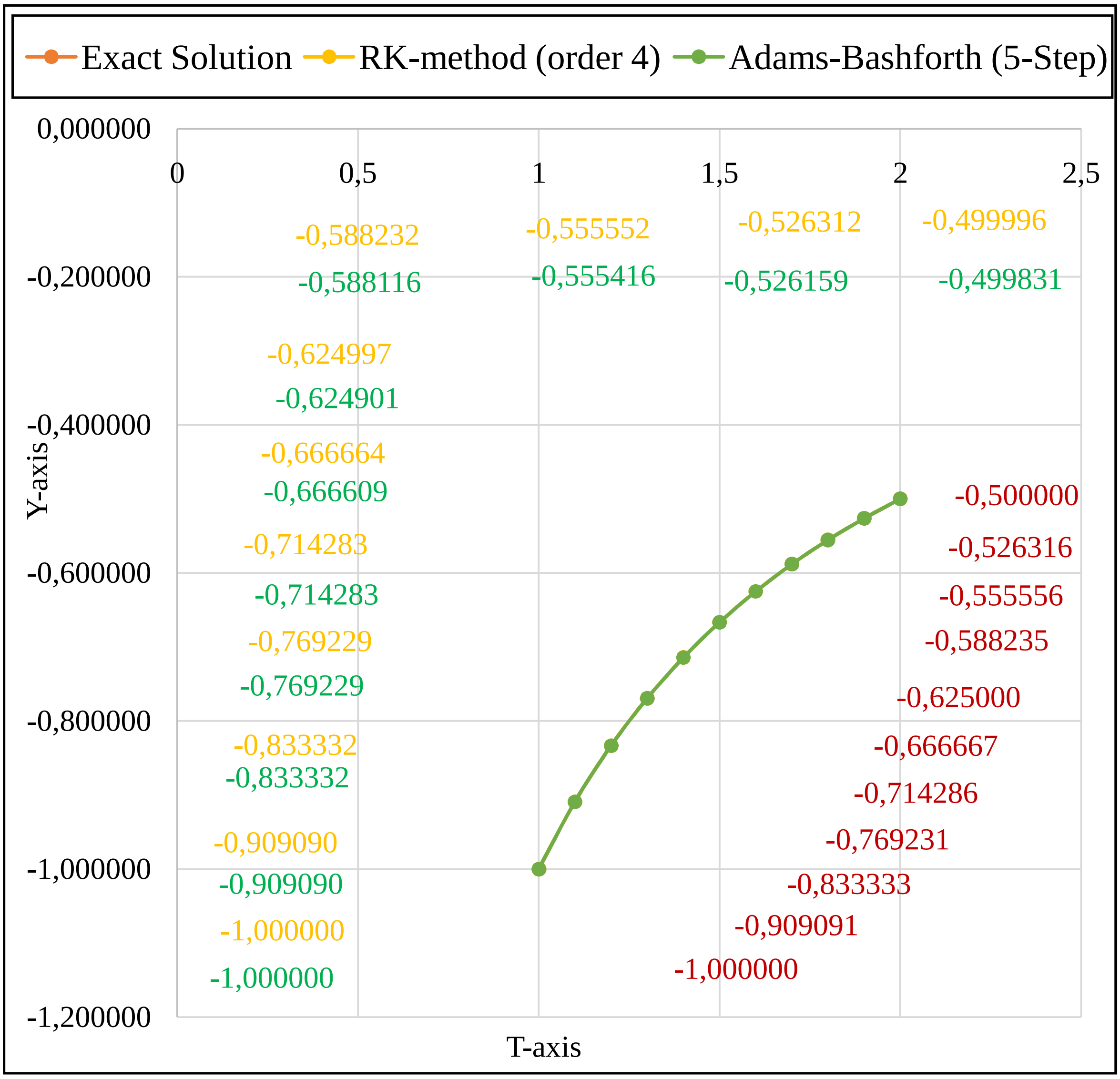


Figure (G.1): Approximate and exact solutions for example (G.1)

when ℎ = 0.1.

NEAR EAST UNIVERSITY

# COMPARISON BETWEEN NUMERICAL AND EXACT SOLUTIONS OF RICCATI DIFFERENTIAL EQUATIONS USING EXCEL

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**NICOSIA 2015**

# Abstract

The thesis deals with the numerical solutions of various forms of nonlinear Riccati Differential Equation. In doing that, several different numerical methods are used and for each numerical method a nonlinear Riccati Differential Equation was used as an illustrative example. The work provides an opportunity to judge and compare the adequacy of the numerical methods compared with the available close form solutions. The use of excel worksheet provides an easy way for implementing the numerical algorithms and also an easy and interactive way to see the effect of the step size $h$ graphically and immediately.

## (Abstract cont.)

In each case a graphical representation for both exact and numerical solutions are presented and the results compare very well for majority of the cases without any need for finer step size $h$.

# Outline of the Presentation

- Introduction
- Literature Review
- Methodology
- Result and discussion
- Conclusion
- References

## ➢ Introduction

This thesis investigates an interesting type of Ordinary Differential Equations (ODEs) known as Riccati Differential Equations (RDE). The general form of the RDE together with its initial condition make a Riccati Initial value problem (RIVP) which is represented by

$$y' = p(t)y^2 + q(t)y + r(t)\ , \qquad y(t_0) = y_0\ ,$$
$$t_i \le t \le t_{i+1}. \qquad (1.1)$$

when $t_i = t_0 + ih$ and $t_{i+1} = t_i + h$ , $h$ is the step size.

## (Introduction cont.)

Provided that $p(t) \not\equiv 0$ , where $p, q, r$ are continuous functions [1], [4], [11], [14]. This equation is nonlinear first order differential equation because it contains $y^2$ and $dy/dt$ . In fact that, to solve the Riccati differential equation by using some known numerical methods that are used for solving Initial Value Problems (IVP) to identify the approximate solution [14], after that I will compare its solution to the exact solution so that we will judge the performance of these methods and judge them accordingly. This way we may gain the experience of judging which method is more suitable for any particular IVP.

## Literature Review

The Riccati Differential Equation (RDE), named after the Italian mathematician and nobleman Jacopo Francesco Riccati (1676-1754) [26]. In actually we have seen some works that a number of mathematicians have studied RDE, involving several of the Bernoulli, Riccati himself, and his son Vincenzo. At the end of 1723, it was known that equation (1.1) cannot be solved in the case of elementary functions, after that, Euler stated that if the particular solution of (1.1) is known then by using the substitution $y = u + 1/v$, converts the RDE in to the linear Differential Equation in $v$ , and then we can get the general solution, he also said that, if the two particular solutions are available then the general solution is considerable in case of simple quadrature [6].

## (Literature Review cont.)

If we don't have at least one particular solution, then it cannot be possible to determine the general solution or no chance to solve such differential equation exactly [27].

## ➢ Methodology

1. Euler’s Method.
2. Taylor’s Method:
   - 2.1 Taylor’s Method for order two.
   - 2.1 Taylor’s Method for order four
3. Runge-Kutta Method:
   - 3.1 Runge-Kutta Method for order two.
   - 3.2 Runge-Kutta Method for order four.
   - 3.3 Runge-Kutta-Fehlberg Method.
   - 3.4 Runge-Kutta-Verner Method.
4. System of Differential Equations.
5. Adams Bashforth Explicit Methods.

## 1. Euler’s Method

Euler’s Method assigned by the basic and common method that used to identify the approximate solution for solving RDE. The general formula of Euler’s Method considered by

$$y_{i+1} = y_i + hf(t_i, y_i) \qquad (1.1)$$

When the RDE expressed in the form of initial value problem then we can use the desired method to determine the numerical solution.

# Example:

An example:

Determine the approximate solutions by Euler's Method over the given Riccati initial value problem

$$y' = \left(1 + \frac{2}{t}\right) - \left(2 + \frac{2}{t}\right)y + y^2 \quad , \; y(1) = \frac{5}{2} \; , \quad 1 \le t \le 2$$

When the actual solution given as

$$y = (3 + 3t - t^2)/(3t - t^2).$$

Solution:

Microsoft Excel Worksheet

## 2. Taylor’s Method

Taylor’s Method is an another method that use for identify the approximate solution for RDE, in this method we must be take the derivatives for the given differential equation as desired in the question.

## 2.1 Taylor's Method for order two

The formula of Taylor's Method for order two is considered by

$$\left.\begin{cases} w_{i+1} = w_i + hT^{(2)}(t_i, w_i) \\ T^{(2)}(t_i, w_i) = f(t_i, w_i) + \dfrac{h}{2} f^{'}(t_i, w_i) \\ \text{When } i = 0,1,2, \dots . N-1. \end{cases}\right\} \quad (2.1)$$

## Example:

Determine the approximate solutions by Taylor's Method of order two over the following initial value problem

$$y' = (1+t) - (1+2t)y + ty^2 \quad , \quad y(0) = 3 \, , \, 0 \le t \le 1.$$

where the actual solution is

$$y = 1 + (1/(t+1))$$

Solution:

Microsoft Excel Worksheet

## 2.2 Taylor's Method for order four

The formula of Taylor's Method for order four is considered by

$$\left\{ \begin{array}{c} w_{i+1} = w_i + hT^{(4)}(t_i, w_i) \\ T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2!} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + + \frac{h^3}{4!} f'''(t_i, w_i) \\ \text{When } i = 0,1,2, \ldots . N - 1. \end{array} \right\} \quad (2.2)$$

## Example:

Determine the approximate solutions by Taylor's Method of order four over the given Riccati initial value problem

$$y' = y^2 - \frac{y}{t} - \frac{1}{t^2} \quad , y(0.5) = 2.36364 \quad \text{for} \quad 0.5 \leq t \leq 1.5$$

When the actual solution given as

$$y(t) = \frac{2t}{3 - t^2} + \frac{1}{t}$$

Solution: Microsoft Excel Worksheet

# 3. Runge-Kutta Method

Since the solution obtained by Euler's Method is not accurate and the results by Taylor's Method need to take the Derivatives then we must be bring the best alternative method that it can be use instead of the preceding methods. Normally Runge-Kutta Method is one of the important method for solving Riccati initial value problems compared to the each other numerical methods because its solution is more extremely accurate and it is not necessary to take mostly small value $h$.

## 3.1 Runge-Kutta Method for order two

The formula of RK-Method for order two (Modified Euler's Method) is given by

$$\left.\begin{array}{c} k_1 = hf(t_i, y_i) \\ k_2 = hf(t_i + h, y_i + hk_1) \\ y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2) \\ \text{When } i = 0,1,2,\ldots.N-1. \end{array}\right\} \quad (3.1)$$

## Example

Determine the approximate solutions by RK-Method of order two over the following initial value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \quad , \qquad y(1) = -1\,, \qquad 1 \le t \le 2$$

When the actual solution given as $y = -1/t$.

Solution: Microsoft Excel Worksheet

## 3.2 Runge-Kutta Method for order four

The formula of RK-Method for order four is given by

$$\left.\begin{aligned}
k_1 &= hf(t_i, y_i)\\
k_2 &= hf\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right)\\
k_3 &= hf\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right)\\
k_4 &= hf(t_i + h, y_i + k_3)\\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&\text{When } i = 0,1,2,\ldots N-1.
\end{aligned}\right\} \quad (3.2)$$

## Example

Determine the approximate solutions by RK-Method of order four over the following initial value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \quad , \qquad y(1) = -1 \, , \qquad 1 \le t \le 2$$

When the actual solution given as $y = -1/t$.

Solution: Microsoft Excel Worksheet

# 3.3 Runge-Kutta-Fehlberg Method

Runge-Kutta method of order five with local truncation error given as

$$\tilde{w}_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \qquad (3.3.1)$$

to estimate the local error in a Runge-Kutta method of order four given as

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5 \qquad (3.3.2)$$

# (Runge-Kutta-Fehlberg Method cont.)

Where

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{1}{4}h, w_i + \frac{1}{4}k_1\right),$$

$$k_3 = hf\left(t_i + \frac{3}{8}h, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right),$$

$$k_4 = hf\left(t_i + \frac{12}{13}h, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right),$$

$$k_5 = hf\left(t_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right),$$

$$k_6 = hf\left(t_i + \frac{1}{2}h, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

## Example

Determine the approximate solutions by RKF-Method of order four and five over the following initial value problem

$$y' = \left(1 + \frac{2}{t}\right) - \left(2 + \frac{2}{t}\right) y + y^2 \quad , y(1) = \frac{5}{2} \ , 1 \le t \le 2$$

When the actual solution given as

$$y = (3 + 3t - t^2)/(3t - t^2).$$

Solution: Microsoft Excel Worksheet

## 3.4 Runge-Kutta-Verner Method

The Runge-Kutta-Verner method for fifth-order is given by

$$w_{i+1} = w_i + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6 \longrightarrow (3.4.1)$$

The Runge-Kutta -Verner method for sixth-order is given by

$$\tilde{w}_{i+1} = w_i + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7 + \frac{43}{616}k_8 ,$$

$$\longrightarrow (3.4.2)$$

# (Runge-Kutta-Verner Method cont.)

$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf\left(t_i + \frac{1}{6}h, w_i + \frac{1}{6}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{4}{15}h, w_i + \frac{4}{75}k_1 + \frac{16}{75}k_2\right),$$
$$k_4 = hf\left(t_i + \frac{2}{3}h, w_i + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right),$$
$$k_5 = hf\left(t_i + \frac{5}{6}h, w_i - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4\right),$$
$$k_6 = hf\left(t_i + h, w_i + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5\right),$$
$$k_7 = hf\left(t_i + \frac{1}{15}h, w_i - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5\right),$$
$$k_8 = hf\left(t_i + h, w_i + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5 + \frac{3850}{26703}k_7\right).$$

## Example

Determine the approximate solutions by RKV-Method of order five and six over the following initial value problem

$$y' = -\frac{2+t}{t(1+t)^2} - \frac{2+t-t^2}{t(1+t)}y + (1+t)y^2 \quad ,$$

$$y(1) = -\frac{1}{2}, \qquad 1 \leq t \leq 2$$

When the actual solution given as $y = -1/(1+t)$.

Solution: Microsoft Excel Worksheet

# 4. System of Differential Equations

The following formulas use for transforming the second-order into the first-order initial value problems by RK-Method for order four and Let $u_1(t) = y(t)$ , $u_2\ (t) = y'(t)$ then

$$\left\{\begin{array}{c} w_{1,i+1} = w_{1,i} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ w_{2,i+1} = w_{2,i} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\ \text{Where } i = 0, 1, 2, \dots, N-1 \end{array}\right\} \quad (4.1)$$

# (system of differential equations cont.)

Where

$$\left.\begin{cases} k_1 = hf_1(t_i, u_{1,i}, u_{2,i}) \\ l_1 = hf_2(t_i, u_{1,i}, u_{2,i}) \\ k_2 = hf_1\left(t_i + \frac{h}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2}\right) \\ l_2 = hf_2\left(t_i + \frac{h}{2}, u_{1,i} + \frac{k_1}{2}, u_{2,i} + \frac{l_1}{2}\right) \\ k_3 = hf_1\left(t_i + \frac{h}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2}\right) \\ l_3 = hf_2\left(t_i + \frac{h}{2}, u_{1,i} + \frac{k_2}{2}, u_{2,i} + \frac{l_2}{2}\right) \\ k_4 = hf_1(t_i + h, u_{1,i} + k_3, u_{2,i} + l_3) \\ l_4 = hf_2(t_i + h, u_{1,i} + k_3, u_{2,i} + l_3) \end{cases}\right\} \quad (4.2)$$

# Example

Use RK-Method for order four to transform the second-order initial value problem

$$ty'' - y' - t^3 y = 0 \quad for \quad 1 \le t \le 2$$

$$y(1) = 2.776347 \quad , \qquad y'(1) = 2.169817.$$

Into the system of the first-order initial value problems, when the actual solutions given as follows

$$y(t) = \frac{3}{2}\exp\left(\frac{t^2}{2}\right) + \frac{1}{2}\exp\left(-\frac{t^2}{2}\right)$$

$$y'(t) = \frac{3}{2}\, t \exp\left(\frac{t^2}{2}\right) - \frac{1}{2}\, t \exp\left(-\frac{t^2}{2}\right).$$

Solution: Microsoft Excel Worksheet

# 5. Adams Bashforth Explicit Methods (4-Step)

The formula of Adams Bashforth Explicit Methods (4-Step) is given by

$$w_{i+1} = w_i + \frac{1}{24}h\{55f(t_i,w_i) - 59f(t_{i-1},w_{i-1}) + 37f(t_{i-2},w_{i-2}) - 9f(t_{i-3},w_{i-3})\},$$

$$where\ i = 3,4,5,\ldots\ldots,N-1$$

## Example

Determine the approximate solutions by Adams Bashforth Explicit Methods (4-Step) over the given initial value problem

$$y' = \frac{t}{1+t^2} + \frac{y}{t} + \frac{y^2}{t(1+t^2)} \quad , y(1) = 0 \quad , 1 \le t \le 2$$

When the actual solution given as

$$y = (t^2 - t)/(t+1).$$

Solution:

Microsoft Excel
Worksheet

# Result and discussion

In this section we will attempt to present an overall two way comparison for a range of known numerical methods and their performance in solving RIVPs. On the other hand we will demonstrate the effect of varying the step size on their performance. The overall results are summarized and presented in a tabulated format and also graphically for ease of comparison.

The one-step numerical methods that are used for this comparison study include; Euler's Method, Taylor's Method of order four, Runge-Kutta Method of order four, Runge-Kutta-Fehlberg Method of order four and Runge-Kutta-Verner Method of order five. Among the multi-step methods, we chose to use the Adams-Bashforth Explicit Method.

Case 1: When the value of step size $h = 0.5$

| Exact solution | Euler's Method | Taylor's Method (order 4) | Runge-Kutta (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (2-step) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 1.166667 | 1.125000 | 1.133789 | 1.166610 | 1.166674 | 1.166669 | 1.166610 |
| 2.000000 | 1.820313 | 1.839480 | 1.999419 | 2.000069 | 2.000019 | 1.937388 |

Case 2: When the value of step size $h = 0.25$

| Exact Solution | Euler's Method | Taylor's Method (order 4) | Runge-Kutta Method (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (3-Step) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 0.821429 | 0.812500 | 0.818176 | 0.821428 | 0.821429 | 0.821429 | 0.821428 |
| 1.166667 | 1.141602 | 1.155969 | 1.166663 | 1.166667 | 1.166667 | 1.166663 |
| 1.550000 | 1.494515 | 1.528939 | 1.549987 | 1.550002 | 1.550000 | 1.546822 |
| 2.000000 | 1.883090 | 1.968497 | 1.999956 | 2.000007 | 2.000001 | 1.986924 |

Case 3: When the value of step size $h = 0.1$

| Exact Solution | Euler's Method | Taylor's Method (order 4) | Runge-Kutta (order 4) | Runge-Kutta-Fehlberg (order 4) | Runge-Kutta-Verner (order 5) | ABEM (4-Step) |
|---|---|---|---|---|---|---|
| 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 | 0.500000 |
| 0.626316 | 0.625000 | 0.626139 | 0.626316 | 0.626316 | 0.626316 | 0.626316 |
| 0.755556 | 0.752563 | 0.755105 | 0.755556 | 0.755556 | 0.755556 | 0.755556 |
| 0.888235 | 0.883095 | 0.887448 | 0.888235 | 0.888235 | 0.888235 | 0.888235 |
| 1.025000 | 1.017095 | 1.023904 | 1.025000 | 1.025000 | 1.025000 | 1.024988 |
| 1.166667 | 1.155176 | 1.165441 | 1.166667 | 1.166667 | 1.166667 | 1.166633 |
| 1.314286 | 1.298101 | 1.313335 | 1.314286 | 1.314286 | 1.314286 | 1.314221 |
| 1.469231 | 1.446836 | 1.469263 | 1.469231 | 1.469231 | 1.469231 | 1.469116 |
| 1.633333 | 1.602612 | 1.635465 | 1.633333 | 1.633333 | 1.633333 | 1.633139 |
| 1.809091 | 1.767031 | 1.814975 | 1.809090 | 1.809091 | 1.809091 | 1.808765 |
| 2.000000 | 1.942205 | 2.012006 | 1.999999 | 2.000000 | 2.000000 | 1.999452 |

## Conclusion

The lessons we have learned from this interesting piece of work can be summarized as follows; The very basic Euler's method is very simple easy to implement therefore, it is useful for learners of numerical analysis and also beginners programmers to use the method as a practice aiming to further improve their numerical knowledge and also their programming skills. Another area that we recommend the use of this method is to compute additional starting values that required when using Multistep methods for solving IVPs. Otherwise if an accurate result is required for academic or scientific purposes, certainly this method is not recommended.

## (Conclusion cont.)

Regarding the Taylor method, Reasonable results can be achieved if higher order Taylor's method is employed, but this is often very costly because the need for evaluating higher order derivatives, where some times can be very tedious and even impossible to obtain. Hence this group of methods are also not recommended for serious scientific or academic work. Coming to Linear Multistep methods it is fairly easy to implement, the drawbacks are that they require additional initial values that are not readily available, therefore by the time you program a single step method to obtain these additional values one will be tempted to continue with this method to produce the full solution of the problem in hand.

## (Conclusion cont.)

The method that stands out among all the numerical methods are the fourth order Runge-Kutte method. These methods produce very good and impressive results, they are easy to implement and do not require additional starting values. Therefore definitely they are our choice of recommendation.

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