

**NUMERICAL SOLUTION OF THE NONLINEAR  
ABEL DIFFERENTIAL EQUATION OF THE  
FIRST KIND USING EXCEL**

**A THESIS SUBMITTED TO THE GRADUATE  
SCHOOL OF APPLIED SCIENCES  
OF  
NEAR EAST UNIVERSITY**

**By  
JUBRAEEL FARIS QADER**

**In Partial Fulfilment of the Requirements for  
the Degree of Master of Science  
in  
Mathematics**

**NICOSIA, 2016**

**JUBRAEEL FARIS  
QADER**

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**To my parents...**

## ABSTRACT

This work mainly concerns with Abel Differential Equations (ADE) of the first kind. Because of their nonlinearity, it is known that the unrestricted forms do not have a closed form analytical solutions. Therefore, a few restricted forms that have exact solutions are studied first using a range of well-known numerical solutions and they are compared with the exact solutions. Once some confident was gained, an unrestricted form of ADE was studied. In the absence of closed form solutions, several high order numerical methods were used and the results were compared with each other. All the results compared well and were very reassuring. Smaller step sizes were used when the results were doubtful in order to gain more accuracy. Excel work sheet was used as a means of computation, because of its availability, ease of use and its graphical capability. Error bound and optimum step size was presented for Euler method and examples are presented to support the theory.

**Keywords:** Abel differential equation; restrict and unrestricted Abel differential equation; Runge-Kutta method; absolute error; graphical illustration



## ÖZET

Bu çalışma temel olarak birinci tip Abel Diferensiyel denklemler ile ilgilidir. bu denklemler linear olmadıklarından, sınırsız forumları analitik çözümlere yakın olmadıkları bilinir. Bununla birlikte, kesin sonuçları olan birkaç sınırsız formu da vardır. Bu çalışmada ilk olarak iyi bilinen numerik çözümlerin görüntüleri ile çalışıldı ve kesin çözümleri ile karşılaştırıldı. Yakın forum çözümler olmadığında, birkaç yüksek mertebeden nümerik çözümler kullanıldı ve bir birleri ile iyi bir şekilde karşılaştırılıp güvenilir sonuçlar elde edilmiştir. Şüpheli sonuçlarda ise daha doğru bir sonuç elde edebilmek için adımlar daha küçük tutuldu.

Kullanım kolaylığı ve grafiksel yeteneğinden dolayı hesaplamalarda Excel programı kullanılmıştır. Hata sınırı ve ideal adım boyutu Euler metodu için, örnekler ise teoriyi desteklemek için verilmiştir.

**Anahtar Kelimeler:** Abel Diferensiyel Denklemler; sınırlı ve sınırsız Abel diferensiyel denklemler; Runge-Kutta metod; mutlak hata; grafiksel örnekleme

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## LIST OF SYMBOLS

$y' = \frac{dy}{dx}$	First derivative of dependent $y$ variable with respect to the independent $x$
$y$	Dependent variable
$x$	Independent variable
$x_0$	Independent initial variable
$y_0$	Dependent initial variable
$f_n(x)$	Function of $x$ where $n=0, 1, 2, 3 \dots$
$f(x, y)$	Ordinary differential equation
$h$	Step size (Increment variable)
$\xi$	Some number between $x_i$ and $x_{i+1}$
$y(x)$	Exact solution
$w_i$	Approximate solution
$k_i$	Coefficients
$\alpha$	Value of initial condition
$i$	Some positive integer number
$M$	Constant number satisfies $ y''(x)  \leq M$ , for all $x \in (a, b)$ .
$L$	The Lipschitz constant
$a$	Lower bound of open interval $(a, b)$
$b$	Upper bound of open interval $(a, b)$
$N$	Positive integer number
$\lambda$	Small positive number between $[0, 1]$
$\infty$	Infinity
$D$	Region in $xy$ -plan
$e$	Exponential function
$\delta$	Round of error

## LIST OF ABBREVIATIONS

<b>ABEM:</b>	Adams Bashforth Explicit Method
<b>ADE:</b>	Abel Differential Equation
<b>AIVPs:</b>	Abel Initial Value Problems
<b>IVPs:</b>	Initial Value Problems
<b>MOE:</b>	Microsoft Office Excel
<b>ODEs:</b>	Ordinary Differential Equations
<b>RKFM:</b>	Runge-Kutta-Fehlberg Method
<b>RKM:</b>	Runge-Kutta Method
<b>RKVM:</b>	Runge-Kutta-Verner Method
<b>TE:</b>	Truncation Error
<b>TM:</b>	Taylor's Method

# CHAPTER 1

## INTRODUCTION

Abel Differential Equation (ADE) classify in two kinds:

Abel differential equation of first kind has the form

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \quad (1)$$

Provided that  $f_3(x) \neq 0$

And, Abel differential equation of the second kind has the form

$$[g_0(x) + g_1(x)y]y' = f_2(x)y^2 + f_1(x)y + f_0(x) \quad (2)$$

Provided that  $f_2(x) \neq 0$

Where  $y = y(x)$  and  $f_0, f_1, f_2, f_3, g_0$  and  $g_1$  are known and arbitrary form of coefficients of the Abel differential equation. These equations arose in the context of the studies of N.H.Abel on the theory of elliptic functions. Note that Abel equations of the first and second kind (1) and (2) are related with each other by a local change of variables (namely, the equation (2) can be reduced to the form (1) by means of the change of variables  $y + g = \frac{1}{u}$ ), (polyanin, and Zaitsev, 2003). Differential equations can be solved by either numerical method or analytically, numerical methods are suitable for general problems but the computational practice is complex, while in most cases, the effectiveness and truth of the results are relatively lower than those of the analytical methods. In this thesis we will investigate ADE of first kind. This equation is not known analytical solution and it is nonlinear first order ordinary differential equation. It represents a natural generalization for most differential equations such as Riccati differential equation, Bernoulli differential equation, separable and linear differential equation.

In the beginning we will work on restricted form of ADE, such as Separable differential equation, linear differential equation, Riccati differential equation and Bernoulli differential equation because they have a closed form solution. So we will use some known numerical methods to solve some examples of these equations, comparing results with given exact solutions. This way we will judge which one is most suitable to use. The best

is the one which has a smaller sized error. Then we will discuss the unrestricted form of ADEs by taking some examples and using different numerical methods for solving those using initial values. A recent development in computer technology has made the task of solving difficult problems, easy to handle and present the results graphically. In this work I depend on Microsoft Excel. Throughout this work, Excel work sheet is used as a computational tool for finding approximate solutions to a range of IVP of ADE. The numerical methods are used include; Euler's Method, Taylor's Method, Runge-Kutta Method and Linear Multistep Methods (Adam Bash fourth Explicit Method). There are several good reasons behind using Excel. One is that Excel is readily available on personal computers, Laptops and smart phones. The second reason is that Excel is user friendly and interactive; the third is that the user does the programming for the computation. Hence, he is involved in the step by step calculations and he is monitoring the progress. Unlike, other software package which acts like a box where the user enters the data from one end and obtains the results from another without knowing what has happened in between. Another benefit of Excel is that the results can be graphically presented for ease of understanding and also for the sake of comparison, for example, comparing exact and numerical solutions or even for comparing the performance of two different numerical solutions. In chapter two, some basic definition and theorems of ODE are discussed. In chapter three some numerical methods are presented. The reader's mind should be focused more on chapter four where comparison between numerical and closed form solutions on restricted form of ADE and using different method to find a numerical solution on unrestricted form of ADE and last chapter contains the conclusions and discussions.

## **1.1 Literature Review**

One of the most active mathematicians in the 18th century was (Niels Henirk Abel). He was born in Norway, and lived between (1802-1829). Although his life was short, he performed remarkable works on differential equations. One of his equations was the equation that I work on. This equation is known as an analytically unsolvable differential equation. It arises in problems such as control theory, cosmology, fluid mechanics, solid mechanics, biology, cancer therapy (Hernandez et al., 2013) and it has an important role in science, engineering, and physics to find solutions for equations describing the development of the universe and in the theory of thin film condensation (Streipert, 2012).

There exists a number of works which presented solutions and methods for solving these equations. Most works were analytic and none of these types of equations discloses general solutions in terms of known functions, except only for very special cases. If  $f_3(x) = 0$  then the equation becomes a Riccati Differential equation which it is known to have an exact solution. If  $f_2(x) = 0$  then the equation becomes a Bernoulli differential equation known with an exact solution. If  $f_3(x) = 0$  and  $f_2(x) = 0$  then the equation becomes to linear differential equation, It is simple to find exact solution. Another way in looking for solutions of ADE is to change of variable  $y = u(x)z(x) + v(x)$  and changing the values to  $z(x)$ , where the coefficients of this equation allow the construction of a system of auxiliary equations with  $Q_1(x), Q_2(x)$  and  $Q_3(x)$  as free functions to the system and  $u, v$  are parameter (Hernandez et al., 2013). Another paper that presented an exact general solution of nonlinear Abel differential Equation, which can be obtained by quadrature's if the four coefficients of the equation satisfy one consistency condition. The constraint imposes severe restrictions, limiting the number of possible solutions that can be obtained in this way (Harko et al., 2013 and MAK and HARKO 2002). A person have presented a solution generating technique for Abel-type ordinary differential equation if  $y = y_1(x)$  is particular solution of (1), then by mean of transformations.

$y = \frac{E(x)}{y - y_1(x)}$ ,  $E(x) = \exp\{\int [3f_3(x)y_1^2 + 2f_2(x)y_1 + f_1(x)]dx\}$  Equation (1) can be

transformed into  $\frac{dy}{dt} + \frac{Q_1}{u} + Q_2 = 0$ , with  $Q_1(x) = f_3(x)E^2(x)$

and  $Q_2(x) = [3f_3(x)y_1(x) + f_2(x)]E(x)$  Therefore, if  $y_1 = -\frac{f_2(x)}{3f_3(x)}$ , then  $Q_2 = 0$

The general solution of equation (1) can be obtained from the integration of a differential equation with separable variables (MAK et al., 2001). ADE can be reduced to a Riccati differential equation or to first order linear differential equation through a change of rational. The change is given explicitly for each class. Moreover, we have found a unified way to find the rational map from the knowledge of the explicitly first integral (Streipert, 2012). Another way gives implicit solutions of first kind Bessel's function second kind Newmann Functions for the canonical form of ADEs of first kind (Panayotounakos, and Zarmoutis. 2011). We can obtain separable ADE by just taking the coefficients  $f_i$  all equal, this means that there are complete Abel classes all of whose members can be transformed into separable ones by  $y = f(x)$ ,  $u = p(x)y + q(x)$ . (Terrab-Cheb, and Roche 1999). In another paper by using appropriate transformation in combination with

specific Abel equations closed form solutions can be obtained containing arbitrary functions. An implicit solution as well as the associated sufficient condition is derived for certain differential equations of the Abel class of first kind (Markakis, 2009). Another work they use short memory principal for ADE of fractional order and he evaluate the approximate solution at the end of required interval, and they used just Adam Bashfourth explicit method (XU, and He, 2013). In fact that all of these works are analytic because they are different and they reduce ADE and under some transformation and some cases they get exact solution. That is to say all above works are correct for Restricted ADE. So still now the general solution for ADE is an open problem because there exist source that tells as what the analytical solutions.

### **1.3 Aim of This Thesis**

The aim of this thesis is to show the details of implementing numerical methods using excel on Abel Differential Equation of first kind. Numerical approximations will be presented and compared with the exact solutions of restricted forms of ADE, since the exact solution are available. For the unrestricted form of ADE, where the exact solutions are not available, various numerical methods will be used to approximate solutions of ADEs of first kind. And compare these numerical results against each other, and presenting them graphically.

## CHAPTER 2

### INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation is a differential equation containing an independent variable and a dependent variable and one or more derivatives of dependent variable. The order of the equation is determined by the order of highest derivative. For example, if the first derivative is the only derivative, the equation is called a first-order ODE. If highest derivative is second order, the equation is called a second-order ODE. Ordinary differential equations are used for mathematical models many branches of calculus, science, engineering and economy that involves the change of variable with respect to another (Suli and Mayers, 2003). Most of these problems require the solution of an initial value problem that is the solution to a differential equation satisfying a given initial condition. In this chapter we will present some definitions, theorems and results from the theory of ordinary differential equations before considering methods for approximating the solutions to initial value problems numerically.

#### 2.1: Lipchitz Condition:

- **Definition:** A function  $f(x, y)$  satisfies a Lipchitz condition on a set  $D \subset \mathbb{R}^2$  if there is a constant  $L > 0$  such that

$$|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2| \quad (2.1)$$

For all  $(x, y_1), (x, y_2) \in D$ . the constant  $L$  is called a Lipschitz constant for  $f$  (Moin, 2010)

- **Example:** Show that  $f(x, y) = x^2 + 2y$  satisfies a lipschitz condition on interval  $D = \mathbb{R}^2$

Solution: for each  $(x, y_1), (x, y_2) \in D$

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |(x^2 + 2y_1) - (x^2 + 2y_2)| \\ &= 2 |y_1 - y_2| \end{aligned}$$

thus  $f$  satisfies a lipschitz condition on  $D$  in the variable  $y$  with lipschitz constant 2.

- **Definition:** A set  $D$  in  $R^2$  is said to be convex, if for all  $(x_1, y_1), (x_2, y_2)$  In  $D$  and for all  $\lambda \in [0, 1]$ , the point

$$((1 - \lambda)x_1 + \lambda x_2, (1 - \lambda)y_1 + \lambda y_2) \in D \quad (2.2)$$

In the geometric sense, this definition states that. A set is convex provided that for all two points belonging to the set, the entire straight-line segments between the points also belong to the set (Burden and Fairs, 2010).

- **Theorem:** Suppose  $f(x, y)$  is defined on a convex set  $D \subset R^2$ . If a constant  $L > 0$  exist with  $\left| \frac{\partial f}{\partial x}(x, y) \right| \leq L$ , for all  $(x, y) \in D$  (2.3)

Then  $f$  satisfies a lipschitz condition on  $D$  in the variable  $y$  with lipschitz constant  $L$  (Burden and Fairs, 2010).

## 2.2: Initial Point

- **Definition:** Initial condition is a condition that the value of  $y$  known at starting point; initial conditions are of the form:  $y(x_0) = y_0$  (Bride, 2004).
- **Definition:** The general form of an ordinary differential equation with initial condition is called initial value problem (IVP) as the form (Dennis and Warren, 2013).

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (2.4)$$

## 2.3: Existence and Uniqueness Theorem

- **Theorem:** let  $f(x, y)$  be continuous and satisfy a lipschitz condition in the variable  $y$  on

$D = \{ (x, y): a \leq x \leq b \text{ and } -\infty \leq y \leq \infty \}$ . Then the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha, \text{ has unique solution } y(x) \text{ for } a \leq x \leq b$$

(Willian, 2013).



- **Example:** Show that there is a unique solution to the IVP

$$y' = y \cos x, \quad 0 \leq x \leq 1, \quad y(0) = 1$$

**Solution:**  $y$  is constant and by theorem of this function

$$f(x, y) = y \cos x$$

$$\text{Then } \left| \frac{\partial f}{\partial y} \right| = \left| \cos x \right| \leq 1$$

It means that  $f$  satisfies lipschitz condition with lipschitz constant  $L = 1$

Then by Theorem  $f$  has a unique solution to this initial value problem

- **Theorem:** Suppose that  $D = \{(x, y) | a \leq x \leq b, -\infty < y < \infty\}$  and that  $f(x, y)$  is Continuous on  $D$ . If  $f$  satisfies a Lipschitz condition in  $y$  on  $D$ , then the initial value problem  $\frac{dy}{dx} = f(x, y), a \leq x \leq b, y(a) = \alpha$  is well-posed (Burden and Fairs, 2010).

- **Example:** Show that the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1, \quad 0 \leq x \leq 2, \quad y(0) = 0.5$$

is well-posed.

**Solution:**

$$\left| \frac{\partial f}{\partial y} \right| = \left| 1 \right| = 1$$

By above Theorem  $f$  satisfy a lipschitz condition in  $y$  on  $D$  with lipschitz constant 1, and because  $f$  is continuous on  $D$ , then the problem is well posed.

## 2.4: Sequences:

- **Definition:** A real sequence is an order set  $(a_1, a_2, a_3, \dots, a_n)$  of real numbers. We write this as  $(a_n)$  or  $(a_n)_{n \in \mathbb{N}}$

We say that the sequence  $(a_n)$  converges to a limit  $\alpha$  if all the terms of the sequence in the end get close to  $\alpha$

Then we write  $(a_n) \rightarrow \alpha$  or  $\lim_{n \rightarrow \infty} (a_n) = \alpha$

If no such number  $\alpha$  exists, then  $(a_n)$  is diverges (Thomas et al. 2005).

- **Examples:**

1. The Sequence  $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \rightarrow 0$  this means that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

2. The sequence defined by  $a_1 = 1$  and  $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$  for  $n \geq 1$  this means that  $\left(1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \dots\right)$  or approximately  $(1, 1.5, 1.41667, 1.4142156 \dots)$  convergence to the number  $\sqrt{2} \approx 1.4142136 \dots$  Through this is not obvious.

## 2.5: Series:

- **Definition:** A real series is of the form  $a_1 + a_2 + a_3 + \dots$  with  $a_i \in \mathbb{R}$ . It is often written by  $\sum_{i=1}^{\infty} a_i$ ,  $a_i$  are terms of the series. An important thing about series is the sequence of partial sums:  $(a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \dots)$  which we can write  $\sigma_n$  with  $\sigma_n = \sum_{i=1}^n a_i$  (Thomas et al. 2005).
- **Definition:** We say that a series  $a_1 + a_2 + a_3 + \dots$  is convergent if its sequence of partial sums is a convergent sequence and we write  $\sum_{i=1}^{\infty} a_i$  for the limit of this sequence.

So  $\sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} (\sum_{i=1}^n a_i)$  provided that this limit is real number. If the series is not convergent we say that it is divergent (Thomas et al. 2005).

- **Definition:** A series of the form  $\sum_{n=0}^{\infty} c_n x^n$  with  $c_n \in \mathbf{R}$  is called a power series.

### Taylor series

- **Definition:** If a function  $f(x)$  on an interval of the real line is the limit of a power series  $\sum_{n=0}^{\infty} c_n x^n$  then this is called the Taylor series.
- **Theorem (Taylor's Theorem):** Suppose  $f$  has finite derivative  $f^{n+1}$  of order  $n + 1$  at every point of an interval  $I$ , and let  $a$  and  $x$  be arbitrary points of  $I$ , then there is a point  $t$  between  $a$  and  $x$  such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k + \frac{f^{(n+1)}(t)}{(n+1)!} (x - a)^{n+1} \quad (2.5)$$

$$= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^n(a)}{n!} (x - a)^n + \frac{f^{(n+1)}(t)}{(n+1)!} (x - a)^{n+1} \quad (2.6)$$

Taylor series have applications for approximate solution of initial value problems; including Euler's Method, Taylor's Method, Runge-Kutta Method and Adams Bashfourth method that we discussed in the next chapter.

## 2.6 Types of Ordinary Differential Equations

- **Linear differential equation:** An Ordinary differential equation of the form

$$\frac{dy}{dx} = f_1(x)y + f_0(x) \quad (2.7)$$

Where  $f_1(x)$  and  $f_0(x)$  are continuous function is called a linear differential equations with a known solution (Willian, 2013).

- **Exact differential equation:** A differential equation

$$M(x, y)dx + N(x, y)dy = 0 \quad (2.8)$$

is said to be exact if there exists a function  $h(x, y)$  such that

$$d [h(x, y)] = M(x, y)dx + N(x, y)dy$$

Where  $d [h(x, y)] = \frac{\partial h}{\partial x}(x, y) + \frac{\partial h}{\partial y}(x, y)$ . (Willian, 2013)

- **Separable differential equation:** A differential equation is said to be separable, if it be separated in two variables  $y$  and  $x$  and its nonlinear ordinary differential equation and it's of the form.

$$f_1(x) \frac{dy}{dx} = f_0(x) \quad (2.9)$$

- **Bernoulli differential equation:** Is a nonlinear ordinary differential equation of the form

$$y' + f_1(x)y = f_0(x)y^n \quad (2.10)$$

With  $n \neq 1$ , and it has exact solution

- **Riccati differential equation:** Is a nonlinear ordinary differential equation and it is of the form  $y' = f_2(x)y^2 + f_1(x)y + f_0(x)$  (2.11)

Provided that  $f_2(x) \neq 0$  (polyanin, and Zaitsev, 2003).

- **Abel differential equation of first kind:** It is a nonlinear differential equation of the form,  $y'(x) = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$ , (2.12)  
Provided that  $f_3(x) \neq 0$

- **Note:** Abel differential equation is not integrable for arbitrary  $f_n(x)$

## 2.7 Reduction ADE to the Canonical Form

The transformation

$$y = N(x)M(p) - \frac{f_2}{3f_3}, \quad (2.13)$$

$$p = \int f_3 N^2 dx, \quad (2.14)$$

Where 
$$N(x) = \exp\left[\int \left(f_1 - \frac{f_2^2}{3f_3}\right) dx\right], \quad (2.15)$$

Brings equation (1) to the canonical (normal) form:

$$M' = M^3 + \emptyset(p) \quad (2.16)$$

Here the function  $\emptyset(p)$  is defined parametrically ( $x$  is the parameter) by the relations:

$$\emptyset = \frac{1}{f_3 N^3} \left( f_0 - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2} + \frac{1}{3} \frac{d}{dx} \frac{f_2}{f_3} \right), \quad (2.17)$$

$$p = \int f_3 N^2 dx$$

(polyanin, and Zaitsev, 2003).

## CHAPTER 3

### NUMERICAL METHODS

Numerical methods are the methods used for finding an approximate solution of mathematical problems, especially those problems that have no closed form. Even when problems have analytical solutions, numerical solutions can be need for comparison purposes. Approximate solutions are mainly suitable for science and mathematics. It also has an important role in applied and pure mathematics, engineering and physics and also it is useful in real life applications. Numerical methods based on iterations for finding answers. Most of nonlinear equations have no analytically solutions. The Abel differential equation is an example of such equations. So we seek methods for solving the ADE.

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \text{ provided that } f_3(x) \neq 0 \quad (3.1)$$

Under certain conditions, that is if one or more of  $f_0, f_1, f_2, f_3$  are zeros, then the ADE is known as a restricted ADE and in general, close form solutions can be found. On the other hand if none of these  $f_0, f_1, f_2, f_3$  are any zeros and depending on the form of the functions, then the ADE is known as unrestricted ADE. In this case generally, there are no known close form solutions. For these unrestricted ADE we can benefit from numerical method for solving initial value problems (IVP). Among the many of numerical methods we start with the simplest and it is Euler's Method.

#### 3.1 Euler's Method

Euler's method is one of methods for solving an initial value problem (IVP) of the form

$$\frac{dy}{dt} = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha \quad (3.2)$$

Although it is not very accurate, its simplicity may reduce it unable for the Abel initial value problem. An Excel work sheet for implementing and using graphical illustration can we need for this purpose.

We have to determine values of  $w_i$  at discrete set of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b,$$

Where  $w_i$  represents the approximate solution to  $y_i = y(x_i)$ . For simplicity, the approximate solution will be sought at equally spaced points;

$$x_i = a + ih, \text{ where } i = 1, 2, \dots, N$$

$$h = \frac{b-a}{N}, \text{ For some positive integer } N$$

Now, we will use Taylor's theorem to derive Euler's method. Suppose that  $y(x)$ , is the unique solution to (3.1), has continuous derivatives on  $[a, b]$ , so that for each  $i = 1, 2, \dots, N - 1$ ,

$$y(x_{i+1}) = y(x_i) + (x_{i+1} - x_i)y'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} y''(\xi_i) \quad (3.3)$$

For some number  $\xi_i$  in  $(x_i, x_{i+1})$ . because  $h = x_{i+1} - x_i$ , we have

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(\xi_i) \quad (3.4)$$

And because  $y(x)$  satisfies the differential equation (3.1),

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2} y''(\xi_i) \quad (3.5)$$

Euler's method arises by dropping the error term and replacing  $y_i$  (exact solution) by  $w_i$  (approximate solution):

$$w_0 = a$$

$$w_{i+1} = w_i + hf(x_i, y_i), \text{ for each } i = 1, 2, \dots, N - 1 \quad (3.6)$$

And for some number  $\xi_i$  in  $(x_i, x_{i+1})$ , then the truncation error =  $\frac{h^2}{2!} y''(\xi_i)$  (Bradie, 2006).

### 3.1.1 Absolute error, Relative error and Error bound

The absolute error is defined by magnitude of the difference between the actual solution and approximate solution; as  $|y(x_i) - w_i|$  (3.7)

Where  $y(x_i)$  is actual solution and  $w_i$  is the approximate solution at  $x = x_i$

It is clear that  $|y(x_i) - w_i| \rightarrow 0$  as  $h \rightarrow 0$ , so Euler's method is convergent.

The relative error is defined as

$$\left. \begin{aligned} \text{Relative error} &= \left| \frac{y(x_i) - w_i}{y(x_i)} \right| \text{ and,} \\ \text{Percentage error} &= \left| \frac{y(x_i) - w_i}{y(x_i)} \right| * 100 \end{aligned} \right\} \quad (3.8)$$

So, if the error is negligible, it shows that the procedure used in obtaining the approximation solution is better, and the outcomes are much close to the exact solution

Suppose  $y(x)$  is an actual solution of IVP.

$$\frac{dy}{dt} = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha$$

In which  $f$  is a continuous function satisfying Lipschitz condition with constant  $L$  on region.

$$D = \{ (x, y) : a \leq x \leq b \text{ and } -\infty \leq y \leq \infty \}$$

Then there exist a constant  $M$ , called the error bound, such that

$$|y''(x)| \leq M, \quad \text{for all } x \in [a, b] \quad (3.9)$$

Suppose  $w_0, w_1, w_2 \dots w_n$  are the approximations generated by the Euler's method for some positive integer  $N$ . Then for each  $i=0, 1, 2 \dots N$

$$|y(x_i) - w_i| \leq \frac{hM}{2L} [e^{L(x_i-a)} - 1] \quad (3.10)$$

The error bound is no longer linear in  $h$ , since



$$\lim_{h \rightarrow 0} \left( \frac{hM}{2} + \frac{\delta}{h} \right) = \infty, \quad (3.11)$$

Where  $\delta$  is a small number defined by  $|\delta_i| < \delta$  and  $\delta_i$  for  $i=1, 2, \dots, N$  denote the round of error associated with each  $w_i$ .

The error would be expected to become large even for some small values of  $h$ . Calculus can be used to determine for the step size  $h$ .

$$\text{Letting } E(h) = \left( \frac{hM}{2} \right) + \left( \frac{\delta}{h} \right) \quad (3.12)$$

$$\text{Implies that } E'(h) = \left( \frac{M}{2} \right) - \left( \frac{\delta}{h^2} \right). \quad (3.13)$$

If  $h < \sqrt{\frac{2\delta}{M}}$ , then  $E'(h) < 0$  and  $E(h)$  is decreasing.

If  $h > \sqrt{\frac{2\delta}{M}}$ , then  $E'(h) > 0$  and  $E(h)$  is increasing.

$$\text{The minimal value of } E(h) \text{ occurs when } h = \sqrt{\frac{2\delta}{M}} \quad (3.14)$$

This is called optimum  $h$ . Decreasing  $h$  beyond this value tends to increase the total error in the approximation. (Burden and Faires, 2010; Bradie, 2006).

### 3.2 Taylor's Method

Although the implementation of the Euler's method is both easy and straightforward, it is not very accurate. Now we will develop several higher-order one-step methods for first-order initial value problems the first one is Taylor's method, this method employs the Taylor series expansion of the solution to the equations, Assume that the solution of  $y(t)$  to the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad (3.15)$$

And  $y$  has  $(n+1)$  continuous derivative. If we develop the solution,  $y(x)$ , in terms of its  $n$ th Taylor polynomial about  $x_i$  and evaluate at  $x_{i+1}$ , we obtain

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i) \quad (3.16)$$

For some number  $\xi_i$  in  $(x_i, x_{i+1})$ .

From the differential equation, we know that we can replace

$$y'(x) = f(x, y(x)), y''(x) = f'(x, y(x)), \text{ And, generally } y^{(k)}(x) = f^{(k-1)}(x, y(x)) \quad (3.17)$$

In equation (3.15), we get

$$y(x_{i+1}) = y(x_i) + hf(x, y(x)) + \frac{h^2}{2}f'(x, y(x)) + \cdots + \frac{h^n}{n!}f^{(n-1)}(x, y(x)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)). \quad (3.18)$$

Note that Euler's method is a special case of Taylor's method of order  $n = 1$ . The local truncation error associated with (3.18) is clearly  $O(h^n)$ . Therefore equation (3.18) represents the general  $n$ th order Taylor's method.

The Abel differential equation of first kind:

$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$  With initial condition  $y(x_0) = y_0$  is showed by Taylor's method. In Chapter four, we will find a good process for achievement in excel program that is calculate  $y'(x), y''(x), y'''(x), \dots$  depending of order's of method and we can find  $T^n$  of the form

$$T^n(x_i, w_i) = f(x_i, w_i) + \frac{h}{2} f'(x_i, w_i) + \frac{h^2}{3!} f''(x_i, w_i) + \dots \frac{h^{n-1}}{n!} f^{n-1}(x_i, w_i) \quad (3.19)$$

For some number  $\xi_i$  in  $(x_i, x_{i+1})$ .

The Taylor's method of order  $n$  is given by

$$w_{i+1} = w_i + h T^n(x_i, w_i) \quad \text{For } i=1, 2, \dots, N-1 \quad (3.20)$$

If the derivative of  $f(x, y)$  exist at  $x = x_0$  and if we have an initial condition then it is easy to compute the solution of ADE at any variety of value  $x$ , so it easy to implement Taylor method for ADE and used graphical illustration (Athinson et al., 2009). If several methods of different order are applied to the same initial value problem and the same step size is used for each method, then, of course the higher order method will produce more accurate approximation. But what happen if we vary the step size from method to method in such a way that each method uses the same total number of function evaluation? Will higher order methods still outperform the lower order methods? Will be find in next chapter.

### 3.2.1 Taylor's method of order two with truncation error

We get Taylor's method of order two from Taylor series when  $n = 2$  then

$$T^2(x_i, w_i) = f(x_i, w_i) + \frac{h}{2} f'(x_i, w_i) \quad (3.21)$$

And the Taylor method of order two as the form

$$w_{i+1} = w_i + h T^2(x_i, w_i) \quad \text{For } i=1, 2, \dots, N-1 \quad (3.22)$$

For some number  $\xi_i$  in  $(x_i, x_{i+1})$

With truncation error  $= \frac{h^3}{3!} y'''(\xi_i)$ ,

To apply Taylor's method on ADE, it is clear that we need to initial condition  $y(x_0) = y_0$ . Since we have  $f'(x, y)$  we need to find  $f''(x, y)$  analytically.

Taking derivative of ADE may be a very hard work since it is nonlinear.

### 3.2.2 Taylor's method of order four with truncation error

We get Taylor's method of order four from Taylor series when  $n=4$  then

$$T^4(x_i, w_i) = f(x_i, w_i) + \frac{h}{2} f'(x_i, w_i) + \frac{h^2}{3!} f''(x_i, w_i) + \frac{h^3}{4!} f'''(x_i, w_i) \quad (3.23)$$

And Taylor method of order four as the form

$$w_{i+1} = w_i + h T^4(x_i, w_i) \quad \text{For } i = 1, 2, \dots, N-1 \quad (3.24)$$

With truncation error  $= \frac{h^5}{5!} y^{(5)}(\xi_i)$ , for some number  $\xi_i$  in  $(x_i, x_{i+1})$

Actually, Taylor method of order four is stressful with maximum effort especially when applied Abel differential equation, since we have to take four derivatives, it is too much hard work, or may be impossible some times.

### 3.3 Runge-Kutta Method

In this section, we will present the most important classes method for numerical solution of ordinary differential equations known as Runge-Kutte method, it is a popular method for solving initial value problem, owing to its accuracy and stability, as well as its ease of implementation, and this method does not need to take any derivatives. Runge-Kutte method classify according to its orders.

#### 3.3.1 Runge- Kutta Method of Order Two

The simplest Runge -Kutta method is an improved version of Euler's method, Where

$$w_{i+1} = w_i + \frac{1}{2} (k_1 + k_2) \quad (3.24)$$

In which

$$\left. \begin{aligned} k_1 &= h f(x_0, y_0) \\ k_2 &= h f(x_1 + h, y_1 + h k_1) \end{aligned} \right\} \quad (3.25)$$

Where:  $h = x_{i+1} - x_i$ ,

We can think of the values  $k_1$  and  $k_2$  as guess of the change in  $y$  when  $x$  advances by  $h$ , because they are the product of the change in  $x$  and a value for the slope of the curve  $\frac{dy}{dx}$ .

### 3.3.2 Runge-kutta method of order four

Accuracy is achieving by the use of the Runge-Kutta methods and gets rid of the involvement of derivatives, which is through evaluation of the function  $f(x, y)$  at a point selected on each subinterval. The most accurate of widely accepted Runge-Kutta method is the 4<sup>th</sup> order which for IVP uses many analytical steps. (Willian, 2013).

Then the fourth order Runge-Kutte method is

$$w_{i+1} = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (3.26)$$

And for step size  $h = x_{i+1} - x_i$ .

Where the coefficients are

$$\left. \begin{aligned} k_1 &= hf(x_i, y_i) \\ k_2 &= hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right) \\ k_4 &= hf(x_i + h, y_i + k_3) \end{aligned} \right\} \quad (3.27)$$

### 3.3.3 Runge-Kutta-Fehlberg method

An extensive of the Runge-Kutta method is called the Runge-Kutta-Fehlberg method. It has a procedure to determine if the proper step size  $h$  is being used. At each step two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size increased (John and Kurtis, 2004). The RKF-Method of order four has the form

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5, \quad (3.28)$$

The four coefficients  $k_1, k_3, k_4,$  and  $k_5$  are used. Note that  $k_2$  is not used. Better approximation for the solution can be determined using a Runge-Kutta method of order five which has the form

$$w_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \quad (3.29)$$

Where the coefficients are given by

$$\left. \begin{aligned} k_1 &= h f(x_i, w_i) \\ k_2 &= h f\left(x_i + \frac{h}{4}, w_i + \frac{1}{4}k_1\right) \\ k_3 &= h f\left(x_i + \frac{3h}{8}, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ k_4 &= h f\left(x_i + \frac{12h}{13}, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ k_5 &= h f\left(x_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \\ k_6 &= h f\left(x_i + \frac{h}{2}, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \end{aligned} \right\} \quad (3.30)$$

### 3.3.4 Runge-kutta-Verener method

This is another family of runge-kuuta method; it is based on the following formulas (Burden and Faires 2010).

The Runge-Kutta-Verner-method of 5th order has the form

$$w_{i+1} = w_i + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6 \quad (3.31)$$

The Runge-Kutta-Verner-method of 6th order has the form

$$w_{i+1} = w_i + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_6 + \frac{43}{616}k_8 \quad (3.32)$$

Notice that in both formulas  $k_2$  is not used

In which the coefficients are

$$\left. \begin{aligned} k_1 &= h f(x_i, w_i) \\ k_2 &= h f\left(x_i + \frac{h}{6}, w_i + \frac{1}{6}k_1\right) \\ k_3 &= h f\left(x_i + \frac{4h}{15}, w_i + \frac{4}{75}k_1 + \frac{16}{75}k_2\right) \\ k_4 &= h f\left(x_i + \frac{2h}{3}, w_i + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right) \\ k_5 &= h f\left(x_i + \frac{5h}{6}, w_i - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 - \frac{85}{96}k_4\right) \\ k_6 &= h f\left(x_i + \frac{h}{2}, w_i + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5\right) \\ k_7 &= h f\left(x_i + \frac{h}{15}, w_i + \frac{8263}{15000}k_1 - \frac{124}{75}k_2 + \frac{643}{680}k_3 - \frac{81}{250}k_4 - \frac{2484}{10625}k_5\right) \\ k_8 &= h f\left(x_i + h, w_i + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 - \frac{24068}{84065}k_5 + \frac{3850}{26703}k_7\right) \end{aligned} \right\} \quad (3.33)$$



### 3.4 Adam- Bashforth Explicit Methods

Another class of methods for approximating solution of IVP's is these called multistep methods (David and Desmond, 2010). The Adam- Bashfourth explicit methods have the following formulas;

#### 3.4.1: Adams-Bashfourth 2-step explicit method

$$w_{i+1} = w_i + \frac{1}{2} h [3f(x_i, y_i) - f(x_{i-1}, y_{i-1})] \quad (3.34)$$

Where  $i=1, 2, \dots, N-1$

#### 3.4.2: Adams-Bashfourth 3-step explicit method

$$w_{i+1} = w_i + \frac{h}{12} [23 f(x_i, y_i) - 16 f(x_{i-1}, y_{i-1}) + 5 f(x_{i-2}, y_{i-2})] \quad (3.35)$$

Where  $i=2, 3, \dots, N-1$

#### 3.4.2: Adams-Bashfourth 4-step explicit method

$$w_{i+1} = w_i + \frac{h}{24} [55 f(x_i, y_i) - 59 f(x_{i-1}, y_{i-1}) + 37 f(x_{i-2}, y_{i-2}) - 9 f(x_{i-3}, y_{i-3})] \quad (3.36)$$

Where  $i=3, 4, 5, \dots, N-1$

#### 3.4.3: Adams-Bashfourth 5-step explicit method

$$w_{i+1} = w_i + \frac{h}{720} [1901 f(x_i, y_i) - 2774 f(x_{i-1}, y_{i-1}) + 2616 f(x_{i-2}, y_{i-2}) - 1274 f(x_{i-3}, y_{i-3}) + 251 f(x_{i-4}, y_{i-4})] \quad (3.37)$$

Where  $i=4, 5, 6, \dots, N-1$

## CHAPTER 4

### METHODOLOGY

#### 4.1 Microsoft Excel Sheet

Microsoft Excel is an application program created by Microsoft Company. It can help to create, edit and calculate numerical data by using rules and formulas added to the system of the program. This program is a great tool to be used for data collection and data entry, Even to use for some numerical manipulation. Excel can show input and output as diagrams, graphs and charts.

Excel can allow users to separate data to show and edit all kinds of changes. Visual Basic is used to created programs that work on Excel, in order to cover be difficulties for some numerical methods that cannot be solved easily. Program creators can directly use Visual Basic Editor to rewrite the codes by Windows for writing codes, code module editor and debugging.

In this chapter, we present applications of excel sheet in numerical analysis and implementing algorithm for solving ordinary differential equation which is include Abel differential equation of first kind

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) , \text{ provided that } f_3(x) \neq 0 ,$$

We classified in to two forms, restricted form and unrestricted form. We start first with restricted form. As mentioned the ability to perform calculation is one of the purposes of using a spreadsheet application. We used different method and for each method we will show the applications by many step and graphical illustrations.

## 4.2 Restricted form of ADEs

To be sure the correctness of the results in unrestricted form of ADEs of first kind we have to present some special cases for restricted form of ADEs, We can obtain under some condition, As we discussed in Chapter two, The general form of ADE can be transformed to a canonical form and also we can obtain some ordinary differential equation that has known closed form solution given below.

1. Separable differential equation if  $f_3(x) = f_1(x) = f_0(x) = 0$
2. Linear differential equation if  $f_3(x) = f_2(x) = 0$
3. Bernoulli differential equation if  $f_2(x) = 0$
4. Ricatti differential equation if  $f_3(x) = 0$

Numerical solutions for each of these above cases were presented by taking an Example and compare with given the known closed form solution. Also we present different order of each method to compare accuracy, and we will use Euler's method to solve separable differential equation, Taylor's Method to solve Linear differential equation, Runge-kutta Method to solve Bernoulli differential equation and Adam bash forth Explicit method to solve Ricatti differential equation, and for all of these we will use different step size "h" and solve each of them using Excel work sheet and graphical illustration to show the results.

### 4.3 Implementing Euler's Method Using Excel

Starting a blank Excel sheet, simply organize the sheet as in Table 4.1 and so far no calculation has taken place; all what has been done is labeling the cells according to the requirement of the problem in hand. In this case Example 1 is used to demonstrate the way Excel is used for the implementation of numerical solution, namely Euler's method.

$$y' = 6xy^2, \quad y(1) = 1/25, \quad 1 \leq x \leq 2. \quad y = 1/(28 - 3x^2)$$

As one can see the number of columns depends on what we want to present and the number of rows depend on how far in the variable  $x$  we need to take the solution.

The following are step by step instructions to implement the procedure of calculation;

Note: What appears inside the curly bracket is what you type in or press on the keyboard

Step 1: Since, the step size  $h$  is constant then we must type in B2 and drag down

Step 2: In the cell C2, simply type :{(=1)} this being the initial value of  $x$  as soon as you press (return) you will see the result 1.

In the cell C3, type: {(=C2+B2)} and return, you will see the result 1.1

In cell C4, either simply type: {(= C3 +B3)} or try this, (click) on cellC3, you will see a small square at the lower right corner of the cell. Carefully bring the curser on this square (the cursor will change shape), click and hold left button on the mouse and carefully (drag) down to the cell c4 and release the mouse button you will see {(= C3 +B3)}. Now click on cell C4 and (hold) the bottom right corner square and (drag) down all the way to cell C12. Now, all the cells up to cell c12 filled with required formula with cell C12 having the value 2 the last value of the variable  $x$ .

Step 3: In cell D2 calculate the exact value of  $y$  at  $x = 1$ . To do this, type {(=1/ (28 – 3\*C2^2)} which is  $1/(28 - 3x^2)$ , if you done this correctly you should get  $1/25 = 0.04$ , which is the initial value of  $y$ .

Step 4.1: In cell E2 type {(=1/25)} the initial value  $w_0$  at  $x = 1$ .

Step 5: In cell F2 calculate  $f(x, y)$  at  $x = 1$ , that is in cell F2 type: {(=6\* C2\*E2^2)} which is  $6xy^2$ .

Step 4.2: In cell E3 type:  $\{= E2+B2*F2\}$  which is  $w_1 = w_0 + h f(x, y)$ , if correct you should get the value 0.0096.

Click cell D2 and drag the corner down to cell D3.

Click cell F2 and drag the corner down to cell F3.

Now you have the row containing the cells D3, E3 and F3, all these cells contain formulas in them, which mean that you can copy them all to the cells below in a very simple manner that is by dragging down the corner square.

Highlight cells D3, E3 and F3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12 and G12. Magic, it is done.

**Table 4.1:** Illustration of the exact solution and Euler's Method by using Excel Sheet

	A	B	C	D	E	F
1	Index i	Step size h	Variable $x_i$	Exact $y_i$	Euler $w_i$	$f(x_i, y_i)$
2	1	0.1				
3	2	0.1				
4	.	.				
5	.	.				
6	11	0.1				

**4.3.1 Example 1:** Consider the IVP  $y'(x) = 6xy^2, y(1) = \frac{1}{25}, 1 \leq x \leq 2$

Use Euler method to find a numerical solution and compare with the actual solution

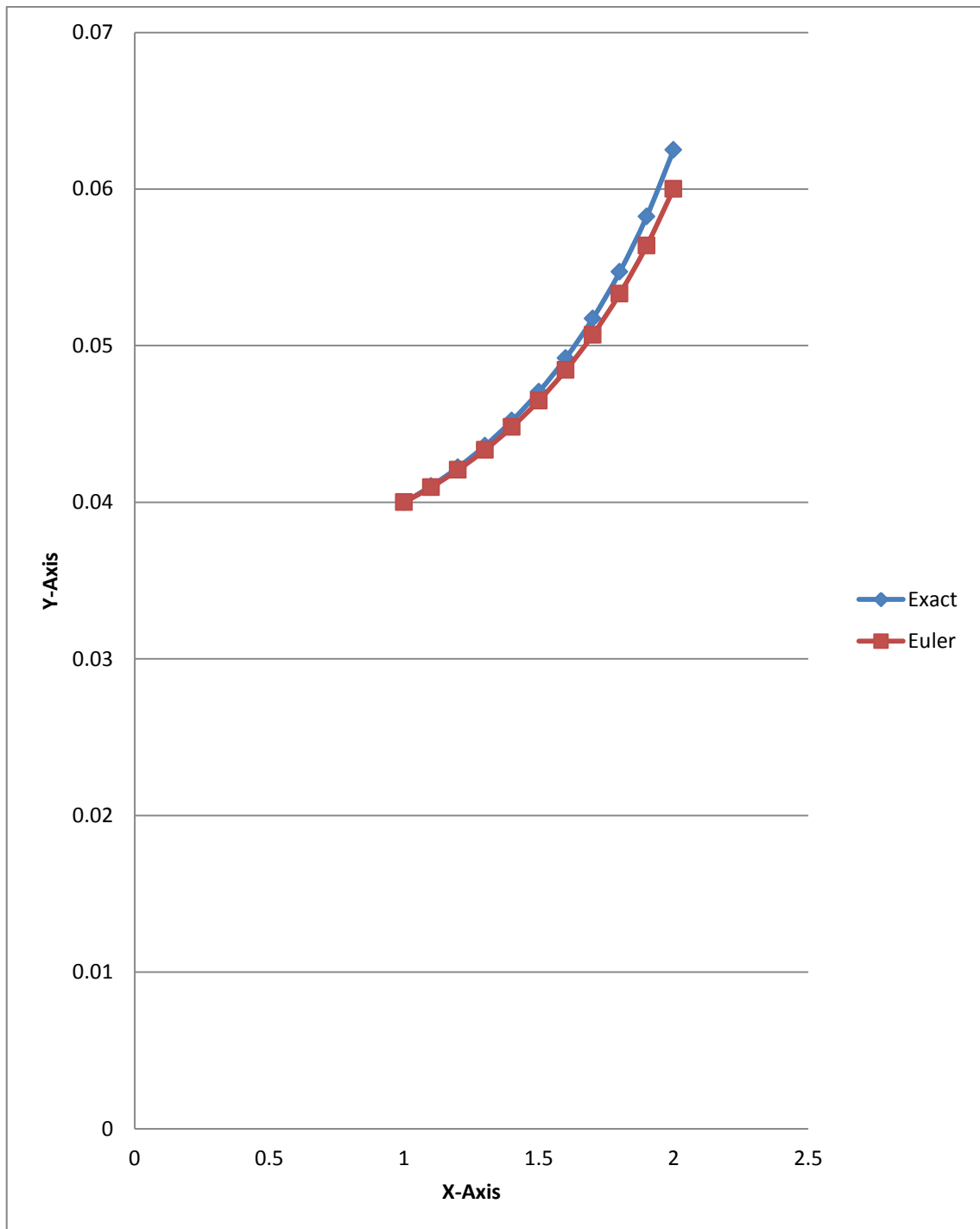
Solution: if we compare this problem with the general form of ADE we can see that this is a simple case of ADE because it seems that  $f_3(x) = 0, f_2(x) = 6x, f_1(x) = 0$  and  $f_0(x) = 0$ , then this differential equation is called Separable Differential equation, The analytical of this problem is.  $y(x) = \frac{1}{28-3x^2}$ . (Dawkins, 2007).

Step by Step solution of Euler method

**Table 4.2:** Illustration of the exact solution and Euler's Method

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Exact</b>	<b>Euler</b>	<b><math>f(x_i, y_i)</math></b>
0	0.1	1	0.04	0.04	0.0096
1	0.1	1.1	0.0410341	0.04096	0.011073
2	0.1	1.2	0.0422297	0.0420673	0.012742
3	0.1	1.3	0.043611	0.04334145	0.014652
4	0.1	1.4	0.045208	0.04480666	0.016864
5	0.1	1.5	0.0470588	0.04649308	0.019454
6	0.1	1.6	0.0492126	0.04843853	0.022524
7	0.1	1.7	0.0517331	0.05069097	0.02621
8	0.1	1.8	0.0547046	0.05331193	0.030695
9	0.1	1.9	0.0582411	0.05638147	0.036239
10	0.1	2	0.0625	0.06000538	0.043208

Although the results are good approximation, these are of low accuracy. This is illustrated in Figure.4.1.



**Figure 4.1:** Euler's method and exact solution where  $h=0.1$

•**Error:**

$$y'(x) = 6xy^2 \text{ Then } y''(x) = 6y^2 + 12xyy'$$

$$\rightarrow y''(x) = \frac{6}{(28-3x^2)} + 72x^2 \left( \frac{1}{28-3x^2} \right)^3$$

Then choose  $M=0.095$  because

$$|y''(x_i)| \leq M, \text{ for all } x \in [1,2]$$

And by theorem

$$\left| \frac{\partial f}{\partial y} \right| = |12xyy'| = \left| \frac{12x}{28-3x^2} \right| < 1.5$$

So  $L=1.5$

$$\text{Error bound} = \frac{hM}{2L} [e^{L(x_i-a)} - 1] = \frac{0.1(0.095)}{2(1.5)} [e^{(1.5)(x_{i+1}-1)} - 1]$$

We can use Excel sheet see the table

**Table 4.3:** Error bound for Euler's method

<b>h</b>	<b>x<sub>i</sub></b>	<b>Error bound</b>
0.1	1	0
0.1	1.1	0.000512
0.1	1.2	0.001108
0.1	1.3	0.0018
0.1	1.4	0.002603
0.1	1.5	0.003537
0.1	1.6	0.004622
0.1	1.7	0.005883
0.1	1.8	0.007347
0.1	1.9	0.009049
0.1	2	0.011025



It is clear from the Table 4.3 that:

$$|y(x_i) - w_i| \leq \frac{0.1(0.095)}{2(1.5)} [e^{1.5(x_i-1)} - 1] \text{ for all } x \in [1,2]$$

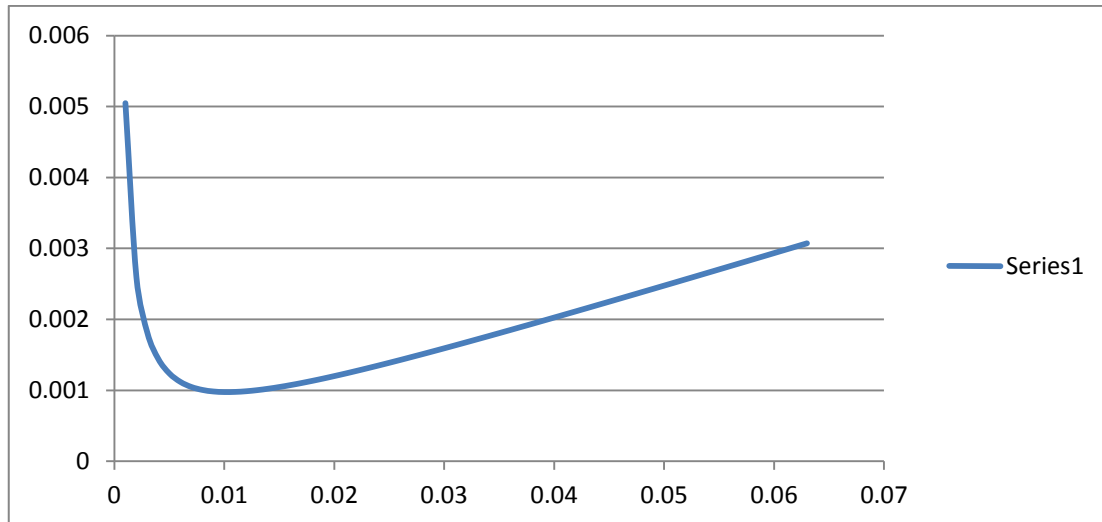
And For Optimum h

$$\text{Opt } h = \sqrt{\frac{2\delta}{M}} = \sqrt{\frac{2(0.00005)}{0.095}} = 0.01026$$

Then the optimum h=0.01

**Table 4.4:** illustration for Optimum h

<b>H</b>	<b>E`(h)</b>	<b>δ</b>	<b>E(h)</b>
0.001	-4.9525	0.000005	0.005048
0.002	-1.2025	0.000005	0.002595
0.003	-0.50806	0.000005	0.001809
0.004	-0.265	0.000005	0.00144
0.005	-0.1525	0.000005	0.001238
0.006	-0.09139	0.000005	0.001118
0.007	-0.05454	0.000005	0.001047
0.008	-0.03063	0.000005	0.001005
0.009	-0.01423	0.000005	0.000983
<b>0.01</b>	<b>-0.0025</b>	<b>0.000005</b>	<b>0.000975</b>
0.011	0.006178	0.000005	0.000977
0.012	0.012778	0.000005	0.000987
0.013	0.017914	0.000005	0.001002
0.014	0.02199	0.000005	0.001022
0.015	0.025278	0.000005	0.001046
0.016	0.027969	0.000005	0.001073
0.017	0.030199	0.000005	0.001102
0.018	0.032068	0.000005	0.001133
0.019	0.03365	0.000005	0.001166
0.02	0.035	0.000005	0.0012
0.021	0.036162	0.000005	0.001236



**Figure 4.2: Optimum  $h$**

#### 4.4 Implementing Taylor's Method Using Excel

##### 4.4.1 Implementing Taylor's method of order two using excel

Starting a new blank Excel sheet, simply organize the sheet as in Table 4.5. In this case Example 2 is used to demonstrate the way Excel is used for the implementation of Taylor's method of order 2.

$$y' = f(x, y) = -\frac{1}{2}y + e^{-x} + 3, \quad 0 \leq x \leq 1, \quad y(0) = 1 \quad y = 6 - 2e^{-x} - 3e^{-\frac{x}{2}}$$

As one can see the number of columns depends on what we want to present and the number of rows depend on how far in the variable  $x$  we need to take the solution.

The following are step by step instructions to implement the procedure of calculation;

Note: What appears inside the curly bracket is what you type in or press on the keyboard

Step 1: Since, the step size  $h=0.1$  is constant then we must type `{=0.1}` in B2 and drag down

Step 2: In the cell C2, type `{(=0)}` as soon as you press (return) you will see the result 0.

In the cell C3, type `{(=C2+B2)}` and return, you will see the result 0.1

In cell C4, either simply type:  $\{ (= C3 + B3) \}$  or try this, (click) on cell C3, you will see a small square at the lower right corner of the cell. Carefully bring the cursor on this square (the cursor will change shape), click and hold left button on the mouse and carefully (drag) down to the cell c4 and release the mouse button you will see  $\{ (= C3 + B3) \}$ . Now click on cell C4 and (hold) the bottom right corner square and (drag) down all the way to cell C12, all the cells up to cell c12 filled with required formula with cell C12 having the value 1 the last value of the variable  $x$ .

Step 3: In cell D2 calculate the exact value of  $y$  at  $x = 0$ . To do this, type  $\{ = (6 - 2 * \text{EXP} (-C2) - 3 * \text{EXP} (-C2/2)) \}$  which is  $6 - 2e^{-x} - 3e^{\frac{-x}{2}}$ , if you done this correctly you should get 1, which is the initial value of  $y$  at  $x_0$ .

Step 4.1: In cell E2 type  $\{ (=1) \}$  the initial value  $w_0$  at  $x = 0$ .

Step 5: In cell F2 calculate  $f(x, y)$  at  $x = 0$ , that is in cell F2 type:  $\{ (= (-1/2) * E2 + \text{EXP} (-C2) + 3) \}$  which is  $-\frac{1}{2}y + e^{-x} + 3$

Step 6: In cell G2 calculate  $f'(x, y)$  at  $x=0$ , that is in cell G2 type:  $\{ (= (1/4) * E2 - (3/2) * \text{EXP} (-C2) - (3/2)) \}$  which is  $y'' = f'(x, y) = \frac{1}{4}y - \frac{3}{2}e^{-x} - \frac{3}{2}$

Step 7: In cell H2 calculate  $T^2$ , that is in cell H2 type:  $\{ (= F2 + (1/2) * B2 * G2) \}$

Step 4.2: In cell E3 type :  $\{ = E2 + B2 * H2 \}$  which is  $w_1 = w_0 + h T^2$  if correct you should get the value 1.

Click cell D2 and drag the corner down to cell D3.

Highlight cells F2, G2 and H2 and drag the corner down to cell F3, G3 and H3.

Now you have the cells D3, E3, F3, G3 and H3 all containing formulas in them, which mean that you can copy them all to the cells below in a very simple manner.

Highlight cells D3, E3, F3, G3 and H3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12, G12 and H12. That is it.

**Table 4.5:** Illustrated the tabulation of Taylor's Method for order two

A	B	C	D	E	F	G	H
$i$	$h$	$x_i$	Exact $y_i$	Taylor $w_i$	$f(x_i, y_i)$	$f'(x_i, y_i)$	$T^2$
1	0.1						
2	0.1						
.	.						
.	.						
5	0.1						

**4.4.1.1 Example 2:** Use Taylor method of order two to find an approximate solution to the following IVP

$$y' = f(x, y) = -\frac{1}{2}y + e^{-x} + 3, \quad 0 \leq x \leq 1$$

$$y(0) = 1$$

Solution: If we compare this above problem with general form of ADE, we see that  $f_3(x) = f_2(x) = 0$  and we have  $f_1(x) = -\frac{1}{2}$ , and  $f_0(x) = e^{-x} + 3$  (David and Desmond, 2010).

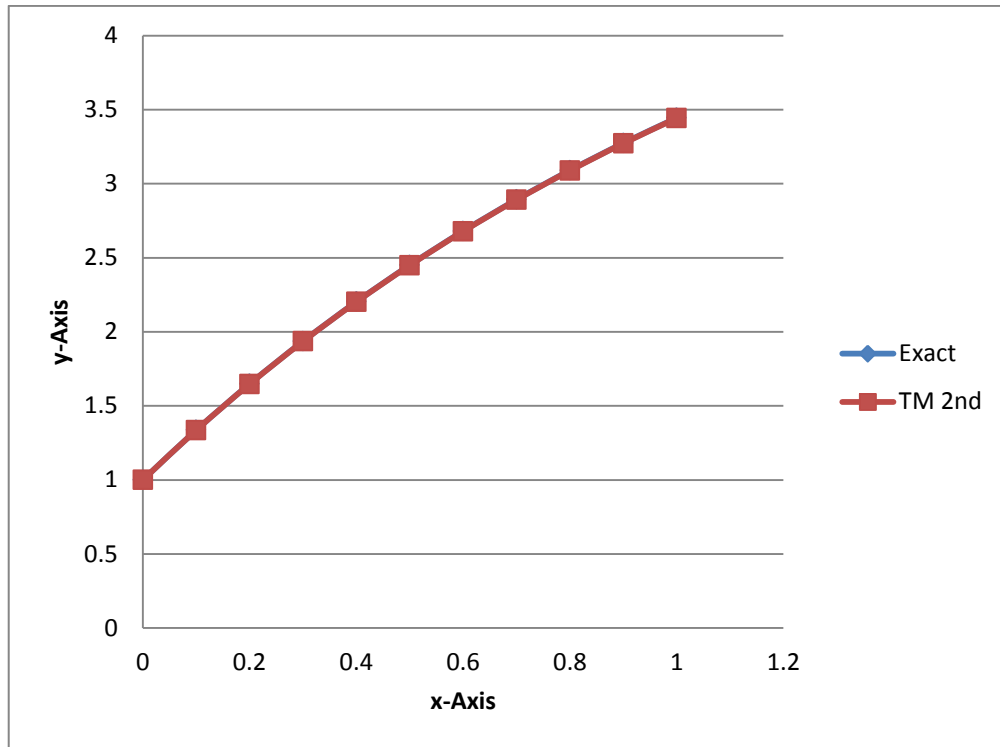
Now, we need to take second derivative to apply Taylor Method of order two,

$$y'' = f'(x, y) = \frac{1}{4}y - \frac{3}{2}e^{-x} - \frac{3}{2}$$

When the exact solution is  $y = 6 - 2e^{-x} - 3e^{-\frac{x}{2}}$

**Table 4.6:** Comparison of the exact and 2<sup>nd</sup> order Taylor's solution

$i$	$h$	$x_i$	Exact	Taylor 2th	$f(x_i, y_i)$	$f'(x_i, y_i)$	T(2)
0	0.1	0	1	1	3.5	-2.75	3.3625
1	0.1	0.1	1.336637	1.33625	3.236712	-2.52319	3.110553
2	0.1	0.2	1.648026	1.647305	2.995078	-2.31627	2.879265
3	0.1	0.3	1.93624	1.935232	2.773202	-2.12742	2.666831
4	0.1	0.4	2.203168	2.201915	2.569363	-1.955	2.471613
5	0.1	0.5	2.450536	2.449076	2.381993	-1.79753	2.292116
6	0.1	0.6	2.679922	2.678288	2.209668	-1.65365	2.126985
7	0.1	0.7	2.892765	2.890986	2.051092	-1.52213	1.974986
8	0.1	0.8	3.090382	3.088485	1.905087	-1.40187	1.834993
9	0.1	0.9	3.273976	3.271984	1.770578	-1.29186	1.705985
10	0.1	1	3.444649	3.442583	1.646588	-1.19117	1.587029



**Figure 4.3:** Taylor method 2<sup>nd</sup> and exact solution

From Figure 4.3, it is clear that there is a good accordance between the exact solution and the results from Taylor's method. If we seek to get a better approximation, we can either choose a higher order method or use the smaller step size  $h$ .

#### 4.4.2 Implementing Taylor's method of order four using excel

Since Taylor's Method of order 4, is an extension of Taylor's method of order 2. That is, in order 4,  $f''(x, y)$  and  $f'''(x, y)$  terms need to be added to  $T^4$  expression. So, two extra columns are required. In a similar manner to order 2 that is repeat steps 1, 2, 3, 4, 5, 6

Step 7: In cell H2 calculate  $f''(x, y)$  at  $x=0$  that is in cell H2 type  $\{=(1/16)*E2+(9/8)*EXP(-C2)-(3/8)\}$  which is  $y''' = f''(x, y) = \frac{1}{16}y - \frac{9}{8}e^{-x} - \frac{3}{8}$

Step 8: In cell I2 calculate  $f'''(x, y)$  at  $x=0$  that is in cell I2 type  $\{=(1/256)*E2+(135/128)*EXP(-C2)-(3/128)\}$  which is  $y'''' = f'''(x, y) = \frac{1}{256}y - \frac{9}{128}e^{-x} + \frac{3}{128}$

Step 9: In cell J2 calculate  $T^4$  that is in cell J2 type  $\{=F2+ (B2/2)*G2+ ((B2^2)/6)*H2+ ((B2^3)/12)*I2\}$  which is  $T^4(x_i, w_i) = f(x_i, w_i) + \frac{h}{2} f'(x_i, w_i) + \frac{h^2}{3!} f''(x_i, w_i) + \frac{h^3}{4!} f'''(x_i, w_i)$ .

Step 4.2: In cell E3 type  $\{=E2+B2*J2\}$  which is  $w_1 = w_0 + h T^4$  if correct you should get the value 1.

Click cell D2 and drag the corner down to cell D3.

Highlight cells F2, G2, H2, I2 and J2 drag the corner down to cell F3, G3, H3, I3 and J3.

Now you have the cells D3, E3, F3, G3, H3, I3 and J3 all containing formulas in them, which mean that you can copy them all to the corresponding cells below in a very simple manner.

Highlight cells D3, E3, F3, G3, H3, I3 and J3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells up to D12, E12, F12, G12, H12, I12 and J12.that is it.

**Table 4.7:** Illustration of the exact solution and Taylor's method 4<sup>th</sup> by using Excel

A	B	C	D	E	F	G	H	I	J
$i$	$h$	$x_i$	Exact $y_i$	Taylor $w_i$	$f(x_i, y_i)$	$f'(x_i, y_i)$	$f''(x_i, y_i)$	$f'''(x_i, y_i)$	$T^4$
1	0.1								
2	0.1								
.	.								
.	.								
5	0.1								

**Example:** Use Taylor method of order four to find an approximate solution to the following IVP

$$y' = f(x, y) = -\frac{1}{2}y + e^{-x} + 3, \quad 0 \leq x \leq 1 \quad y(0) = 1$$

Solution:

We describe it that it is ADE, now we have to take third and fourth derivatives

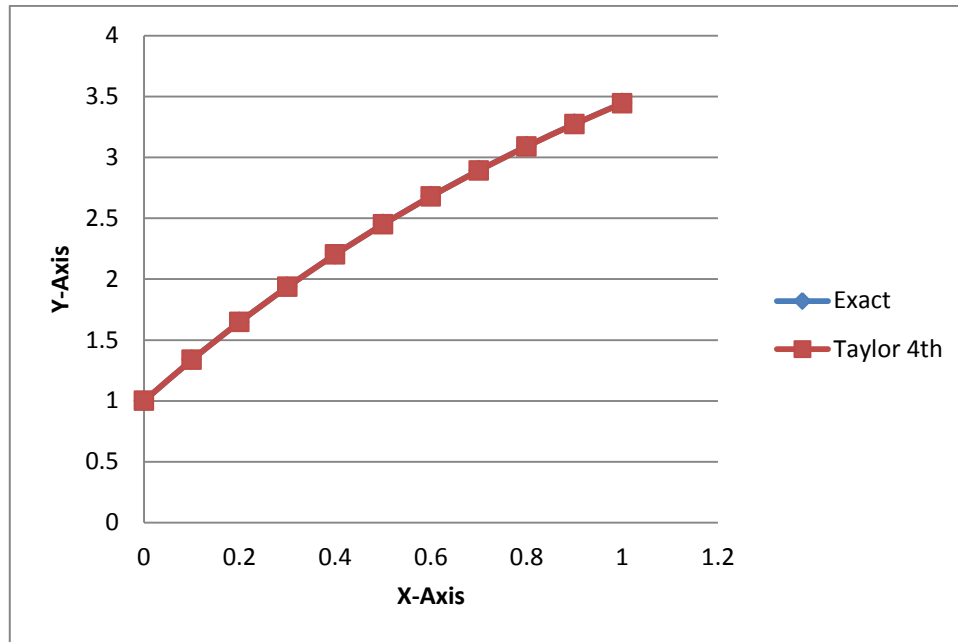
$$y''' = f''(x, y) = \frac{1}{16}y - \frac{9}{8}e^{-x} - \frac{3}{8} \quad \text{and,}$$

$$y'''' = f'''(x, y) = \frac{1}{256}y - \frac{9}{128}e^{-x} + \frac{3}{128}$$

Let  $h=0.1$

**Table 4.8:** Illustration the Taylor method of order four and exact solution

$x_i$	Exact	Taylor 4th	$f(x_i, y_i)$	$f'(x_i, y_i)$	$f''(x_i, y_i)$	$f'''(x_i, y_i)$	T2
0	1	1	3.5	-2.75	0.8125	1.035156	3.36394
0.1	1.336637	1.336394	3.23664	-2.52316	0.726467	0.936104	3.111771
0.2	1.648026	1.647571	2.994945	-2.3162	0.649045	0.846503	2.880287
0.3	1.93624	1.9356	2.773018	-2.12733	0.579395	0.765455	2.667681
0.4	2.203168	2.202368	2.569136	-1.95489	0.516758	0.692144	2.472311
0.5	2.450536	2.449599	2.381731	-1.7974	0.460447	0.625832	2.292681
0.6	2.679922	2.678867	2.209378	-1.6535	0.409842	0.565852	2.127433
0.7	2.892765	2.891611	2.05078	-1.52198	0.364384	0.5116	1.975331
0.8	3.090382	3.089144	1.904757	-1.40171	0.323567	0.462531	1.83525
0.9	3.273976	3.272669	1.770235	-1.29169	0.286933	0.41815	1.706164
1	3.444649	3.443285	1.646237	-1.191	0.25407	0.378011	1.587142



**Figure 4.4:** Taylor's method of order four and Exact solution

## 4.5 Implementing Runge-kutta Method Using Excel.

### 4.5.1 Implementing Runge-Kutta method of order two using excel

Again starting a new blank Excel sheet, simply organize the sheet that as in Table 4.9. In this case Example 3 is used to demonstrate the way Excel is used for the implementation of Runge-Kutta Method of order two.

$$y' = (x + 2x^3)y^3 - xy, \quad 0 \leq x \leq 2, \quad y(0) = \frac{1}{3}, \quad y(x) = (3 + 2x^2 + 6e^{x^2})^{-\frac{1}{2}}$$

The following are step by step instructions to implement the procedure of calculation; in a similar manner to order 2 that is repeat steps 1, 2

Note: What appears inside the curly bracket is what you type in or press on the keyboard.

Step 3: If we have an exact solution then we must devote column D for it. In cell D2 calculate the exact value of  $y$  at  $x=0$ . To do this, type  $\{=(3+2*C2+6*EXP(C2^2))^{(-1/2)}\}$  which is  $(3 + 2x^2 + 6e^{x^2})^{-\frac{1}{2}}$ , if you done this correctly you should get  $1/3 = 0.333333$ , which is the initial value of  $y$ .

Step 4: 1. In cell E2 type  $\{=(1/3)\}$  the initial value  $w_0$  at  $x = 0$



Step 5: In cell F2 calculate  $k_1 = hf(x, y)$  which is  $= h((x + 2x^3)y^3 - xy)$  at  $x = 0$ , that is in cell F2 type:  $\{=(B2*(C2+2*C2^3)*E2^3-C2*E2)\}$ .

Step 6: In cell G2 calculate  $k_2 = hf(x + h, y + hk_1)$  at  $x=0$ , that is in cell G2 type:  $\{=(B2*(((C2+B2)+2*(C2+B2)^3)*(E2+B2*F2)^3-(C2+B2)*(E2+B2*F2)))\}$

Step 4: 2. In cell E3 type  $\{= E2+ (1/2)*(F2+G2)\}$  which is  $w_1 = w_0 + \frac{1}{2}(K_1 + K_2)$ , if correct you should get the value 0.333333.

Click cell D2 and drag the corner down to cell D3.

Highlight cell F2 and G2 and drag the corner down to cell F3 and G3.

Now you have the rows D3, E3, F3 and G3 all containing formulas in them, which mean that you can copy them all to the cells below in a very simple manner.

Finally, Highlight cells D3, E3, F3 and G3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12 and G12. That is it.

**Table 4.9:** Illustration of Runge-Kutta method of order two and Exact solution

	A	B	C	D	E	F	G
1	$i$	$h$	$x_i$	Exact $y_i$	RKM 2nd	$k_1$	$k_2$
2	1	0.1					
3	2	0.1					
4	.	.					
5	.	.					
6	11	0.1					

**4.5.1.1 Example 3:** Consider the following IVP

$$y' = (x + 2x^3)y^3 - xy, \quad 0 \leq x \leq 2$$

$$y(0) = \frac{1}{3}$$

Use Runge-Kutta Method of order two to find an approximate solution

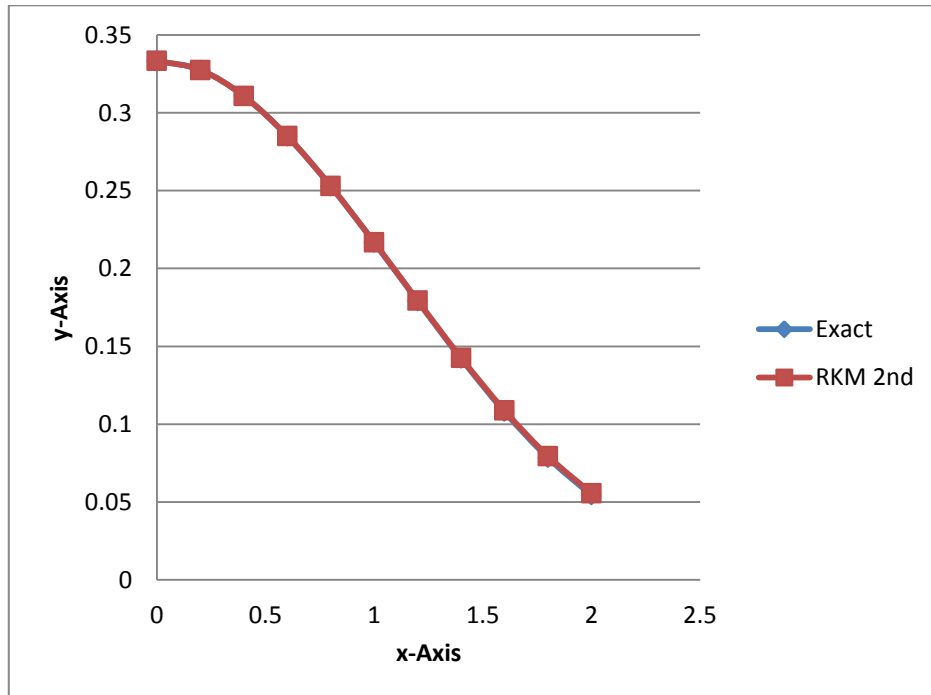
Solution:

If we compare this Example with general form of ADE we see that  $f_3(x) = (x + 2x^3)$ ,  $f_2(x) = -x$  and  $f_1(x) = f_0(x) = 0$  then this Example is changed directly to Bernoulli Deferential Equation because we can write as the form  $y' + xy = (x + 2x^3)y^3$  and the close form solution is  $y(x) = (3 + 2x^2 + 6e^{x^2})^{\frac{-1}{2}}$ . (Burden and Fairs, 2010)

First we use Runge-kutta method of order two by using Microsoft excel sheet

**Table 4.10:** Comparison of the exact and 2<sup>nd</sup> RKMs Solution

h	$x_i$	Exact	RKM 2th	$k_1$	$k_1$
0.2	0	0.333333	0.333333	0	-0.01173
0.2	0.2	0.327475	0.327467	-0.01158	-0.02194
0.2	0.4	0.310669	0.310705	-0.02169	-0.0297
0.2	0.6	0.284902	0.285011	-0.02942	-0.0348
0.2	0.8	0.252708	0.252898	-0.03456	-0.03742
0.2	1	0.216626	0.216905	-0.03726	-0.03772
0.2	1.2	0.179017	0.179418	-0.03768	-0.03576
0.2	1.4	0.142111	0.142695	-0.03595	-0.03178
0.2	1.6	0.107999	0.108831	-0.0323	-0.02634
0.2	1.8	0.078402	0.079508	-0.02727	-0.02038
0.2	2	0.054346	0.055683	-0.02165	-0.01479



**Figure 4.5:** RKM of order two and Exact solution

It is clear from the Table and the Figure that the results were expected, since the method used a low order one and the step size  $h$  used was large.

#### 4.5.2 Implementing Runge-kutta of order four using excel

Since Runge-Kutta Method of order 4, is an extension of RKM of order 2. That is, in order 4,  $k_3$  and  $k_4$  terms need to be added and calculated. So, two extra columns are required. In a similar manner to order 2 that is repeat steps 1, 2, 3, 4.1

Step 5: In cell F2 calculate  $k_1 = h f(x, y)$  which is  $= h((x + 2x^3)y^3 - xy)$  at  $x = 0$ , that is in cell F2 type:  $\{=(B2*(C2+2*C2^3)*E2^3-C2*E2)\}$ .

Step 6: In cell G2 calculate  $k_2 = hf(x + 0.5 * h, y + 0.5 * k_1)$  at  $x=0$ , that is in cell G2 type:  $\{=(B2*(((C2+0.5*B2) +2*(C2+0.5*B2) ^3)*(E2+0.5*F2) ^3-(C2+0.5*B2)*(E2+0.5*F2)))\}$

Step 7: In cell H2 calculate  $K_3 = h f(x + \frac{h}{2}, y + \frac{1}{2}k_2)$  which is  $= h((x + 0.5h + 2(x + 0.5h)^3)(y + 0.5K_2)^3 - (x + 0.5h)(y + 0.5k_2))$  at  $x = 0$ , that is in cell H2 type:  $\{=(B2*(((C2+0.5*B2) +2*(C2+0.5*B2) ^3)*(E2+0.5*G2) ^3)-C2+0.5*B2)*(E2+0.5*G2))\}$ .

Step 8: In cell I2 calculate  $K_4 = h(x + h, y + K_3)$  which is  $= h((x + h) + 2(x + h)^3)(y + k_3)^3 - (x + h)(y + k_3)$ , that is in cell I2 type:  $\{=(B2*(((C2+B2) +2(C2+B2) ^3)*(E2+H2) ^3)-(C2+B2)*(E2+H2)))\}$ .

Step 4: 2. In cell E3 type:  $= E2 + (1/6)*(F2+2*G2+2*H2+I2)$  which is  $w_1 = w_0 + \frac{1}{6}(K_1 + 2K_2 + 2k_3 + k_4)$ , if correct you should get the value 0.333333.

Click cell D2 and drag the corner down to cell D3.

Highlight cells F2, G2, H2, and I2 drag them corner down to cell F3, G3, H3 and I3.

Now you have the rows D3, E3, F3, G3, H3 and I3 all containing formulas in them, which mean that you can copy them all to the cells below in a very simple manner.

Finally, Highlight cells D3, E3, F3, G3, H3 and I3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12, G12, H12 and I12. That is it.

**Table 4.11:** Illustration of the exact solution and RKM of order four by Excel Sheet

	A	B	C	D	E	F	G	H	I
1	i	h	$x_i$	Exact $y_i$	RKM 4 <sup>th</sup>	$k_1$	$k_2$	$k_3$	$k_4$
2	1	0.1							
3	2	0.1							
4	.	.							
5	.	.							
6	11	0.1							

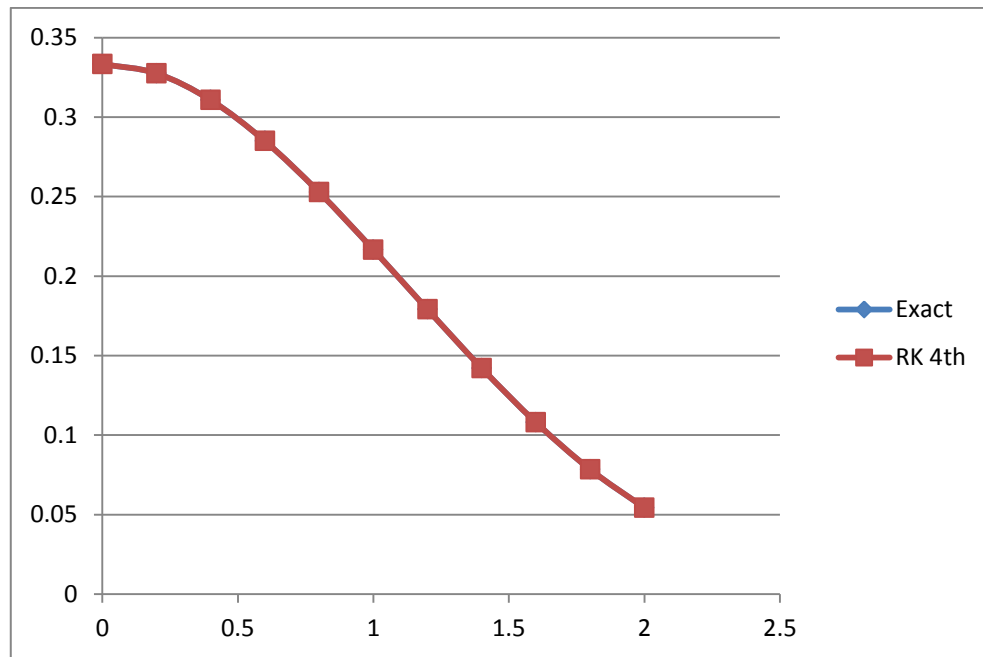
**Example:** Determine the approximate solutions by Runge-kutta method of order four for the following IVP

$$y' = (x + 2x^3)y^3 - xy, \quad 0 \leq x \leq 2, \quad y(0) = \frac{1}{3}$$

Solution: We need to find  $k_3$  and  $k_4$ . We use Microsoft excel sheet, notice that the actual solution is  $y(x) = (3 + 2x^2 + 6e^{x^2})^{-\frac{1}{2}}$  and for  $h=0.2$

**Table 4.12:** Comparison of the exact and RKM of order four

h	$x_i$	Exact	RKM 4th	$k_1$	$k_2$	$k_3$	$k_4$
0.2	0	0.333333	0.333333	0	-0.00591	-0.00587	-0.01158
0.2	0.2	0.327475	0.327475	-0.01158	-0.01694	-0.01684	-0.02169
0.2	0.4	0.310669	0.310669	-0.02169	-0.02594	-0.02581	-0.02941
0.2	0.6	0.284902	0.284902	-0.02942	-0.03236	-0.03224	-0.03454
0.2	0.8	0.252708	0.252708	-0.03455	-0.03622	-0.03614	-0.03722
0.2	1	0.216626	0.216627	-0.03723	-0.03772	-0.03769	-0.03761
0.2	1.2	0.179017	0.179017	-0.03762	-0.03697	-0.03703	-0.03581
0.2	1.4	0.142111	0.142112	-0.03584	-0.0341	-0.03429	-0.03205
0.2	1.6	0.107999	0.108002	-0.03209	-0.02947	-0.02984	-0.02685
0.2	1.8	0.078402	0.078406	-0.02693	-0.02382	-0.02435	-0.02105
0.2	2	0.054346	0.054352	-0.02116	-0.01804	-0.01866	-0.01549



**Figure 4.6: RK-method of order four and Exact solution where  $h=0.2$**

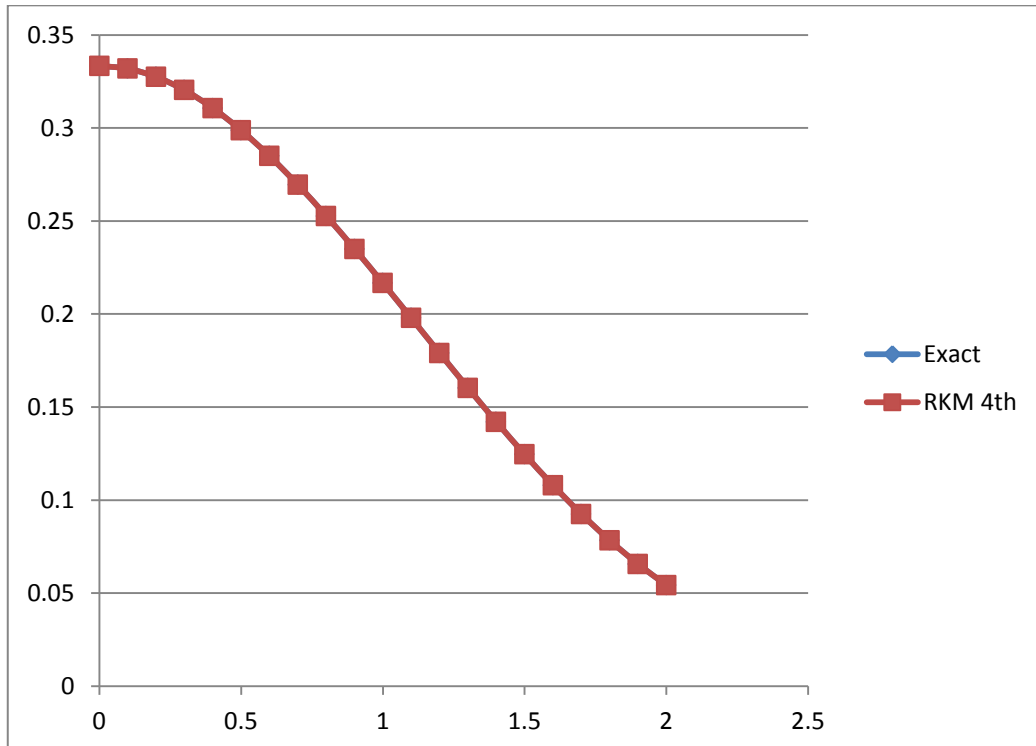
It is clear from the Table and the Figure that the comparison between the numerical and exact solution are much better than the Figure 4.5. To see if this is due to the step size we try using the same RKM of order four, but with smaller step size say  $h=0.1$ .

Let  $h=0.1$

See the results

**Table 4.13:** Comparison of the exact and RKM of order four when  $h=0.1$ 

$x_i$	Exact	RKM 4th	$k_1$	$k_2$	$k_3$	$k_4$
0	0.333333333	0.333333333	0	-0.00148	-0.00148	-0.00295
0.1	0.331856155	0.331856155	-0.00295	-0.00439	-0.00438	-0.00579
0.2	0.327475438	0.327475436	-0.00579	-0.00715	-0.00714	-0.00845
0.3	0.320337705	0.320337701	-0.00845	-0.00969	-0.00967	-0.01084
0.4	0.310668912	0.310668906	-0.01084	-0.01194	-0.01192	-0.01294
0.5	0.298751775	0.298751769	-0.01294	-0.01387	-0.01386	-0.01471
0.6	0.284901713	0.284901708	-0.01471	-0.01548	-0.01546	-0.01615
0.7	0.2694459	0.269445897	-0.01615	-0.01676	-0.01674	-0.01727
0.8	0.252708342	0.252708344	-0.01727	-0.01772	-0.01771	-0.01809
0.9	0.235001902	0.23500191	-0.01809	-0.01839	-0.01838	-0.01861
1	0.216626421	0.216626436	-0.01861	-0.01877	-0.01877	-0.01885
1.1	0.197870857	0.197870881	-0.01885	-0.01887	-0.01887	-0.01881
1.2	0.179016742	0.179016777	-0.01881	-0.01869	-0.01869	-0.0185
1.3	0.160340263	0.160340314	-0.0185	-0.01823	-0.01825	-0.01792
1.4	0.142110899	0.14211097	-0.01792	-0.01753	-0.01755	-0.01709
1.5	0.124585648	0.124585746	-0.01709	-0.01658	-0.01661	-0.01604
1.6	0.107999356	0.107999491	-0.01605	-0.01543	-0.01547	-0.01482
1.7	0.092552983	0.092553164	-0.01482	-0.01413	-0.01418	-0.01346
1.8	0.078402481	0.07840272	-0.01346	-0.01272	-0.01278	-0.01203
1.9	0.065650908	0.065651211	-0.01203	-0.01127	-0.01134	-0.01057
2	0.054345507	0.05434588	-0.01058	-0.00983	-0.0099	-0.00915

**Figure 4.7:** RK-Method of order four and Exact solution when  $h=0.1$

It is clear from the Figure that the exact and the Runge-Kutta method of order four are identical

#### 4.5.3 Implementing Runge-Kutta Feldberg method using excel

Since Runge-Kutta Feldberg Method is an extension of RKM of order 4. That is, RKFM,  $k_5$  and  $k_6$  terms need to be added and calculated. So, two extra columns are required. In a similar manner to order 4 that is repeat steps 1, 2, 3, 4.1

Step 5: In cell F2 calculate  $k_1 = h f(x, y)$  which is  $= h((x + 2x^3)y^3 - xy)$  at  $x = 0$ , that is in cell F2 type:  $\{=(B2*(C2+2*C2^3)*E2^3-C2*E2)\}$ .

Step 6: In cell G2 calculate  $k_2 = hf(x + 0.25 * h, y + 0.25 * k_1)$  at  $x=0$ , that is in cell G2 type :  $\{=(B2*(((C2+0.25*B2)+2*(C2+0.25*B2)^3)*(E2+0.25*F2)^3(C2+0.25*B2)*(E2+0.25*F2)))\}$

Step 7: In cell H2 calculate  $k_3 = h f(x + \frac{3h}{8}, y + \frac{3}{32}k_1 + \frac{9}{32}k_2)$  which is  $= h((x + 0.5h + 2(x + 0.5h)^3)(y + 0.5K_2)^3 - (x + 0.5h)(y + 0.5k_2))$  at  $x = 0$ , that is in cell H2 type:  $\{=B2*(((C2+(3*B2/8))+2*(C2+(3*B2/8))^3)*((E2+(3*F2/32)+(9*G2/32))^3)-((C2+(3*B2/8))*(E2+(3*F2/32)+(9*G2/32))))\}$

Step 8: In cell I2 calculate  $k_4 = h(x + \frac{12h}{13}, y + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$  which is  $= h((x + \frac{12h}{13}) + 2(x + \frac{12h}{13})^3)(y + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)^3 - (x + \frac{12h}{13})(y + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$ , that is in cell I2 type:

$\{=B2*(((C2+(12*B2/13))+2*(C2+(12*B2/13))^3)*((K2+(1932*E2/2197)-(7200*F2/2197)+(7296*G2/2197))^3)-((C2+(12*B2/13))*(K2+(1932*E2/2197)-(7200*F2/2197)+(7296*G2/2197))))\}$

Step 9: In cell J2 calculate  $k_5$ , that is in cell J2 type:

$\{=B5*(((C5+B5)+2*(C5+B5)^3)*((K5+(439*E5/216)-8*F5+(3680*G5/513)-(845*H5/4104))^3)-((C5+B5)*(K5+(439*E5/216)-8*F5+(3680*G5/513)-(845*H5/4104))))\}$

Step 10: In cell K2 calculates  $k_6$ , that is in cell K2 type:

{=B2\*(((C2+(B2/2))+2\*(C2+(B2/2))^3)\*((K2-(8\*E2/27)+2\*F2-(3544\*G2/2565)+(1859\*H2/4104)-(11\*I2/40))^3)-((C2+(B2/2))\*(K2-(8\*E2/27)+2\*F2-(3544\*G2/2565)+(1859\*H2/4104)-(11\*I2/40))))))}

Step 4.2: calculate RKFM of order 4  $w_{i+1} = w_i + \frac{16}{35}k_1 + \frac{6656}{2565}k_3 + \frac{28561}{56430}k_4 + \frac{9}{50}k_5 + \frac{2}{55}k_6$  if you done this correctly you should get  $=1/3=0.333333$

Click cell D2 and drag the corner down to cell D3.

Highlight cells F2, G2, H2, I2, J2 and K2 drag corner down to cell F3, G3, H3, I3, J3 and K3.

Now you have the rows D3, E3, F3, G3, H3, I3, J3 and K3 all containing formulas in them, which mean that you can copy them all to the cells below in a very simple manner.

Finally, Highlight cells D3, E3, F3, G3, H3, I3, J3 and K3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12, G12, H12, I12, J12 and K12.

**Table 4.14:** Illustration of the exact solution and RKFM of order four by Excel Sheet

	A	B	C	D	E	F	G	H	I	J	K
1	i	h	$x_i$	Exact $y_i$	RKFM 4 <sup>th</sup>	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$
2	1	0.1									
3	2	0.1									
4	.	.									
5	.	.									
6	11	0.1									

**Note:** For RKFM of order 5. Repeat the same steps of RKFM of order 4 except Step: 4.2: we must calculate RKFM of order 5.

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 + \frac{1}{5}k_5$$



**Example:** Determine the approximate solutions by Runge-kutta Fehlberg method of order four and five for the following IVP

$$y' = (x + 2x^3)y^3 - xy, \quad 0 \leq x \leq 2$$

$$y(0) = \frac{1}{3}$$

Solution: we need to find  $k_3$  and  $k_4$  we use Microsoft excel sheet, and also the actual solution is  $y(x) = (3 + 2x^2 + 6e^{x^2})^{-\frac{1}{2}}$

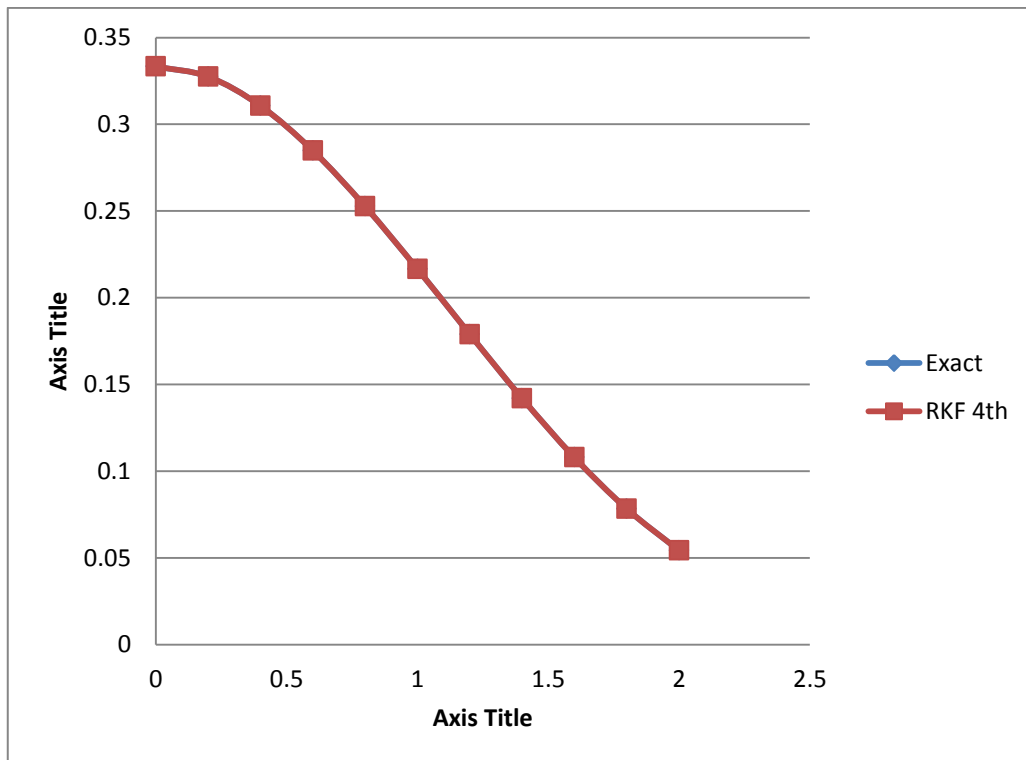
Step by Step Excel Sheet

**Table 4.15:** Illustration of RKF method of order four and five

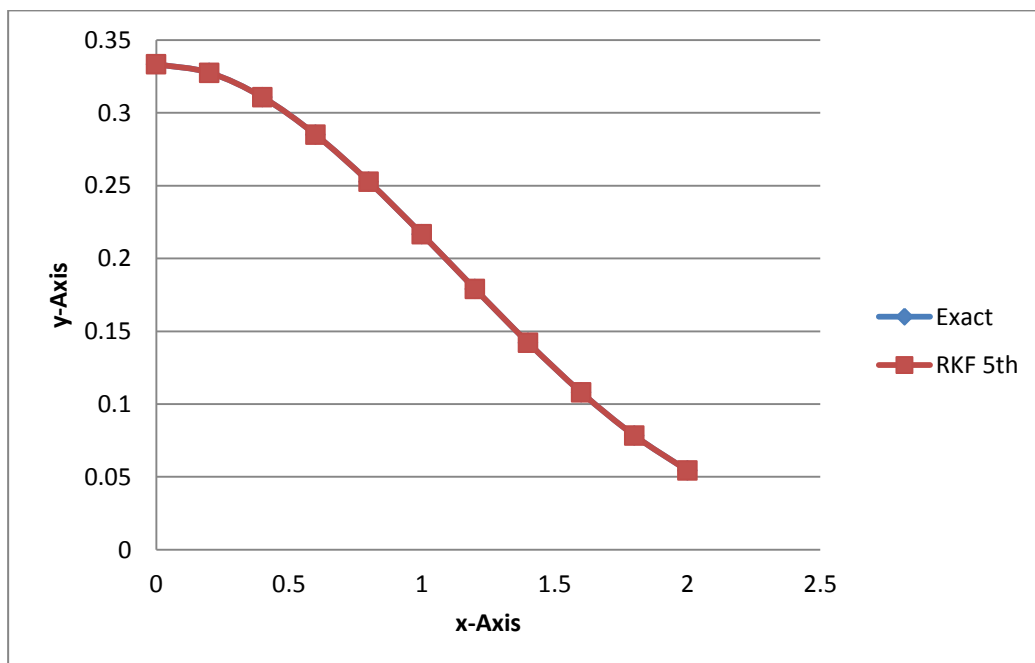
$h$	$x_i$	Exact	RKFM order4	RKforder5
0.2	0	0.33333333	0.33333333	0.33333333
0.2	0.2	0.32747544	0.32747544	0.32747544
0.2	0.4	0.31066891	0.31066891	0.31066893
0.2	0.6	0.28490171	0.284901707	0.28490175
0.2	0.8	0.25270834	0.252708336	0.2527084
0.2	1	0.21662642	0.216626421	0.21662651
0.2	1.2	0.17901674	0.179016752	0.17901685
0.2	1.4	0.1421109	0.142110917	0.14211101
0.2	1.6	0.10799936	0.107999364	0.10799939
0.2	1.8	0.07840248	0.078402435	0.07840226
0.2	2	0.05434551	0.054345343	0.05434475

Where the coefficients are

$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$
0	-0.002961111	-0.0044299	-0.01073	-0.01158	-0.00589
-0.01158	-0.014305518	-0.0156089	-0.02099	-0.02169	-0.01689
-0.02169	-0.023875795	-0.0248881	-0.02892	-0.02942	-0.02587
-0.02942	-0.030953163	-0.0316391	-0.03425	-0.03456	-0.0323
-0.03455	-0.035448818	-0.0358271	-0.03711	-0.03724	-0.03618
-0.03723	-0.037539684	-0.0376359	-0.03767	-0.03763	-0.0377
-0.03762	-0.037370969	-0.0372018	-0.03605	-0.03583	-0.037
-0.03584	-0.035050615	-0.0346431	-0.03242	-0.03205	-0.03419
-0.03209	-0.030866803	-0.0302851	-0.02732	-0.02683	-0.02966
-0.02693	-0.025449533	-0.0247927	-0.02153	-0.02098	-0.0241
-0.02116	-0.019657461	-0.0190259	-0.01592	-0.01537	-0.01837



**Figure 4.8:** RKF- Method of order four and exact solution



**Figure 4.9:** RKF- Method of order five and exact solution

#### 4.5.4 Implementing Runge-Kutta Verner method using excel

Since Runge-Kutta Verner Method is an extension of RKFM of order 4. That is, RKVM,  $k_7$  and  $k_8$  terms need to be added and calculated. So, two extra columns are required. In a similar manner to RKVM that is repeat steps 1, 2, 3, 4.1

Step 5: In cell F2 calculate  $k_1 = h f(x, y)$  which is  $= h((x + 2x^3)y^3 - xy)$  at  $x = 0$ , that is in cell F2 type:  $(=B2*(C2+2*C2^3)*E2^3-C2*E2)$ .

Step 6: In cell G2 calculate  $k_2 = hf(x + \frac{h}{6}, y + \frac{1}{6}k_1)$  at  $x=0$ , that is in cell G2 type:  $\{=B2*(((C2+(B2/6))+2*(C2+(B2/6))^3)*((E2+(1/6)*F2)^3)-(C2+(B2/6))*(E2+(F2/6))))\}$

Step 7: In cell H2 calculate  $k_3 = hf(x + \frac{4h}{15}, y + \frac{4}{75}k_1 + \frac{16}{75}k_2)$  which is  $= h((x + 0.5h + 2(x + 0.5h)^3)(y + 0.5K_2)^3 - (x + 0.5h)(y + 0.5k_2))$  at  $x = 0$ , that is in cell H2 type:  $\{=B2*(((C2+(4*B2/15))+2*(C2+(4*B2/15))^3)*((E2+(4/75)*F2+(16/75)*G2)^3)-(C2+(4*B2/15))*(E2+(4/75)*F2+(16/75)*G2)))\}$ .

Step 8: In cell I2 calculate  $k_4 = h(x + \frac{2h}{3}, y + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3)$  which is  $= h((x + \frac{12h}{13}) + 2(x + \frac{12h}{13})^3)(y + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)^3 - (x + \frac{12h}{13})(y + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$ , that is in cell I2 type:  $\{=B2*(((C2+(2*B2/3))+2*(C2+(2*B2/3))^3)*((E2+(5/6)*F2-(8/3)*G2+(5/2)*H2)^3)-(C2+(2*B2/3))*(E2+(5/6)*F2-(8/3)*G2+(5/2)*H2)))\}$ .

Step 9: In cell J2 calculate  $k_5$ , that is in cell J2 type:

$\{=B2*(((C2+(5*B2/6))+2*(C2+(5*B2/6))^3)*((E2-(165/64)*F2+(55/6)*G2-(425/64)*H2+(85/96)*I2)^3)-(C2+(5*B2/6))*(E2-(165/64)*F2+(55/6)*G2-(425/64)*H2+(85/96)*I2)))\}$

Step 10: In cell K2 calculates  $k_6$ , that is in cell K2 type:

$\{=B2*(((C2+B2)+2*(C2+B2)^3)*(E7+(12/5)*F2-8*G2+(4015/612)*H2-(11/36)*I2+(88/255)*J2)^3)-(C2+B2)*(E2+(12/5)*F2-8*G2+(4015/612)*H2-(11/36)*I2+(88/255)*J2)))\}$ .

Step 11: In cell L2 calculate  $k_7$ , that is in cell L2 type:

$$\{=B2*(((C7+(B2/15))+2*(C2+(B2/15))^3)*(E2-(8263/15000)*F2+(124/75)*G2-(643/680)*H2-(81/250)*I2+(2484/10625)*J2))-(C2+(B2/15))*(E2-(8263/15000)*F2+(124/75)*G2-(643/680)*H2-(81/250)*I2+(2484/10625)*J2))\}.$$

Step 12: In cell M2 calculate  $k_8$ , that is in cell M2 type:

$$\{=B7*(((C7+B7)+2*(C7+B7)^3)*(E2+(3501/1720)*F2-(300/43)*G2+(297275/52632)*H2-(319/2322)*I2+(24068/84065)*J2+(3850/26703)*L2))-(C2+B2)*(E2+(3501/1720)*F2-(300/43)*G2+(297275/52632)*H2-(319/2322)*I2+(24068/84065)*J2+(3850/26703)*L2))\}$$

Step 4.2: In cell E3 calculate RKVM of order 5  $w_{i+1} = w_i + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6$  if you done this correctly you should get  $=1/3=0.333333$

Click cell D2 and drag the corner down to cell D3.

Highlight cells F2, G2, H2, I2, J2, K2, L2 and M2 drag them corner down to cell F3, G3, H3, I3, J3, K3, L3 and M2.

Now you have the rows D3, E3, F3, G3, H3, I3, J3, K3, L3 and M3 all containing formulas in them, which mean that you can copy them all to the cells below in a very simple manner.

Finally, Highlight cells D3, E3, F3, G3, H3, I3, J3, K3, L3 and M3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, E12, F12, G12, H12, I12, J12, K12, L12 and M12.

**Table 4.16:** Illustration of the exact solution and RKVM of order 5 by Excel Sheet

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	i	h	$x_i$	Exact $y_i$	RKVM 5 <sup>nd</sup>	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$
2	1	0.1											
3	2	0.1											
4	.	.											
5	.	.											
6	11	0.1											

Note: For RKFM of order 6. Repeat the same steps of RKVM of order 5 except Step: 4.2: we must calculate RVFM of order 6.

$$w_{i+1} = w_i + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_6 + \frac{43}{616}k_8$$

**Example:** Determine the approximate solutions by Runge-kutta method of order four for the following IVP

$$y' = (x + 2x^3)y^3 - xy, \quad 0 \leq x \leq 2, \quad y(0) = \frac{1}{3}$$

Solution: We need to find  $k_7$  and  $k_8$  we use Microsoft excel sheet, and also the actual solution is  $y(x) = (3 + 2x^2 + 6e^{x^2})^{-\frac{1}{2}}$ , Now, for  $h=0.2$  see the results

**Table 4.17:** Illustration of RK-Vernr Method of order 5

$x_i$	Exact	RKV order 5	$k_1$	$k_2$	$k_3$
0	0.33333333	0.33333333	0	-0.00197476	-0.0031553
0.2	0.32747544	0.327475437	-0.011582	-0.013406735	-0.0144714
0.4	0.31066891	0.31066891	-0.021687	-0.023160132	-0.024
0.6	0.28490171	0.284901711	-0.029415	-0.030455372	-0.0310343
0.8	0.25270834	0.252708339	-0.034546	-0.035162255	-0.0354918
1	0.21662642	0.216626417	-0.037226	-0.037449708	-0.03755
1.2	0.17901674	0.179016736	-0.037622	-0.03747085	-0.0373542
1.4	0.1421109	0.142110895	-0.035837	-0.035330995	-0.0350151
1.6	0.10799936	0.107999362	-0.032093	-0.031293932	-0.0308247
1.8	0.07840248	0.078402521	-0.026927	-0.025958624	-0.0254131
2	0.05434551	0.054345613	-0.02116	-0.020171643	-0.0196339

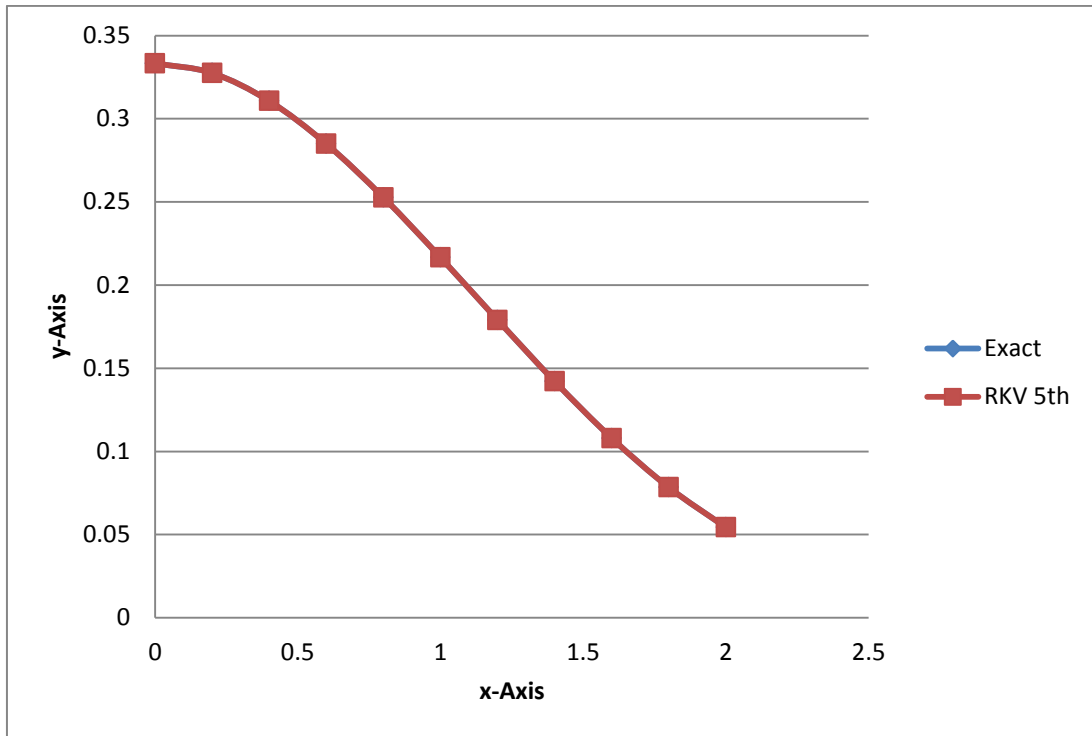
$k_4$	$k_5$	$k_6$	$k_7$	$k_8$
-0.0078201	-0.00972	-0.011582	1.47705E-06	0.00104826
-0.0185544	-0.02015	-0.021689	0.001284021	0.00800419
-0.0271299	-0.0283	-0.02942	0.008755097	0.02500743
-0.0331203	-0.03386	-0.034553	0.026129671	0.05342074
-0.0365949	-0.03693	-0.037233	0.053897469	0.09178771
-0.0377388	-0.0377	-0.037628	0.089122696	0.13636114
-0.0366662	-0.03628	-0.035838	0.126090364	0.18188335
-0.033529	-0.03285	-0.032079	0.157742816	0.22255003
-0.0287475	-0.0279	-0.026883	0.177766905	0.25309488
-0.0230713	-0.02222	-0.021073	0.182666054	0.26978511
-0.0173655	-0.01663	-0.015491	0.1727612	0.27106793

And for RKV of order 6

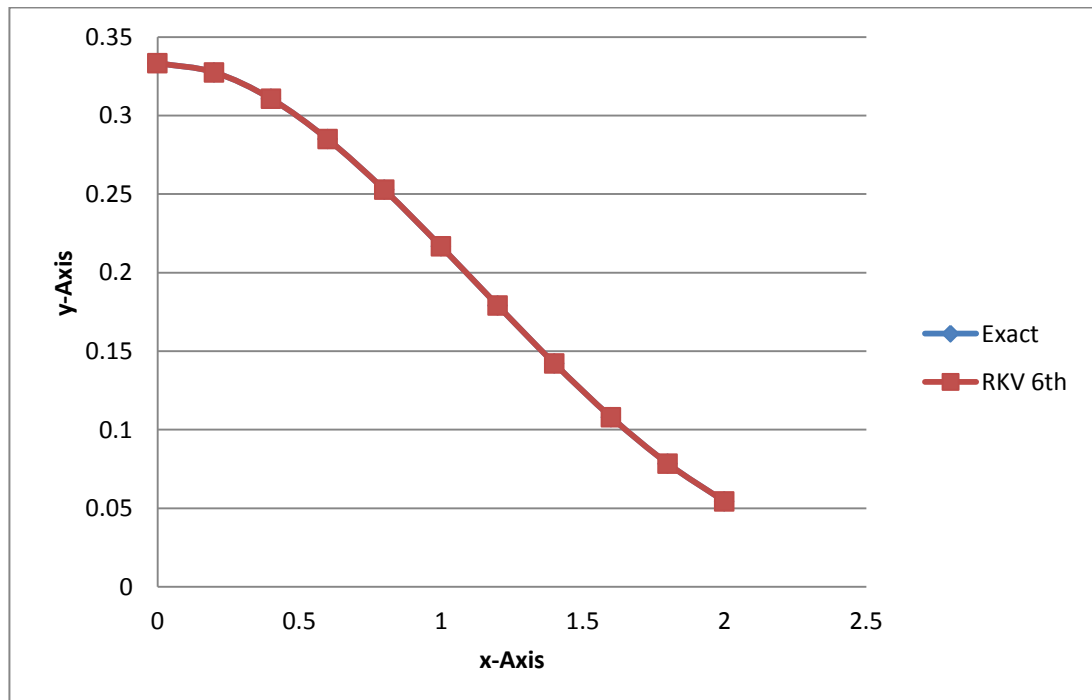
**Table 4.18:** Illustration of RKV-method of order six

$x_i$	Exact	RKV order 6	$k_1$	$k_2$	$k_3$
0	0.3333333	0.3333333	0	-0.00197	-0.003155
0.2	0.3274754	0.3274754	-0.011582	-0.01341	-0.014471
0.4	0.3106689	0.3106689	-0.021687	-0.02316	-0.024
0.6	0.2849017	0.2849017	-0.029415	-0.03046	-0.031034
0.8	0.2527083	0.2527083	-0.034546	-0.03516	-0.035492
1	0.2166264	0.2166264	-0.037226	-0.03745	-0.03755
1.2	0.1790167	0.1790167	-0.037622	-0.03747	-0.037354
1.4	0.1421109	0.1421109	-0.035837	-0.03533	-0.035015
1.6	0.1079994	0.1079994	-0.032093	-0.03129	-0.030825
1.8	0.0784025	0.0784025	-0.026927	-0.02596	-0.025413
2	0.0543455	0.0543455	-0.02116	-0.02017	-0.019634

$k_4$	$k_5$	$k_6$	$k_7$	$k_8$
-0.0078201	-0.00972	-0.01158	-0.00079	-0.01158
-0.0185544	-0.02015	-0.02169	-0.01232	-0.02169
-0.0271299	-0.0283	-0.02942	-0.02228	-0.02942
-0.0331203	-0.03386	-0.03455	-0.02984	-0.03455
-0.0365949	-0.03693	-0.03723	-0.0348	-0.03723
-0.0377388	-0.0377	-0.03763	-0.03732	-0.03763
-0.0366662	-0.03628	-0.03584	-0.03757	-0.03584
-0.033529	-0.03285	-0.03208	-0.03565	-0.03208
-0.0287475	-0.0279	-0.02688	-0.03179	-0.02689
-0.0230713	-0.02222	-0.02107	-0.02655	-0.02109
-0.0173655	-0.01663	-0.01549	-0.02078	-0.01551



**Figure 4.10:** RKV method of order five and exact solution



**Figure 4.11:** RKV- Method of order six and exact solution

## 4.6 Implementing Adam Bashfourth Explicit Method Using Excel

### 4.6.1 Implementing Adam Bashfourth explicit 2-step method using excel

Again starting a new blank Excel sheet, simply organize the sheet that as in Table 4.19. In this case Example 4 is used to demonstrate the way Excel is used for the implementation of Adam Bashfourth explicit 2-step method.

$$y' = y^2 - 2xy + x^2 + 1, \quad 2 \leq x \leq 3 \quad y(2) = 1, \quad y = x + \frac{1}{1-x}$$

The following are step by step instructions to implement the procedure of calculation; in a similar manner Runge-Kutta method of order 2 that is repeat steps 1, 2

Note: What appears inside the curly bracket is what you type in or press on the keyboard.

Step 3: If we have an exact solution then we must devote column D for it. In cell D2 calculate the exact value of  $y$  at  $x=0$ . To do this, type:  $\{=(C19+(1/(1-C19)))\}$  which is  $x + \frac{1}{1-x}$ , if you done this correctly you should get 1, which is the initial value of  $y$ .

Step 4: Since, this method needs to find  $w_1$  and  $w_2$  we must take from exact solution or other numerical methods. In our case we take in RKM of order two, for that devote column E.

Step 5.1: In cell F2 type  $\{=(1)\}$  the initial value  $w_0$  at  $x = 2$  and F3 taken from RKM 2<sup>nd</sup>

Step 6: In cell G2 calculate  $f(x, y)$  at  $x=2$ , that is in cell G2 type:  $\{=F2^2-2*C2*F2+C2^2+1\}$  which is  $y^2 - 2xy + x^2 + 1$  and drag the corner down to get G3

Step 5.2: in cell F4 type:  $\{=F3+0.5*B3*(3*G3-G2)\}$  which

is  $w_2 = w_1 + \frac{1}{2} h [3f(x_1, y_1) - f(x_2, y_2)]$ , if it is correct you should get the value 1

Click cell D2 and drag the corner down to cell D3 and D4.

Click cell G3 and drag the corner down to cell G4.

Now you have the row containing the cells D3, F3 and G3, all these cells contain formulas in them, which mean that you can copy them all to the cells below in a very simple manner that is by dragging down the corner square.



Highlight cells D3, F3 and G3 then release, hold the small square on the bottom right of the highlighted cells, drag down all the way to cover the cells D12, F12 and G12. That is it.

**Table 4.19:** Illustration ABEM 2-step with exact solution by using Excel sheet

	A	B	C	D	E	F	G
1	Index i	Step size h	Variable $x_i$	Exact $y_i$	RKM 2 <sup>nd</sup>	ABEM 2st	$f(x, y)$
2	1	0.1					
3	2	0.1					
4	.	.					
5	.	.					
6	11	0.1					

**Note:** Repeat all above steps for ABEM 3st except column F, that is in cell F calculate formula 3.35 , for ABEM 4st calculate formula 3.36 and for ABEM 5st calculate formula 3.37.

**4.6.1.1 Example 4:** Determine an approximate solution to the following IVP

$$y' = y^2 - 2xy + x^2 + 1, \quad 2 \leq x \leq 3 \quad y(2) = 1$$

Use Adam Bash fourth explicit method, when the actual solution is

$$y = x + \frac{1}{1-x}. \text{ (Burden and Fairs, 2010)}$$

**Solution:**

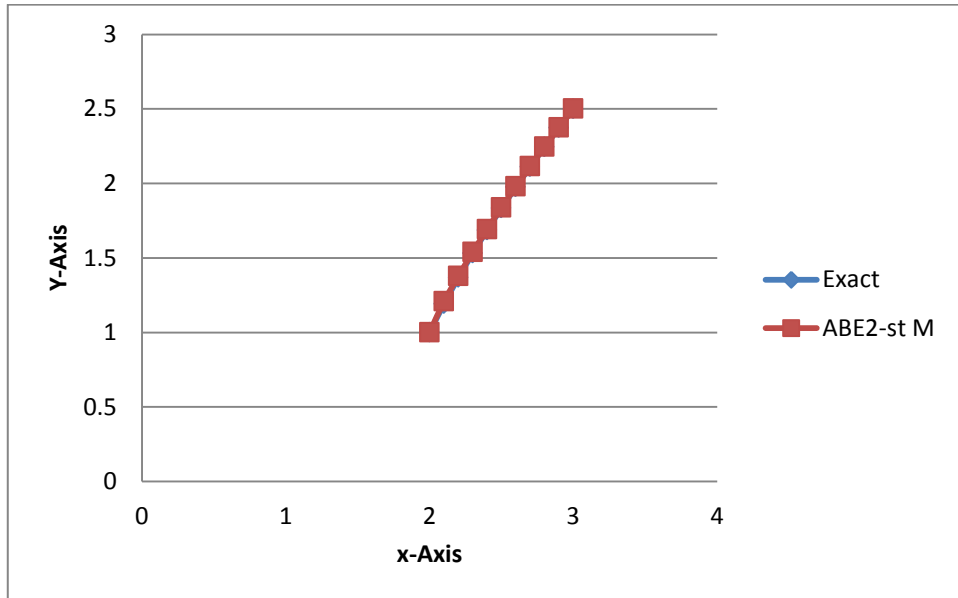
1. Adam Bashfourth explicit 2-st method: If we compare this Example with the general form of ADE, we see that  $f_3(x) = 0$  then the equation transform directly to Riccati differential equation, this is a case of Abel differential equation that we discuss in literature review

Now, we are ready to applying step by step excel work sheet According to ABEM 2st formula we need to find  $w_1, w_2$  and  $f_1, f_2$ , we can take from another method like Euler, Taylor, Runge kutta methods

Here we use Runge-kutta method of order two to find  $w_1, w_2$  and then we use the ABEM to find an approximate solution

**Table 4.20:** Illustration of ABEM 2-step

<b>h</b>	<b><math>x_i</math></b>	<b>Exact</b>	<b>ABEM 2st</b>	<b>RK order 2</b>	<b><math>f(x_i, y_i)</math></b>
0.1	2	1	1	1	2
0.1	2.1	1.190909	1.20832	1.20832	1.795093
0.1	2.2	1.366667	1.377584	1.395482	1.676368
0.1	2.3	1.530769	1.539285	1.567276	1.578688
0.1	2.4	1.685714	1.692269	1.727523	1.500883
0.1	2.5	1.833333	1.838467	1.878859	1.437625
0.1	2.6	1.975	1.979067	2.023163	1.385558
0.1	2.7	2.111765	2.115019	2.161812	1.342202
0.1	2.8	2.244444	2.247072	2.295844	1.30573
0.1	2.9	2.373684	2.375821	2.426053	1.274763
0.1	3	2.5	2.501749	2.55306	1.248254

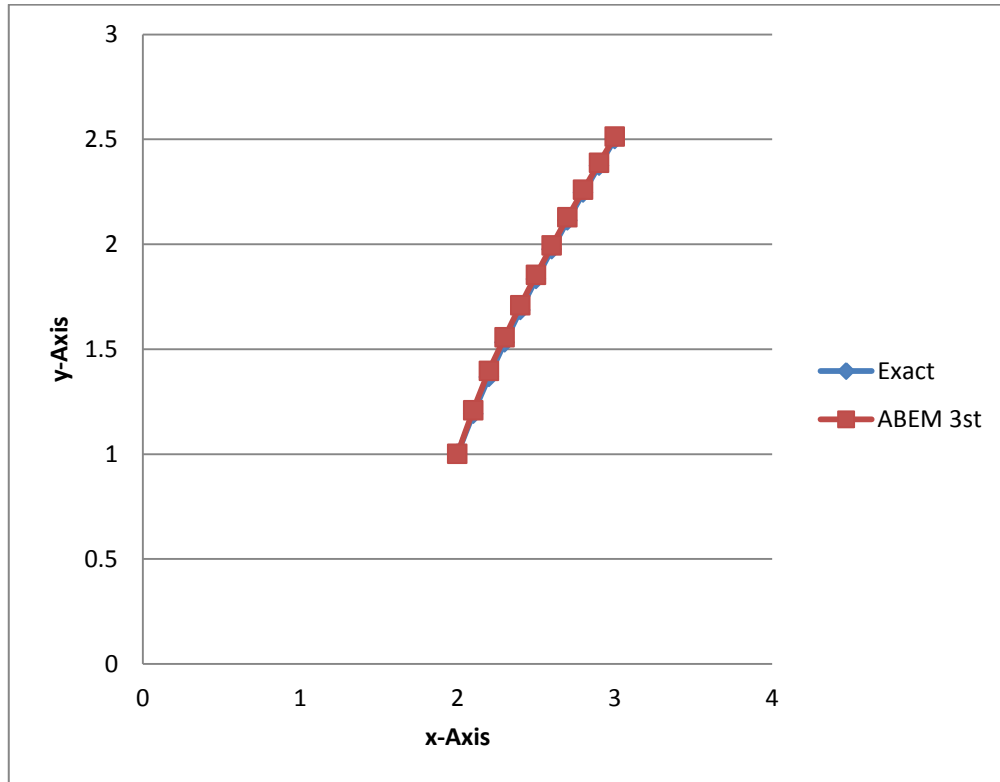
**Figure 4.12:** ABEM 2-steps and RKM of order 2 and exact solution

## 2-Adam Bashfourth Explicit 3-step Method

See the results

**Table 4.21:** Illustration of ABEM 3-step

<b>h</b>	<b>x</b>	<b>Exact</b>	<b>ABEM3st</b>	<b>RK order 2</b>	<b>F(xi,yi)</b>
0.1	2	1	1	1	2
0.1	2.1	1.190909	1.20832	1.20832	1.795093
0.1	2.2	1.366667	1.395482	1.395482	1.647249
0.1	2.3	1.530769	1.555192	1.567276	1.554738
0.1	2.4	1.685714	1.708346	1.727523	1.478385
0.1	2.5	1.833333	1.85304	1.878859	1.418557
0.1	2.6	1.975	1.992593	2.023163	1.368943
0.1	2.7	2.111765	2.127432	2.161812	1.327834
0.1	2.8	2.244444	2.258515	2.295844	1.293206
0.1	2.9	2.373684	2.386374	2.426053	1.263812
0.1	3	2.5	2.511503	2.55306	1.238629



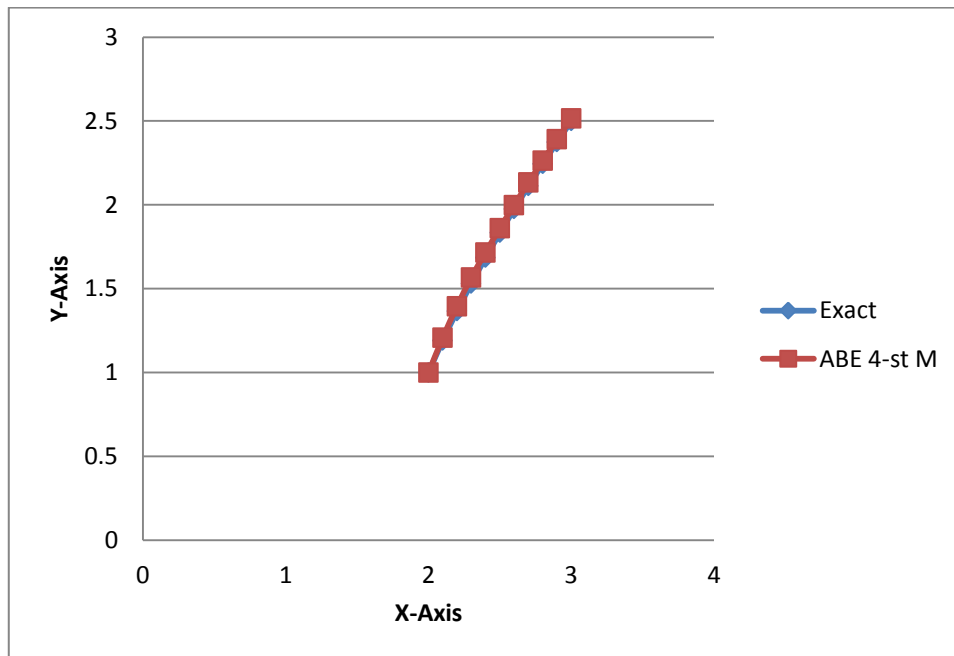
**Figure 4.13:** ABEM 3-steps and RKM of order 2 and exact solution

### 3-Adam Bashfourth Explicit 4-step Method

See the Results

**Table 4.22:** Illustration of ABEM 4-step

$h$	$x_i$	Exact	ABEM 4st	RK order 2	$f$
0.1	2	1	1	1	2
0.1	2.1	1.190909	1.20832	1.20832	1.795093
0.1	2.2	1.366667	1.395482	1.395482	1.647249
0.1	2.3	1.530769	1.567276	1.567276	1.536885
0.1	2.4	1.685714	1.716273	1.727523	1.467482
0.1	2.5	1.833333	1.861389	1.878859	1.407824
0.1	2.6	1.975	1.998424	2.023163	1.361894
0.1	2.7	2.111765	2.133038	2.161812	1.321446
0.1	2.8	2.244444	2.263079	2.295844	1.288284
0.1	2.9	2.373684	2.390621	2.426053	1.259467
0.1	3	2.5	2.515197	2.55306	1.235033



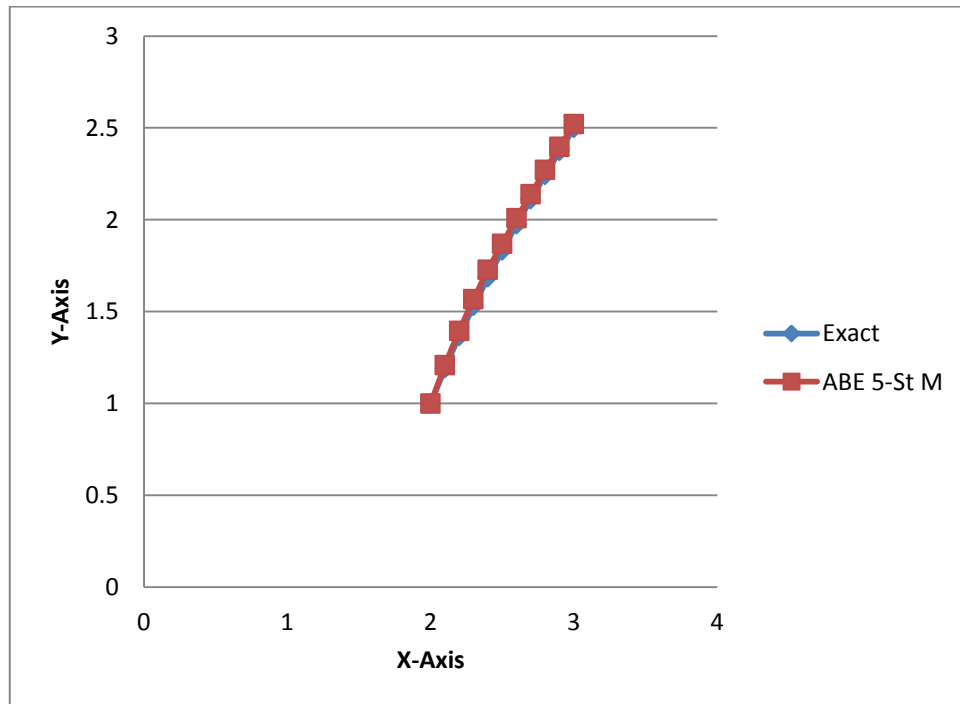
**Figure 4.14:** ABEM 4-steps and RKM of order 2 and exact solution

#### 4-For Adam Bashfourth 5-Step Method

See the Results

**Table 4.23:** Illustration of ABEM 5-step

<b>h</b>	<b><math>x_i</math></b>	<b>exact</b>	<b>ABEM 5st</b>	<b>RKM 2nd</b>	<b><math>f(x, y)</math></b>
0.1	2	1	1	1	2
0.1	2.1	1.190909	1.20832	1.20832	1.795093
0.1	2.2	1.366667	1.395482	1.395482	1.647249
0.1	2.3	1.530769	1.567276	1.567276	1.536885
0.1	2.4	1.685714	1.727523	1.727523	1.452225
0.1	2.5	1.833333	1.869415	1.878859	1.397638
0.1	2.6	1.975	2.008429	2.023163	1.349957
0.1	2.7	2.111765	2.139499	2.161812	1.314161
0.1	2.8	2.244444	2.270789	2.295844	1.280064
0.1	2.9	2.373684	2.39625	2.426053	1.253764
0.1	3	2.5	2.521433	2.55306	1.229026



**Figure 4.15:** ABEM 5-steps and RKM of order 2 and exact solution

## 4.7 Unrestricted Form of ADEs

The following Examples are called Abel Differential equation of first kind in full unrestricted because they have all coefficients  $f_3$ ,  $f_2$ ,  $f_1$  and  $f_0$ . Since there are no known analytical solution forms for this differential equation. So we don't have any choices except using suitable numerical methods. We have already discussed several of numerical methods for solving ordinary differential equation and for each method we took a special form of Abel differential equation. Now, for the two ADE that are known to be unsolvable analytically, we would want to use several available numerical methods with different step sizes  $h$ , and compare the results. In the absence of exact solution we think, this is the only way to ensure that the approximate solutions similar to each other. These results are better compared graphically.

**4.7.1 Example 1:** Consider the nonlinear Abel differential equation of first kind

$$y'(x) = \sin x y^3 - xy^2 + x^2y - x^3, \quad x \in (0,1]$$

$$y(0) = 0$$

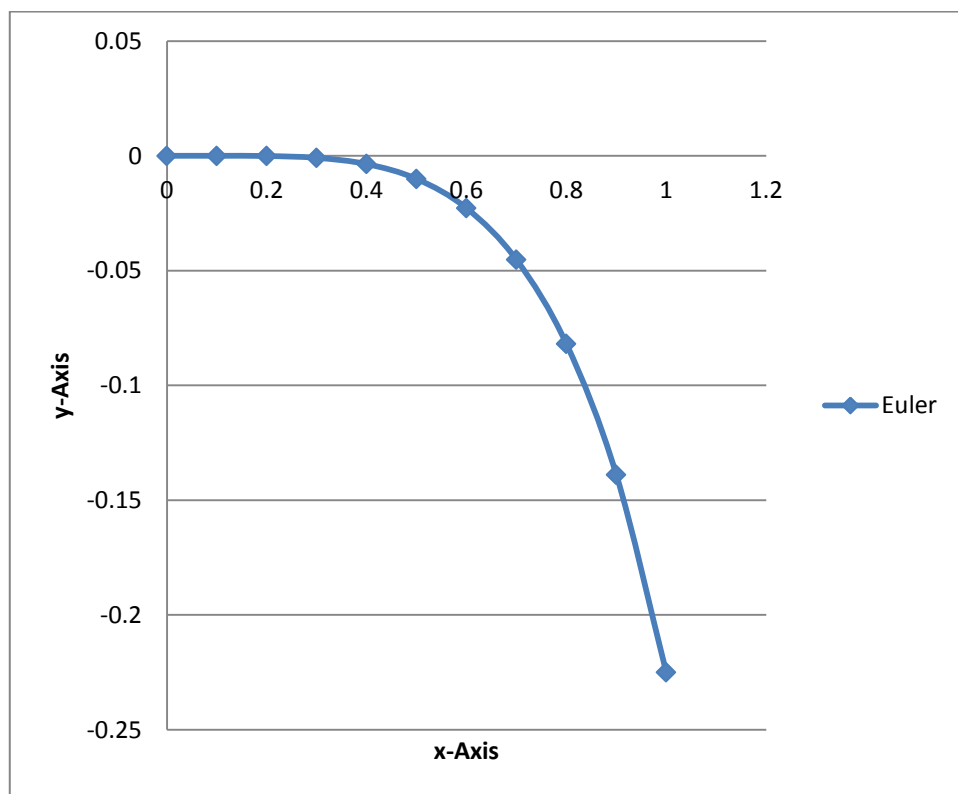
Solution:

In this example we have  $f_3(x) = \sin(x)$ ,  $f_2(x) = -x$ ,  $f_1(x) = x^2$  and  $f_0(x) = -x^3$ , here we can't find close form solution. Now we oblige to use numerical methods for finding approximate solutions. Apply each of above methods on this example and also we have initial point and clo-open interval. (XU and He, 2013)

First for Euler Method

**Table 4.24:** Illustration of Abel differential equation with Euler's method

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Euler</b>	<b><math>f(x_i, w_i)</math></b>
1	0.1	0	0	0
2	0.1	0.1	0	-0.001
3	0.1	0.2	-0.0001	-0.0080040
4	0.1	0.3	-0.0009004	-0.0270813
5	0.1	0.4	-0.0036085	-0.0645826
6	0.1	0.5	-0.0100668	-0.1275679
7	0.1	0.6	-0.0228236	-0.2245357
8	0.1	0.7	-0.0452771	-0.3666806
9	0.1	0.8	-0.0819452	-0.5702117
10	0.1	0.9	-0.1389664	-0.8610454
11	0.1	1	-0.2250709	-1.2853218

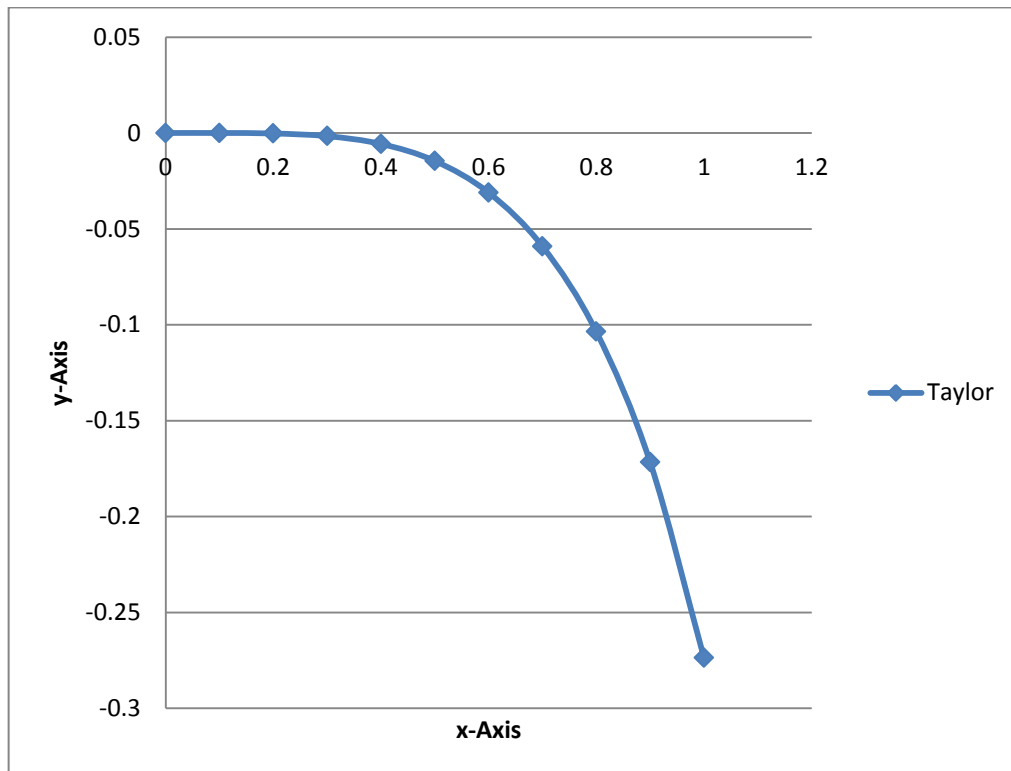


**Figure 4.16:** Graph of ADE by Euler method

We presented Taylor's Method of order two and of order four in restrict form of ADEs. Since these results are close to each other, we choose one of them, here Taylor Method of order two see the results

**Table 4.25:** Tabulation of ADE by Taylor's method of order two

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Taylor 2th</b>	<b><math>f(x_i, w_i)</math></b>	<b><math>f'(x_i, w_i)</math></b>	<b>T(2)</b>
1	0.1	0	0	0	0	0
2	0.1	0.1	0	-0.001	-0.02999	-0.0025
3	0.1	0.2	-0.00025	-0.00801	-0.11978	-0.014
4	0.1	0.3	-0.0016499	-0.027149	-0.26858	-0.04058
5	0.1	0.4	-0.0057077	-0.064926	-0.47451	-0.08865
6	0.1	0.5	-0.0145729	-0.128751	-0.73457	-0.16548
7	0.1	0.6	-0.0311208	-0.227802	-1.04564	-0.28008
8	0.1	0.7	-0.0591292	-0.374554	-1.40897	-0.445
9	0.1	0.8	-0.1036294	-0.587712	-1.84423	-0.67992
10	0.1	0.9	-0.1716218	-0.898482	-2.43285	-1.02012
11	0.1	1	-0.2736343	-1.365751	-3.45728	-1.53861



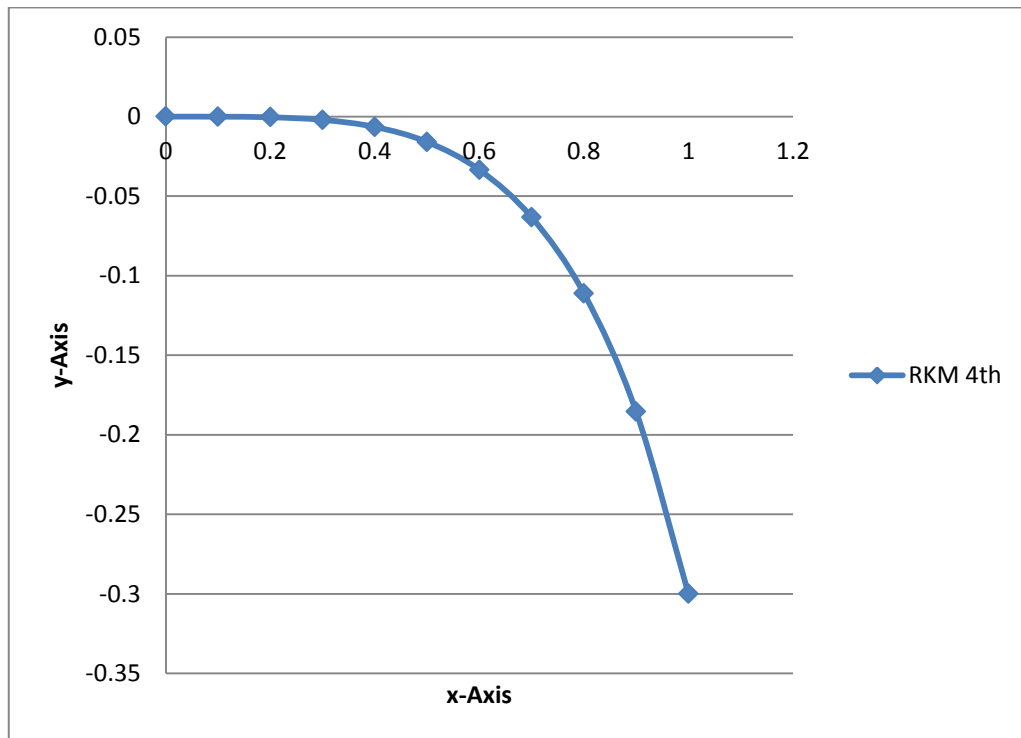
**Figure 4.17:** Graph of ADE by Taylor's method of order two



In restricted form of ADEs we are used different order of Runge-Kutte Methods the results are close with smaller step size  $h$ , then here we use Runge-Kutta method of order four with smaller step size  $h=0.1$ .

**Table 4.26:** Tabulation of ADE by RK method of order four

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>RKM 4th</b>	<b><math>k_1</math></b>	<b><math>k_2</math></b>	<b><math>k_3</math></b>	<b><math>k_4</math></b>
1	0.1	0	0	0	-1.3E-05	-1.3E-05	-0.0001
2	0.1	0.1	-2.5E-05	-0.0001	-0.00034	-0.00034	-0.0008
3	0.1	0.2	-0.0004005	-0.0008	-0.00157	-0.00157	-0.00272
4	0.1	0.3	-0.0020328	-0.00272	-0.00433	-0.00434	-0.0065
5	0.1	0.4	-0.0064595	-0.00651	-0.00931	-0.00934	-0.01291
6	0.1	0.5	-0.0159139	-0.01291	-0.01734	-0.01742	-0.02287
7	0.1	0.6	-0.0334629	-0.02287	-0.0295	-0.02966	-0.03769
8	0.1	0.7	-0.0632743	-0.0377	-0.04735	-0.04769	-0.05938
9	0.1	0.8	-0.1111349	-0.0594	-0.07348	-0.0742	-0.0915
10	0.1	0.9	-0.1855124	-0.09152	-0.1127	-0.11427	-0.14123
11	0.1	1	-0.2999622	-0.14127	-0.17546	-0.17934	-0.22618



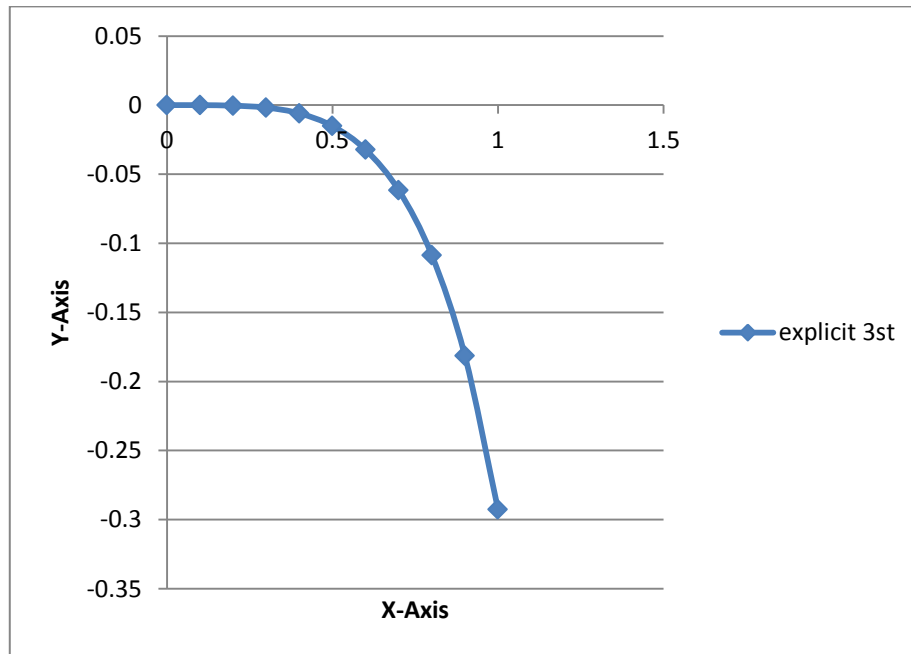
**Figure 4.18:** Graph of ADE by RKM of order four

Also we presented different steps of ABEM in restrict form of ADEs and the results are close to each other, we choose one of them. Here, we use Adam Bash Fourth Explicit 3-step Method

Not: I took  $f_1, f_2, f_3$  and  $w_1, w_2, w_3$  from runge - kutte method order four

**Table 4.27:** Tabulation of ADE by ABEM 3-step

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Explicit 3st</b>	<b><math>f(x_i, w_i)</math></b>
1	0.1	0	0	0
2	0.1	0.1	-2.50E-05	-0.001
3	0.1	0.2	-0.0004005	-0.00802
4	0.1	0.3	-0.0018035	-0.02716
5	0.1	0.4	-0.0059827	-0.06497
6	0.1	0.5	-0.0151478	-0.1289
7	0.1	0.6	-0.0323232	-0.22828
8	0.1	0.7	-0.0615973	-0.37599
9	0.1	0.8	-0.1085953	-0.59185
10	0.1	0.9	-0.1814138	-0.91024
11	0.1	1	-0.2926292	-1.39935



**Figure 4.19:** Graph of ADE by ABEM 3-step

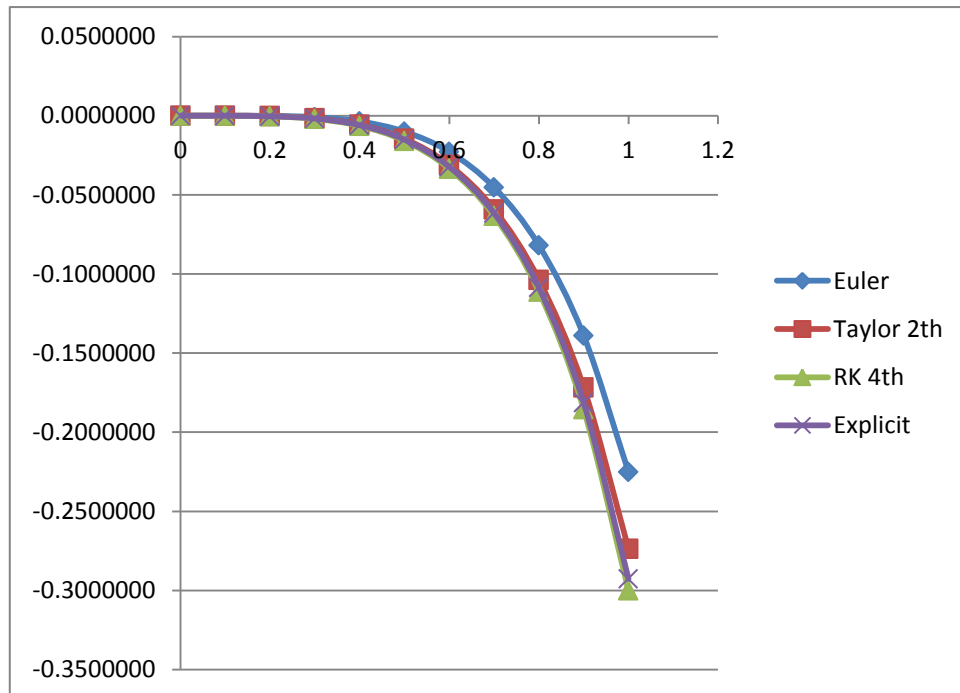
See all results together

**Table 4.28** Tabulation of Euler, Taylor, RK 4th and ABEM 3-steps when  $h=0.1$

<b>h</b>	<b>xi</b>	<b>Euler</b>	<b>Taylor 2nd</b>	<b>RK 4th</b>	<b>explicit 3st</b>
0.1	0	0.0000000	0	0	0
0.1	0.1	0.0000000	0	-2.5E-05	-2.5E-05
0.1	0.2	-0.0001000	-0.000025	-0.0004005	-0.0004005
0.1	0.3	-0.0009004	-0.0016499	-0.0020328	-0.0018035
0.1	0.4	-0.0036085	-0.0057077	-0.0064595	-0.0059827
0.1	0.5	-0.0100668	-0.0145729	-0.0159139	-0.0151478
0.1	0.6	-0.0228236	-0.0311208	-0.0334629	-0.0323232
0.1	0.7	-0.0452771	-0.0591292	-0.0632743	-0.0615973
0.1	0.8	-0.0819452	-0.1036294	-0.1111349	-0.1085953
0.1	0.9	-0.1389664	-0.1716218	-0.1855124	-0.1814138
0.1	1	-0.2250709	-0.2736343	-0.2999622	-0.2926292

Note that, if we compare between the results for each way, we can see that the results are close to each other.

Also the graphs together



**Figure 4.20:** Illustration of Euler, Taylor 2nd, RKM 4th, and ABEM 3st

It is clear from the figure that the graphs of each method are not identical. Therefore we discuss the causes of this problem.

### • Result and Discussion

For find a better approximation for the above example. We must generally attempt to get the approximate results with a smaller  $h$ .

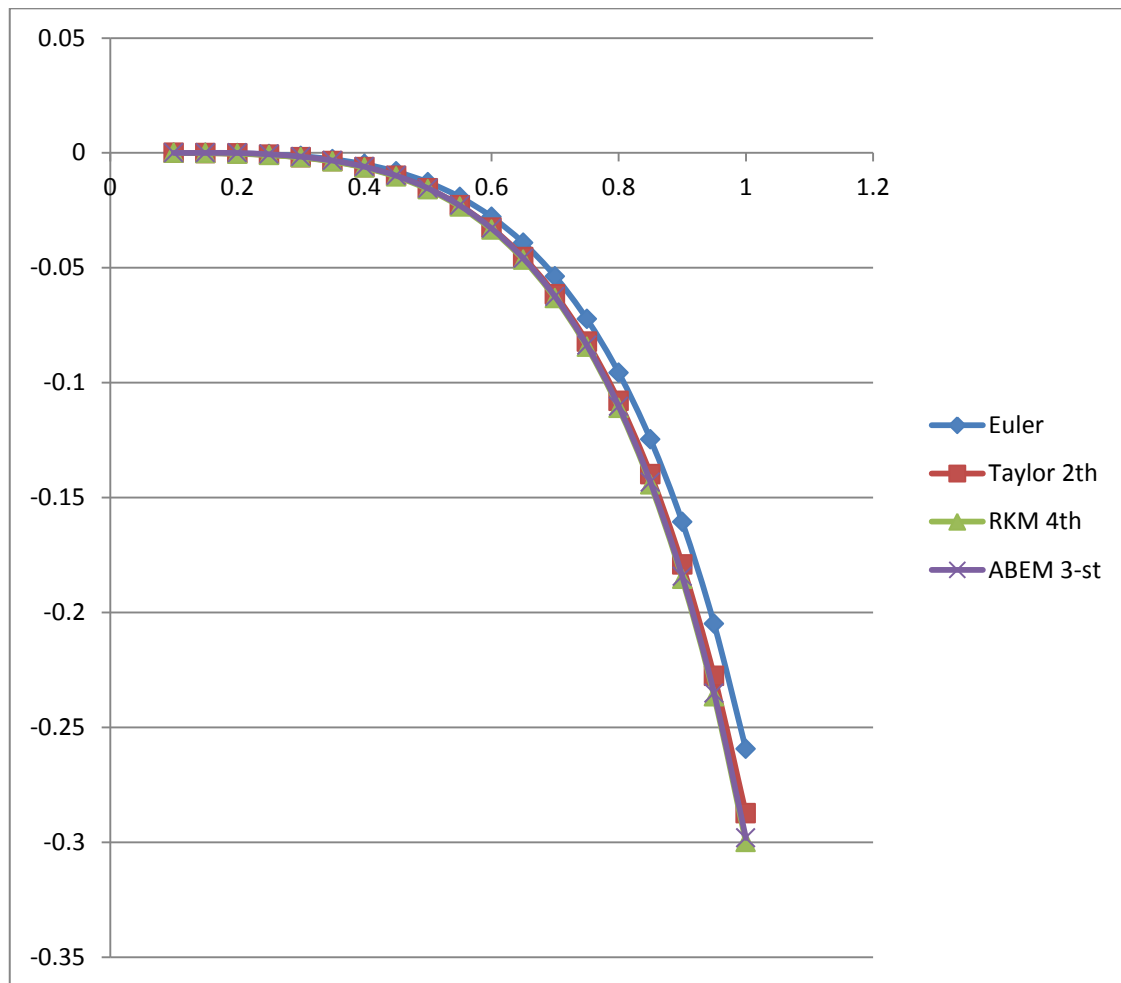
Let  $h=0.05$

The results are shown in Table 4.25

**Table 4.29:** Illustrations Euler, Taylor, RKM 4th, ABEM 3st when  $h=0.05$

<b>I</b>	<b>h</b>	<b>xi</b>	<b>Euler</b>	<b>Taylor 2th</b>	<b>RKM 4th</b>	<b>ABEM 3-st</b>
1	0.05	0.1	0	0	0	0
2	0.05	0.15	-0.00005	-8.7488E-05	-0.0001016	-1.5625E-06
3	0.05	0.2	-0.00022	-0.00034065	-0.0003754	-2.5004E-05
4	0.05	0.25	-0.00062	-0.0008911	-0.00095362	-0.0005876
5	0.05	0.3	-0.0014	-0.00190886	-0.00200765	-0.00162543
6	0.05	0.35	-0.00276	-0.00360343	-0.00374945	-0.0033496
7	0.05	0.4	-0.00492	-0.00622553	-0.00643402	-0.00601463
8	0.05	0.45	-0.00816	-0.0100698	-0.01036266	-0.0099212
9	0.05	0.5	-0.0128	-0.01547864	-0.01588785	-0.0154209
10	0.05	0.55	-0.01921	-0.02284767	-0.02341988	-0.02292285
11	0.05	0.6	-0.02783	-0.03263333	-0.03343603	-0.03290278
12	0.05	0.65	-0.03916	-0.04536348	-0.04649328	-0.04591547
13	0.05	0.7	-0.05377	-0.06165222	-0.06324604	-0.06261218
14	0.05	0.75	-0.07234	-0.08222082	-0.08447117	-0.08376516
15	0.05	0.8	-0.09568	-0.1079279	-0.11110417	-0.11030282
16	0.05	0.85	-0.12474	-0.13981374	-0.14429257	-0.14336149
17	0.05	0.9	-0.16069	-0.17916708	-0.18547727	-0.18436375
18	0.05	0.95	-0.20497	-0.22762905	-0.23652081	-0.23514062
19	0.05	1	-0.25943	-0.28736051	-0.29991808	-0.29813014

Also see each of graphs together



**Figure 4.21:** graphs of Euler, Taylor 2<sup>nd</sup>, RKM 4<sup>th</sup>, and ABEM 3st when  $h=0.05$

Also there is a different between each of graphs. So we have to change a step size  $h$

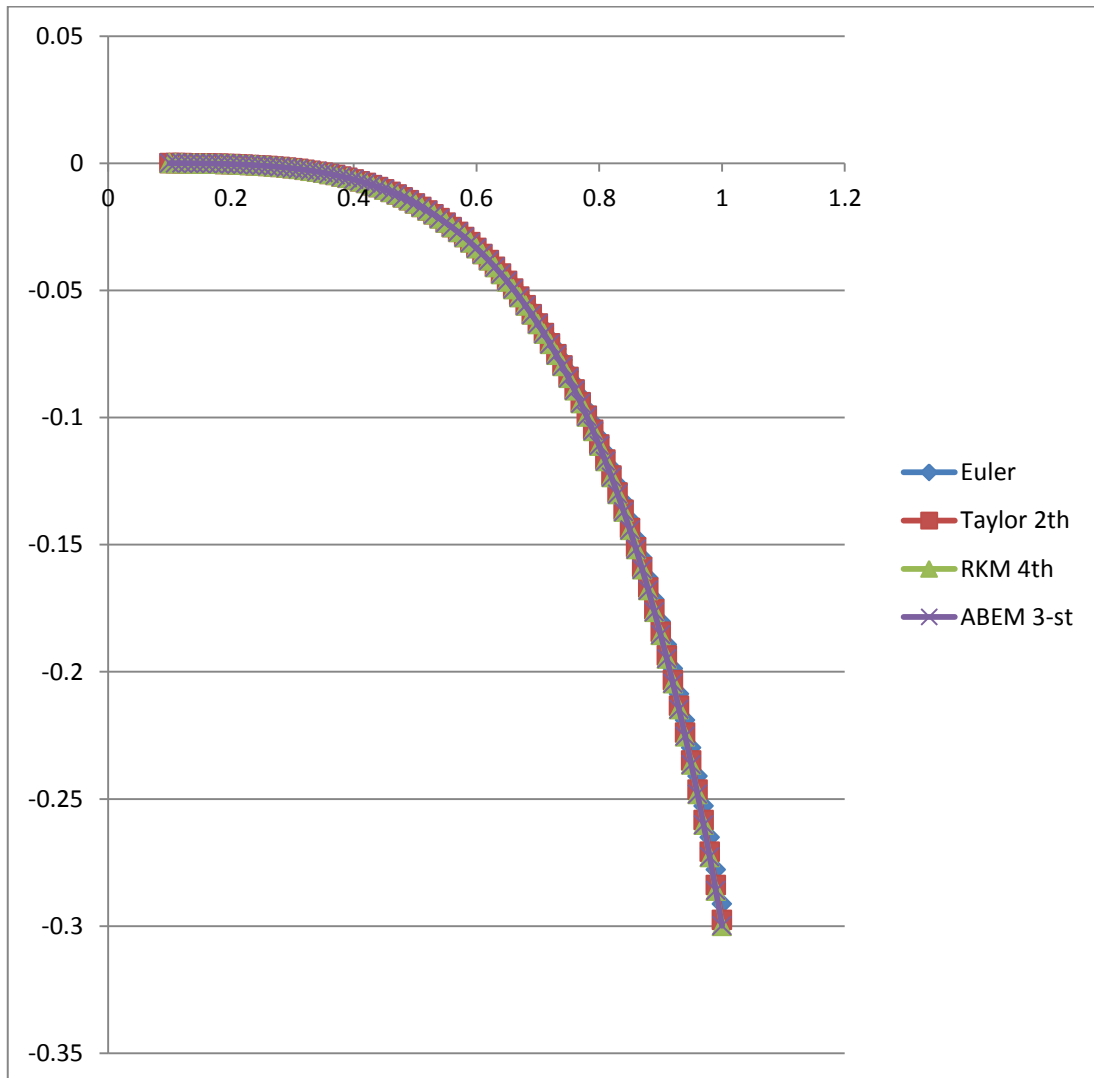
Let  $h=0.01$  Then see all results together

**Table 4.30:** Illustrations Euler, Taylor, RKM 4th, ABEM 3st when  $h=0.01$ 

<b>i</b>	<b>h</b>	<b>xi</b>	<b>Euler</b>	<b>Taylor 2th</b>	<b>RKM 4th</b>	<b>ABEM 3-st</b>
0	0.01	0.1	0	0	0	0
1	0.01	0.11	-0.00001	-1.1E-05	-1.16E-05	-2.5E-09
2	0.01	0.12	-2.3E-05	-2.7E-05	-2.68E-05	-4E-08
3	0.01	0.13	-4.1E-05	-4.6E-05	-4.64E-05	-2E-05
4	0.01	0.14	-6.3E-05	-7.1E-05	-7.11E-05	-4.4E-05
5	0.01	0.15	-9E-05	-0.0001	-0.000102	-7.5E-05
6	0.01	0.16	-0.00012	-0.00014	-0.000139	-0.00011
7	0.01	0.17	-0.00016	-0.00018	-0.000184	-0.00016
8	0.01	0.18	-0.00021	-0.00024	-0.000238	-0.00021
9	0.01	0.19	-0.00027	-0.0003	-0.000301	-0.00027
10	0.01	0.2	-0.00034	-0.00037	-0.000375	-0.00035
11	0.01	0.21	-0.00042	-0.00046	-0.000462	-0.00043
12	0.01	0.22	-0.00051	-0.00056	-0.000561	-0.00053
13	0.01	0.23	-0.00062	-0.00067	-0.000676	-0.00065
14	0.01	0.24	-0.00074	-0.0008	-0.000806	-0.00078
15	0.01	0.25	-0.00088	-0.00095	-0.000954	-0.00093
16	0.01	0.26	-0.00104	-0.00112	-0.00112	-0.00109
17	0.01	0.27	-0.00121	-0.0013	-0.001307	-0.00128
18	0.01	0.28	-0.00141	-0.00151	-0.001516	-0.00149
19	0.01	0.29	-0.00163	-0.00174	-0.001749	-0.00172
20	0.01	0.3	-0.00188	-0.002	-0.002008	-0.00198
21	0.01	0.31	-0.00215	-0.00229	-0.002293	-0.00227
22	0.01	0.32	-0.00245	-0.0026	-0.002609	-0.00258
23	0.01	0.33	-0.00278	-0.00295	-0.002955	-0.00293
24	0.01	0.34	-0.00314	-0.00333	-0.003334	-0.00331
25	0.01	0.35	-0.00354	-0.00374	-0.003749	-0.00372
26	0.01	0.36	-0.00397	-0.00419	-0.004202	-0.00417
27	0.01	0.37	-0.00444	-0.00468	-0.004694	-0.00467
28	0.01	0.38	-0.00496	-0.00522	-0.005229	-0.0052
29	0.01	0.39	-0.00551	-0.0058	-0.005808	-0.00578
30	0.01	0.4	-0.00611	-0.00642	-0.006434	-0.00641
31	0.01	0.41	-0.00676	-0.00709	-0.00711	-0.00708
32	0.01	0.42	-0.00747	-0.00782	-0.007838	-0.00781
33	0.01	0.43	-0.00822	-0.0086	-0.008621	-0.00859
34	0.01	0.44	-0.00903	-0.00944	-0.009461	-0.00943
35	0.01	0.45	-0.0099	-0.01034	-0.010363	-0.01033
36	0.01	0.46	-0.01083	-0.0113	-0.011328	-0.0113
37	0.01	0.47	-0.01183	-0.01233	-0.01236	-0.01233
38	0.01	0.48	-0.01289	-0.01343	-0.013461	-0.01343
39	0.01	0.49	-0.01403	-0.0146	-0.014636	-0.01461
40	0.01	0.5	-0.01524	-0.01585	-0.015888	-0.01586
41	0.01	0.51	-0.01653	-0.01718	-0.017219	-0.01719
42	0.01	0.52	-0.0179	-0.01859	-0.018635	-0.01861
43	0.01	0.53	-0.01936	-0.02009	-0.020137	-0.02011
44	0.01	0.54	-0.0209	-0.02167	-0.021731	-0.0217

45	0.01	0.55	-0.02254	-0.02336	-0.02342	-0.02339
46	0.01	0.56	-0.02428	-0.02514	-0.025208	-0.02518
47	0.01	0.57	-0.02611	-0.02702	-0.027099	-0.02707
48	0.01	0.58	-0.02805	-0.02902	-0.029098	-0.02907
49	0.01	0.59	-0.0301	-0.03112	-0.031209	-0.03118
50	0.01	0.6	-0.03227	-0.03334	-0.033436	-0.03341
51	0.01	0.61	-0.03455	-0.03568	-0.035785	-0.03575
52	0.01	0.62	-0.03695	-0.03814	-0.03826	-0.03823
53	0.01	0.63	-0.03949	-0.04074	-0.040866	-0.04083
54	0.01	0.64	-0.04216	-0.04347	-0.043609	-0.04358
55	0.01	0.65	-0.04496	-0.04634	-0.046493	-0.04646
56	0.01	0.66	-0.04791	-0.04936	-0.049525	-0.04949
57	0.01	0.67	-0.05101	-0.05253	-0.05271	-0.05268
58	0.01	0.68	-0.05427	-0.05585	-0.056055	-0.05602
59	0.01	0.69	-0.05768	-0.05935	-0.059565	-0.05953
60	0.01	0.7	-0.06127	-0.06301	-0.063246	-0.06321
61	0.01	0.71	-0.06502	-0.06685	-0.067106	-0.06707
62	0.01	0.72	-0.06896	-0.07087	-0.071151	-0.07112
63	0.01	0.73	-0.07309	-0.07508	-0.075389	-0.07536
64	0.01	0.74	-0.07741	-0.07949	-0.079826	-0.07979
65	0.01	0.75	-0.08193	-0.08411	-0.084471	-0.08444
66	0.01	0.76	-0.08667	-0.08894	-0.089332	-0.0893
67	0.01	0.77	-0.09162	-0.09399	-0.094417	-0.09438
68	0.01	0.78	-0.0968	-0.09928	-0.099734	-0.0997
69	0.01	0.79	-0.10221	-0.1048	-0.105293	-0.10526
70	0.01	0.8	-0.10787	-0.11057	-0.111104	-0.11107
71	0.01	0.81	-0.11378	-0.1166	-0.117176	-0.11714
72	0.01	0.82	-0.11996	-0.12289	-0.12352	-0.12348
73	0.01	0.83	-0.12641	-0.12947	-0.130146	-0.13011
74	0.01	0.84	-0.13315	-0.13633	-0.137066	-0.13703
75	0.01	0.85	-0.14018	-0.1435	-0.144293	-0.14425
76	0.01	0.86	-0.14752	-0.15099	-0.151837	-0.1518
77	0.01	0.87	-0.15518	-0.1588	-0.159714	-0.15967
78	0.01	0.88	-0.16318	-0.16695	-0.167936	-0.16789
79	0.01	0.89	-0.17153	-0.17545	-0.176519	-0.17648
80	0.01	0.9	-0.18024	-0.18433	-0.185477	-0.18543
81	0.01	0.91	-0.18933	-0.19359	-0.194828	-0.19478
82	0.01	0.92	-0.19881	-0.20326	-0.204589	-0.20454
83	0.01	0.93	-0.20871	-0.21334	-0.214778	-0.21473
84	0.01	0.94	-0.21903	-0.22387	-0.225415	-0.22537
85	0.01	0.95	-0.22981	-0.23486	-0.236521	-0.23647
86	0.01	0.96	-0.24106	-0.24633	-0.248117	-0.24807
87	0.01	0.97	-0.2528	-0.25831	-0.260229	-0.26018
88	0.01	0.98	-0.26506	-0.27081	-0.272881	-0.27283
89	0.01	0.99	-0.27786	-0.28388	-0.286101	-0.28605
90	0.01	1	-0.29123	-0.29753	-0.299918	-0.29986

Also see each of graphs together



**Figure 4.22:** Graphs of Euler, Taylor 2<sup>nd</sup>, RKM 4<sup>th</sup>, and ABEM 3st when  $h=0.0$



**4.7.2 Example 2:** Consider the nonlinear Abel differential equation of first kind

$$y'(x) = x^2y^3 + xy^2 + \sqrt{x}y + \tan x, \quad x \in (0,1]$$

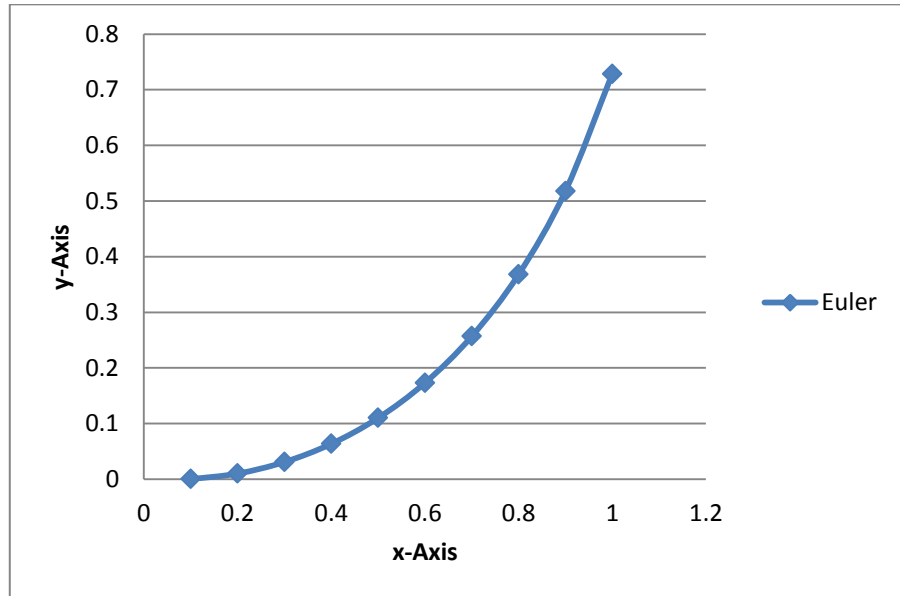
$$y(0) = 0$$

Solution: In this Example we have  $f_3(x) = x^3$ ,  $f_2(x) = x$ ,  $f_1(x) = \sqrt{x}$  and  $f_0(x) = \tan x$ . Also there is no known closed form solution. Now, we apply each of above methods on it . (Jafari et al, 2013)

First for Euler Method

**Table 4.31:** Tabulation of Abel differential equation by Euler's method

i	h	$x_i$	Euler	$f(x_i, w_i)$
1	0.1	0.1	0	0.1003347
2	0.1	0.2	0.0100335	0.2072173
3	0.1	0.3	0.0307552	0.3264679
4	0.1	0.4	0.063402	0.4645409
5	0.1	0.5	0.1098561	0.6303481
6	0.1	0.6	0.1728909	0.8378527
7	0.1	0.7	0.2566762	1.1114431
8	0.1	0.8	0.3678205	1.4987091
9	0.1	0.9	0.5176914	2.1048696
10	0.1	1	0.7281783	3.2019417

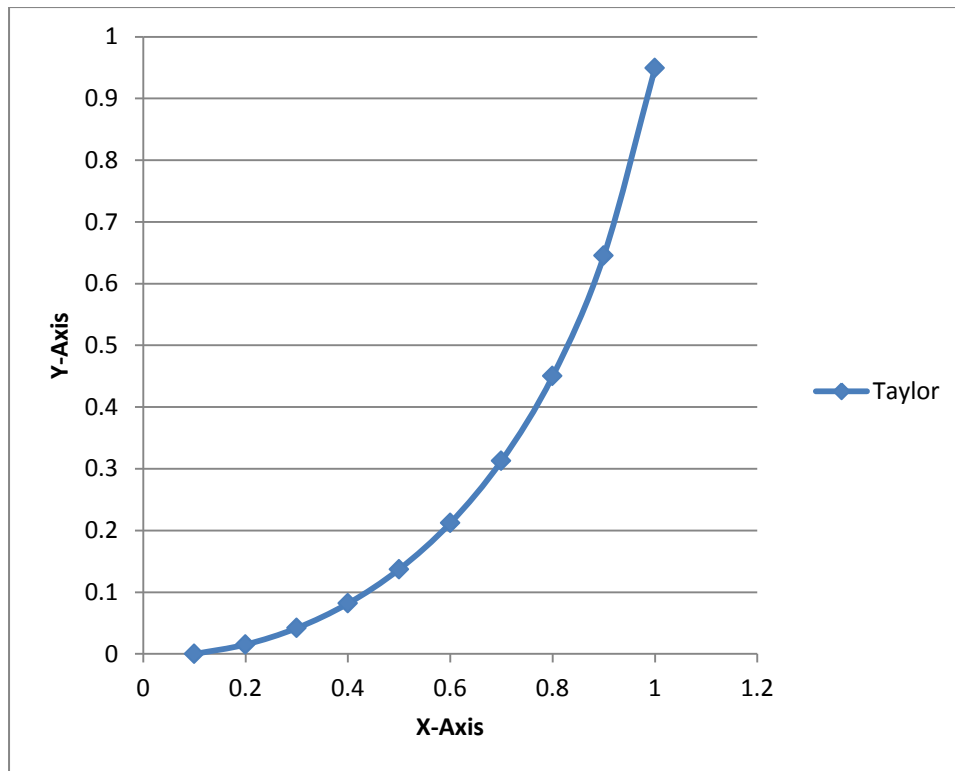


**Figure 4.23:** Graph of ADE by Euler's method

For Taylor Method of order two

**Table 4.32:** Illustration of Abel differential equation by Taylor method

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Taylor</b>	<b><math>f(x_i, w_i)</math></b>	<b><math>f'(x_i, w_i)</math></b>	<b>T(2)</b>
1	0.1	0.1	0	0.100335	1.041796	0.152424
2	0.1	0.2	0.015242	0.209573	1.139742	0.26656
3	0.1	0.3	0.041898	0.332818	1.299805	0.397809
4	0.1	0.4	0.081679	0.477208	1.546462	0.554531
5	0.1	0.5	0.137132	0.653317	1.930452	0.74984
6	0.1	0.6	0.212116	0.878873	2.559151	1.006831
7	0.1	0.7	0.312799	1.187482	3.681462	1.371555
8	0.1	0.8	0.449955	1.652361	5.958514	1.950286
9	0.1	0.9	0.644984	2.463783	11.54998	3.041282
10	0.1	1	0.949112	4.262305	30.38257	5.781433

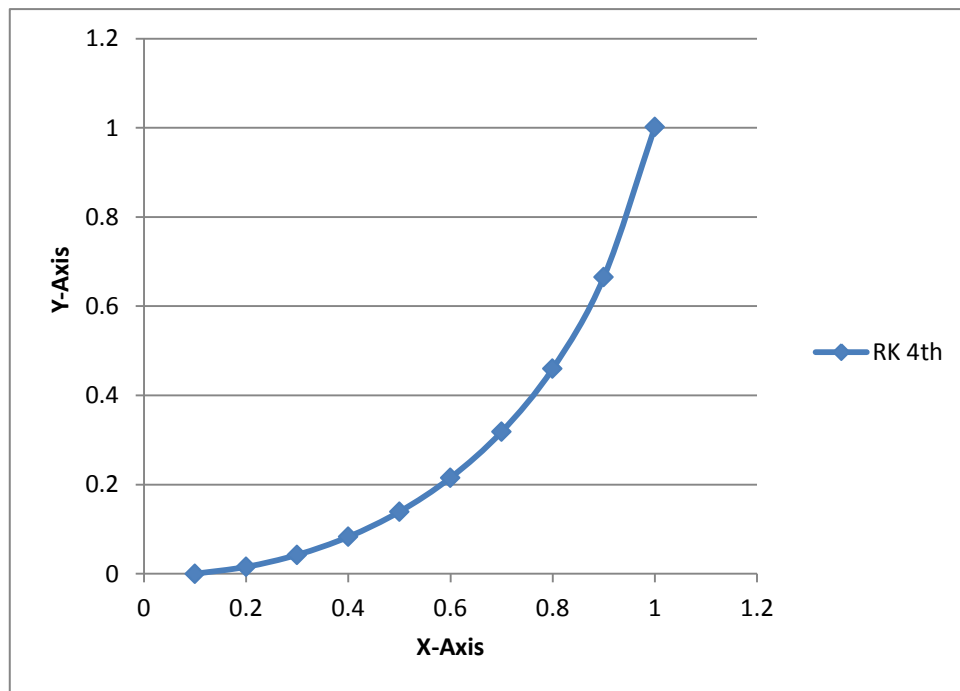


**Figure 4.24:** graph of ADE by Taylor method of order two

For Runge-Kutta method of order four

**Table 4.33:** Illustration of Abel Differential equation by RK-Method of order 4

i	h	$x_i$	RK 4th	$k_1$	$k_2$	$k_3$	$k_4$
1	0.1	0.1	0	0.010033	0.015308	0.015411	0.020965
2	0.1	0.2	0.015406	0.020965	0.026845	0.026997	0.033311
3	0.1	0.3	0.042399	0.033311	0.040121	0.040338	0.047795
4	0.1	0.4	0.082736	0.047795	0.055995	0.056313	0.065496
5	0.1	0.5	0.139054	0.065497	0.075829	0.076325	0.08824
6	0.1	0.6	0.215395	0.088242	0.102059	0.102913	0.119533
7	0.1	0.7	0.318348	0.119539	0.139669	0.141369	0.167207
8	0.1	0.8	0.459819	0.167228	0.200633	0.204842	0.252615
9	0.1	0.9	0.664951	0.252708	0.321168	0.33582	0.456203
10	0.1	1	1.002099	0.457002	0.664901	0.760138	1.385087

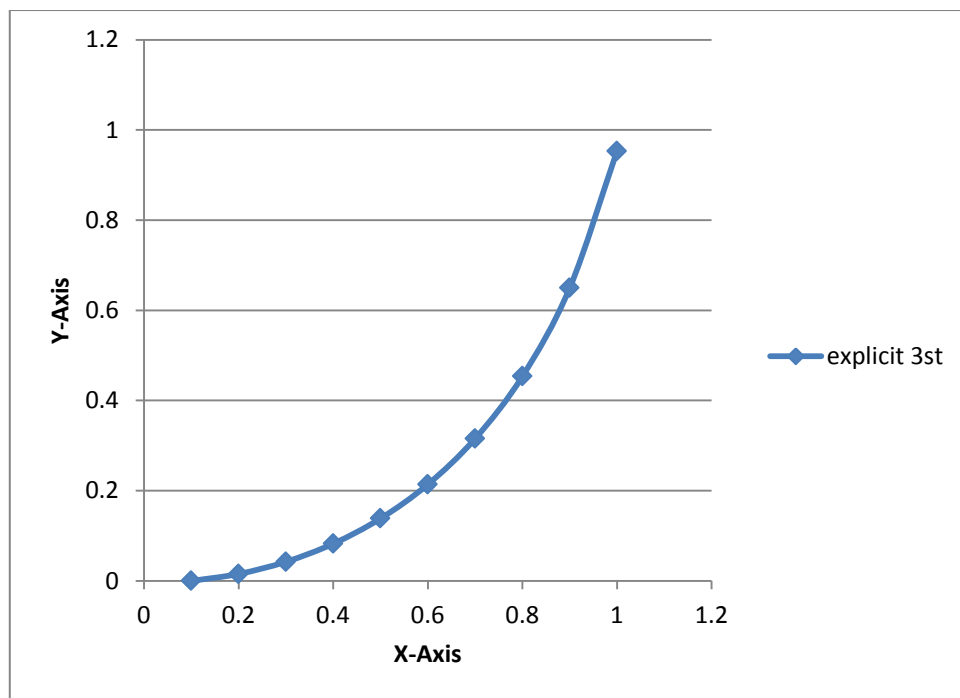


**Figure 4.25:** graph of ADE by Runge-Kutta Method of 4th order

And For Adam Bash Fourth Explicit 3-Step method

**Table 4.34:** Illustration of Abel Differential equation by ABE 3-step Method

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Explicit 3st</b>	<b><math>f(x_i, w_i)_{exp}</math></b>
1	0.1	0.1	0	0.100335
2	0.1	0.2	0.015406	0.209647
3	0.1	0.3	0.042399	0.333105
4	0.1	0.4	0.082472	0.477763
5	0.1	0.5	0.138364	0.654375
6	0.1	0.6	0.213964	0.880867
7	0.1	0.7	0.315454	1.191255
8	0.1	0.8	0.453594	1.659672
9	0.1	0.9	0.649567	2.478136
10	0.1	1	0.952889	4.283515

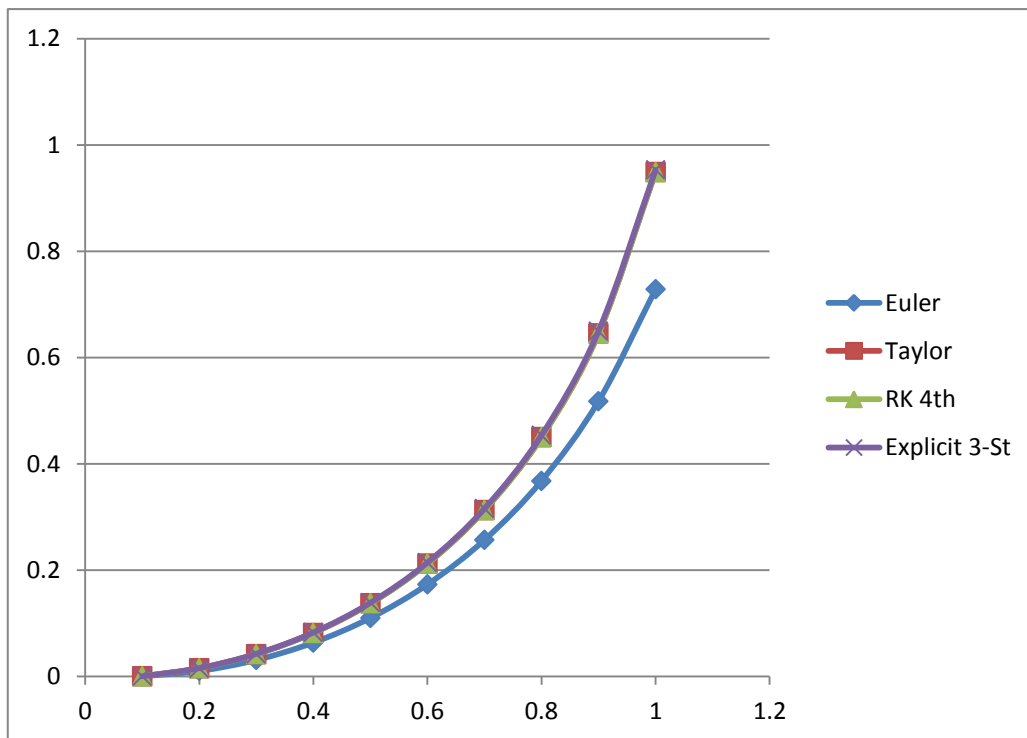


**Figure 4.26:** graph of ADE by ABEM 3-steps

And All Methods together when  $h=0.1$

**Table 4.35:** Illustration of ADE by Euler, Taylor 2th, RK4th and ABE 3-step Methods

i	h	$x_i$	Euler	Taylor	RK 4th	explicit 3st
1	0.1	0.1	0	0	0	0
2	0.1	0.2	0.010033	0.015242	0.015406	0.015406
3	0.1	0.3	0.030755	0.041898	0.042399	0.042399
4	0.1	0.4	0.063402	0.081679	0.082736	0.082472
5	0.1	0.5	0.109856	0.137132	0.139054	0.138364
6	0.1	0.6	0.172891	0.212116	0.215395	0.213964
7	0.1	0.7	0.256676	0.312799	0.318348	0.315454
8	0.1	0.8	0.36782	0.449955	0.459819	0.453594
9	0.1	0.9	0.517691	0.644984	0.664951	0.649567
10	0.1	1	0.728178	0.949112	1.002099	0.952889



**Figure 4.27:** Illustrations of Euler, Taylor, RKM 4th, and ABEM 3st

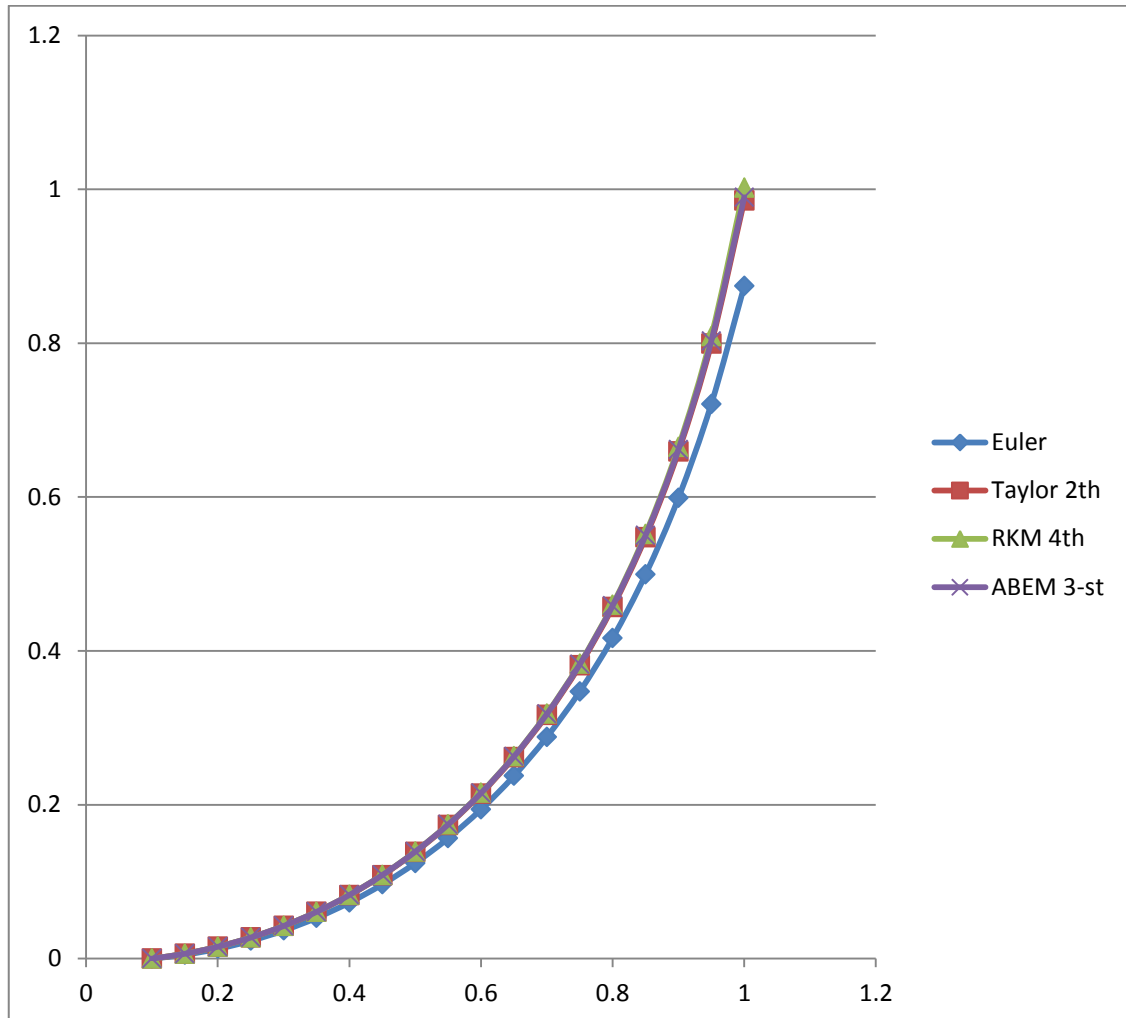
**Note:** If we compare between the results for each way, we can see that the results are close to each other. And all figures are the same graphs.

The results of all methods together illustrated in the table below where  $h=0.05$

**Table 4.36:** Illustrations Euler, Taylor, RKM 4th, ABEM 3st when  $h=0.05$

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Euler</b>	<b>Taylor</b>	<b>RKM 2th</b>	<b>explicit 3st</b>
0	0.05	0.1	0	0	0	0
1	0.05	0.15	0.005017	0.006319	0.006338	0.006338
2	0.05	0.2	0.012671	0.015353	0.015406	0.015406
3	0.05	0.25	0.023092	0.027259	0.027361	0.027348
4	0.05	0.3	0.036444	0.042231	0.042399	0.04237
5	0.05	0.35	0.052935	0.060507	0.06076	0.060712
6	0.05	0.4	0.072819	0.082377	0.082736	0.082664
7	0.05	0.45	0.096409	0.108197	0.108687	0.108585
8	0.05	0.5	0.124099	0.138404	0.139054	0.138912
9	0.05	0.55	0.156379	0.173545	0.174389	0.174194
10	0.05	0.6	0.193876	0.21431	0.215394	0.215123
11	0.05	0.65	0.237395	0.261592	0.262976	0.262597
12	0.05	0.7	0.287998	0.316577	0.318346	0.317805
13	0.05	0.75	0.347095	0.380882	0.383166	0.382374
14	0.05	0.8	0.41661	0.456799	0.459813	0.45861
15	0.05	0.85	0.49922	0.547723	0.551847	0.549932
16	0.05	0.9	0.598745	0.658953	0.664932	0.661687
17	0.05	0.95	0.720805	0.799352	0.808795	0.802811
18	0.05	1	0.873976	0.985134	1.002034	0.989602

And all graphs together showed in the figure below



**Figure 4.28:** Illustrations of Euler, Taylor, RKM 4th, and ABEM 3st

To get best result for this problem we use finer step size  $h$ , Now for  $h=0.01$  see all the results of Methods together.

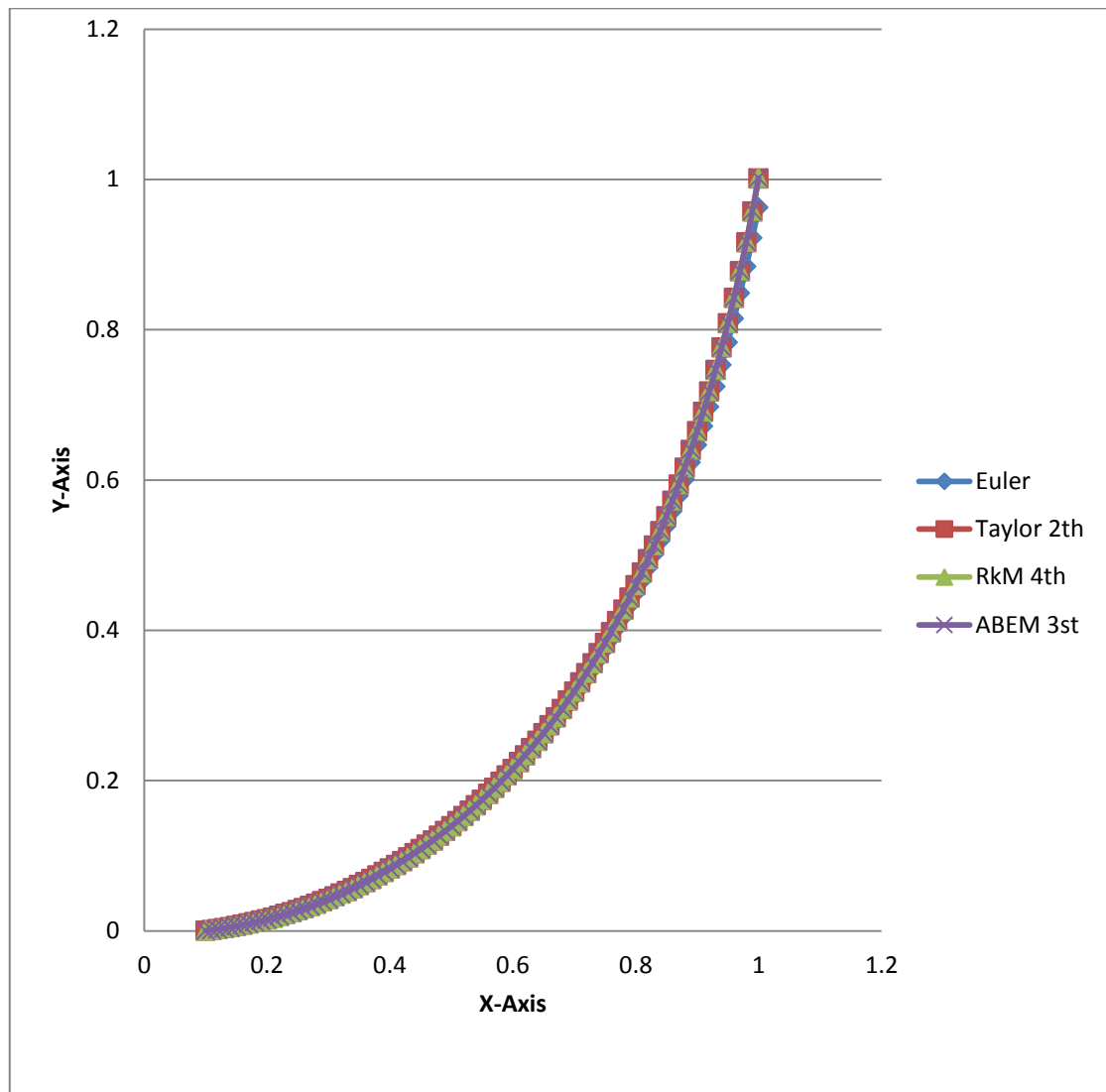
**Table 4.37:** Illustrations Euler, Taylor, RKM 4th, ABEM 3st when  $h=0.01$ 

<b>i</b>	<b>h</b>	<b><math>x_i</math></b>	<b>Euler</b>	<b>Taylor</b>	<b>RKM 4th</b>	<b>ABEM 3st</b>
0	0.01	0.1	0	0	0	0
1	0.01	0.11	0.001003	0.001055	0.001056	0.001056
2	0.01	0.12	0.002111	0.002216	0.002216	0.002216
3	0.01	0.13	0.003324	0.003482	0.003483	0.003483
4	0.01	0.14	0.004644	0.004855	0.004856	0.004856
5	0.01	0.15	0.00607	0.006337	0.006338	0.006338
6	0.01	0.16	0.007605	0.007927	0.007929	0.007928
7	0.01	0.17	0.009249	0.009627	0.00963	0.009629
8	0.01	0.18	0.011004	0.011439	0.011442	0.011442
9	0.01	0.19	0.012871	0.013363	0.013367	0.013367
10	0.01	0.2	0.014851	0.015401	0.015406	0.015406
11	0.01	0.21	0.016944	0.017554	0.01756	0.01756
12	0.01	0.22	0.019154	0.019825	0.019832	0.019831
13	0.01	0.23	0.021481	0.022213	0.022221	0.022221
14	0.01	0.24	0.023927	0.024721	0.02473	0.02473
15	0.01	0.25	0.026492	0.027351	0.027361	0.027361
16	0.01	0.26	0.02918	0.030103	0.030115	0.030115
17	0.01	0.27	0.031991	0.032981	0.032994	0.032994
18	0.01	0.28	0.034928	0.035985	0.036	0.035999
19	0.01	0.29	0.037992	0.039118	0.039134	0.039134
20	0.01	0.3	0.041185	0.042381	0.042399	0.042399
21	0.01	0.31	0.044509	0.045777	0.045797	0.045796
22	0.01	0.32	0.047966	0.049308	0.04933	0.049329
23	0.01	0.33	0.051559	0.052975	0.053	0.052999
24	0.01	0.34	0.055289	0.056783	0.056809	0.056808
25	0.01	0.35	0.059159	0.060732	0.06076	0.060759
26	0.01	0.36	0.063172	0.064825	0.064856	0.064855
27	0.01	0.37	0.06733	0.069065	0.069098	0.069097
28	0.01	0.38	0.071635	0.073455	0.073491	0.07349
29	0.01	0.39	0.076091	0.077998	0.078036	0.078035
30	0.01	0.4	0.0807	0.082696	0.082736	0.082735
31	0.01	0.41	0.085465	0.087552	0.087596	0.087594
32	0.01	0.42	0.09039	0.092571	0.092617	0.092616
33	0.01	0.43	0.095477	0.097754	0.097804	0.097802
34	0.01	0.44	0.10073	0.103107	0.103159	0.103158
35	0.01	0.45	0.106153	0.108631	0.108687	0.108685
36	0.01	0.46	0.111748	0.114332	0.114391	0.11439
37	0.01	0.47	0.117521	0.120213	0.120276	0.120274
38	0.01	0.48	0.123475	0.126279	0.126345	0.126343
39	0.01	0.49	0.129614	0.132533	0.132603	0.132601
40	0.01	0.5	0.135943	0.138981	0.139054	0.139052
41	0.01	0.51	0.142466	0.145627	0.145704	0.145702
42	0.01	0.52	0.149188	0.152476	0.152557	0.152554
43	0.01	0.53	0.156114	0.159533	0.159618	0.159616
44	0.01	0.54	0.16325	0.166804	0.166894	0.166891



45	0.01	0.55	0.1706	0.174296	0.174389	0.174387
46	0.01	0.56	0.178171	0.182013	0.182111	0.182109
47	0.01	0.57	0.18597	0.189963	0.190066	0.190063
48	0.01	0.58	0.194001	0.198152	0.198259	0.198256
49	0.01	0.59	0.202273	0.206587	0.2067	0.206696
50	0.01	0.6	0.210793	0.215277	0.215394	0.215391
51	0.01	0.61	0.219567	0.224228	0.224351	0.224347
52	0.01	0.62	0.228605	0.23345	0.233578	0.233574
53	0.01	0.63	0.237914	0.242951	0.243085	0.243081
54	0.01	0.64	0.247504	0.252742	0.252881	0.252877
55	0.01	0.65	0.257383	0.262831	0.262976	0.262972
56	0.01	0.66	0.267563	0.273231	0.273382	0.273377
57	0.01	0.67	0.278054	0.283951	0.284109	0.284104
58	0.01	0.68	0.288867	0.295006	0.29517	0.295164
59	0.01	0.69	0.300014	0.306407	0.306577	0.306571
60	0.01	0.7	0.311509	0.318168	0.318346	0.318339
61	0.01	0.71	0.323366	0.330305	0.33049	0.330483
62	0.01	0.72	0.335599	0.342834	0.343026	0.343019
63	0.01	0.73	0.348224	0.355772	0.355972	0.355964
64	0.01	0.74	0.361259	0.369136	0.369345	0.369336
65	0.01	0.75	0.374721	0.382949	0.383166	0.383156
66	0.01	0.76	0.388631	0.39723	0.397456	0.397446
67	0.01	0.77	0.403011	0.412004	0.412239	0.412228
68	0.01	0.78	0.417882	0.427295	0.427541	0.427529
69	0.01	0.79	0.433272	0.443133	0.443388	0.443375
70	0.01	0.8	0.449206	0.459545	0.459812	0.459798
71	0.01	0.81	0.465714	0.476566	0.476845	0.476829
72	0.01	0.82	0.48283	0.494232	0.494523	0.494506
73	0.01	0.83	0.500588	0.512581	0.512886	0.512867
74	0.01	0.84	0.519027	0.531658	0.531978	0.531957
75	0.01	0.85	0.538189	0.55151	0.551846	0.551823
76	0.01	0.86	0.558123	0.57219	0.572544	0.572518
77	0.01	0.87	0.578879	0.593758	0.594132	0.594103
78	0.01	0.88	0.600515	0.616279	0.616674	0.616642
79	0.01	0.89	0.623096	0.639827	0.640246	0.640209
80	0.01	0.9	0.646692	0.664485	0.66493	0.664888
81	0.01	0.91	0.671383	0.690344	0.69082	0.690772
82	0.01	0.92	0.697259	0.717511	0.718021	0.717966
83	0.01	0.93	0.724422	0.746106	0.746654	0.746592
84	0.01	0.94	0.752985	0.776265	0.776858	0.776786
85	0.01	0.95	0.78308	0.808146	0.808791	0.808707
86	0.01	0.96	0.814856	0.841931	0.842636	0.842539
87	0.01	0.97	0.848484	0.877831	0.878608	0.878493
88	0.01	0.98	0.884163	0.916094	0.916956	0.91682
89	0.01	0.99	0.922125	0.957014	0.957978	0.957815
90	0.01	1	0.962639	1.000939	1.002026	1.001829

And also the graphs of all methods together showed in the figure below where  $h=0.01$



**Figure 4.29:** Illustrations of Euler, Taylor, RKM 4th, and ABEM 3st

## CHAPTER 5

### CONCLUSIONS

In Chapter 2, some basic definitions and theorems to examine the numerical methods were presented. Then some known numerical methods for solving the Abel differential equation of first kind were discussed. For restrict Form of Abel differential equation of first kind it was concluded that the comparison between numerical and closed form solution were in very close agreement. In this comparison we have learned that Euler's Method is the simplest and most elementary method for solving IVP, for this reason it is not reliable in terms of accuracy. But it's so helpful for those who want to learn the basic principles of numerical methods for solving ODEs and their derivations. This also true for those who want to learn programming and implementing these methods using EXCEL.

With respect to Taylor's Method, although it is reasonably accurate with high order of local truncation error, but again it is very rarely used in practice since it requires the calculation of the derivatives of the functions  $f(x, y)$ .

For Adam Bash forth multi step method, however we have obtained the accuracy required. The drawback here is that as the name multi step indicates, we require several starting points depending on the order of the method used. This is usually taken from numerical methods that require only one starting point like Euler method or Runge-Kutta method. Because of this the method is not widely used in practice.

For Runge-kutta Method is very accurate, because of its higher order of its local truncation error and it convergence faster to close form solution and also it does not need the derivative of  $f(x, y)$ . Another important reason for its popularity it only requires the initial value as a starting point. Therefore they are the most popular methods for solving ODEs.

Note that sometimes higher order of methods does not mean higher accuracy. But for step size  $h$ , we used different step size but the best one is one it has a smaller step size  $h$ . Those Methods are presented by Excel work sheet and graphical illustrated, usually the close form solutions are not always available to compare the numerical methods with. Two examples on unrestricted form of Abel differential equation of first kind have been taken. Figure 4.20 shows all methods together for step size  $h=0.1$ , It can be seen that there are some discrepancy between them but in Figure 4.21 a step size  $h=0.05$  was used and results

compared much better than those with  $h=0.1$ , in Figure 4.22 shows even comparison the curves are almost juxtaposed indicating an excellent comparison. This is also true for Figure 4.27, Figure 4.28, and Figure 4.29.

Finally, we concluded that every differential equation however it is not known closed form solution; the numerical solution is the best way to solve it approximately.

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