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## d-ORTHOGONAL POLYNOMIALS

# A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY 

By MUSA DAN-AZUMI MOHAMMED

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In Partial Fulfillment of the Requirements for theDegree of Master of Science in
Mathematics
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Mathematics

# Musa Dan-azumi Mohammed: d-ORTHOGONAL POLYNOMIALS 

## Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Ikay SAL HOĞLU

We certify that, this thesis is satisfactory for the award of the degree of Master of Science In Mathematics.

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my parents...


#### Abstract

This work consists of definitions and basic properties of Orthogonal Polynomials, dOrthogonal Polynomials and some examples of the d-Orthogonal Polynomial families. d-Orthogonal polynomials are extensions of Orthogonal Polynomials. For specific values of some constants which are used in the definitions of d-orthogonal polynomials, they give the classical d-Orthogonal Polynomials.

Several properties such as generating functions, differential equations and recurrence relations for these Orthogonal/ d-Orthogonal polynomial families are obtained.

Keywords:Orthogonal Polynomials, d-Orthogonal Polynomials, Laguerre Polynomials, Jacobi Polynomials, d-Laguerre Polynomials, d-Hermite Polynomials, Laguerre-Pólya class and Hyper-Bessel functions.


## ÖZET

Bu çalışma ortogonal polinomlar, d-Ortogonal polinomlar vedortogonalpolinomlarailesinintanımve bazıtemel özellikleriniiçermektedir. dOrtogonalpolinomlar Ortogonal polinomlarıngenişletilmişhalidir. d-

Ortogonal polinomlarıntanımlarındakullanılanbazısabitlerinözeldeğerleriiçinklasikdOrtogonalpolinomlarıvermektedir.Üretenfonksiyonlar,diferensiyeldenklemlervetekrarbağıntıla rıgibiOrtogonal/d-Ortogonalpolinomailesininbazıözelliklerieldeedilmiştir.

AnahtarKelimeler :Ortogonalpolinomlar, d - Ortogonalpolinomları, Laguerre polinomları,Jacobipolinomları, d - Laguerre polinomları, d - Hermite polinomları,Laguerre Pólyasınıfve Hyper- Besselfonksiyonları.

## TABLE OF CONTENTS

ACKNOWLEDGEMENTS ..... i
ABSTRACT ..... iii
ÖZET ..... iii
TABLE OF CONTENT ..... v
LIST OF FIGURES ..... vii
LIST OFSYMBOLS ..... vii
CHAPTER 1: INTRODUCTION AND BASIC DEFINITIONS
1.1 Introduction ..... 1
1.2 Gamma Function ..... 2
1.2.1 Definition .....  2
1.2.2 Definition ..... 3
1.2.3 Lemma. ..... 4
1.3 Orthogonal Polynomials ..... 4
1.3.1 Definition ..... 4
1.3.2 Definition (Orthogonal Polynomials) ..... 5
1.3.2.1 Properties of Polynomials ..... 5
1.4 Orthogonality of Functions ..... 5
1.4.1 Theorem .....  6
1.5 Properties of Orthogonal Polynomials ..... 7
1.5.1 Definition Recurrence Relation for Orthogonal Polynomial ..... 7
1.5.2 Definition ..... 7
1.5.3 Definition ..... 8
1.6 Some Special Orthogonal Polynomial Families ..... 8
1.6.1 Hermite Polynomials ..... 8
1.6.1.1 Rodrigues Formula for Hermite Polynomials .....  9
1.6.1.2 The generating Function for the Hermite Polynomials ..... 9
1.6.1.3 The Norm of the Hermite Polynomials ..... 9
1.6.2 Laguerre Polynomials ..... 11
1.6.3 Jacobi Polynomials ..... 15
1.6.3.1 Differential Equations for Jacobi Polynomials ..... 18
1.6.3.2 Generating Function ..... 18
1.7 Vector Space ..... 18
1.7.1 Definition ..... 18
1.7.2 Definition (Inner Product) ..... 19
1.7.2.1 Definition (Inner Productand Orthogonal Functions) ..... 20
CHAPTER 2: d-ORTGOHONAL POLYNOMIALS
2.1 Definition ..... 22
2.2 Definition ..... 22
2.2.1 Lemma ..... 22
CHAPTER 3: SOME EXAMPLES AND PROPERTIES OF D-ORTHOGONAL POLYNOMIALS
3.1 d-Orthogonalof Hermite Type Polynomials ..... 24
3.2 d-Orthogonal $\Delta \omega$ - Appell Polynomial: ..... 24
3.2.1 $\Delta \omega$ - Appell Polynomial Sets ..... 25
3.2.1.1 d-Orthogonal $\Delta \omega$ - Appell Polynomial: ..... 27
3.3 Hypergeometric d-Orthogonal Polynomial ..... 28
3.3.1 Some Propertiesofthe Polynomials $\ell n \alpha d$ ..... 29
3.4 Jensen d - Orthogonal Polynomial ..... 30
3.4.1 Laguerre-Pólya class $\square \mathcal{P}$ ..... 30
3.5 Continuous d - Orthogonal Polynomials ..... 33
3.5.1 d - OPS of Hermite - Type ..... 33
3.6 Discrete d-Orthogonal Polynomials ..... 35
3.6.1 d-OPS of Meixner type ..... 35
3.6.2 d-OPS of Charlier Type ..... 37
3.7 The Hyper-Bessel Function, The D-Orthogonal Laguerre Polynomials And The Relation Between Them ..... 40
CHAPTER 4: CONCLUSION
4.1 Conclusion ..... 42

REFERENCES................................................................................................ 43

## LIST OF FIGURES

Figure 1.1:The Graph of Hermite Polynomials ..... 9
Figure 1.2:The Graph of Laguerre Polynomials ..... 12
Figure 1.3:The Graph of I. Type Chebyshev Polynomials ..... 16
Figure 1.3:The Graph of Legendre Polynomials ..... 17

## LIST OF SYMBOLS

| $\Gamma(x)$ | Gamma Functions |
| :---: | :---: |
| $(x)_{n}$ | Pohammer Symbol |
| $H_{n} x$ | Hermite Polynomials of degree $n$ on $[-\infty, \infty$ ] |
| $L_{n}^{\alpha}{ }^{\text {a }}$ | Laguerre Polynomials of degree $n$ on $[0, \infty$ ] |
| $L_{n}{ }^{\alpha_{d}} x$ | d-Laguerre Polynomials of degree $n$ on $[0, \infty]$ |
| $L_{n}^{(\alpha)}(x)$ | $)^{2}$ Norm of Laguerre Polynomials of degree $n$ on $[0, \infty$ ] |
| $P_{n}{ }^{\alpha, \beta} \quad x$ Jacobi Polynomials of degree $n$ on the interval [-1,1] |  |
| $\frac{n}{2}$ | Integer $n$ |
| $\langle u, v\rangle$ | Inner Product between two vectors $u$ and v |
| $\delta_{n, m}$ | Kronecker symbol |
| $u$ | d-dimensional functional vector |
| $\Delta_{\omega}$ | Operator |
| ${ }_{p} F_{q}$ z | Generalized Hypergeometric Function |
| $I_{\alpha}$ | Bessel function of the first kind |
| ® $\mathcal{P}$ | Laguerre-Pólya class |
| $g_{n} \varphi, x$ | Jensen polynomial associated to $\varphi x$ |
| $D_{t}$ | D - operator |
| $\begin{array}{ll}:= & \text { Definition } \\ & \text { Double brackets (or white square brackets) }\end{array}$ |  |
|  |  |

$\mathrm{V}^{(f)}$ Operators acting on formal power series that increase the order of every formal power series by exactly one. $j \geq 0$
$\mathrm{V}^{-j}$
Operators acting on formal power series that decrease the order of every formal power series by exactly one. $j \geq 0$
$\mathcal{J}_{\alpha_{d}}$ z Hyper - Bessel's function

## CHAPTER 1

## INTRODUCTION AND BASIC DEFINITIONS

### 1.1 Introduction

This Chapter gives several basic definitions, theorems and some special cases about orthogonal polynomial theories.

In the field of applied mathematics and physics, orthogonal polynomials have played a very important role. Moreover, geometrically, orthogonal polynomials are the basis of vector spaces. Therefore meaning any member of this vector space can be expanding a series of orthogonal polynomials.

The subject had its origin in the investigation of P. L. Tchebyshev (beginning in 1874) and his pupil A. Markov on the "theory of limiting new ideas", among them was a general concept of orthogonal polynomials.

Almost four decade ago, Konhauser (1965-1967) found a pair of orthogonal polynomials which satisfied additional conditions, which are generalization of orthogonality conditions. Such polynomials are called biorthogonal polynomials. After Konhauser's study, several properties of these polynomials and other biorthogonal polynomial pairs were found.

In this work, general and basic properties of orthogonal polynomials are given, then dorthogonal polynomials. Later then types of d-Orthogonal polynomials namely the Hermite, Laguerre polynomials were investigated.

In the first chapter, several basic definitions, theorems and some special cases about orthogonal polynomial theories are given.

In the second chapter, definition and some theorems of d-Orthogonal polynomials were obtained.

In the third chapter, some examples and special cases of d-Orthogonal polynomials are given with several properties of the d-Orthogonal family.

In the fourth chapter, we see the definition of the Hyper Bessel function and the Laguerre Polynomial and the similarities between them.

In the fifth chapter we give conclusion.

### 1.2 Gamma Function

Many important functions in applied sciences are defined via improper integrals. One of the most famous amongst them is the Gamma functions.

It has several applications in Mathematics and Mathematical Physics.

### 1.2.1 Definition (Rainville, 1965)

The improper integral

$$
\begin{equation*}
{ }_{0}^{\infty} t^{x-1} e^{-t} d t \tag{1.1}
\end{equation*}
$$

converges for any $x>0$ is called "Gamma function" and is denoted by $\Gamma$ :

$$
\begin{equation*}
\Gamma \mathrm{x}={ }_{0}^{\infty} t^{x-1} e^{-t} d t \tag{1.2}
\end{equation*}
$$

Some basic properties of Gamma function are given without their proofs (Rainville, 1965)

$$
\begin{equation*}
{ }_{0}^{\infty} t^{x-1} e^{-t} d t=n!=\Gamma(n+) \tag{1.3}
\end{equation*}
$$

where n is a positive integer.

$$
\begin{equation*}
\mathrm{n} \Gamma \mathrm{n}=\Gamma(\mathrm{n}+1) \tag{1.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma 2 b \sqrt{\pi}=2^{1-2 b} \Gamma(b) \Gamma\left(b+\frac{1}{2}\right) \tag{1.5}
\end{equation*}
$$

where $\operatorname{Re}(b) \quad 0$

$$
\begin{equation*}
\Gamma 2 \mathrm{~b}+\mathrm{n} \sqrt{\pi} 2^{1-2 b-n}=\Gamma\left(\mathrm{b}+\frac{1}{2} \mathrm{n}\right) \Gamma\left(\mathrm{b}+\frac{1}{2}, n+\frac{1}{2}\right) \tag{1.6}
\end{equation*}
$$

where $\operatorname{Re}(\mathrm{b}) \quad 0$ and n is non - negative

$$
\begin{equation*}
\Gamma \mathrm{a}=(a)^{n} \frac{n-1!}{(a)_{n}} \tag{1.7}
\end{equation*}
$$

### 1.2.2 Definition (Askey, 1999)

Let x be a real or complex number and n be a positive integer,

$$
\begin{equation*}
(x)_{n}=\frac{\mathrm{r}(x+\mathrm{n})}{\Gamma(x)}=x x+1 \ldots(x+n-1) \tag{1.8}
\end{equation*}
$$

is known as "Pochhammer Symbol" where $x_{n}$ is used to represent the falling factorial (sometimes called the "descending factorial", "falling sequential product", "lower factorial")

There are some properties of Pochhammer symbol below:
$1 \quad c+n_{k}=\frac{(c)_{n+k}}{(c)_{n}}$,
where c is a real or complex number and $\mathrm{n}, \mathrm{k}$ are natural numbers
$2 \frac{n!}{n-k!}=\frac{(-n)_{k}}{(-1)^{k}}$,
where n and k are natural numbers
$3 \quad \frac{c_{2 k}}{2^{2 k}}=\frac{c}{2}{ }_{k} \frac{c}{2}+\frac{1}{2}{ }_{k}$,
where c is a complex number and k is a natural number
$4 \quad \frac{(2 k)!}{2^{2 k} k!}=\frac{1}{2}_{k}$
where k is a natural number.
Sequel to this, there is a useful Lemma for Pochhammer symbol. The proof can be obtained as follows:

### 1.2.3 Lemma

$$
\begin{equation*}
(\alpha)_{2 n}=2^{2 n} \frac{\alpha}{2} n^{\frac{\alpha+1}{2}} n \tag{1.9}
\end{equation*}
$$

## Proof:

$$
\begin{gathered}
\alpha 2 n=\alpha \alpha+1 \quad \alpha+2 \ldots \alpha+2 n-1 \\
=2^{2 n} \frac{\alpha}{2} \frac{\alpha+1}{2} \quad \frac{\alpha}{2}+1 \ldots \frac{\alpha}{2}+n-1 \frac{\alpha+1}{2}+n-1 \\
=2^{2 n} \frac{\alpha}{2} \frac{\alpha+1}{2} \quad \frac{\alpha}{2}+1 \ldots \frac{\alpha}{2}+n-1 \frac{\alpha+1}{2} \quad \frac{\alpha+1}{2}+1 \ldots \frac{\alpha+1}{2}+n-1 \\
=2^{2 n} \frac{\alpha}{2} \frac{\alpha+1}{2}
\end{gathered}
$$

### 1.3 Orthogonal Polynomials

In this section we are going to talk on polynomials, definition and some main properties of orthogonal polynomials which are special case of the biorthogonal polynomials (Askey, 1999)

### 1.3.1 Definition

A function p: $\rightarrow$ is a polynomial if $P x=a_{n} x^{n}+a_{n-1} x^{n-1}+a_{n-2} x^{n-2}+\cdots+$ $a_{1} x+a_{0}$, where n is a non - negative integer and the numbers $a_{0}, a_{1}, a_{2}, \ldots, a_{n}$ are real constants (called the coefficients of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the "leading coefficient" $a_{n} \neq 0$ and $n \quad 0$, then $n$ is called "degree" of the polynomial. (The degree of the zero polynomial is not defined).

If the variable $x$ and the coefficients $a_{n}(n=0,1,2,3, \ldots)$ are real, then $\mathrm{P}(\mathrm{x})$ is called a real polynomial, if the variable $x$ and the coefficients $a_{n}(n=0,1,2,3, \ldots)$ are complex, then $\mathrm{P}(\mathrm{x})$ is called a complex polynomial.

### 1.3.2 Definition (Orthogonal Polynomials)

Let $\mathscr{P}$ be the linear space of polynomials with complex coefficients and let $\mathscr{P}$ be its algebraic dual. A polynomial sequence $P_{n} n \geq 0$ in $\mathscr{P}_{\text {is called polynomial set if and only if } \operatorname{deg}\left(P_{n}\right)=n}^{n}$ for all non - negative integer $n$.

### 1.3.2.1 Properties of Polynomials:

a) A constant multiplying a polynomial is also a polynomial.
b) The sum, difference and product of two or more polynomials is also a polynomial.
c) The composition of two polynomials is also a polynomial.
d) If the number r is a root of a polynomial of degree $n$ with multiplicity $m$, then there is polynomial $q$ with degree $(n-m)$ such that for all $x \in$, $P x=x-r^{m} q(x)$
e) A polynomial of degree $n$ and with real coefficients $a_{n}$, has at most $n$ real roots (Zeros).

### 1.4 Orthogonality of Functions:

If $A . B=0$, then the vectors $A$ and $B$ are called "Orthogonal". If the vectors $A$ and $B$ are given by the form $A=a_{1} \vec{\imath}+a_{2} \vec{l}+a_{3} k, \bar{B}=b_{1} \vec{\imath}+b_{2} \vec{\jmath}+b_{3} k$, then the orthogonality of $A$ and $\bar{B}$ implies
that $A \cdot \bar{B}=\sum_{i=1}^{3} a_{i} \cdot b_{i}=0$.
Any function $A(x)$ can be thought like a vector. Let the values of functions $A(x)$ and $B(x)$ at the points $x_{1}, x_{2}$ and $x_{3}$ be important. If $A x_{1}=a_{1}, A x_{2}=a_{2}, A x_{3}=a_{3}, B x_{1}=b_{1}$, $B x_{2}=b_{2}, B x_{3}=b_{3}$ and $\sum_{i=1}^{3} a_{i} \cdot b_{i}=0$, then the function $A(x)$ is called Orthogonal to function $B(x)$.

For a function which is defined on any interval $a, b=I \subset$, all points of the interval $I$ are important. Let $x_{i} \in l$.

The function $A(x)$ can be thought a vector with infinite dimension and with the coefficients $A\left(x_{i}\right)$. If the points of $x_{i}$ take all values of points from the interval $I$, when $i$ changes, and if $\sum_{i} A x_{i} \cdot B x_{i}=0$ which means ${ }_{a}^{b} A x B x d x=0$ is satisfied, then the function $A x$ is called Orthogonal to the function $B(x)$ on the interval $a, b$.

The (.), which means the scalar product of vectors, is changed by the integral. Both of them are special types of inner product.

### 1.4.1 Theorem (Askey, 1999)

It is sufficient for the orthogonality of the polynomials on the interval [a,b] with respect to the weight function $\omega(\mathrm{x})$ to satisfy the condition:

$$
{ }_{a}^{b} \omega x \phi_{n} x x^{i} d x=0, i=0,1,2, \ldots, n-1
$$

Here, $\phi_{n}(x)$ is a polynomial of degree n .

## Proof:

If the polynomial $\phi_{n}(x)$ and $\phi_{m}(x)$ are orthogonal on the interval $[\mathrm{a}, \mathrm{b}]$ with respect to $\omega x$, then

$$
{ }_{a}^{b} \omega x \phi_{i} x \phi_{j} x d x=0, \quad m \neq n
$$

$x^{i}$ can be written a linear combinations,


Substituting $x^{i}$ into $\quad{ }_{a}^{b} \omega x \phi_{n} x x^{i} d x=0, i=0,1,2, \ldots, n-1$, we have:

$$
{ }_{a}^{b} \omega x \phi_{n} x x^{i} d x={ }_{a}^{b} \omega x \phi_{n} x{ }_{\overline{m=0}}^{i} a_{m} \phi_{m} x d x
$$

$$
={ }_{\bar{m}=0}^{i} a_{m}{ }_{a}^{b} \omega x \phi_{n} x \phi_{m} x d x=0
$$

for $0 \leq m \leq i, \phi_{n} x$ and $\phi_{m} x$ w马ere $0 \leq m \leq n$. Hence,

$$
{ }_{a}^{b} \omega x \phi_{n} x x^{i} d x=0, i=0,1,2, \ldots, n-1
$$

### 1.5 Properties of Orthogonal Polynomials

Orthogonal polynomials have several important properties. In this section, we are going to be looking at the general definitions of these properties and obtain special forms of them from well-known orthogonal polynomial families.

### 1.5.1 Definition Recurrence Relation for Orthogonal Polynomial

In Mathematics, a recurrence relation is an equation that recursively defines a sequence or multidimensional array of values once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding term.

Any polynomial family $\phi_{n} x$, which is orthogonal on the closed interval [a, b] with respect to the weight function $\omega(\mathrm{x})$, satisfies the recurrence formula:
$\phi_{N+1} x-x A_{N}+B_{N} \phi_{N} x+C_{N} \phi_{N-1} x=0$
here $A_{N} B_{N}$ and $C_{N}$ are constants which depend on N

### 1.5.2 Definition (Askey, 1999)

The Rodrigues formula for orthogonal polynomials are written as:
$\phi_{n} x=A_{n} \frac{1}{\omega(x)} \frac{d^{n}}{d x^{n}} \omega x u^{n} x, \quad n=0,1,2, \ldots$.
Here, $A_{n}$ is a constant, $\phi_{n} x$ polynomials are with respect to the weight function $\omega(x)$ and $u^{n} x$ is a function which vanishes at a and b and is equivalent to the coefficient of $y^{\prime \prime}$ in the second order linear differential equation of orthogonal polynomial family.

### 1.5.3 Definition (Askey, 1999)

If the two variable function $\mathrm{F}(\mathrm{x}, \mathrm{t})$ has a Taylor series as in the form :

$$
F x, t={ }_{\overline{n=0}}^{t} a_{n} \phi_{n} x t^{n}
$$

With respect to one of its variables t , then the function $\mathrm{F}(\mathrm{x}, \mathrm{t})$ is called the generating function for the polynomials $\phi_{n}(x)$.

### 1.6 Some Special Orthogonal Polynomial Families

Some well-known orthogonal families having several applications in Applied Mathematics and Physics are given in this section. These polynomial families have several properties which are common and obtainable for any orthogonal polynomial for any polynomial family.

### 1.6.1 Hermite Polynomials (Askey, 1999)

The Hermite polynomial denoted by $H_{n}(x)$, which are orthogonal on the interval $-\infty<x<$ $\infty$ with respect to the weight function $\omega x=e^{-x^{2}}$ is given by:

$$
\phi_{n} x=H_{n} x={ }_{\overline{k=0}}^{n / 2} \frac{-1^{k} n!}{k!n-2 k!} 2 x^{n-2 k} 1.13
$$

$\mathrm{n}=0,1,2, \ldots$
some of the polynomials $H_{n} x$ are;
$H_{0} x=1$
$H_{1} x=2 x$
$H_{2} x=4 x^{2}-2$
$H_{3} x=8 x^{3}-12 x$
$H_{4} x=16 x^{4}-48 x^{2}+12$
$H_{5} x=32 x^{5}-160 x^{3}+120 x$
The graphs of the first six Hermite polynomials $H_{0} x, H_{1} x, H_{2} x, H_{3} x, H_{4} x$ and $H_{5} x$ are shown in the figure below:


Figure 1.1:The Graph of Hermite Polynomials

### 1.6.1.1 Rodrigues Formula for Hermite Polynomials

$$
\begin{equation*}
H_{n}(x)=(-1)^{n} e^{x^{2} / 2} \frac{d^{n}}{d x^{2}}\left(e^{-x^{2}}\right)( \tag{1.14}
\end{equation*}
$$

1.6.1.2 The Generating Function for the Hermite Polynomials

$$
\begin{equation*}
e^{2 t x-t^{2}}=\sum_{\frac{1}{n=0}}^{\infty} \frac{H_{n}(x)}{n!} t^{n}(1 \tag{1.15}
\end{equation*}
$$

### 1.6.1.3 The Norm of the Hermite Polynomials

$$
\begin{equation*}
\left\|H_{n}(x)\right\|^{2}=\left.\right|_{-\infty} ^{\infty} e^{-x^{2}} H_{n}(x) d x=2^{n} \sqrt{\pi} n! \tag{1.16}
\end{equation*}
$$

From the equation

$$
\frac{d}{d x} e^{x^{2}} \frac{d}{d x} H_{n} x
$$

The Hermite differential equation can be obtained as :

$$
\begin{equation*}
y^{\prime \prime}-2 x y^{\prime}+2 n y=0 \tag{1.17}
\end{equation*}
$$

which has the solution as Hermite Polynomials.

Finally, the recurrence relation for the Hermite polynomial is given as

$$
\begin{equation*}
H_{n+1} x-2 x H_{n} x+2 n H_{n-1} x=0 \tag{1.18}
\end{equation*}
$$

By using the generating function (1.15), we can obtain the recurrence relation above by following steps.

Taking the derivative of both sides in (1.15) with respect to .

$$
\begin{array}{r}
(2 x-2 t) e^{2 t x-t^{2}}={ }_{\overline{n=0}}^{\infty} \frac{H_{n} x}{n!} n t^{n-1} \\
(2 x-2 t)_{\overline{n=0}}^{\infty} \frac{H_{n}(x)}{n!}={ }_{\overline{n=0}}^{\infty} \frac{H_{n} x}{(n-1)!} t^{n-1} \\
\infty \quad \frac{H_{n}(x)}{n!} t^{n}-\underset{\substack{\overline{n=0} \\
n \rightarrow n-1}}{\infty} \frac{2 H_{n}(x)}{n!} t^{n+1}={ }_{\overline{n=0}}^{\infty} \frac{H_{n} x}{n!} n t^{n-1}
\end{array}
$$

If the indices are manipulated to make all the powers of $t$ as $t^{n}$,

$$
\frac{2 x H_{n}(x)}{n!} t^{n=0}-{ }_{\overline{n=0}}^{\infty} \frac{2 H_{n-1}(x)}{n-1!} t^{n}={ }_{\overline{n=0}}^{\infty} \frac{H_{n+1}(x)}{n+1} t^{n}
$$

and open some terms to start the summations from 1,

$$
2 x H_{0} x-\overline{\bar{n}=1}_{\infty}^{\infty} 2 x H_{n} x-2 n H_{n-1} x \frac{t^{n}}{n!}=H_{0} x+{ }_{\overline{n=1}}^{\infty} H_{n+1} x \frac{t^{n}}{n!}
$$

is obtained. By the equality of the coefficients of the term $\frac{t^{n}}{n!}$,
$2 x H_{n} x-2 n H_{n-1} x=H_{n+1} x$,
can be written, which gives the recurrence relation (1.18)

### 1.6.2 Laguerre Polynomials (Rainville, 1965)

The Laguerre polynomial denoted by $L_{n}^{\alpha}(x)$ is orthogonal on the closed interval $0 \leq x \leq \infty$ for $\alpha>-1$ with respect to the weight function $\omega x=x^{\alpha} e^{-x}$ given by:

$$
\phi_{n} x=L_{n}^{\alpha} x=\sum_{\overline{k=0}}^{n}(-1)^{k} \begin{aligned}
& n+\alpha \\
& n-k
\end{aligned} \frac{x^{k}}{k!}, \quad n=0,1,2, \ldots
$$

The special case $\alpha=0$ is $L_{n}^{\alpha} \quad x=L_{n} \quad x$. Let us give the first five Laguerre polynomials,

$$
\begin{gathered}
L_{0} x=1 \\
L_{1} x=-x+1 \\
L_{2} x=\frac{1}{2} x^{2}-4 x+2 \\
L_{3} x=\frac{1}{6}-x^{3}-16 x^{2}-18 x+6 \\
L_{4} x=\frac{1}{24} x^{4}-16 x^{3}+72 x^{2}-96 x+24 \\
L_{5} x=\frac{1}{120}-x^{5}+25 x^{4}-200 x^{3}+600 x^{2}-600 x+120
\end{gathered}
$$

The graphs of first six Laguerre polynomials $L_{0} x, L_{1} x, L_{2} x, L_{3} x, L_{4} x$ and $L_{5}(x)$ are shown in the figure below:


Figure 1.2:The Graph of Laguerre Polynomials

Several properties of Laguerre polynomials similar to orthogonal polynomials can be obtained. One of these properties is that it satisfies second order differential equations.

Starting from $\frac{d}{d x}\left[x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n}(x)\right]$, we obtain Laguerre differential equation $x y^{\prime \prime}+(\alpha+1-x) y^{\prime}+n y=0$, where the solution of this is the differential equation are Laguerre polynomials can be obtained.

Let's start with the equation below:
$\frac{d}{d x}\left|x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n}(x)\right|=x^{\alpha} e^{-x}\left|x \frac{d^{2}}{d x^{2}} L_{n}(x)+(\alpha+1-x) \frac{d}{d x} L_{n}(x)\right|$
It can be written as linear combinations,

$$
x \frac{d^{2}}{d x^{2}} L_{n}(x)+(\alpha+1-x) \frac{d}{d x} L_{n}(x)=\sum_{\frac{i=1}{n}} \alpha_{i} L_{i}(x)
$$

Therefore,

$$
\frac{d}{d x} x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n}(x)=x^{\alpha} e^{-x} \alpha_{i=1}^{n} \alpha_{i} x
$$

by integrating over the interval $(0, \infty)$, it is deduced that

$$
\begin{aligned}
& { }_{0}^{\infty} L_{j} x \frac{d}{d x} x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n} x \quad d x=\int_{0}^{\infty} L_{j} x x^{\alpha} e^{-x}{ }_{\overline{i=1}}^{n} \alpha_{i} L_{i} x \\
& \quad=\alpha_{j} \int_{0}^{\infty} x^{\alpha} e^{-x} L_{j}^{2}(x) d x+\prod_{\substack{i=1 \\
j=1 \\
i \neq j}}^{n} \alpha_{0}^{\infty} e^{-x} x^{\alpha} L_{j} x L_{i} x d x
\end{aligned}
$$

It is known that the Laguerre polynomial are orthogonal, then

$$
e_{0}^{\infty} e^{-x} x^{\alpha} L_{j} x L_{i} x d x=0, \quad i \neq j
$$

Consequently,

$$
\begin{gathered}
{ }_{0}^{\infty} L_{j} x \frac{d}{d x} x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n} x \quad d x=\alpha_{j}{ }_{0}^{\infty} x^{\alpha} e^{-x} L_{j}^{2} x d x=0 \\
\alpha_{j}=\frac{{ }_{0}^{\infty} L_{j}(x) \frac{d}{d x}\left[x^{\alpha+1} e^{-x} \frac{d}{d x} L_{n}(x)\right] d x}{0_{0}^{\infty} x^{\alpha} e^{-x} L_{j}^{2} x d x}
\end{gathered}
$$

and $\frac{d}{d x} L_{n} x=y^{\prime}, \frac{d^{2}}{d x^{2}} L_{n} x=y^{\prime \prime}$, so

$$
\begin{gathered}
x y^{\prime \prime}+\alpha+1-x y^{\prime}+\lambda y=0 \\
\lambda_{n}=-n \frac{n-1}{2} x^{\prime \prime}+\alpha+1-x^{\prime}=n
\end{gathered}
$$

Then the following differential equation is obtained
$x y^{\prime \prime}+\alpha+1-x y^{\prime}+n y=0$
The generating function for the Laguerre polynomials

$$
\begin{equation*}
\frac{L_{n=0}^{\alpha}}{\infty} x t^{n}=\frac{1}{1-t} e^{\frac{-x t}{1-t}} \tag{1.19}
\end{equation*}
$$

can be written. For obtaining the norm $L_{n}^{(\alpha)}(x)$ norm of Laguerre polynomials, the generating function (1.19) is rewritten as the form of

$$
\begin{equation*}
\frac{\infty}{m=0} e^{-x} L_{m}^{\alpha} x t^{m}=e^{-x} \frac{1}{(1-t)} e^{\frac{-x t}{1-t}} \tag{1.20}
\end{equation*}
$$

by multiplying both sides of (1.19) by $\omega x=e^{-x}$ where $m \neq n$. If (1.19) and (1.20) are multiplied side by side and integrated over the interval ( $0, \infty$ )

$$
{ }_{n, \overline{m=0}}^{\infty} \quad 0^{\infty} e^{-x} L_{n}^{\alpha}(x) L_{m}^{(\alpha)}(x) d x t^{n+m}=\frac{1}{(1-t)^{2}} e^{\infty} e^{\frac{x(1+t)}{t-1}}
$$

is obtained. If the left hand side of the last equation is separated for $\mathrm{m}=\mathrm{n}$ and $\mathrm{m} \neq \mathrm{n}$, and take the integral at the right hand side,

$$
\sum_{n, m=0}^{\infty} e^{-x} L_{n}^{2}(x) d x t^{2 n}+{ }_{n, m=0}^{\infty} \int_{0}^{\infty} e^{-x} L_{n}^{\alpha}(x) L_{m}^{\alpha}(x) d x t^{n+m}=\frac{1}{(1-t)^{2}} \frac{1-t}{1+t}=\frac{1}{1-t^{2}} .
$$

is obtained. By using the orthogonality of Laguerre polynomials, for $n=m$, second integral at the left hand side is equal to zero.

If the Taylor series
$\frac{1}{(1-t)}={ }_{\overline{n=0}}^{\infty} t^{2 n}$
is used on the right hand side of the last equality, then

$$
{ }_{n, \overline{m=0}}^{\infty} \quad 0 \quad e^{-x} L_{n}^{2}(x) d x t^{2 n}={ }_{\overline{n=0}}^{\infty} t^{2 n},
$$

is obtained. Thus, equality of the coefficient of $t^{2 n}$ in both sides give the norm of Laguerre polynomials as

$$
L_{n}^{(\alpha)}(x)^{2}=\quad e^{-x} L_{n}^{2} x d x=1
$$

Finally, the recurrence relation for Laguerre polynomial $L_{n}^{\alpha}(x)$ is given as,

$$
n+1 L_{n+1}^{\alpha} x+x-2 n-1-\alpha L_{n}^{\alpha} x+n+\alpha L_{n-1}^{\alpha} x=0
$$

### 1.6.3 Jacobi Polynomials (Askey, 1999)

The Jacobi polynomial denoted by $P_{n}^{\alpha, \beta}(x)$ is orthogonal for $\alpha>-1, \beta>-1$, on the interval $[-1,1]$ with respect to the weight function $\omega x=(1-x)^{\alpha}(1+x)^{\beta}$, by the formula $P_{n}^{\alpha, \beta} x={\frac{1}{2^{n}}}_{\overline{k=0}}^{n} \begin{array}{cc}n+\alpha & n+\beta \\ k-k\end{array} x^{n+1^{k} x-1^{n-k}, \quad n=0,1,2, \ldots}$

If $\alpha=\beta$, the polynomials $P_{n}^{\alpha, \beta}(x)$, are called "Ultraspherical Polynomials".
Some special case of Jacobi polynomials which depends on the value of $\alpha$ and $\beta$ are given below:

1 For $\alpha=\beta=-\frac{1}{2}$, the polynomials
are called "I. Type Chenyshev Polynomials".
Some of the polynomials $P_{n}(x)$ are

$$
\begin{gathered}
P_{0} x=1 \\
P_{1} x=x \\
P_{2} x=2 x^{2}-1 \\
P_{3} x=4 x^{3}-3 x \\
P_{4} x=8 x^{4}-8 x^{2}+1 \\
P_{5} x=16 x^{5}-20 x^{3}+5 x
\end{gathered}
$$

The graphs of this first six I. Type Chebyshev Polynomials $P_{0} x, P_{1} x, P_{2} x, P_{3} x, P_{4}(x)$ and $P_{5}(x)$ are shown in the figure below:


Figure 1.3:The Graph of I. Type Chebyshev Polynomials
$2 \quad$ For $\alpha=\beta=0$

$$
P_{n}^{(0,0)}(x)=2^{-n} \frac{[n / 2]}{\frac{1}{k=0}}(-1)^{k}\binom{n}{k}\binom{2 n-2 k}{n} x^{n-2 k}=P_{n}(x),
$$

are called "Legendre Polynomials". Let us give the first five Legendre polynomials;

$$
\begin{gathered}
P_{0}(x)=1 \\
P_{1}(x)=x \\
P_{2}(x)=\frac{1}{2}\left(3 x^{2}-1\right) \\
P_{3}(x)=\frac{1}{2}\left(3 x^{2}-3 x\right) \\
P_{4}(x)=\frac{1}{8}\left(35 x^{4}-30 x^{2}+3\right) \\
P_{5}(x)=\frac{1}{8}\left(63 x^{5}-70 x^{3}+15 x\right)
\end{gathered}
$$

The graphs of the first six Legendre polynomials $P_{0}(x), P_{1}(x), P_{2}(x), P_{3}(x), P_{4}(x)$ and $\left[P_{5}(x)\right.$ are shown below:


Figure 1.31:The Graph of Legendre Polynomials

Here:

### 1.6.3.1 Differential Equations for Jacobi Polynomials

If $\frac{d}{d x}\left(1-x^{2}\right) 1-x^{\alpha}(1+x)^{\beta} \frac{d}{d x} P_{n}^{\alpha, \beta} \quad x \quad$ is used to start, the Jacobi differential equation can be obtained as:

$$
1-x^{2} y^{\prime \prime}+\beta-\alpha-\alpha+\beta+2 y^{\prime}+n n+\beta+\alpha+1 y=0, \square
$$

which has the solutions as a Jacobi polynomial.

### 1.6.3.2 Generating Function

The generating function for the Jacobi Polynomial is given as :

$$
P_{\bar{n}=0}^{\infty} P_{n}^{\alpha, \beta} x t^{n}=\frac{2^{\alpha+\beta}}{\sqrt{1-2} t x \overline{+} t^{\overline{2}} 1-t+\sqrt{1-2} x \overline{t+} t^{\overline{2}}} 1+t+\sqrt{1-2} x \overline{+} t^{\overline{2}} \quad .
$$

Finally, the recurrence relation for the Jacobi Polynomials are given as:
$2 n+1 \quad n+\alpha+\beta-1 \quad 2 n+\beta+\alpha P_{n+1}^{\alpha, \beta} \quad x-\left[(2 n+\alpha+\beta+1)\left(\alpha^{2}-\beta^{2}\right)(2 n+\alpha \beta\right.$ $+$
$\beta) x] P_{n}^{\alpha, \beta} x+2 n+\alpha \quad \alpha+\beta \quad 2 n+\alpha+\beta+2 P_{n-1}^{\alpha, \beta} \quad x=0$.

### 1.7 Vector Space

### 1.7.1 Definition

A vector space (over [0) consists a set $V$ by which two operations " + " and " $\bullet$ " are defined, called vectors addition and scalar multiplication respectively.

The operation " + " which is known as vector addition must satisfy the following conditions:
Closure: if $u$ and $v$ are any vectors in $V$, then the sum of $u+v$ belongs to $V$

1) Commutative law: For all vectors $u$ and $v$ in $V, u+v=v+u$
2) Associative law: For all vectors $u, v$ and $\operatorname{win} V, u+v+w=u+v+w$
3) Additive identity: The set $V$ contains an additive identity element, denoted by 0 , such that for any vector $v$ in $V, 0+v=v$ and $v+0=v$.
4) Additive inverse: For each vector $v$ in $V$, the equation $v+x=0$ and $x+v=0$ have a solution $x$ in $V$, called an additive inverse of $v$ and is denoted by $-v$

The operation "•" which is known as scalar multiplication is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

Closure: If $v$ is any vector in $V$, and c is any real number, then the product $c \cdot v$ belongs to $V$.
5) Distributive law: For all real numbers $c$ and all vectors $u, v$ in $V, c \bullet u+v=c \bullet u+c \bullet$ $v$
6) Distributive law: For all real numbers $c, d$ and all vectors $v i n V, c+d \bullet v=c \bullet v+$ $d \cdot v$
7) Associative law: For all real numbers[c,d and all vectors $v$ in $V, c \cdot d \bullet v=c d \bullet v$
8) Unitary law: For all vectors $v$ in $V, 1 \cdot v=v$

### 1.7.2 Definition (Inner Product)

Let $V$ be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is a real number assigned, denoted by $[u, v \square$ The function is called a (real) inner product on $V$ if it satisfies the following axioms:
$A_{1}$ : (Linear property) $\left[a u_{1}+b u_{2}, v \square=a\left[u_{1}, v \square+b\left\lceil u_{2}, v \square\right.\right.\right.$
$A_{2}:$ (Symmetric Property) $[u, v \square+[v, u \square$
$A_{3}:$ (Positive Definite Property) $[u, v \square \geq 0$; and $[u, v \square=0$ if and only if $u=0$.

The vector space $V$ with an inner product is called a (real) inner product space.
$A_{1}$ is equivalent to two conditions as follows:

$$
i\left[u_{1}+u_{2}, v \square=\left[u_{1}, v \square+\left[u_{2}, v\lceil\text { and } i i \quad[k u, v \square=k\lceil u, v \square\right.\right.\right.
$$

Using $A_{1}$ and $A_{2}$ (Symmetric axiom), we have

$$
\left[u, c v_{1}+d v_{2} \square=\left[c v_{1}+d v_{2}, u \square=c\left[v_{1}, u \square+d\left[v_{2}, u \square=c\left[u, v_{1} \square+d\left[u, v_{2} \square\right.\right.\right.\right.\right.\right.
$$

Then equivalently, we obtain from the two conditions as follows:

$$
i\left[u, v_{1}+v_{2} \square=\left[u, v_{1} \square+\left[u, v_{2} \text { and } i i[u, k v \square=k[u, v \square\right.\right.\right.
$$

Which means that the inner product function is also linear in its second position (variable).
Using induction, we obtain:

$$
\left[a_{1} u_{1}+a_{2} u_{2}+\cdots+a_{n} u_{n}, v \square=a_{1}\left[u_{1}, v \square+a_{2}\left[u_{2}, v \square+\cdots+a_{n}\left[u_{n}, v \square\right] \frac{1}{}\right.\right.\right.
$$

and

Combining the (1.22) and (1.23), yields the general formula


### 1.7.2.1 Definition (Inner Productand Orthogonal Functions)

The inner product of two functions $f x, g(x)$ on the interval $a, b$ is a number denoted by $\left[f, g \square\right.$ given by $\left[f, g \square={ }_{a}^{b} f \times g x d x\right.$

If the inner product is zero, we say $f$ and $g$ are orthogonal.

## CHAPTER 2

## 2.0 d-Orthogonal Polynomials

In this chapter, we are going to see definition and some theorems of d-Orthogonal polynomials obtained.

During the past two decades, there has been increase interest in some extension of the concept of standard orthogonality. One of them is the so - called multiple orthogonality. This notation has many applications in various domains of mathematics as analytic number theory, approximation theory, special functions theory and spectral theory of operators. A convenient frame work to discuss explicit examples of multiple orthogonal polynomials known as dorthogonal polynomials. This notion was introduced as follows:

Let $\mathscr{P}$ be a vector space of polynomials with coefficients in 0 and $\mathscr{P}$ ' be its algebraic dual. We denote by $[u, f \square$ the effect of the linear functional $u \in \mathscr{P}$ ' on the polynomial $f \in \mathscr{P}$. Let $P_{n} n \geq 0$ be a sequence of polynomial set. The corresponding monic polynomial sequence $\hat{P}_{n}{ }_{n \geq 0}$ is given by $P_{n}=\lambda_{n} \hat{P}_{n}$,
$n \geq 0$, where $\lambda_{n}$ is the normalization coefficient and its dual sequence $u_{n} n \geq 0$ is defined by:
$\left[u_{n}, \widehat{P}_{m} \square=\delta_{n, m} \square \mathrm{~d}\|\mathrm{~d}\| n, m \geq 0\right.$
$\delta_{n, m}$ being the Kronecker symbol.

## Note:

The Kronecker symbol in Mathematics also known as Kronecker's delta named after Leopold Kronecker, is a function of two variables, positive integers. The function is 1 when the variables are equal and 0 when otherwise:

$$
\delta_{n, m}=\begin{aligned}
& 0 \llbracket i f[m \neq n \\
& 1 \llbracket i f[m=n
\end{aligned}
$$

where the Kronecker delta $\delta_{n, m}$ is a piece wise function of variables $i$ and $j$. For example $\delta_{3,4}=0$ where as $\delta_{2,2}=1$.

### 2.1 Definition

Let ' $d$ ' be an arbitrary positive integer. The polynomial sequence $P_{n}{ }_{n \geq 0}$ is called a dorthogonal polynomial sequence (d-OPS, for shorter) with respect to the d-dimensional functional vector

$$
\begin{aligned}
& u={ }^{t}\left(u_{0}, \ldots, u_{d-1}\right) \\
& \text { if }
\end{aligned}
$$

for each integer k belonging to $0,1,2, \ldots, d-1$

### 2.2 Definition

Let $\omega_{1}, \omega_{2}, \omega_{3}, \ldots, \omega_{d}$ be ' $d$ ' linear functionals $\square d \geq 1$ ). The polynomial sequence $P_{n} n \geq 0$ is called a d-orthogonal sequence (d-OPS) with respect to $\Omega={ }^{t}\left(\omega_{1}, \omega_{2}, \omega_{3}, \ldots, \omega_{d}\right)$ if it fulfills:
for each integer $\alpha$ with $\alpha=1,2, \ldots, d$.

### 2.2.1 Lemma

For any linear function $u$ and $p \square \geq 1$ integer, the following two statements are equivalent:
a. $\quad\left[u, P_{p-1} \square \neq 0 ; \llbracket u, P_{n} \square=0 \square \| n \geq p\right.$
b. $\quad \exists\left[\lambda_{v} \in 0,0 \leq v \leq p-1,\left[\lambda_{p-1} \neq 0\right.\right.$ such that $\left[u=\sum_{v=0}^{p-1} u_{v} \lambda_{v}\right.$.

The notation of the d-orthogonality for polynomials, define and studied, appears as a particular case of the general notation of biorthogonality.

## Remark

1 When $d=1$, we meet again the ordinary regular orthogonality. In this case, $P_{n} n \geq 0$ is an orthogonal polynomial sequence (OPS).

2 The inequalities (2.2) are the regularity conditions (for equivalent conditions of regularity). In this case, the d-dimensional function $\Omega$ is called regular. It is not unique. Indeed, according to Lemma 2.1, we have:

$$
\begin{gathered}
{\left[\omega_{\alpha}={ }_{\overline{v=0}}^{\alpha-1} \lambda_{v}^{\alpha} u_{v}, \lambda_{\alpha-1}^{\alpha} \neq 0, \quad 1 \leq \alpha \leq d\right.} \\
\Leftrightarrow u_{v}={ }_{\overline{\alpha=1}}^{v+1} \tau_{\alpha}^{v} \omega_{\alpha} \square \tau_{v+1}^{v} \neq 0 \mathbb{W} 0 \leq v \leq d-1
\end{gathered}
$$

## CHAPTER 3

## SOME EXAMPLES AND PROPERTIES OF D-ORTHOGONAL POLYNOMIALS

## 3.1 d-Orthogonalof Hermite Type Polynomials

In this third chapter, some examples and special cases of d-Orthogonal polynomials will be given with several properties of the d-Orthogonal family.

Al-Salam (1990) did a wide research on the subject (Hermite polynomials). The Hermite polynomials:

since $\frac{\gamma_{n, 2}(k+1)}{\gamma_{n, 2}(k)}=\frac{-1}{4} \frac{(-n+2 k)_{2}}{k+1}$ is a rational function in $k$, we consider a polynomial set $P_{n} n \geq 0$ of the form

$$
P_{n} x=\gamma_{\overline{k=0}}^{\frac{n}{m}} \gamma_{n, m} k \mathfrak{B}_{n-m k} x
$$

where $m \geq 2, \mathfrak{B}_{k k \geq 0} \quad$ is the basis given by $\mathfrak{B}_{0} x=1$ and

$$
\mathfrak{B}_{k}(x)={ }_{\bar{r}=\bar{o}}^{k-1}(x+\pi(r)), k=1,2, \ldots
$$

## 3.2 d-Orthogonal $\Delta_{\omega}$ - Appell Polynomial

Before this d-orthogonal polynomial of $\Delta_{\omega}$, we need to know what is the definition of $\Delta_{\omega^{-}}$ Appell Polynomial.

### 3.2.1 $\Delta_{\omega}$ - Appell Polynomial Sets

As an example of the $\Delta_{\omega}$ - Appell Polynomial Sets, we first mention the polynomial set
is a $\Delta_{\omega}$ - Appell Polynomial set since we have $\Delta_{\omega} \boldsymbol{x}^{(n, \llbracket \omega)}=n x^{(n-1, \llbracket \omega)}$. Such type of polynomial sets are generated by:

Now this example can be used in characterizing all the $\Delta_{\omega}$ - Appell Polynomial set. We have a theorem supporting this:

### 3.2.1.1 Theorem

Let $P_{n}(\omega ; .)_{n \geq 0}$ be a polynomial set. The following assertions are equivalent:
i) $\quad P_{n}(\omega ; \text {. })_{n \geq 0}{ }_{n}$ is a $\Delta_{\omega}$ - Appell polynomial set.
ii) There exist a sequence $a_{k} k \geq 0$ is independent of $\mathrm{n} ; a_{o} \neq 0$ such that

$$
P_{n} \omega ; x={ }_{\overline{k=0}}^{n} \frac{n!}{(n-k)!} x^{(n-k, \omega) \square}
$$

iii) $\quad P_{n}(\omega ; \text {. })_{n \geq 0}$ is generated by

where

## Proof:

The implications $i i \Rightarrow(i i i)$ and $i i i \quad \Rightarrow(i)$ are evident.

Next, we prove that $i \Rightarrow(i i)$. Since $P_{n}(\omega ; .)_{n \geq 0}$ and $P_{n}(\omega ; .)_{n \geq 0} x^{(n, w)}{ }_{n \geq 0}$ are two polynomials to write:
where the coefficients $a_{n, k}$ depend on n and k , and $a_{n, 0} \neq 0$. We want to prove these coefficients are independent of $n$. By applying the operator $\Delta_{\omega}$ to each of the member of (3.5), also having the fact that $P_{n}(\omega ; \text {. })_{n \geq 0}$ and $x^{(n, w)}{ }_{n \geq 0}$ are $\Delta_{\omega}$ - Appell polynomials. If we want to obtain:

Since $\Delta_{\omega} x^{(0, \omega)}=0$. By shifting the index $n \rightarrow n+1$ in (3.6), we have

Comparing the equations (3.5) and (3.6) and noting that $a_{n, k}=a_{n+1, k}$ for all k and n which means that $a_{n, k}=a_{k}$ is independent of n and this finishes the proof.

Having known the definition of the $\Delta_{\omega}$ - Appell polynomials, we now see the definition of the d-orthogonal $\Delta_{\omega^{-}}$Appell polynomial.

### 3.2.1' d-Orthogonal $\Delta_{\omega}$ - Appell Polynomial

We start with a Lemma.

## Lemma 3.1

The polynomial sequence $P_{n}(\omega ; \text {. })_{n \geq 0}$ generated by (3.3)-(3.4) is a d-OPS if an only if the coefficient $\beta_{k}=0$ for $k \geq 0$; given by
where $\beta_{n}=\alpha_{n}+\omega \alpha_{n-1} ; n \geq 0$ : with $\alpha_{-1}=0$
satisfying the conditions:

$$
\beta_{k}=0\left[\text { for } \left[k \geq d+1\left[\operatorname { a n d } \left[\beta_{d} \neq 0\right.\right.\right.\right.
$$

## Proof:

According to definition (2.1), the polynomial by (3.3) - (3.4) is a d-OPS if and only if these polynomials satisfy a recurrence relation of type:


$$
\hat{P}_{0} x=x-\beta_{n-1} \hat{P}_{n-1} x-\sum_{v=0}^{n-2} \gamma_{n-1-v}^{d-1-v} \bar{P}_{n-2-v} x, \| \mathbb{l} \leq n \leq d
$$

such conditions, by virtue are equivalent to the fact that the coefficients $\beta_{k} ; k \geq 0$; given.
Let $P_{n}(\omega ; \text {. })_{n \geq 0}$ be a polynomial set generated by (3.3) - (3.4), if

$$
A t=\exp { }_{\frac{d}{v=1}}^{d} \delta_{v} t^{\gamma}
$$

with $\delta_{d} \neq 0$

$$
\frac{A^{\prime}(t)}{A(t)}={ }_{\overline{v=1}}^{d} v \delta_{\gamma} t^{v-1}
$$

### 3.3 Hypergeometricd-Orthogonal Polynomial

The generalized hypergeometric functions are defined by:

$$
{ }_{p} F_{q} z={ }_{p} F_{q} \quad a_{p} ; b_{q} ; z={ }_{\overline{m=0}}^{+\infty} \frac{a_{1 m} \ldots a_{p}{ }_{m}}{b_{1 m} \ldots b_{q}{ }_{m}^{m}} \frac{z^{m}}{m!} 3.10
$$

where
i) $\quad \mathrm{p}$ and q are positive integers or 0 (Interpreting an empty product as 1 )
ii) $\quad \mathrm{z}$ is the complex variable
iii) $\quad a_{p}$ abbreviates the set of p complex parameters $a_{1}, a_{2}, a_{3}, \ldots, a_{p}$
iv) $\quad a_{m}, a \neq 0,-1,-2, .$. , 1 s the Pochhammer symbol given below
v) The numerator parameters $a_{p}$ and then the denominator parameter $b_{q}$ take on complex values provided $b_{j} \neq 0,-1,-2, \ldots$ 皿 $\llbracket j=1, \ldots, q \square$.

Thus, if a numerator parameter is a negative integer or a zero, then the ${ }_{p} F_{q}$ series terminates and this leads us to a generalized hyper geometric polynomial of the type:
where $\alpha_{j} \neq-1,-2, \ldots, ; \llbracket j=1, \ldots, q$. Also setting $D_{x}={ }^{d} d x$, we have the identity :

$$
D_{x p} F_{q} \quad a_{p} ; b_{q} ; x=\frac{a_{p} \ldots a_{1}}{b_{q} \ldots b_{1}}{ }_{p} F_{q} \quad a_{p}+1 ; b_{q}+1 ; x
$$

Two great Mathematicians: Abdul - Halim and Al - Salam (1990) dealt with the problem of finding all orthogonal polynomials of the form (3.11). They proved that the only orthogonal polynomial of this type are essentially the Laguerre polynomials $(p=0, q=1)$ for which we have the representation:

Obtaining the following using a different approach from Abdul-Halim - Al-Salam that the hypergeometric polynomials given by (3.11) are d-Orthogonal if and only if $\mathrm{p}=0$ and $\mathrm{q}=\mathrm{d}$. That is, the polynomials of type:

$$
\mathbb{m}_{n}^{a_{d}} x={ }_{1} F_{d}-n ;\left[\alpha_{d}+1 ; x, \llbracket \alpha_{j} \neq \square-1,-2, \ldots, \llbracket j=1, \ldots, d \llbracket\right)_{\|}
$$

### 3.3.1 Some Propertiesofthe Polynomials $\left[\ell_{n}{ }^{a_{d}}\right.$

i) The Explicit Formula- As we can see from (3.14) that $\mathbb{U}_{n}^{a_{d}}$ has the explicit formular:
ii) The Generating Function- from the Brafman identity (1951):

$$
e_{p}^{t} F_{q} \quad a_{p} ; \alpha_{q}+1 ;-x t={\frac{t^{n}}{n}}_{n!}^{p+1} F_{q}-n, a_{p} ; \alpha_{q}+1 ; x,
$$

we deduce that the generating function for the polynomials $\mathbb{W}_{n}^{\alpha_{d}}, \llbracket n \geq 0$

If we multiply the two members of the identity by $e^{-t}$ and equalizing the coefficient of $t^{n}$ we obtain:
which leads us to expand any analytic function in terms of $\mathbb{m}_{n}^{\alpha_{d}}$ using the generating function techniques or the series rearrangement technique when ever we can.

We can note that if we put $\mathrm{d}=1$, the generating function (3.15) reduces to the classical generating function for Laguerre polynomials
where $\left[J_{\alpha}\right.$ is the Bessel function of the first kind.

### 3.4 Jensen d - Orthogonal Polynomial

First we look at the definition below:

### 3.4.1 Laguerre-PólyaClass $\mathcal{L P}$

An entire function $\varphi x$ is said to be in the Laguerre-Pólya class, i.e $\varphi x \in \square \mathcal{P}$ if it can be represented in the form:
where $\alpha \geq 0, c, \beta\left[\epsilon \square \square\right.$ and $x_{n}$ is a finite or infinite sequence of non - zero real numbers with $\sum_{n} x_{n}^{-2}<\infty$. If $\omega=0$, then by convention, the product is defined to be 1 .

A real entire function $\varphi x$ is of type I in the Laguerre-Pólya class, $\because \mathcal{P} l$, if $\varphi x$ or $\varphi-x$ can be represented in the form:
where $\sigma \geq 0, c\left[1 \mathrm{~s}\right.$ real, m is a non - negative integer, $x_{n}>0$ and $\sum_{n} x_{n}^{-1}<\infty$
if
with $\gamma_{n} \geq 0\left(\right.$ or $\gamma_{n} \leq 0\left[\operatorname{or} \square-1{ }^{n} \gamma_{n} \geq 0\right)$ for all $n=0,1,2, . .$, then $\varphi x \in \square \mathcal{P} 1$.

The significance of the Laguerre-Pólya class in the theory of entire functions stem from the fact that functions in this class, and only thses, are the uniform limits on compact subsets of 0 , of polynomials with only real zeros.

The class $\square^{\mathcal{P}} 1$ consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials whose zeros are either all positive or all negative.

The class $\mathfrak{T P}$ and $\mathcal{P} 1$ attracted the interest of great masters of the classical analysis. The main reason is the fact that they are closely related to the Riemann Hypothesis.

Now we define the Jensen polynomials associated to the entire function $\varphi x$ with $\gamma_{n} \neq 0$ for all $n=0,1,2, \ldots$ by:

The Jensen polynomials enjoy a number of important properties. For example, they are generated by $x, t=e^{t} \varphi(x t)$, that is :

$$
e^{t} \varphi x t={ }_{\overline{n=0}}^{+\infty} g_{n}(x) \frac{t^{n}}{n!}
$$

The Jensen polynomials associated with an arbitrary entire function form a natural sequence of approximating polynomials. In fact, the sequence $\left(g_{n}\left(\varphi, x^{\prime} n\right)\right)_{n}$ converges locally uniformly to $\varphi x$.

Jensen established the following interesting characterization of functions in $\mathbb{T P}$.

## Theorem 3.4

The function $\varphi$ belongs to $\because \mathcal{P}$ if and only if the associated Jensen polynomials $g_{n} \varphi, x, n=$ $1,2,$. have only real zeros. Moreover, the sequence $g_{n} \varphi:[z / n$ converges locally uniformly to $\varphi(z)$

Finally, the Jensen $d$ - Orthogonal polynomials are new characterization of the $d$ Orthogonal Laguerre polynomials.

We have:

$$
\exp t \psi x t=P_{\bar{n}=0}^{+\infty} P_{n}(x) \frac{t^{n}}{n!}
$$

With

$$
\psi z=c_{\overline{n=0}}^{+\infty} c_{n} z^{n}, c_{n} \neq 0
$$

are the polynomials $L_{n}^{\alpha_{d}}$.

In view of this, we have the theorem below

## Theorem 3.5

The only d-Orthogonal Jensen polynomials are the d-Orthogonal Laguerre polynomials $L_{n}^{\alpha_{d}}$.

### 3.5 Continuous d - Orthogonal Polynomials

### 3.5.1 d - OPS of Hermite - Type

We consider the polynomial set $P_{n}(. ; d)_{n \geq 0}$ of Sheffer type zero generated by
(Douak) showed that $P_{n}(. ; d)_{n \geq 0}$ is the only polynomial set which is both Appell and d Orthogonal and derived the corresponding d-dimensional functional vetor.

### 3.5.1.1 Theorem (Douak)

The polynomial set $P_{n}(. ; d){ }_{n \geq 0}$ generated by (3.22) is d-orthogonal.

## Proof:

By application of the operator $D_{t}$ to each members of the equality

$$
G_{d} x, t=\exp x t-\pi_{d+1} t
$$

we obtain

$$
x G_{d} x, t=D_{t} G_{d} x, t+\pi_{d+1}^{\prime} t G_{d} x, t .
$$

It follows by virtue of the Lemma and (3.23),

## Lemma 3.5 (Freeman)

Let $F x, t=\sum_{n=0}^{\infty} P_{n}(x) e_{n}(t)$ where $P_{n} n \geq 0$ is a polynomial set in $\mathscr{P}_{\text {and }} e_{n n \geq 0}$ is a



$$
L_{x} F x, t=\hat{L}_{t} F x, t \quad \operatorname{resp} . M_{t} F x, t=\widehat{M}_{x} F x, t .
$$

where we let $\bigvee^{(1)}$ be the set of operators acting on formal power series that increase the order of every formal power series by exactly one.

Let $V^{-1}$ be the set of operators acting on formal power series that decrease the order of every formal power series by exactly one.

The operator $\bar{L}$ (resp. $\bar{M}$ ) is called the transform operator of $L$ (resp. $M$ ) relative to the generating function $F(x, t)$. Next we limit ourselves to the case:

$$
e_{n} t=\frac{t^{n}}{n!} d \mathbb{d} L_{x}=X \llbracket \operatorname{and}\left[M_{t}=T \square\right.
$$

where $X$ (resp. $T$ ) is the multiplicative operator by $x$ (resp. the multiplicative operator by $t$ ). For this case, the generating function $\left[G x, t=\sum_{n=0}^{\infty} P_{n}(x) \frac{t^{n}}{n!}\right.$ appears as the Eigen function of the operator $\hat{X} \mp \hat{X}_{t}$ (resp. $\widehat{T} \not \widehat{T}_{x}$ associated with the eigenvalue $x$ (resp. $t$ ). We have in fact:

Using Lemma 3.5 and (3.23) that the transform $\hat{X}_{t}$ of $X$ relative to $G_{d} x, t$ is given by

$$
\tilde{X}_{t}=D_{t}+\pi_{d+1}^{\prime} t \in \square \quad{ }_{d+2}^{-1}
$$

Taking into account the theorem below, we conclude that $P_{n}(. ; d){ }_{n \geq 0}$ is d-Orthogonal.

## Theorem 3.6 (Douak)

Let $P_{n} \quad n \geq 0$ be a polynomial set generated by
where $G_{0} 0, t=1$. Let $\hat{X}$ and $\sigma\lceil\widehat{T}$ be, respectively, the transform of $X$ and $T$ the multiplicative operator by $x$ and the multiplicative operator by $t$ according to the generating function $G(x, t)$. Then
i) The following assertions are equivalent :
(a) $P_{n}{ }_{n \geq 0}$ is a d-OPS.
(b) $\hat{X} \in V_{d+2}^{-1}$
ii) If $P_{n}{ }_{n \geq 0}$ is a d-OPS, the d-dimensional functional vector $\mathcal{U}={ }^{t}\left(u_{0}, \ldots, u_{d-1}\right)$ for which the d-Orthogonality holds is given by
$\left[u_{r}, f \square=\frac{1}{r!} \frac{\sigma^{r}}{A(\sigma)} \square f\right)(x)_{x=0}=\frac{\sigma^{r}}{r!A(\sigma)}[f 0, \| \amalg] r=0,1, \ldots, d-1, f \in \mathscr{P}(3.25)$

### 3.6 Discrete d-Orthogonal Polynomials

Here we are going to see some multiple Orthogonal sets generalizing the Charlier and Meixner polynomials following an approach based on Theorem 3.6, to investigate some discrete d-OPS generalizing the Charlier and Meixner.

### 3.6.1 d-OPS of MeixnerType

Here we are considering a new d-OPS. Let us consider the polynomial set $P_{n}(. ; \beta, c){ }_{n \geq 0}$ of Sheffer type zero generated by

$$
G_{d} x, t=\exp \pi_{d-1} t \quad 1-t^{-\beta} 1+\frac{c-1}{c} \frac{t}{1-t}{ }^{x}=\frac{P_{n}(x ; \beta, c)}{n!} t^{n} \text { سाسा( 3.26) }
$$

$\pi_{d-1} t$ being the polynomial of degree $d-1$.
$P_{n}(. ; \beta, c){ }_{n \geq 0}$ is reduced to Meixner-Polynomials $m_{n}^{(\alpha)}{ }_{n \geq 0}$ for $\mathrm{d}=1$. First we state

### 3.6.1.1 Theorem

The Polynomial set $P_{n}(. ; \beta, c)_{n \geq 0}$ is d-Orthogonal.

## Proof:

The application of the operator $D_{t}$ to each member of (3.26) gives

$$
D_{t} G_{d} x, t=\pi_{d-1}^{\prime} t G_{d} x, t+\frac{\beta}{1-t} G_{d} x, t+\frac{c-1}{(1-t)(c-t)}\left[x G_{d} x, t\right. \text { आயायआ(3.26) }
$$

From (3.26), we obtain after some computations

$$
\begin{aligned}
x G_{d} x, t= & \frac{-c}{1-c} D_{t} \\
& +\frac{1}{1-c} 1+c t D_{t}+c \beta+t t D_{t}-\beta+1-t \quad c-t \pi_{d-1}^{\prime} t \quad G_{d}(x, t)
\end{aligned}
$$

So, by virtue of Lemma 3.5 and (3.23), the transform operator $\tilde{X}_{t}$ of $X_{x}$ relative to $G_{d}(x, t)$ is given by

$$
\begin{aligned}
\hat{X}_{t}=\frac{-c}{1-c} D_{t} & +\frac{1}{1-c} 1+c t D_{t}+c \beta+t t D_{t}-\beta+1-t \quad c-t \pi_{d-1}^{\prime} t \\
& \in \quad-1
\end{aligned}
$$

According to theorem 3.6, $P_{n}(. ; \beta, c)_{n \geq 0}$ is orthogonal.

For a case: $d=2$, बnd $\pi_{1} t=-t$, we express the two-dimensional functional vector
$\mathcal{U}={ }^{t}\left(u_{0}, u_{1}\right)$ for which the polynomial set ${ }_{n}(. ; \beta, c)_{n \geq 0}$ of Meixner type generated by
is 2-Orthogonal.

### 3.6.2 d-OPS of CharlierType

The Polynomial set $P_{n}(. ; d, \omega)_{n \geq 0}$ of Sheffer type zero generated by

$$
\left.G_{d} x, t=1+\omega t^{\frac{x}{\omega}} \exp \pi_{d} t=P_{\overline{n=0}}^{\infty} P_{n}(x ; d, \omega) \frac{t^{n}}{n!} \square 山\left\|\pi_{d} t={ }_{\bar{k}=0}^{d} \gamma_{k} t^{k} \square \gamma_{k} \neq 0\right\| 3.28\right)
$$

was considered in a paper by (Cheikh) and (Zaghouani). They showed that this polynomial set is the only polynomial set with both d-Orthogonal and $\Delta_{\omega^{-}}$Appell, where $\Delta_{\omega}$ is the difference operator defined by

$$
\Delta_{\omega} f x=\frac{f x+\omega-f(x)}{\omega} \square \omega \neq 0 .
$$

The d-Orthogonality property was obtained using the following:

$$
\begin{aligned}
& \text { d }
\end{aligned}
$$

where $\beta_{n+1} \alpha_{0, n-d} \neq 0$ and $P_{-n}$,
next we give a new proof of the property using Theorem 3.6.
Applying the operator $D_{t}$ to each member of (3.28), we obtain

$$
D_{t} G_{d} x, t=\pi_{d}^{\prime} t+\frac{x}{1+\omega t} G_{d} x, t, \square
$$

which leads to

$$
x G_{d} x, t=\left(1+\omega t D_{t}-(1+\omega t) \pi_{d}^{\prime}(t)\right.
$$

This means that $\hat{X}_{t} \in \mathrm{~V}_{d+2}^{(-1)}$. Taking into account Theorem 3.6, the sequence $P_{n}(. ; d, \omega)_{n \geq 0}$ is d-Orthogonal. (3.25) was used to express explicitly the d-dimensional functional $U$ for which the polynomial set $P_{n}(. ; d)_{n \geq 0}$ of Charlier type generated by

It is also stated that

### 3.6.2.1 Theorem (Cheikh and Zaghouani)

The polynomial set $P_{n}(. ; d)_{n \geq 0}$ generated by (3.30) is a d-OPS with respect to the ddimensional functional vector $U={ }^{t}\left(u_{0}, \ldots, u_{d-1}\right)$ given by

$$
\begin{aligned}
& d-1 \infty
\end{aligned}
$$

where

$$
\Psi d, a ; d j=\frac{a^{\prime}}{j!}{ }^{d-1} F_{d-1} \begin{gather*}
j+\frac{1}{d}, j+\frac{2}{d}, \ldots, j+\frac{d-1}{d} ; \\
\frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d} ;
\end{gather*}
$$

and

$$
\begin{aligned}
& \Psi d, a ; d j+s \\
& \begin{aligned}
&=\frac{(-1)^{d+s} a^{j+1}}{j+1!} \frac{d j+d}{d j+s}{ }^{d-1} F_{d-1} \begin{array}{c}
j+1+\frac{1}{d}, j+1+\frac{2}{d}+\cdots+j+1+\frac{d-1}{d} ; \\
1+\frac{1-s}{d}, 1+\frac{2-s}{d}, \ldots, 1-\frac{1}{d}, 1+\frac{1}{d}, \ldots 1+\frac{d-s}{d} ;
\end{array} \llbracket^{\left[a(-1)^{d}\right.} \\
& s \square=[1,2,3, \ldots, d \square-1 \square
\end{aligned}
\end{aligned}
$$

and

$$
\left[u_{r}, f \square=\frac{1}{r!}\left[u_{r}, \Delta^{r} f \square \square \| \frac{d ा}{} r=1,2, \ldots, d-1, \quad f \in[\mathcal{P} .\right.\right.
$$

Some examples are worthy to note:

## Example:

In the case where $d=1$ corresponds to Charlier polynomials $C_{n}^{\alpha}(x)$ generated by

$$
\exp -a t \quad 1+t^{x}={\frac{C_{n}^{\alpha}}{\overline{n=0}}}_{n!}^{n!} t^{n}, a \neq 0
$$

For this case, identity (3.31) is reduced to

$$
\Psi 1,1, a ; j=\frac{a^{j}}{j!}{ }^{5} F_{0} \xrightarrow{-; \text {;欰 } a} a=\frac{e^{-a} a^{j}}{j!}
$$

From which it has been deduced the well - known result: The Charlier polynomials are orthogonal with respect to the linear functional:

$$
\left[u_{0}, f \square={ }_{j=0}^{\infty} \frac{e^{-a} a^{j}}{j!} f j, \text { 监 } f \square \square[\mathcal{P} \mathbb{W}]\right.
$$

### 3.7The Hyper-Bessel Function, the D-Orthogonal Laguerre Polynomials and the

## Relation Between them:

In the fourth chapter, we see the definition of the Hyper Bessel function and the Laguerre Polynomial and the similarities between them.

The hyper - Bessel's function is defined by:
where the standard notion for hypergeometric function is:
the shifted factorials are defined by $(\beta)_{0}=1,(\beta)_{n}=\beta \beta+1 \ldots \beta+n-1, n \geq 1$, and the contracted notation $\left(\alpha_{d}\right)$ is used to abbreviate the array of $d$ parameters $\alpha_{1}, \ldots, \alpha_{d}, \ldots$ The function $\mathcal{J}_{\alpha_{d}}$ was introduced by Delerue and later studied by Kiryakova as multi index analogues of the Bessel function $\llbracket \mathcal{J}_{\alpha}$. A close relation between the components with respect to the cyclic group of order $n$ of the Bessel's function and the Hyper - Bessel's function.

The normalized Hyper - Bessel's function
appears as a generating function of the polynomial set $L_{n}^{\alpha_{d}}$
where $L_{n}^{\alpha_{d}}(x)$ denotes the $\mathrm{d}-$ Laguerre polynomials $L_{n}^{\alpha_{d}}$
where $d=1$, the Hyper - Bessel's function and the d - Laguerre polynomials are respectively reduced to the well-known Bessel function and Laguerre Polynomials.

## CHAPTER 4

## CONCLUSION

### 4.1 Conclusion

In this research work, definitions and examples of d-Orthogonal Polynomials are given and the d-Orthogonal Hermite and d-Orthogonal Laguerre were defined.

Moreover, some generalization of the d-Orthogonal Polynomials were obtained using the generating functions, and based on these, new ideas can be applied like we saw the similarity between Laguerre d-Orthogonal and Hyper-Bessel function.

However, new d-Orthogonal families can be investigated using generation functions, hypergeometric functions and much more.

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