BESSEL'S DIFFERENTIAL EQUATION

AND

APPLICATION OF SCHRODINGER EQUATION TO

NEUMANN AND HANKEL FUNCTIONS

A THESIS SUBMITTED TO THE GRADUATE
SCHOOL OF APPLIED SCIENCES
OF
NEAR EAST UNIVERSITY

By
SALIHU SABIU MUSA

In Partial Fulfilment of the Requirements for
The Degree of Master of Science
in
Mathematics

NICOSIA, 2016

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Salihu Sabiu Musa: BESSEL'S DIFFERENTIAL EQUATION AND APPLICATION OF SCHRODINGER EQUATION TO NEUMANN AND HANKEL FUNCTIONS

Approval of Director of Graduate School of Applied Sciences

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We certify this thesis is satisfactory for the award of the degree of Master of Sciences In Mathematics.

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my supervisor Ass. Prof. Dr. Evren Hincal for the continuous support of my study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me a lot in all the time of my research and writing of this thesis.

I would also like to express my special thanks of gratitude to all the lecturers of the mathematics department at NEU for their encouragement and an insightful comments. Their excitement and willingness to provide feedback made the completion of this research an enjoyable experience.

I will also not forget to acknowledge my sponsor in person of Dr. Rabiu Musa Kwankwaso for his wonderful support and encouragement throughout the entire program. A special thanks also goes to my brothers and friends for their support directly or indirectly.

Above all, my unlimited thanks and heartfelt love would be dedicated to my dearest wife Rahama and my parents late Alh Sabiu Musa and Haj Maryam for their support physically and spiritually throughout my life.

To my parents and my wife...

ABSTRACT

This research consists of three chapters. In the first chapter, we consider the Historical background of the study, also some essential definitions were given. In the second chapter, Bessel's differential equation were obtained via the cylindrical coordinates of Laplace equation. In addition, Bessel functions which are the solutions of Bessel's differential equation and their properties were studied. In the third chapter, applications of Bessel functions which are solutions of Schrödinger equation to Neumann and Hankel functions were examined and the solutions were obtained.

Keywords: Bessel's differential equation, Bessel functions, Hankel functions, Neumann functions and Schrödinger equation..

ÖZET

Bu tez üç bölümden oluşmaktadır. Birinci bölümde konunun tarihsel gelişimi ve bazı temel kavramlar verilmiştir. İkinci bölümde Laplace denkleminin silindirik koordinatlardaki ifadesinden yararlanılarak Bessel denklemi elde edilmiştir. Ayrıca, Bessel denkleminin çözümleri olan Bessel fonksiyonları ve onların özellikleri üzerinde durulmuştur. Üçüncü bölümde ise, Neumann ve Hankel fonksiyonları Schrödinger denkleminin çözümünü olan Bessel fonksiyonlarının uygulamaları incelenmiş ve çözümleri elde edilmiştir.

Anahtar sözcükler: Bessel diferansiyel denklemi, Bessel fonksiyonları, Hankel fonksiyonları, Neumann fonksiyonları, Schrödinger denklemi..

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$\boldsymbol{J_v(x)}$	Bessel Function of order v
$\boldsymbol{\Gamma(x)}$	Gamma function
$\boldsymbol{H_v^{(1)}(x)}$	Hankel function of the First kind
$\boldsymbol{H_v^{(2)}(x)}$	Hankel function of the Second kind
$\boldsymbol{\nabla^2}$	Laplacian operator
$\boldsymbol{I_v(x)}$	Modified Bessel function of the first kind
$\boldsymbol{K_v(x)}$	Modified Bessel function of the second kind
$\boldsymbol{Y_v(x)}$ **or** $\boldsymbol{N_v(x)}$	Neumann or Weber function
$\hbar$	Planck's constant
$\boldsymbol{u(x)}$	Potential energy
$\boldsymbol{\Psi(x,t)}$	Wave function
$\boldsymbol{a_M^n}$	Zeros of Bessel function

CHAPTER 1

INTRODUCTION AND DEFINITIONS

This chapter gives historical background of Bessel's equation, Bessel functions and Schrödinger equation, and also some basic definitions were also stated.

Bessel function were studied by Euler, Lagrange and the Bernoulli. The Bessel functions were first used by Friedrich Wilhelm Bessel to explain the three body motion, with the Bessel function which emerge in the series expansion of planetary perturbation. Bessel function are named for Friedrich Wilhelm Bessel (1784-1846), after all, Daniel Bernoulli is generally attributed with being the first to present the idea of Bessel functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain hanging at one end. By the year 1764, Leonhard Euler employed Bessel functions of both the integral orders and zero orders in an analysis of vibrations of a stretched membrane, a research that was further developed by Lord Rayleigh in 1878, where he proved that Bessel functions are particular case of Laplace functions (Niedziela, 2008).

Bessel's differential equation arises as a result of determining separable solutions to Laplace's equation and the Helmholtz equation in spherical and cylindrical coordinates. Therefore, Bessel functions are of great important for many problems of wave propagation and static potentials.

Bessel equation were also obtained in solving various classical physics problems. Historically, the equation with $v = 0$ was first experience and solved by Daniel Bernoulli in1732 in his research of the hanging chain problem. Similar equations emerged later in1770 in the work of Lagrange on astronomical problems. In 1824, the German mathematician and astronomer F.W.Bessel in his research of the problem of elliptic planetary motion come across a special form of equation (9). Influenced by the great work of Fourier that had just emerged in 1822, Bessel conducted an efficient research of equation (9) (Asmar, 2005).

Bessel while accepting named credit for these functions, did not in participate them into his research as an astronomer until 1817. The Bessel function was the outcome of Bessel research of problem of Kepler for finding the motion of three bodies travelling under mutual gravitation. In 1824, he integrated Bessel functions in a research of planetary perturbations where the Bessel functions emerged as a coefficients in a series expansion of the indirect perturbation of a planet, that is, the motion of the sun induced by the perturbing body. It was like the Lagrange's work on elliptical orbits that were first proposed to Bessel to study on the Bessel functions.

The notation $J_{v,n}$ was first used by Hansen (1843) and afterwards by Schlomilch (1857) and later modified to $J_n(2v)$ by Watson (1922). Subsequent research of Bessel functions included the works of Mathews in 1895, "A treatise on Bessel functions and their applications to physics" written in joint effort with Andrew Gray. It was the first major dissertation on Bessel functions in English and covered topics such as, application of Bessel functions to electricity, hydrodynamics and diffraction. In 1922, Watson first presented his comprehensive analysis of Bessel functions "A dissertation on the theory of Bessel functions".

Intermittently, the key to solving such a problems is to identify the form of this equations. Thus, leaving employment of the Bessel functions as solutions. The Frobenius method is used to obtain a Bessel functions which is a solution to Bessel differential equations with variable coefficients. Also we can obtained the Laplace equation in polar coordinates with Bessel equation by using the expression, which is the key equation in mathematical physics, engineering science and basic science and other related fields are common in finding the problems of this equation.

Applications of Bessel functions to the theory of heat conduction, which include dynamical system and heat conduction in spherical or cylindrical objects, which are very large. In the theory of elasticity, the solutions of Bessel functions are efficient for all special problems, which are the solutions of cylindrical or spherical coordinates, and also for various problems relating to the oscillation of plates and equilibrium of plates on an electric foundation, for a series of the questions of theory of shells, for the problems on concentration of the stress near cracks and others. In each of these fields there are many applications of Bessel functions.

Different parts of the theory of Bessel functions are extensively used when solving problems of hydrodynamics, acoustics, radio physics, atomic and nuclear physics, quantum physics and so on.

Bessel functions made their first emergence by relating the angular position of a planet travelling along a Keplerian ellipse to elapsed time. Though the integral and power series appears in other places, generally regarding the radial variable after separating the Laplace's equation in polar or spherical polar coordinates. In diverse problems of mathematical physics, whose solution is highly connected with the application of cylindrical and spherical coordinates.

The constant v in the Bessel differential equation determines the order of the Bessel functions and can take any real numbered value ($v = n + \frac{1}{2}$) while for cylindrical problems the order of the Bessel function is an integer value ($v = n$). Bessel functions are also applicable for many problems of wave propagation, static potentials and its applications. Heat conduction in a cylindrical objects, electromagnetic waves in a cylindrical waveguide, modes of vibration of a thin circular or annular artificial membrane, diffusion problems on a lattice and solution to the radial Schrodinger equation (in spherical and cylindrical coordinates for a free particle). We are going to consider only the last application which is the application of radial Schrodinger equation in cylindrical coordinates for a free particle (zero potential) to Neumann and Hankel functions respectively (Nuriye, 2012).

The Schrodinger equation which requires the idea of electromagnetic wave equation and the basic of Einstein's special theory of relativity is a new criterion in physics which appeared at the beginning of the last century and now popularly known as quantum mechanics, and was motivated by two types of experimental observations: The "Lumpiness", or quantization of energy transfer in light-matter interactions, and the dual wave-particle nature of both light and matter.

It has been well acknowledged that photon show (exhibits) both wave-like properties, the so-called wave particle duality in physics. In order to express particle-like nature of light, Einstein suggested that the energy E and momentum p of a photon can be expressed as follows:

$E = hv = \hbar\omega, p = \frac{E}{c} = \frac{h}{\lambda} = \hbar k$

Where v is the frequency of a photon, $\omega = 2\pi v$ is the angular frequency, λ is the wavelength of a photon, $k = |K| = \frac{2\pi}{\lambda}$ is the wave number (k is the wave vector) and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant.

In 1923, de Broglie postulate that all matter not just photon, possess (acquire) the wave-like nature. For a free particle material, de Broglie assumed that the associated wave of the particle also has a frequency and wavelength as given by:

$v_d = \frac{E}{h}, \lambda_d = \frac{h}{p}$

Where h is the Planck constant, E is the energy of the particle and p is the momentum of the particle. Without considering relativistic effects, the de Broglie wavelength of a particle with a mass m and a velocity v can be easily obtained from the above second equation as follows;

$$\lambda_d = \frac{h}{mv} = \frac{h}{\sqrt{2mE_k}}$$

Where $E = \frac{mv^2}{2}$ is the kinetic energy of the particle.

In 1926, Erwin Schrödinger as a result of his interest by the de Broglie hypothesis created an equation as a way of expressing the wave behavior of matter particle, for example, the electron. The equation was later named as Schrödinger equation which can be written as:

$$\left(\frac{-\hbar}{2m}\nabla^2 + U(r,t)\right)\psi(r,t) = i\hbar\psi(r,t)$$

Where m is the mass of the particle, $U(r,t)$ is the potential energy, ∇^2 is the Laplacian, and $\psi(r,t)$ is the wave function. Indeed, the Schrödinger equation given above is of most important and fundamental equation of the modern physics, the time dependent Schrödinger equation for a quantum system is introduced as a powerful analog of Newton's second law of motion for a classical system. However, we consider only the time independent Schrödinger equation for a free particle (Griffiths, 1995).

Definition 1: *(Ordinary and singular point) If the coefficients* $P(x)$ *and* $Q(x)$ *of an equation of the form* $y''(x) + P(x)y' + Q(x)y = 0$ *are both analytic at the point* x_0*, then* x_0 *is called an ordinary point for the equation. A point which is not an ordinary point is called a singular point.*

Definition 2: *(Linear dependent and Linear independent) Two functions* u *and* v *are said to be linearly independent on the interval* (α, β) *if neither is a constant multiple of the other on that interval. If one is a constant multiple of the other on* (α, β) *they are said to be linearly dependent there.*

Definition 3: *(Wronskian determinant) Let* f *and* g *be two differentiable functions. Then, the wronskians of* f *and* g *is defined by;*

$$W(f, g) = fg' - f'g$$

Definition 4: *(Orthogonal functions) A functions is orthogonal if a defined inner product vanishes between two unlike components of a particular inner product space (an inner product) between a function* $\Psi(a)$ *and* $\Psi(b)$ *shall be depicted mathematically by* $\langle\Psi(a)\,|\Psi(b)\,\rangle$*. It is common to use the following inner product for two functions* f *and* g*:*

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$$

Here we introduce a nonnegative weight functions $w(x)$ *in the definition of this inner product. We say those functions are orthogonal if that inner product is zero.*

$$\int_a^b f(x)g(x)w(x)dx = 0$$

Definition 5: *(Norm of function) The norm of a function defined by* $\|f\|$ *which is equal to*

$$\left(\int_0^1 f^2(x)dx\right)^{1/2}$$

Definition 6: *(Frequency) Frequency describes the number of waves that pass a fixed place in a given amount of time.*

Definition 7: *(superposition principle) For a linear homogeneous ordinary differential equation, if* $y_1(x)$ *and* $y_2(x)$ *are solutions, then so is* $k_1 y_1(x) + k_2 y_2(x)$.

Definition 8: *(Heisenberg's Uncertainty principle) Heisenberg's uncertainty principle is one of the fundamental concepts of quantum physics, and is the basis for the initial realization of fundamental uncertainties in the ability of an experimenter to measure more than one quantum variable at a time. Attempting to measure an elementary particle's position to the highest degree of accuracy, for example, leads to an increasing uncertainty in being able to measure the particle's momentum to an equally high degree of accuracy. Heisenberg's uncertainty principle is typically written mathematically in either of the two forms:*

$\Delta E \Delta t \geq {h}/{4\pi}$ *and* $\Delta x \Delta p \geq {h}/{4\pi}$

In essence, the uncertainty in the energy (Δt) *times the uncertainty in the time* (Δt) *or alternatively, the uncertainty in the position* (Δx) *multiplied by the uncertainty in the momentum* (Δp) *is greater or equal to a constant* $\left({h}/{4\pi}\right)$. *The constant h, is called Planck's constant. (where* ${h}/{4\pi} = 0.527 \times 10^{-34} Js$*). (Nuriye)*

Definition 9: *(The generating function for* $J_n(x)$*) Let* $f(x,t)$ *be two variables function and its Taylor expansion for one of its variables could be as follows:*

$$f(x,t) = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

The function $f(x,t)$ *with* $\{f_n(x)\}$, $n = integer$ *called the generating function for* $J_n(x)$. *This series of functions are not necessarily converge for all* $x's$ *and* $t's$. *Let* I *be a closed interval and* r *be a positive constant and let* $|t| < r$ *and* $x \in I$ *is enough for convergence.*

CHAPTER 2

BESSEL'S EQUATION AND BESSEL FUNCTIONS

This chapter explains the concept of Bessel's differential equation and some properties of Bessel functions and its application.

2.1 Bessel's Differential Equation

Bessel's equation and Bessel's function occurs in relation with many problems of engineering and physics also there is an extensive literature that deals with the theory and application of this equation and its solution.

Bessel's equation can be used to find a solution of Laplace's equation (that is the key equation in the field of mathematical physics) related with the circular cylinder functions.

In Cartesian coordinates, the Laplace's equation is given by:

$$\nabla^2 K = \frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} + \frac{\partial^2 K}{\partial z^2} = 0 \tag{2.1}$$

Where ∇^2 is the Laplacian operator. Now we are more concerned in finding the solution of Laplace's equation using cylindrical coordinates. In such a coordinate system the equation can be written as follows:

$$\frac{1}{q}\frac{\partial}{\partial q}\left(q\frac{\partial K}{\partial q}\right) + \frac{1}{q^2}\frac{\partial^2 K}{\partial h} + \frac{\partial^2 K}{\partial z^2} = 0$$

Implies;

$$\frac{\partial^2 K}{\partial q^2} + \frac{1}{q}\frac{\partial K}{\partial q} + \frac{1}{q^2}\frac{\partial^2 K}{\partial h^2} + \frac{\partial^2 K}{\partial Z^2} = 0 \tag{2.2}$$

We use separation of variables method to solve this equation, which is a method used to solve many kind of partial differential equations.

We suppose the solution as follows:

$$K(q,h,z) = Q(q)H(h)Z(z)$$

By taking the derivatives appropriately, the following equations are obtained:

$$\frac{\partial K}{\partial q} = HZ\frac{dQ}{dq}, \quad \frac{\partial^2 K}{\partial q^2} = HZ\frac{d^2Q}{dq^2},$$

$$\frac{\partial K}{\partial h} = QZ\frac{dH}{dh}, \quad \frac{\partial^2 K}{\partial h^2} = QZ\frac{d^2H}{dh^2},$$

$$\frac{\partial k}{\partial z} = QH\frac{dZ}{dz}, \quad \frac{\partial^2 K}{\partial z^2} = QH\frac{d^2Z}{dz^2}$$

Substituting these derivatives into equation (2.2), yield the intermediate result as:

$$HZ\frac{d^2Q}{dq^2} + \frac{1}{q}HZ\frac{dQ}{dq} + \frac{1}{q^2}QZ\frac{d^2H}{dh^2} + QH\frac{d^2Z}{dz^2} = 0$$

$Q(q)H(h)Z(z) \neq 0$, and dividing the above equation by QHZ for the two sides, we have:

$$\frac{1}{Q}\frac{d^2Q}{dq^2} + \frac{1}{qQ}\frac{dQ}{dq} + \frac{1}{q^2H}\frac{d^2H}{dh^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Implies:

$$\frac{Q''}{Q} + \frac{1}{q}\frac{Q'}{Q} + \frac{1}{q^2}\frac{H''}{H} + \frac{Z''}{Z} = 0$$

Implies:

$$\frac{Q''}{Q} + \frac{1}{q}\frac{Q''}{Q} + \frac{1}{q^2}\frac{H''}{H} = -\frac{Z''}{Z} \tag{2.3}$$

In the equation above, the left hand side depends on q and h, while the right hand side depends on z. The only way these sides will be equal for all values of q, h and z is when both of them are equal to some constant. Let us defined such a constant as γ^2, for this choice of the constant by considering the left hand side of equation (2.3),

i.e.

$$\frac{Q''}{Q} + \frac{1}{q}\frac{Q'}{Q} + \frac{1}{q^2}\frac{H''}{H} = -\gamma^2 \tag{2.4}$$

And since $\frac{Z''}{Z} = +\gamma^2$, the following equation is obtained:

$$Z'' - \gamma^2 Z = 0 \tag{2.5}$$

And the general solution of equation (2.5) is:

$$Z(z) = \zeta_1 e^{\gamma z} + \zeta_2 e^{-\gamma z}$$

For this solution, when we consider the specific boundary conditions, will allow $Z(z)$ to go to zero for z going to $\pm\infty$, that make physical sense. But if we had taken a constant as negative, we would have had periodic trigonometric functions, which will not tend to zero for z going to infinity.

Once solved the z-dependency, we need to take care of q and h. Equation (2.3) will now reads as:

$$\frac{1}{Q}\frac{d^2Q}{dq^2} + \frac{1}{qQ}\frac{dQ}{dq} + \frac{1}{q^2H}\frac{d^2H}{dh^2} = -\gamma^2$$

Implies:

$$\frac{q^2}{Q}\frac{d^2Q}{dq^2} + \frac{q}{Q}\frac{dQ}{dq} + \gamma^2 q^2 = -\frac{1}{H}\frac{d^2H}{dh^2} \tag{2.6}$$

Again, the only way this equation can be equal is when both sides are equal to some constant. This time around we choose a positive constant, which we called v^2,

The equation for H will becomes:

$$-\frac{1}{H}\frac{d^2H}{dh^2} = v^2$$

Implies:

$$\frac{d^2H}{dh^2} + v^2 H = 0 \tag{2.7}$$

And the general solution of equation (2.7) can be written as:

$$H(h) = t_1 sin(vh) + t_2 cos(vh)$$

This solution is appropriate to explain the variation for an angular coordinate like h. Had we decided to set both members of equation (2.6) equal to a negative number, we would have finished up with exponential functions with a different value assigned to $H(h)$ for each $360°$ turn, which is clearly nonphysical solution.

The q-dependency. From equation (2.6) we have:

$$\frac{q^2}{Q}\frac{d^2Q}{dq^2} + \frac{q}{Q}\frac{dQ}{dq} + \gamma^2 q^2 = v^2$$

Which implies:

$$q^2 \frac{d^2Q}{dq^2} + q\frac{dQ}{dq} + (\gamma^2 q^2 - v^2)Q = 0 \tag{2.8}$$

Equation (2.8) is a popular equation of mathematical physics called parametric Bessel's equation. By using a simple linear transformation of variable $x = \gamma q$, equation (2.8) is changed into a Bessel's equation of index v, and its solution is called cylindrical or Bessel's function.

That is,

$$x^2 Q''(x) + xQ'(x) + (x^2 - v^2)Q(x) = 0 \tag{2.9}$$

Where $Q''(x)$ and $Q'(x)$ represent first and second derivatives with respect to x and we assume that v to be real, non-negative number.

2.2 Frobenius Method Applied to Bessel's Differential Equations

Consider the Bessel's differential equation with order v.

i.e.

$$x^2 Q''(x) + xQ'(x) + (x^2 - v^2)Q(x) = 0 \tag{2.9}$$

Equation (2.9) is a linear second order differential equation, thus it is general solution can be written in the form:

$$u(x) = c_1 Q_1(x) + c_2 Q_2(x)$$

Where $Q_1(x)$ and $Q_2(x)$ are linearly independent partial solutions of equation (2.9). We checked that $x = 0$ is a regular singular point. In some application of Bessel's differential equation the parameter x will be distance of a point from the starting point in polar coordinates. It will be vital to see how the solution acts when x is closed to zero, and the point is closed to the origin. So that, we shall try to find a solution of equation (2.9) in the form of a generalized power series, that is, a Frobenius method in increasing powers of argument x.

$$Q(x) = \sum_{n=0}^{\infty} a_n \, x^{s+n} \tag{2.10}$$

Where $a_0 \neq 0$.

Taking the derivatives of the first and second series, we have:

$$Q'(x) = \sum_{n=0}^{\infty} (s+n) a_n x^{s+n-1} \tag{2.11}$$

And

$$Q''(x) = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{s+n-2} \tag{2.12}$$

Replacing equation (2.10), (2.11) and (2.12) with equation (2.9), we obtain:

$$\sum_{n=0}^{\infty} a_n (s+n)(s+n-1) x^{s+n} + \sum_{n=0}^{\infty} a_n (s+n) x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n+2} - \sum_{n=0}^{\infty} a_n v^2 x^{s+n} = 0$$

Our next target is to collect equal powers of x and set the corresponding coefficients to zero:

$$n = 0 \Rightarrow a_0 s(s-1) + a_0 s - a_0 v^2 = 0$$

$$n = 1 \Rightarrow a_1 (s+1)s + a_1 (s+1) - a_1 v^2 = 0$$

$$n = 2 \Rightarrow a_2 (s+2)(s+1) + a_2 (s+2) + a_0 - a_2 v^2 = 0$$

$$\vdots$$

$$n = k \Rightarrow a_k (s+k)(s+k-1) + a_k (s+k) + a_{k-2} - a_k v^2 = 0$$

After some simplification, we have:

$$\begin{cases} a_0(s^2 - v^2) = 0 \\ a_1[(s+1)^2 - v^2] = 0 \\ a_2 = -{}^{a_0}\!/_{[(s+2)^2 - v^2]} \\ \vdots \\ a_k = -{}^{a_{k-2}}\!/_{[(s+k)^2]} \\ \vdots \\ \vdots \end{cases} \tag{2.13}$$

The term corresponding to $n = 0$ is the so-called indicial equation. Thus, the roots are $s = \pm v$.

The Frobenius method show us that two different solutions each one having form (2.10), can be found for equation (2.9) if the difference between these two roots, i.e. $v - (-v) = 2v$, is neither zero no an integer. Now, let us consider those cases where v is different from a multiple of $\frac{1}{2}$. For $s = v$, from the second of equation (2.13), we can find $a_1 = 0$. For the remaining equations we can obtain:

$$a_k = -\frac{a_{k-2}}{k(k+2v)}, \quad k = 1,2,3 \ldots \tag{2.14}$$

Given that $a_1 = 0$, equation (2.14) yields:

$$a_2 = -\frac{a_0}{[2(2+2v)]}$$

$$a_3 = -\frac{a_1}{[3(3+2v)]} = 0$$

$$a_4 = -\frac{a_2}{[4(4+2v)]}$$

$$a_5 = -\frac{a_3}{[5(5+2v)]} = 0$$

$$a_6 = -\frac{a_4}{[6(6+2v)]}$$

$$\vdots$$

Thus, all odd coefficients are zero. We can re-write even coefficients with an integer value n ranging from 1 to ∞ as;

$$a_{2n} = -\frac{a_{2n-2}}{[2n(2n+2v)]} = -\frac{a_{2n-2}}{[2^2 n(v+n)]}$$

Therefore, the first few coefficients will be;

$$a_2 = -\frac{a_4}{2^2 \cdot 1(v+1)}$$

$$a_4 = -\frac{a_2}{2^2 \cdot 2(v+2)} = -\frac{1}{2^2 \cdot 2(v+2)}\left[-\frac{a_0}{2^2 \cdot 1(v+1)}\right]$$
$$= (-1)^2 \frac{a_0}{2^{2\cdot 2}(2\cdot 1)(v+2)(v+1)}$$

$$a_6 = -\frac{a_4}{2^2 \cdot 3(v+3)} = \cdots = (-1)^3 \frac{a_0}{2^{2\cdot 3}(3\cdot 2\cdot 1)(v+3)(v+2)(v+1)}$$

$$\vdots$$

Finally, extrapolating to the n-th term:

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(v+1)(v+2)\cdots(v+n)}, \quad n = 1,2,3\cdots \tag{2.15}$$

As of right now we can't give a specific value to coefficient a_0, in light of the fact that we are not dealing with any particular issue and have no limit conditions which would give us the likelihood to ascertain it. Historically, however, it has been discovered helpful to standardize solutions of Bessel's equation by assigning a particular value to a_0, and express all its specific solution as a function of a standardized ones.

Let us choose a_0 to be;

$$a_0 = \frac{1}{2^v \Gamma(v+1)} \tag{2.16}$$

Where $\Gamma(x)$ is the gamma function.

With this choice of a_0 equation (2.15) will now be written as:

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!\,(v+1)(v+2)\cdots(v+n)} \frac{1}{2^v \Gamma(v+1)}, \quad n = 1,2,3 \cdots$$

Using recursive property, the above equation is transformed into:

$$a_{2n} = \frac{(-1)^n}{2^{2n+v} n! \Gamma(v+n+1)}, \; n = 1,2,3 \cdots \tag{2.17}$$

And so an independent solution of Bessel's differential equation is given by the following expression:

$$J_v(x) = x^v \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{v+2n} n! \Gamma(v+n+1)} \tag{2.18}$$

$J_v(x)$ is called Bessel's function of the first kind of order v. Here we just need to find the general solution of Bessel's differential equation for v different from an integer or a semi integer. Using Frobenius method we know that, with these values for v, a second solution for Bessel's function is given by $J_{-v}(x)$:

$$J_{-v}(x) = \frac{1}{x^v} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{-v+2n} n! \Gamma(-v+n+1)} \tag{2.19}$$

Therefore, the general solution of Bessel's differential equation, with v different from an integer or a semi-integer, is given by:

$$Q(x) = c_1 J_v(x) + c_2 J_{-v}(x), \qquad v \geq 0, v \neq k\frac{1}{2}, k = 0,1,2 \cdots \tag{2.20}$$

The presence of x^v in equation (2.19) implies that some caution has to be utilized when calculating both $J_v(x)$ and $J_{-v}(x)$. First of all, $x = 0$ is ruled out from the general solution range because x^v appears at the denominator. Secondly, powers of negative numbers give real numbers only for integer values of the power. No real values are, in general, assigned to non-integer powers of negative numbers. For example, $-2^{0\cdot 2}$ is real, negative number equal to $\sqrt[5]{-2}$, while $-2^{0\cdot 7} = \sqrt[10]{(-2)^7}$ is a complex number. For this reason it is safer to defined solution (2.20) only for positive values of x, i.e. for $x > 0$.

2.2.1 Bessel's Equation of Order Zero ($v = 0$)

For $v = 0$, the Bessel's differential Equation is equivalent to the equation given by;

$$xQ''(x) + Q'(x) + xQ = 0 \tag{2.21}$$

Which is called Bessel's differential equation of index zero. Now, we find the solutions of this equation that are useful in an interval $0 < x < R$. Clearly, $x = 0$ is a regular singular point, and hence, we shall assume a solution of the form:

$$Q(x) = \sum_{m=0}^{\infty} c_m x^{m+r}$$

By taking the derivatives of the above series twice and substituting into equation (2.21), we obtain:

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^{m+r-1} + \sum_{m=0}^{\infty}(m+r)c_m x^{m+r-1} + \sum_{m=0}^{\infty} c_m x^{m+r+1} = 0$$

Simplifying, we have;

$$\sum_{m=0}^{\infty}(m+r)^2 c_m x^{m+r-1} + \sum_{m=0}^{\infty} c_{m-2} x^{m+r-1} = 0$$

Implies;

$$r^2 c_0 x^{r-1} + (1+r)^2 c_1 x^r + \sum_{m=2}^{\infty}[(m+r)^2 c_m + c_{m-2}]x^{m+r-1} = 0$$

Equating the coefficient of the lowest power of x to zero in this equation, we have the indicial equation $r^2 = 0$ which has the roots as $r_1 = r_2 = 0$. Again, equating the coefficients of the higher power of x to zero in the above equation, we have;

$$(1+r)^2 c_1 = 0 \tag{2.22}$$

And we can have the recurrence relation as follows:

$$(m+r)^2 c_m + c_{m-2} = 0, \quad m \geq 2 \tag{2.23}$$

We let $r = 0$ in equation (2.22), we find $c_1 = 0$.

Also if we let $r = 0$ in equation (2.23), we obtained the new recurrence relation which is written as;

$$m^2 c_m + c_{m-2} = 0, \quad m \geq 2$$

Which implies:

$$c_m = -\frac{c_{m-2}}{m^2}, \quad m \geq 2.$$

From this we can obtain;

$$c_2 = \frac{c_0}{2^2},\ c_3 = -\frac{c_1}{3^2} = 0 \text{ (since } c_1 = 0),\ c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 \cdot 4^2},\ \ldots$$

Now, we note that all of the odd coefficients are equals to zero and that the even coefficients may be written in general as:

$$c_{2m} = \frac{(-1)^m c_0}{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots \cdot (2m)^2}, \quad m \geq 1$$

We let $r = 0$ in equation (2) and using these values of c_{2m}, we have the solution

$$Q_1(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

If we set $c_0 = 1$, we obtain a particular solution of equation (2.21). This solution define a function which denoted by $J_0(x)$ and is called the Bessel function of the first kind of order zero. i.e., $J_0(x)$ is a particular solution of equation (2.21) which is defined by:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

Writing out some few terms of the above series, we have:

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$

Therefore, since the indicial equation has equal roots. The general solution of equation (2.21) must be of the form:

$$Q = x\sum_{m=0}^{\infty} c_m^* x^m + J_0(x)lnx, \quad \text{for } 0 < x < R$$

Therefore, after some simplification, we obtain the second solution as:

Let $A_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. Then;

$$Q_2(x) = J_0(x)lnx + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}A_m}{2^{2m}(m!)^2} x^{2m}$$

Since the solution $Q_2(x)$ which is defined in the second solution is linearly independent of $J_0(x)$, we would write the general solution of the equation (2.21) as a general linear combination of $J_0(x)$ and $Q_2(x)$. However, this is unusual, instead, we must choose a certain special linear combination of $J_0(x)$, and $Q_2(x)$ and we call this special linear combination as the "second solution of the differential equation (2.21).

This special linear combination is defined as:

$$Y_0(x) = \frac{2}{\pi}[Q_2(x) + (\gamma - ln2)J_0(x)]$$

Where,

$$\gamma = \lim_{m\to\infty}(A_m - lnm) \approx 0.5772 \quad \text{Euler's constant}$$

Therefore,

$$Q(x) = c_1 J_0(x) + c_2 Y_0(x) \tag{2.24}$$

Where c_1 and c_2 are arbitrary constant.

Also, if we use equation (2.18), the solution will becomes:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\cdot n!\,\Gamma(n+1)} \cdot (x)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots \frac{(-1)^n}{(n!)^2}$$

This shows that the series we have derived above defined an important Bessel function $J_0(x)$.

Again for $v = 1$ using the same equation (2.18), implies:

$$J_1(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+1} n!\,\Gamma(n+2)} = x \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2) x^{2n}}{2^{2n+2} n!\,\Gamma(n+2)}$$

$$= \frac{x}{2} - \frac{1}{1!\,2!}\frac{x^3}{2^3} + \frac{1}{2!\,3!}\frac{x^5}{2^5} - \cdots + \frac{(-1)^n}{n!\,(n+1)!}\frac{x^{2n+1}}{2^{2n+1}} + \cdots$$

The relation between the above series can be summarize as follows,

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

The roots of these series $J_0(x) = 0$ and $J_1(x) = 0$ can be obtained by equalizing them to zero. That is, by using Frobenius series (power series expansion) and strum theory. Base on the fact that each equation has infinitely many real roots. Since the different between these roots are getting bigger, the results converging to the number π. For such a reason the function $J_0(x)$ and $J_1(x)$ are called periodic functions $J_v(x)$ and $J_{-v}(x)$ are linearly independent. If $v = m$ is an integer, then

$$\Gamma(m) = (m-1)!$$

$$\Gamma(m+v+1) = (m+v)!$$

And the function $J_n(x)$ can be re-written in the form:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(m+v)!} \cdot \left(\frac{x}{2}\right)^{2n+m}$$

Re-written equation (2.19), starting from (n+1)-th term, we obtained the following equation as follows:

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-m+n+1)} \cdot \left(\frac{x}{2}\right)^{2n-m} = J_0(x) \qquad = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-m+n+1)} \cdot \left(\frac{x}{2}\right)^{2n-m}$$

$$= \frac{(-1)^m}{m!\,\Gamma(-m+m+1)} \left(\frac{x}{2}\right)^{2m-m} + \frac{(-1)^{m+1}}{(m+1)!\,\Gamma(-m+m+2)} \left(\frac{x}{2}\right)^{-m+2m+2}$$

$$+ \cdots = (-1)^m \left[\frac{\left(\frac{x}{2}\right)^m}{0!\,m!} - \frac{\left(\frac{x}{2}\right)^{m+2}}{1!\,(m+1)!} + \frac{\left(\frac{x}{2}\right)^{m+4}}{2!\,(m+2)!} - \cdots \right] = (-1)^m J_m(x)$$

Therefore,

$$J_{-m}(x) = (-1)^m J_m(x)$$

As we can see, $J_m(x)$ and $J_{-m}(x)$ are linearly dependent when n is an integer.

Indeed,

$$Q(x) = C_1 J_v(x) + C_2 J_{-v}(x) = [C_1 + (-1)^v C_2] J_v(x) = C J_v(x) \text{ for } v = n \text{ integer.}$$

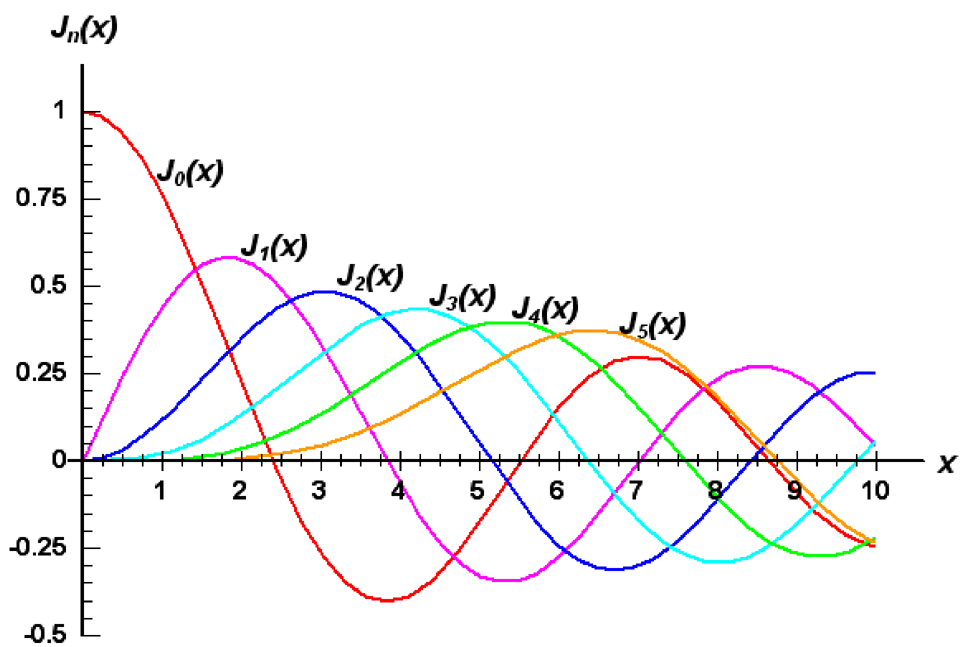

Figure 2.1: Bessel Function of the First Kind

2.2.2 Bessel Function of the First Kind for m Equal to Semi-integers

The confinement at equation (2.20) can be considered less strong when we prove that J_v and J_{-v} are independent when v is equal to semi-integer. For such values of v, the equation can be expressed as a combination of algebraic and trigonometric functions.

Now, consider $J_{\frac{1}{2}}$, from equation (2.18) we obtained:

$$J_{\frac{1}{2}}(x) = \sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+1/2}n!\Gamma(n+3/2)} = \frac{\sqrt{2}}{\sqrt{x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}n!\Gamma(n+3/2)} \tag{2.25}$$

To obtain a solution of the above equation we need to simplify the denominator. Firstly, the gamma function can be written as:

$$\Gamma(n+3/2) = (n+1/2)\cdot(n-1/2)\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)$$

Since

$$\Gamma(1/2) = \sqrt{\pi}$$

Thus:

$$\Gamma(n+3/2) = \frac{1}{2^{n+1}}(2n+1)\cdot(2n-1)\cdot\cdots\cdot 3\cdot 1\cdot\sqrt{\Pi} \tag{2.26}$$

From the denominator of equation (2.25) we can also have:

$$2^{2n+1}n! = 2\cdot 2^n\cdot 2^n\cdot n\cdot(n-1)\cdot\cdots\cdot 2\cdot 1 = 2^{2n+1}\cdot(2n)\cdot(2n-2)\cdot\cdots\cdot 4\cdot 2 \tag{2.27}$$

By putting equation (2.26) and (2.27) into (2.25), we obtained:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

The expression inside the bracket is McLaurin Series for $\sin(x)$. Thus, we have:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \tag{2.28}$$

Similarly,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \tag{2.29}$$

From equation (2.28) and (2.29) we see that $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ are independent functions. Also, by using the recurrence relations, we can find the Bessel function for any index of the form $n + {}^{1}/_{2}$, where n is an integer, and prove that for all integer n the following formulae holds:

$$J_{n+\frac{1}{2}}(x) = \frac{(-1)^n (2x)^{n+\frac{1}{2}}}{\sqrt{\Pi}} \frac{d^n}{(dx^2)^n}\left(\frac{\sin(x)}{x}\right) \tag{2.30}$$

$$J_{-n+\frac{1}{2}}(x) = \frac{(-1)^n (2x)^{n+\frac{1}{2}}}{\sqrt{\Pi}} \frac{d^n}{(dx^2)^n}\left(\frac{\cos(x)}{x}\right) \tag{2.31}$$

For the modified Bessel function, we use the same method and we can have:

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$$

And

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}e^{-x}$$

We can also use

$$Q(x) = C_1 J_{{}^{1}/_{2}}(x) + C_2 J_{-{}^{1}/_{2}}(x)$$

As the general solution for Bessel's differential equation with $v = {}^{1}/_{2}$. Without a doubt, all Bessel function with v equal to a half-integer, could be expressed in terms of elementary algebraic and trigonometric functions, and for these values of v, J_v will always be independent of J_{-v} . Sometimes we can called Bessel functions for semi- integer values of v as spherical Bessel functions. Thus, we can re-write the general solution (2.20), as:

$$Q(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x) \,, \qquad v \geq 0, \neq k \,,\, k = 0,1,2,3,\cdots \tag{2.32}$$

And it converges for all real $x > 0$.

2.3 Modified Bessel Function (Cylindrical Functions of a Pure Imaginary Arguments)

Modified Bessel functions are solutions of the modified Bessel's differential equation.

Now, consider the Bessel's differential equation:

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dQ}{dx}\right) - \left(1 + \frac{v^2}{x^2}\right)Q = 0 \tag{2.33}$$

This equation will shows up if we make a simple transformation $x \to ix$ because we have to observe not only asymptotic at $x \to 0$, but also asymptotic at $x \to \infty$.

$$(ix)^2\left(\frac{-d^2Q}{dx^2}\right) + \frac{(ix)}{i}\left(\frac{dQ}{dx}\right) + ((ix)^2 - v^2)Q = 0$$

Implies:

$$-x^2\left(\frac{-d^2Q}{dx^2}\right) + x\left(\frac{dQ}{dx}\right) + (-x^2 - v^2)Q = 0$$

$$x^2\left(\frac{d^2Q}{dx^2}\right) + x\left(\frac{dQ}{dx}\right) - (x^2 + v^2)Q = 0$$

$$x^2Q'' + xQ' - (x^2 + v^2) = 0$$

Which is called the modified Bessel function. And has a regular singular point at $x = 0$. We also use Frobenius method to obtain a solution of Modified Bessel function. One solution $I_v(x)$ of equation (2.33) is defined by the series

$$I_v(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+v+1)} \cdot \left(\frac{x}{2}\right)^{2n} \tag{2.34}$$

At $x \to 0$

$$I_v \approx \frac{x^v}{2^v \Gamma(v+1)} \tag{2.35}$$

At $x \to \infty$

$$I_v(x) = \sqrt{\frac{2}{\pi x}} e^x \tag{2.36}$$

$I_v(x)$ is the real function of real argument. They are related with Bessel functions of the first kind by:

$$I_v(x) = e^{-\frac{\pi}{2} v i} J_v(ix) \tag{2.37}$$

In particular,

$$I_m(x) = -i^m J_m(ix) \tag{2.38}$$

Modified Bessel functions of second kind are defined by the relation

$$K_v(x) = \frac{\pi i}{2} e^{\frac{\pi}{2} v i} H_v^{(1)}(ix) \tag{2.39}$$

$$K_v(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad , \quad x \to \infty \tag{2.40}$$

They have asymptotic at both $I_v(x)$ and $K_v(x)$.

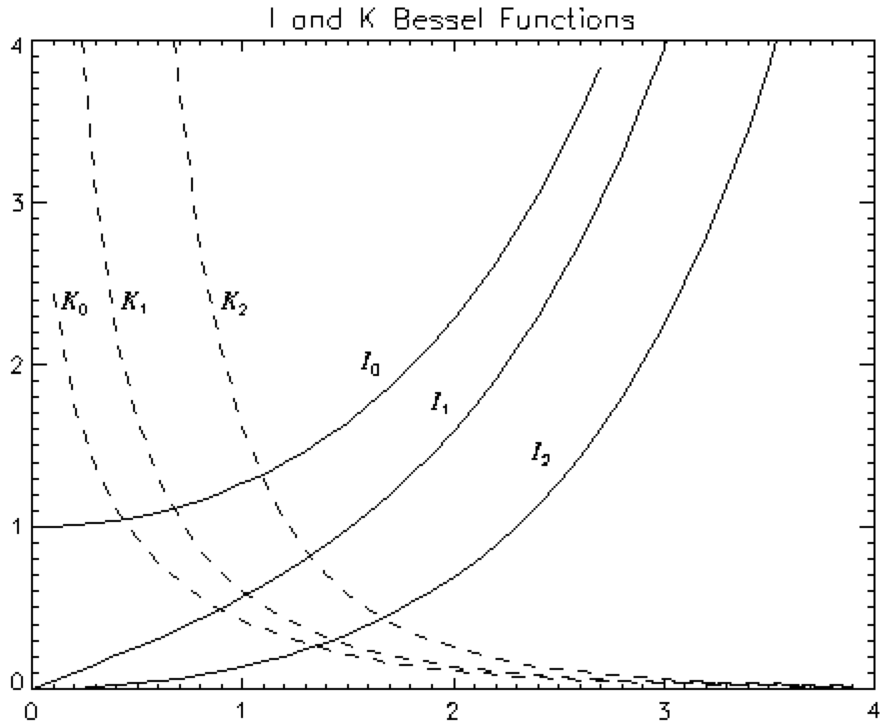

Figure 2.2: Modified Bessel Function

2.4 Cylindrical Function of the Second Kind (Neumann or Weber's Function)

At whatever point v is not an integer, a fundamental system for a solution of Bessel's differential equation for functions of order v is formed by a pair $J_v(x)$ and $J_{-v}(x)$. In case $v = m$ (m an integer), the functions $J_m(x)$ and $J_{-m}(x)$ are linearly dependent, so that $J_{-m}(x)$ is not a second solution of the equation. The second solution can be obtained as a combination of $J_v(x)$ and $J_{-v}(x)$ as follows:

$$Y_v(x) = \frac{J_v(x)\cos(\pi v) - J_{-v}(x)}{\sin(\pi v)} \tag{2.41}$$

This is weber's function (Neumann function) which satisfy Bessel's differential equation because it is linear combination of $J_v(x)$ and $J_{-v}(x)$. When $v = m$, the second solution is given by:

$$Y_m(x) = \lim_{v \to m} \frac{J_v(x)\cos(\pi v) - J_{-v}(x)}{\sin(\pi v)} \tag{2.42}$$

Also the general form of equation (2.42) above has been given by Neumann as:

$$Y_m(x) = J_m(x)\{\log x - S_m\} - \sum_{n=0}^{m-1} \frac{2^{(m-n-1)} m!\, J_n(x)}{(m-n)!\, n!\, Z^{(m-n)}} + \sum_{n=0}^{m-1} \frac{(-1)^{(n-1)}(m+2n) J_{m+2n}(x)}{n(m+n)} \tag{2.43}$$

Where $S_m = 1 + {}^1\!/_2 + {}^1\!/_3 + \cdots + {}^1\!/_m$, $\quad S_0 = 0$.

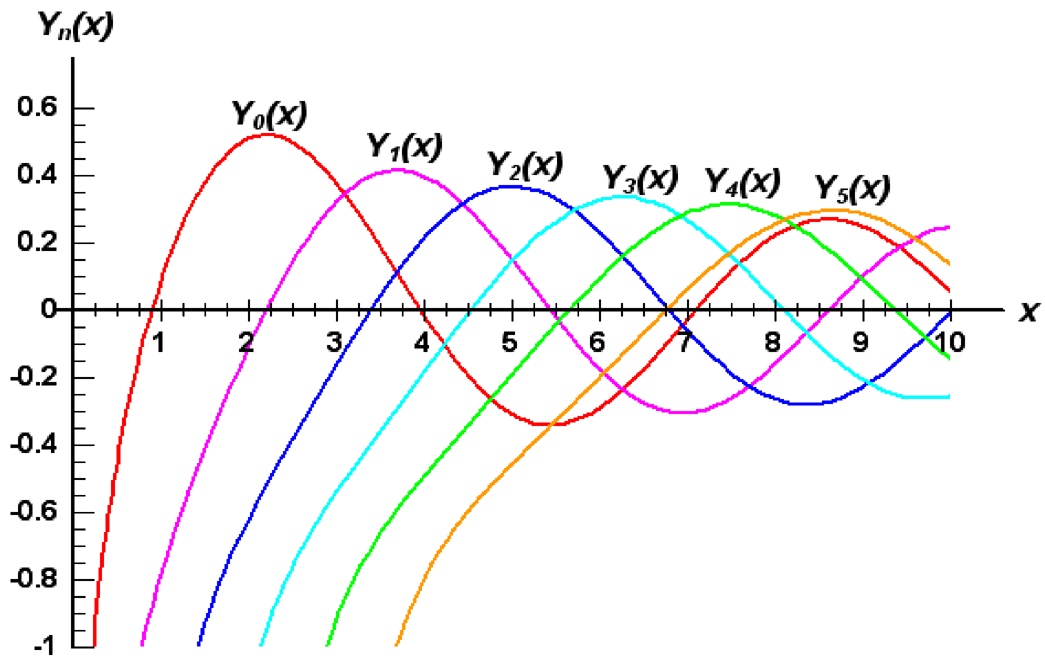

Figure 2.3: Bessel Function of the Second Kind.

2.5 Cylindrical Function of the Third Kind (Hankel Function)

Hankel function is a combination of Bessel's functions of the first kind ($J_v(x)$) and second kind ($Y_v(x)$). That is

$$H_v^{(1)}(x) = J_v(x) + jY_v(x) \tag{2.44}$$

$$H_v^{(2)}(x) = J_v(x) - jY_v(x)$$

Where $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$ represents Hankel functions of the first kind and second kind, respectively. Since the functions of the third kind, are linear combination of:

$$H_v^{(1)}(x) = J_v(x) + jY_v(x) = j\,\frac{e^{-v\pi j}J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

And

$$H_v^{(2)}(x) = J_v(x) - jY_v(x) = -j\,\frac{e^{v\pi j}J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

(2.45)

So that, as $x \to \infty$ they have the following asymptotic;

$$H_v^{(1)}(x) \to \sqrt{\frac{2}{\pi x}}\,e^{j\left(x-\frac{\pi v}{2}-\frac{\pi}{4}\right)}$$

And

$$H_v^{(2)}(x) \to \sqrt{\frac{2}{\pi x}}\,e^{-j\left(x-\frac{\pi v}{2}-\frac{\pi}{4}\right)}$$

Apparently,

$$H_v^{-(1)}(x) = H_v^{(2)}(x)$$

The above functions are linearly independent solutions of Bessel equations. Whereby v represents the degree of the Hankel functions of the first and second kind. When we add $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$ side by side, we obtained:

$$H_v^{(1)}(x) + H_v^{(2)}(x) = 2J_v(x)$$

$$J_v(x) = {}^{1}\!/_{2}\left[H_v^{(1)}(x) + H_v^{(2)}(x)\right] \qquad (2.46)$$

Again, when we subtract the same equation, we can have,

$$H_v^{(1)}(x) - H_v^{(2)}(x) = 2jY_v(x)$$

$$Y_v(x) = {}^1\!/_{2j}\left[H_v^{(1)}(x) - H_v^{(2)}(x)\right] \tag{2.47}$$

Therefore, the first and second kind Hankel functions are multiplied by $e^{jv\pi}$ and $e^{-jv\Pi}$ respectively, and then adding them side by side, we obtained:

$$e^{jv\pi}H_v^{(1)}(x) + e^{-jv\Pi}H_v^{(2)}(x) = 2J_{-v}(x)$$

$$J_{-v}(x) = {}^1\!/_2\left[e^{jv\Pi}H_v^{(1)}(x) + e^{-jv\Pi}H_v^{(2)}(x)\right] \tag{2.48}$$

2.6 Relations Between the Three Kinds of Bessel Functions

The relations express each of the function in terms of functions of other two kinds:

$$J_v(x) = \frac{H_v^{(1)}(x)+H_v^{(2)}(x)}{2} = \frac{Y_{-v}(x)+Y_v(x)\cos(\pi v)}{\sin(\pi v)} \tag{2.49}$$

$$J_{-v}(x) = \frac{e^{j\pi v}H_v^{(1)}(x)+e^{-j\pi v}H_v^{(2)}(x)}{2} = \frac{Y_{-v}(x)\cos(\pi v)-Y_v(x)}{\sin(\pi v)} \tag{2.50}$$

$$Y_v(x) = \frac{J_v(x)\cos(\pi v)-J_{-v}(x)}{\sin(\pi v)} = \frac{H_v^{(1)}-H_v^{(2)}(x)}{2j} \tag{2.51}$$

$$Y_{-v}(x) = \frac{J_v(x)-J_{-v}(x)\cos(\pi v)}{\sin(\pi v)} = \frac{e^{j\pi v}H_v^{(1)}-e^{-j\pi v}H_v^{(2)}(x)}{2j} \tag{2.52}$$

$$H_v^{(1)}(x) = \frac{J_{-v}(x)-e^{-j\pi v}J_v(x)}{j\sin(\pi v)} = \frac{Y_{-v}(x)-e^{-j\pi v}Y_v(x)}{\sin(\pi v)} \tag{2.53}$$

$$H_v^{(2)}(x) = \frac{e^{j\pi v}J_v(x)-J_{-v}(x)}{j\sin(\pi v)} = \frac{Y_{-v}(x)-e^{j\pi v}Y_v(x)}{\sin(\pi v)} \tag{2.54}$$

2.7 Formulae of Differentiation and Recurrence Relations

Let us divide equation (2.18) by x^v, we have:

$$\frac{J_v(x)}{x^v} = \frac{1}{2^v}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!\,\Gamma(v+n+1)}$$

After differentiation with respect to x, we obtain:

$$\frac{d}{dx}\frac{J_v(x)}{x^v} = \frac{1}{2^v}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(v+n+1)}\left(\frac{x}{2}\right)^{2n-1} = -\frac{J_{v+1}}{x^v}$$

Which implies:

$$\frac{1}{x}\frac{d}{dx}\frac{J_v(x)}{x^v} = \frac{-J_{v+1}}{x^{v+1}}(x) \tag{2.55}$$

Similarly,

$$\frac{1}{x}\frac{d}{dx}[x^v J_v(x)] = x^{v-1}J_{v-1}(x) \tag{2.56}$$

After differentiating equation (2.55) and (2.56), we can obtain:

$$\frac{d}{dx}J_v(x) = -J_{v+1}(x) + \frac{vJ_v}{x}(x) \tag{2.57}$$

Similarly,

$$\frac{d}{dx}J_v(x) = J_{v-1}(x) - \frac{vJ_v}{x}(x) \tag{2.58}$$

Which implies the following recurrence formulae:

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2vJ_v}{v} \tag{2.59}$$

And

$$J_{v-1}(x) - J_{v+1}(x) = 2\frac{d}{dx}J_v(x) \tag{2.60}$$

In equation (2.18), we substitute x with kx, and obtain:

$$J_v(kx) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(v+n+1)}\left(\frac{kx}{2}\right)^{v+2n}$$

Also multiplying the above equation by x^v, we have:

$$x^v J_v(kx) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n+1)} \left(\frac{kx}{2}\right)^{v+2n} x^v = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n+1)} \left(\frac{k}{2}\right)^{v+2n} x^{2(v+n)}$$

And then differentiating side by side as follows;

$$\frac{d}{dx}(x^v J_v(kx)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n+1)} \left(\frac{k}{2}\right)^{v+2n} \cdot 2(n+v) \cdot x^{2(n+v)-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+v)}{n!\,(n+v)\Gamma(v+n)} \left(\frac{k}{2}\right)^{v+2n} . x^{2n+2v-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n)} \left(\frac{kx}{2}\right)^{v+2n-1} . x^v k$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma((v-)n+1)} \left(\frac{kx}{2}\right)^{(v-1)+2n} . x^v k$$

Therefore, we have:

$$\frac{d}{dx}[x^v J_v(kx)] = kx^v J_{v-1}(kx) \tag{2.61}$$

Similarly,

$$\frac{d}{dx}[x^{-v} J_v(kx)] = -kx^{-v} J_{v+1}(kx) \tag{2.62}$$

Differentiating equation (61) and (62), we get:

$$\frac{d}{dx}[J_v(kx)] = kJ_{v-1}(kx) - \frac{v}{x} J_v(x) \tag{2.63}$$

And

$$\frac{d}{dx}[J_v(kx)] = -kJ_{v+1}(kx) + \frac{v}{x} J_v(kx) \tag{2.64}$$

So, we can replace $J_v(x)$ in the above formulae by any of the functions; $Y_v(x)$, $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$. Again, if we differentiate equation (2.55) and (2.56), we can have;

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k [x^v J_v(x)] = x^{v-k} J_{v-k}(x) \tag{2.65}$$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k [x^{-v}J_v(x)] = (-1)^k x^{-v-k} J_{v-k}(x) \quad (2.66)$$

For the modified functions, we can have the following relations of differentiation, that are obtained as a result of the change of the variable (argument) x by ix and the representation of the functions $J_v(x)$ and $H_v^{(1)}(x)$ through the functions $I_v(x)$ and $L_v(x)$:

$$\frac{d}{dx} I_v(x) = \frac{1}{2}[I_{v-1}(x) + I_{v+1}(x)] \quad (2.67)$$

$$\frac{d}{dx} L_v(x) = -\frac{1}{2}[I_{v-1}(x) + I_{v+1}(x)] \quad (2.68)$$

The corresponding recurrence relations has the form:

$$I_{v-1}(x) - I_{v+1}(x) = \frac{2v}{x} I_v(x) \quad (2.69)$$

$$L_{v-1}(x) - L_{v+1}(x) = -\frac{2v}{x} L_v(x) \quad (2.70)$$

2.8 Wronskian Determinant

The wronskian determinant must be non-zero since $J_v(x)$ and $J_{-v}(x)$ are linearly independent solutions of the Bessel equation.

Let $y_1 = J_v(x)$ and $y_2 = J_{-v}(x)$, then the wronskian can be obtain as follows;

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2'' \end{vmatrix} = W\big(J_v(x), J_{-v}(x)\big)$$

$$= \begin{vmatrix} J_v(x) & J_{-v}(x) \\ J_v'(x) & J_{-v}'(x) \end{vmatrix} = J_v(x)J_{-v}'(x) - J_{-v}(x)J_v'(x) \quad (2.71)$$

Substituting equation (2.71) into equation (2.9), we obtain:

$$J_{-v}''(x) + \frac{1}{x}J_{-v}'(x) + \left(1 - \frac{v^2}{x^2}\right)J_{-v}(x) = 0 \quad (2.72)$$

$$J_v''(x) + \frac{1}{x}J_v'(x) + \left(1 - \frac{v^2}{x^2}\right)J_v(x) = 0 \quad (2.73)$$

If we multiply (2.72) and (2.73) by $J_v(x)$ and $J_{-v}(x)$ respectively, we obtain:

$$J''_{-v}(x)J_v(x) + \frac{1}{x}J'_{-v}(x)J_v(x) + \left(1 - \frac{v^2}{x^2}\right)J_{-v}(x)J_v(x) = 0$$

And

$$J''_v(x)J_{-v}(x) + \frac{1}{x}J'_v(x)J_{-v}(x) + \left(1 - \frac{v^2}{x^2}\right)J_v(x)J_{-v}(x) = 0$$

If we subtract the above equation side by side, we obtain:

$$J_v(x)J''_{-v}(x) - J_{-v}(x)J''_v(x) + \frac{1}{x}[J_v(x)J'_{-v}(x) - J_{-v}(x)J'_v(x)] = 0$$

Implies that;

$$\frac{d}{dx}[J_v(x)J'_{-v}(x) - J_{-v}(x)J'_v(x)] + \frac{1}{x}[J_v(x)J'_{-v}(x) - J_{-v}(x)J'_v(x)] = 0 \qquad (2.74)$$

By substituting

$$W = J_v(x)J'_{-v}(x) - J_{-v}(x)J'_v(x)$$

This implies;

$$\frac{dW}{dx} + \frac{W}{x} = 0$$

By using separation of variables, we get

$$W(x) = \frac{k(v)}{x} \qquad (2.75)$$

Suppose that the above equation has a non-integer index. Now, we should obtain the Wronskian as follows:

$$W(J_v(x), J_{-v}(x)) = \frac{k(v)}{x} \qquad (2.76)$$

$$k(v) = x[J_v(x)J'_{-v}(x) - J_{-v}(x)J'_v(x)] \qquad (2.77)$$

The value of the constant $k(v)$ can easily be obtained, if we pass to the limit as $x \to 0$ in equation (2.71) and using the expansions of the Bessel functions obtained in section [2.2]. Notice that, if v is non-integer index, and by using equation (2.18) and (2.19), we have:

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n+1)} \left(\frac{x}{2}\right)^{v+2n} =$$

$$\left(\frac{x}{2}\right)^{v} \frac{1}{\Gamma(v+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+n+1)} \left(\frac{x}{2}\right)^{v+2n}$$

And

$$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-v+n+1)} \left(\frac{x}{2}\right)^{-v+2n} =$$

$$\left(\frac{x}{2}\right)^{-v} \frac{1}{\Gamma(-v+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,\Gamma(-v+n+1)} \left(\frac{x}{2}\right)^{-v+2n}$$

This implies,

$$J_v(x) = \left(\frac{x}{2}\right)^{v} \frac{1}{\Gamma(v+1)} \left(1 + O(x^2)\right) \tag{2.78}$$

$$J_v'(x) = \left(\frac{x}{2}\right)^{v-1} \frac{1}{2\Gamma(v)} \left(1 + O(x^2)\right) \tag{2.79}$$

Similarly,

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \frac{1}{\Gamma(-v+1)} \left(1 + O(x^2)\right) \tag{2.80}$$

$$J_{-v}'(x) = \left(\frac{x}{2}\right)^{-v-1} \frac{1}{2\Gamma(-v)} \left(1 + O(x^2)\right) \tag{2.81}$$

As $x \to 0$, and $O(x^2)$ denotes a quantity, whose ratio to x^2 is bounded as $x \to 0$.

Substituting equations (2.78), (2.79), (2.80) and (2.81) into equation (2.77), we obtain:

$$k(v) = x\left[\left(\frac{x}{2}\right)^{v} \frac{1}{\Gamma(v+1)} \left(1 + O(x^2)\right) \left(\frac{x}{2}\right)^{-v-1} \frac{1}{2\Gamma(v)} \left(1 + O(x^2)\right)\right] +$$

$$x\left[-\left(\frac{x}{2}\right)^{-v} \frac{1}{\Gamma(-v+1)} \left(1 + O(x^2)\right) \left(\frac{x}{2}\right)^{v-1} \frac{1}{2\Gamma(v)} \left(1 + O(x^2)\right)\right]$$

As $x \to 0$, $O(x^2) = 0$, therefore

$$k(v) = \left[\frac{1}{\Gamma(v+1)}\frac{1}{\Gamma(v)} - \frac{1}{\Gamma(-v+1)}\frac{1}{\Gamma(v)}\right] \qquad (2.82)$$

By using the formula of gamma function in (2.82), which is

$$\Gamma(v)\Gamma(-v+1) = \frac{\pi}{\sin(\pi v)}$$

This implies:

$$k(v) = -\frac{\sin(v\pi)}{\pi} - \frac{\sin(v\pi)}{\pi} = -2\frac{\sin(v\pi)}{\pi} \qquad (2.83)$$

Substituting equation (2.83) into equation (2.76), we obtain:

$$W[J_v(x), J_{-v}(x)] = -2\frac{\sin(v\pi)}{\pi x} \qquad (2.84)$$

$\sin(v\pi) \neq 0$, since v is not an integer.

Therefore,

$$W[J_v(x), J_{-v}(x)] \neq 0 \qquad (2.85)$$

Therefore, the functions $J_v(x)$ and $J_{-v}(x)$ are linearly independent solutions of the Bessel equation.

2.9 Integral Representation

Firstly, we have to consider the integral:

$$A_s(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - is\theta} d\theta \qquad (2.86)$$

To simplify this, we have to use the Taylor expansion of the exponent:

$$e^{ix\sin\theta} = \sum_{m=0}^{\infty}\frac{1}{m!}(ix\sin(\theta))^m = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{x}{2}\right)^m (e^{i\theta} - e^{-i\theta})^m \qquad (2.87)$$

Note that, the integral:

$$I_{m,\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{i\theta - i\theta}\right)^m e^{-is\theta} d\theta \qquad \text{if} \quad m < 0 \qquad (2.88)$$

Then, we represent m$= s + p$. The integrand in the equation (2.88) can be written in the form:

$$\frac{1}{2\pi}(e^{i\theta} - e^{-i\theta})^{s+p}e^{-is\theta} = (1 - e^{-2i\theta})^s(e^{i\theta} - e^{-i\theta})^p$$

Suppose p is odd $(p = 2q + 1)$. All the terms in the first bracket are even powers of $e^{-i\theta}$, while all the terms in the second bracket are odd powers $(+ \; or \; -)$ on $e^{-i\theta}$. Therefore the integral is zero, and we can let $p = 2q$. We obtained the following:

$$A_s(x) = \left(\frac{x}{2}\right)^s \sum_{s=0}^{\infty} \frac{1}{(s+2q)!}\left(\frac{x}{2}\right)^q I_{q,s} \tag{2.89}$$

Where

$$I_{q,s} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{i\theta} - e^{-i\theta}\right)^{s+2q} e^{-is\theta} d\theta \tag{2.90}$$

To evaluate $I_{q,s}$, we have to use the binomial expansion in the bracket. In this expansion, we are only interested in the single term proportional to $e^{is\theta}$. All the other terms after the multiplication to equation (2.90) and integration over θ are cancelled.

Hence,

$$\left(e^{i\theta} - e^{-i\theta}\right)^{s+2q} \approx \frac{(s + 2q)!}{q!\,(s + q)!}\left(e^{i\theta}\right)^{s+q}\left(-e^{-i\theta}\right)^q = \frac{(-1)^q(s + 2q)!}{q!\,(s + q)!}e^{is\theta}$$

And

$$I_{s,q} = \frac{(-1)^q(s+2q)!}{q!(s+q)!} \tag{2.91}$$

By substituting (2.91) into equation (2.89), we get:

$$A_s(x) = \left(\frac{x}{2}\right)^s \sum_{q=0}^{\infty} \frac{(-1)^q}{q!\,(s + q)!}\left(\frac{x}{2}\right)^q = J_s(x)$$

We can now obtain the integral representation for $J_s(x)$:

$$J_s(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - is\theta} d\theta \tag{2.92}$$

The result of the above equation is correct for $+s$.

Note that;

$$J_s(-x) = (-1)^s J_s(x) \tag{2.93}$$

Bessel functions of even order are even function on x, while Bessel functions of odd order are odd. Now, we can obtain $A_s(x)$ at $-s$. Let us simultaneously change the signs on x and s.

$$A_{-s}(-x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta + is\theta} d\theta$$

Replacing $\theta \to -\theta$, we restore the previous result.

Therefore,

$$A_{-s}(-x) = A_s(x) = J_s(x)$$

$$A_{-s}(x) = J_s(-x) = (-1)^s J_s(x) \tag{2.94}$$

Finally, for all integrals on,

$$A_s(x) = (-1)^s J_s(x)$$

Note that $J_s(x)$ is real. Then equation (7) can be re-written as:

$$J_s(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - s\theta)\, d\theta \tag{2.95}$$

Now, taking a look at $e^{ix\sin\theta}$. This is a periodic function that can be expanded in Fourier series.

Apparently,

$$e^{ix\sin\theta} = \sum_{s=-\infty}^{\infty} J_s(x) e^{is\theta} = J_0(x) + \sum_{s=1}^{\infty} J_s(x)(e^{is\theta} + (-1)^s e^{-is\theta}) \tag{2.96}$$

After separating the real and imaginary parts, we obtain:

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{q=1}^{\infty} J_{2q}(x)\cos(2q\theta)$$

$$\sin(x\sin\theta) = 2\sum_{q=-\infty}^{\infty} J_{2q+1}(x)\sin((2q+1)\theta) \tag{2.97}$$

By introducing $ce^{i\theta}$, we can transform (2.96) to the following:

$$e^{\frac{x}{2}(c-\frac{1}{c})} = \sum_{s=-\infty}^{\infty} J_s(x)c^s \tag{2.98}$$

This means that $F(x,c) = e^{\frac{x}{2}(c-\frac{1}{c})}$ is a "generating function" for all Bessel functions of integral orders.

2.10 Asymptotic Behavior at $x \to \infty$

To get the asymptotic behavior of the Bessel functions at $x \to \infty$, we can use the device similar to the one used to obtained the Sterling formula. We present an integral;

$$J_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta - im\theta}d\theta$$

In the form:

$$J_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\Phi(x,\theta)}d\theta \tag{2.99}$$

$$\Phi(x,\theta) = x\sin\theta - m\theta \tag{2.100}$$

If $x \to \infty$, the integral is the fast oscillation function everywhere except the two points where

$\frac{d\Phi}{d\theta} = 0$. These points are defined by the equation:

$x\cos\theta = m$ at $x \to \infty$

$\cos\theta \to 0$ at $\theta \to \pm\frac{\pi}{2}$.

The contributions of points $\theta^{\pm} = \pm\frac{\pi}{2}$ give the complex conjugated results. Therefore, it is enough to study the neighborhood of the point $\theta = \frac{\pi}{2}$. Now, let us introduce $\theta = \frac{\pi}{2} + \tau$ for small τ,

$$\Phi(x,\theta) \approx x - \frac{m\pi}{2} - \frac{1}{2}x\tau^2 \tag{2.101}$$

The integral (2.99) can be replace approximately by the following integral:

$$J_m(x) = \frac{1}{\pi} \Re e^{i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)} \int_{-\infty}^{\infty} e^{\frac{-ix}{2}\tau^2}\, d\tau$$

Where $\Re = real\ parts.$

Let us make the transformations:

$\tau = \sqrt{\frac{2}{ix}} y, \frac{1}{\sqrt{i}} = e^{\frac{-\pi i}{4}}$

Then,

$$J_m(x) = \frac{\sqrt{2}}{\pi\sqrt{x}} \Re e^{i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)} \int_{\frac{-i\pi}{4}\cdot\infty}^{\frac{i\pi}{4}\cdot\infty} e^{-y^2}\, dy \qquad (2.102)$$

Integration is going in the complex plane on a straight line at an angle of 45^0 with respect to the real axis. As shown in the figure above. Therefore, the contour of integration can turned back to the real axis (to verify this, we have to use some elements of complex analysis. But this is true). In the other hand, the integral in equation (2.102) can be replaced by $\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$.

So, we have:

$$J_m(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \qquad (2.103)$$

We derived this expression only for integral m. In fact, we need to use a more sophisticated integral representation for $J_n(x)$ which is valid not only for integral,

$$J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad (2.104)$$

In particular,

$$J_{\frac{1}{2}}(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\right) \to \sqrt{\frac{2}{\pi x}} \sin x$$

This is unique Bessel function coinciding with its own asymptotic behavior.

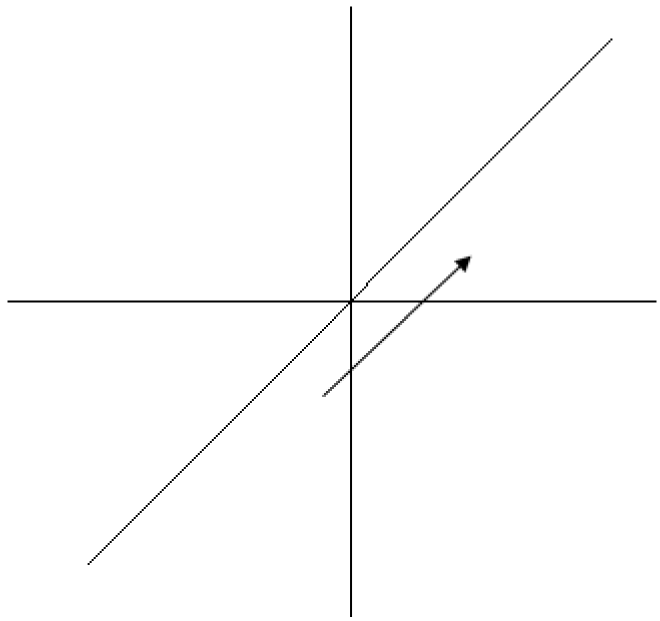

Figure 2.4: Contour of Integration.

2.11 Orthogonality and Fourier-Bessel Series

Let $J_n(x)$ be the Bessel function of index n. let a_M^n be its zeros, so that

$J_n(a_M^n) = 0$. Suppose that $0 < r < a$ is an interval on the real axis. Now, we consider the set of function $R_M^n(r) = J_n\left(\frac{r}{R} a_M^n\right)$. This is the set of functions against the weight r. In the other hands;

$$\int_0^a R_M^n(r) R_N^n(r) r dr = 0 \qquad \text{if} \quad M \neq N \tag{2.105}$$

To verify this fact, we first of all mention that,

$$R_M^n(r) = J_n(a_M^n) = 0 \tag{2.106}$$

We say that, these functions satisfies the following equations:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R_M^n}{\partial r}+\left(K_M-\frac{n^2}{r^2}\right)R_M^n=0 \quad , \qquad K_M=\frac{a_M}{a} \tag{2.107}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R_N^n}{\partial r}+\left(K_N-\frac{n^2}{r^2}\right)R_N^n=0 \quad , \qquad K_N=\frac{a_N}{a} \tag{2.108}$$

When we multiply these equations by rR_N and rR_M respectively, and subtracting the results, we can get:

$$R_N^n\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R_M^n}{\partial r}-R_M^n\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R_N^n}{\partial r}=(K_N^2-K_N^2)rR_MR_N \tag{2.109}$$

Re-written the left hand side as:

$$\frac{\partial}{\partial r}r[R_NR_M]=(K_N^2-K_N^2)rR_MR_N \tag{2.110}$$

$$[R_NR_M]=R_N\frac{\partial}{\partial r}R_N-R_M\frac{\partial}{\partial r}R_M \tag{2.111}$$

$$[R_NR_M]\mathrm{I}_{r=a}=0 \tag{2.112}$$

Then if $K_N^2\neq K_N^2$, integral from 0 to a will leads to the condition of equation (2.105). Note that, we can replace the functions $R_M^n(r)$ by $\widetilde{R_M^n}(r)=J_n\left(\frac{r}{a}b_M^n\right)$. They will satisfy the condition $\widetilde{R_M'}(a)=0$. Then, the Wronskian $[R_NR_M]=0$ at $r=a$.

Therefore, the function $\widetilde{R_M^n}(r)$ satisfies the orthogonality conditions (2.105).

Suppose that $f(r)$, $0<r<a$, is some real or complex function defined on the interval (0,r). We can represent this function as a linear combination of $R_M^n(r)$.

Let

$$f(r)=\sum_{M=1}^{\infty}f_MR_M^n(r) \tag{2.113}$$

Multiplying this to $rR_N^n(r)$ and integrating we have;

$$f_M=\frac{1}{{\lambda^{n^2}}_M}\int_0^a f(r)r\,R_M^n(r)dr=\frac{1}{{\lambda^{n^2}}_M}\int_0^a f(r)r\,J_M^n\left(\frac{r}{a}a_M\right)dr$$

Here,

$$\lambda^2{}_M = \int_0^a r\, R_M^5(r)dr = \frac{1}{2}\frac{a^2}{a^2{}_M}\int_0^{a_M} xJ^2{}_x(x)dx = \frac{1}{2}a^2J^2{}_{n\pm 1}(a_M) \qquad (2.114)$$

Remarks, all functions $R_M^n(r) \to \left(\frac{r}{2a}a_M\right)^n$ at $r \to 0$. It means that the series (2.106) converges if the function $f(r)$ behaves at $r \to a_n$

$$f(r) \to kr^n \qquad (2.115)$$

If the asymptotic of equation (2.108) holds, the conditions for the convergence of the series are similar to the corresponding conditions for the standard Fourier series. In particular, if $(a) = 0$, $|f'(r)| < k$, where k is some arbitrary constant, this series converges absolutely and uniformly on $0 < r < a$. A function $f(r,\theta)$ defined in the disk in the Bessel Fourier series. First of all, we present $f(r,\theta)$ as a Fourier series in angles.

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} f_m(r)e^{im\theta} \qquad (2.116)$$

$$f_m(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\, e^{-im\theta}d\theta \qquad (2.117)$$

What is asymptotic of $f_m(r)$ if $r \to 0$? We now go back to the Cartesian coordinates ($x = r\cos\theta$, $y = r\sin\theta$). Let $f_m(\theta)$ be written as follows:

$$f_m = f_0(\theta) + rf_1(\theta) + \frac{1}{2}r^2f_2(\theta) + \cdots + \frac{1}{r}R^{n-1}f_{n-1}(\theta) \qquad (2.118)$$

$$f_1(\theta) = f_x\cos\theta + f_y\sin\theta$$

All other $f_m(\theta)$ are trigonometric polynomials of order $m-1$.

So;

$$\int_0^{2\pi} f_c(\theta)e^{-im\theta}d\theta \qquad \text{If} \quad c < m$$

Hence $f_m(r) \to P_m r^m$ as $r \to 0$, with P_m some constant, and functions $f_m(r)$ are good for the expansion of series in the Fourier function of order m.

Finally,

$$f(r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{M=1}^{\infty} f_m M e^{im\theta} J_m\left(\frac{r}{a}a_M\right) \qquad (2.119)$$

$$f_{mM} = \frac{1}{2\pi\lambda_{mN}^2}\int_0^{2\pi} e^{-im\theta}d\theta \int_0^a rJ_m\left(\frac{r}{a}a_M\right)f(r,\theta)dr \tag{2.120}$$

In particular, if $f(x,y) = \delta(x-x_0)\delta(y-y_0) = r_0\delta(\theta-\theta_0)$ then,

$$f_{mM} = \frac{1}{2\pi\lambda_{mn}^2}r_0e^{-im\theta}J_m\left(\frac{r_0}{a}a_M\right) \tag{2.121}$$

Equation (2.112) is good and fast converging if $f(r,\theta)$ satisfies the Drichlet condition $f(a,\theta) = 0$. If this function satisfies the Weber or Neumann function $(f_r(a,\theta) = 0)$, we can use the following orthogonal functions:

$$1, J_m\left(\frac{r}{a}b_1\right), \cdots, J_m\left(\frac{r}{a}b_M\right), \cdots$$

2.12 Zeros of Bessel Functions

It is clear from equation (2.104) that is;

$$J_n(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

That the Bessel function $J_n(x)$ has an infinite amount of zeros for half axis $0 < x < \infty$. Let us represent these zeros as a_M^n, where $M = 1,2,3,\cdots,\infty$. From the equation above, we can conclude that the distance between two neighboring zeros will tends to π.

$$a_{M+1}^n - a_M^n \to \pi \quad \text{as} \quad M \to \infty \tag{2.122}$$

The first five Bessel functions of integral order are plotted on figure2.5. The first five of each zeros are represented in table1. Note that;

$$a_5^5 - a_4^5 = 3.2377$$

While:

$$a_5^0 - a_4^0 = 3.1394$$

As we can see both values are close to π. The derivatives of Bessel functions have the following asymptotic behavior:

$$J_n'(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{2.123}$$

The derivatives of $J_n(x)$ also have an infinite amount of zeros b_M^n.

Also'

$$b_{M+1}^n - b_M^n \rightarrow \pi \quad \text{if} \quad M \rightarrow \infty \tag{2.124}$$

Let us re-write the Bessel equation as follows:

$$\frac{d}{dx} x J_n' + x J_n - \frac{n^2}{x} J_n = 0 \tag{2.125}$$

Multiplying the above equation by $2xJ'$, we have;

$$\frac{d}{dx}(x^2 J_2'^2 - n^2 J_n^2) + 2x^2 JJ' = 0$$

$$2x^2 JJ' = x^2 \frac{d}{dx} J^2 = \frac{d}{dx} x^2 J^2 - 2xJ^2$$

Finally,

$$2xJ_n^2 = \frac{d}{dx}[n^2 J_n'^2 + (x^2 - n^2) J_n^2] \tag{2.126}$$

Integrating equation (2.126) w.r.t. x from 0 to a_m, we obtained:

$$\int_0^{a_m} x J_n^2(x) dx = \frac{1}{2} a_M^2 J_n'^2(a_M) = \frac{1}{2} a_M^2 J_{n\pm1}^2(a_M) \tag{2.127}$$

The last part of equation (2.126) follows from the following:

$$J_{n-1}(x) = \frac{n}{x} J_n + J_n'(x) \tag{2.128}$$

And

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n \tag{2.129}$$

In virtue of equation (2.128), $J'_n(a_M) = J_{n-1}(a_M)$. In virtue of equation (2.129), that is $J_{n+1}(a_M) = -J_{n-1}(a_M)$.

Thus:

$$J_{n+1}^2(a_M) = J_{n-1}^2(a_M) = J_n'^2(a_M) \tag{9(2.130)}$$

From table1, we can see that the first zero a_0^m grows with m. The following statement is correct: the number of zeros of $J_n(x)$ on the interval

$$0 < x < \left(c + \frac{n}{x} + \frac{1}{4}\right)\pi \tag{2.131}$$

Is exactly c. Putting $c = 1$ into equation (2.131), we get:

$$a_1^n < \left(\frac{3}{4} + \frac{n}{2}\right)\pi \tag{2.132}$$

For $n = 5$, we get $a_1^5 < 10.35$. In reality $a_1^5 < 8.7715$. We can see that this estimation is rather accurate.

Table 2.1: Roots of Bessel Function

Zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2.2: Roots of the Derivative of Bessel Function

Zero	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

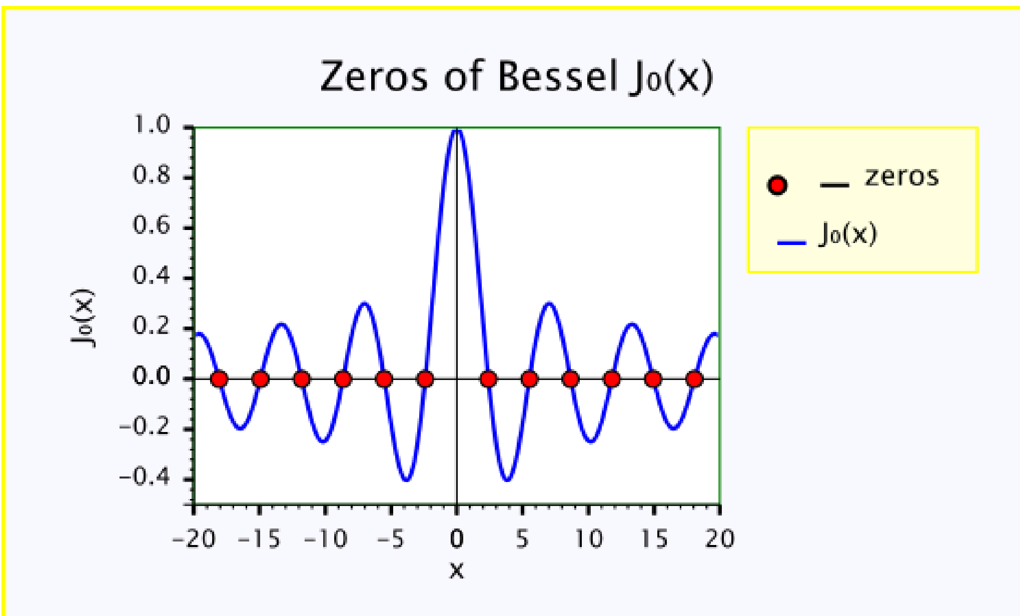

Figure 2.5: Zeros of Bessel Function

2.13 Heavy Chain

The future of Bessel function showed up in mathematics in 1732, when Daniel Bernoulli solved the problem on oscillations of the hung (heavy chain). Let the heavy chain of length l and the linear density ρ be a hung such that $x = 0$ at the free end of the chain. Then the deviation from equilibrium state $a = a(x, t)$ satisfies the equation:

$$\frac{\partial^2 a}{\partial t^2} = g\left(x\frac{\partial^2 a}{\partial x^2} + \frac{\partial a}{\partial x}\right) \tag{2.133}$$

$$a|_{x=l} = 0$$

By separating of variables:

$$a = X(x)T(t)$$

Where $T(t) = \sin(wt + R)$ leads to the equation:

$$xX''(x) + X'(x) + \frac{\omega^2}{g}X(x) = 0 \tag{2.134}$$

With the boundary condition:

$X(l) = 0\ ,\quad X(0) < \infty$

By introducing the new variable:

$$Z = 2\omega\sqrt{\frac{x}{g}}$$

We change equation (2.134) to the Bessel's equation:

$$\frac{d^2X}{dz^2} + \frac{1}{z}\frac{dX}{dz} + X = 0 \tag{2.135}$$

This is the equation for Bessel functions at zero index. Thus, the solution is:

$$a = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) \tag{2.136}$$

The characteristic frequency ω_k can take consequences of discrete values $(\omega_k, k = 1,2,3,\cdots,\infty)$. They can be found from the boundary condition:

$$a(l) = 0 \qquad J_0\left(2\omega_k\sqrt{\frac{l}{z}}\right) = 0$$

Hence,

$$2\omega_k\sqrt{\frac{l}{z}} = a_k^0 \quad J_0(a_k^0) = 0$$

$$\omega_k = \frac{1}{2}\sqrt{\frac{g}{l}}\, a_k^0$$

a_m^0 zeros of the Bessel function J_0.

2.14 Some Differential Equations Reducible to Bessel's Equation

1. The modified Bessel's equation is one of the well-known Bessel's equation that are reduce to differential equation by replacing x to $-ix$. Thus;

$$x^2Q'' + xQ' - (x^2 + v^2)Q = 0$$

The solution of the above equation are expressed via the so-called modified Bessel functions of the first and second kind:

$$Q(x) = c_1J_v(ix) + c_2Y_v(-ix) = c_1I_v(x) + c_2K_v(x)$$

Where $I_v(x)$ and $K_v(x)$ are the modified Bessel functions of the first and second kind respectively.

2. The airy differential equation known in astronomy and physics has the form:

$$Q''(x) - xQ(x) = 0$$

It can be reduced to Bessel's differential equation. Its solution is given by the Bessel functions of the fractional order $\frac{1}{3}$:

$$Q(x) = c_1\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right) + c_2\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right)$$

Also, the one dimensional Schrödinger equation for a constant force are airy functions that can be transformed into Bessel functions of order $\frac{1}{3}$.

3. The differential equation of type

$$x^2Q'' + xQ' + (n^2x^2 - v^2)Q = 0$$

Differs from the Bessel's equation only by a factor n^2 before x^2 and has the general solution as:

$$Q(x) = c_1 J_v(nx) + c_2 Y_v(nx)$$

CHAPTER 3

APPLICATION OF BESSEL FUNCTIONS: SOLUTION TO SCHRODINGER EQUATION IN A NEUMANN AND HANKEL FUNCTIONS

This chapter discussed some applications of Bessel functions to mathematical physics and engineering and the solutions of Schrödinger equation to Neumann and Hankel functions were obtained.

Bessel's equation arises as a result of determining separable solutions to Laplace's equation and the Helmholtz equation in spherical and cylindrical coordinates. Bessel's functions made their first appearance by relating the angular position of a planet travelling along a keplerian ellipse to elapsed time. Though the integral and power series appears in their places, generally regarding the radial variable after separating the Laplace's equation in polar or spherical polar coordinates. In diverse problems of mathematical physics whose solution is highly connected with the application of cylindrical and spherical coordinates.

The constant v in the Bessel differential equation determines the order of the Bessel functions and can take any real numbered value $(v = n + \frac{1}{2})$ for spherical coordinates, while for cylindrical problems the order of the Bessel function is an integer value $(v = n)$. Bessel functions are also applicable for many problems of wave propagation, static potentials and its applications. Heat conduction in a cylindrical objects, electromagnetic waves in a cylindrical waveguide, modes of vibration of a thin circular or annular artificial membrane, diffusion problems on a lattice and solution to the radial Schrodinger equation (in spherical and cylindrical coordinates for a free particle) (Asmar, 2005). In this chapter we are going to consider only the last application which is the application of radial Schrodinger equation in cylindrical coordinates for a free particle (zero potential) to Neumann and Hankel functions respectively.

3.1 Derivation of Time Independent From the Time Dependent Schrodinger Equation

The Schrödinger equation is the analog of the second law of motion and describes the motion and behavior of systems on the atomic and subatomic levels using the wave function $\Psi(x,t)$.

Consider a particle of mass m moving along the x-axis. At any position (x) and momentum (p) in time (t), the behavior and motion of the particle is given by the wave function $\Psi(x,t)$.

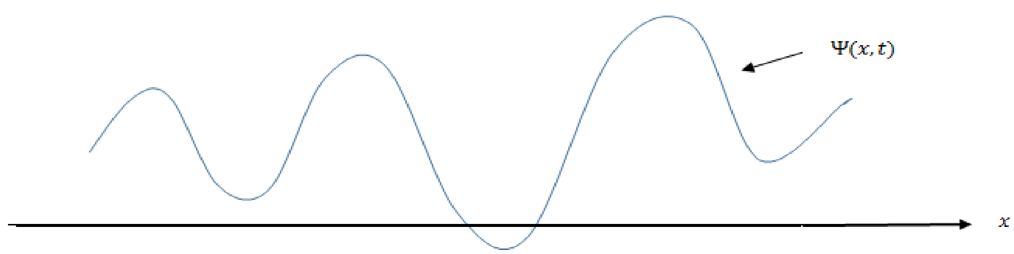

Figure 3.1: Wave Function

Where $\Psi(x,t)$ could take the form of any continues function that can be squared and integrated to get a finite answer. It is a fact from quantum mechanics that the wave function $\Psi(x,t)$ is a solution of the Schrödinger equation. (Tarasov, 2016).

The one dimensional time dependent Schrödinger equation is given by:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = u(x)\Psi(x,t) - \frac{\hbar^2}{2m}\cdot\frac{\partial^2\Psi(x,t)}{\partial x^2} \tag{3.1}$$

From the above equation we can see that if we know what the wave function is at same initial time (say $t = 0$), we can use that to determine the behavior of that particle at some future time. i.e. if we know that $\Psi(x,0)$ looks like, we can predict the future of the motion of the particle.

In quantum mechanics, we describe systems using wave functions. When we treat a system as a wave, the wave function represent the displacement of the wave. If we treat the system as a particle, then the wave function is use to give the probability of finding the particle at same point $(|\Psi|^2)$. In order to describe any system in quantum mechanics, we must be able to determine what the wave function is numerically. The Schrödinger equation is a differential equation that we can use to solve for the wave function quantitatively. In the same way that Isaac Newton invented the second law of motion ($\sum f = \overrightarrow{ma}$). The Schrödinger equation was invented and confirmed using experiments. In order to determine what form the equation takes, we will use the conservation of energy. We will also assume that the wave function does not depend on time and only depends on the spatial position of the system $\Psi(x, t)$ (Griffiths, 1995).

Given;

$$\Psi(x, t) = A \sin(kx - \omega t) + B \cos(kx - \omega t)$$

Which is a classical wave equation. Since we are assuming time-independent Schrödinger equation.

Let $t = 0$,

Implies;

$$\Psi(x) = A \sin kx + B \cos kx \tag{3.2}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{p}{h/_{2\pi}} = \frac{p}{\hbar}$$

Which implies,

$$k = \frac{p}{\hbar}$$

For a particle u, mass m and velocity v, the total energy is:

$$E = k + u = \frac{1}{2}mv^2 + u = \frac{p^2}{2m} + u$$

$p = mv \quad \Rightarrow x = \frac{p}{m}$

$$\text{Since } p = k\hbar \ \ \Rightarrow E = \frac{k^2\hbar^2}{2m} + u \tag{3.3}$$

Since we are looking for a differential equation that look like equation (3.3) and has a solution that look like equation (3.2);

$$\Rightarrow \frac{d^2\Psi}{dx^2} = -k^2[A\sin(kx) + B\cos(kx)]$$

Which implies,

$$\frac{d^2\Psi}{dx^2} = -k^2\Psi(x) \tag{3.4}$$

Now, we multiply equation (3.4) by $\frac{-\hbar}{2m}$ and we obtain;

$$\frac{-\hbar}{2m}\frac{d^2\Psi}{dx^2} = \frac{\hbar^2k^2}{2m}\Psi(x) \tag{3.5}$$

If we take equation (3.3) and multiply both sides by $\Psi(x)$, we have;

$$E\cdot\Psi(x) = \frac{\hbar^2k^2}{2m}\Psi(x) + u\cdot\Psi(x)$$

Implies;

$$E\cdot\Psi(x) = u\cdot\Psi(x) - \frac{-\hbar}{2m}\cdot\frac{d^2\Psi}{dx^2} \tag{3.6}$$

The above equation is called one-dimensional time-independent Schrödinger equation.

Also, we can obtain the time-independent Schrödinger equation from the more general time-dependent equation using the method of separation of variable as follows:

Given the time independent Schrödinger equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = u(x) - \frac{\hbar^2k^2}{2m}\frac{d^2\Psi(\mathrm{x},\mathrm{t})}{dx^2}$$

Let

$$\Psi(x,t) = \Psi(x)\cdot f(t) \tag{3.7}$$

We now substitute equation (3.7) into (3.1), we have:

$$i\hbar\frac{\partial\Psi(x)f(t)}{\partial t} = u(x)\Psi(x)f(t) - \frac{\hbar^2k^2}{2m}\frac{d^2\Psi(\mathrm{x})\cdot \mathrm{f(t)}}{dx^2}$$

Dividing both sides of the above equation by $\Psi(\mathrm{x})\cdot \mathrm{f(t)}$, we have;

$$i\hbar\frac{1}{f(t)}\cdot\frac{\partial f(t)}{\partial t} = u(x) - \frac{\hbar^2k^2}{2m}\cdot\frac{1}{\Psi(x)}\cdot\frac{d^2\Psi(\mathrm{x})}{dx^2} \quad (3.8)$$

So, the only way this equation can be equal is when both of them are equal to some constant, that is;

$$i\hbar\frac{1}{f(t)}\cdot\frac{\partial f(t)}{\partial t} = E \quad (3.9)$$

And

$$u(x) - \frac{\hbar^2k^2}{2m}\cdot\frac{1}{\Psi(x)}\cdot\frac{d^2\Psi(\mathrm{x})}{dx^2} = E \quad (3.10)$$

Now, let us take equation (3.10) and multiply both sides by $\Psi(x)$, we obtain;

$$u(x)\Psi(x) - \frac{\hbar^2k^2}{2m}\cdot\frac{d^2\Psi(\mathrm{x})}{dx^2} = E\Psi(x)$$

So that, for a free particle (zero potential) solutions, we have;

$$-\frac{\hbar^2k^2}{2m}\cdot\frac{d^2\Psi(\mathrm{x})}{dx^2} = E\Psi(x)$$

3.2 Solution to Schrödinger Equation in a Cylindrical Functions of the Second Kind (Neumann Functions)

Consider the functions J_v and J_{-v} which are two linearly independent solutions of the Bessel's equation as representatives of the Neumann or Weber's functions. That is,

$$Y_v(x) = N_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v} \quad (3.11)$$

Which in the Schrödinger equation presents:

$$-\frac{\hbar^2 k^2}{2m}\cdot\frac{d^2Y_v}{dx^2} = EY_v(x) \tag{3.12}$$

Now, we differentiate equation (3.11) for the second times and substitute into equation (3.12), as follows:

$$\frac{dY_v(x)}{dx} = \frac{d}{dx}\left[\frac{J_v(x)\cos\pi v}{\sin\pi v} - \frac{J_{-v}(x)}{\sin\pi v}\right]$$

$$= \frac{\cos\pi v}{\sin\pi v}J_v'(x) - \frac{1}{\sin\pi v}J_{-v}'(x)$$

Again,

$$\frac{d^2Y_v}{dx^2} = \frac{d}{dx}\left[\frac{\cos\pi v}{\sin\pi v}J_v'(x) - \frac{1}{\sin\pi v}J_{-v}'(x)\right]$$

Implies;

$$\frac{d^2Y_v}{dx^2} = \frac{\cos\pi v}{\sin\pi v}J_v''(x) - \frac{1}{\sin\pi v}J_{-v}''(x) \tag{3.13}$$

Therefore, we substitutes equation (3.13) into equation (3.12), we have;

$$-\frac{\hbar^2 k^2}{2m}\cdot\frac{d^2Y_v}{dx^2} = EY_v(x)$$

Implies,

$$-\frac{\hbar^2}{2m}\left[\frac{\cos\pi v}{\sin\pi v}J_v''(x) - \frac{1}{\sin\pi v}J_{-v}''(x)\right] = E\left[\frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v}\right]$$

$$\frac{1}{\sin\pi v}\left[\cos\pi v\, J_v''(x) - J_{-v}''(x)\right] = \frac{-2mE}{\hbar^2}\left[\frac{1}{\sin\pi v}\right]\left[J_v(x)\cos\pi v - J_{-v}(x)\right]$$

$$\cos\pi v\, J_v''(x) - J_{-v}''(x) = \frac{-2mE}{\hbar^2}\left[J_v(x)\cos\pi v - J_{-v}(x)\right]$$

By letting $r^2 = \frac{2mE}{\hbar^2}$, we have:

$$\cos\pi v\, J_v''(x) - J_{-v}''(x) = -r^2\cos\pi v\, J_v(x) + r^2 J_{-v}(x)$$

So that we can obtain;

$$\cos \pi v \, J_v''(x) + r^2 \cos \pi v \, J_v(x) = J_{-v}''(x) + r^2 J_{-v}(x)$$

And we can re-write the above equation as:

$$\cos \pi v \, [J_v''(x) + r^2 J_v(x)] = J_{-v}''(x) + r^2 J_{-v}(x) \tag{3.14}$$

Therefore, the only way this equation can be equal is when both of them is equal to some constant. That is;

$$J_{-v}''(x) + r^2 J_{-v}(x) = k \tag{3.15}$$

$$\cos \pi v \, [J_v''(x) + r^2 J_v(x)] = k \tag{3.16}$$

To simplify equation (3.15) and equation (3.16), we follow the method of undetermined coefficient and obtain the solution as follows;

$$J_{-v}(x)_{genaral} = J_{-v}(x)_{complementary} + J_{-v}(x)_{particular}$$

So, for equation (3.15), we have:

$J_{-v}(x)_{complementary}$: $\quad J_{-v}''(x) + r^2 J_{-v}(x) = 0$

The characteristics equation is $\lambda^2 + r^2 = 0$, which implies $\lambda = \pm ir$.

Therefore,

$$J_{-v}(x)_{complementary} = c_1 \cos rx + c_2 \sin rx \tag{3.17}$$

And for $J_{-v}(x)_{particular}$, we have:

By letting $J_{-v}(x) = A_1 \Rightarrow J_{-v}'(x) = 0$ and $J_{-v}''(x) = 0$

$$\therefore J_{-v}(x) = A_1 \tag{3.18}$$

So, the general solution is now written as:

$$J_{-v}(x) = c_1 \cos rx + c_2 \sin rx + A_1$$

But, we know that $r = \frac{\sqrt{2mE}}{\hbar}$, therefore;

$$J_{-v}(x) = c_1 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_2 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_1 \tag{3.19}$$

Similarly, for equation (3.16), we have:

$$J_v(x) = c_3 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_4 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_2 \tag{3.20}$$

Remark:

Clearly, equation (3.19) and equation (3.20) are similar, this shows that $J_v(x)$ and $J_{-v}(x)$ are the two linearly independent solutions of the Bessel's differential equation which also appears in the Neumann (Weber's) functions.

3.3 Solutions to Schrödinger Equation in a Cylindrical Functions of the Third Kind (Hankel Functions)

Here, also we are going to apply the Schrödinger equation to cylindrical functions of the third kind (Hankel functions) and obtain the solution of Bessel's differential equation. The Hankel function of the first and second kind are respectively given by:

$$H_v^{(1)}(x) = J_v(x) + iY_v(x) = i\,\frac{e^{-v\pi i}J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{3.21}$$

And

$$H_v^{(2)}(x) = J_v(x) - iY_v(x) = -i\,\frac{e^{v\pi i}J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{3.22}$$

Again, on applying equation (3.21) and equation (3.22) in the Schrödinger equation, that is;

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2 H_v^{(1)}(x)}{dx^2} = EH_v^{(1)}(x) \tag{3.23}$$

We obtain the solutions as follows:

Now, we differentiate equation (3.21) for the second time and substitute into equation (3.23), we have;

$$H_v^{(1)\prime}(x) = \frac{d}{dx}\left(\frac{ie^{-v\pi i}J_v(x)}{\sin \pi v}\right) - \frac{d}{dx}\left(\frac{iJ_{-v}(x)}{\sin \pi v}\right)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}\frac{d}{dx}J_v(x) - \frac{i}{\sin \pi v}\frac{d}{dx}\left(J_{-v}(x)\right)$$

Which implies,

$$H_v^{(1)\prime\prime}(x) = \frac{d}{dx}\left(\frac{ie^{-v\pi i}}{\sin \pi v}\frac{d}{dx}J_v(x)\right) - \frac{d}{dx}\left(\frac{i}{\sin \pi v}\frac{d}{dx}J_{-v}(x)\right)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}\frac{d^2}{dx^2}J_v(x) - \frac{i}{\sin \pi v}\frac{d^2}{dx^2}J_{-v}(x)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) - \frac{i}{\sin \pi v}J_{-v}''(x) \tag{3.24}$$

By replacing equation (3.24) into equation (3.23), we have;

$$-\frac{\hbar^2}{2m}\left(\frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) - \frac{i}{\sin \pi v}J_{-v}''(x)\right) = E\left(\frac{ie^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{i}{\sin \pi v}J_{-v}(x)\right)$$

Letting $a^2 = \frac{2mE}{\hbar^2}$

Implies;

$$\frac{i}{\sin \pi v}J_{-v}''(x) - \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) = a^2\left[\frac{ie^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{i}{\sin \pi v}J_{-v}(x)\right]$$

Implies;

$$\frac{i}{\sin \pi v}J_{-v}''(x) - \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) = \frac{ia^2e^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{ia^2}{\sin \pi v}J_{-v}(x)$$

$$\frac{i}{\sin \pi v}J_{-v}''(x) + \frac{ia^2}{\sin \pi v}J_{-v}(x) = \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) + \frac{ia^2e^{-v\pi i}}{\sin \pi v}J_v(x)$$

Therefore,

$$J''_{-v}(x) + a^2 J_{-v}(x) = e^{-v\pi i}\left(J''_v(x) + a^2 J_v(x)\right) \tag{3.25}$$

Again, the only way this equation can be equal is when both of them equal to some constant. That is:

Suppose;

$$J''_{-v}(x) + a^2 J_{-v}(x) = k_1 \tag{3.26}$$

And

$$e^{-v\pi i}\left(J''_v(x) + a^2 J_v(x)\right) = k_1 \tag{3.27}$$

So, we can obtain the solution as follows:

For equation (3.26), we have;

$$J_{-v}(x) = c_5 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_6 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_3 \tag{3.28}$$

And from equation (3.27), we have;

$$J_v(x) = c_7 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_8 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_4 \tag{3.29}$$

Remark:

Clearly, $J_v(x)$ and $J_{-v}(x)$ which are presents in equation (3.28) and equation (3.29) are linearly independent solutions of Bessel's differential equation that appears in the Hankel functions of the first kind.

Similarly, for the Hankel functions of the second kind in equation (3.22), we have the solution as follows:

$$J_{-v}(x) = c_9 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{10} \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_5 \tag{3.30}$$

And

$$J_v(x) = c_{11}\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{12}\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_6 \tag{3.31}$$

Remark:

Lastly, the $J_v(x)$ and $J_{-v}(x)$ which are presents in equation (3.30) and equation (3.31) are linearly independent solutions of Bessel's differential equation that appears in the Hankel functions of the second kind.

CHAPTER 4

CONCLUSION

We have discussed the solution of a free particle (zero potential) time-independent Schrödinger equation as applied to cylindrical function of the second kind (Neumann functions) and cylindrical function of the third kind (Hankel functions of the first and second kind). It has been find out that, the solution in each case which are presents in the solution of Bessel differential equation are the same. The constants in each of the solution are to be determined using application of boundary conditions. This shows that the Bessel function appeared in many diverse scenarios, more especially in a situation involving cylindrical symmetry.

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APPENDICES

Appendix 1: Gamma Function

The gamma function is defined for $v > 0$ by;

$$\Gamma(v) = \int_0^\infty t^{v-1} e^{-t}\, dt \tag{4.1}$$

This integral is improper and converges for all $v > 0$. The basic property of gamma function is

$$\Gamma(v+1) = v\Gamma(v)$$

To prove this we use integration by parts as follows:

$$\Gamma(v+1) = \int_0^\infty t^v e^{-t}\, dt = -t^v e^{-t}|_0^\infty + v\int_0^\infty t^{v-1} e^{-t}\, dt = v\Gamma(v)$$

Where in the first integral we let $u(t) = t^v$, $dv = e^{-t}dt$, $du = vt^{v-1}dt$ and $v(t) = -e^{-t}$.

We can easily find the value of gamma function at the positive integers. For example,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

The basic property now gives

$$\Gamma(2) = 1\Gamma(1) = 1!$$

$$\Gamma(3) = 2\Gamma(2) = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3!$$

$$\vdots$$

Continuing in this manner, we see that

$$\Gamma(n+1) = n! \tag{4.2}$$

For all $n = 0, 1, 2, 3, \ldots$ where we have set $0! = 1$. For this reason the gamma function is sometimes called the generalized factorial function. Other values of the gamma function can be found with various degrees of difficulty.

From the value of

$$\Gamma\left({}^{1}/_{2}\right) = \sqrt{\pi} \tag{4.3}$$

And the basic property we find

$\Gamma\left({}^{1}/_{2}\right) = {}^{1}/_{2}\,\Gamma\left({}^{1}/_{2}\right) = {}^{\sqrt{\pi}}/_{2}$ and $\Gamma\left({}^{5}/_{2}\right) = {}^{3}/_{2}\,\Gamma\left({}^{3}/_{2}\right) = {}^{3}/_{2}\,{}^{\sqrt{\pi}}/_{2} = {}^{3}/_{4}\sqrt{\pi}$

Although we have defined the gamma function for $v > 0$, it is possible to extent its definition to all real numbers other than $0, -1, -2, -3, \ldots$ in such a way that the basic property continues to hold. To do so, we write the basic property as;

$$\Gamma(v) = \frac{1}{v}\Gamma(v+1)$$

And then defined the value of the gamma function at v from its value at $v + 1$. For example, we have;

$$\Gamma\left(-{}^{1}/_{2}\right) = -2\Gamma\left({}^{1}/_{2}\right) = -2\sqrt{\pi}$$

And

$$\Gamma\left(-{}^{3}/_{2}\right) = -{}^{2}/_{3}\,\Gamma\left(-{}^{1}/_{2}\right) = {}^{4}/_{3}\sqrt{\pi}$$

This clearly extends the definition of the gamma function to negative numbers other than $-1, -2, -3, \ldots$

The graph of the gamma function is sketched in the figure below. Notice that the vertical asymptotes at $x = 0, -1, -2, \ldots$ Also, notice the alternating sign of the gamma function over negative integers.

For $n = 0, 1, 2, \ldots$

$\Gamma(n + 1) = n!$

$\Gamma(1) = 0! = 1$

$\Gamma(-n)$ is not defined

$\Gamma(v) > 0$ for $v > 0$

$\Gamma(v)$ alternates signs on the negative axis.

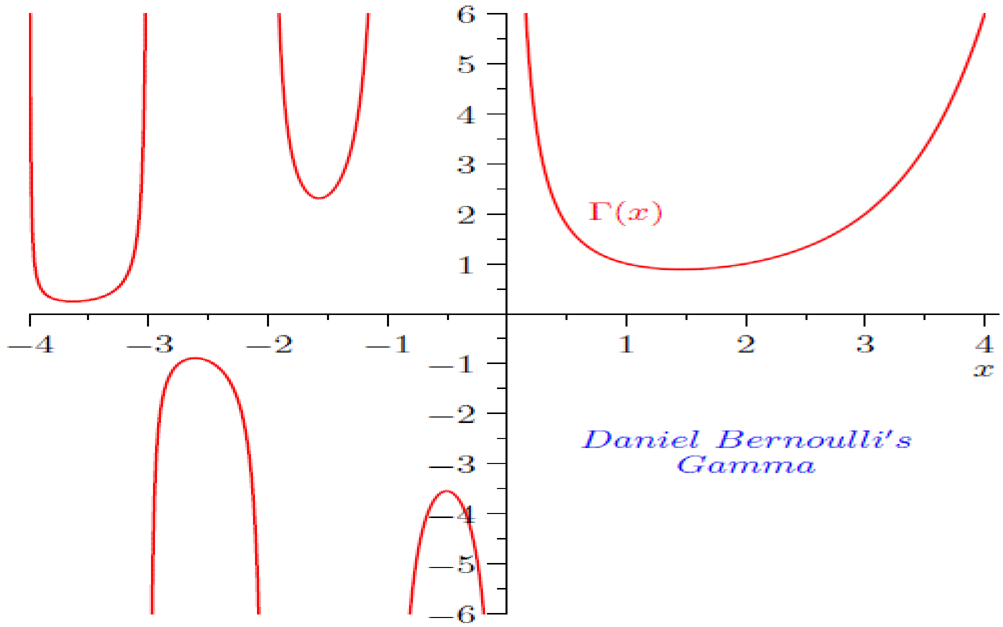

Figure 4.1: The generalized factorial function (Gamma function)

Appendix 2: The Method of Frobenius

We obtained the solutions of Bessel equation using the method of Frobenius. It is motivated by Euler and power series. We will be solving equation of the form:

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y = 0 \tag{4.4}$$

Putting it in the standard form, we can see that $x = 0$ is not an ordinary point. Therefore, we cannot apply the power series method. For application, it is of particular importance to know the behavior of the solutions at $x = 0$. To end this, we will develop a generalization of the power series method, known as the Frobenius method.

Consider the homogeneous differential equation

$$y''(x) + p(x)y'(x) + q(x)y = 0 \tag{4.5}$$

Notice that, a is an ordinary point of the differential equation if p and q have power series expansions at a. Otherwise, a is called a singular point.

Now, we say that $x = 0$ is a regular singular point of the equation if both of the functions $xp(x)$ and $x^2q(x)$ have power series expansions at $x = 0$. The Frobenius method that we now describe applies to equations for which $x = 0$ is a regular singular point. We now tri the series solution of the form:

$$y = \sum_{m=0}^{\infty} a_m x^{r+m} \tag{4.6}$$

Where $a_0 \neq 0$. Such a series is called a Frobenius series. By differentiating equation (4.6) twice, we obtain;

$$y'(x) = \sum_{m=0}^{\infty} a_m(r+m)x^{r+m-1}$$

And

$$y''(x) = \sum_{m=0}^{\infty} a_m(r+m)(r+m-1)x^{r+m-2}$$

Substituting the above series into equation (4.5), we have;

$$\sum_{m=0}^{\infty} a_m(r+m)(r+m-1)x^{r+m-2} + p(x)\sum_{m=0}^{\infty} a_m(r+m)x^{r+m-1} + q(x)\sum_{m=0}^{\infty} a_m x^{r+m} = 0$$

We factor x from the second series and x^2 from the third to make all exponents the same and get

$$\sum_{m=0}^{\infty} a_m(r+m)(r+m-1)x^{r+m-2} + xp(x)\sum_{m=0}^{\infty} a_m(r+m)x^{r+m-2} + x^2q(x)\sum_{m=0}^{\infty} a_m x^{r+m-2} = 0 \qquad (4.7)$$

Since by assumption $x = 0$ is a regular singular point, the function $xp(x)$ and $x^2q(x)$ have power series expansion about 0, say

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots$$

And

$$x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots$$

Substituting these into equation (4.7), we have;

$$\sum_{m=0}^{\infty} a_m(r+m)(r+m-1)x^{r+m-2} + (p_0 + p_1x + p_2x^2 + \cdots)\sum_{m=0}^{\infty} a_m(r+m)x^{r+m-2} + (q_0 + q_1x + q_2x^2 + \cdots)\sum_{m=0}^{\infty} a_m x^{r+m-2} = 0$$

The total coefficient of each power of x on the left sied of this equation must be 0, since the right side is zero. The lowest power of x that appears in the equation is x^{r-2}. Its coefficient is $a_0r(r-1) + p_0a_0r + q_0a_0 = a_0[r(r-1) + p_0r + q_0] = 0$

Since $a_0 \neq 0, r$ must be a root of the indicial equation. That is;

$$r(r-1)+p_0 r+q_0=0 \tag{4.8}$$

The roots of this equation are called the indicial roots and are denoted by $r_1\ and\ r_2$ with the convention that $r_1 \geq r_2$ whenever they are real. Note that, p_0 and q_0 are easily determined, since they are the values of $xp(x)$ and $x^2q(x)$ at $x=0$. Once we have determined r_1 and r_2, we have substitute r_1 in equation (4.7) and solve for the unknown coefficients a_n as we would do with the power series method. This will determined a first solution of equation (4.5). Summing up, we have the following result;

Theorem 1:

If $x=0$ is a regular singular point of the equation

$$y''(x)+p(x)y'(x)+q(x)y=0$$

Then one solution is of the form;

$$y_1=|x|^{r_1}(a_0+a_1x+a_2x^2+\cdots),\ \ a_0\neq 0$$

Where r_1 is a root of the indicial equation (4.8), with the convention that r_1 is the larger of the two roots when both roots are real.

Theorem 2:

Suppose that $x=0$ is a regular singular point of the differential equation

$$y''(x)+p(x)y'(x)+q(x)y=0$$

And let r_1 and r_2 denoted the indicial roots. The differential equation has two linearly independent solutions y_1 and y_2, as we now describe;

Case 1:

If r_1-r_2 is not an integer, then

$$y_1=|x|^{r_1}\sum_{m=0}^{\infty}a_m x^m$$

And

$$y_2 = |x|^{r_2} \sum_{m=0}^{\infty} b_m x^m$$

Where $a_0 \neq 0$ and $b_0 \neq 0$.

Case 2:

If $r = r_1 = r_2$, then

$$y_1 = |x|^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

And

$$y_2 = y_1 ln|x| + |x|^{r} \sum_{m=1}^{\infty} b_m x^m$$

Where $a_0 \neq 0$.

Case 3:

If $r_1 - r_2$ is a positive integer, with $r_1 \geq r_2$, then

$$y_1 = |x|^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

And

$$y_2 = k y_1 ln|x| + |x|^{r_2} \sum_{m=0}^{\infty} b_m x^m$$

Where $a_0 \neq 0, b_0 \neq 0$ (k may or may not be zero).