SOLVING LINEAR FIRST ORDER DELAY DIFFERENTIAL EQUATIONS BY MOC AND STEPS METHOD COMPARING WITH MATLAB SOLVER

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

SAAD IDREES JUMAA

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2017

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Approval of Director of Graduate School of
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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To those who believed in me…

ABSTRACT

This research concentrates on some elementary methods to solving linear first order delay differential equations (DDEs) with a single constant delay and constant coefficient, such as characteristic method and the method of steps and comparing the methods solution with some codes from Matlab solver such as DDE23 and DDESD. The study discussed the compare solution by merging algebraic solution and approximate solution in one graph for each problem. We used Matlab program in this thesis because is very powerful language program to deal with complex problem in mathematics and obtain the solution faster than many language programs and to obviate miscalculation. We interested in this thesis to find solution for this kind of linear delay equation, $\dot{u}(t) = c_1 u(t) + c_2 u(t - \beta)$, with single constant delay and constant coefficients c_1 and c_2.

***Keywords*:** Delay differential equation; Linear delay differential equation ; Constant delay; Characteristic method; Method of steps; Matlab codes; DDE23 solver; DDESD solver; time delay; Functional differential equation; Boundary value problem

ÖZET

Bu tezde, birinci derece Gecikmeli linear diferensiyel denklemlerin, karakteristik method ve adım metodu gibi bazı çözüm metodları üzerine ve DDE23 ve DDESD Matlam çözücü kodları ile metodların karşılaştırılması üzerinde çalışılmıştır. Bu çalışmada, her bir problem için cebirsel ve sayısal çözümler bir grafik üzerinde birleştirilerek karşılaştırlıdı. Matematikte karmaşık problemlerle başa çıkabilmek için güçlü bir programlama diline sahip olduğu ve bir çok programa göre daha hızlı sonuçlar elde ettiği ve yanlış hesaplamayı önlediği için Matlab programı kullanılmıştır. Metodlar, c_1 ve c_2 sabit sayılar olmak üzere, $\dot{u}(t) = c_1u(t) + c_2u(t-\beta)$denklemini içerecek şekilde genişletilmiştir.

***AnahtarKelimeler*:** Gecikmeli diferensiyel denklemler; Lineer gecikmeli diferensiyel denklmler; Sabir gecikme; Karakteristik metod; Adımlar Metodu; Matlab kodları; DDE23 çözücü; DDESD çözücü; Gecikmeli zaman; Kesirli diferensiyel denklemler; Sınır değer problemleri

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DDE: Delay differential Equation

LDDE: Linear Delay Differential Equation

DE: Differential Equation

ODE: Ordinary Differential Equation

FDE: Functional Differential Equation

RFDE: Retarded Functional Differential Equation

BVP: Boundary Value Problem

IV: Initial Value

MOC: Method Of Characteristic

LUB: Least Upper Bound

GLB: Greatest Lower Bound

NDFE: Neutral Functional Differential Equation

AFDE: Advanced Functional Differential Equation

SDDE: Stochastic Delay Differential Equation

NDDE: Neutral Delay Differential Equations

RCDS: Remote Control Dynamical System

LIST OF SYMBOLS

$\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}, \boldsymbol{u}^{(i)}$	Total derivatives of $u(t)$ with respect to t
$\boldsymbol{m}$	Number of equations
$\boldsymbol{n}$	Number of unknowns
$\boldsymbol{\mu} + \boldsymbol{i\gamma}$	Complex number
$[\boldsymbol{t}]$	Integer part of t
$\boldsymbol{\theta}$	Pre-function
$\boldsymbol{\beta}$	Delay
$\boldsymbol{D_{in}}$	Arbitrary constants
$\boldsymbol{DDE23}$	Matlab code solver
$\boldsymbol{DDESD}$	Matlab code solver
$\boldsymbol{IF}$	Integrating factor
$[\boldsymbol{0}, \boldsymbol{t}]$	Time interval
$[-\boldsymbol{\beta}, \boldsymbol{0}]$	Pre-interval
$[-\boldsymbol{\beta}, \boldsymbol{t}]$	Time interval including history

CHAPTER 1
INTRODUCTION

One of the mathematic students' common questions is ' why don't we study Ordinary Differential Equation (ODEs) or Partial Differential Equation (PDEs) instead of studying Delay Differential Equation? Since we have more information about them and they are much easier to handle. The simple answer is because of the crucial impact of the time delay on everything related to human life encompassing variety of domains and applications such as biology, economics, microbiology, ecology, distributed networks, mechanics, nuclear reactors physiology, engineering systems, epidemiology and heat flow (Gopalsamy, 1992). We have many examples of time delay in our life. A vivid example of a time delay is when forests are destroyed by human through cutting trees, this action will be done in a short span of time or when the forests are destroyed because of natural catastrophes such as fires and hurricanes and floods, and in a short time the forests deceases. Forest destruction takes short time, but it might take at least 25 years of cultivation and planting to give life back to the forest. Delay time will be included in any mathematical model to renew and harvest the forest. Time delay is a vital component of any dynamic process in life sciences.

There are different species of delay differential equation; such as linear delay differential equations (LDDEs), nonlinear delay differential equations (Non-LDDEs), neutral delay differential equations (NDDEs), stochastic delay differential equations (SDDEs)…etc. We will concentrate in this thesis on one type namely linear first order delay differential equation with a single delay and constant coefficients: $\dot{u}(t) = a(t)u(t) + b(t)u(t - \beta)$; for $\beta \geq 0,\ t \geq 0$ and $u(t) = p(t);\ t \leq 0$.In this thesis, we discussed an algebraic solution of linear first order delay differential equation. We give a detailed description of two methods, characteristic method and the method of steps, we shown how to solve the delay equation by this two methods step by step. The reader must have a good background in the differential equation to understand everything in this study because we used some techniques course of Ordinary differential equations (ODEs).

The method of characteristic to solve the linear firs order differential equation, $\dot{u}(t) = bu(t-\beta),\ \beta > 0,\ on\ [0,d], u(t) = \ \theta(t),\ on\ [-\beta, 0]$. When the value of $a = 0$, depends on some important notes such as the history function $u(t)$ has the form $u(t) = De^{st}$.Therefore this form of solution have four cases of solutions when each case have different real roots, for example case one when $b < -\frac{1}{\beta e} < 0$, has not any root, case two when $b = -\frac{1}{\beta}$, has one real roots $-\frac{1}{\beta}$, case three when $-\frac{1}{\beta e} < b < 0$ has two non-positive real roots s_1 and s_2, and case four when $b > 0$, has exactly one real roots, $s > 0$. As well as we need some numerical methods in steps of approximate solution form like Newton's Method (Falbo, 1995), so if we partition the interval $[-\beta, 0]$ to some interval for solving the given $j \times j$ non-singular system of constant coefficient, D_{in}. Then the approximate solution for the linear first order delay differential equation by using the method of characteristic has the form $u_m(t) = D_0 e^{(-1/\beta)t} + D_1 e^{(s_2)t} + D_2 e^{(s_1)t} + D_3 e^{(s)t} + \sum_{n=1}^{m} e^{\mu_n t} \left(D_{1n} \cos(\gamma_n t) \ \ + D_{2n} \sin(\gamma_n t)\right)$ .The general idea of the method of steps is converting the linear first order differential equation (DDE) on a given interval to ordinary differential equation (ODE) over that interval, (El'sgol'ts and Norkin, 1973), so this process make given (DDEs) as (ODEs) and we can solve it by some techniques from (ODE).So this thesis sheds light on algebraic solution of (LDDEs) and comparing with numerical solution by using Matlab solver such as DDE23 solver and DDESD solver by merging algebraic solution and approximate solution in one graph, the meaning and the definition of the two methods and the algorithm program of Matlab solver will be presented later.

1.1 Aims of the Study

The aim of this study focuses on how to find algebraic solutions of linear first order differential equations and comparing with approximate solutions, by using some elementary method for solving delay equations such as MOC and the method of steps, as well as in this research we uses the most powerful language mathematics program namely Matlab for given approximate solution by using some special codes such as DDE23 and DDESD. Since Matlab has great power to deal with very complex problems in various mathematics fields to give best answer for any problem.

1.2 Thesis Outline

This thesis is divided into five chapters; the first chapter focuses on introduction and the aim of study.

Chapter two contains a background and literature review; in literature review we showed a short history of delay differential equation, and we introduced some important terminologies, concepts and definitions. And we gave some problems containing time delay such as control theory. We explained each kind of delay differential equations (DDEs) and its area applications in our daily life, the algorithm of language Matlab program have been presented with illustrative examples in Chapter two.

Chapter three consists of methods and methodology for solving linear first order delay differential equations (DDEs) with single delay and constant coefficient; we discussed two methods for solving delay equations and methodology for the two methods is also given with step by step. Moreover, we explain the algorithm codes in Matlab program such as DDE23 solver and DDESD solver.

Chapter four discusses algebraic solutions of linear first order delay differential equation by using MOC and the method of steps. And also comparing algebraic solutions with approxima-te solutions by using Matlab program, the special codes in Matlab program to find numerical solutions have been used such as DDE23 and DDESD.

In Chapter 5, the conclusion of this work is presented; it summarizes and analyses the entire work conducted in this thesis.

CHAPTER 2
LITERATURE REVIEW

When someone tries to find the solutions of differential equations, it is certain that he will try to know which kind of differential equations in his hand. Usually we know more things in ordinary differential equations (ODEs) and partial differential equations (PDEs).But if we have a special class of differential equations, such as delay differential equations (DDEs). Likewise for reading this topic, the delay differential equations, if you do not have background knowledge of the differential equations, it will be difficult for you to understand all aspects of the DDEs and consequently this thesis. Thus the main aim of this chapter is to give the reader an easy to comprehend background and history of delay DDEs, from where it began? How did it start from the beginning? By whom it was developed? In which field it has been used and for what purpose? Etc… Also to illustrate some concepts and definitions of DDEs, classify DDEs and which methods we will use to solve the DDEs.

2.1 History of Delay Differential Equations

Researchers had been preoccupied with Differential Integral Equations, Functional Differential Equations (FDEs) and Difference Differential Equations (DDEs) for at least two centuries. The progress of human learning and reliance on automatic control system after the World War I gave birth to different type of equation named Delay Differential Equation (DDEs). The last 60 years, researchers have been concerned about the theory of DDEs and FDEs. This theory has become an indispensable part in any researchers' glossary who deal with particular applications(implementations) such as biology, microbiology, heat flow, engineering mechanics, nuclear reaction, physiology... etc. (Kolmanovski and Mshkis, 1999). Laplace and Condorcet are the pioneers of this study; it appeared in the 18^{th} century (Fuksa et al., 1989). The stability's main theory of basic DDEs was developed (elaborated) by Pontryagin in 1942, however, after the World War II, there was rapid growth of the theory and its applications (after the World War II, the theory grow rapidly). Bellman and Cooke are credited with writing significant works about DDEs in 1963 (Bellman and Cooke, 1963).

The DDEs studies witnessed massive movement(growth) in 1950 regarding DDEs studies resulting in publishing many important works such as Myshkis in 1951, Krasovskii in 1959, Bellman and Cooke in 1963, Halanay in 1966, Norkin in 1971, Hale in 1977, Yanushevski in 1978, Marshal in 1979, these researches and publications lasted until this day in a variety of domains

2.2 Delay Differential Equations

The more general kind of DEs is called a functional differential equations (FDEs), as well as the delay differential equations is a simplest maybe most natural class of functional differential equations (Driver, 1977). If we look at various fields and its applications we will see the time delay are normal ingredients of the dynamic process of various life sciences such as biology, economics, microbiology, ecology, distributed networks, mechanics, nuclear reactors, physiology, engineering systems, epidemiology and heat flow (Gopalsamy, 1992) and "to ignore them is to ignore reality" (Kuang, 1993). Delay differential equations (DDEs) is of the form

$$u'(t) = g\left(t, u(t), u\left(t - \beta_1\big(t, u(t)\big)\right), u\left(t - \beta_2\big(t, u(t)\big)\right), \dots\right) \qquad (2.1)$$

For $t \geq 0$ and $\beta_i > 0$, the delays, $\beta_i, i = 1, 2, \dots$ are commensurable physical quantities and may be constant. In DDEs the derivative at any time relies on the solution at previous times (and in the situation of neutral equations on the derivative at previous times), more generally that is $\beta_i = \beta_i\big(t, u(t)\big)$. Example of familiar delay problem such as Remote Control, images are sent to Earth and a signal is sent back. For the Moon, the time delay in the control loop is 2-10 s and for the Mars, it is 40 minutes! (Erneux, 2014) For many years Ordinary differential equations were an essential tool of the mathematical models. However, the delay has been ignored in ordinary differential equation models. DDEs model is better than ODE model because DDE model used to approximate a high-dimensional model without delay by a lower dimensional model with delay, the analysis of which is more easily carried out. This approach has been used extensively in the process control industry (Kolmanoviskii and Myshkis, 1999).

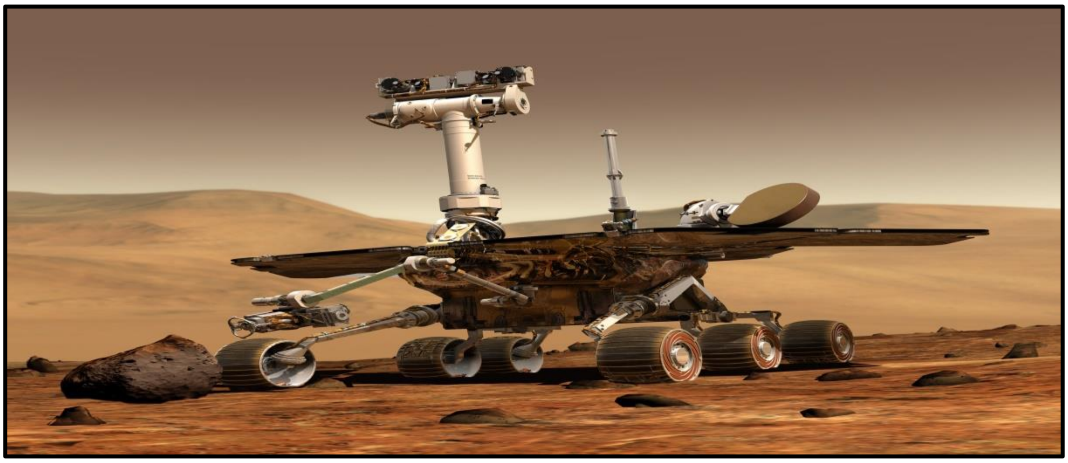


Figure 2.1: when the Robot sent images to Earth

DDE model depends on the initial function to determine a unique solution, because $u'(t)$ depends on the solution at prior times. Then it is necessary to supply an initial auxiliary function sometimes called the "history" function, before t=0, the auxiliary function in many models is constant, $\beta: \max \beta_i$.

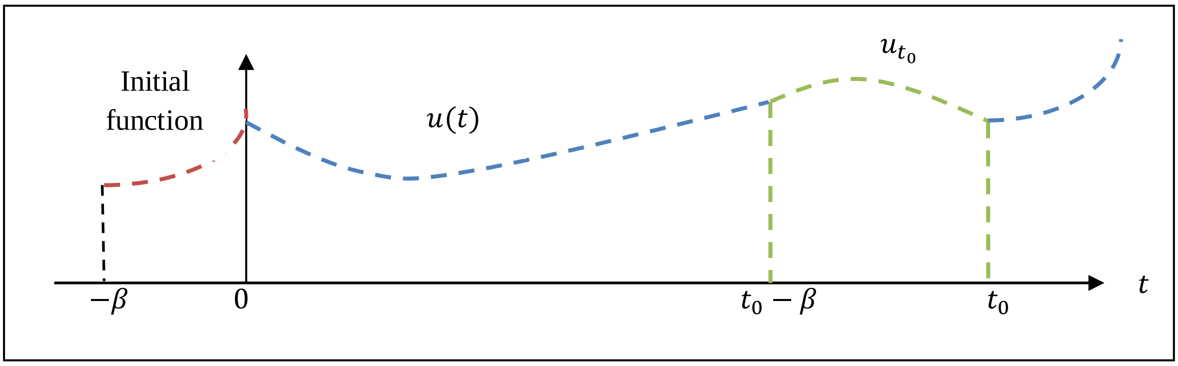


Figure 2.2: The initial function defined over the interval $[-\beta, 0]$ is mapped into a solution curve on the interval$[0, t_0 - \beta]$. Initial function segment $\emptyset(\sigma), \sigma \in [-\beta, 0]$ has to be specified and t $= t_0$, function segment$u_{t_0}(\sigma), \sigma \in [-\beta, 0]$

There are no many differences between properties of Delay differential equation and ordinary differential equation, sometimes analytical method of ODEs have been used in DDEs when it is possible to apply. The order of the DDEs is the highest derivative include in the equation (Driver, 1977), in Table 2.1 we have shown some examples about the order of delay differential equation (DDE).

Table 2.1: The order of DDE and ODE

ODE	Order of ODE	DDE	Order of DDE
$u''(x) + vuu' = 0$	Second order linear	$u'(t) = \mu u(t) + \alpha u(t - \beta)$	First order Linear
$\dfrac{d^4u}{dv^4} + 5\dfrac{d^2u}{dv^2} + 3u = -2vu^3$	Forth order Nonlinear	$u^{(3)}(t) = u(t - \beta)[1 - u(t)]$	Third order Nonlinear
$u^{(7)} + 25u^{(8)} - 34vu = sinu$	Eighth order Linear	$cu''(t) + bu'(t - \beta) = \sin t$	Second order Linear

We have shown the substantial difference between DDEs and ODEs in Table 2.2

Table 2.2: Substantial difference between DDEs and ODEs

Delay Differential Equations	Ordinary Differential Equations
Supposed to take into account the history of the past due to the influence of the changes on the system is not instantaneous	Supposed to take into account the principle of causality due to the influence of the changes on the system is instantaneous (Hale, 1993)
Depends on initial function to define a unique solution	Depends on initial value to define a unique solution
Give a system that is infinite dimensional	Give a system that is finite dimensional
Analytical theory is well less developed	Analytical theory is well developed (Lumb, 2004)

2.3 Classification of (FDEs) and (RFDEs)

In this section we introduce some nomenclature and definitions about DDEs that will be required from the reader in order to understand this topic well, as we said before the DDEs is class of FDEs, therefore we will try to explain the power relation between DDEs and FDEs. Suppose, $\beta_{max} = constant \in [0, \infty)$, and let $u(t)$ be an n-dimensional variable portraying the conduct of a operation in the time period $t \in [t_0 - \beta_{max}, t_1]$. FDE is formulated as follows, let $\psi_1(t)$ and $\psi_2(t)$ be time-dependent sets of real number, $\forall\ t \in [t_0, t_1]$. Suppose that u is continuous function in $[t_0, t_1]$, and $\dot{u}(t)$ for $t \in [t_0, t_1]$ is the right-hand derivatives of u. For each, $\in [t_0, t_1]$, u_t is defined by $u_t(r) = u(t + r)$, where $r \in \psi_1(t)$ and analogously $\dot{u}_t$ is defined by $\dot{u}_t(r) = \dot{u}(t + r)$ where $r \in \psi_2(t)$. We say that u satisfies an FDE in $[t_0, t_1]$ if $\forall\ t \in [t_0, t_1]$ the following equation holds.

$$\dot{u}(t) = g(t, u_t, \dot{u}_t, v(t)) \tag{2.2}$$

$v(t)$ is given for the whole time interval necessary, the equation (2.2) have three kind of differential equations (DEs)

i) If $\psi_1(t) \subset (-\infty, 0]$ and $\psi_2(t) = \phi$ for $t \in [t_0, t_1]$, we say that FDE is retarded functional differential equation (**RFDE**), therefor the right-hand side of (2.2) does not depend on the derivative of u.

$$\dot{u}(t) = g(t, u_t, v(t)) \tag{2.3}$$

In other words, the rate of change of the state of an RFDE is determined by the inputs $v(t)$, as well as the present and past states of the system. An RFDE is sometimes also designated as a hereditary differential equation or, in control theory as a time-delay system.

ii) If $\psi_1 \subset (-\infty, 0]$ and $\psi_2(t) \subset (-\infty, 0]$ for, $t \in [t_0, t_1]$, we say that FDE is a neutral functional differential equation (**NDFE**), that is mean the rate of change of the state depends on its own past values as well.

iii) An FDE is called an advanced functional differential equation (**AFDE**) if $\psi_1(t) \subset [0, \infty)$ and $\psi_2(t) = \varnothing$ for $t \in [t_0, t_1]$. An equation of the advanced type may represent a system in which the rate of change of a quantity depends on its present and future values of the quantity and of the input signal $v(t)$.

Note: And retarded functional differential equation (**RFDE**) classify to another kind of differential equations.

1) Retarded difference equation or sometimes called functional differential equation with discrete delay.
2) Functional differential equation contains distributed delays.
3) If delays are constant are called fixed point delays, systems which have only multiple constant time delay can be classified as, if the delays related by integer will be called linear commensurate time delay system.

If the delays are not related by integer will be called linear non commensurate time delay system, in Figure 2.3 the diagram below functional differential equation and their branches are classified.

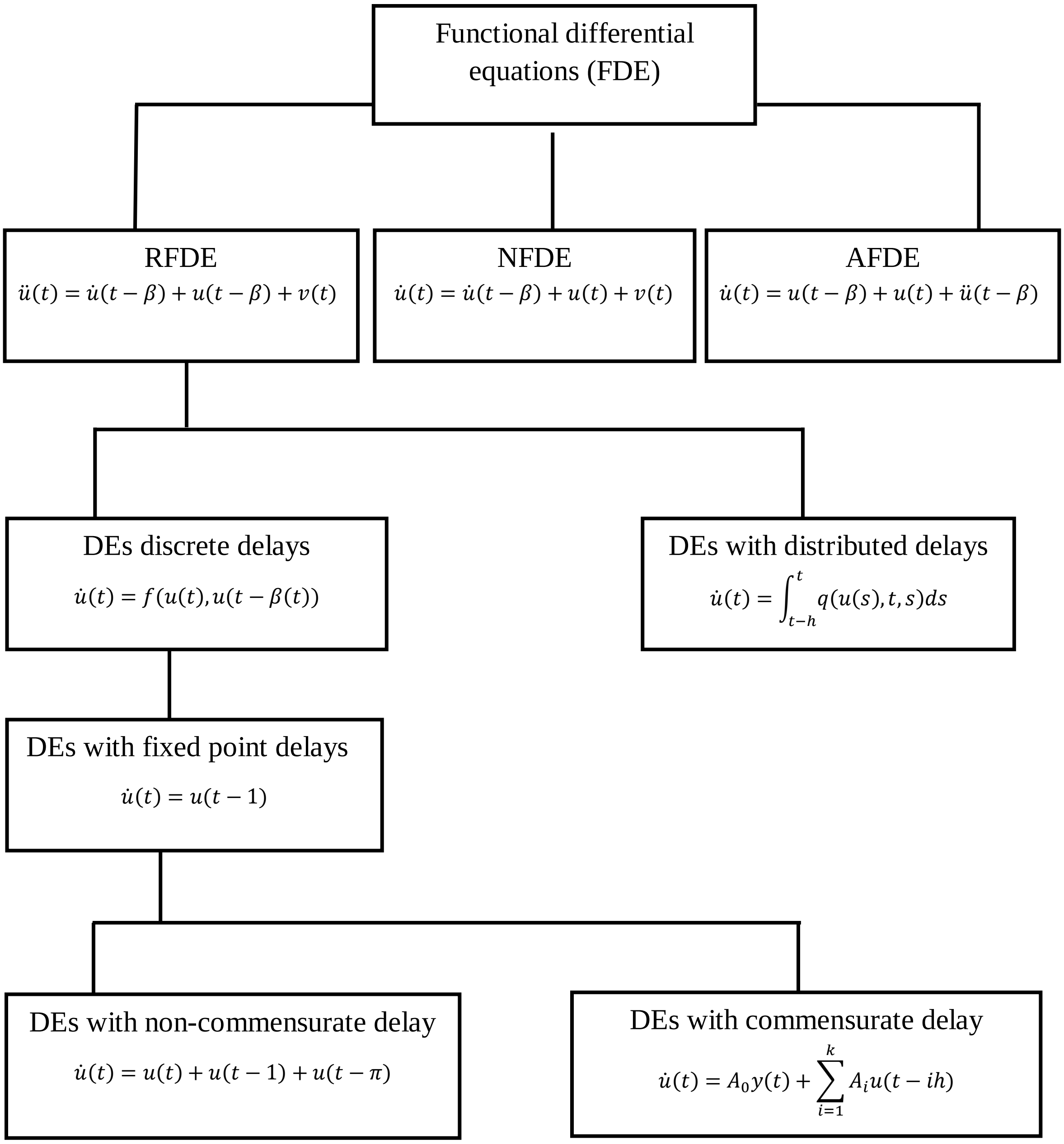


Figure 2.3: Classification of FDEs and RFDEs, (Schoen, 1995)

2.4 Classification of Delay Differential Equations (DDEs)

Delay differential equations can be classified as (Lumb, 2004):-

- Linear delay differential equations (LDDEs).
- Nonlinear delay differential equations (Non-LDDEs).
- Stochastic delay differential equations (SDDEs)
- Neutral delay differential equations (NDDEs).
- Autonomous delay differential equations (never changing under the chang t).
- Non-autonomous delay differential equations.

2.5 Types of Delay Differential Equation and its Applications

The fact that the ordinary differential equation models are replaced by the delay differential equation models led to the rapid growth of delay differential equation models in a variety of fields and each field has its scope of applications. The first mathematical modeler is Hutchinson; he introduced delay in biological model (Driver, 1977). Various classes of delay differential equation have various range of application (Lumb, 2004). For instance, retarded differential equation (RDDE) is applied in radiation damping (Chicone et al., 2001), modeling tumor growth (Buric and Todorovic, 2002), the application area of distributed delay differential equation is in model of HIV infection (Nelsonand Perelson, 2002), Biomodeling,, neutral delay differential equations (NDDE) application area is distributed network (Kolmanoviskii and Myshkis, 1999), Fixed differential equation is applied in Cancer chemotherapy (Kolmanoviskii, 1999) and infectious disease modeling (Harer et al., 2010), and another model, Single fixed delay application is in Immunology ((Luzyanina et al., 2001) and Nicholson blowflies model (Kolmanoviskii and Myshkis, 1999).

2.6 Linear Delay Differential Equations (LDDEs)

We consider the linear first order delay differential equations, with single constant-delay and constant coefficients

$$\dot{u}(t) = a(t)u(t) + b(t)u(t-\beta); \;\; for \quad t > 0 \qquad (2.4)$$
$$u(p) = \alpha(p); \quad -\beta \le p \le 0$$

Where $\alpha(p)$ is the initial history function and, $a(t)$ and, $b(t)$ are any constant functions, with$\beta > 0$. β, Is constant function In general the solution $u(t)$ of equation (2.4) has a jump discontinuity in $\dot{u}(t)$ at the initial point. The left and right derivatives are not equal.

$$\lim_{t\to 0-} \dot{u}\,(t) = p'(0) \neq \lim_{t\to 0+} \dot{u}\,(t)$$

For example, the simple delay differential equation $\dot{u}(t) = u(t-1), t \ge 0$ with history function $u(t) = 1, t \le 0$, it is easy to verify that, $\dot{u}(0^+) = 1 \neq \dot{u}(0^-) = 0$. Another example: $\dot{u}(t) = -u(t-1), t \ge 0$ with history function $u(t) = 1, t \le 0$, it is easy to verify that, $\dot{u}(0^+) = -1 \neq \dot{u}(0^-) = 0$.The second derivative $\ddot{u}(t)$ is given by $\ddot{u}(t) = -\dot{u}(t-1)$ and therefor it has a jump at $t = 1 = \beta$, the third derivative $\dddot{u}(t)$ is given by $\dddot{u}(t) = -\ddot{u}(t-1) = -\dot{u}(t-2)$, and hence it has jump at $t = 2 = 2\beta$, in general, the jump in $\dot{u}(t)$ at $t = 0$ propagates to a jump in $u^{n+1}(t)$ at time $t = n$. The propagation of discontinuities is a feature of DDEs that does not occur in ODEs and …etc. This propagates becomes subsequence discontinuity points (Bellen and Zennaro, 2013).

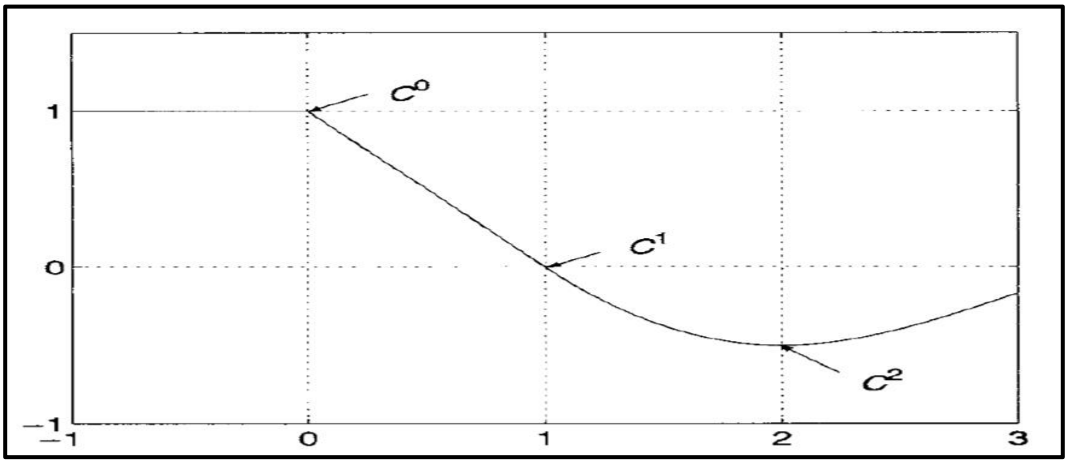


Figure 2.4: The propagation of discontinuities

2.7 Uniqueness and Existence of DDEs

Delay differential equation (DDE) as Ordinary differential equation (ODE), have the theorem of uniqueness and existence. The Boundary Value Problem (BVP)

$$\dot{u}(t) = au(t-\beta),\ \beta > 0\ , on\ [0, d] \tag{2.5}$$

$$u(t) = \theta(t), on\ [-\beta, 0]$$

Where a and β are any real numbers, with $\beta > 0$ and $d > 0$, $\theta \in C^1[-\beta, 0]$. As we stated before that, the delay differential equations is a special class of functional differential equations, (Falbo, 1995), the interval $[-\beta, 0]]$ is called the (pre-interval) and the function θ is called (pre-function).

2.7.1 Existence Theorem

$$\dot{u}(t) = au(t-\beta),\ \beta > 0\ , on\ [0, d], d > 0 \tag{2.6}$$
$$u(t) = 0 \quad on\ [-\beta, 0]$$

Has unique solution $u(t) \equiv 0$ on the interval$[-\beta, 0]$.

Note: If $d > \beta$ this implies that $u \equiv 0$ is the solution on the interval$[0, \beta]$, then if $d > 2\beta$ we transfer the DE to the interval$[\beta, 2\beta]$, then we have new interval$[0, \beta]$, on which $u = 0$. This implies that we can solve the problem only on $[0,2\beta]$. If $\beta < d < 2\beta$, then the solution expanded on $[0, d]$. So that if we continue this way, the solution moved along cover $[0, d]$, for any positive real number d.

Proof: we observe that the DE itself is linear first order delay differential equation with single constant-delay and constant coefficient, and we observe that by plugging the function $u \equiv 0$ is the solution on the interval $[0, \beta]$. Now if $v(t)$ and $u(t)$ are any two solution, then $\dot{v}(t) = av(t-\beta)$ and $\dot{u}(t) = au(t-\beta)$. As well, if we define a function $z(t) = J_1u(t) + J_2v(t)$ for ant two constants J_1, J_2, then $\dot{z}(t) = az(t-\beta)$. This mean that, $z(t)$ is also a solution to the DE. As we know the function $u(t) \equiv 0$ is one solution, now by contradiction, there exists another function $v(t)$ not identically zero that satisfies the equation (2.6). Thus $v(t)$ satisfies the DE on the interval $[0, \beta]$, and the function 0 (zero) on the interval $[-\beta, 0]$.

But if we take on a nonzero value at least once somewhere in semi-open interval $(0, \beta]$. This implies we are supposing that $v(r) \neq 0$ for some $r \in (0, \beta]$.Let H be the set of reals such that $\tau \in H$ if and only if either $\tau = -\beta$ or $\tau > -\beta$ and $v(t) = 0$ for all $t \in [-\beta, \tau]$.

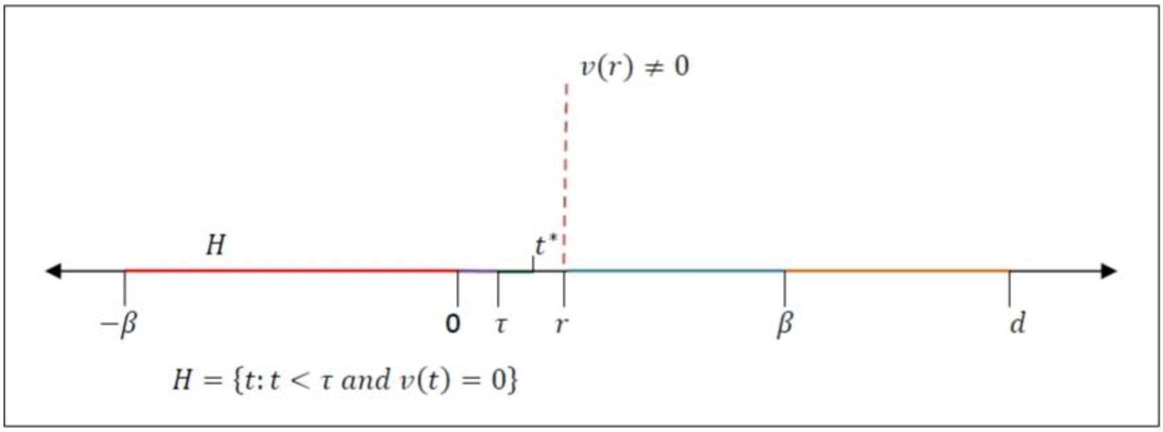


Figure 2.5: The set, H

The set H exist since it contains all of the points in the interval $[-\beta, 0]$. H is bounded above, since r is one of its upper bounds. Suppose t^* be the Least Upper Bound (LUB) of H. Note that $v(t^*) = 0$, otherwise there exist a positive number, c such that $v(t) \neq 0$ on $(t^* - c, t^* + c)$, making $t^* - c$ an upper bound of H, less than the least upper bound of H.We assume that, $t^{**} = t^* + \frac{\beta}{2}$, then $\exists$ a number t_0 between t^* and t^{**} such that $v(t_0) \neq 0$. If there is not any t_0, then $v(t) = 0, \forall \, t$ between t^* and t^{**}, making t^* not UB of H. Since v is continuous then $\exists$ an interval $[e, r]$ containing t_0 as an interior point and such that for all $t \in [e, r]$, $v(t) \neq 0$. Let ε be the minimum of r and t^{**}. Therefore $v(t) \neq 0$ on the interval $[e, \varepsilon], \varepsilon \leq t^{**}$.Now, let K be the number set such that $\tau \in K$ if and only if either $\tau = \varepsilon$ or $\tau < \varepsilon$ and $v(t) \neq 0$ for all $t \in (\tau, \varepsilon]$. We can note that K exists since $t_0 \in K$. Since $v(t^*) = 0$, K is bounded below because t^* is one of its lower bounds, assume x be the Greatest Lower Bound (GLB) of K. Since v is continuous at x then, $v(x) = 0$ otherwise would be nonzero throughout the open interval $(x - c^*, x + c^*)$, making x not a lower bound of K. Denote K by $(x, e]$, since for all $t \in K$, $t < t^{**} = t^* + \frac{\beta}{2}$, then $t - \beta \in H$ and $v(t - \beta) = 0$, so from the DE $\dot{v}(t) = av(t - \beta)H$. Hence, $\dot{v}(t) \equiv 0$ on $(x, e]$. This mean that $v(t) =$ a constant, J on $(x, e]$. But $v(x) = 0$, so by continuity of v at x, the constant must be zero.

Therefore $v(t) \equiv 0$ on $(x, e]$ contradiction the assumption that $v(t_0) \neq 0$ at some point in $[t^*, d]$.

2.7.2 Uniqueness Theorem

If $v(t)$ and $u(t)$ is a solution to the Boundary Value Problem (BVP) (2.5), then $v(t) \equiv u(t)$ on $[-\beta, d]$.

Proof: Let $z(t) = v(t) - u(t)$, then

$$\begin{aligned}
\dot{z}(t) &= \dot{v}(t) - \dot{u}(t) \\
&= av(t-\beta) - au(t-\beta) \\
&= az(t-\beta) \text{ on } (0, d].
\end{aligned}$$

As well, on $[-\beta, 0]$, $v(t) = u(t) = \theta(t)$; so $z(t) = 0$. Therefore $z(t)$ is the trivial solution satisfying equation (2.6), then $v(t) \equiv u(t)$ on $[-\beta, d]$.

2.8 Software Packages for Solving DDEs

Matlab is one of the best software programs to solve different class in mathematics, such as, optimization, graph theory, linear algebra, differential equations …etc. In (Bellen and Zennaro, 2003), they used a package continuous-time model simulation (CTMS) for solving delay differential equations. Today many codes for the numerical integration of delay differential equations are available, these involve, DDE23, DDESD…etc. we will show that how to use the Matlab solver DDE23 and DDESD to solve linear first order delay differential equations (DDEs) with constant delays to obtain the graph of DDEs.

2.8.1 Matlab illustrate one

Computing and plotting the solution of DDEs, on $[0,5]$, by using solver DDE23.

$$\begin{cases} \dot{u}(t) = -u(t-1.25), \mathrm{t} \geq 0 \\ u(t) = 1, t \leq 0 \end{cases}$$

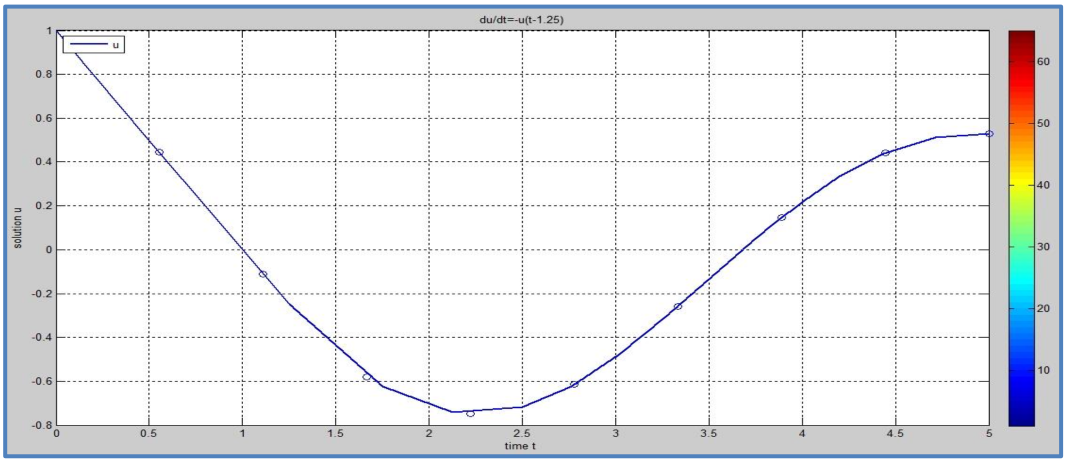


Figure 2.6: Solution of DDEs

Table 2.3: Value of u and t in Figure2.6 from Matlab illustrate one

Value of $u\ \&\ t$	**Columns 1 through 7**	**Value of $u\ \&\ t$**	**Columns 8 through 10**
('o')	$u = 1.0000, t = 0$ $u = 0.4444, t = 0.6$ $u = -0.1111, t = 1.3$ $u = -0.5799, t = 1.7$ $u = -0.7496, t = 2.3$ $u = -0.6143, t = 2.8$ $u = -0.2596, t = 3.4$	('o')	$u = 0.1465, t = 3.9$ $u = 0.4422, t = 4.9$ $u = 0.5287, t = 5$

Algorithm of DDEs in Matlab illustrate one

```
function VDde23
% solving DDEs
clear;
clc;
function dydt = ddex1de(t,y,Z)
ylag1 = Z(:,1);
dydt = ylag1(1);
end
function S = ddex1hist(t)
S = 1;
End lags = 1.25;
```

```
sol =
dde23(@ddex1de,lags,@ddex1hist,[0,
5]);
plot(sol.x,sol.y);
title('dy/dt=-y(t-1.25)');
xlabel('time t');
ylabel('solution y');
legend('y','Location','NorthWest')
;
tint = linspace(0,5,10);
Sint = deval(sol,tint) hold on
plot(tint,Sint,'o');
```

2.8.2 Matlab illustrate two

Computing and plotting the solution of DDEs, on $[0,5]$, by using solver DDE23.

$$\begin{cases} \dot{u}_1(t) = u_1(t-2), t \geq 0 \\ \dot{u}_2(t) = u_1(t-2) + u_2(t-0.5), t \geq 0 \\ u_1(t) = 1, u_2(t) = 1, t \leq 0 \end{cases}$$

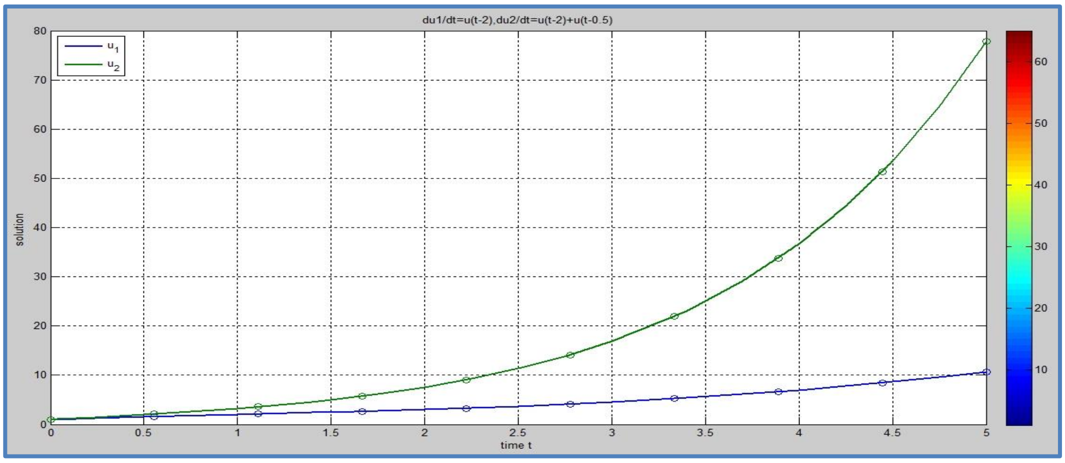


Figure 2.7: Solution of DDEs

Table 2.4: Value of u_1, u_2, and t in Figure2.7 from Matlab illustrate two

Value of u & t	**Columns 1↔7 (t , u_1)**	**Columns 1↔7 (t , u_2)**	**Value of u & t**	**Columns 8↔10 (t , u_1)**	**Columns 8↔10 (t , u_2)**
('o')	(0, 1.0000) (0.6, 1.5556) (1.2, 2.1111) (1.7, 2.6667) (2.3, 3.2469) (2.8, 4.0802) (3.4, 5.2222)	(0, 1.0000) (0.6, 2.1142) (1.23, 3.596) (1.7, 5.7932) (2.31, 9.066) (2.8, 14.149) (3.41, 21.926)	('o')	(3.9, 6.6728) (4.4, 8.4467) (5, 10.6667)	(3.9, 33.6886) (4.9, 51.3555) (5, 77.8691)

Algorithm of DDEs in Matlab illustrate two

```
function VDde23
% solving DDEs
clear;
clc;
function dydt = ddex1de(t,y,Z)
  ylag1 = Z(:,1);
  ylag2 = Z(:,2);
 dydt = [ylag1(1);ylag1(1)+ylag2(2)];
end
function S = ddex1hist(t)
  S = ones(2,1);end lags = [2,0.5];
```

```
sol =
dde23(@ddex1de,lags,@ddex1hist,[
0,5]); plot(sol.x,sol.y);
title('dy1/dt=y(t-2),dy2/dt=y(t-
2)+y(t-0.5)');
xlabel('time t');
ylabel('solution y');
legend('y_1','y_2','Location','N
orthWest');
tint = linspace(0,5,10);
Sint = deval(sol,tint)on end
```

2.8.3 Matlab illustrate three

Computing and plotting the solution of DDEs, on [0,5], by using solver DDE23.

$$\begin{cases} \dot{u}(t) = u(t-3) + u(t-0.5), t \geq 0 \\ u(t) = 1, t \leq 0 \end{cases}$$

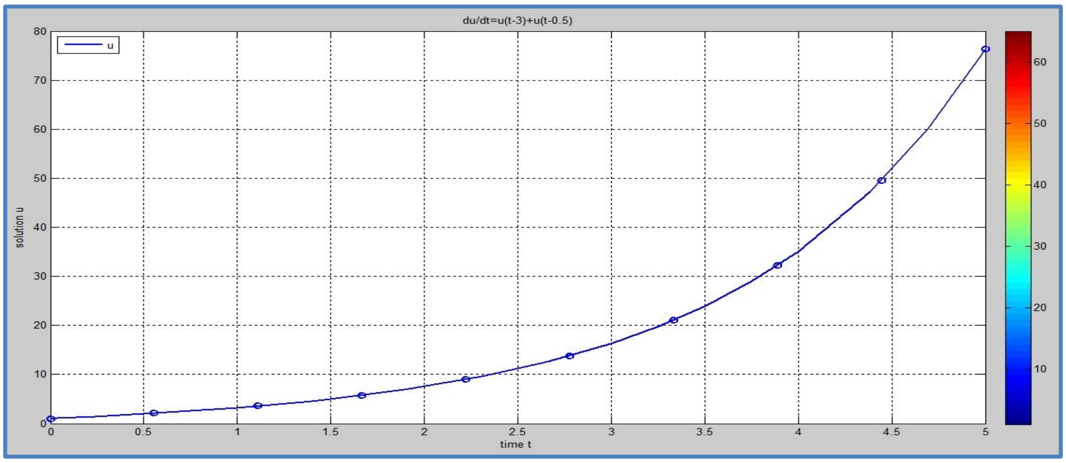


Figure 2.8: Solution of DDEs

Table 2.5: Value of u and t in Figure2.8 from Matlab illustrate three

Value of u & t	Columns 1 through 7	Value of u & t	Columns 8 through 10
('o')	$u = 1.0000, t = 0$ $u = 2.1142, t = 0.6$ $u = 3.5961, t = 1.2$ $u = 5.7931, t = 1.7$ $u = 9.0413, t = 2.3$ $u = 13.8427, t = 2.8$ $u = 21.0513, t = 3.4$	('o')	$u = 32.2607, t = 3.9$ $u = 49.5961, t = 4.4$ $u = 76.3627, t = 5$

Algorithm of DDEs in Matlab illustrate three

```
function VDde23
% solving DDEs
clear;
clc;
function dydt = ddex1de(t,y,Z)
ylag1 = Z(:,1)+Z(:,2);
dydt = ylag1(1);
end
function S = ddex1hist(t)
S = 1;
end
```

```
sol =
dde23(@ddex1de,lags,@ddex1hist,
[0,5]); plot(sol.x,sol.y);
title('dy/dt=y(t-3)+y(t-0.5)');
xlabel('time t');
ylabel('solution y');
legend('y','Location','NorthWes
t');
tint = linspace(0,5,10);
Sint = deval(sol,tint)
hold on plot(tint,Sint,'o');
```

2.8.4 Matlab illustrate four

Computing and plotting the solution of DDEs on [0,5], by using solver DDE23, (Shampi and Thompson, 2000).

$$\begin{cases} \dot{u}_1(t) = u_1(t-0.5), t \geq 0 \\ \dot{u}_2(t) = u_1(t-0.5) + u_2(t-0.8), t \geq 0 \\ \dot{u}_3(t) = u_2(t), t \geq 0 \\ u_1(t) = 1, u_2(t) = 1, t \leq 0 \end{cases}$$

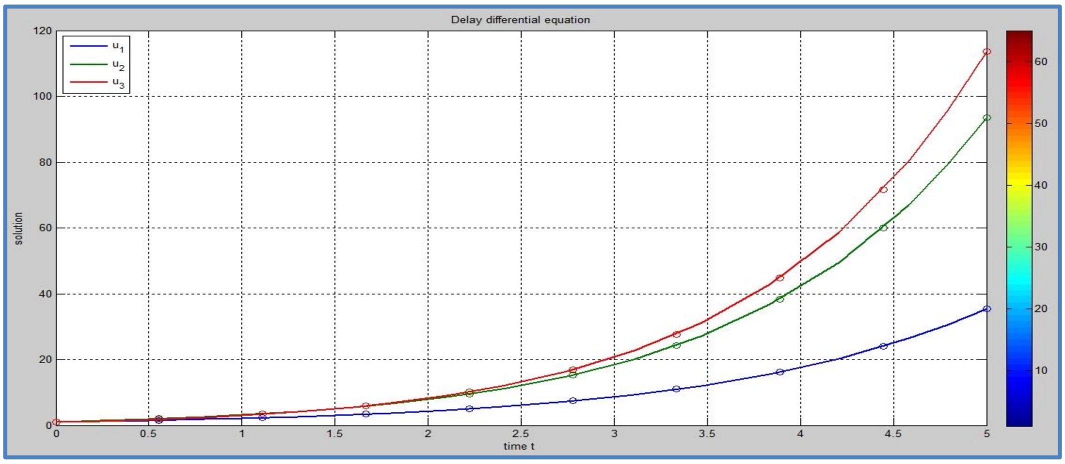


Figure 2.9: Solution of DDEs

Table 2.6: Value of u_1, u_2, u_3, and t in Figure2.9 from Matlab illustrate four

Value of $\boldsymbol{u \,\&\, t}$	**Columns 1 through 7** (t, u_1) (t, u_2) (t, u_3)	**Value of** $\boldsymbol{u \,\&\, t}$	**Columns 8 through 10** (t, u_1) (t, u_2) (t, u_3)
('o')	(0, 1.0000), (0, 1.00000), (0, 1.0000) (0.6, 1.557), (0.6, 2.112), (0.6, 1.864) (1.2, 2.298), (1.2, 3.506), (1.2, 3.393) (1.7, 3.396), (1.7, 5.822), (1.7, 5.935) (2.2, 5.020), (2.2, 9.478), (2.2, 10.10) (2.8, 7.421), (2.8, 15.24), (2.8, 16.85) (3.4, 10.97), (3.4, 24.25), (3.4, 27.64)	('o')	(3.8, 16.21), (3.8, 38.28), (3.8, 44.7) (4.4, 23.96), (4.4, 60.01), (4.4,71.60) (5.0, 38.43), (5.0, 97.51), (5., 117.58)

Algorithm of DDEs in Matlab illustrate four

```
function VDde23
% solving DDEs
clear;
clc;
function dydt = ddex1de(t,y,Z)
  ylag1 = Z(:,1);
  ylag2 = Z(:,2);
 dydt = [ylag1(1); ylag1(1)+ylag2(2)
 y(2)];
end
function S = ddex1hist(t)
  S = ones(3,1);
```

```
sol =
dde23(@ddex1de,lags,@ddex1hist,[
0,5]); plot(sol.x,sol.y);
title('Delay differential
equation');
xlabel('time t');
ylabel('solution y');
legend('y_1','y_2','y_3','Locati
on','NorthWest');
tint = linspace(0,5,10);
Sint = deval(sol,tint)
hold on plot(tint,Sint,'o');
```

CHAPTER 3
METHODS AND METHODOLOGY FOR SOLVING LDDE

In this chapter methods for solving linear first order delay differential equations (LDDEs) will be discussed; there are many methods for solving DDEs: Characteristic, Steps, Matrix Lambert Function, Differential transform, a domain e-composition, Multistep Block, Theta, and Laplace transform …etc. We will use some of these methods to solve linear first order delay differential equations, with single constant-delay and constant coefficients**.** Graph-Matica and Matlab will be used to plotting the graph in this chapter, to understanding this chapter well; the reader must have a good background in differential equations and knowing how to use Matlab codes, because Matlab is very smooth to solve many problems in various class of mathematics.

3.1 Characteristic Method

Consider the linear first order delay differential equation, with single constant-delay and constant coefficient, with Boundary Value Problem (BVP), (Falbo, 1995).

$$\begin{cases} \dot{u}(t) = \delta u(t-\beta),\ \beta > 0,\ on\,[0,d] \\ u(t) = \theta(t),\ on\,[-\beta,0] \end{cases} \tag{3.1}$$

To solve linear first order delay differential equation (3.1) by method of characteristic (MOC), following, (Hale and Lunel, 1993). Recall that in the case of n linear homogenous ordinary differential equations with constant coefficients there are n linearly independent solutions. And we know that the general solution is expressible as an arbitrary linear combination of these n solutions. But the situation is more complicated for linear first order delay differential equation with single constant-delay and constant coefficients, because this equation has infinitely many linearly independent solutions. The characteristic equation for a homogeneous linear delay differential equation with constant coefficients is obtained from the equation by looking for nontrivial solutions of the form De^{st} where D is constant. Suppose (3.1) has non trivial solution $u(t) = De^{st}$, if and only if $g(s) = Se^{s\beta} - \delta = 0$.

If we plugging De^{st} into equation (3.1), $\dot{u}(t) = \delta u(t-\beta), \beta \neq 0$, then we obtain the nonlinear characteristic equation $Se^{s\beta} - \delta = 0$. When β is a single constant non-negative number, and the function $g(s)$ is defined as

$$g(s) = Se^{s\beta} - \delta \qquad (3.2)$$

Where, δ is the parameter. Figure (3.1) shows the graph of equation (3.2), which we sketch a few member of this δ-parameter set of curves. Then we get four various cases when β is a single constant-delay and different value of parameter δ.

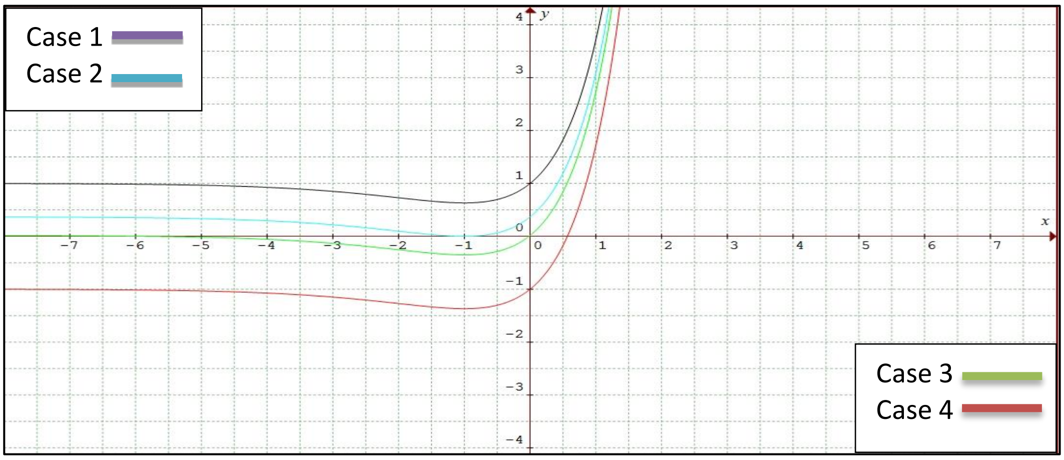


Figure 3.1:$g(s) = Se^{s\beta} - \delta$ for fixed β and different δ

Now we need to show the complex roots of $g(s) = 0$, this implies that

$$Se^{s\beta} - \delta = 0 \qquad (3.3)$$

If $\delta = 0$, in this situation, the delay differential equation $\dot{u}(t) = 0$ and equation (3.3) has only one root $s = 0$, then the solution is the constant $\theta(0)$. The our aim here is when $\delta \neq 0$, therefor we have four cases. This equation has infinite many complex (non-real) solutions, and then we describe roots of $g(s)$ belongs to these four possibility cases:

Case one: If $\delta < -\frac{1}{\beta e} < 0$, then $g(s)$ has no real roots.
Case two: If $\delta = -\frac{1}{\beta e}$, then $g(s)$ has exactly one real root, $s = -\frac{1}{\beta}$.
Case three: If $-\frac{1}{\beta e} < \delta < 0$, then $g(s)$ has exactly two real roots, both non-positive, and
Case four: If $\delta > 0$, then $g(s)$ has exactly one real root, s, and $s > 0$.

3.2 The Method Solution

In this section we will show conditions for each cases and write the general formal solutions, to solve Boundary Value Problem (3.1)

$$\begin{cases} \dot{u}(t) = \delta u(t-\beta), \qquad \beta > 0, \qquad on\ [0,d] \\ u(t) = \ \theta(t), \qquad on\ [-\beta, 0] \end{cases}$$

3.2.1 Case one

$\delta < -\frac{1}{\beta e} < 0$, this mean $g(s)$ has no real roots. But in order to start the first step of solution, we can order complex number $w = \mu + i\gamma$, such that $we^{w\beta} - \delta = 0$. If

$$(\mu + i\gamma)e^{(\mu+i\gamma)\beta} - \delta = 0, \text{ then}$$

$$(\mu + i\gamma)e^{i\gamma\beta} = \delta e^{-\mu\beta}$$

$$(\mu + i\gamma)(\cos(\gamma\beta) + i\sin(\gamma\beta)) = \delta e^{-\mu\beta}$$

This implies that

$$\mu\cos(\gamma\beta) - \gamma\sin(\gamma\beta) = \delta e^{-\mu\beta} \tag{3.4}$$

$$\gamma\cos(\gamma\beta) + \mu\sin(\gamma\beta) = 0 \tag{3.5}$$

Or

$$\mu = -\gamma\cot(\gamma\beta)\,,\ \gamma \neq 0 \tag{3.6}$$

Then we can note that

$$\lim_{\gamma\to 0} -\gamma\cot(\gamma\beta) = \lim_{\gamma\to 0}\frac{-\gamma\beta\cos(\gamma\beta)}{\beta\sin(\gamma\beta)} = -\frac{1}{\beta}$$

Apply L'Hopital's Theorem: For $\lim_{\gamma\to a}\left(\frac{q(\gamma)}{p(\gamma)}\right)$, if

$$\lim_{\gamma\to a}\left(\frac{q(\gamma)}{p(\gamma)}\right) = \frac{0}{0}$$

Or

$$\lim_{\gamma\to a}\left(\frac{q(\gamma)}{p(\gamma)}\right) = \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{\gamma \to a} \left(\frac{q(y)}{p(\gamma)} \right) = \lim_{\gamma \to a} \left(\frac{q(y)'}{p(\gamma)'} \right)$$

Test L'Hopital's condition: $\frac{0}{0}$

$$\lim_{\gamma \to 0} \frac{-\gamma\beta \cos(\gamma\beta)}{\beta \sin(\gamma\beta)} = \lim_{\gamma \to 0} \frac{(-\gamma\beta \cos(\gamma\beta))'}{(\beta \sin(\gamma\beta))'}$$

Apply product rule: $(q.p)' = q'.p + q.p'$

$$\lim_{\gamma \to 0} \frac{(-\gamma\beta \cos(\gamma\beta))'}{(\beta \sin(\gamma\beta))'} = \lim_{\gamma \to 0} \left(\frac{-\beta(\cos(\beta x) - \beta x \sin(\beta x))}{\beta^2 \cos(\beta x)} \right)$$

$$= \lim_{\gamma \to 0} \left(\frac{\beta x \sin(\beta x) - \cos(\beta x)}{\beta \cos(\beta x)} \right)$$

$$= \frac{\beta(0) \sin(\beta.0) - \cos(\beta.0)}{\beta \cos(\beta.0)} = -\frac{1}{\beta}$$

when $\gamma \neq 0$, substitute μ from equation (3.6) into equation (3.4), then we get.

$$\gamma = -\delta \sin(\gamma\beta)\, e^{\gamma\beta \cot(\gamma\beta)} \tag{3.7}$$

Now, let $X = \gamma\beta$, then

$$X = -\delta\beta \sin(X)\, e^{X \cot(X)} \text{ , where } -\beta\delta > \frac{1}{e} \tag{3.8}$$

If we find the intersection of the line $Y = X$, for solving the equation (3.8) with the one-parameter set of curves.

$$Y = -\delta\beta \sin(X)\, e^{X \cot(X)} \tag{3.9}$$

As we say that before, β is single constant-delay and δ is the coefficient, Figure (3.2) shows that equation (3.8) has infinitely many solutions, denoted by, $X_i, i = 1,2,3, \dots$, this for case one, and we can use some of Numerical Methods to obtain solutions for different given values of δ, such as Newton's Method, (Falbo, 1995).

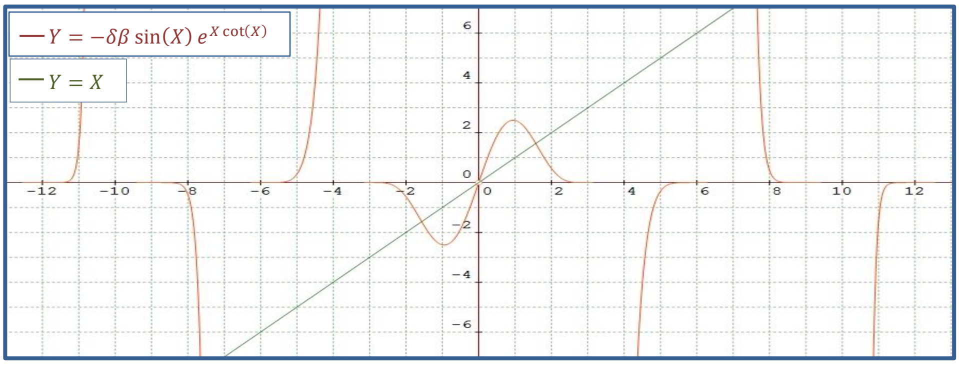


Figure 3.2: $Y = X$ and $Y = -\delta\beta \sin(X)\, e^{X \cot(X)}$

We know, $\gamma = X/\beta$, this implies that $\gamma_n = X_n/\beta$, now from equation (3.6) we obtain μ_n, then the roots of equation (3.8) are $\mu_n + i\gamma_n$, and the characteristic solutions are $e^{\mu_n t}\cos(\gamma_n t)$ and $e^{\mu_n t}\sin(\gamma_n t)$, so the formal solution to the linear first order delay differential equations, (LDDEs) is

$$u(t) = \sum_{n=1}^{\infty} e^{\mu_n t}\left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right) \tag{3.10}$$

Because the Boundary Value Problem (3.1) is linear, and $\delta < -\frac{1}{\beta e}$, where D_{1n} and D_{2n} are arbitrary constant, if we observe the point (X, Y) is that, when $X > 0$, the set of curves defined by equation (3.9) are intersected to the right of the vertical asymptotes that are non-even multiples of π .Then the values of μ_n are negative at all these points of intersection, so that when $|X| \to \infty$, the values of μ_n are decrease, as well as:

If we are thinking for some non-negative integers r and n, $\delta = -\frac{(4r+1)\pi}{2\beta} = \gamma_n$, then $\mu_n = 0$, for that n: so, the solutions are vacillate and undamped, but $\mu_n < 0, \forall$ other values of γ_n, and the vacillations in equation (3.10) are damped by the fullness $e^{\mu_n t}$.

3.2.2 Case two

From equation (3.6) when $\lim_{\gamma\to 0}\mu = -\frac{1}{\beta}$, which is mean that $\mu \to -\frac{1}{\beta}$ as $\gamma \to 0$, continuity at $\gamma = 0$, this implies that equation (3.4) and (3.5) are satisfied by $(\mu, \gamma) = \left(-\frac{1}{\beta}, 0\right)$, and so $\delta = -\frac{1}{\beta e}$, when $\gamma = 0$, then $g(s)$ has one real root $s = -\frac{1}{\beta}$, and we can found the real root $\mu = -\frac{1}{\beta}$, from equations (3.4) and (3.5) when $\gamma = 0$.

So we will add a new part characteristic solution $e^{(-1/\beta)t}$ to the formal solution of linear first order delay differential equations, (LDDEs), with Boundary Value Problem (3.1) which is of the form

$$u(t) = D_0 e^{(-1/\beta)t} + \sum_{n=1}^{\infty} e^{\mu_n t} \left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right) \qquad (3.11)$$

Where μ_n and γ_n are roots of equations (3.4) and (3.5) for this δ.

3.2.3 Case three

If $-\frac{1}{\beta e} < \delta < 0$, then $g(s)$ has two non-positive real roots, $s_1 < -\frac{1}{\beta} < s_2$. To solve for s_2 use Newton's Method, with initial value, start point $h_0 = -\frac{1}{2\beta}$ and for s_1, the start point is $h_0 = -\frac{2}{\beta}$, and for each positive integer k, define

$$h_{k+1} = h_k - \frac{g(h_k)}{g'(h_k)}$$

Then, $s_2 = \lim_{k\to\infty} h_k$, the two new characteristic solutions $e^{s_1 t}$ and $e^{s_2 t}$, obtained from equations (3.4) and (3.5). When, $\delta \in \left(-\frac{1}{\beta}, 0\right)$, so the formal solution to the linear first order delay differential equations, (LDDEs), with Boundary Value Problem (3.1) is

$$u(t) = D_1 e^{(s_2)t} + D_2 e^{(s_1)t} + \sum_{n=1}^{\infty} e^{\mu_n t} \left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right) \qquad (3.12)$$

3.2.4 Case four

If $a > 0$, the equation (3.3), $Se^{s\beta} - \delta = 0$ has exactly one positive root s, we can use Newto-n's Method to find it with initial value, start point $h_0 = 1$, so when $\delta > 0$ the formal solution to linear first order delay differential equations (LDDEs), with Boundary Value Problem (3.1) is

$$u(t) = D_3 e^{(s)t} + \sum_{n=1}^{\infty} e^{\mu_n t} \left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right) \qquad (3.13)$$

Note: so we can solve any equation which is linear first order delay differential equations (LDDEs) with Boundary Value Problems (BVPs), by one of these four cases , but the important thing here to show and write the general formal solution to the Boundary Value

Problems, we will talking about the general solution and the approximate solution in the next section.

3.3 The General Solution

The values of $\mu_n < 0$ for all cases and all the infinite series solutions in each of the equations (3.10) through (3.13) are convergent. Now we summarize the formal solutions to the linear first order delay differential equations (LDDEs) with Boundary Value Problems (BVPs)

3.3.1 Theorem

Assume β be any non-negative number, $\delta \in \mathbb{R} \setminus [0]$, and equation (3.3) has complex roots $\mu_n + i\gamma_n$ obtained from equation (3.4) and (3.5), then for arbitrary constants D_{1n} and D_{2n} the function $u(t)$ defined as follows

$$u(t) = D_0 e^{(-1/\beta)t} + D_1 e^{(s_2)t} + D_2 e^{(s_1)t} + D_3 e^{(s)t} \tag{3.14}$$

$$+\sum_{n=1}^{\infty} e^{\mu_n t} \left(D_{1n} \cos(\gamma_n t) + D_{2n} \sin(\gamma_n t)\right)$$

Satisfies the equation $\dot{u}(t) = \delta u(t-\beta),\ \beta > 0,\ on\ [0, d], d > 0$

Provided that

i. $D_0 = D_1 = D_2 = D_3 = 0$, when $\delta < -\frac{1}{\beta e}$,
ii. $D_1 = D_2 = D_3 = 0$ and D_0 is arbitrary when $\delta = -\frac{1}{\beta e}$,
iii. $D_0 = D_3 = 0$ and D_1 & D_2 are arbitrary and s_1 and s_2 are the real roots of equation (3.3), when $-\frac{1}{\beta e} < \delta < 0$.
iv. $D_0 = D_1 = D_2 = 0$ and D_3 is arbitrary and s is the real root of equation (3.3) when $\delta > 0$.

Now to solve equation (3.1), we must use equation (3.14) for a given pair δ, β and a given function $\theta(t)$ with condition for $t \in [-\beta, 0]$.

$$\theta(t) = D_0 e^{(-1/\beta)t} + D_1 e^{(s_2)t} + D_2 e^{(s_1)t} + D_3 e^{(s)t}$$

$$+\sum_{n=1}^{\infty} e^{\mu_n t} \left(D_{1n} \cos(\gamma_n t) + D_{2n} \sin(\gamma_n t)\right) \tag{3.15}$$

3.3.2 Approximate solutions

To approximate the solution of equation (3.15), we define the function $u_m(t)$ as follows

$$u_m(t) = D_0 e^{(-1/\beta)t} + D_1 e^{(s_2)t} + D_2 e^{(s_1)t} + D_3 e^{(s)t} \qquad (3.16)$$
$$+ \textstyle\sum_{n=1}^{m} e^{\mu_n t} \left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right)$$

Because the characteristic functions $\{e^{\mu_n t}\cos(\gamma_n t), e^{\mu_n t}\sin(\gamma_n t)\}$ are linearly independent so, to prove the two characteristic functions are linearly independent, we need to take the Wronskian for these two solutions and show that it is not zero.

$$W = \begin{vmatrix} e^{\mu_n t}\cos(\gamma_n t) & e^{\mu_n t}\sin(\gamma_n t) \\ \mu_n e^{\mu_n t}\cos(\gamma_n t) - \gamma_n e^{\mu_n t}\sin(\gamma_n t) & \mu_n e^{\mu_n t}\sin(\gamma_n t) + \gamma_n e^{\mu_n t}\cos(\gamma_n t) \end{vmatrix}$$

$$= e^{\mu_n t}\cos(\gamma_n t)\left(\mu_n e^{\mu_n t}\sin(\gamma_n t) + \gamma_n e^{\mu_n t}\cos(\gamma_n t)\right)$$
$$- e^{\mu_n t}\sin(\gamma_n t)\left(\mu_n e^{\mu_n t}\cos(\gamma_n t) - \gamma_n e^{\mu_n t}\sin(\gamma_n t)\right)$$

$$= \gamma_n e^{2\mu_n t} cos^2(\gamma_n t) + \gamma_n e^{2\mu_n t} sin^2(\gamma_n t)$$

$$= \gamma_n e^{2\mu_n t}\left(cos^2(\gamma_n t) + sin^2(\gamma_n t)\right)$$

$$= \gamma_n e^{2\mu_n t}$$

Now, the exponential will never be zero and $\gamma_n \neq 0$, (if it were we wouldn't have complex roots !) and so $W \neq 0$. Therefore, these two solutions are in fact a fundamental set of solutions and so the approximate solution is equation (3.16). Therefore $u_m(0) = \theta(0)$ for continuity at 0. If we uniformly partition $[-\beta, 0]$ into j subintervals where $j = 2m - 1 + b$ points, here b is depend on the number of arbitrary coefficients through the first four which b is either $0,1,2,3$ or 4. We denote the partition of $[-\beta, 0]$ by σ_j so this implies that its points are: $-\beta = t_0 < t_1 < \cdots < t_j = 0$ then $u_m(t_i)$= $\theta(t_i)$, for$i = 0,1, \dots, j-1$, is a $j \times j$ non-singular linear system that can be solved for its coefficient.

Note: We can apply the Characteristic Method to the Boundary Value Problem (3.17), for given $\beta > 0, d > 0$, (Hale and Lunel, 1993).

$$\begin{cases} \dot{u}(t) = c_1 u(t) + c_2 u(t-\beta),\ \beta > 0,\ on\ [0,d] \\ u(t) = \ \theta(t),\ on\ [-\beta, 0] \end{cases} \tag{3.17}$$

As we assumed in the equation (3.1) , we will assume that the solution to (3.17) has the form $u(t) = De^{zt}$, with D arbitrary for some z, (real or complex)

$$Dze^{zt} = c_1 De^{zt} + c_2 De^{zt-z\beta} \tag{3.18}$$

$$(z - c_1)e^{z\beta} - c_2 = 0 \tag{3.19}$$

Now, suppose $z - c_1 = k$, this becomes $ke^{k\beta} - c_2 e^{-c_1\beta} = 0$. Since c_1, c_2, and β are given, we can write $c_2 e^{-c_1\beta}$ as a single number, φ, obtaining

$$ke^{k\beta} - \varphi = 0 \tag{3.20}$$

Now we can solve equation (3.20) for k as we solved equation (3.3) for s.

3.4 Method of Steps

In this section we will show how to use the method of steps to solve linear first order delay differential equations, the method of steps is one of the rudimentary methods that can solve some delay differential equation such as lineal first order delay differential equations, with single constant delay and constant coefficients analytically. The general idea in this method is change the delay differential equation (DDE) on a given interval to ordinary differential equation (ODE) over that interval, and this process is repeated in the next interval. Consider the following general linear delay differential equation:

$$\dot{u}(t) = r_0 u(t) + r_1 u(t-\beta_1) + r_2 u(t-\beta_2) + \cdots + r_n u(t-\beta_n) \tag{3.21}$$
$$u(t) = \theta_0(t) \qquad t_0 - \beta \le t \le t_0,\ \beta > 0$$

The most natural solution for equation (3.21) is called the method of steps or " The method of successive integrations ", (El'sgol'ts and Norkin, 1973). The function $u(t)$ is the given function $\theta_0(t)$ so that $u(t)$ is known in the interval $[t_0 - \beta, t_0]$, $\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n$ the

$$\dot{u}(t) = r_0 u(t) + r_1\theta_0(t-\beta_1) + r_2\theta_0(t-\beta_2) + \cdots + r_n\theta_0(t-\beta_n) \tag{3.22}$$
$$u(t_0) = \theta_0(t_0) \qquad t_0 \le t \le t_0 + \beta, \qquad \beta > 0$$

Since for $t_0 \le t \le t_0 + \beta$, arguments $\{(t-\beta_1),(t-\beta_2),\dots (t-\beta_n)\}$, and $\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n$,varies in the initial interval set $[t_0 - \beta, t_0]$, so we get:

$$\dot{u}(t) = r_0 u(t) + r_1\theta_1(t-\beta_1) + r_2\theta_1(t-\beta_2) + \cdots + r_n\theta_1(t-\beta_n) \tag{3.23}$$
$$u(t_0+\beta) = \theta_1(t_0+\beta) \quad t_0+\beta \le t \le t_0 + 2\beta, \qquad \beta > 0$$

Then if we continue in this way

$$\dot{u}(t) = r_0 u(t) + r_1\theta_n(t-\beta_1) + r_2\theta_n(t-\beta_2) + \cdots + r_n\theta_n(t-\beta_n) \tag{3.24}$$
$$u(t_0+n\beta) = \theta_n(t_0+n\beta) \quad t_0+n\beta \le t \le t_0 + (n+1)\beta,\ \beta > 0$$

Note 1: we can apply the method of steps to solve the linear first order delay differential equation by another way, but have the same idea of this method, especially if the history function is constant, consider the lineal first order delay differential equations, with single constant-delay and constant coefficient

$$\begin{cases} \dot{u}(t) = u(t-\beta), & 0 \le t \le \beta \\ u(t) = a, & -\beta \le t \le 0 \end{cases} \tag{3.25}$$

When a is arbitrary constant, assume that we have $u(t) = g_{k-1}(t)$ over some interval$[t_k - 1, t_k]$. Then over the interval$[t_k, t_k + 1]$, we have by separation of variables, (Heffernan and Corless, 2006).

$$\int_{g_{k-1}(t_k)}^{u(t)} dx^* = \int_{t_k}^{t} g_{k-1}(t^*-\beta)dt^* \tag{3.26}$$

$$\therefore u(t) = g_k(t) = g_{k-1}(t_k) + \int_{t_k}^{t} g_{k-1}(t^*-\beta)dt^* \tag{3.27}$$

Note 2: if we have this kind of linear delay first order differential equation (LDDE), with single constant-delay and constant coefficients:

$$\dot{u}(t) = r_1(t)u(t) + r_2(t)u(t-\beta), for\ t \in\ [0,\beta] \tag{3.28}$$
$$u(t) = H(t), for\ t \in [-\beta, 0]$$

$r_1(t) \neq 0$ and $r_2(t) \neq 0$, are constant functions, $\beta > 0$. Again to solve the equation (3.28), we will use the method of steps and apply its condition. On the interval $[-\beta, 0]$ the history function is the given function $H(t)$, so the history function is known there. Thus we can say the equation is solved for the interval $[-\beta, 0]$, now when $t \in [0, \beta], t - \beta \in [-\beta, 0]$, so $u(t-\beta)$ becomes $u_0(t-\beta)$ on $[0, \beta]$. So the equation (3.28) in the interval $[0, \beta]$ becomes (Falbo, 2006).

$$\dot{u}(t) = r_1(t)u(t) + r_2(t)u_0(t-\beta), for\ t \in \ [0, \beta] \qquad (3.29)$$
$$u(0) = H(0)$$

Then the equation (3.29) is an ordinary differential equation and not a delay differential equat-ion because $u_0(t-\beta)$ is known; it is $H(t-\beta)$. Thus we solve it on the interval $[0, \beta]$ and using intial condition, $u(0) = H(0)$.

$$\dot{u}(t) - r_1(t)u(t) = r_2(t)H(t-\beta), for\ t \in \ [0, \beta] \qquad (3.30)$$
$$u(0) = H(0)$$

The general solution of equation (3.30) is

$$u(t) = \frac{r_2}{e^{\int -r_1 dt}} \int e^{\int -r_1 dt} H(t-\beta)dt\, , on\ [0, \beta] \qquad (3.31)$$

Again on the interval $[\beta, 2\beta]$, the equation becomes

$$\dot{u}(t) - r_1(t)u(t) = r_2(t)u_1(t-\beta), for\ t \in \ [\beta, 2\beta] \qquad (3.32)$$
$$u(\beta) = u_1(\beta)$$

Note 3:

$$\dot{u}(t) = bu(t-\beta) \qquad (3.33)$$
$$u(t) = d,\ t_0 - \beta \leq t \leq t_0$$

Where d and β are constant, $\beta > 0$, applying the method of steps, we get

$$u(t) = d \sum_{k=0}^{\left[\frac{t-t_0}{\beta}\right]} t^k \frac{(t - t_0 - (k-1)\beta)^k}{k!}$$

where $[t]$ is the integer part of t , (El'sgol'ts and Norkin, 1973).

3.5 How to Use Matlab Codes

3.5.1 DDE23 solver

In this section we will show that how to use DDE23 solver in Matlab for solving linear first order delay differential equations, with constant single delay and constant coefficient, Our aim is to solve delay differential equations (DDEs) by easier way such as using DDE23 solver, whereas ordinary differential equations include derivatives which rely on the solution at the present value of the autonomous variable ("time") and delay differential equations include in addition derivatives which rely on the solution at previous times. The purpose of using the Matlabe codes such as DDE23, for both ODEs and DDEs that many problems its solutions have several continuous derivatives, and the discontinuities in low order derivatives require special attention because this is very serious matter for delay differential equations. For important things that the discontinuities are not uncommon for ordinary differential equations, but they are almost always present for delay differential equations. Then generally the discontinuity is appear in the first derivatives of the solution at the initial point (Thompson, 2000). To know how discontinuities propagate and smooth out, let us solve $u(t) = u(t-1)$ for $0 \leq t$ with history $\theta(t) = 1$ f or $t \leq 0$. With this history, the problem reduces on the interval $0 \leq t \leq 1$ to the ODE $\dot{u}(t) = 1$ with initial value $u(0) = 1$. Solving this problem we find that $u(t) = t + 1$ for $0 \leq t \leq 1$. Notice that the solution has a discontinuity in its first derivative at $t = 0$. In the same way we find that $u(t) = \frac{(t^2+1)}{2}$ for $1 \leq t \leq 2$. The first derivative is continuous at t = 1, but there is a discontinuity in the second derivative. In general the solution on the interval $[k, k + 1]$ is a polynomial of degree $k + 1$ and there is a discontinuity of order k + 1 at t = k, (Thompson, 2000).

A popular approach to solving DDEs is to extend one of the methods used to solve ODEs. Most of the codes are based on explicit Runge-Kutta methods. DDE23 takes this approach by extending the method of the Matlab ODE solver ODE23. The idea is the same as the so-called "method of steps" for solving DDEs that was used to solve an example in the last section.

Maybe another methods it will be used on Matlab to find approximate solutions, to be concrete, we describe the idea as applied to this example. In solving our example for $0 \leq t \leq 1$, the DDE reduces to an initial value problem for an ODE with $u(t-1)$ equal to the given history $\theta(t-1)$ and initial value $u(0) = 1$. We can solve this ODE numerically using any of the popular methods for the purpose. Analytical solution of the DDE on the next interval $1 \leq t \leq 2$ is handled the same way as the first interval, but the numerical solution is somewhat complicated, and the complications are present for each of the subsequent intervals. The first complication is that we must keep track of how the discontinuity at the initial point propagates because of the delays. Another is that at each discontinuity we start the solution of an initial value problem for an ODE. Runge-Kutta methods are attractive because they are much easier to start than other popular numerical methods for ODEs. Still another issue is the term $u(t-1)$ that is in principle known because we have already found $u(t)$ f or $0 \leq t \leq 1$. This has been a serious obstacle to applying Runge-Kutta methods to DDEs, so we need to discuss the matter more fully. Runge-Kutta methods, like all discrete variable methods for ODEs, produce approximations u_n to $u(v_n)$ on a mesh $\{v_n\}$ in the interval of interest, here $[0,1]$.

They do this by starting with the given initial value, $u_0 = u(a)$ at $v_0 = a$, and stepping from $u_n \approx u(v_n)$a distance of h_n to $u_{n+1} \approx u(v_{n+1})$ at $v_n + 1 = v_n + h_n$. The step size h_n is chosen as small as necessary to get an accurate approximation. It is chosen as big as possible so as to reach the end of the interval in as few steps as possible, which is to say, as cheaply as possible. In the case of solving our example on the interval $[1,2]$, we have values of the solution only on a mesh in $[0,1]$. So, where do the values $u(t-1)$ come from? In their original form Runge-Kutta methods produce answers only at mesh points, but it is now known how to obtain "continuous extensions" that yield an approximate solution between mesh points. The trick is to get values between mesh points that are just as accurate and to do this cheaply. In some cases the continuous extensions can be viewed as interpolants. The Runge-Kutta methods mentioned are all explicit recipes for computing u_{n+1} given u_n and the ability to evaluate the equation, (Thompson, 2000).

For reasons of efficiency, a solver tries to use the biggest step size u_n that will yield the specified accuracy, but what if it is bigger than the shortest delay β? In taking a step to $v_n + h_n$, we would then need values of the solution at points in the span of the step, but we are trying to compute the solution at the end of the step and do not yet know these values. A good many solvers restrict the step size to avoid this issue. Some solvers, including DDE23, use whatever step size appears appropriate and iterate to evaluate the implicit formula that arises in this way. We illustrate the straightforward solution of a DDE by computing and plotting the solution of Example, (Thompson, 2000). The equations

$$\dot{u}_1(t) = u_1(t-0.5),\, t \in [0,5]$$
$$\dot{u}_2(t) = u_1(t-0.5) + y_2(t-0.8),$$
$$\dot{u}_3(t) = u_2(t)$$
$$u_1(t) = 1, u_2(t) = 1,\ \ t \leq 0$$

The syntax has the form

$$sol = dde23(ddefile, lags, history, tspan);$$

The interval [0,5] is the interval of integration which is denote by (" tspan"), the history argu-ment is the name of a function that evaluates the solution at the input value of β and returns it as a column vector, the function for evaluating the DDEs is denoted by ("ddefile").

Here exam1h.m can be coded as:

$$function\, v \ = \ exam1h(t)$$
$$v \ = \ ones(3{,}1)$$

Quite often the history is a constant vector. A simpler way to provide the history then is to supply the vector itself s the history argument. The delays are provided as a vector lags, here $[0.5, 0.8]$. ddefile is the name of a function for evaluating the DDEs. Here exam1f.m can be coded as:

$$function\, v = \ exam1f(t, u, Z)$$

$$ulag1 \ = \ Z(:,1);$$

$$ulag2 = Z(:,2);$$

$$v = zeros(3,1);$$

$$v(1) = ulag1(1);$$

$$v(2) = ulag1(1) + ulag2(2);$$

$$v(3) = u(2);$$

The input t is the current t and y, an approximation to $u(t)$. The input array Z contains approximations to the solution at all the delayed arguments. Specifically, $Z(:,j)$ approximates $u(t-\beta_j)$ f or τj given as $lags(j)$. It is not necessary to define local vectors $ulag1$, $ulag2$ as we have done here, but often this makes the coding of the DDEs clearer. The ddefile must return a column vector, (Thompson, 2000).This is perhaps a good place to point out that DDE23 does not assume that terms like $u(t-\beta_j)$ actually appear in the equations. Because of this, you can use DDE23 to solve ODEs. If you do, it is best to input an empty array, [ ], f or lags because any delay specified affects the computation even when it does not appear in the equations. The input arguments of dde23 are much like those of ODE23, but the output differs formally in that it is one structure, here called sol, rather than several arrays $[t, u, \dots] = ode23(\ldots)$. The field $sol.x$ corresponds to the array t of values of the independent variable returned by ODE23 and the field $sol.u$, to the array u of solution values. So, one way to plot the solution is: $plot(sol.x, sol.u)$; After defining the equations in exam1f.m, the complete program exam1.m to compute and plot the solution is:

$$plot(sol.t, sol.u);$$

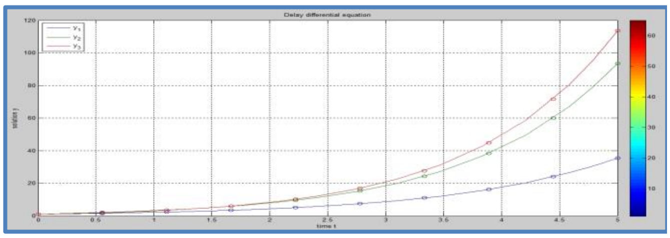
$$title('Figure\ 1. Example\ of\ DDEs')$$

$$xlabel('time\ t');$$

$$ylabel('u(t)');$$

Note that we must supply the name of the ddefile to the solver, i.e., the string 'exam1f' rather than exam1f. Also, we have taken advantage of the easy way to specify a constant history.

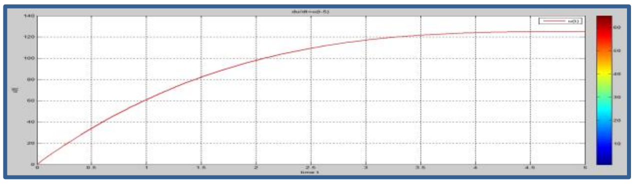
Table 3.1: explain the DDE23 solver to solve delay differential equation

$function\ v\ =\ exam1h(t)$ $v\ =\ ones(3,1)$	The history function is a constant vector. A simpler way to provide the history then is to equipping the vector itself as the history argument. The delays are provided as a vector lags, here $[0.5, 0.8]$.
$function\ v =$ $exam1f(t, u, Z)$ $ulag1\ =\ Z(:,1);$ $ulag2\ =\ Z(:,2);$ $v\ =\ zeros(3,1);$ $v(1)\ =\ ulag1(1);$ $v(2) = ulag1(1)$ $+ ulag2(2);$ $v(3)\ =\ u(2);$	The input t is the current t and u, an approximation to $u(t)$. The input array Z contains approximations to the solution at all the delayed arguments. Specifically, $Z(:, i)$ approximates $u(t-\beta_i)$ for β_j given as $lags(i)$. Define $ulag1$ and $ulag2$, but often this makes the coding of the DDEs clearer. The ddefile must return a column vector.
$\boldsymbol{sol\ =\ dde23('exam1f',}$ $\boldsymbol{[0.5, 0.8], ones(3, 1), [0, 5]);}$	DDE23 solver
$plot(sol.t, sol.u);$ $title('Figure\ 1.Example$ $of\ DDEs')$ $xlabel('time\ t');$ $ylabel('u(t)');$	We must define the equations in exam1f.m, the complete program exam1.m to compute and plot the solution. Note that we must supply the name of the $\boldsymbol{ddefile}$ to the solver.
Solution of delay differential equations	

3.5.2 DDESD solver

DDESD solver for solving delay differential equation (DDEs) with general delays, this code is like the DDE23 in some properties.

Table 3.2: explain the DDESD solver to solve delay differential equation

function $VDddesd2$ % *solving DDEs* *clear*; *clc*;	Define the m-file in the local function.
function $yp = ddefun(t,u,z)$ $up = z;$	The delay equation which is denoted by z.
function $d = delay(t,y)$ $d = t - \beta;$ *end* *function* $u = historu(t)$ $u = history;$ *end* $sol = ddesd(@ddefun,$ $@delay, @history,$ [0 a]);	Define the time delay and the hisory function , specify the history function in one of three ways. • A function of t such that $u =$ history $u(t)$ returns the solution $u(t)$ for $t \leq t_0$ as a column vector. • A constant column vector, if $u(t)$ is constant. • The solution sol from a previous integration, if this call continues that integration.
$tn = linspace(0,a);$ $yn = deval(sol,tn);$ $plot(tn,yn,'color','r','linewidth',2);$ $title('du/dt = z')$ $xlabel('time\ t'), ylabel('u(t)')$	The $\boldsymbol{ddefun}$ is function handel that evaluates the rghit sids of the differential eqautions, again $\boldsymbol{tspan}$ is interv-al of integration from $t_0 = tspan$ to $tf = tspan(end)$ with $t_0 < tf$.
Example: $\begin{cases} \dot{u}(t) = u(t-5), & t \in [0,5] \\ u(t) = t^2, & on\ [-5,0] \end{cases}$	

CHAPTER 4
SOLVING LDDE BY MOC AND METHOD OF STEPS

In this chapter we will give some examples of linear first order delay differential equations, with single constant delay (DDEs), and solving these examples by using the method of characteristic and method of steps, we will compare algebraic solutions with approximate solutions by using Matlab. For comparing the solutions we used two cods from the Matlab, DDE23 and DDESD solver, we will start algebraic solutions to every problem with drawing the solution in a graph, and then compare them with approximate solutions at another graph, therefore the best program for solving many types of delay differential equation is Matlab because it is deal with complicated problem by easier way, we have some another program to solve DDEs such as Maple…etc. But Matlab is the best, we have another program for drawing the solutions named Graph-Matica we used it.

4.1 MOC Examples

In this section we will solve some examples of linear first order delay differential equation, with single constant-delay and constant coefficients.

4.1.1 Example of case one

Use Characteristic Method to solve the (BVP) and sketch the graph, (Falbo, 1995) with given an approximate solution $\boldsymbol{u_2(t)}$.

$$\begin{cases} \dot{u}(t) = -1.25u(t-1.25), & on\ [0,30] \\ \theta(t) = e^{-t^2}, & on\ [-\beta, 0] \end{cases}$$

Solution: $d = 30 > 0, \beta = 1.25 > 0, \delta = -1.25 < 0$, now if we check the value of δ and β its belongs to case one, which $\delta < -\frac{1}{\beta e} < 0 \to -1.25 < -\frac{1}{1.25e} < 0$. Now substitute value of a and β into equation (3.8) we get

$$X = 1.5625 \sin(X)\, e^{X \cot(X)}$$

Solving this equation by Newton's Method for a few roots, we will obtain the values of X, and $X = \pm 1.5684, \pm 7.6465, \pm 13.9808, \ldots,$ since $X = \gamma\beta$ from this we get values of γ, which is equal to $\gamma: \gamma = 1.2547, 6.1172, 11.1847$. From equation (3.6) we get the values of μ, which is equal to $\mu: \mu = -0.0031, -1.2877, -1.7629$.

Now, we know our approximate solution is $u_2(0) = \theta(0)$, which $m = 2$, we must divide the interval $[-1.25,0]$ to subintervals depend on this $j = 2m - 1 + b$, so that $j = 2(2) - 1 + 0 = 3$, $[-1.25,0] = [-1.25, -0.8333] \cup [-0.8333, -0.41666] \cup [-0.41666, 0]$, this implies that $t = 0, t_1 = -0.41666, t_2 = -0.8333, t_3 = -1.25$.

Now since $u_m(t) = \theta(t) = e^{-t^2}, m = 2$, then

$$u_2(t) = \theta(t) = \sum_{n=1}^{2} e^{\mu_n t} \left(D_{1n}\cos(\gamma_n t) + D_{2n}\sin(\gamma_n t)\right)$$

$$e^{-t^2} = e^{\mu_1 t}(D_{11}\cos(\gamma_1 t) + D_{21}\sin(\gamma_1 t)) + e^{\mu_2 t}(D_{12}\cos(\gamma_2 t) + D_{22}\sin(\gamma_2 t))$$

$$= e^{-0.0031t} D_{11}\cos(1.2547t) + e^{-0.0031t} D_{21}\sin(1.2547t) + e^{-1.2877t} D_{12}\cos(6.1172t) + e^{-1.2877t} D_{22}\sin(6.1172t)$$

To obtain the value of arbitrary coefficients, $(D_{11}, D_{21}, D_{12}, D_{22})$, we need the values of t, where $t = 0, t_1 = -0.41666, t_2 = -0.8333, t_3 = -1.25$. We will get the 4×4 system of linear equations.

When $t = 0$

$$e^{-(0)^2} = e^{-0.0031(0)} D_{11}\cos(1.2547(0)) + e^{-0.0031(0)} D_{21}\sin(1.2547(0)) + e^{-1.2877(0)} D_{12}\cos(6.1172(0)) + e^{-1.2877(0)} D_{22}\sin(6.1172(0))$$

$$D_{11} + D_{12} = 1 \tag{4.1}$$

When $t_1 = -0.41666$

$$e^{-(-0.41666)^2} = e^{-0.0031(-0.41666)}D_{11}\cos\big(1.2547(-0.41666)\big) + e^{-0.0031(-0.41666)}D_{21}\sin\big(1.2547(-0.41666)\big) + e^{-1.2877(-0.41666)}D_{12}\cos\big(6.1172(-0.41666)\big) + e^{-1.2877(-0.41666)}D_{22}\sin\big(6.1172(-0.41666)\big)$$

$$0.867755D_{11} - 1.4829D_{12} - 0.49993D_{21} + 0.95539D_{22} = 0.840062 \qquad (4.2)$$

When $t_2 = -0.8333$

$$e^{-(-0.8333)^2} = e^{-0.0031(-0.8333)}D_{11}\cos\big(1.2547(-0.8333)\big) + e^{-0.0031(-0.8333)}D_{21}\sin\big(1.2547(-0.8333)\big) + e^{-1.2877(-0.8333)}D_{12}\cos\big(6.1172(-0.8333)\big) + e^{-1.2877(-0.8333)}D_{22}\sin\big(6.1172(-0.8333)\big)$$

$$0.50273D_{11} + 1.09843D_{12} - 0.86743D_{21} + 2.71011D_{22} = 0.499379 \qquad (4.3)$$

When $t_3 = -1.25$

$$e^{-(-1.25)^2} = e^{-0.0031(-1.25)}D_{11}\cos(1.2547(-1.25)) + e^{-0.0031(-1.25)}D_{21}\sin(1.2547(-1.25)) + e^{-1.2877(-1.25)}D_{12}\cos(6.1172(-1.25)) + e^{-1.2877(-1.25)}D_{22}\sin(6.1172(-1.25))$$

$$0.00243D_{11} + 1.03017D_{12} - 1.00387D_{21} - 4.89367D_{22} = 0.2096 \qquad (4.4)$$

Now we have 4×4 system of linear equations

$$D_{11} + D_{12} = 1 \tag{4.1}$$

$$0.867755D_{11} - 1.4829D_{12} - 0.49993D_{21} + 0.95539D_{22} = 0.840062 \tag{4.2}$$

$$0.50273D_{11} + 1.09843D_{12} - 0.86743D_{21} + 2.71011D_{22} = 0.499379 \tag{4.3}$$

$$0.00243D_{11} + 1.03017D_{12} - 1.00387D_{21} - 4.89367D_{22} = 0.2096 \tag{4.4}$$

Solving 4×4 system of linear equations by Gaussian Elimination, rewrite the system in mat-rix form and solving by Gaussian Elimination (Gauss-Jordan elimination)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 1 \\ 0.87755 & -1.4829 & -0.49993 & 0.95539 & \vdots & 0.87755 \\ 0.50273 & 1.09843 & -0.86743 & 2.71011 & \vdots & 0.50273 \\ 0.00243 & 1.03017 & -1.00387 & -4.989367 & \vdots & 0.00243 \end{bmatrix}$$

$R_2 - 0.87755R_1 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_3 - 0.50273\ R_1 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_4 - 0.00243R_1 \rightarrow R_n$, mul-tiply (k) row by (m) and subtract it from (n) row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 1 \\ 0 & -2.36045 & -0.49993 & 0.95539 & \vdots & -0.037488 \\ 0 & 0.5957 & -0.86743 & 2.71011 & \vdots & -0.003351 \\ 0 & 1.02774 & -1.00387 & -4.989367 & \vdots & 0.20717 \end{bmatrix}$$

$R_2/-2.36045 \rightarrow R_{(i)}$, divide the (i) row by (n)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0.2117943612 & -0.4047490944 & \vdots & 0.15399097 \\ 0 & 0.5957 & -0.86743 & 2.71011 & \vdots & -0.003351 \\ 0 & 1.02774 & -1.00387 & -4.989367 & \vdots & 0.20717 \end{bmatrix}$$

$R_1 - 1R_1 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_3 - 0.5957\ R_2 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_4 - 1.02774R_2 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row.

$$\begin{bmatrix} 1 & 0 & -0.2117943612 & 0.4047490944 & \vdots & 0.98411828253 \\ 0 & 1 & 0.2117943612 & -0.4047490944 & \vdots & 0.15399097 \\ 0 & 0 & -0.993595900 & 2.95121903556 & \vdots & -0.0128117505 \\ 0 & 0 & -1.221539537 & -4.4776931656 & \vdots & 0.190847723688 \end{bmatrix}$$

$R_3/-0.993595900 \rightarrow R_{(i)}$, divide the (i) row by (n)

$$\begin{bmatrix} 1 & 0 & -0.2117943612 & 0.4047490944 & \vdots & 0.98411828253 \\ 0 & 1 & 0.2117943612 & -0.4047490944 & \vdots & 0.15399097 \\ 0 & 0 & 1 & -2.97024075 & \vdots & 0.012894315 \\ 0 & 0 & -1.221539537 & -4.4776931656 & \vdots & 0.190847723688 \end{bmatrix}$$

$0.2117943612R_3 + R_1 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_3 - 0.2117943612\ R_2 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $1.221539537R_3 + R_4 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row.

$$\begin{bmatrix} 1 & 0 & 0 & -0.224331148 & \vdots & 0.98684905 \\ 0 & 1 & 0 & 0.224331148 & \vdots & 0.013150774 \\ 0 & 0 & 1 & -2.97024075 & \vdots & 0.012894315 \\ 0 & 0 & 0 & -8.105959677 & \vdots & 2.0659864 \end{bmatrix}$$

$R_4/-8.105959677 \rightarrow R_{(i)}$, divide the (i) row by (n)

$$\begin{bmatrix} 1 & 0 & 0 & -0.224331148 & \vdots & 0.98684905 \\ 0 & 1 & 0 & 0.224331148 & \vdots & 0.013150774 \\ 0 & 0 & 1 & -2.97024075 & \vdots & 0.012894315 \\ 0 & 0 & 0 & 1 & \vdots & 0.025487252 \end{bmatrix}$$

$0.224331148R_4 + R_1 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $2.97024075\ R_4 + R_3 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row; $R_2 - 0.224331148R_4 \rightarrow R_n$, multiply (k) row by (m) and subtract it from (n) row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0.9666 \\ 0 & 1 & 0 & 0 & \vdots & 0.033353 \\ 0 & 0 & 1 & 0 & \vdots & -0.05108 \\ 0 & 0 & 0 & 1 & \vdots & -0.0248704 \end{bmatrix}$$

So that the values of arbitrary coefficients ($D_{11} = 0.9666, D_{21} = -0.05108, D_{12} = 0.033353\,, D_{22} = -0.0248706$) the characteristic solutions:

$$u_2(t) = e^{-0.0031t}0.9666\cos(1.2547t) - e^{-0.0031t}0.05108\sin(1.2547t) \\ + e^{-1.2877t}0.033353\cos(6.1172t) - e^{-1.2877t}0.02488\sin(6.1172t)$$

Figure (4.1) shows the graph of characteristic solution and Figure (4.2) shows the numerical solution by using solver DDE23 in Matlab.

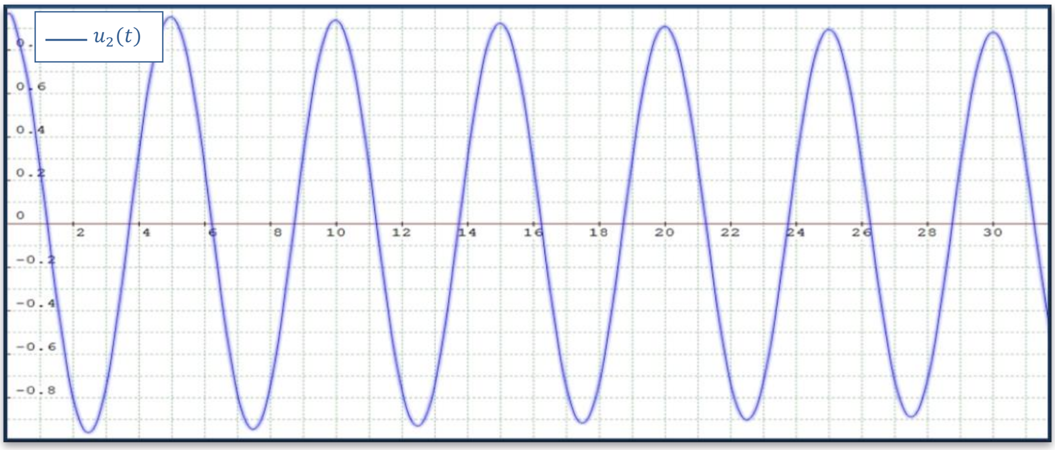


Figure 4.1: Characteristic solution of $u_2(t)$

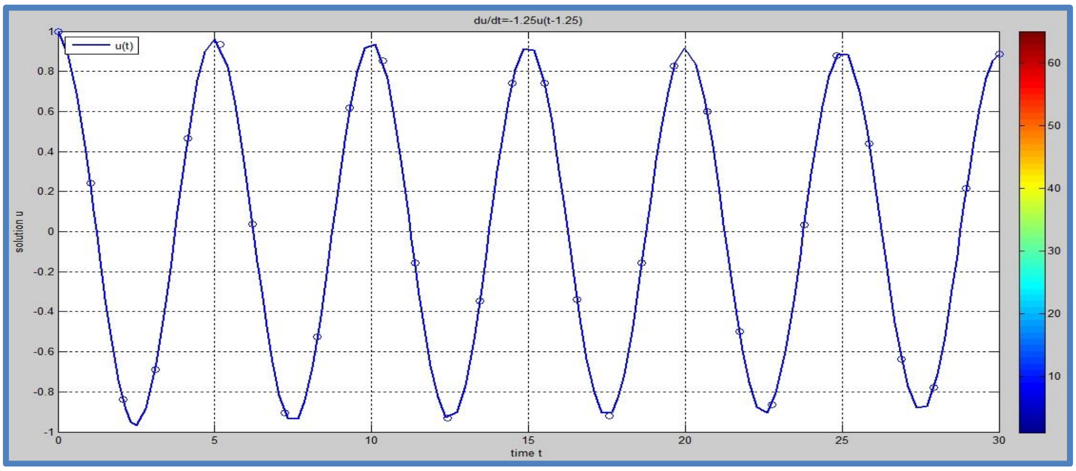


Figure 4.2: Approximate solution by using solver DDE23

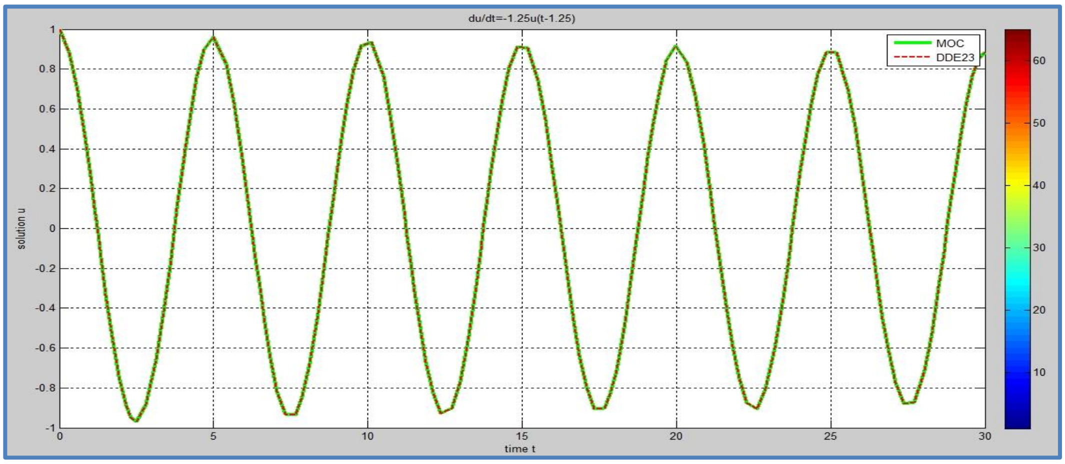


Figure 4.3: Comparing the two solutions Characteristic and Approximate

4.1.2 Example of case two

Consider the Boundary Value Problem for case two

$$\begin{cases} \dot{u}(t) = -\frac{1}{1.25e}u(t-1.25), & on\ [0,10] \\ \theta(t) = e^{-t^2}, & on\ [-\beta,0] \end{cases}$$

The steps of solution is like case one , we must apply the condition of case two, to solve this equation, Figure (4.4) shows the approximate solution.

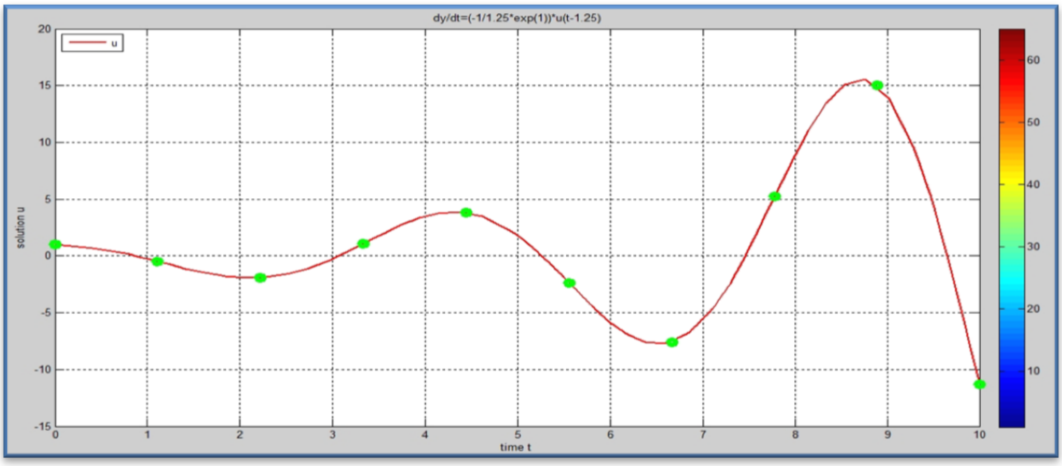


Figure 4.4: Approximate solution of case two

4.1.3 Example of case three

Consider the Boundary Value Problem for case three

$$\begin{cases} \dot{u}(t) = -0.02u(t-1.25), & on\ [0,15] \\ \theta(t) = e^{-t}, & on\ [-\beta, 0] \end{cases}$$

The steps of solution is like case one , we must apply the condition of case three, to solve this equation, Figure (4.5) shows the approximate solution.

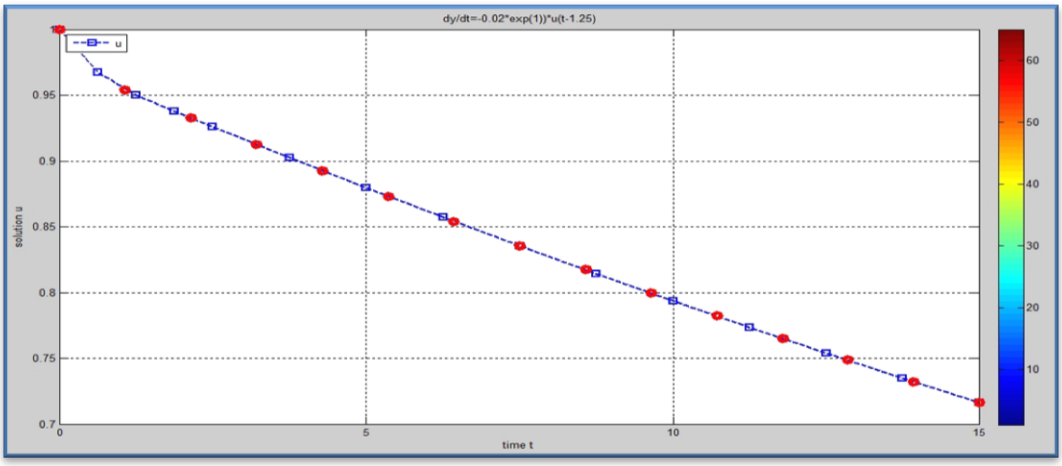


Figure 4.5: Approximate solution of case three

4.1. Example of case four

Consider the Boundary Value Problem for case four

$$\begin{cases} \dot{u}(t) = 1.25u(t-1.25), & on\ [0,10] \\ \theta(t) = e^{-t^2}, & on\ [-\beta, 0] \end{cases}$$

The steps of solution is like case one , we must apply the condition of case four, to solve this equation, Figure (4.6) shows the approximate solution.

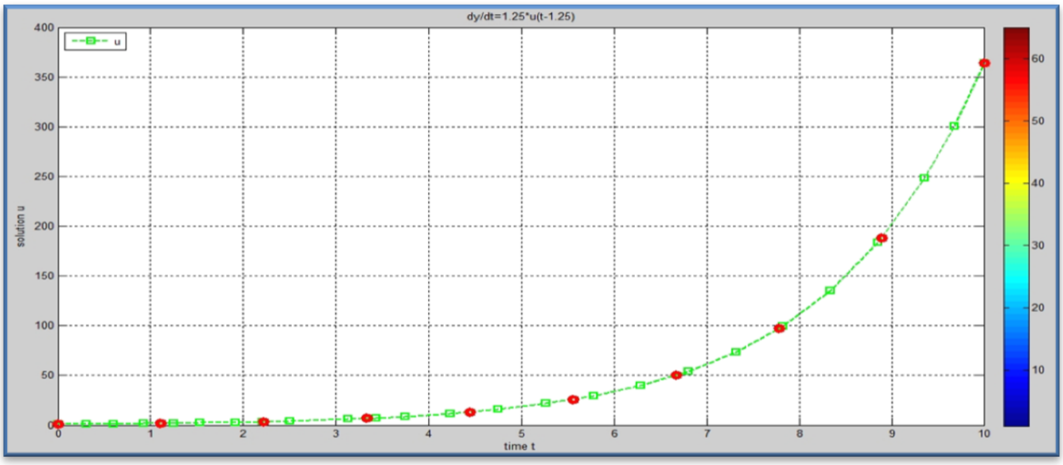


Figure 4.6: Approximate solution of case four

4.2 STEPS Examples

In this section we will solve some examples of linear first order delay differential equation, with single constant delay and constant coefficients.

4.2.1 Polynomial problems

If the history function is polynomial for example use the method of steps to solve linear delay differential equation and sketch the graph, for $1 \le t \le 5$, $\dot{u}(t) = 4u(t-1), \ \ t > 1$ with $u(t) = (t-2)^2 - 1, \ \ 0 \le t \le 1$.

Solution: the initial interval is $0 \le t \le 1$, so we will start from the interval $1 \le t \le 2$ to solve this equation.

$1 \le t \le 2 : \rightarrow 0 \le t-1 \le 1$, so $u(t-1) = (t-3)^2 - 1$ and $u(t) = (t-2)^2 - 1$, with initial condition $u(1) = 0$, $\frac{du}{dt} = 4[(t-3)^2 - 1] = 4(t-3)^2 - 4$ by separation of variable we get $du = [4(t-3)^2 - 4\,]dt$ integrate both side

$$\int du = \int [4(t-3)^2 - 4\,]dt$$

$u(t) = \frac{4}{3}(t-3)^3 - 4t + c$ and we have initial condition $u(1) = 0$, so $\frac{4}{3}(1-3)^3 - 4t + c = 0$, then value of $c = \frac{44}{3}$

$$u_1(t) = \frac{4}{3}(t-3)^3 - 4t + \frac{44}{3} \tag{4.5}$$

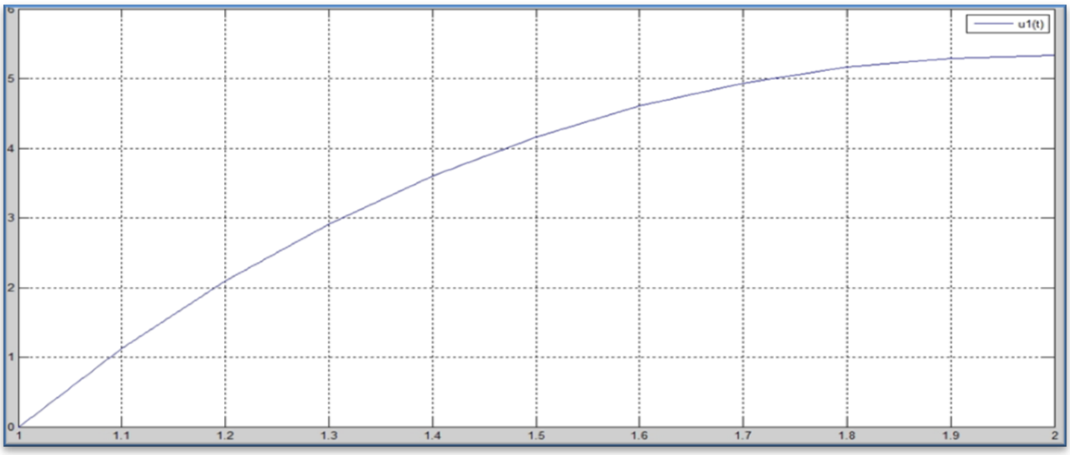


Figure 4.7: Graph of Equation 4.5

Now $2 \leq t \leq 3: \rightarrow 1 \leq t-1 \leq 2$, so

$$u(t-1) = \frac{4}{3}(t-4)^3 - 4(t-1) + \frac{44}{3}$$

And $u(t) = \frac{4}{3}(t-3)^3 - 4t + \frac{44}{3}$,with initial condition $u(2) = \frac{16}{3}$

$$\frac{du}{dt} = 4\left[\frac{4}{3}(t-4)^3 - 4(t-1) + \frac{44}{3}\right]$$

$$= \frac{16}{3}(t-4)^3 - 16(t-1) + \frac{176}{3}$$

By separation of variable we get

$$du = [\frac{16}{3}(t-4)^3 - 16(t-1) + \frac{176}{3}]dt$$

Integrate both side

$$\int du = \int [\frac{16}{3}(t-4)^3 - 16(t-1) + \frac{176}{3}]dt$$

$$u(t) = \frac{4}{3}(t-4)^4 - 8(t-1)^2 + \frac{176}{3}t + c$$

And we have initial condition $u(2) = \frac{16}{3}$, so

$$\frac{4}{3}(2-4)^4 - 8(2-1)^2 + \frac{176}{3}(2) + c = \frac{16}{3}$$

Then value of $c = -\frac{376}{3}$

$$u_2(t) = \frac{4}{3}(t-4)^4 - 8(t-1)^2 + \frac{176}{3}t - \frac{376}{3} \tag{4.6}$$

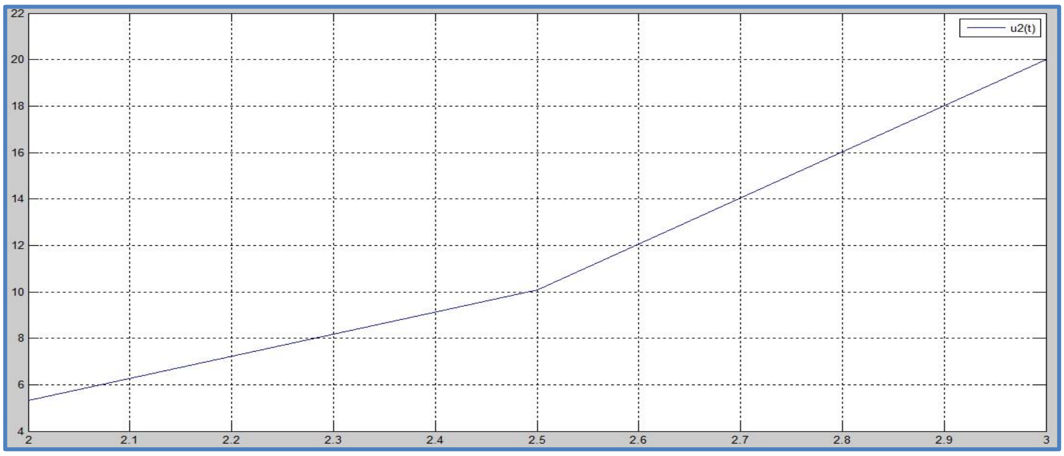


Figure 4.8: Graph of Equation 4.6

Now $3 \le t \le 4 : \rightarrow 2 \le t-1 \le 3$, so

$$u(t-1) = \frac{4}{3}(t-5)^4 - 8(t-2)^2 + \frac{176}{3}(t-1) - \frac{376}{3}$$

And

$$u(t) = \frac{4}{3}(t-4)^4 - 8(t-1)^2 + \frac{176}{3}t - \frac{376}{3}$$

With initial condition $u(3) = 20$

$$\frac{du}{dt} = 4\left[\frac{4}{3}(t-5)^4 - 8(t-2)^2 + \frac{176}{3}(t-1) - \frac{376}{3}\right]$$

$$\frac{du}{dt} = \frac{16}{3}(t-5)^4 - 32(t-2)^2 + \frac{704}{3}(t-1) - \frac{1504}{3}$$

By separation of variable we get

$$du = [\frac{16}{3}(t-5)^4 - 32(t-2)^2 + \frac{704}{3}(t-1) - \frac{1504}{3}]dt$$

Integrate both side:

$$\int du = \int [\frac{16}{3}(t-5)^4 - 32(t-2)^2 + \frac{704}{3}(t-1) - \frac{1504}{3}]dt$$

$$u(t) = \frac{16}{15}(t-5)^5 - \frac{32}{3}(t-2)^3 + \frac{352}{3}(t-1)^2 - \frac{1504}{3}t + c$$

And we have initial condition $u(3) = \frac{100}{3}$, so

$$\frac{16}{15}(3-5)^5 - \frac{32}{3}(3-2)^3 + \frac{352}{3}(3-1)^2 - \frac{1504}{3}(3) + c = 20$$

Then value of $c = \frac{49476}{45}$

$$u_3(t) = \frac{16}{15}(t-5)^5 - \frac{32}{3}(t-2)^3 + \frac{352}{3}(t-1)^2 - \frac{1504}{3}t + \frac{49476}{45} \quad (4.7)$$

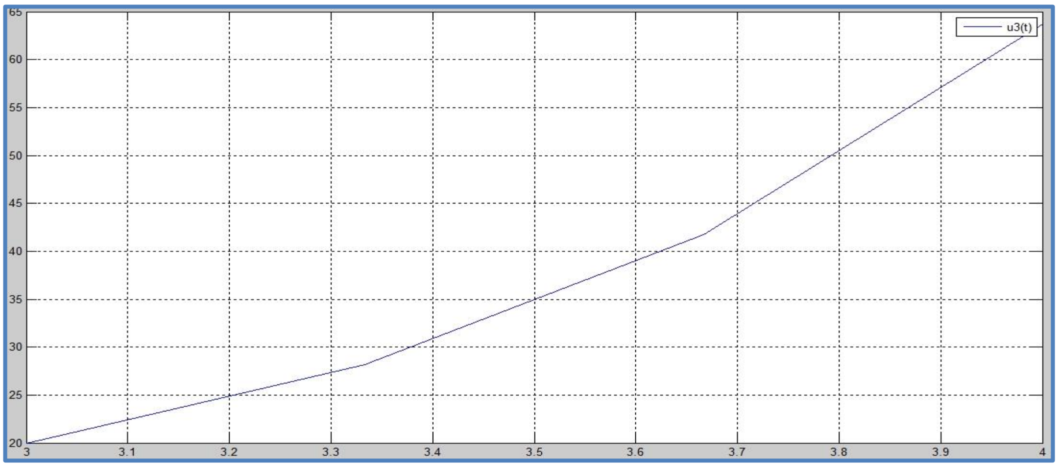


Figure 4.9: Graph of Equation 4.7

Now $4 \leq t \leq 5 : \rightarrow 3 \leq t-1 \leq 4$, so

$$u(t-1) = \frac{16}{15}(t-6)^5 - \frac{32}{3}(t-3)^3 + \frac{352}{3}(t-2)^2 - \frac{1504}{3}(t-1) + \frac{49476}{45}$$

And

$$u(t) = \frac{16}{15}(t-5)^5 - \frac{32}{3}(t-2)^3 + \frac{352}{3}(t-1)^2 - \frac{1504}{3}t + \frac{49476}{45}$$

With initial condition $u(4) = \frac{2868}{45}$,

$$\frac{du}{dt} = 4\left[\frac{16}{15}(t-6)^5 - \frac{32}{3}(t-3)^3 + \frac{352}{3}(t-2)^2 - \frac{1504}{3}(t-1) + \frac{49476}{45}\right]$$

$$\frac{du}{dt} = \frac{64}{15}(t-6)^5 - \frac{128}{3}(t-3)^3 + \frac{1408}{3}(t-2)^2 - \frac{6016}{3}(t-1) + \frac{197904}{45}$$

By separation of variable we get

$$du = [\frac{64}{15}(t-6)^5 - \frac{128}{3}(t-3)^3 + \frac{1408}{3}(t-2)^2 - \frac{6016}{3}(t-1) + \frac{197904}{45}]dt$$

Integrate both side

$$\int du = \int [\frac{64}{15}(t-6)^5 - \frac{128}{3}(t-3)^3 + \frac{1408}{3}(t-2)^2 - \frac{6016}{3}(t-1) + \frac{197904}{45}]dt$$

$$u(t) = \frac{64}{90}(t-6)^6 - \frac{32}{3}(t-3)^4 + \frac{1408}{9}(t-2)^3 - \frac{3008}{3}(t-1)^2 + \frac{197904}{45}t + c$$

And we have initial condition $u(4) = \frac{454098}{45}$, so

$$\frac{64}{90}(4-6)^6 - \frac{32}{3}(4-3)^4 + \frac{1408}{9}(4-2)^3 - \frac{3008}{3}(4-1)^2 + \frac{197904}{45} + c = \frac{2868}{45}$$

Then value of $c = 9790.1333$.

$$u_4(t) = \frac{64}{90}(t-6)^6 - \frac{32}{3}(t-3)^4 + \frac{1408}{9}(t-2)^3$$

$$-\frac{3008}{3}(t-1)^2 + \frac{197904}{45} + 9790.1333 \qquad (4.8)$$

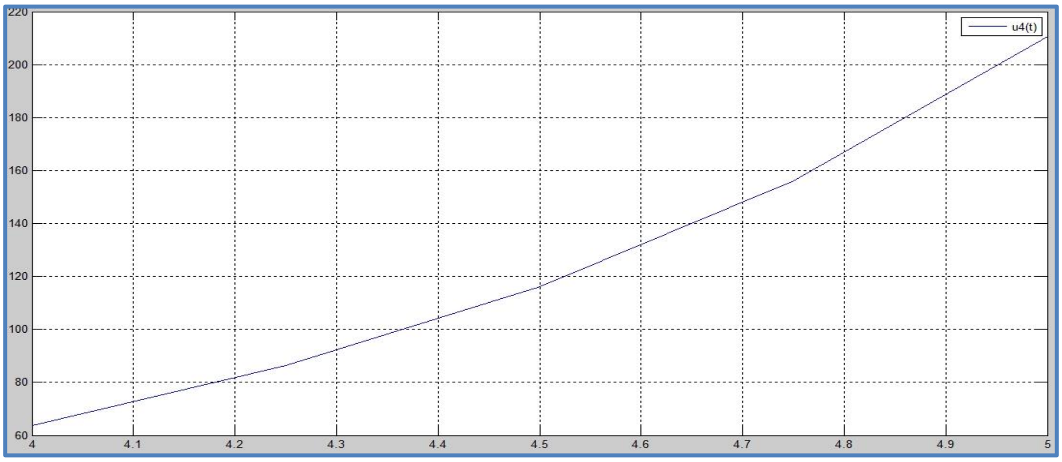


Figure 4.10: Graph of Equation 4.8

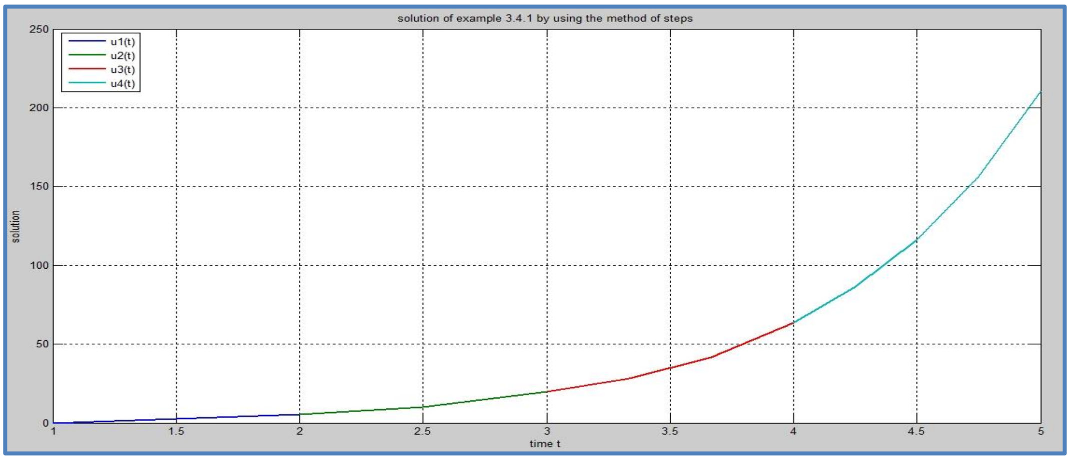


Figure 4.11: Steps solutions

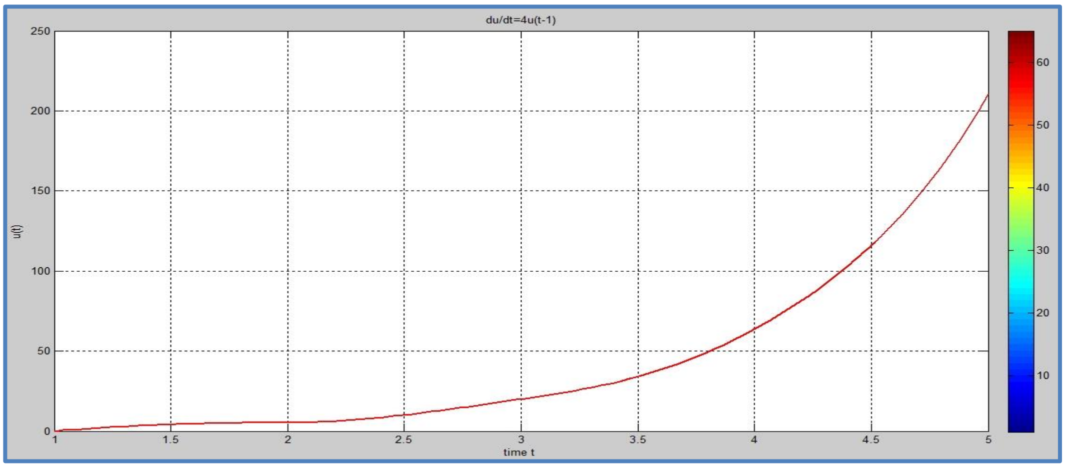


Figure 4.12: Approximate solution by using DDESD

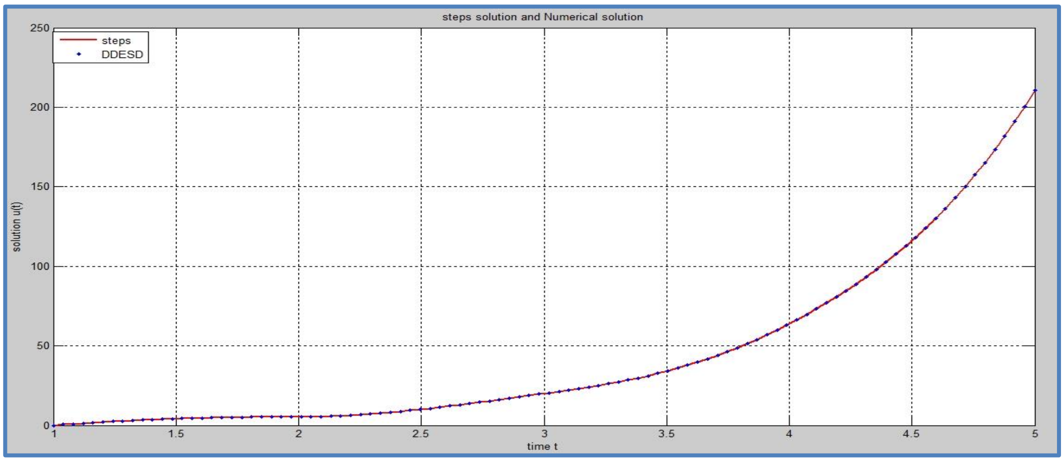


Figure 4.13: Comparing the two solutions Steps and Approximate

Another example about history polynomial function**,** use the method of steps to solve linear delay differential equation and sketch the graph, for $1 \leq t \leq 5$, $\dot{u}(t) = 4u(t-1),\ \ t \geq 1$ with $u(t) = t,\ \ 0 \leq t \leq 1$.

Solution: the initial interval is $0 \leq t \leq 1$, so we will start from the interval $1 \leq t \leq 2$ to solve this equation.

$1 \leq t \leq 2 : \rightarrow 0 \leq t-1 \leq 1$, so $u(t-1) = (t-1)$ and $u(t) = t$, with initial condition $u(1) = 1$, $\frac{du}{dt} = 4(t-1)$ by separation of variable we get $du = 4(t-1)dt$ integrate both side

$$\int du = \int 4(t-1)dt$$

$u(t) = 2(t-1)^2 + c$ and we have initial condition $u(1) = 1$, so $2(1-1)^2 + c = 1$, then value of $c = 1$

$$u_1(t) = 2(t-1)^2 + 1 \tag{4.9}$$

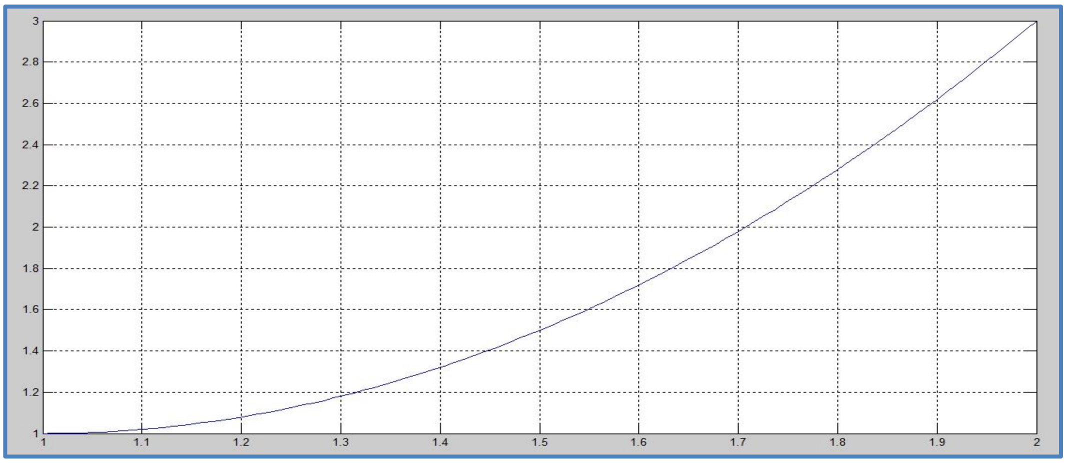


Figure 4.14: Graph of Equation 4.9

Now $2 \leq t \leq 3 : \to 1 \leq t-1 \leq 2$, so

$$u(t-1) = 2(t-2)^2 + 1$$

And

$$u(t) = 2(t-1)^2 + 1$$

With initial condition $u(2) = 3$

$$\frac{du}{dt} = 8(t-2)^2 + 4$$

By separation of variable we get $du = [8(t-2)^2 + 4\,]dt$ integrate both side

$$\int du = \int [8(t-2)^2 + 4\,]dt$$

$u(t) = \frac{8}{3}(t-2)^3 + 4t + c$ and we have initial condition $u(2) = 3$, so

$$\frac{8}{3}(2-2)^3 + 4(2) + c = 3$$

Then value of $c = -5$

$$u_2(t) = \frac{8}{3}(t-2)^3 + 4t - 5 \qquad (4.10)$$

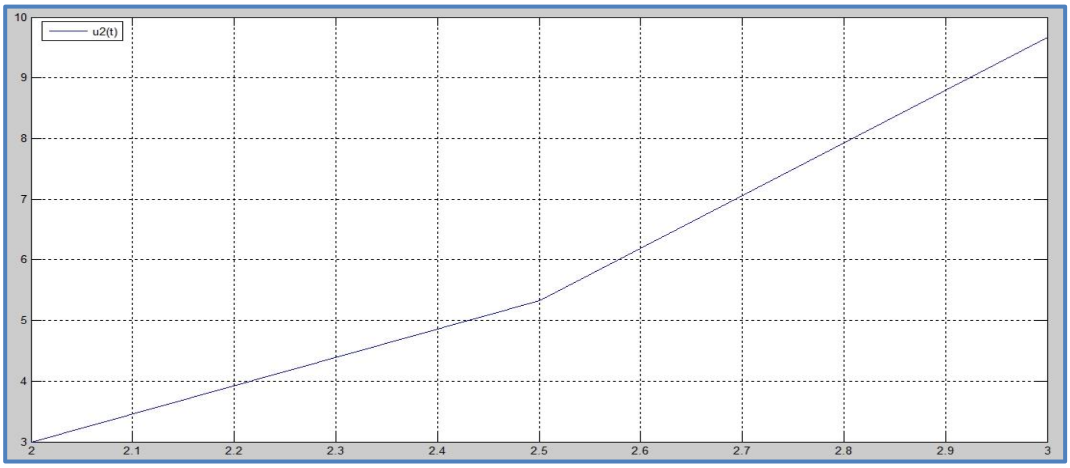


Figure 4.15: Graph of Equation 4.10

Now $3 \leq t \leq 4 : \rightarrow 2 \leq t-1 \leq 3$, so

$$u(t-1) = \frac{8}{3}(t-3)^3 + 4(t-1) - 5$$

And

$$u(t) = u(t) = \frac{8}{3}(t-2)^3 + 4t - 5$$

With initial condition $u(3) = \frac{29}{3}$

$$\frac{du}{dt} = \frac{32}{3}(t-3)^3 + 16(t-1) - 20$$

By separation of variable we get $du = [8(t-2)^2 + 4\,]dt$ integrate both side

$$\int du = \int [\frac{32}{3}(t-3)^3 + 16(t-1) - 20 \quad]dt$$

$$u(t) = \frac{8}{3}(t-3)^4 + 8(t-1)^2 - 20t + c$$

And we have initial condition $u(3) = \frac{29}{3}$, so

$$\frac{8}{3}(3-3)^4 + 8(3-1)^2 - 20(3) + c = \frac{29}{3}$$

Then value of $c = \frac{113}{3}$

$$u_3(t) = \frac{8}{3}(t-3)^4 + 8(t-1)^2 - 20t + \frac{113}{3} \tag{4.11}$$

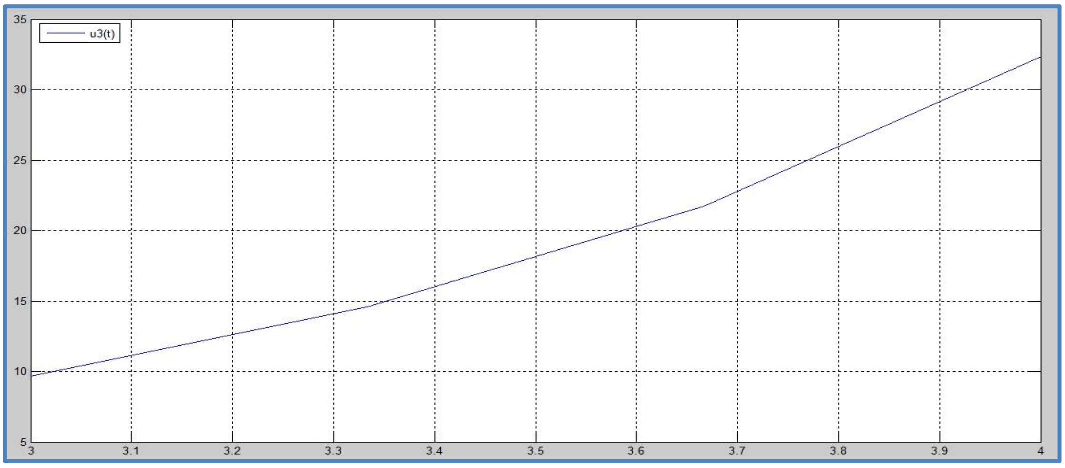


Figure 4.16: Graph of Equation 4.11

Now $4 \le t \le 5 : \rightarrow 3 \le t-1 \le 4$, so

$$u(t-1) = \frac{8}{3}(t-4)^4 + 8(t-2)^2 - 20(t-1) + \frac{113}{3}$$

And

$$u(t) = \frac{8}{3}(t-3)^4 + 8(t-1)^2 - 20t + \frac{113}{3}$$

Again with initial condition $u(4) = \frac{97}{3}$

$$\frac{du}{dt} = \frac{32}{3}(t-4)^4 + 32(t-2)^2 - 80(t-1) + \frac{452}{3}$$

By separation of variable we get

$$du = [\frac{32}{3}(t-4)^4 + 32(t-2)^2 - 80(t-1) + \frac{452}{3}]dt$$

Integrate both side

$$\int du = \int [\frac{32}{3}(t-4)^4 + 32(t-2)^2 - 80(t-1) + \frac{452}{3}]dt$$

$$u(t) = \frac{32}{15}(t-4)^5 + \frac{32}{15}(t-2)^3 - 40(t-1)^2 + \frac{452}{3}t + c$$

And we have initial condition $u(3) = \frac{97}{3}$, so

$$\frac{32}{15}(4-4)^5 + \frac{32}{15}4-2)^3 - 40(4-1)^2 + \frac{452}{3}(4) + c = \frac{97}{3}$$

Then value of $c = -\frac{887}{3}$

$$u_4(t) = \frac{32}{15}(t-4)^5 + \frac{32}{15}(t-2)^3 - 40(t-1)^2 + \frac{452}{3}t - \frac{887}{3} \qquad (4.12)$$

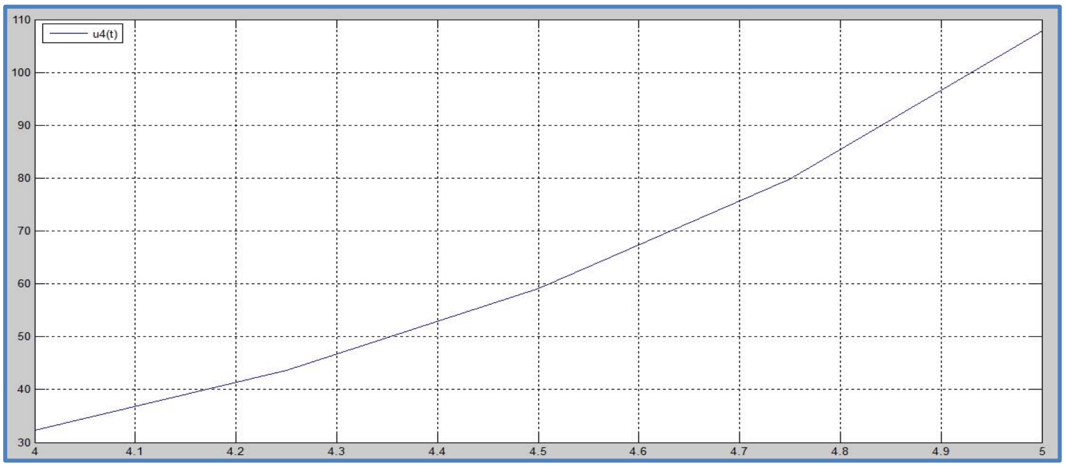


Figure 4.17: Graph of Equation 4.12

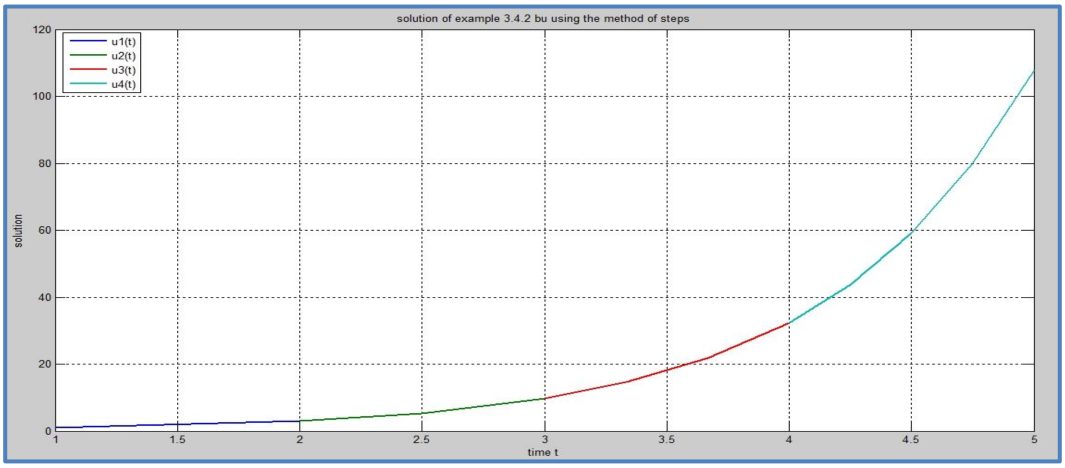


Figure 4.18: Steps solution

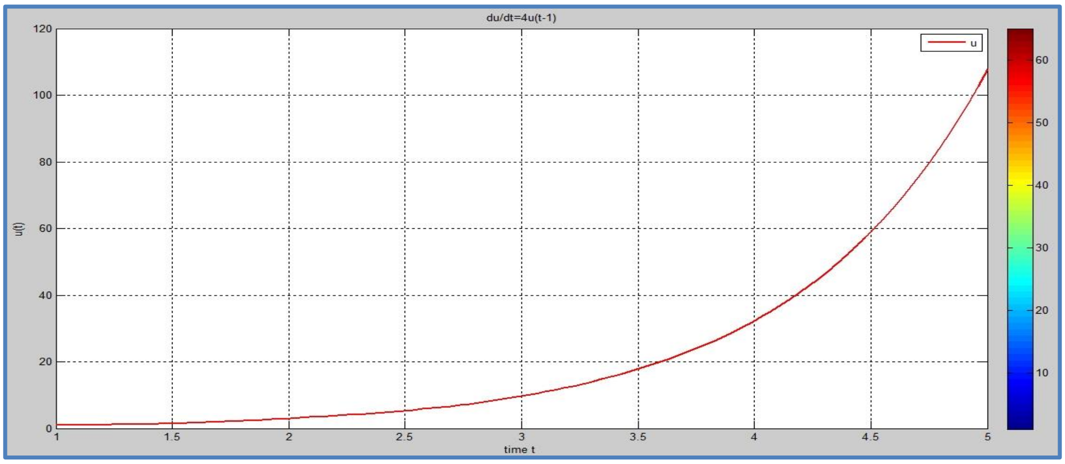


Figure 4.19: Approximate solution by using DDESD

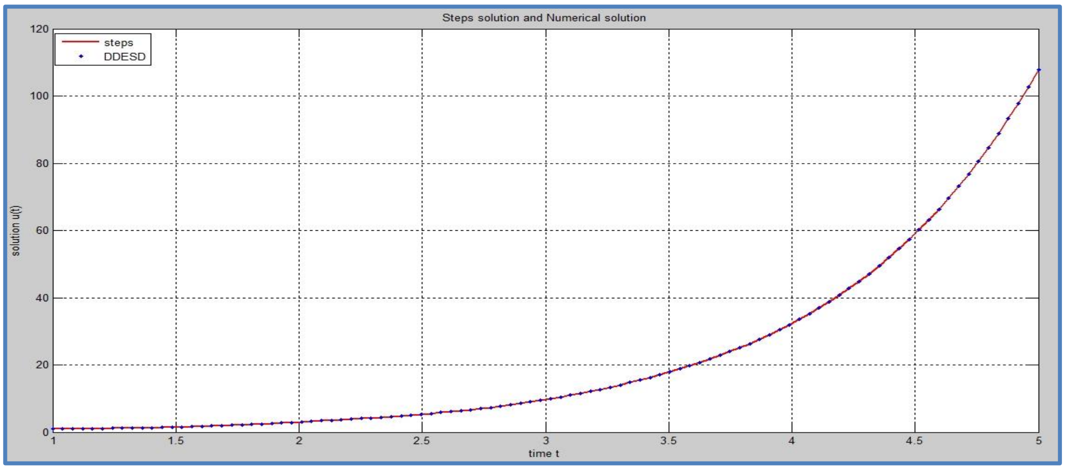


Figure 4.20: Comparing the two solutions Steps and Approximate

4.2.3 Constant problem

If the history functions is constant, for example, use the method of steps to solve linear delay differential equation and sketch the graph, for $0 \leq t \leq 5$, $\dot{u}(t) = -u(t-1),\ \ t > 1$ with $u(t) = 8,\ -1 \leq t \leq 0$.

Solution: the initial interval is $-1 \leq t \leq 0$, so we will start from the interval $0 \leq t \leq 1$ to solve this equation. In the interval $[0,1]$, by equation (3.27) we have :

$$u_1(t) = 8 - \int_0^t 8dt^*$$

$$u_1(t) = 8 - 8t$$

Now in the interval [1,2], then we have :

$$u_2(t) = 0 - \int_1^t [8 - 8(t^* - 1)]\, dt^* = -[8t^* - 4(t^* - 1)^2]_1^t$$

$$u_2(t) = 4(t - 1)^2 - 8(t - 1)$$

Now in the interval [2,3], the solution is :

$$u_3(t) = -4 - \int_2^t [4(t^* - 2)^2 - 8(t^* - 2)]\, dt^* = -4 - \left[\frac{4}{3}(t^* - 2)^3 - 4(t^* - 2)^2\right]_2^t$$

$$= -4 - \left[\frac{4}{3}(t - 2)^3 - 4(t - 2)^2 - (\frac{4}{3}(2 - 2)^3 - 4(2 - 2)^2)\right]$$

$$u_3(t) = -4 - \frac{4}{3}(t - 2)^3 + 4(t - 2)^2$$

In the interval [3,4], the solution is :

$$u_4(t) = -\frac{4}{3} - \int_3^t \left[-4 - \frac{4}{3}(t^* - 3)^3 + 4(t^* - 3)^2\right] dt^*$$

$$= -\frac{4}{3} — 4t^* - \frac{1}{3}(t^* - 3)^4 + \frac{4}{3}(t^* - 3)^3 {}_3^t$$

$$= -\frac{4}{3} — 4t - \frac{1}{3}(t - 3)^4 + \frac{4}{3}(t - 3)^3 — 12$$

$$= -\frac{4}{3} + 4t - \frac{4}{3}(t - 3)^3 + \frac{1}{3}(t - 3)^4 - 12$$

$$u_4(t) = -\frac{40}{3} + 4t - \frac{4}{3}(t - 3)^3 + \frac{1}{3}(t - 3)^4$$

Now in the interval [4,5], the solution is :

$$u_5(t) = \frac{5}{3} - \int_4^t \left[-\frac{40}{3} + 4(t^* - 1) - \frac{4}{3}(t^* - 4)^3 + \frac{1}{3}(t^* - 4)^4\right] dt^*$$

$$= \frac{5}{3} - \left[-\frac{40}{3} t^* + 2(t^* - 1)^2 - \frac{1}{3}(t^* - 4)^4 + \frac{1}{15}(t^* - 4)^5 \right]_4^t$$

$$= \frac{5}{3} - \left[-\frac{40}{3} t + 2(t - 1)^2 - \frac{1}{3}(t - 4)^4 + \frac{1}{15}(t - 4)^5 - \left(-\frac{160}{3} + 18 \right) \right]$$

$$= \frac{5}{3} + \frac{40}{3} t - 2(t - 1)^2 + \frac{1}{3}(t - 4)^4 - \frac{1}{15}(t - 4)^5 - \frac{106}{3}$$

$$u_5(t) = -\frac{101}{3} + \frac{40}{3} t - 2(t - 1)^2 + \frac{1}{3}(t - 4)^4 - \frac{1}{15}(t - 4)^5$$

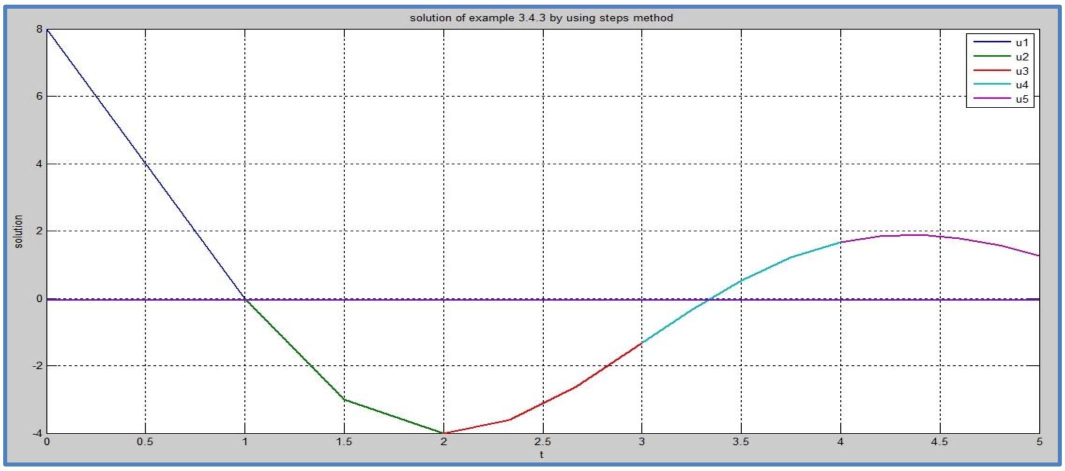


Figure 4.21: Steps solution

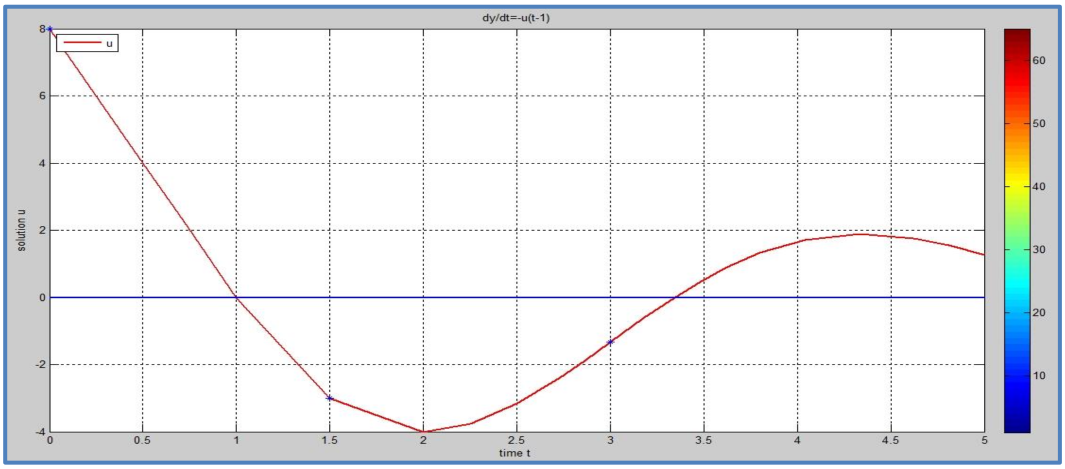


Figure 4.22: Approximate solution by using DDESD

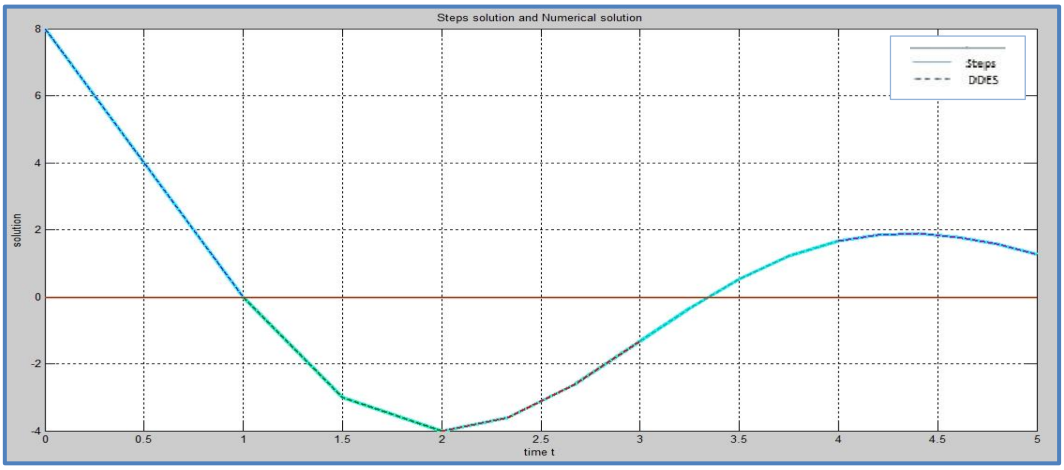


Figure 4.23: Comparing the two solutions Steps and Numerical

4.2.4 Trigonometric problem

If the history function is trigonometric, for example**,** use the method of steps to solve linear delay differential equation and sketch the graph, for $0 \leq t \leq 2$, $\dot{u}(t) = 3 - 0.5u(t-1)$, $0 \leq t \leq 1$ with $u(t) = -\sin(t)$, $-1 \leq t \leq 0$.

Solution: the initial interval is $-1 \leq t \leq 0$, so we will start from the interval $0 \leq t \leq 1$ to solve this equation. $0 \leq t \leq 1 :\rightarrow -1 \leq t-1 \leq 0$, so $u(t-1) = -\sin(t-1)$ and $u(t) = -\sin(t)$, with initial condition $u(0) = 0$, $\frac{du}{dt} = 3 + 0.5\sin(t-1)$ by separation of variable and integrate both side

$$\int du = \int [3 + 0.5\sin(t-1)]dt$$

Apply integration by parts:$\int xy' = xy' - \int x'y$, so $x = 3 + 0.5\sin(t-1)$ and $y' = 1$, then this implies that $dx = 0.5\cos(t-1)dt$ and $y = t$.

$$\int [3 + 0.5\sin(t-1)]dt = t(3 + 0.5\sin(t-1)) - 0.5\int t\cos(t-1)\,dt$$

$$= t(3 - 0.5\sin(1-t)) - 0.5\int t\cos(1-t)\,dt$$

Now, apply integral substitution

$$\int f\big(g(t)\big).g'(t)dt = f(r)dr, r = g(t)$$

So $r = 1 - t, dr = -dt$, $dt = -dr$ then $0.5 \int (1-r)(-\cos(r))dr$

Refine

$$= 0.5\left(-\int (1-r)\cos(r)\,dr\right)$$

$$= 0.5\left(-\left(-\int (r-1)\cos(r)\,dr\right)\right)$$

Apply integration by parts:$\int xy' = xy' - \int x'y$, so $x = (r-1)$ and $y' = \cos(r)dr$, then this implies that $dx = dr$ and $y = \sin(r)$, therefore

$$= 0.5\left(-\left(-\left((r-1)\sin(r) - \int \sin(r)\,dr\right)\right)\right)$$

Now use the common integral $\int \sin(r)\,dr = (-\cos r)$

$$= 0.5\left(-\left(-\big((r-1)\sin(r) - (-\cos(r))\big)\right)\right)$$

Substitute back $r = 1 - t$,

$$= 0.5\left(-\left(-\big((1-t-1)\sin(1-t) - (-\cos(1-t))\big)\right)\right)$$

$$= \frac{1}{2}(\cos(1-t) - t\sin(1-t))$$

$$u(t) = t(3 - 0.5\sin(1-t)) - 0.5(\cos(1-t) - t\sin(1-t)) + c$$

And we have $u(0) = 0$, then

$$(0)(3 - 0.5\sin(1-0)) - 0.5(\cos(1-0) - (0)\sin(1-t)) + c = 0$$

$c = 0.5\cos(1)$

$$u_1(t) = t(3 - 0.5\sin(1-t)) + 0.5(t\sin(1-t) - \cos(1-t)) + 0.5\cos(1) \quad (4.13)$$

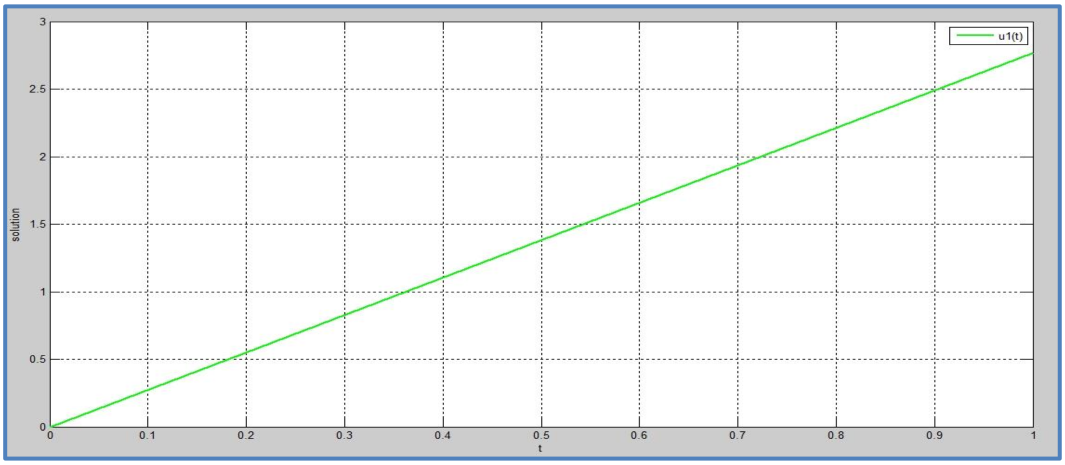


Figure 4.24: Graph of Equation 4.13

Now, $1 \le t \le 2: \to 0 \le t-1 \le 1$, so

$$u(t-1) = (t-1)(3 - 0.5\sin(2-t)) + 0.5\big((t-1)\sin(2-t) - \cos(2-t)\big) + 0.5\cos(1)$$

And

$$u(t) = t(3 - 0.5\sin(1-t)) + 0.5(t\sin(1-t) - \cos(1-t)) + 0.5\cos(1)$$

With initial condition, $u(1) = \frac{\cos(1)+5}{2}$

$$\frac{du}{dt} = 3 - 0.5(t-1)(3 - 0.5\sin(2-t)) + 0.25(t-1)\sin(2-t) - 0.25\cos(2-t) + 0.25\cos(1)$$

By separation of variable we get:

$$\int du = \int [3 - 0.5(t-1)(3 - 0.5\sin(2-t)) + 0.25(t-1)\sin(2-t) - 0.25\cos(2-t) \quad + 0.25\cos(1)]dt$$

Now, apply integral substitution:

$$\int f(g(t)).g'(t)dt = f(r)dr, r = g(t)$$

so $r = 2 - t, dr = -dt, dt = -dr$ so

$$\int -[3 - 0.5(t-1)(3 - 0.5\sin(r)) + 0.25(t-1)\sin(r) - 0.25\cos(r) + 0.25\cos(1)]dt$$

$$= \int [1.5t - 0.25\cos(r) - 4.36492]\, dr$$

$$= \int [1.5(2-r) - 0.25\cos(r)\, 4.36492]\, dr$$

$$= \int [-1.5r - 0.25\cos(r) - 1.36492]\, dr$$

$$= -\frac{3r^2}{4} - \frac{\sin(r)}{4} - \frac{4350r}{3187} + c$$

$$= -\frac{3(2-t)^2}{4} - \frac{\sin(2-t)}{4} - \frac{4350(2-t)}{3187} + c$$

$$= -\frac{3}{4}(2-t)^2 - \frac{4350}{3187}(2-t) - \frac{1}{4}\sin(2-t) + c$$

The initial condition is $u(1) = \frac{\cos(1)+5}{2}$, then

$$-\frac{3}{4}(1)^2 - \frac{4350}{3187}(1) - \frac{1}{4}\sin(1) + c = \frac{\cos(1)+5}{2}$$

$c = 5.119206937$

$$u_2(t) = -\frac{3}{4}(2-t)^2 - \frac{4350}{3187}(2-t) - \frac{1}{4}\sin(2-t) + 5.119206937 \qquad (4.14)$$

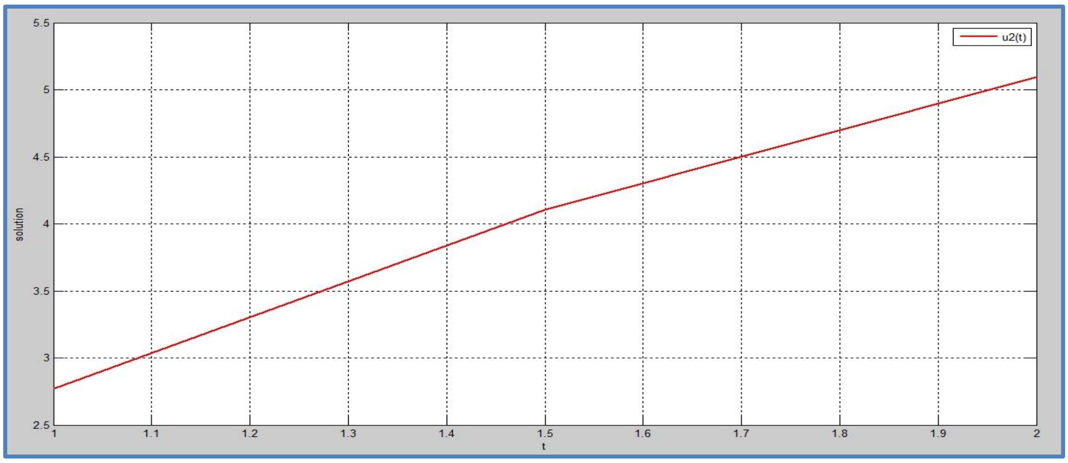


Figure 4.25: Graph of Equation 4.14

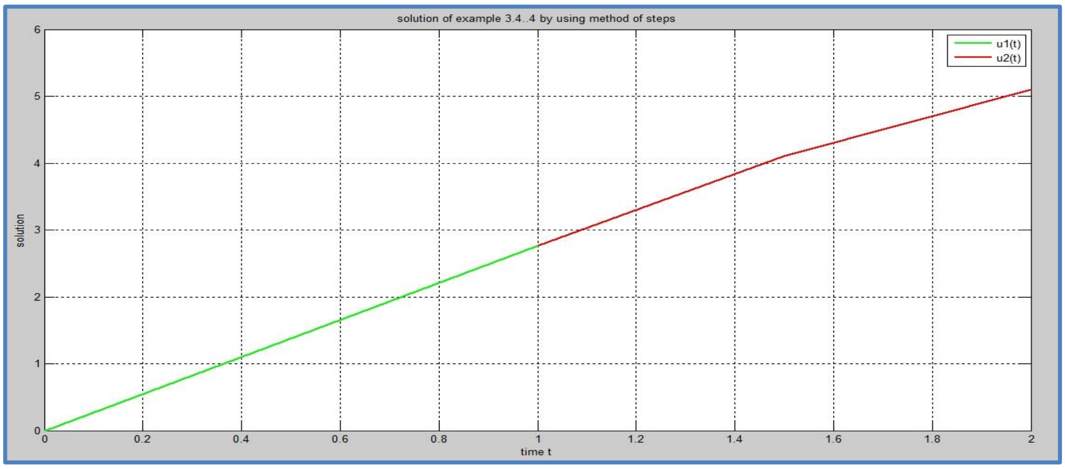


Figure 4.26: Steps solution

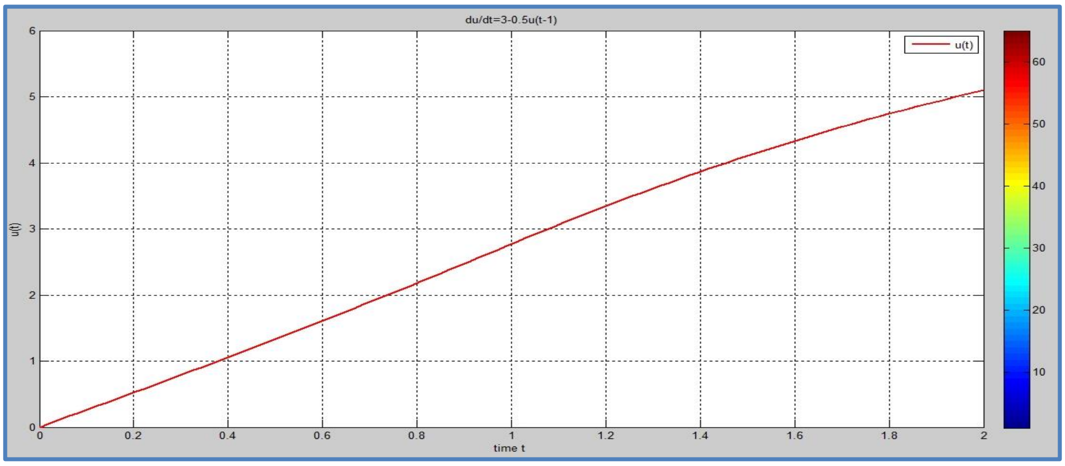


Figure 4.27: Approximate solution by using DDESD

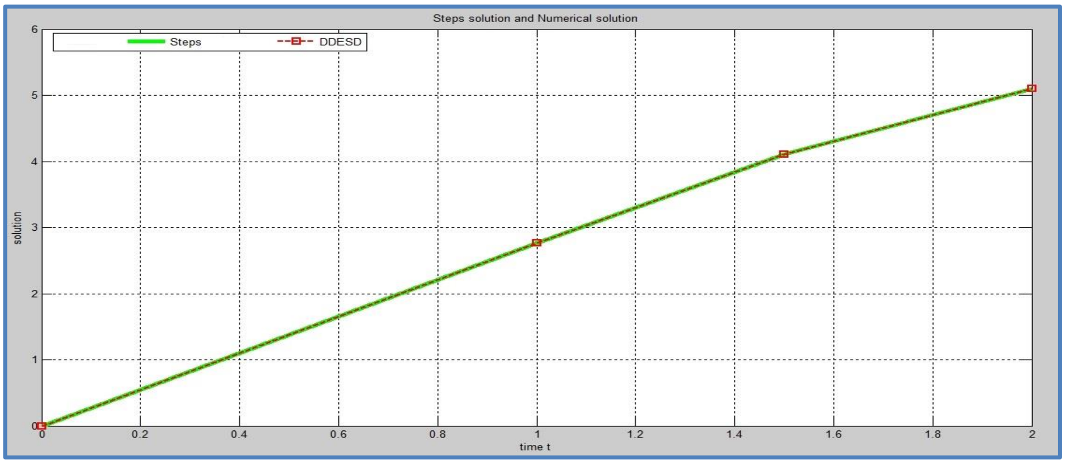


Figure 4.28: Comparing the two solutions Steps and Approximate

4.2.5 One step example

Find one step of the solution to the linear delay differential equation.

$$\dot{u}(t) = r_1(t)u(t) + r_2(t)u(t-\beta), for\ t \in \ [0,\beta]$$
$$u(t) = H(t), for\ t \in [-\beta, 0]$$

For the following data ($\beta = 5, H(t) = 5 - t(t+5), r_1 = -1, r_2 = 0.5$)

Solution: substituting these values, we obtained

$$\dot{u}(t) = -u(t) + 0.5u(t-5), for\ t \in \ [0,5]$$
$$u(t) = 5 - t(t+5), for\ t \in [-5,0]$$

Then by replacing $u(t-\beta)$ to $H(t-\beta)$, and use the history function to obtain the initial condition $u(0) = H(0)$. On the first interval the solution $u_1(t)$ will be the function satisfying

$$\dot{u}(t) = -u(t) + 0.5H(t-5)$$
$$u(0) = H(0)$$

Then this implise that $u(0) = H(0) = 5$ and

$$\frac{du}{dt} + u = 2.5t - 0.5t^2 + 2.5$$

This equation is a first order linear ODE has the form of $u'(t) + p(t)u = q(t)$ So the general solution is

$$u(t) = \frac{1}{e^{\int p(t)dt}} \int e^{\int p(t)dt} q(t)dt + c$$

$p(t) = 1, q(t) = 2.5t - 0.5t^2 + 2.5$, so $1. u + \frac{d}{dt}(u) = 2.5t - 0.5t^2 + 2.5$, find the integrat- ing factor $\mu(t). p(t) = \mu'(t)$, we know $p(t) = 1$

$$\frac{d}{dt}\big(\mu(t)\big) = \mu(t).$$

Divide both side by $\mu(t)$

$$\frac{\frac{d}{dt}\big(\mu(t)\big)}{\mu(t)} = \frac{\mu(t). 1}{\mu(t)}$$

$$\frac{\frac{d}{dt}\big(\mu(t)\big)}{\mu(t)} = \frac{d}{dt}\Big(\ln\big(\mu(t)\big)\Big)$$

$$\frac{d}{dt}\Big(\ln\big(\mu(t)\big)\Big) = 1$$

Solve

$$\frac{d}{dt}\Big(\ln\big(\mu(t)\big)\Big) = 1$$

$$\frac{d}{dt}\Big(\ln\big(\mu(t)\big)\Big)dt = \int 1dt = t + c_1$$

$$\int \left(\frac{d}{dt}\Big(\ln\big(\mu(t)\big)\Big)\right)dt = \ln\big(\mu(t)\big) + c_2$$

$$\ln\big(\mu(t)\big) + c_2 = t + c_1$$

Combine the constants

$$\ln\big(\mu(t)\big) = t + c_1$$

Therefore the final solution for $\ln\big(\mu(t)\big) = t + c_1$ is:

$$\mu(t) = e^{t+c_1} = e^t e^{c_1}$$

Put the equation in the form

$$(\mu(t).u)' = \mu(t).q(t)$$

Multiply by integration factor $\mu(t)$ and rewrite the equation as:

$$1.u + \frac{d}{dt}(u) = 2.5t - 0.5t^2 + 2.5$$

$$e^t \frac{d}{dt}(u) + e^t u = 2.5te^t - 0.5t^2 e^t + 2.5e^t$$

Apply the product rule, $(f.g)' = f'g + fg', f = e^t, g = u$:

$$\frac{d}{dt}(e^t u) = 2.5te^t - 0.5t^2 e^t + 2.5e^t$$

Integrating both side

$$\int \frac{d}{dt}(e^t u)dt = \int (2.5te^t - 0.5t^2 e^t + 2.5e^t)\, dt$$

$$= 2.5 \int te^t dt - 0.5 \int t^2 e^t dt + 2.5 \int e^t dt$$

Apply integration by part $\int rs' = rs - \int r's$

$$2.5 \int te^t dt : r = t, r' = 1dt, s' = e^t dt, s = e^t$$

$$2.5 \int te^t dt = \frac{5}{2}(e^t t - e^t)$$

$$-0.5 \int t^2 e^t dt : r = t^2, r' = 2tdt, s' = e^t dt, s = e^t$$

$$-0.5 \int t^2 e^t dt = -0.5\left(t^2 e^t - 2\int te^t dt\right)$$

$$= \frac{1}{2}(2(e^t t - e^t) - e^t t^2)$$

$$e^t u + c_2 = \frac{1}{2}(2(e^t t - e^t) - e^t t^2) + \frac{5}{2}(e^t t - e^t) + \frac{5}{2}e^t + c_1$$

Combine the constants, then the solution is

$$u = \frac{-2e^t + 7te^t - t^2 e^t + c_1}{2e^t}$$

The initial condition $u(0) = 5$, then $c_1 = 12$, so the general solution is:

$$u = \frac{-2e^t + 7te^t - t^2 e^t + 12}{2e^t}$$

$$u = \frac{-2 + 7t - t^2 + 12e^{-t}}{2}$$

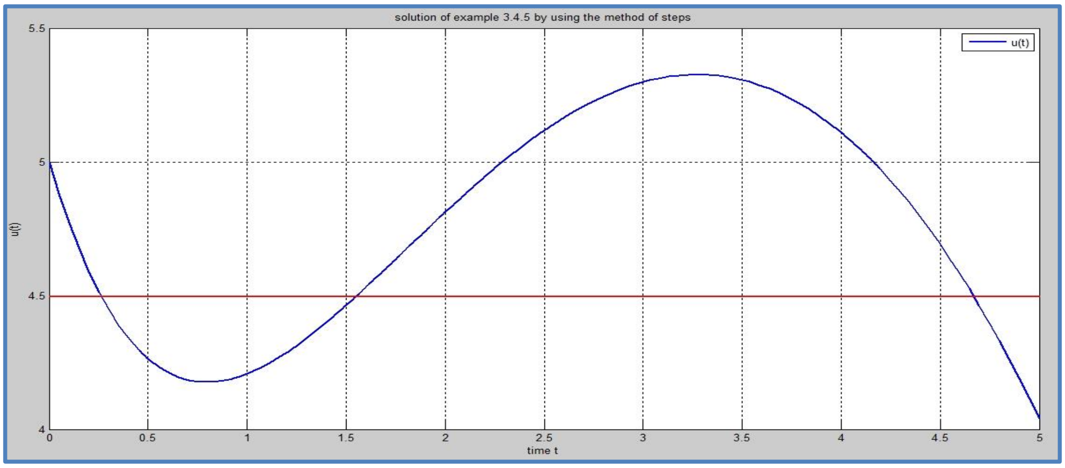


Figure 4.29: Steps solution

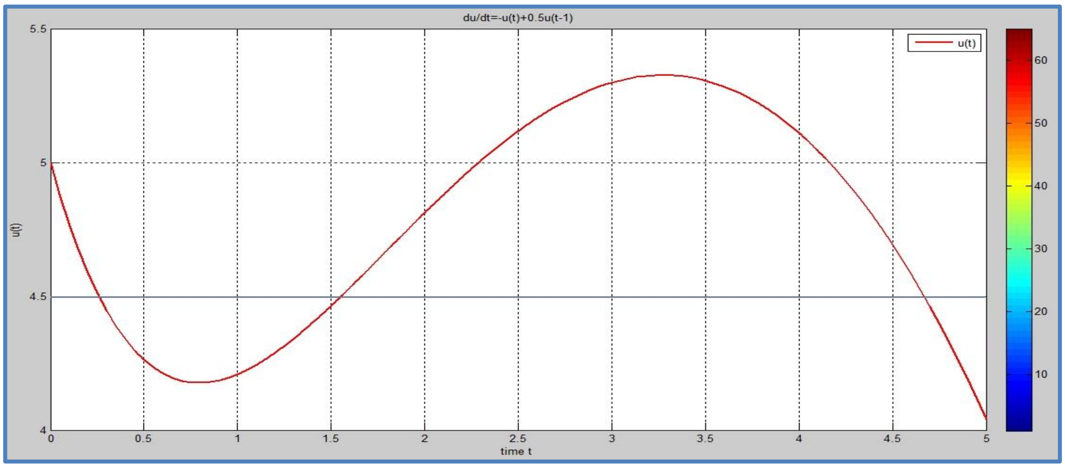


Figure 4.30: Approximate solution by using DDESD

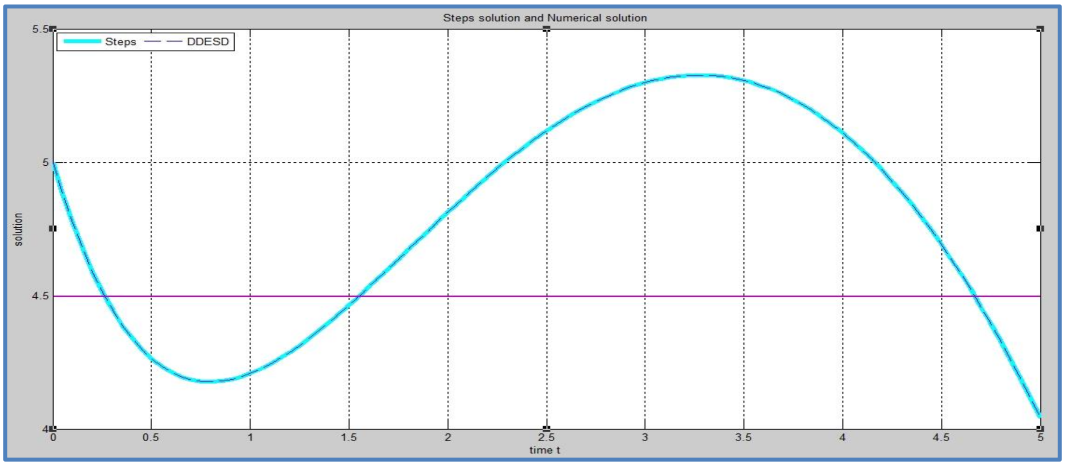


Figure 4.31: Comparing the two solutions Steps and Approximate

4.2.6 Exponential problem

If the history function is exponential, for example, use the method of steps to solve linear delay differential equation and sketch the graph, for $0 \leq t \leq 2.5$, $\dot{u}(t) = -0.02u(t - 1.25),\ on\ [0,1.25]$, when history function is $u(t) = e^{-t},\ on\ [-1.25,0]$.

Solution: when $t \in [0,1.25]$, this implies that $t - 1.25 \in [-1.25,0]$, so $u(t - 1.25)$ becomes $H(t - 1.25)$ on $[0,1.25]$, so $H(t - 1.25) = e^{-(t-1.25)}$, then $\dot{u}(t) = -0.02e^{-(t-1.25)}$, with initial condition, $u(0) = 1$.

$$du = -0.02 \int e^{-(t-1.25)} dt$$

$$u = 0.02e^{1.25-t} + c$$

We know $u(0) = 1$, so $0.02e^{1.25-t} + c = 1$, then the value of $c = 1 - 0.02e^{1.25}$, therefore

$$u_1(t) = 0.02e^{1.25-t} - 0.02e^{1.25} + 1 \qquad (4.15)$$

Now, we have a new system of delay equation in interval $[\beta, 2\beta] = [1.25,2.5]$

$$\begin{cases} \dot{u}(t) = -0.02u(t-1.25), \quad on\ [1.25,2.5] \\ u_1(t) = 0.02e^{1.25-t} - 0.02e^{1.25} + 1, on\ [0,1.25] \end{cases}$$

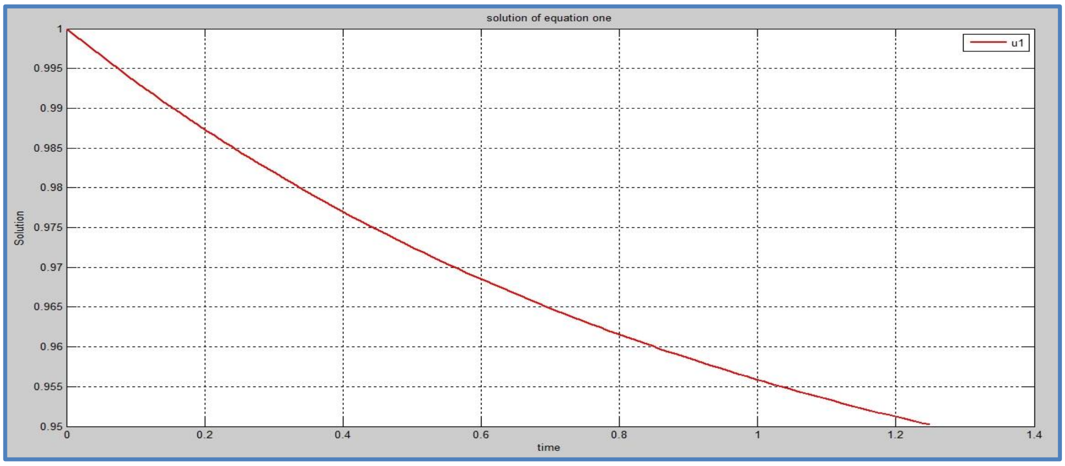


Figure 4.32: Graph of Equation 4.15

Again, when $t \in [1.25,2.5]$, this implies that $t - 1.25 \in [0,1.25]$, so $u(t-1.25)$ becomes $H(t-1.25)$ on $[1.25,2.5]$, so

$$H(t-1.25) = 0.02e^{2.5-t} - 0.02e^{1.25} + 1,$$

with initial condititon $u(1.25) = 0.02(1 - e^{1.25}) + 1$, and

$$\dot{u}(t) = -0.02(e^{2.5-t} - 0.02e^{1.25} + 1)$$

$$du = \int \left(-\frac{1}{2500}e^{2.5-t} + \frac{1}{2500}e^{1.25} - \frac{1}{50}\right)$$

$$u = \frac{1}{2500}e^{2.5-t} - \frac{2475727t}{133075971} + c$$

We know $u(1.25) = 0.02(1 - e^{1.25}) + 1$, so

$$\frac{1}{2500}e^{2.5-t} - \frac{2475727t}{133075971} + c = 0.02(1 - e^{1.25}) + 1$$

then the value of $c = \frac{1}{50}(1 - e^{1.25}) + 1.02325483\frac{1}{2500}e^{1.25}$, therefore

$$u_2(t) = \frac{1}{2500}e^{2.5-t} - \frac{2475727t}{133075971} + \frac{1}{50}(1 - e^{1.25}) + 1.02325483\frac{1}{2500}e^{1.25} \quad (4.16)$$

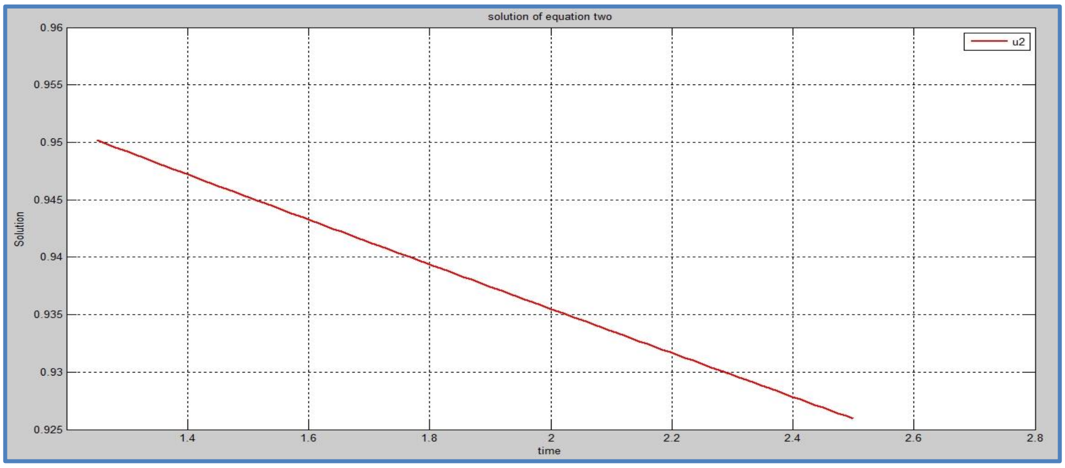


Figure 4.33: Graph of Equation 4.16

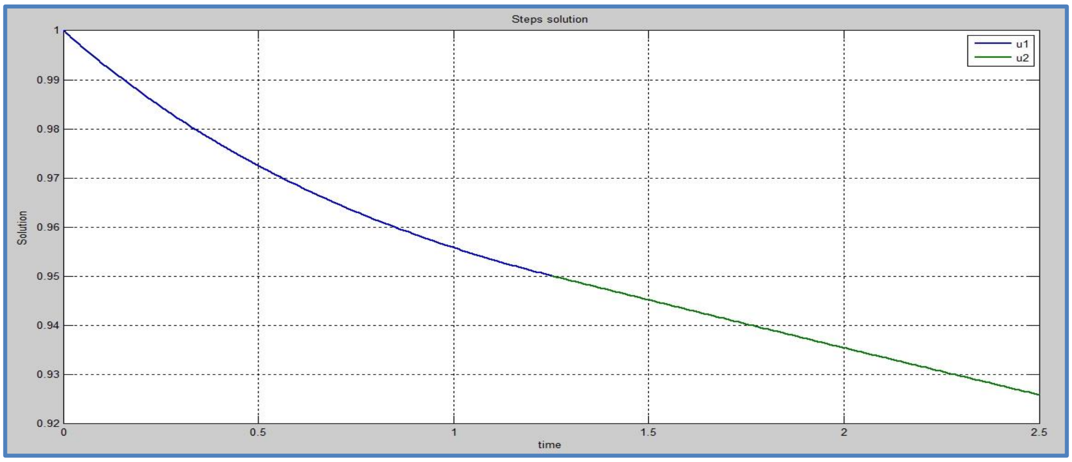


Figure 4.34: Steps solution

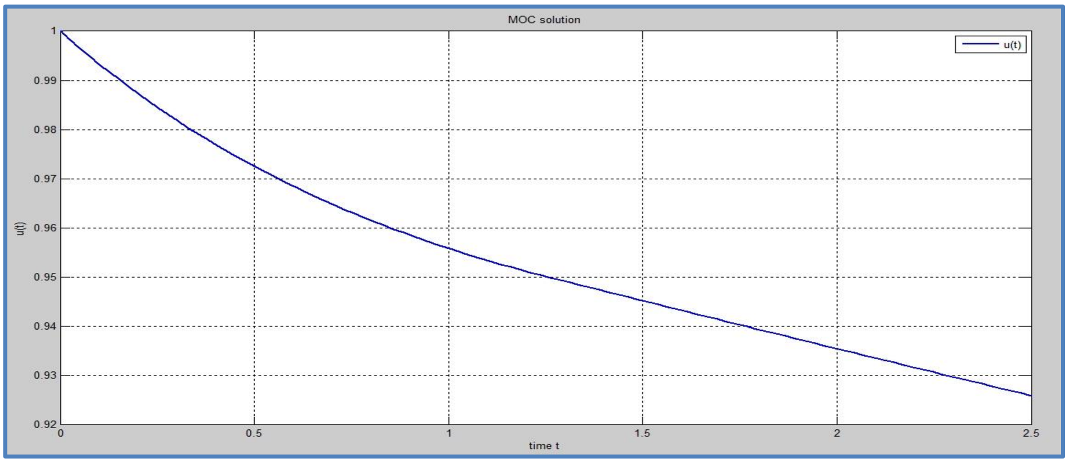


Figure 4.35: MOC solution

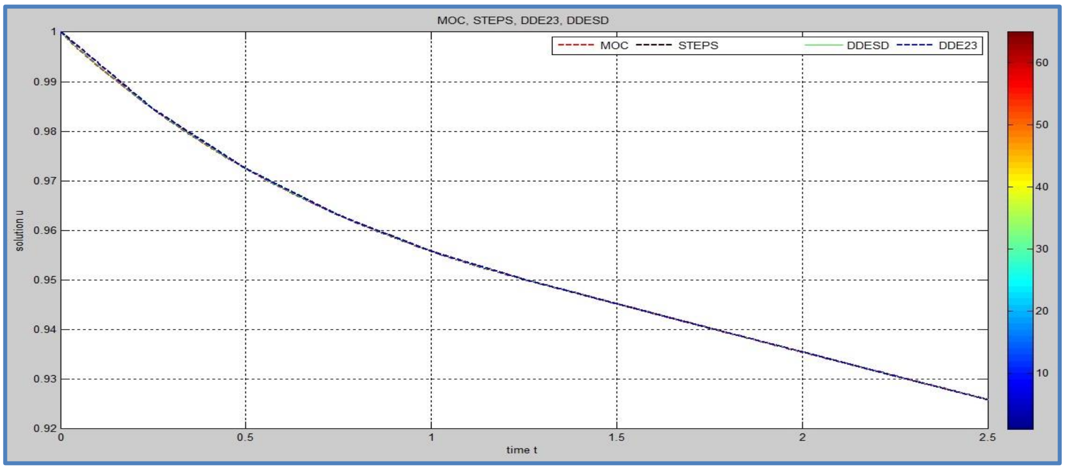


Figure 4.36: Comparing the four solutions MOC, Steps, DDE23 and DDESD

CHAPTER 5

5.1 Conclusion

In this thesis we have introduce two important methods to solving linear first order delay differential equations with a single delay and constant coefficient namely characteristic method and the method of steps. We discussed the basic definitions of the concepts. How and when the problems of delay time arise in our daily lives in various fields and their applications is well covered. We also explained the formulation of Matlab program and its codes solver such as DDE23 and DDESD.

We interested in this thesis to find solution for this kind of linear delay equation,$\dot{u}(t) = c_1u(t) + c_2u(t-\beta)$, with single constant delay and constant coefficients $c_1$and c_2. It is noted that, characteristic method (MOC) and the method of steps (STEPS) may be very effective in solving linear first order delay differential equation with a single constant delay and constant coefficient, we also observed that algebraic solution and approximate solution is very closed to each other by merging them in one graph for each problem, and we can also say that one of the best language programme is Matlab because is very powerful to deal with very complex problem in various mathematic fields, especially in differential equation such as the special kind of functional differential equation (FDE), namely delay differential equation (DDE), further, the Matlab program is very fast to give the result for our problem in this thesis. Finally we can confirm here is the biggest problems faced human beings is delay time because delay time has effect in everything, therefore to know how to find solution for this kind of delay equation is very important because many equation of applications have relation with life, contains delay time, so we can make it as a models contains delay time and solve this model by one of these methods such as MOC and STEPS, as well as the Matlab program can solve these models very quickly.

5.2 Recommendations

There are numerous possible open research problems related to the work in this study, and we list some of these kinds of problems and challenges in the following, which may be of interest in the future, we can generalize the following problems in future study .

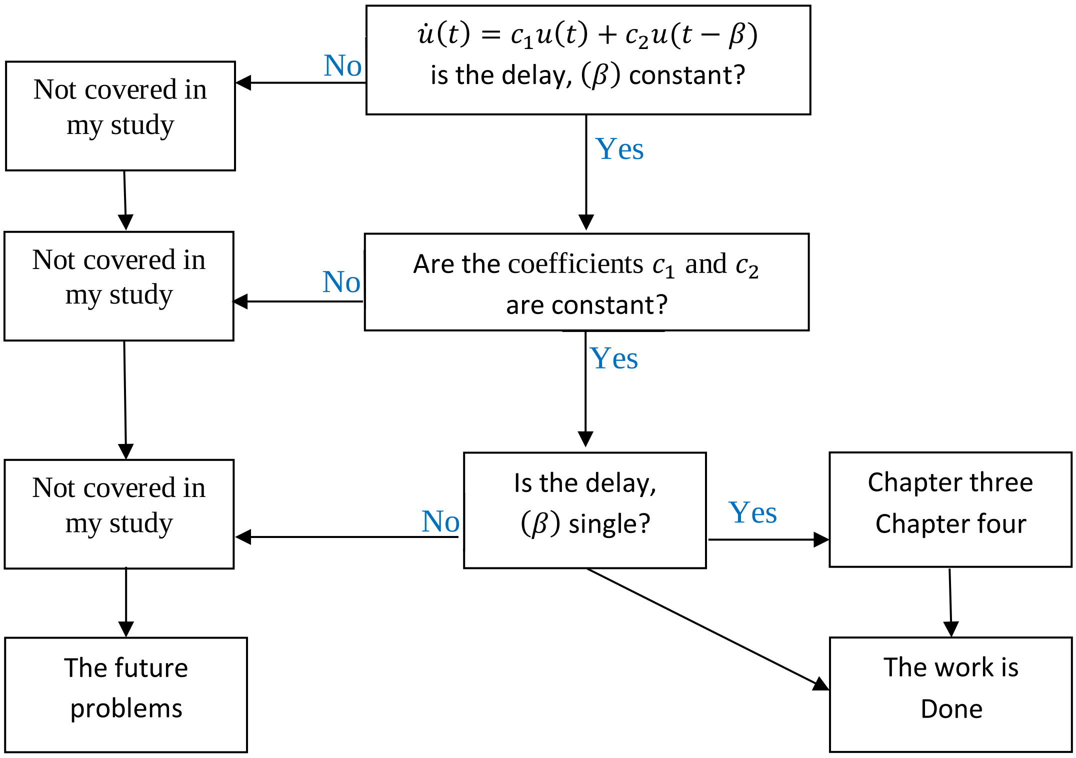


Figure 5.1: The diagram of my work in this thesis

So the done work in my study is to find the solutions of linear first order delay differential equations, with single constant delay and constant coefficients, by using two elementary methods namely MOC and STEPS.

How can we find algebraic solution and approximate solution for these linear first order delay differential equations (DDEs?)

Case one: If delays $\beta_i, i = 1, 2, \dots, n$ are non-single, with constant coefficients, this means that we have these kinds of linear delay differential equations (LDDEs)

- Linear first order delay differential equation with non-single constant delay and the delays are not equal $\beta_1 \neq \beta_2 \neq \cdots \neq \beta_n$, with constant coefficients $c_1, c_2, \dots, c_n$.

$$\dot{u}(t) = c_0 u(t) + c_1 u(t - \beta_1) + c_2 u(t - \beta_2) + \cdots + c_m u(t - \beta_n), \beta_1, \dots, \beta_n \geq 0$$

- Linear first order delay differential equation with non-single non constant delay, which means the delays are variable functions, $\beta_1(t) = \beta_2(t) = \cdots = \beta_n(t)$ with constant coefficients, $c_1, c_2, \dots, c_n$.

$$\dot{u}(t) = c_0 u(t) + c_1 u\big(t - \beta_1(t)\big) + c_2 u\big(t - \beta_2(t)\big) + \cdots + c_m u\big(t - \beta_n(t)\big), \beta_i(t) \geq 0$$

- Linear first order delay differential equation with non-single non constant delay, which means the delays are variable functions, $\beta_1(t) \neq \beta_2(t) \neq \cdots \neq \beta_n(t)$ with constant coefficients, $c_1, c_2, \dots, c_n$.

$$\dot{u}(t) = c_0 u(t) + c_1 u\big(t - \beta_1(t)\big) + c_2 u\big(t - \beta_2(t)\big) + \cdots + c_m u\big(t - \beta_n(t)\big), \beta_i(t) \geq 0$$

Case two: If delays $\beta_i, i = 1, 2, \dots, n$ are non-single, with variable coefficients, this means that we have these kinds of linear delay differential equations (LDDEs)

- Linear first order delay differential equation with non-single constant delay and the delays are not equal $\beta_1 \neq \beta_2 \neq \cdots \neq \beta_n$, with variable coefficients, $c_1(t), \dots, c_n(t)$.

$$\dot{u}(t) = c_0(t) u(t) + c_1(t) u(t - \beta_1) + \cdots + c_m(t) u(t - \beta_n), \beta_1, \dots, \beta_n \geq 0$$

- Linear first order delay differential equation with non-single non constant delay, which means the delays are variable functions, $\beta_1(t) = \beta_2(t) = \cdots = \beta_n(t)$ with variable coefficients, $c_1(t), c_2(t), \dots, c_n(t)$.

$$\dot{u}(t) = c_0(t) u(t) + c_1(t) u\big(t - \beta_1(t)\big) + \cdots + c_m(t) u\big(t - \beta_n(t)\big), \beta_i(t) \geq 0$$

- Linear first order delay differential equation with non-single non constant delay, which means the delays are variable functions, $\beta_1(t) \neq \beta_2(t) \neq \cdots \neq \beta_n(t)$ with variable coefficients$c_1(t), c_2(t), \dots, c_n(t)$.

$$\dot{u}(t) = c_0(t)u(t) + c_1(t)u\big(t - \beta_1(t)\big) + \cdots + c_m(t)u\big(t - \beta_n(t)\big), \beta_i(t) \geq 0$$

Case three: If delays $\beta_i, i = 1, 2, \dots, n$ constants, with variable coefficients, this means that we have these kinds of linear delay differential equations (LDDEs)

- Linear first order delay differential equation with single constant delay β, with variable coefficients, $c_1(t), c_2(t), \dots, c_n(t)$.

$$\dot{u}(t) = c_0 u(t) + c_1 u(t - \beta), \qquad \beta \geq 0$$

- Linear first order delay differential equation with non-single constant delay, and $\beta_1 = \beta_2 = \cdots = \beta_n$ with variable coefficients, $c_1(t), c_2(t), \dots, c_n(t)$.

$$\dot{u}(t) = c_0 u(t) + c_1 u\big(t - \beta_1(t)\big) + c_2 u\big(t - \beta_2(t)\big) + \cdots + c_m u\big(t - \beta_n(t)\big), \beta_i(t) \geq 0$$

Applications of delay differential equations have the wide area in various life fields such as, biology, economics, microbiology, ecology, distributed networks, mechanics, nuclear reactors, physiology, engineering systems, and epidemiology and heat flow, so I interested in my study to solve linear first order delay differential equations $\dot{u}(t) = c_1 u(t) + c_2 u(t - \beta)$, which means we can find a models from various field like this equation and solve it by one of this methods such as MOC and STEPS in future study.

Climate modeling

El Niño–Southern Oscillation (ENSO): is an irregularly periodical variation in winds and sea surface temperatures over the tropical eastern Pacific Ocean, affecting much of the tropics and subtropics. The warming phase is known as *El Niño* and the cooling phase as *La Niña*. *Southern Oscillation* is the accompanying atmospheric component, coupled with the sea temperature change: *El Niño* is accompanied with high and *La Niña* with low air surface pressure in the tropical western Pacific. The two periods last several months each (typically occur every few years) and their effects vary in intensity, an early model of the El Niño–Southern Oscillation phenomenon with physical parameter $\alpha > 0$ is:

$$\dot{T}(t) = T(t) - \alpha T(t - \beta)$$

Recruitment Models

(Blythe et al, 1982) proposed a general single species population model with a time delay

$$\frac{dx}{dt} = R\big(x(t - \tau)\big) - Dx(t)$$

Where R and D represent the rates of recruitment to, and death rate from, an adult population of size x; and $\tau > 0$ is the maturation period. For a linear analysis of the model, see (Brauer and Castillo, 2001).

Remote Control Dynamical System

If you send a signal to a robot telling it to turn, stop, or perform some other task, there will be some lag between the time you initiate the signal and the time the robot responds. It takes another delay for you to see what the robot did and then to make use of this feedback to influence your next decision about what new signal to send. For another example, if you are trying to row a boat you may push your oar through the water and then wait to see the heading of the boat before dipping the oar again. However, if you are heading for a dangerous obstacle, you may not wait after each stroke, but simply decide to execute a series of pre-planned back strokes before getting feedback. Typically, controls are not sent as individual signals, one at a time, but rather as a pre-set pattern, a template.

Almost any "automated" process works from a template. For instance, a pre-determined design can be programmed into a weaving machine to produce a desired pattern in a rug. Specially, we want to solve a Remote Control Dynamical System (RCDS), which is denned as one whose dynamical equation is the FDE

$$y'(t) = f\left(t, y(t), y\big(h(t)\big)\right), t \in I$$

Where $I,$ is an open interval, called the operational interval. The deviating argument, $h(t) \in C^1[I]$; is one whose range,$h[I]$, (called the remote domain or remote interval) is disjoint from I. Initially, the system is assumed to be subject to a control function, $p(t) \in C^1[h[I]]$ defined on the remote domain. Thus, the output function, $y(t)$, is the solution of Equation (Ryder, 1969) on I, and $y(t) = p(t)$ on $h(I)$ .When d > 0 and the deviating argument h is defined by $h(t) = t - d$ we get the DDE. The function $t - d$ is called the argument.

$$y'(t) = a_1 y(t) + a_t y\big(h(t)\big), on\ [0, d]$$

$$y(t) = p(t), on\ [-d, 0]$$

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