CONFLUENT HYPERGEOMTRIC FUNCTION WITH KUMMER'S FIRST FORMULA

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

SHWAN SWARA FATAH

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2016

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Approval of Director of Graduate School of

Applied Sciences

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To my parents…

ABSTRACT

The study is an examination of the definitions and basic properties of hypergeometric function, confluent hypergeometric function, The main objective of the study confluent hypergeomtric function with Kummer’s first formula. Several properties such as contiguous function relations**.**, differential equations and Elementary series manipulation for these hypergeomtric and confluent hypergeomtric families are obtained. Was to ascertain an approximation of solution of confluent hypergeometric function. Were therefore drawn from the study that the Kummer’s function has wide application in various subjects and hence proving stability or other properties were also drawn to be of paramount importance.

Keywords: Hypergeomitric function; Confluent hypergeomtric function; Kummer’s first formula; Gamma function; Pochhammer function

ÖZET

çalışma,Hipergeometrik fonksiyonların ve Birleşik hyperbolic fonksiyonların tanımlarını ve temel özelliklerini inceler. Çalışmanın temel amacı birinci Kummer formula ile birleşik hipergeometrik fonksiyonları çalışmaktır. Hipergeometrik ve birleşik hipergeometrik ailelerinin differensiyel denklemleri, bitişik fonksiyon ilişkileri, ve temel seri manipulasyonları gibi bası özellikler elde edilmiştir. Birleşik hipergeomertik fonksiyon çözümlerinin yaklaşımları bulunmuştur. Kummer's fonksiyonları çeşitli konlarda geniş uygulama alanlarına sahiptir ve kararlılığı ve diğer özelliklerinin de oldukça önemli olduğu bu tezde vurgulanmıştır.

Anahtar Kelimeler: Hipergeometrik fonksiyon; Birleşik Hipergeometrik fonksiyon; Kummer in birinci formula; Gamma fonksiyon; Pochmmar fonksiyon

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LIST OF SYMBOLS

$\boldsymbol{\Gamma}(\boldsymbol{x})$	Gamma Functions
$(\boldsymbol{x})_{\boldsymbol{n}}$	Pohammer Symbol
${}_{2}\boldsymbol{F}_{1}(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c};\ \boldsymbol{z})$	Hypergeomtric function
${}_{p}\boldsymbol{F}_{q}\left(\begin{matrix}\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_p \\ \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_q\end{matrix}; \boldsymbol{z}\right)$	General hypergeomtric function
${}_{1}\boldsymbol{F}_{1}(\boldsymbol{a}; \boldsymbol{c};\ \boldsymbol{z})$	Confluent hypergeomtric function
${}_{0}\boldsymbol{F}_{0}\left(\begin{matrix}- \\ -\end{matrix}, \boldsymbol{z}\right)$	Expositional function
${}_{1}\boldsymbol{F}_{0}\left(\begin{matrix}\boldsymbol{a} \\ -\end{matrix}, \boldsymbol{z}\right)$	BinoType equation here.mial function
$\boldsymbol{J}_{\boldsymbol{\alpha}}$	Bessel function of the first kind
$\prod$	Product
[ ]	Bracket

CHAPTER 1

INTRODUCTION

This chapter outlines several basic definitions, theorems and some properties of special functions. This study thrives to proffer insights about the confluent hypergeometric function with by employing the Kummer's formula. The notion behind the Kummer confluent hypergeometric function (CHF) stems from an essential category of special functions of mathematical physics. Kummer's formula in (CHF) can be decomposed into the following;

The initial Kummer's formula assumes the following form:

$$e^{-z}\ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(c-a)_n(-z)^n}{(c)_n n!} = {}_1F_1(c-a;c;-z\ ), c \neq \{o\} \cup \{-1,-2,-3,..\}$$

And, kummer's second formula

$$e^{-z}\ {}_1F_1(a,2a;2z) = \sum_{n=0}^{\infty} \frac{(\frac{1}{4}z^2)^n}{\left(a+\frac{1}{2}\right)_n n!} = {}_0F_1\left(-;a+\frac{1}{2};\frac{1}{4}z^2\right).$$

if a is not odd positive integer.

This study will therefore offer further explanations about the Kummer's first formula in confluent hyper geometric functions. Despite the fact that Gauss played an essential role in the systematic study of the hypergeometric function, (Kummer, 1837) assumed a critical role in the development of properties of confluent hypergeometric functions. Kummer published his work on this function in 1836 and since that time it has been commonly referred to as the Kummer's function (Andrews, 1998). Under the hypergeometric function, the confluent hypergeometric function is related to a countless number of different functions.

This work therefore outlines the general and basic properties of hypergeometric and confluent hypergeometric function and the Kummer's first formula. This study will also extend to incorporate the related examples and theorems.

The first chapter deals with the synopsis of basic definitions, theorems and exceptions of the hyper geometric functions while the second chapter is a blueprint of definitions, properties and theorems of confluent hyper geometric functions. Meanwhile, chapter three lays out examples and special cases of Kummer's first formula coupled with reinforcing explanations. A recapitulation of properties of the hypergeometric functions is given in the fourth chapter while the fifth chapter concludes this study by looking at conclusions that can be drawn from this study.

1.2 Gamma Function

It is undoubtable that most essential functions in applied sciences are defined via improper integrals. Of notable effect is Gamma functions. Such functions have several applications in Mathematics and Mathematical Physics.

1.2.1 Definition

The elementary definition of the gamma function is Euler's integral (Gogolin, 2013)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

is converges for any $z>0$

1.2.2 Some basic properties of Gamma function with their proofs(Özergin, 2011).

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t} \, |_0^\infty = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt[2]{t}} dt = 2\Gamma(1) = 2\int_0^\infty e^{-u^2}\, du = 2\frac{\sqrt[2]{\pi}}{2} = \sqrt[2]{\pi}$$

$$\Gamma(x+1)=\int_0^\infty t^x e^{-t}\,dt=-t^x\,e^{-t}\,|_0^\infty-\int_0^\infty xt^{x-1}(-e^{-t})dt=x\int_0^\infty t^{x-1}e^{-t}\,dt=x\Gamma(x)$$

1.2.3 Lemma

The Gamma function satisfies the functional equation

$$\Gamma(x+1)=x\Gamma(x)\ ,\quad x>0$$

Moreover, by iteration for x > 0 and n ∈ N

$$\Gamma(x+n)=\Gamma(x+n-1).\ .\ .\ .(x+1)x\Gamma(x)=\prod_{i=1}^{n}(x+1-i)\Gamma(x)$$

$$\Gamma(n+1)=(\prod_{i=1}^{n}i)\Gamma(x)=\prod_{i=1}^{n}(i)=n!$$

In other words, the Gamma function can be interpreted as an extension of factorials.

1.3 Definition (Sebah, 2002)

The beta function or Eulerian integral of the first kind is given by

$$B(x,y)=\int_0^\infty t^{x-1}(1-t)^{y-1}\,dt\ ,\qquad where\ x,y>0$$

This definition is also valid for complex numbers x and y such as

$R(x)>0$ amd $R(y)>0$

1.3.1 Theorem (Gronan, 2003)

$$if\ \ R(x)>0\ and\ R(y)>0\ then\ B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=B(y,x)$$

1.3.2 Some special values for Beta function

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\pi\ ,$$

$$B(x,1)=\frac{1}{x}\ ,$$

$$B(x,n)=\frac{(n-1)!}{x(x+1)\ldots(x+n-1)}\quad n\geq 1$$

1.4 Definition

Let x be a real or complex number and n be a positive integer,

$$(x)_n=\frac{\Gamma(x+\mathrm{n})}{\Gamma(x)}=x(x+1)\ldots.(x+n-1)$$

"Pochhammer Symbol" is where $(x)_n$ is used to represent the falling factorial sometimes called the descending factorial, falling sequential product, lower factorial (Freeden, 2013).

1.4.1 Some properties of Pochhammer symbol

$i)\ \ (a+n)_k=\dfrac{(a)_{n+k}}{(a)_n},$

Where a is a real or complex number and n, k are natural numbers

$ii)\ \dfrac{(a)_{2k}}{2^{2k}}=\left(\dfrac{a}{2}\right)_k\left(\dfrac{a}{2}+\dfrac{1}{2}\right)_k$

Where a is a complex number and k is a natural number

$iii)\ \ \dfrac{(2k)!}{2^{2k}k!}=\left(\dfrac{1}{2}\right)_k$

Where k: is a natural number.

Note

If $a = 1$ $then\ we\ have$ $(a)_n = (1)_n = 1 \times 2 \times 3 \times \ldots \times n = n!$

If $a = 2$ then $(2)_n$=$(n+1)$ and also we have

$(a)n = (-N)n = (-N)(-N+1)(-N+2)\cdots(-N+n-1) = 0$ $if\ a = -n$,

$n = \{0,1,2,. . .\}$.

$(a)_0 = 1$, $\quad a \neq 0$

1.4.2 Theorem

Show that for 0≤ $k \leq n$,

$$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}$$

Note particularly the special case α = 1

Proof:

Consider $(a)_{n-k}$ for 0≤ $k \leq n$,

$$(a)_{n-k} = a(a+1)\ .\ .\ .\ (a+n-k-1)$$

$$= \frac{a(a+1)\ .\ .\ .\ (a+n-k-1)[(a+n-k)(a+n-k+1)\ .\ .\ .\ (a+n-1)}{(a+n-k)(a+n-2)\ .\ .\ .\ (a+n-k)}$$

$$= \frac{(a)_n}{(a+n-k)_k}$$

$$= \frac{(a)_n}{(-1)^k (1-a-n)_k}$$

$$= \frac{(-1)^k (a)_n}{(1-a-n)_k}$$

Not for $a = 1, (n-k)! = \frac{(-1)^k n!}{(-n)_k}$.

1.4.3 Lemma

$$(\alpha)_{2n} \;= 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

Proof:

$$(\alpha)_{2n} = \alpha(\alpha+1)(\alpha+2)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+1}{2}\right)\quad\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+n-1\right)\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha}{2}+1\right)\left(\frac{\alpha}{2}+n-1\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}+1\right)\dots\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

CHAPTER 2

CONFLUENT HYPERGEOMETRIC FUNCTION

This section draws attention on the confluent hypergeometric function, its definition and inherent properties. Due to the importance that is attached to the confluent hypergeometric function in hypergeometric function; this study will therefore draw attention to the examination of the hyper geometric function.

2.1 Hypergeometric Function

The function ${}_2F_1(a, b; c; x)$ corresponding to p=2, q=1 is the first hyper geometric function to be examined (and, in general, emerges in prominence especially in physical problems), as is synonymously referred to as "the" hyper geometric equation or, more explicitly, Gauss's hyper geometric function (Gauss, 1812; Barnes 1908). To confound matters much more, the term "hyper geometric function" is less usually used to mean shut structure, and "hyper geometric series" is sometimes used to mean hyper geometric function.
Hyper geometric functions are solutions to the hyper geometric differential equation, which has a regular singular point at the starting point. A hyper geometric function can be derived from the hyper geometric differential equation.

2.1.1 Definition

(Rainville,1965). Asserts that a hyper geometric function can be defined as follows;

$$F(a\ b; c; z) = {}_2F_1(a, b; c;\ z) = F\ (b, a.c;\ z) = \sum_{n=}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n\ , |z| \le 1 \qquad (2.1)$$

For c neither zero nor negative integer. In 2.1, the notation

1 - Refers to number of parameters in denominator

2 - Refers to number of parameters in numerator

2.1.2 Functions with representations like Hypergeometric series

$$F(1,b,b,z) = \sum_{n=0}^{\infty} \frac{(1)_n}{n!} \cdot z^n = \sum_{n=0}^{\infty} z^n$$

$$\arcsin z = F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right)$$

$$ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^n = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n} \frac{(-1)^n z^{n+1}}{n!} = zF\begin{pmatrix}1\ ,1 \\ 2\end{pmatrix};-z\Bigg)$$

2.1.3 Properties of Hypergeometric functions

2.1.3.1 Differential representation

The Differential representation of the hypergeometric function is given by

$$\frac{d}{dz}F(a,b;c;z) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}z^n}{(c)_{n+1}n!}$$

$$= \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n z^n}{(c+1)_n n!}$$

$$= \frac{ab}{c}F(a+1,b+1;c+1;z)$$

2.1.3.2 Integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^a\,dt \quad c > b > 0$$

Where Gamma is defined by

$$\Gamma(x) = \int_0^{\infty} t^x\, e^{-t} dt\, , x > 0$$

2.1.3.3 The Hypergeometric equation

The linear second-order DE

$$z(1 - z)\frac{dw^2}{dz^2} + (c\ -\ (\alpha\ +\ b\ +\ 1)z)\frac{dw}{dz} - abw = 0$$

is called the hypergeometric equation

These functions were studied by numerous mathematicians including Riemann who gathered in their conduct as functions of a complex variable, also, concentrated on its analytic continuation regarding it as a solution to the differential equation (Campos, 2001).

$$z(1 - z)\frac{dw^2}{dz^2} + (c\ -\ (\alpha\ +\ b\ +\ 1)z)\frac{dw}{dz} - abw = 0 \qquad (2.1)$$

or, multiplying equation (4) by z and denoting $\theta = z\ \frac{d}{dz}$,

$$[\theta(\theta\ +\ c\ -\ 1) - \ z(\theta\ +\ \alpha)(\theta\ +\ b)](z) = 0 \qquad (2.2)$$

Equation 2.1 or 2.2, has three regular singular points at 0, 1 and ∞, and it is

Up to standardization the general form of a second order linear differential equation with this conduct.

Note if one of the numerator parameters a or b are equal to the denominator parameter c we get

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a\,,\ b \\ b \end{matrix}\ ; z\right) &= \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(b)_n n!} z^n \\ &= {}_1F_0\left(\begin{matrix} a \\ \_ \end{matrix}\ ; z\right) \\ &= \sum\nolimits_{n=0}^{\infty} \binom{-a}{n} (-z)^n \\ &= (1 - z)^{-a}\, , \qquad |z|\ < 1 \end{aligned}$$

2.1.4 Problem

Which results in

$$F\begin{bmatrix} -n\,,\ b; & \\ c; & 1 \end{bmatrix} = \frac{(c-b)_n}{(c)_n}$$

Solution

Consider $F(-n, b; c; 1)$.at once, if $R(c-b) > 0$,

$$F(-n, b; c; 1) = \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-b)} = \frac{(c-b)_n}{(c)_n}$$

Actually the condition $R(c-b) > 0$ is not necessary because of the termination of the series involved.

2.2. Generalized Hypergeometric Function

As outlined in the definition (1) there are two numerator parameters, a and b; and one denominator, c. it is a natural generalization to move from the definition (1) to a similar function with any number of numerator and denominator parameters.

We define a generalized hyper geometric function by

$${}_pF_q\begin{pmatrix} a_1\,, a_2\ \,, a_3. \ .\ .\ .\ , a_p \\ b_1\,, b_2\,, b_3\,,\ .\ .\ ,. b_q \end{pmatrix} ; z \Bigg) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n .\ .\ .\ (a_p)_n z^n}{(b1)_n (b2)_n (b3)_n .\ .\ .\ .(b_q)_n n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (ai)_n z^n}{\prod_{i=1}^{q} (ai)_n\, n!} \quad i = 1\,, 2\,, 3\,, .\ .\ . n$$

The parameters must be such that the denominator factors in the terms of the series are never zero. When one of the numerator parameters ai equals $-N$, where N is a nonnegative integer, the hypergeometric function is a polynomial in z (see below). Otherwise, the radius of convergence p of the hypergeometric series is given by

$$p = \begin{cases} \infty & if\ p < q+1 \\ 0 & if\ p > q+1 \\ 1 & if\ p = q+1 \end{cases}$$

This follows directly from the ratio test. In fact, we have

$$\lim_{n\to\infty} |\frac{c_{n+1}}{c_m}| \quad p = \begin{cases} \infty & if\ p < q+1 \\ 0 & if\ p > q+1 \\ 1 & if\ p = q+1 \end{cases}$$

In the case that $p = q + 1$ the situation that $|z| = 1$ is of special interest.

The hypergeometric series ${}_{q+1}F_{p}$ $(a_1 \ a_2 ,\dots, a_{q+1} ;\ b_{1,} b_2 ,\dots\ b_p, z)$

with $|z| = 1$ converges absolutely if Re $(\sum b_i - \sum a_j) \leq 0$

The series converges conditionally if $|z| = 1$ with $z \neq 1$ and $-1 <$ Re $(\sum b_i - \sum a_j) \leq 0$

And the series diverges if Re $(\sum b_i - \sum a_j) \leq -1$.

Two elementary instances of the ${}_pF_q$ follow if no numerator or denominator parameters are present.

Which results to

$${}_0F_0 \left(\begin{matrix} - \\ - \end{matrix} ,z \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

Which is called the exponential function where $z \in \mathbb{C}$

And also if we have one numerator parameter without denominator parameter, we obtain

$${}_1F_0 \left(\begin{matrix} a \\ -; \end{matrix} z \right) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = (1-z)^a, \quad z \in \mathbb{C}$$

is called a binomial function

2.3 Bessel Function

We already know that the ${}_0F_0$ is an exponential and that ${}_1F_0$ is a binomial. It is natural to examine next the most general 0F1, the only other ${}_pF_q$ with less than two parameters. The function we shall study is not precisely the ${}_0F_1$ but one that has an extra factor definition below (Dickenstein,2004).

2.3.1 Definition

If n is not a negative integer

$$J_n(z) = \frac{\left(\frac{z}{2}\right)^n}{\Gamma(1+n)}\ {}_1F_0\left(-;1+n;-\frac{z^2}{4}\right).$$

2.4 Confluent Hypergeometric Function

This section provides an examination of the most powerful methods implemented to accurately and efficiently evaluate the confluent hypergeometric function, Kummer's (confluent hypergeometric) function M (a, b, z), introduced by (Kummer, 1837),

The term confluent refers to the merging of singular points of families of differential equations; confluent is Latin for "to flow together.

2.4.1 Definition

The Kummer confluent hypergeometric function is defined by the absolutely convergent infinite power series"

$$M(a,c,z) = \ {}_1F_1\left({a \atop c}\ ;\ z\right) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}\ , \qquad -\infty < z < \infty$$

It is analytic, regular at zero entire single-valued transcendental function of all a, c, x, (real or complex) except c$\neq$ 0 or a negative integer.

Note

The confluent hypergeometric function it is related to the hypergeometric function according to

$$\lim_{b\to\infty}\binom{a\ ,b}{c}\ ;\ \frac{z}{b}\Big) = \lim_{b\to\infty}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(\frac{z}{b})^n}{(c)_n n!} = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(c)_n n!}\lim_{n\to\infty}\frac{(b)_n}{b^n} = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(c)_n n!}$$

So that $\lim_{b\to\infty}\binom{a\ ,b}{c}\ ;\ \frac{z}{b}\Big) = m(a,c,z)$

2.4.2 Relation to other functions

$i)\ m(-n;1;z) = \ln(z)$

$ii)\ m(a;a.z) = e^z$

$iii)\ m(1,;2;2z) = \dfrac{e^z}{z}sinhz$

2.4.3 Theorem

$m(a;a.z) = e^z$

Proof:

$$m(a,a,z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(a)_n n!} = \sum_{n=0}^{\infty}\frac{z^n}{n!} = e^z$$

2.4.4 Elementary properties of Confluent Hypergeometric Function

2.4.4.1 Differential representation

Because of the similarity of definition to that of F (a, b; c; z), the function M (a; c; z) obviously has many properties analogous to those of the hypergeometric function (ko, 2011). For example, it is easy to show that;

$$i)\ \frac{d}{dz}m(a,c;z) = \frac{a}{c}m(a+1,c+1;z).$$

Since

$$\frac{d}{dz}m(a,c;z) = \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{(a)_{n+1}z^n}{(c)_{n+1}n!}$$

$$= \frac{a}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n z^n}{(c+1)_n n!}$$

$$= \frac{a}{c}m(a+1;c+1;z)\cdot$$

Also in general

$$ii)\ \frac{d^k}{dx^k}m(a,c;z) = \frac{(a)_k}{(c)_k}m(a+k,c+k;z), \quad k = 1,2,3,\ldots$$

2.4.4.2 Integral representation

Based on Euler's integral representation for the ${}_2F_1$ hypergeometric function, one might expect that the confluent hypergeometric function satisfies

$$m(a;c;z) = {}_1F_1\left({a \atop c};\, z\right)$$

$$= \lim_{b\to\infty}\left({a\,,b \atop c};\, \frac{z}{b}\right)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}\, t^{a-1}(1-t)^{c-a-1}\, dt, \quad c > a > 0$$

2.4.4.3 Theorem

For Re c > Re a > 0 *we have*

$${}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}\, t^{a-1}(1-t)^{c-a-1}\, dt$$

Proof: note that we have

$$\int_0^1 e^{xt}\, t^{a-1}(1-t)^{c-a-1}\, dt \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_0^1 t^{n+a-1}(1-t)^{c-a-1}\, dt$$

And Re a$>$ *R*

$e(c-a\,) > 0$

$$\int_0^1 t^{n+a-1}(1-t)^{c-a-1} dt \;= B(n\,+\,a, c\,-\,a)$$

$$= \frac{\Gamma(n+a)\Gamma(c-a)}{\Gamma(n+c)}$$

$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \frac{(a)_n}{(c)_n}$$

$for\ n = 0,1,2,.,.,.\ this\ implies\ that$

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}\, t^{a-1}(1-t)^{c-a-1} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} = \;{}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\,;\, z\right).$$

2.4.4.5 Confluent Hypergeometric equation

The Confluent hypergeometric equation established by (Buchholz, 2013) defines the hypergeometric function $y\; = F\; (a, b;\; c;\; z)$ as a solution of Gauss' equation

$$z(1 - z)\frac{d^2w}{dz^2} \;+\; (c \;-\; (\alpha \;+\; b \;+\; 1)z)\frac{dw}{dz} \;-\; \alpha bw \;=\; 0 \qquad (2.3)$$

By making the change of variable $z = \frac{x}{b}$ (2.6) becomes

$$\left(1 - \frac{x}{b}\right) w'' + \left(c - x - \frac{a+1}{b} x\right) w' - aw = 0$$

and then allowing b→ ∞ we find

$$x\, w'' + (c - x)w' - \alpha w = 0 \tag{2.4}$$

For C ∉ Z the general solution of the confluent hypergeometric differential equation (2.4) can be written as

$$w(z) = A\, {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; z\right) + Bz^{1-c}\, {}_1F_1\left(\begin{matrix} a+1-c \\ 2-c \end{matrix} ; z\right)$$

with A and B arbitrary constants

2.4.4.6 Multiplication formula

A known formula, given by (Luke, 2014) can be utilized to determine the value of the confluent hypergeometric function in terms of another confluent hypergeometric function with the same parameters but with the variable of opposite sign. This formula can be specified as follows;

$${}_1F_1(a; b; z) \times {}_1F_1(a; b; -z) = {}_2F_3(a, b - a; b, \tfrac{1}{2}b + \tfrac{1}{2}; \tfrac{z^2}{4})$$

2.4.4.7 The Contiguous function relation

The function $m(a; c; z)$ also satisfies recurrence relations involving the contiguous functions $m(a \pm 1; c; z)$and $m(a; c \pm 1; z)$. from these four contiguous functions, taken two at a time, we find six recurrence relations with coefficients at most linear in z (Pearson, 2009).

$i)$ $(c - a - 1)m(a; c; z) + am(a + 1; c; z) = (c - 1)m(a; c - 1; z)$

$ii)$ $cm(a; c; z) - cm(a - 1; c; z) = zm(a; c + 1; z)$

$iii)$ $(a - 1 + c)m(a; c; z) + (c - a)m(a - 1; c; z) = (c - 1)m(a; c - 1; z)$

$iv)$ $c(a + z)m(a; c; z) - acm(a + 1; c; z) = (c - a)zm(a; c + 1; z)$

$v)\ \ (c-a)m(a-1;c;z)+(2a-c+z)m(a;c;z)=am(a+1;c;z)$

$vi)\ \ c(c-1)m(a;c-1;z)-c(c-1+z)m(a;c;z)=(a-c)zm(a;c+1\colon z)$

2.4.5 Some example of confluent Hypergeometric function

Example 1

The function

$$erF(z)=\frac{2}{\sqrt{\pi}}\int_{0}^{z} exp(-t^2)\,dt$$

as defined by (Rainville, 1965) exhibits that

$$erF(z)=\frac{2z}{\sqrt{\pi}}1F1\left(\frac{1}{2};\frac{3}{2};-z^2\right).$$

Solution. Let

$$erF(z)=\frac{2}{\sqrt{\pi}}\int_{0}^{z} exp(-t^2)\,dt.$$

Then,

$$erF(z)=\frac{2}{\sqrt{\pi}}\sum_{n=o}^{\infty}\frac{(-1)^n\int_0^z t^{2n}}{n!}$$

$$=\frac{2}{\sqrt{\pi}}\sum_{n=o}^{\infty}\frac{(-1)^n\int_0^z z^{2n+1}}{n!\,(2n+1)}$$

$$=\frac{2z}{\sqrt{\pi}}\sum_{n=o}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)_n z^{2n}}{n!\left(\frac{3}{2}\right)_n}$$

$$=\frac{2z}{\sqrt{\pi}}1F1\left(\frac{1}{2};\frac{3}{2};-z^2\right)$$

Example 2

The incomplete gamma function may be defined by the equation

$$y(a,z) = \int_0^z e^{-t} t^{a-1} dt, R(a) > 0.$$

So that

$$y(a,z) = a^{-1} x^{-a} 1F1(a; a+1; -z).$$

Solution: Let

$$y(a,z) = \int_0^z e^{-t} t^{a-1} dt, R(a) > 0.$$

Then

$$y(a,z) = \int_0^z \sum_{n=o}^{\infty} \frac{(-1)^n t^{n+a-1}}{n!}$$

$$= \sum_{n=o}^{\infty} \frac{(-1)^n z^{n+a-1}}{n!\,(a+n)}$$

$$now, (a+n) = \frac{a(a+1)_n}{(a)_n}.$$

Hence

$$y(a,z) = a^{-1} x^{a} \sum_{n=o}^{\infty} \frac{(-1)^n (a)_n z^n}{n!\,(a+1)_n} = a^{-1} x^{-a}\ {}_1F_1(a; a+1; -z).$$

CHAPTER 3

CONFLUENT HYPERGEOMTRIC FUNCTIONWITH KUMMER'S FIRST FORMULA

This section introduces the Kummer's first formula and impact with both hypergeometric and confluent hypergeometric function

We can explain the product $e^z.\ {}_1F_1\binom{a}{c}; z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}) \left(\sum_{k=0}^{\infty} \frac{(a)_k z^k}{(c)_k k!}\right)$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n z^n}{n!} \frac{(a)_k z^k}{(c)_k k!}$$

When we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(\,k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(\,k, n-) \tag{3.1}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} z^{n-k}}{(n-k)!} \frac{(a)_k z^k}{(c)_k k!}$$

and since $(n-k)! = \frac{(-n)^k k!}{(-n)_k}, \quad 0 \le k \le n.$ (3.2)

We may write

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n z^n}{n!} \frac{(-n)_k (a)_k z^k}{(c)_k k!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \cdot {}_2F_1(-n, a; c; 1)$$

But we already know that

$${}_2F_1\ (-n, a; c; 1) = \frac{\Gamma(c)\Gamma(c-a+n)}{\Gamma(c-a)\Gamma(c+n)}\ , \frac{\Gamma(c-a+n)}{\Gamma(c-a)} = (b-a)_n\ , \frac{\Gamma(c+n)}{\Gamma(c)} = (b)_n$$

So that,

$$ {}_2F_1\ (-n, a; c; 1) = \frac{(c-a)_n}{(c)_n} $$

then,

$$ e^{-z}\ {}_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(c-a)_n(-z)^n}{(c)_n n!} = \ {}_1F_1(c-a; c; -z\ ). \qquad (3.3) $$

This is Kummer first formula, c∉ {-1,-2,-3. . . 0}

Now under this definition , we will prove the following theorems,

3.1 Theorem

$e^{-t}F(-k, a+n; a; 1) = \ {}_1F_1(-n; a; -t)$, where k,n are non-negative integer

Proof

$$ e^{-t}F(-k, a+n; a; 1) = e^{-t} \sum_{s=0}^{k} \frac{(-k)_s(a+n)_s}{(a)_s s!} $$

We know that $(a+n)_s = \frac{(a)_{n+s}}{(a)_n}$ $(pochhmmer\ property)$

So

$$ e^{-t}F(-k, a+n; a; 1) = e^{-t} \sum_{s=0}^{k} \frac{(-k)_s(a)_{n+s}}{s!\,(a)_s(a)_n} $$

$$ = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{(-k)_s(a)_{n+s}}{s!\,(a)_s(a)_n} \frac{(-1)^k t^k}{k!} \ , by\ (3.2) $$

We obtain

$$ = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{(-1)^s k!\ (a)_{n+s}}{(k-s)!\ s!\,(a)_s(a)_n} \frac{(-1)^k t^k}{k!} $$

$$ = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{(-1)^s\ (a)_{n+s}}{s!\,(a)_s(a)_n} \frac{(-1)^k t^k}{(k-s)!} $$

Hence

$$\sum_{k=0}^{\infty}\sum_{s=0}^{k} A(s,k) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} A(s,+s) \tag{3.4}$$

So

$$\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s\,(a)_{n+s}}{s!\,(a)_s(a)_n}\frac{(-1)^{k+s}t^{k+s}}{k!} = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a+n)_s t^s}{s!\,(a)_s}\frac{(-t)^k}{k!}$$

$$= \sum_{s=0}^{\infty} e^{-t}\frac{(a+n)_s t^s}{(a)_s s!}$$

$$= e^{-t}\ {}_1F_1\,(+n;a;t)$$

And since ${}_1\mathrm{F}_1\ (a+n;a;t) = {}_1F_1 e^t(-n;a;-t)$ by Kummer's first formula 3.3,

$$\sum_{s=0}^{\infty} e^{-t}\frac{(a+n)_s}{(a)_s s!} = e^{-t}\,e^t\ {}_1F_1(-n;a;-t) = {}_1F_1(-n;a;-t)$$

$$\because \quad e^{-t}F(-k,a+n;a;1) = {}_1F_1(-n;a;-t)$$

3.2 Theorem

To prove that

$$\frac{d^k}{dz^k}[\,{}_1F_1(a+n,a,-t)] = \frac{(-n)_k}{(a)_k}\ {}_1F_1(a+n,a+k,-t)$$

Proof:

$$\frac{d^k}{dz^k}[\,{}_1F_1(a+n,a,-t)] = \frac{d^k}{dz^k}[e^{-t}\ {}_1F_1(-n,a,t)] \quad by\ (3.3)$$

$$= e^{-t}\left[\frac{d^k}{dz^k}\ {}_1F_1(-n,a,t)\right]$$

$$= e^{-t}\left[\frac{(-n)_k}{(a)_k}\ {}_1F_1(-n+k,a+k,t)\right] \tag{3.5}$$

$$= e^{-t}\left[e^t\,\frac{(-n)_k}{(a)_k}\ {}_1F_1(a+k+n-k,a+k,-t)\right]\ by\ (3.3)$$

$$= \frac{(-n)_k}{(a)_k}\ {}_1F_1(a+n, a+k, -t)$$

Not that theorem (3.5) is equal to zero if k=n

$\frac{d^n}{dz^n}[\,{}_1F_1(a+n, a, -t)] = \frac{d^n}{dz^n}[e^{-t}\ {}_1F_1(-n, a, t)]$ by Kummer first formula

$$= e^{-t}\left[\frac{d^n}{dz^n}\ {}_1F_1(-n, a, t)\right]$$

$$= e^{-t}\left[\frac{(-n)_n}{(a)_n}\ \ {}_1F_1(-n+n, a+n, t)\right] = 0$$

Note

Examination of Kummer's first formula soon arouses interest in the special case when the two (CHF) have the same parameters. This happens when $b - a = a$, $b = 2a.$ we then obtain

$${}_1F_1(a; 2a; z) = e^{z}\ {}_1F_1(a; 2a; -z),$$

or

$$e^{-\frac{z}{2}}\ {}_1F_1(a; 2a; z) = e^{\frac{z}{2}}\ {}_1F_1(a; 2a; -z) \qquad (3.6)$$

More pleasantly, (3.5) may be expressed by saying that the function

$$e^{-z}\ {}_1F_1(a; 2a; 2z)$$

Is an even function of z. After some step in (Rainiville 1967) we get

$$e^{z}\ {}_1F_1(a; 2a; -z) = \ {}_0F_1\left(-; a+\frac{1}{2}; -\frac{1}{4}z^2\right). \qquad (3.7)$$

If $2a$ is not an odd integer < 0

Equation (3.7) is known as Kummer's second formula.

3.3 Some example of Kummer's formula with (CHF)

Problem 3.1

Show that

$$ {}_1F_1(a;b;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \; {}_0F_1(-;b;zt)dt $$

Solution 3.1

We know that

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{a-1} dt, \qquad Re(a) > 0. $$

Then

$$ {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!\,(b)_n} $$

$$ = \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) z^n}{n!\,(b)_n} $$

$$ = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{t^{a+n-1} z^n}{n!\,(b)_n} dt $$

$$ = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \, 0F1(-;b;zt)dt\,, \qquad Re(a) > 0 $$

Problem 3.2

Show that the aid of the result in problem 1.3, that

$$ \int_0^\infty \exp(-t^2)\, t^{2a-n-1} J_n(zt)dt = \frac{\Gamma(a) z^n}{2^{n+1}\Gamma(n+1)} \; {}_1F_1\left(a; n+1; -\frac{z^2}{4}\right). $$

Solution 3.2

we obtain

$$A = \int_0^\infty exp(-t^2)\, t^{2a-n-1} J_n(zt)dt = \int_0^\infty \frac{e^{-t^2} t^{2a-n-1} z^n t^n}{2^n \Gamma(1+n)}\ {}_0F_1\left(-; 1+n; -\frac{z^2t^2}{4}\right) dt$$

Put $t^2 = \beta$. Then

$$A = \frac{z^n}{2^n \Gamma(1+n)} \cdot \frac{1}{2} \int_0^\infty e^{-\beta} \beta^{a-1}\ {}_0F_1\left(-; 1+n; \frac{z^2\beta^2}{4}\right) d\beta$$

$$= \frac{z^n}{2^{n+1}\Gamma(1+n)} \cdot \frac{\Gamma(a)}{1}\ {}_1F_1\left(a; n+1; -\frac{z^2}{4}\right)$$

CHAPTER 4

SEVERAL PROPERTIES OF HYPERGEOMTRIC FUNCTION

This chapter proffers an outline of the several properties of hypergeometric function and a detailed discussion of the results.

4.1 Properties

4.1.1 The Contiguous Function relations.

Gauss defined as contiguous to $F(a,b;c;z)$ each of the six function obtained by increasing or decreasing one of the parameters by unity. For simplicity in printing we use the notation,

$$F = F(a,b;c;z)$$

$$F(a+) = F(a+1,b;c;z) \tag{4.1}$$

$$F(a-) = F(a-1,b;c;z) \tag{4.2}$$

Together with similar notations $F(b+), F(b-), F(c+) and\ F(c-)$ for the other four of the six functions contiguous to F. After some step in (Rainville, 1965) we get this contiguous function relations.

$i)$ $(a-b)F = aF(a+) - bF(b+)$

$ii)$ $(a-c+1)F = aF(a+) - (c-1)F(c-)$

$iii)$ $[a + (b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+)$

$iv)$ $(1-z)F = F(a-) - c^{-1}(c-b)zF(c+)$

$v)$ $(1-z)F = F(b-) - c^{-1}(c-a)zF(c+),$

Example 4.1

From these contiguous functions we can obtain other relations

1) from (iii) and (iv) we get

$$[a + (b - c)z - (c - a)(1 - z)]F = a(1 - z)F(a+) - (c - a)F(a-),$$

In the left hand We get

$$[a + bz - cz - [c - cz - a + az]F = a(1 - z)F(a+) - (c - a)F(a-),$$

So

$$[2a - c + (b - a)z]F = a(1 - z)F(a +) - (c - a)F(a -). \qquad (4.3)$$

2) from (iii) and (vi) we get

$$[a + (b - c)z - (c - b)(1 - z)]F = a(1 - z)F(a+) - (c - b)F(b-)$$

So

$$[a + b - c]F = a(1 - z)F(a +) - (c - b)F(b -). \qquad (4.4)$$

3) from (2) and (3) we get

$$[a + (b - c)z - (a - c + 1)(1 - z)]F$$
$$= (c - 1)(1 - z)F(c -) - c^{-1}(c - a)(c - b)zF(c +)$$

Then

$$[c - 1 + (a + b - 2c + 1)z]F$$
$$= (c - 1)(1 - z)F(c -) - c^{-1}(c - a)(c - b)zF(c +). \qquad (4.5)$$

4) from (1) and (4.1) we get

$$[(a - b)(1 - z) - 2a + c - (b - a)z]F$$
$$= (c - a)F(a-) - b(1 - z)F(b+),$$

Then

$$[c - a - b]F = (c - a)F(a -) - b(1 - z)F(b +). \qquad (4.6)$$

4.1.2 Hypergeometric differential equation:

The operator $\theta = z\left(\frac{d}{dz}\right)$, already used in the chapter two of section (2.1.2.3) we resultantly obtained this equation,

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0 \qquad (4.7)$$

Example 4.2

In the deferential equation (4.7) for $w = F(a,b;c;z)$ introduce a new dependent variable u by $w = (1-z)^{-a}u$, thus obtaining

$$z(1-z)^2u'' + (1-z)[c+(a-b-1)z]u' + a(c-b)u = 0.$$

Next change the independent variable to x by putting $x = \frac{-z}{1-z}$ Show that the equation for u in terms of x is ,

$$x(1-x)\frac{d^2u}{dx^2} + [c-(a+c-b+1)x]\frac{du}{dx} - a(c-b)u = 0, \qquad (4.8)$$

And thus derive the solution

$$w = (1-z)^{-1}F\left[\begin{matrix} a\ ,\ c-b; & \frac{-z}{1-z} \\ c; & \end{matrix}\right]$$

Solution

We know that $w = F(a,b;c;z)$ is a solution of the equation (4.7) in this equation we put $w = (1-z)^{-a}u$ then

$$w' = (1-z)^{-a}u' + a(1-z)^{-a-1}u, \qquad (4.9)$$

$$w'' = (1-z)^{-a}u'' + 2a(1-z)^{-a-1}u' + a(a+1)(1-z)^{-a-2}u. \qquad (4.10)$$

Now we get the new equation from the eq (4.8),(4.9)and (4.7)

$$z(1-z)u'' + 2azu' + a(a+1)z(1-z)^{-1}u + cu' + ca(1-z)^{-1}u - (a+b+1)zu'$$

$$-a(a+b+1)z(1-z)^{-1}u - abu = 0,$$

Then

$$z(1-z)^2 + (1 - z)[c + (a - b - 1)z]u' + a(c - b)u = 0. \qquad (4.11)$$

Now put $x = \frac{-z}{1-z}$ .then $z = \frac{-x}{1-x}$, $x = \frac{-z}{1-z}$, $1-z = \frac{1}{1-x}$ so $\frac{dx}{dz} = \frac{-1}{(1-z)^2} = -(1-x)^2$

, $$\frac{d^2x}{dz^2} = \frac{-2}{(1-z)^3} = -2(1-x)^3$$

The old equation (4.11) above may be written

$$\frac{d^2u}{dz^2} + \left[\frac{c}{z(1-z)} + \frac{a-b-1}{1-z}\right]\frac{du}{dz} + \frac{a(c-b)}{z(1-z)^2}u = 0,$$

Which then leads to the new equation

$$(1-x)^4\frac{d^2u}{dx^2} + \left[-2(1-x)^3 - (1-x)^2\left\{\frac{c(1-x)^2}{-x} + (a-b-1)(1-x)\right\}\right]\frac{du}{dx}$$

$$-\frac{a(c-b)(1-x)^3}{x}u = 0$$

or

$$x(1-x)\frac{d^2u}{dx^2} + [-2x - \{-c(1-x) + (a-b-1)x\}]\frac{du}{dx} - a(c-b)u = 0,$$

Or

$$x(1-x)\frac{d^2u}{dx^2} + [x - (a-b+c+1)x]\frac{du}{dx} - a(c-b)u = 0 \qquad (4.12)$$

Now (4.12) is a hypergeomtric equation with parameters $\gamma = c, \alpha + beta + 1$

$a - b + c + 1, \alpha\beta = a(c - b). Hence\ \alpha = a, \beta = c - b, \gamma = c$. One solution of (4.12) is

$$u = F(a, c-b;\ c;\ x),$$

So one solution of equation (4.7) is

$$w = (1-z)^{-1}F\begin{bmatrix} a\ , & c-b; & \frac{-z}{1-z} \\ & c; & \end{bmatrix}$$

4.1.3 Elementary series manipulation

(Choi, 2003) established some generalized principles of double series manipulations some special cases of which are also written for easy reference in their use. Not that $A_{x,y}$ denotes a function of two variables x and y, and N is the set of positive integers

1) $\sum_{n=0}^{\infty}\sum_{k=o}^{\infty} A_{k;n} = \sum_{n=0}^{\infty}\sum_{k=o}^{n} A_{k;n-k}$;

2) $\sum_{n=0}^{\infty}\sum_{k=o}^{n} A_{k;n} = \sum_{n=0}^{\infty}\sum_{k=o}^{\infty} A_{k;n+k}$;

3) $\sum_{n=0}^{\infty}\sum_{k=o}^{\infty} A_{k;n} = \sum_{n=0}^{\infty}\sum_{k=o}^{\frac{n}{2}} A_{k;n-2k}$

4) $\sum_{n=0}^{\infty}\sum_{k=o}^{\frac{n}{2}} A_{k;n} = \sum_{n=0}^{\infty}\sum_{k=o}^{n} A_{k;n+2k}$

Example 4.2

Prove that if g_n= F $(-n, \alpha;\ 1\ + \alpha - n;\ 1)$ and α is not an integer, then g_n = 0 for n ≥ 1, g_0 = 1.

Solution

Let g_n= F (−n, α; 1 +α−n; 1).

Then

$$g_n = \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{k!\,(1+a-n)_k} = \sum_{k=0}^{n} \frac{n!\,(-a)_k (a)_k}{n!\,(k-1)!\,(a)_n}$$

Hence compute the series

$$\sum_{n=0}^{\infty}\frac{(-a)_n g_n t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=o}^{n}\frac{(a)_k(-a)_{n-k}t^n}{k!\,(n-k)!}$$

$$=\left(\sum_{n=0}^{\infty}\frac{(a)_n t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-a)_n t^n}{n!}\right)$$

$$=(1-t)^a(1-t)^{-a}$$

$$=1$$

Therefore, $g_0 = 1$ and $g_n = 0$ for n ≥ 1. (Note: easiest to choose α≠ integer, can actually do better than that probably).

4.1.5 A quadratic transformation

A quadratic transformation as established by(Rainville, 1965) is based on the following;

4.1.5.1 Theorem

If $2b$ is neither zero nor negative integer and if both $|x| < 1$ and $|4x(1+x)^{-2}| < 1$

$$(1+x)^{-2a}F\begin{bmatrix} a\,,\ b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{bmatrix} = F\begin{bmatrix} a\,,\ a-b+\dfrac{1}{2}; & x^2 \\ b+\dfrac{1}{2}; & \end{bmatrix}.$$

Example 4.5

In this theorem put $b = \propto, a = \propto + \frac{1}{2}$, $4x(1+x)^{-2} = z$ and thus prove that

$$\begin{bmatrix} \propto,\ \propto+\dfrac{1}{2}; & z \\ 2\propto; & \end{bmatrix} = (1-z)^{\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\propto-1}$$

Solution

Theorem 2 gives us

$$(1+x)^{-2a} F\left[\begin{matrix} a\,,\ b; & \dfrac{4x}{(1+x)^2} \\ ab; & \end{matrix}\right] = F\left[\begin{matrix} a\,,\ a-b+\dfrac{1}{2}; & \\ b+\dfrac{1}{2}; & x^2 \end{matrix}\right].$$

Then put $b = \propto, a = \propto + \frac{1}{2}$, $4x(1+x)^{-2} = z$

then

$$zx^2 + 2(z-2)x + z = 0$$

$$zx = 2 - 3 \pm \sqrt{z^2 - 4z + 4 - 3^2} = 2 - z \pm 2\sqrt{1-z}$$

Now $x = 0$ when $z = 0$, so

$$zx = 2 - z - 2\sqrt{1 - z = 1 - z + 1 - 2\sqrt{1-z}}$$

Therefore

$$x = \frac{(1-\sqrt{1-z}\,)^2}{z} = \frac{\left(1-\sqrt{1-z}\right)[1-(1-z)]}{z\left(1+\sqrt{1-z}\right)}.$$

Thus

$$x = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$$

And

$$x + 1 = \frac{2}{\left(1+\sqrt{1-z}\right)}$$

Then we obtain

$$\frac{4x}{(1+x)^2} = \frac{\left(1-\sqrt{1-z}\right)}{\left(1+\sqrt{1-z}\right)} \cdot \frac{\left(1-\sqrt{1-z}\,\right)^2}{4} = z$$

a check. Now with $b = \propto, a = \propto + 1$ theorem 4 yields

$$\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\propto-1} F\left[\begin{matrix}\propto+\frac{1}{2},\propto; \\ 2\propto;\end{matrix}\quad z\right]=\left[\begin{matrix}\propto+\frac{1}{2},1; \\ \propto+\frac{1}{2};\end{matrix}\quad x^2\right]=F\left[\begin{matrix}1; \\ -;\end{matrix}\quad x^2\right]=(1-x^2)^{-1}\cdot$$

Since

$$1-x=\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\ and\ 1+x=\frac{2}{1+\sqrt{1-z}}\cdot$$

$$(1-x^2)=\frac{4\sqrt{1-z}}{\left(1+\sqrt{1-z}\right)^2}$$

Thus we have

$$F\left[\begin{matrix}\propto,\ \propto+\frac{1}{2}; \\ 2\propto;\end{matrix}\quad z\right]=\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\propto+1}\cdot\left[\frac{2}{1+\sqrt{1-z}}\right]^{-2}(1-z)^{-\frac{1}{2}}$$

$$=(1-z)^{\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\propto-1},$$

as defined. Now we use theorem 3 to see that

$$F\left[\begin{matrix}\propto,\ \propto+\frac{1}{2}; \\ 2\propto;\end{matrix}\quad z\right]=(1-z)^{-\frac{1}{2}}F\left[\begin{matrix}\propto,\ \propto-\frac{1}{2}; \\ 2\propto;\end{matrix}\quad z\right]$$

So that we also get

$$F\left[\begin{matrix}\propto,\ \propto-\frac{1}{2}; \\ 2\propto;\end{matrix}\quad z\right]=\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\propto-1},$$

as desired.

4.1.6 Additional properties

We will obtain one more identity as an example of those resulting from combination of the theorem proved earlier in this chapter. In the Identity of theorem 3, replace $a\ by\ (\frac{1}{2}c - \frac{1}{2}a)$and $b\ by\ (\frac{1}{2}c + \frac{1}{2}a - \frac{1}{2})$ to get

$$F\left[\begin{matrix}\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{1}{2}; \\ c;\end{matrix} 4x(1-x)\right] = F\left[\begin{matrix}c - a, c + a - 1; \\ c;\end{matrix} x\right].$$

Theorem 1 yields

$$F\left[\begin{matrix}c - a, c + a - 1; \\ c;\end{matrix} x\right] = (1-x)^{1-c}\, F\left[\begin{matrix}a, 1 - a; \\ c;\end{matrix} x\right],$$

Which leads to the desired result.

4.1.6.1 Theorem

If c is nether zero nor negative integer and if both $|x| < 1$ and $|4x(1-x)| < 1$

$$F\left[\begin{matrix}a, 1 - a; \\ c;\end{matrix} x\right] = (1-x)^{1-c} F\left[\begin{matrix}\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c + \frac{1}{2}a - \frac{1}{2}; \\ c;\end{matrix} 4x(1-x)\right].$$

Example

Use this theorem to show that

$$(1-x)^{1-c} F\left[\begin{matrix}a, 1 - a; \\ c;\end{matrix} x\right] = (1-2x)^{a-c} F\left[\begin{matrix}\frac{1}{2}c - \frac{1}{2}a, \frac{1}{2}c - \frac{1}{2}a + \frac{1}{2}; \\ c;\end{matrix} \frac{4x(1-x)}{(1-2x)^2}\right]$$

Solution

$$(1-x)^{1-c} F\left[\begin{matrix}a, 1 - a; \\ c;\end{matrix} x\right] = F\left[\begin{matrix}\frac{c-a}{2}, \frac{c+a-1}{2}; \\ c;\end{matrix} 4x(1-x)\right]$$

$$= F\left[\begin{matrix}\frac{c-a}{2}, \frac{c+a-1}{2}; \\ c;\end{matrix}\; 1-(1-2x)^2\right]$$

$$= (1-2x)^{2\frac{a-c}{2}} F\left[\begin{matrix}\frac{c-a}{2}, c-\frac{c-a+1}{2}; \\ c;\end{matrix}\; \frac{1-(1-2x)^2}{(1-2x)^2}\right]$$

$$= (1-2x)^{a-c} F\left[\begin{matrix}\frac{c-a}{2}, \frac{c-a+1}{2}; \\ c;\end{matrix}\; \frac{4x(1-x)}{(1-2x)^2}\right].$$

4.2 Some theorem without proof

4.2.1 Theorem

If $|z| < 1$,

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

4.2.2 Theorem

If 2b is nether zero nor a negative integer and if $|y| < \frac{1}{2}$ and $\left|\frac{y}{1-y}\right| < 1$,

$$(1-y)^{-a} F\left[\begin{matrix}\frac{a}{2}, \frac{a+1}{2}; \\ c;\end{matrix}\; \frac{y^2}{(1-y)^2}\right] = F\left[\begin{matrix}a,\ b; \\ c;\end{matrix}\; 2y\right]$$

4.2.3 Theorem

If $a + b + \frac{1}{2}$ is neither zero nor a negative integer and if both $|x| < 1$ and

$|4x(1-x)| < 1$

$$F\left[\begin{matrix}a, b; \\ a+b+\frac{1}{2};\end{matrix}\; 4x(1-x)\right] = F\left[\begin{matrix}2a,\ 2b; \\ a+b+\frac{1}{2};\end{matrix}\; x\right].$$

CHAPTER 5

CONCLUSION AND SUGGESTIONS FOR FUTURE

This study had presented definitions and examples of hypergeometric function; confluent hypergeometric function, and Kummer confluent hypergeometric function. It can therefore be concluded that theorems and some properties. Moreover, it can also be concluded that the Kummer function has wide application in various subjects and hence proving stability or other properties were drawn to be of paramount importance. This study centered on the Kummer's first formula with confluent hypergeometric function. Future studies can endeavor to extend insights on this area in depth.

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. İlkay SALİHOĞLU

We certify that, this thesis is satisfactory for the award of the degree of Master of Sciences in Mathematics.

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