**STUDY ON CLASS OF IMPROVED Q-BERNOULLI MATRIX AND ITS PROPERTIES**

**IBRAHIM YUSUF KAKANGI, A STUDYON A CLASSES OF Q-BERNOULLI MATRIX AND ITS PROPERTIES, NEU, 2017**

**A THESIS SUBMITTED TO THE GRADUATE**

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**OF**

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**By**

**IBRAHIM YUSUF KAKANGI**

**In Partial Fulfilment of the Requirements for**

**the Degree of Master of Science**

**in**

**Mathematics**

**NICOSIA, 2017**

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**Approval of Director of Graduate School of**

**Applied Sciences**

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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# 

**To Dr. Ramalan Yero ….**

# ABSTRACT

Since 19th century, a lot of q-Bernoulli numbers and polynomials has been introduced. Carlitz was the first who made a generation of q-Bernoulli numbers, afterwards, a lot of researcher’s works on a new form of q-Bernoulli numbers and matrices. In this thesis, we introduce ordinary Bernoulli and q-Bernoulli matrices and their related Pascal matrices and their relations. At the end by using generating function and improved q-exponential function we work on a new class of q-Bernoulli matrix and related properties are given. Our definition is more significant since it demonstrates a better definition of q-Bernoulli matrix and the properties are convinced the ordinary case as well.

**KEYWORDS**: Bernoulli Number; Bernoulli Matrices; q-Bernoulli Number; q-Bernoulli Matrices; Improved q-Bernoulli number; and Improved q-Bernoulli Matrices

# ÖZET

19. yüzyıldan beri, bir sürü q-Bernoulli sayısı ve polinomları tanıtıldı. Carlitz, daha sonra, q-Bernoulli sayılarının ve matrislerinin yeni bir formuyla ilgili birçok araştırmacı tarafından q-Bernoulli sayıları üreten ilk kişiydi. Bu tezde sıradan Bernoulli ve q-Bernoulli matrislerini ve ilgili Pascal matrislerini ve bunların ilişkilerini tanıtmaktayız. Sonunda üretme fonksiyonu ve geliştirilmiş q-üstel fonksiyonu kullanılarak q-Bernoulli matrisinin yeni bir sınıfında çalıştık ve ilgili özellikler verildi. Tanımımız, q-Bernoulli matrisinin daha iyi tanımlanmasını gösterdiği için daha belirgindir ve özellikler olağan durumu ikna eder.

**ANAHTAR KELİMELER:** Bernoulli sayısı; Bernoulli matrisleri; q-Bernoulli sayısı; q- Bernoulli matrisleri; Geliştirilmiş q-Bernoulli sayısı; ve Geliştirilmiş q-Bernoulli Matrisleri

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# CHAPTER 1

# INTRODUCTION

Majority of the work in this chapter was presented from The book ‘ A comprehensive treatment of q-Calculus’ (Ernst.T, 2012) , ‘Quantum Calculus’ (Kac.V, and Cheung.P,2002) and any other information that is not from there was cited by means of reference.

Among the most important sequence in mathematics is the sequence of Bernoulli numbers it has a quiet good relationship to the number theories, for example you can express the value of 𝜁(2𝑛) using the Bernoulli number, where 𝑛 is a positive integer and 𝜁(𝑛) is a Riemann zeta function [1] you can also find the uses of Bernoulli number in analysis, for instance, they also find it in the Euler-Maclaurins formula, the formula that is very useful in physics and mathematics, in asymptotic of a q-special functions, the Bernoulli numbers is very essential. Bernoulli matrix, Pascal matrix are some example of matrix with binomial coefficients as their element which are very important in matrix theory and combinatory. So many researchers has been showing interest in this related area, for that reason we also want to relate this kind of matrix with quantum calculus (q-calculus), but just before then here are some terminologies that one needs to know about q-calculus.

## 1.1 Quantum Calculus

If

exists, that gives us the well known definition of the derivative of a function f(x) at a point . However if we assume that or , where q is a fixed constant not equal to 1 and h is a fixed constant not equal to 0, and do not take the limit, we enter into different concept of mathematics called the quantum calculus.

Quantum calculus involves two types of derivative which are; q-derivative and h-derivative that leads to the study of q-calculus and h-calculus respectively. In the course of studying quantum calculus in relation to the ordinary calculus, so many important results and notions in number theory, combinatory and different area of mathematics have been discovered.

For instance, a q-derivative of , where

and represent the ordinary 'n' in the ordinary derivative of .

### 1.1.1 Definition q and h-Differentiation

A q-differential and h-differential of an arbitrary function say on the set of real numbers are defined as (Kac.V, and Cheung.P,2002)

And

Respectively.

Particularly keeping in mind that, also that .

Considering the q-differential and the h-differential we defined their corresponding quantum derivative as

### 1.1.2 Definition q and h-Derivative

Supposed is an arbitrary function on the set of real numbers . Its q-derivative and h-derivative are defined as

where,

with

referred to as the q-derivative and h-derivative respectively, of the arbitrary function

If we noticed that

provided that the function is differentiable. Looking at notation of Leibniz which has to do with the ratio of two ‘infinitesimals’ is somewhat difficult to understand, because there is need to give further detail of the notion of the differential . But on the other hand, one can easily see on the notion of q-calculus and h- calculus also the q-derivative and the h-derivative are plain ratios.

### 1.1.3 Lemma Linearity of q and h-Derivative

Just like in the concept of ordinary derivative, the linear operator behaves in the same way while finding the q-derivative or h-derivative of a function. In essence, if and are q-derivative and h-derivative, then for any constants a and b, the following property hold:

**Example:** if , and n is an integer greater than zero, then the q-derivative and h-derivative can be find as

but there is a frequent appearance of in the q-derivative so therefore we used the notation

And it is referred to as the q-analogue of n, for any integer n greater than zero, and hence (1.9) becomes

## 1.2 q-Taylor’s Formula For Polynomial

Before going to q-Taylor formula, lets recall the generalized Taylor formula in the ordinary calculus.

Taylor theorem says

is the power series of any function which has derivative of all kind of order is analytic at , provided we can write it as a power series about a point .

We can increase the definition of a function to a more interesting domain by Taylor expansion of an analytic function. For instance, if we defined the exponentials as a square matrices and a complex number by using the Taylor expansion of , with which then we express the q-analogue of the following expression where the q-Taylor formula follows

### 1.2.1 q-Analogue of Some q-Combinatory

**Definition 1.2.1** (Kac.V, and Cheung.P,2002) If is a positive integer. we defined the q-analogue of as:

**Definition 1.2.2** The q- binomial coefficients of any integer , is defined as

and (1.13) satisfies also

for (Naim and KUS, 2015)

and

the q-analogue of binomial function is defined as:

**Definition 1.2.3** (Kac.V, and Cheung.P,2002)

For , -analogue of is defined as:

**Definition 1.2.4** (Kac.V, and Cheung.P,2002)

Let the -analogue of is defined as:

**Definition 1.2.5** (Kac.V, and Cheung.P,2002)

Generalize q-polynomial function is defined as

Where is a polynomial.

### 1.2.2 Some Properties of q-Calculus Functions

**Proposition 1.2.1** (Kac.V, and Cheung.P,2002)

The following properties hold for any integer

1. .

By using the above definitions and proposition we eventually come up with the **q-Taylor binomial formula for polynomial** (Kac.V, and Cheung.P,2002) as

## 1.3 q-Exponential Function

Before we study the Euler identity and the q-exponential function, there is need to understand the concept of Gauss’s binomial formula and Hein’s binomial formula which were both derived from the q-Taylor binomial formula, in this case we assumed that .

With as a variable and using (1.17) of definition 1.2.2 we obtain the Gauss binomial formula

### 1.3.1 Gauss binomial formula

Where,

Is the q-binomial coefficient

And the Heine’s binomial formula

### 1.3.2 Heine’s Binomial Formula

(Kac.V, and Cheung.P,2002)

Now considering (1.22) by replacing and by 1 and respectively i.e

and (1.22)

What will happen if we take the limit of n as in both the expression? Depending on the value of , the result is infinitely small or infinitely large so therefore producing not interesting result in the ordinary calculus i.e when But in q-calculus it is entirely different because, an example is, assuming the infinite product

will eventually converge to some finite limit. Therefore if we let we have

and

So therefore there is difference in the behaviour between the q-analogues of integer and binomial coefficients for a larger integer to their ordinary counterparts.

Taking the limits as and substituting (1.24) and (1.25) in the Heine’s and Gauss’s binomial formula we develop two identities of formal power series in (with the assumption that ). (Kac.V, and Cheung.P,2002)

### 1.4.3 q-Euler Identities

and call (1.26) and (1.27) Euler’s first and second identities or E1 and E2 respectively (Kac.V, and Cheung.P,2002) because he was the one that reveals them at the time of his live before Gauss’s and Heine. Also the identities relate infinite product and infinite sums but they don’t have classical analogue because each and every term in the sum don’t have meaning when .

### 1.3.4 q-Exponential Functions

Studying those identities helps us to define the q-analogue of the exponential function, but before then, lets recall the Taylors’s exponential function expansion. i.e

From dividing both the numerator and the denominator of the R.H.S by we got

**Definition**: (Kac.V, and Cheung.P,2002) The classical exponential function has a q-analogue as

By using and we get

or its equivalent

That is the case of E2, we can also use E1 to defined another q-exponential function.

**Definition** (Kac.V, and Cheung.P,2002)

We can relate (1.31) and (1.32) as

### 1.3.5 Relationship Between and

From the above property we can say

### 1.3.6 q-Derivative Of The q-Exponential Functions

And the q-derivative of the two q-exponential function is given as

### 1.3.7 Convergence Of q-Exponential Functions

The series of non-negative terms in q-calculus converges if a bounded sequence is formed by its partial sums, so for two classical q- exponential functions we can find interval of convergence as follows

Let

Then by using De-Alembert theory

using

Hence converges and the interval of convergence is

Similarly we can prove the other q-exponential function as

Let

By using De-Alembert we see that

## 1.4 q-Trigonometric Functions

By using the well-known Euler formula in terms of exponential function, we can define the q- analogues of the two trigonometric functions.

**Proposition 1.3.1** (Kac.V, and Cheung.P,2002) The sine and cosine q-analogue function are given by

### 1.4.1 Properties Of q-Trigonometric Functions

We can see from (1.37) and (1.38) that

### 1.4.2 q-Derivative Of q-Trigonometric Functions

The q-derivative of the q-trigonometric function is given by

And (1.39), (1.40) and (1.41) are being proved by proposition 1.3.1

## 1.5 Improved q-Exponential Function

There are two exponential functions that are define by Euler in the previous section, both there are some properties that are lost, for example

Which allows us to defined the improved q-exponential function as

### 1.5.1 Definition

Let be new q-exponential function, and defined as (Jan L. & Cieśliński, 2011)

Where are the standard q-exponential functions. Classical Cayley transformation motivated the above definition. (the infinite product representation is valid for .

### 1.5.2 Basic Definitions on Improved q-Exponential Function

**Definition 1.5.2:** If is any real or complex number, then we defined the following terms as

Therefore

**Definition 1.5.2:** Bernoulli number can be demonstrated in term of improved q-Bernoulli number by the following recurrence relation:

Where is the Bernoulli number.

* **Definition 1.5.4:** (Wikipedia) If and are real or complex parameter, then the summation by Newton expansion in an ordinary case as

In the same manner, the following q-addition of the expression is define as (Zhang.Z and JunWang, 2006)

### 1.5.3 Unification Of q-Exponential Functions

The following statement holds true

**Proof**

…

As required.

### 1.5.4 Improved q-Trigonometric Functions

We can use the natural way to define the new q-sine and q-cosine functions as

## 1.6 Bernoulli Numbers

In this work Bernoulli numbers will be defined by the exponential generating function

We see that the first Bernoulli number is easy to find, i.e.

### 1.6.1 Recurrence Formula for Ordinary Bernoulli Numbers

Continuing in this way we will use the tool that we have i.e the ordinary exponential function in order to derive the recurrence formula for Bernoulli numbers

By using the Cauchy product of two series (Rudin, 1964) i.e

Given the two series and we write

Then is said to be the multiplication of the two series.

Going back to our work we see that

we obtain

By comparing the power of we have

which is the ***recurrence formula*** for Bernoulli numbers.

### 1.6.2 Kronecker Delta

**Proposition 1.6.1** (Riordan, 1968) If is a Bernoulli number number then,

Where is called Kronecker delta

**Proof**

Prove by Cauchy product on generating function.

Since Kronecker delta is defined as

Then we can write (1.6.2) as

Since

when we assume that , i.e

we will then have

by opening the summation we have

Why? Because

Therefore we have

Which is the same as proposition 1.6.1

### 1.6.3 Lemma Explicit Definition of Bernoulli Number (Arakaya.T & et.al, 2014)

Bernoulli number satisfy the recurrence

### 1.6.4 Proposition Bernoulli Numbers as Rational numbers (Arakaya.T & et.al, 2014)

The Bernoulli numbers are rational numbers.

Solving for the first seven of the Bernoulli numbers using the above recurrence

…

### 1.6.5 Bernoulli Polynomials

When we multiply the left hand side of (1.51) with and rise it to the power of some arbitrary constant say a real or complex parameter.

It is called the generating function for Bernoulli polynomial.

The generalized Bernoulli polynomials are given as

Then by using the above expression we obtain the few Bernoulli polynomials:

### 1.6.6 Some Properties Of Bernoulli Polynomials

**Proposition1.6.2** (Zhang.Z and JunWang, 2006) for all integers greater than or equal to one, then the following holds

**Proof** (Kac.V and Cheung.P, 2002)

Let

It is quiet simple to see that (a) . (b)

Since these characters uniquely characterize you see that the (a) is so simple to find out because

But for (b) and using the fact that if

Differentiating both side and for , we see that

But

Multiplying and dividing by we get

as required.

Now from the Bernoulli polynomial generating function we deduced the following proposition:

**Proposition (1.6.3)** (Zhang.Z and JunWang, 2006) If and are two real or complex parameters then we say that

**Proof**

Here we see that there are two polynomial with variables of with index multiplying themselves.

It follows from (1.54)

Therefore we have

by comparing the coefficient of ,

we have

as required.

But

when we interchange x and y in the addition (1.54) and put the equation yields

As a special case by putting we have

# CHAPTER 2

# BERNOULLI MATRIX AND SOME PROPERTIES

Majority of the work in this chapter was presented from a Journal Bernoulli matrix and its algebraic properties (Zhizheng & JunWang, 2006) and any other information was cited by means of reference.

## 2.1 Bernoulli Matrix

Before we talk about the Bernoulli matrix, let recall the definition of the algebra matrix that says:

### 2.1.1 Definition Ordinary Matrix and some Properties

The rectangular arrangement of numbers is what we referred to as Matrix. A matrix that has number of rows and number of columns is said to be of size and can be displayed as:

,

where, the entries are real numbers and they can also be complex in some other kind of matrix.

Matrix has some properties as follows:

Supposes a matrix and a matrix are matrix and is a scalar with another matrix , then,

* Matrix addition is defined as:
* Matrix subtraction is defined as:
* Scalar multiplication is defined as:
* Multiplication of matrix is defined as: ,

a.nd so many other properties. Now lets go back to the Bernoulli matrix.

### 2.1.2 Definition Bernoulli Matrix and Bernoulli Polynomials

Supposed is an Bernoulli number and is a Bernoulli polynomial and , then the generalized Bernoulli matrix and Bernoulli polynomial matrix . are defined respectively as (Zhang.Z and JunWang, 2006)

is the Bernoulli polynomial matrix, and is the Bernoulli matrix

**Example:** the Bernoulli polynomial is given by



### 2.1.3 Theorem Bernoulli Polynomial Matrix of x and y

The following relation for generalized Bernoulli matrix holds true

**Proof** (Zhang.Z and JunWang, 2006)

In general case we need to show that the sum of the two Bernoulli matrices with the index from the L.H.S is equal to their product at the R.H.S.

Now, the Bernoulli matrix of the L.H.S has the element of the form

But on the R.H.S since it has to do with the multiplication of two matrices, therefore the number of the row of the first Bernoulli matrix has to be the same as the number of column of the other Bernoulli matrix. So that the elements are

Now multiplying those two matrices we see that

Opening that summation we get

On the index over there up to the point that we are going to reach index the result is not equal to zero but after that where the index become greater than index it becomes zero. These are the zeros of index, what about the zeros of index? When ever the result will be zero up to the point where ,

now assume that , we have

All the terms with the becomes zero leaving first and the last term with

Now, it follows from (1.6.6), by simplifying the combinatorial

Which implies (2.3)

**2.1.4 Corollary**

**Proof**  Using mathematical induction

And taking

we see that for

we assume that it is true for ,

For gives

Therefore the right hand side becomes

which satisfies theorem (2.1).

We can further assume that or then the Bernoulli matrix will have a simple powers.

**2.1.5 Corollary** (Zhang.Z and JunWang, 2006)

If

and specially,

**Proof:**

By applying mathematical induction on in all the above expression we see that it is true.

### 2.1.4 Definition Inverse Of Bernoulli Matrix

Let be matrix which is defined as (Zhang.Z and JunWang, 2006)

**Theorem 2.1.4:** Inverse of Bernoulli matrix can be defined by the previous definition of . That means

Also,

**Proof**

We need to show that is the inverse of .

Now if we take their matrix we have

which is going to be a of delta kronecker matrix

expanding the summation we have

Let

Treating the two matrices in terms of their component

Satisfying proposition (1.49)

The result becomes i.e., Looking at this result and corollary 2.1.3 we noticed that

as required.(Zhang.Z and JunWang, 2006)

## 2.2 Bernoulli Matrix and Generalized Pascal Matrix

### 2.2.1 Definition Pascal Matrix

Supposed is an unknown variable and is an integer that is not equal to zero, then the generalized Pascal matrix is denoted as and defined as (Zhang.Z and JunWang, 2006)

### 2.2.2 Theorem Relationship Between Bernoulli Polynomial Matrix And Pascal Matrix

For a Bernoulli matrix and Pascal matrix of a non zero real number

Specially (Zhang.Z and JunWang, 2006)

**Proof.**

The matrix on the R.H.S in and component can be written as

and L.H.S is

multiplying the two matrices on the L.H.S

expanding the summation

assume

By putting in terms of their respective matrix

All the terms with component disappear leaving the ones with

Since for we have

Which give

In the same manner we can obtain the other part of (2.8). i.e.

**Proof.**

The matrix on the R.H.S in and component can be written as

and L.H.S is

multiplying the two matrices on the L.H.S

expanding the summation

assume

by putting in terms of their respective matrix

all the terms with component disappear leaving the ones with

since for we have

by considering the terms where we see that by using (1.6.8),

which give In the same manner, we can obtain

**Example**





.

### 2.2.3 Theorem Inverse Of Bernoulli Polynomial Matrix and Pascal Matrix

(Zhang.Z and JunWang, 2006)

**Proof:**

Lets try and see how it operate before getting into the details of the proof

Pascal matrix is given as

i.e

P[x]= 

and its inverse as

I.e

= 

multiplying them we have

P[x]P[-x] = 



There are three cases involved

first

assume

Second

assume

The last case,

assume that

changing the boundary of summation we get

Now from Newton expansion formula of binomial that says

If or then

the final expression becomes

According to Theorem 2.2.1, we have

if , we have

According to the discussion that we have above that

multiplying from the left side by

now multiplying by

Since

**Example**





.

# CHAPTER 3

# Q-BERNOULLI MATRIX AND ITS PROPERTIES

Majority of the work in this Chapter was presented from a journal ‘Q-Bernoulli Matrices and Their Some Properties’ (Naim and KUS, 2015) and any other information was cited by means of reference.

## 3.1 Q-Bernoulli Matrix

Having studying the basic concept of q-calculus in Chapter 1 and Bernoulli matrix with some of its properties in Chapter 2 of this theses, we now continue to see how the q-Bernoulli matrix is being defined by using the q-Bernoulli polynomials and then some other properties of it,

### 3.1.1 Definition q-Bernoulli numbers

For integer and the Bernoulli numbers . The q-Bernoulli numbers are defined as (Naim and KUS, 2015)

By using the above definition we see that the first six q-Bernoulli numbers are:

### 3.1.2 Definition q-Bernoulli Polynomials

The q-Bernoulli polynomials as is defined as (Naim and KUS, 2015)

The first six q-Bernoulli polynomials also can be seen to be:

**Theorem 3.1.3** (Naim and KUS, 2015) When we apply the commutativity property on and such that , then

with the same approached on the theorem we get

### 3.1.3 Definition q-Bernoulli Matrix

If is an q-Bernoulli number, then the q-Bernoulli matrix is defined as (Naim and KUS, 2015)

where

For example



### 3.1.4 Definition q-Bernoulli Polynomials Matrix

Suppose is a -Bernoulli polynomial. The -Bernoulli polynomial matrix as (Naim and KUS, 2015)

### 3.1.5 Theorem Inverse of q-Bernoulli Matrix

Let be matrix that is defined as (Naim and KUS, 2015)

Then is called the inverse of the -Bernoulli matrix.

**Proof** (Naim and KUS, 2015) If is the -Bernoulli matrix, and using the definition of above, then

to prove this there is need to show that the multiplication is equal to q-kronecker delta.

Now

Putting them in their matrix form

expanding the summation we have

assume that

All the component with becomes zero leaving the terms with

Then by considering the terms which are not equal to zero and one

Let

Shifting the summation to start from to

and multiplying by

By using the orthogonality relation for Bernoulli numbers i.e Proposition 1.6.1

gives

And (3.8) is of Kronecker delta.

## 

## 3.2 -Bernoulli Matrix and -Pascal Matrices

### 3.2.1 Definition Pascal Matrix and Inverse of Pascal Matrix

The generalized -Pascal matrix is defined as (Naim and KUS, 2015)

and the inverse of the generalized -Pascal matrix as

Now, the factorization of -Pascal matrix can be generalized by the following theorem

### 3.2.2 Theorem Relationship Between q-Bernoulli Polynomial Matrix and Pascal Matrix

(Naim and KUS, 2015) Supposed the -Bernoulli polynomial matrix and the generalized -Pascal matrix , then

the interchanging occur as a result of commutative property

And specially

**Proof**  (Naim and KUS, 2015) Consider as the -Bernoulli polynomial matrix and as generalized -Pascal matrix. Then we see that

Expanding

Supposed

by expressing them in their matrix form:

and

now

let

shifting the summation to start from

comparing it with , we have the equivalent

Similarly we can obtain the second part of (3.2.3). i.e.

**Proof**  (Naim and KUS, 2015)

Opening the summation

Supposed

Expressing them in their matrix form:

and

Now

Let

Shifting the summation to start from

Comparing it with , we have the equivalent

and also poof (3.12) in similar way with (3.11). i.e.

**Proof.** (Naim and KUS, 2015)

expanding the summation we have

assume that

writing them in their matrix form

all the terms with the combination where will tends to zero leaving

then the other part will be

Shifting the summation to start from

**Example**

33 -Bernoulli polynomial matrix

= 







### 3.2.3 Definition Inverse of q-Bernoulli Polynomials Matrix

(Naim and KUS, 2015) If is a -Bernoulli polynomial matrix, then , where

## 3.2.4 Corollary

(Naim and KUS, 2015) Let be the generalized -Pascal matrix and be -Bernoulli matrix. Then we can use the factorization of in (3.12)

With the inverse of generalized -Pascal matrix (3.10), and considering the R.H.S

assume that

but for

considering the other part

Let

Shifting the summation to start at

# CHAPTER 4

# IMPROVED Q-BERNOULLI MATRIX AND ITS PROPERTIES

In this chapter, together with the knowledge that we obtained in the previous Books and Journals, we used some properties in other related materials as they are been cited by means of reference and at the end we developed another form of Bernoulli matrix called ‘ **THE IMPROVED BERNOULLI MATRIX**’ where the Bernoulli polynomials are generated with the improved q-exponential function. But before then lets us give a brief history of q-Bernoulli numbers.

## 4.1 History of q-Bernoulli Numbers

Carlitz was the first person to studied the q-analogue of Bernoulli numbers together with Bernoulli polynomials in the middle of last century where he introduced a new sequence as , and relationship between Bernoulli polynomials and Euler polynomials are been proved in (H. M. Srivastava & Pint´er, 2004) . and they also presented the generalized polynomials. Properties of Genocchi polynomials and Euler polynomials are been investigated by kim et al. in (T., 2006)- (Kim, 2007), some recurrence relationship are also given there, the q-extension of Genocchi numbers are presented in different manner in (Cenkci & et.al, q-extensions of Genocchi numbers, 2006) by Cenkci et al. The new concept of the q-Genocchi number and polynomials are presented by Kim in (Kim, 2007). In (Cenkci & et.al, q-extensions of Genocchi numbers, 2006), The q-Genocchi zeta function and function through the use of generating functions and Mellin transformation are been discuss by Simsek et al. in (Simsek & et.al, 2008), There so many recent interesting research on this related area by so many authors as in: Kurt V. (Kurt V. , 2014), Gabuarry and Kurt B. (Gabaury & Kur, 2012) , Kurt in (Kurt & et.al, 2013), Srivastava in (Srivastava & et.al, 2004), (Srivastava & Vignat, 2012), Choi in (Choi & et.al, 2008), Nalci and Pashaev in (Nalci & Pashaev, 2012), Luo in (Luo, 2010), and Srivastava in (Srivastava & Luo, 2006), (Srivastava & Luo, 2011), (Cenkci & et.al, 2008), and Cheon in (Cheon, 2003).

### 4.1.1 Definition Carlitz q-Bernoulli Number

We first present here the initial recurrence q-Bernoulli number by Carlitz as:

From the Bernoulli generating function. i.e

There are a lot definition of the quantum form of Bernoulli number and the Bernoulli polynomials, we find their differences according to their application, for example we can defined it by generating function, so because we have a several types of quantum exponential function, so we have the several types of the q-Bernoulli numbers as well, we can also defined it arbitrary like its been done in the previous chapter.

## 4.2 Improved q-Bernoulli Numbers

In this chapter, the classical definition of quantum calculus concept will be used, by recalling (1.41) of Definition 1.4.2 we can have the following lemma.

### 4.2.1 Lemma Recurrence Formula For Improved q-Bernoulli Number

We can equivalently define by means of the generating function as:

**Proof**

Let

Cross and multiply

By using the Cauchy product of series we have

By comparing the power of , we get

Which is the recurrence formula for the improved q-Bernoulli number as required.

By using the expression in (4.2) we can have the first few improved q-Bernoulli numbers as:

### 4.2.3 Lemma Advantage Of Improved q-Exponential Function

All the coefficient of the improved quantum Bernoulli numbers are zero except the initial one. i.e

**Proof**

Let

Subtracting the first term in the above expression

But

We assume that is an even function, i.e

Now,

From the definition of implies that

Therefore,

Multiplying by , we have

As required.

And the previous lemma is one of the advantages of the improved q-exponential function over the ordinary q-exponential function.

### 4.2.4 Improved q-Bernoulli Polynomials

We can also use the means of generating function to defined the improved q-Bernoulli polynomials as

Where is a real or complex parameter.

We can observed that, and are the classical improved Bernoulli polynomials and classical Bernoulli numbers, respectively.

The Bernoulli polynomials can also be defined with respect to and as

### 4.2.5 Theorem Additive Theory

supposed , then

**Proof**

By using (4.4)

Comparing the power we have

As required. For the second equation we must use 4.3.2, then it follows

By using Cauchy product we lead to

Thus the last equation holds true.

**4.2.6** **Theorem** for a real or complex parameter , the following holds is true

**Proof**

Considering the right hand side

Multiplying them we obtain

By using the Cauchy product of series formula

But the left hand side is

By comparing the power of from the both side

As required base on the consequence of lemma 4.3 which is equivalent to (1.55)

**lemma 4.2.7** The improved Bernoulli polynomials can also be demonstrated as

**Proof.** Put at additive theorem 4.3.1, then we lead to this equality.

By using the above expression we obtain the first few Bernoulli polynomials as

## 4.3 Improved q-Bernoulli Matrix And Its Properties

### 4.3.1 Definition Improved q-Bernoulli Matrix and Improved q-Bernoulli Polynomials Matrix

The generalized q-improved Bernoulli polynomial matrix as

is defined by the following formular

And are called the improved q-Bernoulli polynomial matrix and improved Bernoulli matrix respectively.

Example:

 And 

Are the example of improved q-Bernoulli polynomial and Bernoulli matrix respectively. When we tend q to 1 from the left side, we reach to the form of ordinary Bernoulli matrix.

### 4.3.2 Theorem Improved q-Bernoulli Polynomial Matrix in terms of x and y

The following relation for improved q-Bernoulli matrix with respect to and holds true:

**Proof**

The case for follows the same way as the previous proof in the generalized Bernoulli matrix and for we have:

And the last expression is additive theory as required.

**Lemma 4.3.3** The following relation for improved q-Bernoulli numbers hold true

**Proof.** At lemma (1.50) we proved this relation for Bernoulli numbers. The proof is exactly similar to the ordinary case.

**Definition 4.3.4** the inverse of improved q-Bernoulli matrix is defined by a matrix D, which is

### 4.3.3 Theorem Inverse of Improved q-Bernoulli matrix

Inverse of Improves q-Bernoulli matrix can be defined by the previous definition of . That means

**Proof**

Since both of the matrices are lower triangle, their multiplications are also lower triangle. For the another entries we may use the similar calculation as follow

As required.

Satisfying theorem 4.3.3.

# CHAPTER 5

# SUMMARY AND CONCLUSION

There are several definition for q-Bernoulli matrix and we works on a classes of improved q-Bernoulli matrix which is more suitable to make a q-analogue of the same concept. Some properties of improved q-Bernoulli numbers and polynomials allowed us to work this concept easily. At the end of the day, we may defined these numbers by using different generating functions. But the improved one works better and convinced the ordinary case better.

# 

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