DISCRETE GREEN'S FUNCTION

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
ALHAM MUSTAFA AL-REFAI

In Partial Fulfilment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2017

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Name, Last name: ALHAM MUSTAFA AL-REFAI

Signature:

Date:

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To my parents and family…

ABSTRACT

A priori estimations play one of the central roles in investigating stability, existence and uniqueness of the solutions of the differential equations.

Green's function method is one of the effective methods to get this type of estimations. However, construction of the Green's function in explicit form for many problems is problematic.

Similar problems arise in the investigation of the finite difference equations. In addition to continuous problems Green's function method in the discrete problems are very effective in the determination (in solving convergence problem) of the rate of convergence of finite difference solution to the exact solution of the differential equation as discretization parameter approaches zero.

In this thesis the existing in the literature techniques of the Green's function method for the Laplace difference operator are reviewed and investigated. As it follows from the existing results the obtained by discrete Green's function method error estimations the maximum order was $O(h^4)$.

Also in this thesis, in the case of discrete Dirichlet problem for Poisson's equation on the square grid with step size h by using discrete Green's function $O(h^6)$ order of error estimation is obtained.

Keywords: Green's function; Laplace and Poisson's equation; Dirichlet problem; finite difference method; error estimations

ÖZET

Differensiyel denklemlerin çözümleri için öncül tahminlerde en önemli rol çözümün kararlılığı, varlığı ve tekliği oynamaktadır.

Green fonksiyon metodu bu tip tahminleri elde etmek için etkili bir yöntemdir. Bununla birlikte, pek çok problem için kapalı formdaki Green fonksiyonu oluşturmak problemlidir.

Benzer problemler sonlu fark denklemlerinin araştırılmasındada ortaya çıkmaktadır. Sürekli problemlere ek olarak, ayrık problemlerde Green fonksiyon yöntemi sonlu fark çözümünün diferensiyel denklemin kesin çözümüne yakınsaklık hızı ayrıklaştırma parametresinin sıfıra yaklaşması probleminde çok etkili bir çözüm oldu.

Bu tezde, Laplace Fark operatörü için Green fonksiyon metodunun mevcut olan teknikleri gözden geçirilmiş ve araştırılmıştır. Ayrık Green fonksiyonu yöntemi ile elde edilen mevcut sonuçlarda yakınsaklık hatasının $O\left(h^4\right)$olduğu elde edilmiştir.

Ayrıca bu tez çalışmasında, Poisson denklemi için ayrık Dirichlet problem durumunda ızgara üzerinde ızgara adımı h olmak üzere ayrık Green fonksiyon kullanılarak yakınsaklık hatası $O(h^6)$ elde edilmiştir.

Anahtar Kelimeler: Green funksiyon; Laplace ve Poisson denklemleri; Dirichlet problem; sonlu farklar metodu; hata tahmini

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CHAPTER 1
INTRODUCTION

The finite- difference method is one of the most widely applied methods for the approximation of ordinary and partial differential equations.

We can practice this discretization method in many science applications such as in dynamical meteorology, aerodynamic, mathematical physics, oceanography, and many other disciplines. Therefore, the convergence analysis and the error estimation of this scheme hold practical, as well as theoretical importance.

An example of the application of finite-difference can also be seen in Richardson's extrapolation method. We use the finite-difference analogue of an equation in this method to improve the order of convergence, so resulting in a more accurate method. Then we can show that the finite-difference is the first step for the improvement of error estimation.

When analyzing the error estimation and the convergence of the applied finite-difference scheme, the determination of the order of accuracy by the suggested scheme is important. Moreover, with investigation of the scheme, it might be possible to structure schemes with increased accuracy.

In the usual study of the discretization error resulting from approximating boundary value problems for elliptic equations by finite difference methods, for the error estimation in maximum norm, there are three effective methods:

(i) The methods which based on maximum principle.

(ii) The methods which based on discrete Green's function.

(iii) The methods which based on energy inequalities with embedding theorems.

In 1930 S. Gershgorin gave a method for estimating the order of convergence of the solution to a certain class of finite difference analogues to the solution of the Dirichlet problem for elliptic equations of order $O(h)$. His method was based on a maximum principle for the finite difference analogue. In 1933 L.Collatz proposed a certain boundary approximation and using the techniques of Gershgorin, showed that this approximation gives rise to an $O(h^2)$estimate for the truncation error. The estimates of both Gershgorin and Collatz assume the knowledge of bounds for certain higher derivatives of the solution of the Dirichlet problem.

From an analogy to probability theory Courant, Friedrichs, and Lewy give a finite difference Green's function for the Dirichlet problem for Poisson's equation. Using this Green's function they give an analogue of Green's third identity. Wasow studies the asymptotic behavior of the finite difference Green's function and Laasonen uses an explicit representation of the finite difference Green's function for the rectangle to obtain bounds in that case.

A.Samarskii obtained a priori estimates for the solution of finite difference problems by the method of energy inequalities. This estimation is used to get error estimation in maximum norm by applying the discrete forms of the embedding theorems.

In this thesis, the error analysis for two different finite-difference schemes have been reviewed. Furthermore, the discrete Green's function method in the case of square grids to get $O(h^6)$ is improved.

CHAPTER 2
LITERATURE REVIEW

The global convergence as mesh step $h \to 0$ was proved first for the Laplace equation on a square mesh by R.G.D. Richardson in (1917) and by Phillips and Wiener in (1922); the aim of these authors was to establish existence theorems for solutions of the Dirichlet problem for $\nabla^2 u = 0$ from algebraic existence theorems for $\nabla_h^2 u = 0$. In (Courant, Friedrichs, and Lewy, 1928) it was proved that, all difference quotients of given order converge to the appropriate derivatives, as $h \to 0$.

The maximum principle was applied to the Poisson equation by (Gerschgorin, 1930) to prove $O(h)$ global accuracy. (Collatz, 1933), proved this result by using linear interpolation on the boundary, under appropriate differentiability assumptions to prove $O(h^2)$ accuracy. Also by (Wasow, 1952), and by (P. Laasonen, 1957) the loss of accuracy introduced by corners is discussed.

There was a study by (Walsh and Young, 1954), for the effect on the error of the smoothness of the boundary values. They proved for the Dirichlet problem, by using Fourier series, that $|U - u| \leq Mh$ for continuous and piecewise differentiable boundary values $g(s)$, provided that $g''(s)$is bounded except where $g'(s)$ has jumps, M is a constant independent of h.

Also (Collatz, 1933) gives a recipe for fitting boundary values on a general domain by approximate values at nodes of a square mesh.

The complete subject was carefully reconsidered by Bramble and Hubbard, who used the Green's function approach systematically. The accuracy of the five-point difference approximation with variable coefficients has been studied by (Bramble, Hubbard, Kellogg, and Thomee, 1968), under weakened assumptions of smoothness on the boundary. Finally, the $O(h^2)$ convergence of all difference quotients to the appropriate derivatives was proved for the Laplace differential equation on a square mesh by V.Thomee in Birkhoff-Varga, and by Achi Brandt. Making stronger smoothness assumptions, also Thomee showed that difference quotients converge at the same rate as the solution in the interior.

Also (Bramble, Hubbard, and Zlamal, 1968) studied the effect of singularities, and they obtained error bounds for the Poisson equation. Thomee, also has proved convergence of order $O(h^{1/2})$ for simple difference approximations to the Dirichlet problem for any linear, constant-coefficient equation of elliptic type, and the global error bounds for difference approximations to certain mildly nonlinear elliptic problems was obtained by (McAllister, 1969). Hence, Bramble has shown that one can reduce the error of difference approximations to $L[u] = f$ for uniformly elliptic L, by appropriately smoothing f.

By (Bahvalov,1959) it was proved that the regularity demands on the solution u of the continuous problem in some cases can be relaxed by essentially two derivatives at the boundary without losing the convergence estimate and that for still less regular u one can obtain correspondingly weaker convergence estimates. (Bahvalov, 1959) was using his error bounds to estimate the number of arithmetic operations needed to obtain u to a prescribed accuracy. Also related results were obtained in special cases by (Wasow, 1952), (Laasonen, 1958), and by (Volkov, 1966) and references there in.

2.1 Green's Function for the Differential Equations

Further estimation of a solution of the boundary-value problem for a second-order difference equation will involve its representation in terms of Green's function. The boundary-value problem for the differential equation

$$Lu = \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)u = -f(x), \qquad 0 < x < 1\,,$$

$$u(0) = 0\,, \quad u(1) = 0\,, \quad k(x) \geq c_1 > 0, \quad q(x) \geq 0, \tag{2.1}$$

can add interest and aid in understanding. As known, the solution of this problem arranges itself as an integral

$$u(x) = \int_0^1 G(x,\xi) f(\xi) d\xi \ , \tag{2.2}$$

where $G(x,\xi)$ is the source function or Green's function. Function (2.2) is a solution to equation (2.1) subject to the boundary conditions $u(0) = 0$ and $u(1) = 0$ if Green's function $G(x,\xi)$ as a function of x for fixed ξ satisfies the conditions

$$L_x G(x,\xi) = \frac{d}{dx}\left(k(x)\frac{dG(x,\xi)}{dx}\right) - q(x)G(x,\xi) = 0$$

$$x \neq \xi \ , \quad 0 < x < 1 \ , \quad G(0,\xi) = \ G(1,\xi) = 0 \tag{2.3}$$

$$[G] = G(\xi + 0, \xi) - G(\xi - 0, \xi) = 0 \quad , \quad \left[k\frac{dG}{dx}\right] = -1 \quad \text{for} \quad x = \xi \ .$$

It's proved that this type of defined Green's function is nonnegative and symmetric:

$$G(x,\xi) \geq 0 \quad , \quad G(x,\xi) = G(\xi,x),$$

and $G(x,\xi)$ can be written in the explicit form

$$G(x,\xi) = \begin{cases} \frac{\alpha(x)\beta(\xi)}{\alpha(1)} for x \leq \xi \\ \frac{\alpha(\xi)\beta(x)}{\alpha(1)} for x \geq \xi \end{cases} \quad , \tag{2.4}$$

where $\alpha(x)$ and $\beta(x)$ are solutions of the following problems:

$$L\alpha = 0 \ , \quad 0 < x < 1 \ , \quad \alpha(0) = 0 \ , \quad k(0)\alpha^{'}(0) = 1 \ ,$$

$$L\beta = 0 \ , \quad 0 < x < 1 \ , \quad \beta(1) = 0 \ , \quad k(1)\beta^{'}(1) = -1. \tag{2.5}$$

From this analysis follows the difficulties of the construction of the exact form of Green's function.

2.2 Green's Function for the Difference Equations

Consider the closed rectangle

$$\bar{R} = \{(x, y): 0 \leqq x \leqq a\,, 0 \leqq y \leqq b\}\,,$$

such that the ratio a/b is rational. The square grid on which the difference equation will be considered consists of the node points (x_m, y_n) :

$$\begin{aligned} x = x_m = mh\,, \qquad & (m = 0,1, \dots, M), \quad (Mh = a), \\ y = y_n = nh, \qquad & (n = 0,1, \dots, N)\,, \quad (Nh = b). \end{aligned} \tag{2.6}$$

Denote a parameter point by (ξ, η) or

$$\xi = \mu h, \quad \eta = vh, \ (0 \leqq \mu \leqq M \ \ , \ 0 \leqq v \leqq N). \tag{2.7}$$

For the sake of simplicity set

$$n' = N - n\,, \qquad v' = N - v \ \ .$$

Replace Laplace's equation by its simplest analogue, namely,

$$\begin{aligned} \Delta_h u(x, y) = \frac{1}{h^2}[&u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) \\ &- 4u(x, y)] = 0\,. \end{aligned}$$

Green's function $G_h(x, y; \xi, \eta)$ is now defined on the grid by the difference equations

$$\Delta_h G_h(x,y;\xi,\eta) = \begin{cases} 0, & when \quad (x,y) \neq (\xi,\eta) \\ h^{-2}, & when \quad x = \xi and y = \eta \end{cases}, \tag{2.8}$$

and by the condition that it must vanish on the boundary of the rectangle. This function can be represented by the following expressions:

$$G_h(mh,nh;\mu h,vh)$$
$$= \begin{cases} -\frac{2}{M}\sum_{k=1}^{M-1}\frac{\sin\mu\alpha_k \sin m\alpha_k \sin hv'\beta_k \sin hn\beta_k}{\sin h\beta_k \sin hN\beta_k} & (n \leqq v) \\ -\frac{2}{M}\sum_{k=1}^{M-1}\frac{\sin\mu\alpha_k \sin m\alpha_k \sin hv\beta_k \sin hn'\beta_k}{\sin h\beta_k \sin hN\beta_k} & (n \geqq v) \end{cases}, \tag{2.9}$$

with

$$\alpha_k = \frac{k\pi}{M} = \frac{k\pi h}{a}, \quad \cos h\,\beta_k = 2 - cos\alpha_k \quad , \tag{2.10}$$

From the expression (2.9) follows the symmetry of the discrete Green's function with respect to its two kind of variables (x, y) and (ξ, η).

If M increases indefinitely and, correspondingly, h decreases, then the factors α_k and β_k approach zero; but the terms in these sums converge to the related terms in the following infinite series:

$$G(x,y;\xi,\eta) = \begin{cases} -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin\frac{k\pi\xi}{a}\sin\frac{k\pi x}{a}sh\frac{k\pi\eta'}{a}sh\frac{k\pi y}{a}}{ksh\frac{k\pi b}{a}} & (y \leqq \eta), \\ -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin\frac{k\pi\xi}{a}\sin\frac{k\pi x}{a}sh\frac{k\pi\eta}{a}sh\frac{k\pi y'}{a}}{ksh\frac{k\pi b}{a}} & (y \geqq \eta). \end{cases} \tag{2.11}$$

In (Pentti Laansonen, 1958), estimate for the rate of convergence of $G_h(x, y; \xi, \eta)$ to $G(x, y; \xi, \eta)$ for a decreasing h was established by the following inequality

$$|G_h(x, y; \xi, \eta) - G(x, y; \xi, \eta)| \leqq 2.15 \left(\frac{h}{\rho}\right)^2 \ , \tag{2.12}$$

where ρ is the distance

$$\rho = \sqrt{(x-\xi)^2 + (y-\eta)^2} \ .$$

2.3 Effective error estimation in rectangular domain

By means of estimate (2.12) it is now possible to compute some bounds for the error made in approximating the solution of Poisson's equation by the finite difference analogue. The solution u_h of Poisson's difference equation

$$\Delta_h u_h = f(x, y) \ ,$$

where

$$\Delta_h u(x, y) = \frac{1}{h^2}[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)],$$

and $u_h = 0$ on the boundary nodes.

The solution of this finite-difference problem by using the above defined discrete Green's function can be represented as follows:

$$u_h(\mu h, \nu h) = h^2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} G_h(mh, nh; \mu h, \nu h) f(mh, nh)$$

The corresponding formula for the solution of Poisson's differential equation is

$$u(\xi,\eta) = \int_0^a \int_0^b G(x,y;\xi,\eta) f(x,y) dxdy \quad .$$

The difference $u_h - u$ at a node point $\xi = \mu h, \eta = vh$ may be decomposed into the following terms:

$$u_h - u = h^2 G_h(\mu h, vh; \mu h, vh) f(\mu h, vh) - \iint_{S_{m,n}} G(x,y;\xi,\eta) f(x,y) dxdy +$$
$$h^2 \sum{}' \sum{}' [G_h(mh,nh;\mu h,vh) - G(mh,nh;\mu h,vh)] f(mh,nh) -$$
$$\sum{}' \sum{}' G(mh,nh;\mu h,vh) \iint_{S_{m,n}} [(f(x,y) - f(mh,nh)] dxdy -$$
$$\sum{}' \sum{}' f(mh,nh) \iint_{S_{m,n}} [(G(x,y;\mu h,vh) - G(mh,nh;\mu h,vh)] dxdy -$$
$$\sum{}' \sum{}' \iint_{S_{m,n}} [(G(x,y;\mu h,vh) - G(mh,nh;\mu h,vh)] \times [f(x,y) -$$
$$f(mh,nh)] dxdy - \sum{}'' \sum{}'' \iint_{S_{m,n}} G(x,y;\xi,\eta) f(x,y) dxdy. \qquad (2.13)$$

All double sums affixed with primes range over all $(h \times h)$ squares $S_{m,n}$ with the interior node points (mh, nh) as their centers, with the exception of the square about h, vh . The sums affixed with double primes range over all those parts of the boundary squares $S_{m,n}$ (where m is either 0 or M, or n is either 0 or N) which are inside the rectangle.

If f is continuous , $\mathfrak{W}$ the maximum of $|f|$, $\epsilon(r)$ the modulus of continuity , i.e., the maximum variation of f between any two points with distance less than or equal to r , and d is the largest of the two sides a and b , then an estimate for the total error reads :

$$|u_h - u|$$
$$\leqq \left(21.4 + 18.8 \log \frac{d}{h}\right) h^2 \mathfrak{W} + 2.83 d^2 \epsilon\left(\frac{h}{\sqrt{2}}\right) + 11.4\, hd\epsilon\left(\frac{h}{\sqrt{2}}\right). \qquad (2.14)$$

This estimate proves that the truncation error tends to zero for decreasing h. Furthermore construction of this result is not essentially impaired if discontinuous of bounded variation are allowed on certain rectifiable curves whose total length is bounded, because these only generate an additional term of magnitude $O(h^2 \log\left({d}/{h}\right))$.

In order to have a check on the accuracy obtainable by the assume method applied, on the contrary, that f is not only continuous but also has continuous first order and bounded second order derivatives. In this case the result is

$$|u_h - u| = \left[\left(21.4 + 18.8\ \log\frac{d}{h}\right)\mathfrak{W} + 8.1\ d\mathfrak{W}' + 2.7\ d^2\mathfrak{W}''\right]h^2. \qquad (2.15)$$

Where $\mathfrak{W}'$ is the maximum of grad f and $\mathfrak{W}''$ the maximum of the second order derivatives. This result may now be compared with a previously known error estimate.

If the function $f(x,y)$ of the Poisson's equation is analytic and if the boundary of a domain is an analytic curve, then, of course, the solution with vanishing boundary values is analytic in the closed domain. The results of Gerschgorin show in this case, that if the grid can be chosen so that all boundary nodes are on the curve, then the truncation error of the associated discrete approximations is of the order $O(h^2)$. Now, the result (2.15) gives the rate $O(h^2 \log h^{-1})$ for the rectangular domain. This lower result than $O(h^2)$ of the convergence rate is the presence of the corners at the boundary.

CHAPTER 3
GREEN'S FUNCTION ON THE DOMAINS WITH CURVED BOUNDARIES

The approach taken here is to define an appropriate related finite difference Green's function for various finite difference analogues. In each case the analogue of Green's third identity is given and used to obtain estimates for the truncation error.

In the second order estimate the truncation error is studied for a finite difference approximation. Although at points near the boundary the finite difference operator approximates the Laplace operator only to $O(h)$ it is seen that the resulting contribution to the truncation error is $O(h^3)$.

3.1 Second Order Estimates

Consider the finite – difference approximation of the boundary value problem

$$\begin{aligned} &\Delta u(x,y) = F(x,y), \qquad (x,y) \in R \ , \\ &u(x,y) = f(x,y), \qquad (x,y) \in C \ . \end{aligned} \tag{3.1}$$

We assume that R is a bounded region in the (x, y) plane with boundary C .

Let R_h be the set of mesh points in R whose nearest neighbours in the x and y directions lie in R. Those grid points in R which do not belong to R_h will makeup the set called C_h^* . The points of intersection of the grid with the boundary C form the set C_h.

For any point P belonging to $R_h + C_h^* + C_h$ we define the neighbors $N(P)$ to be those nearest points in $R_h + C_h^* + C_h$ lying along grid lines.

If $V(x, y)$ is an arbitrary mesh function defined on $R_h + C_h^* + C_h$ then for such vectors we define the finite difference operator Δ_h.

If $(x, y) \in R_h$, then

$$\Delta_h V(x,y) = h^{-2}\{V(x+h,y) + V(x,y+h) + V(x-h,y) + V(x,y-h) - 4V(x,y)\}. \quad (3.2)$$

This is the usual $O(h^2)$ approximation of Δ for functions $V(x,y) \in C^4(\bar{R})$. In fact,

$$|\Delta V(x,y) - \Delta_h V(x,y)| \leq \frac{h^2}{6} M_4 \quad , \qquad (x,y) \in R_h \quad , \quad (3.3)$$

where we have used the notation

$$M_j = sup_{P\in R}\left\{\left|\frac{\partial^j U(P)}{\partial x^i \partial y^{j-i}}\right| : i = 0,1,\ldots,j\right\}. \quad (3.4)$$

At points of C_h^*, Δ_h is defined to be the 5-point divided difference approximation to Δ .

For example, if $(\bar{x},\bar{y}) \in C_h^*$, we use for the approximation $\frac{\partial^2 v}{\partial x^2}$ and $\frac{\partial^2 v}{\partial y^2}$ the following:

$$\frac{\partial^2 v}{\partial x^2} \cong V_{\bar{x}x} = \frac{1}{\left(\frac{h+\alpha h}{2}\right)}\left(\frac{V(\bar{x}+h,\bar{y}) - V(\bar{x},\bar{y})}{h} - \frac{V(\bar{x},\bar{y}) - V(\bar{x}-\alpha h,\bar{y})}{\alpha h}\right)$$

$$= \frac{2}{h^2(1+\alpha)}\{V(\bar{x}+h,\bar{y}) - \left(1+\frac{1}{\alpha}\right)V(\bar{x},\bar{y}) + \frac{1}{\alpha}V(\bar{x}-\alpha h,\bar{y})\}$$

$$= \frac{2}{h^2(1+\alpha)}V(\bar{x}+h,\bar{y}) - \frac{2}{h^2(1+\alpha)}\cdot\left(\frac{\alpha+1}{\alpha}\right)V(\bar{x},\bar{y}) + \frac{2}{h^2\alpha(1+\alpha)}V(\bar{x}-\alpha h,\bar{y}).$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} \cong V_{\bar{y}y} = \frac{2}{h^2(1+\beta)}\left\{V(\bar{x},\bar{y}+\beta) - \left(1+\frac{1}{\beta}\right)V(\bar{x},\bar{y}) + \frac{1}{\beta}V(\bar{x},\bar{y}-\beta h)\right\}$$

$$= \frac{2}{h^2(1+\beta)}V(\bar{x},\bar{y}+\beta) - \frac{2}{h^2(1+\beta)}\cdot\left(\frac{\beta+1}{\beta}\right)V(\bar{x},\bar{y})$$
$$+ \frac{2}{h^2\beta(1+\beta)}V(\bar{x},\bar{y}-\beta h)$$

$$\Delta_h V(\bar{x},\bar{y}) = 2h^{-2}\left\{\left(\frac{1}{\alpha+1}\right)V(\bar{x}+h,\bar{y}) + \frac{1}{\alpha(\alpha+1)}V(\bar{x}-\alpha h,\bar{y}) + \left(\frac{1}{\beta+1}\right)V(\bar{x},\bar{y}+h) + \frac{1}{\beta(\beta+1)}V(\bar{x},\bar{y}-\beta h) - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)V(\bar{x},\bar{y})\right\}. \quad (3.5)$$

Combining these, as $\Delta_h V = V_{\bar{x}x} + V_{\bar{y}y}$, we obtain (3.5).

If $\alpha = \beta = 1$, then Δ_h takes the same form as in (3.2).

We note that Δ_h as defined in (3.5) approximates Δ to $O(h)$ for $V(x,y) \in C^3$ in R, i.e.

$$|\Delta V(\bar{x},\bar{y}) - \Delta_h V(\bar{x},\bar{y})| \le \frac{2M_3 h}{3} \quad . \quad (3.6)$$

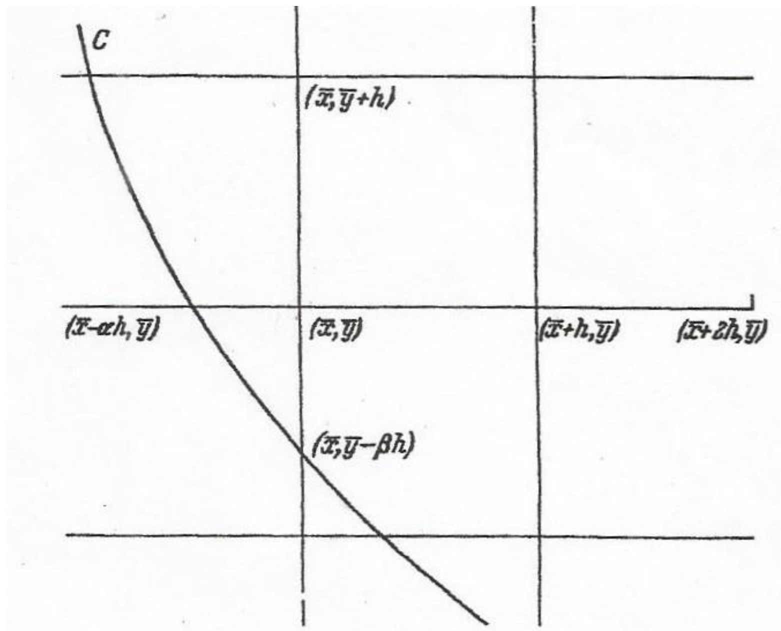

Figure 3.1: Regular and Irregular Points

The following is finite difference analogues of (3.1),

$$\Delta_h U(x,y) = F(x,y), \qquad (x,y) \in R_h + C_h^* \quad ,$$
$$U(x,y) = f(x,y), \qquad (x,y) \in C_h \,. \tag{3.7}$$

This is a system of simultaneous linear equations for the determination of the mesh function $U(x,y)$.

The truncation error $\varepsilon(P) \equiv u(P) - U(P)$, $\quad P \in R_h + C_h^* + C_h$ satisfies an inequality of the type

$$|\varepsilon|_M \leq Kh^2, \tag{3.8}$$

where K is a constant independent of P and h . In (3.8) we have used the notation

$$\psi_M = sup_{P \in S \subset \bar{R}} \psi(P) \, , \tag{3.9}$$

for any function ψ defined on a subset S of $\bar{R}$.

Finite Difference Analogue of Green's function $G_h(P,Q)$ is

$$\begin{aligned} &\Delta_{h,P} G_h(P,Q) = -\delta(P,Q)h^{-2} \quad , \qquad P\epsilon\, R_h + C_h^* \,, \\ &G_h(P,Q) = \delta(P,Q) \qquad , \quad P \in C_h \ , \end{aligned} \tag{3.10}$$

for $Q \in R_h + C_h^* + C_h$

$$\delta(P,Q) = \begin{cases} 1 & , \quad P = Q \ , \\ 0 & , \quad P \neq Q \ . \end{cases} \tag{3.11}$$

Lemma 1. (Maximum Principle)

For any mesh function $V(P)$ defined on $R_h + C_h^* + C_h$ if $\Delta_h V(P) \geq 0$ for $P\epsilon\, R_h + C_h^*$ then $V(P)$ takes on its maximum on C_h.

Lemma 2. (Green's Third Identity)

Let $V(P)$ be any arbitrary mesh function defined on $R_h + C_h^* + C_h$. Then for any $P \in R_h + C_h^* + C_h$

$$V(P) = h^2 \sum_{Q\in R_h + C_h^*} G_h(P,Q)[-\Delta_h V(Q)] + \sum_{Q\in C_h} G_h(P,Q)V(Q) \ . \tag{3.12}$$

Proof: Let $P \in R_h + C_h^*$

$$\begin{aligned} &\Delta_h W(P) = h^2 \Delta_h G_h(P,P)[-\Delta_h V(P)] \\ &= h^2 \cdot \left(-h^{-2}\right)(-\Delta_h V(P)) \\ &= \Delta_h V(P). \end{aligned}$$

Let $P \in C_h$ then

$$W(P) = G_h(P,P)V(P) = V(P).$$

It follows that

$$\Delta_h W(P) = \Delta_h V(P)\,, \qquad P \epsilon R_h + C_h^*\,, \tag{3.13}$$

$$W(P) = V(P). \qquad P \in C_h\,. \tag{3.14}$$

Lemma 3. $\qquad G_h(P,Q) \geq 0\,,\, Q \in R_h + C_h^* + C_h\,.$ (3.15)

Proof: Substitute $-G_h(P,Q)$ into Green's operator

i.e.

$$\Delta_{h,P}(-G_h(P,Q)) = \delta(P,Q)h^{-2} \geq 0 \quad \text{on} \quad R_h + C_h^*,$$

$$-G_h(P,Q) = -\delta(P,Q) \leq 0 \qquad \text{on} \quad C_h\,.$$

By the maximum principle, it can obtain its maximum on C_h .

Hence

$$-G_h(P,Q) \leq 0 P \in R_h + C_h^*\,,$$

$$G_h(P,Q) \geq 0.$$

Lemma 4. $\qquad \sum_{Q \in C_h^*} G_h(P,Q) \leq 1,\;\; P \in R_h + C_h^* + C_h\,.$ (3.16)

Proof: Let the mesh function $W(P)$ be given by

$$W(Q) = \begin{cases} 1 & , \quad Q \in R_h + C_h^* , \\ 0 & , \quad Q \in C_h . \end{cases} \tag{3.17}$$

Then $\Delta_h W(P) = 0$, $Q \in R_h$. It is easily seen from the definition of Δ_h on C_h^* that $-\Delta_h W(P) \geq h^{-2}$.

Applying lemma 2.2 it follows that for $P \in R_h + C_h^*$

$$1 = h^2 \sum_{Q \in C_h^*} G_h(P,Q)[-\Delta_h W(Q)] \geq \sum_{Q \in C_h^*} G_h(P,Q).$$

If $P \in C_h$, then

$$\sum_{Q \in C_h^*} G_h(P,Q) \leq 1 \; .$$

Lemma 5. If d is the diameter of the smallest circumscribed circle containing R then

$$h^2 \sum_{Q \in R_h + C_h^*} G_h(P,Q) \leq \frac{d^2}{16} \quad , \quad P \in R_h + C_h^* + C_h \, . \tag{3.18}$$

Proof: Let 0 be the center of the circumscribed circle about R of diameter d .

Let $W(P) = \frac{r(P)^2}{4}$ for $P \in R_h + C_h^* + C_h$, where $r(P)$ is the Euclidean distance from 0 to P.

Then,

$$\Delta_h W(P) = 1 \quad , \quad P \in R_h + C_h^* ,$$

$$W(P) = \frac{x_1^2 + x_2^2}{4} \quad \text{as} \quad \Delta_{h,\alpha} x_\alpha^2 = 2 \, , \quad \frac{1}{4}\left(\Delta_h (x_1^2 + x_2^2)\right) = \frac{4}{4} = 1 \, .$$

Now define the mesh function

$$V(P) = h^2 \sum_{Q \in R_h + C_h^*} G_h(P,Q).$$

We see from (3.10) that

$$\Delta_h V(P) = -1 \ , \ P \in R_h + C_h^* \quad (\text{as } \Delta_h V(P) = h^2 \sum_{Q \in R_h + C_h^*} G_h(P,Q) = 1),$$
$$V(P) = 0 \qquad , \quad P \in C_h \ . \tag{3.19}$$

Hence $\Delta_h[V(P) + W] = 0$ for $P \in R_h + C_h^*$ and $V(P) + W(P) \leq \frac{d^2}{16} = \frac{r(P)^2}{4} = \frac{(d/2)^2}{4}$ for $P \in C_h$.

By the maximum principle, since $W \geq 0$, it follows that

$$V(P) \leq \frac{d^2}{16} \quad , \quad P \in R_h + C_h^* + C_h \ ,$$

i.e.

$$h^2 \sum_{Q \in R_h + C_h^*} G_h(P,Q) \leq \frac{d^2}{16} \qquad , \quad P \in R_h + C_h^* + C_h$$

Theorem 1. Let $u(x,y)$ be the solution of (3.1) and $U(x,y)$ the solution of (3.7). Then the truncation error $\varepsilon(P) = u(P) - U(P)$ satisfies the inequality

$$|\varepsilon|_M \leq \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3 \ . \tag{3.20}$$

Proof: Since $\varepsilon(P) = 0$, $P \in C_h$ we see from Lemma (2) that

$$\varepsilon(P) = h^2 \sum_{Q \in R_h + C_h^*} G_h(P,Q)[-\Delta_h \varepsilon(Q)]. \tag{3.21}$$

Since

$$|-\Delta_h \varepsilon(Q)| = |\Delta_h u(Q) - \Delta u(Q)| \,, \tag{3.22}$$

we have that

$$|\varepsilon(P)| = h^2 \sum_{Q\in R_h} G_h(P,Q)[-\Delta_h \varepsilon(Q)] + h^2 \sum_{Q\in C_h^*} G_h(P,Q)[-\Delta_h \varepsilon(Q)]$$
$$\leq \left| h^2 \sum_{Q\in R_h} G_h(P,Q) \right| \frac{h^2 M_4}{6} + h^2 \left| \sum_{Q\in C_h^*} G_h(P,Q) \right| \frac{2M_3 h^3}{3}$$
$$\leq \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3.$$

3.2 Other Boundary Approximations

3.2.1 Zero Order Interpolation

Let $G_h^*(P,Q)$ be the finite difference Green's function for R_h with boundary C_h^* . This is given by

$$\begin{aligned} \Delta_{h,P} G_h^*(P,Q) &= -\delta(P,Q) h^{-2} \quad , \quad P \in R_h \,, \\ G_h^*(P,Q) &= \delta(P,Q) \quad , \quad P \in C_h^* \end{aligned} \tag{3.23}$$

for all $P \in R_h + C_h^*$.

Just as in Lemma 2 we have the identity

$$V(P) = h^2 \textstyle\sum_{Q\in R_h} G_h^*(P,Q)[-\Delta_h V(Q)] + \sum_{Q\in C_h^*} G_h^*(P,Q) V(Q) \,. \tag{3.24}$$

In addition all of the other lemmas of section (3.1) are valid if we make the substitutions

$$G_h \to G_h^* ,$$
$$R_h + C_h^* \to R_h ,$$
$$C_h \to C_h^* . \tag{3.25}$$

We shall also need the following Lemma:

Lemma 1. For $P \in R_h + C_h^*$,

$$\sum_{Q \in C_h^*} G_h^*(P,Q) = 1 \ . \tag{3.26}$$

Proof: Apply (3.24) to $V(P) \equiv 1$.
Let $V(P) \equiv 1$. Then

$$\Delta_{h,P} G_h^*(P,Q) = -h^{-2} \ \text{ if } P \in R_h ,$$

$$V(P) = 1 = h^2 \cdot h^{-2} + 0 = 1 \quad \text{if} \quad P \in C_h^* ,$$

$$V(P) = 1 = 0 + \sum_{Q \in C_h^*} G_h^*(P,Q) V(P) ,$$

$$\sum_{Q \in C_h^*} G_h^*(P,Q) = 1 \ .$$

Let $V(P)$ satisfy

$$\Delta_h U(P) = F(P) \quad , \quad P \in R_h ,$$
$$U(P) = f(P') \quad , \quad P \in C_h^* \ , \tag{3.27}$$

where P' is one of the neighbours of P in C_h .

Theorem 2. Let $u(x,y)$ be the solution of (3.1) and $U(x,y)$ the solution of (3.27) . Then the truncation error $\varepsilon(P) = u(P) - U(P)$ satisfies the inequality

$$|\varepsilon|_M \leq hM_1 + \frac{M_4 d^2}{96} h^2 \ . \tag{3.28}$$

Proof: From (3.24)

$$\varepsilon(P) = h^2 \sum_{Q\in R_h} G_h^*(P,Q)[-\Delta_h \varepsilon(P,Q)] + \sum_{Q\in C_h^*} G_h^*(P,Q)\varepsilon(Q). \tag{3.29}$$

We note that for $Q \in C_h^*$

$$|\varepsilon(Q)| = |u(Q) - U(Q)| = |u(Q) - U(Q')| \leq hM_1 \ . \tag{3.30}$$

We have that

$$|\Delta_h \varepsilon(Q)| \leq \frac{h^2}{6} M_4 \quad , \qquad Q \in R_h \ . \tag{3.31}$$

Taking absolute values of both sides of (3.29) and substituting the inequalities obtained we end up with

$$|\varepsilon|_M \leq hM_1 + \frac{M_4 d^2}{96} h^2.$$

Since in (3.30) we have Q' as the closest point to Q on the boundary

Hence

$$U(Q') = f(Q') = u(Q').$$

Thus

$$|u(Q) - u(Q')| \leq \left|u'(\xi)\right| \cdot |Q - Q'| \leq M_1 h .$$

3.2.2The first order interpolation

We consider here the finite difference analogue of (3.1) given in (Collatz, 1933). He defines the following approximation to (3.1)

$$\Delta_h U(P) = F(P), \qquad P \in R_h ,$$
$$U(P) = f(P), \qquad P \in C_h \quad . \tag{3.32}$$

At a point P of C_h^* he prescribes that $U(P)$ lie on a straight line between the values of U at two neighbours of P, one of which is in R_h , the other in C_h. For example for the point $(\bar{x}, \bar{y})$ of Fig.1 we have

$$U(\bar{x}, \bar{y}) = \frac{\alpha}{\alpha+1} U(\bar{x} + h, \bar{y}) + \frac{1}{\alpha+1} U(\bar{x} - \alpha h, \bar{y}). \tag{3.33}$$

Alternatively we could have interpolated in the y direction.

As (Collatz, 1933) has shown this method gives rise to an estimate of the truncation error which is $O(h^2)$. The contribution to the truncation error arising from the points of C_h^* is also $O(h^2)$. The following analysis again yields similar results.

Theorem 3. (Collatz): Let $u(x, y)$ be the solution of (3.1) and $U(x, y)$ the solution of (3.32) and (3.33) . Then the truncation error $\varepsilon(P) = u(P) - U(P)$ satisfies

$$|\varepsilon|_M \leq \left[M_2 + \frac{M_4 d^2}{48}\right] h^2 . \tag{3.34}$$

Proof: For $Q \in C_h^*$

$$|\varepsilon(\bar{x},\bar{y})| = |u(\bar{x},\bar{y}) - U(\bar{x},\bar{y})|$$
$$= \left|u(\bar{x},\bar{y}) - \tfrac{\alpha}{\alpha+1}U(\bar{x}+h,\bar{y}) - \tfrac{1}{\alpha+1}U(\bar{x}-\alpha h,\bar{y})\right|. \qquad (3.35)$$

Using the triangle inequality,

$$|\varepsilon(\bar{x},\bar{y})| = \left|u(\bar{x},\bar{y}) - \frac{\alpha}{\alpha+1}\left(u(\bar{x}+h,\bar{y}) - \varepsilon(\bar{x}+h,\bar{y})\right) - \frac{1}{\alpha+1}u(\bar{x} - \alpha h,\bar{y})\right|$$

$$\leq \left|u(\bar{x},\bar{y}) - \tfrac{\alpha}{\alpha+1}u(\bar{x}+h,\bar{y}) - \tfrac{1}{\alpha+1}u(\bar{x}-\alpha h,\bar{y})\right| + \tfrac{\alpha}{\alpha+1}|\varepsilon|_M\ . \qquad (3.36)$$

[$\varepsilon(\bar{x}-\alpha h,\bar{y}) = 0$ as point is on boundary]

Expanding this using Taylor series, and keeping in mind that $0 < \alpha \leq 1$, we obtain

$$|\varepsilon(Q)| \leq \tfrac{M_2}{2}h^2 + \tfrac{1}{2}|\varepsilon|_M\ . \qquad (3.37)$$

Combining with earlier results, we obtain

$$|\varepsilon(Q)| \leq \tfrac{1}{2}|\varepsilon|_M + \left(\tfrac{M_2}{2} + \tfrac{M_4 d^2}{96}\right)h^2\ .$$

Then

$$|\varepsilon|_M = max_{R_h}|\varepsilon(p)| \leq \tfrac{1}{2}|\varepsilon|_M + \left(\tfrac{M_2}{2} + \tfrac{M_4 d^2}{96}\right)h^2,$$

$$2|\varepsilon|_M \leq |\varepsilon|_M + \left(M + \tfrac{M_4 d^2}{48}\right)h^2.$$

Therefore,

$$|\varepsilon|_M \leq \left(M_2 + \frac{M_4 d^2}{48}\right) h^2. \tag{3.38}$$

3.2.3 The second order interpolation.

We can show an example of a finite difference analogue of (3.1) which fails to be of positive type at points of C_h^*.

Let $U(P)$ satisfy the system

$$\begin{aligned} &\Delta_h U(P) = F(P), \qquad P \in R_h\ , \\ &U(P) = f(P)\ , \qquad P \in C_h\ . \end{aligned} \tag{3.39}$$

At a point P of C_h^* let $U(P)$ lie on a parabola through value of $U(P)$ at a neighboring point of C_h and two points of $R_h + C_h^*$. All four points involved must of course be collinear. In addition we require one of the points of $R_h + C_h^*$ to be a neighbour of P and the other to be taken at a distance $3h$ from P. For example, for the point $(\bar{x}, \bar{y})$ in Figure (3.1).

$$U(\bar{x}, \bar{y}) = \frac{3}{3+\alpha(\alpha+4)}\Big\{U(\bar{x} - \alpha h, \bar{y}) + \frac{\alpha}{2}(\alpha + 3)U(\bar{x} + h, \bar{y}) - \frac{\alpha}{6}(\alpha + 1)U(\bar{x} + 3h, \bar{y})\Big\}. \tag{3.40}$$

From Taylor's formula it is easy to see that for a sufficiency smooth function $U(P)$ in R we have an inequality of the type

$$\left|u(\bar{x}, \bar{y}) - \frac{3}{3+\alpha(\alpha+4)}\Big\{u(\bar{x} - \alpha h, \bar{y}) + \frac{\alpha}{2}(\alpha + 3)u(\bar{x} + h, \bar{y}) - \frac{\alpha}{6}(\alpha + 1)U(\bar{x} + 3h, \bar{y})\Big\}\right| \leq \frac{14 h^3 M_3}{3}\ , \tag{3.41}$$

where $(\bar{x}, \bar{y}) \in C_h^*$ .In some cases the interpolation will be in the y direction.

Theorem 4. Let $u(x, y)$ be the solution of (3.1) and $U(x, y)$ the solution of (3.39) and (3.40). Then the truncation error $\varepsilon(P) = u(P) - U(P)$ satisfies

$$|\varepsilon|_M \leq \frac{d^2 M_4}{12} h^2 + \frac{112}{3} M_3 h^3 \ . \tag{3.42}$$

Proof: The proof follows in a manner analogous to that of Theorem 3.

We have the inequality

$$|\varepsilon(\bar{x}, \bar{y})| \leq \Big| u(\bar{x}, \bar{y}) - \frac{3}{3+\alpha(\alpha+4)} \{u(\bar{x} - \alpha h, \bar{y}) + \frac{\alpha}{2}(\alpha + 3) u(\bar{x} + h, \bar{y}) - \frac{\alpha}{6}(\alpha + 1) u(\bar{x} + 3h, \bar{y}) \Big| + \frac{7}{8} |\varepsilon|_M. \tag{3.43}$$

For the point $(\bar{x}, \bar{y})$ of Fig (1), it follows that

$$|\varepsilon(Q)| \leq \frac{14}{3} M_3 h^3 + \frac{7}{8} |\varepsilon|_M \quad , \tag{3.44}$$

where $Q \in C_h^*$.

The inequality follows,

$$|\varepsilon|_M - \frac{7}{8} |\varepsilon|_M = \frac{1}{8} |\varepsilon|_M \leq \frac{d^2 M_4}{96} h^2 + \frac{14 M_3}{3} h^3$$

$$|\varepsilon|_M \leq \frac{d^2 M_4}{12} h^2 + \frac{112}{3} M_3 h^3.$$

CHAPTER 4
HIGHER – ACCURATE SCHEMES

In this chapter we will consider two methods for the construction of sixth order approximation for the Dirichlet problem for Poisson's equation on rectangular domains. Moreover, we will define Discrete Green's function for the constructed sixth order difference operator to prove the sixth order convergence theorem in the maximum norm.

4.1 The Dirichlet Poisson Problem on the Rectangle

The structure of difference schemes for the numerical solution of Poisson problem with Dirichlet conditions on the rectangular sides is analyzed. We obtain the system of 9-point difference equations by using the 5-point stencils.

Let

$$R = \{(x, y): 0 < x < a, 0 < y < b\}$$

be an open rectangle γ^j , $j = 1,2,3,4$ be the sides of this rectangle including the vertices. Let the numbering be in counter clockwise direction starting from the side which lies on the x-axis.

The Dirichlet Poisson equation on a rectangle is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \; on \; R \; , \qquad (4.1)$$

$$u = \varphi^m on \; \gamma^m \quad , \quad m = 1,2,3,4 \; .$$

4.2The First Method

Let us draw two systems as shown in Figure (4.1) of parallel lines on the plane:

$$x = x_0 + ih = x_i \ ,$$
$$y = y_0 + kh = y_k \ . \qquad (4.2)$$

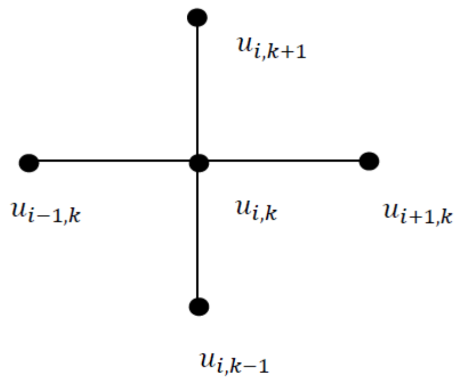

Figure 4.1: 5-point stencil

Consider the node (i, k) of the net, and take the nodes closest to it which are $(i + 1, k)$, $(i, k + 1)$, $(i - 1, k)$, $(i, k - 1)$, $(i + 1, k + 1)$, $(i + 1, k - 1)$, $(i - 1, k - 1)$, $(i - 1, k + 1)$ as shown in Figure (4.2), and expand them about the point $u_{i,k}$ using Taylor's formula. The expressions for the neighboring points of $u_{i,k}$ are as follows :

$$u_{i+1,k} - u_{i,k} = hu_x + \frac{h^2}{2!}u_{x^2} + \frac{h^3}{3!}u_{x^3} + \frac{h^4}{4!}u_{x^4} + \cdots$$
$$u_{i-1,k} - u_{i,k} = -hu_x + \frac{h^2}{2!}u_{x^2} - \frac{h^3}{3!}u_{x^3} + \frac{h^4}{4!}u_{x^4} + \cdots$$
$$u_{i,k+1} - u_{i,k} = hu_y + \frac{h^2}{2!}u_{y^2} + \frac{h^3}{3!}u_{y^3} + \frac{h^4}{4!}u_{y^4} + \cdots$$
$$u_{i,k-1} - u_{i,k} = -hu_y + \frac{h^2}{2!}u_{y^2} + \frac{h^3}{3!}u_{y^3} + \frac{h^4}{4!}u_{y^4} + \cdots \qquad (4.3)$$

$$u_{i+1,k+1} - u_{i,k} = h\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u + \frac{h^2}{2!}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u + \frac{h^3}{3!}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 u + \cdots$$

$$u_{i-1,k+1} - u_{i,k} = h\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u + \frac{h^2}{2!}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u + \frac{h^3}{3!}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 u + \cdots$$

$$u_{i-1,k-1} - u_{i,k} = h\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + \frac{h^2}{2!}\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \frac{h^3}{3!}\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^3 u + \cdots$$

$$u_{i+1,k-1} - u_{i,k} = h\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + \frac{h^2}{2!}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \frac{h^3}{3!}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^3 u + \cdots \quad (4.4)$$

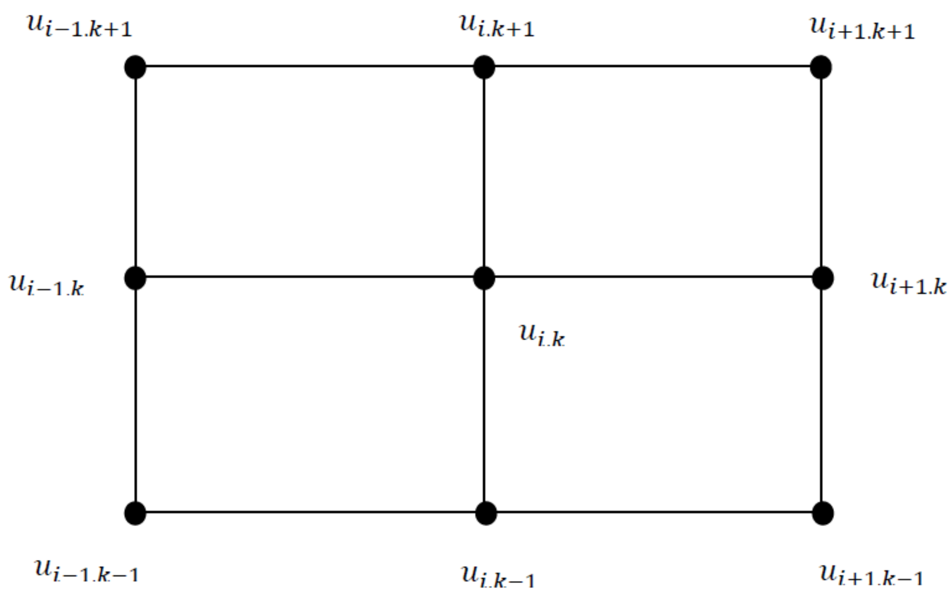

Figure 4.2 : 9-point stencil

With the above differences we form the sums $\boxdot\, u_{i,k}$ and $\boxplus\, u_{i,k}$ which gives

$$\boxdot\, u_{i,k} = u_{i+1,k} + u_{i,k+1} + u_{i-1,k} + u_{i,k-1} - 4u_{i,k} = 2\left[\frac{h^2}{2!}\left(u_{x^2} + u_{y^2}\right) + \frac{h^4}{4!}\left(u_{x^4} + u_{y^4}\right) + \frac{h^6}{6!}\left(u_{x^6} + u_{y^6}\right) + \cdots\right], \tag{4.5}$$

and

$$\boxplus\, u_{i,k} = u_{i+1,k+1} + u_{i-1,k+1} + u_{i-1,k-1} + u_{i+1,k-1} - 4u_{i,k} = 4\left[\frac{h^2}{2!}\left(u_{x^2} + u_{y^2}\right) + \frac{h^4}{4!}\left(u_{x^4} + 6u_{x^2y^2} + u_{y^4}\right) + \frac{h^6}{6!}\left(u_{x^6} + 15u_{x^4y^2} + 15u_{x^2y^4} + u_{y^6}\right) + \cdots\right]. \tag{4.6}$$

Finally we will look for the combination $c_1 \boxdot u_{i,k} + c_2 \boxplus u_{i,k}$ to get an approximate expression for Δu . There is no way to choose c_1 and c_2 such that the fourth order derivatives will vanish, however by choosing $c_1 = \frac{2}{3h^2}$ and $c_2 = \frac{1}{6h^2}$ the term with the fourth order derivatives form an operator

$$\Delta\Delta u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4},$$

which is known since $\Delta u = f(x,y)$ and $\Delta\Delta u = \Delta f(x,y)$.
Therefore we get the high accurate scheme

$$\frac{1}{6h^2}\left(4 \boxdot u_{i,k} + \boxplus u_{i,k}\right) = \Delta u + \frac{2h^2}{4!}\Delta^2 u + \frac{2h^4}{6!}\left(\Delta^3 u + 2\frac{\partial^4}{\partial x^2 \partial y^2}\Delta u\right) + R_{i,k} \quad ,$$

$$R_{i,k} = \frac{2}{3}\frac{h^6}{8!}\left[3\Delta^4 u + 16\frac{\partial^4}{\partial x^2 \partial y^2}\Delta^2 u + 20\frac{\partial^8 u}{\partial x^4 \partial y^4}\right] + \cdots \tag{4.7}$$

If we had expanded the equations (4.4) by Taylors formula with reminder term, by taking derivatives of up to the seventh order at the point (i,k), and derivatives of the eighth order at some mean points , including them in the reminder term of the formula , we obtain for $R_{i,k}$ an expression of the following type :

$$R_{i,k} = \frac{520h^6}{3 \cdot 8!} M_8 \, . \qquad (4.8)$$

Let $u_{i,k}$ be the point in Figure (4.1) and we define Δ_h to be the usual nine point operator there, i.e.

$$\Delta_h^{(9)} u_h \equiv \frac{1}{6h^2} \left[4 \sum_{i=1}^{4} u_i + \sum_{i=5}^{8} u_i - 20u_h \right] .$$

Hence

$$\left| \Delta_h^{(9)} u - \Delta u \right| \le \frac{520h^6}{3 \cdot 8!} M_8 \, . \qquad (4.9)$$

4.3 The Second Method

On the basis of the 5-point scheme, we can construct operators giving an error approximation of $O(|h^4|)$ or $O(|h^6|)$ for a solution within the square (cube) grid.

Consider $u = u(x)$ satisfiying the equation

$$\Delta w = \sum_{\alpha=1}^{P} \frac{\partial^2 u}{\partial x_\alpha^2} = -f(x) \qquad (4.10)$$

For $P = 2$ (2D case) we have

$$\Delta u = (L_1 + L_2)u = L_1 u + L_2 u \quad , \quad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2} \quad , \quad \alpha = 1,2 \, .$$

By appealing to the difference operator

$$\Lambda u = (\Lambda_1 + \Lambda_2)u = \Lambda_1 u + \Lambda_2 \mathrm{u} \ , \qquad \Lambda_\alpha u = u_{\bar{x}_\alpha x_\alpha} \ , \qquad \alpha = 1,2 \ .$$

Let $u = u(x)$ possess all necessary derivatives. So that

$$\Lambda u - Lu = \frac{h_1^2}{12} L_1^2 u + \frac{h_2^2}{12} L_2^2 u + O\left(\left|h^4\right|\right). \tag{4.11}$$

By the equation $L_1 u + L_2 u = -f(x)$ we find that

$$L_1^2 u = -L_1 f - L_1 L_2 u \ , \qquad L_2^2 u = -L_2 f - L_1 L_2 u \ .$$

In order that

$$\Lambda u = Lu - \frac{h_1^2}{12} L_1 f - \frac{h_2^2}{12} L_2 f - \frac{h_1^2 + h_2^2}{12} L_1 L_2 u + O\left(\left|h^4\right|\right) . \tag{4.12}$$

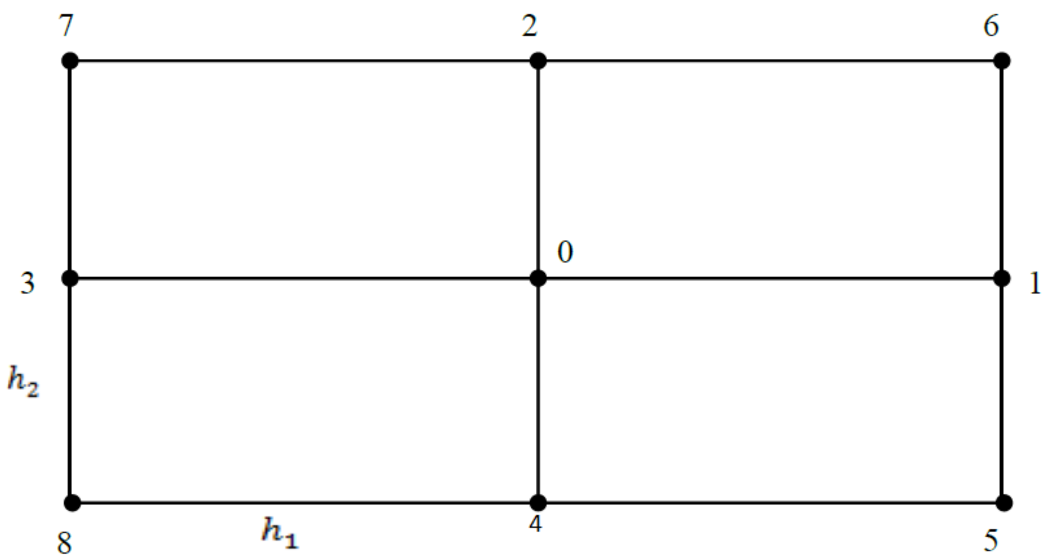

Figure 4.3 : 9-Point on Rectangle Domain

We substitute here $-f$ in place of Lu and change L_1L_2u by the difference operator,

$$\Lambda_1\Lambda_2 u = u_{\bar{x}_1 x_1 \bar{x}_2 x_2} - L_1L_2u = \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} .$$

This operator is defined on the 9-point pattern given in figure and we have $\Lambda_1\Lambda_2 u$, as follows,

$$\Lambda_1\Lambda_2 u = \Lambda_1 \left[\frac{u(x_1, x_2 - h_2) - 2u(x_1, x_2) + u(x_1, x_2 + h_2)}{h_2^2}\right]$$

$$= \frac{1}{h_1^2 h_2^2}\{u(x_1 - h_1, x_2 - h_2) - 2u(x_1, x_2 - h_2) + u(x_1 + h_1, x_2 - h_2) + 4u(x_1, x_2) - 2u(x_1 - h_1, x_2) + u(x_1 - h_1, x_2 + h_2) - 2u(x_1, x_2 + h_2) - 2u(x_1 + h_1, x_2) + u(x_1 + h_1, x_2 + h_2)\} .$$

Is required within the estimation of the error of approximation to $\Lambda_1\Lambda_2 u - L_1L_2u$ through advantage of the good-established expansion

$$\Lambda r = r_{\bar{x}x} = \frac{r(x+h) - 2r(x) + r(x-h)}{h^2} r(\lambda) , \quad \lambda = x + \theta h \ , \quad |\theta| \leq 1 . \qquad (4.13)$$

Suppose that $r(x) \in C^2[x - h, x + h]$, so that

$$\Lambda r = r_{\bar{x}x} = r''(x) + \frac{h^2}{12} r^{(4)}(\lambda^*) , \qquad \lambda^* = x + \theta^* h , \quad |\theta^*| \leq 1 , \qquad (4.14)$$

$$r(x) \in C^4[x - h, x + h] .$$

By taking x_1 to be fixed we have

$$\Lambda_2 r = L_2 r(x_1, x_2) + \frac{h_2^2}{12}\frac{\partial^4 r}{\partial x_2^4}(x_1, \lambda_2) \ , \quad \lambda_2 = x_2 + \theta_2 h_2 \ , \quad |\theta_2| \leq 1 \ ,$$

$$\Lambda_1\Lambda_2 u(x_1,x_2) = \Lambda_1 L_2 u(x_1,x_2) + \tfrac{h_2^2}{12}\Lambda_1 \tfrac{\partial^4 u}{\partial x_2^4}(x_1,\lambda_2) \ \ .$$

Applying equation (4.14) with $r = L_2 u$ and $x = x_1$ to the first summand yields

$$\Lambda_1 L_2 u(x_1,x_2) = L_1 L_2 u(x_1,x_2) + \tfrac{h_1^2}{12}\Lambda_1 \tfrac{\partial^4 u}{\partial x_1^4}(\lambda_1^*,x_2)\ ,\quad \lambda_1^* = x_1 + \theta_1^* h_1\ \ ,$$
$$|\theta_1^*| \le 1.$$

By the similar method for the second summand with respect to equation (4.12)

$$\tfrac{h_2^2}{12}\Lambda_1 \tfrac{\partial^4 u}{\partial x_2^4}(x_1,\lambda_2) = \tfrac{h_2^2}{12}\Lambda_1 \tfrac{\partial^6 u}{\partial x_1^2 \partial x_2^4}(\lambda_1,\lambda_2),\quad \lambda_1 = x_1 + \theta_1 h_1\ \ ,\ \ |\theta_2| \le 1\ \ .$$

What must be done is to bring together the outcomes acquired:

$$(\Lambda_1\Lambda_2 - L_1 L_2)u(x_1,x_2) = \Lambda_1\Lambda_2 u(x_1,x_2) - L_1 L_2 u(x_1,x_2) = O\left(h_1^2\right) +$$
$$O\left(h_2^2\right) = O(|h|^2)\ .$$

Substituting into equation (4.12) the difference operator $\Lambda_1\Lambda_2 u$ into place of $L_1 L_2 u$,

$$L_1 L_2 u = \Lambda_1\Lambda_2 u + O(|h|^2)\ ,$$

and $-f(x)$ into place of Lu , we finally obtain

$$\Lambda u = Lu - \frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 u - \frac{h_1^2}{12}L_1 f - \frac{h_2^2}{12}L_2 f + O\left(\left|h^4\right|\right)$$
$$= \left(f + \tfrac{h_1^2}{12}L_1 f + \tfrac{h_2^2}{12}L_2 f\right) - \tfrac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 u + O\left(\left|h^4\right|\right). \qquad (4.15)$$

Since, the equation

$$\Lambda^{'} y = -\phi \ , \quad \Lambda^{'} y = \Lambda y + \frac{h_1^2 + h_2^2}{12} \Lambda_1 \Lambda_2 y \ ,$$

$$\phi = f + \frac{h_1^2}{12} L_1 f + \frac{h_2^2}{12} L_2 f \ , \tag{4.16}$$

provides an approximation of order 4 for a solution $u = u(x)$ of Poisson's equation (4.10). In fact, equation (4.15) gives

$$\Lambda^{'} u + \phi = \Lambda^{'} u + \phi - Lu - f = O(|h^4|) \ , \qquad L = L_1 + L_2 \ .$$

The operator $\Lambda^{'}$ formed using the nodes in Figure (4.3) $(x_1 + m_1 h_1, x_2 + m_2 h_2)$; $m_1, m_2 = -1,0,1$, and used in (4.16) is represented by

$$\frac{5}{3}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right) u = \frac{1}{6}\left(\frac{5}{h_1^2} - \frac{1}{h_2^2}\right)(u^{+1_1} + u^{-1_1}) + \frac{1}{6}\left(\frac{5}{h_2^2} - \frac{1}{h_1^2}\right)(u^{+1_2} + u^{-1_2}) +$$

$$\frac{1}{12}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right)\left(u^{(+1_1,+1_2)} + u^{(-1_1,-1_2)}\right) + \left(u^{(-1_1,-1_2)} + u^{(-1_1,+1_2)}\right) + \varphi. \tag{4.17}$$

Here,

$$u^{+1_1} = u(x_1 + h_1, x_2) \ , u^{-1_1} = u(x_1 - h_1, x_2) \ , u^{(+1_1,-1_2)} = u(x_1 + h_1, x_2 - h_2).$$

When the equidistant grid is considered in all directions (If $h_1 = h_2 = h$) the equation is obtained as :

$$\frac{5}{3}\cdot\frac{2}{h^2}u_h(x,y)=\frac{1}{6}\left(\frac{4}{h^2}\right)\left(u_h(x+h,y)+u_h(x-h,y)\right)+\frac{1}{6}\left(\frac{4}{h^2}\right)\left(u(x,y+h)+\right.$$
$$\left.u_h(x,y-h)\right)+\frac{1}{12}\left(\frac{2}{h^2}\right)\left(u_h(x+h,y+h)+u_h(x-h,y-h)\right)+$$
$$\left(u_h(x-h,y-h)+u_h(x-h,y+h)\right)+\varphi\ .$$

$$\frac{2}{3h^2}\left(u_h(x+h,y)+u_h(x-h,y)+u_h(x,y+h)+u_h(x,y-h)\right)+$$
$$\frac{1}{6h^2}\left(u(x+h,y+h)+u_h(x+h,y-h)+u_h(x-h,y-h)+u_h(x-h,y+\right.$$
$$\left.h)\right)-\frac{10}{3h^2}u_h(x,y)+\varphi=0\ .$$

Therefore,

$$u_0=\frac{4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8}{20}+\frac{3}{10}h^2\phi\ .$$

(See Figure 4.1)

To avoid exhaustive computations, we put $\Lambda_1 f$ in place of $L_1 f$ and $\Lambda_2 f$ in place of $L_2 f$ into the equation of ϕ and replace ϕ by $O\left(\left|h^4\right|\right)$, as $\psi=\Lambda' u+\phi=O\left(\left|h^4\right|\right)$, so that

$$\phi=f+\frac{h_1^2}{12}\Lambda_1 f+\frac{h_2^2}{12}\Lambda_2 f.$$

4.4 Discrete Green's Function for the Six Order Error Estimation

From the sections (4.2) and (4.3) follows:

$$\left|\Delta_h^{(9)}u(x,y)-\Delta u(x,y)\right|\le\frac{520h^6}{3\cdot 8!}M_8. \qquad (4.18)$$

For the Dirichlet problem for Poisson's equation we have the following finite difference problem

$$\Delta_h^{(9)}U(x,y) = F(x,y), \qquad (x,y) \in R_h \quad ,$$
$$U(x,y) = f(x,y), \qquad (x,y) \in C_h \; . \tag{4.19}$$

Lemma 1. For any mesh function $V(P)$ defined on $R_h + C_h$ if $\Delta_h^{(9)}V(P) \geq 0$ for $P \epsilon R_h$ then $V(P)$ takes on its maximum on C_h.

From Lemma 1 follows that the solution of problem (4.19) exists and unique.

We define the Finite difference analogue of Green's function $G_h(P,Q)$ as

$$\Delta_{h,P}^{(9)}G_h(P,Q) = -\delta(P,Q)h^{-2} \quad , \qquad P \epsilon R_h \, ,$$
$$G_h(P,Q) = \delta(P,Q) \qquad , \qquad P \in C_h \; , \tag{4.20}$$

for $\quad Q \in R_h + C_h$, and

$$\delta(P,Q) = \begin{cases} 1 & , \quad P = Q \, , \\ 0 & , \quad P \neq Q \, . \end{cases} \tag{4.21}$$

Lemma 2. (Green's Third Identity).
Let $V(P)$ be any arbitrary mesh function defined on $R_h + C_h$. Then for any $P \in R_h + C_h$

$$V(P) = h^2 \textstyle\sum_{Q \in R_h} G_h(P,Q)[-\Delta_h^{(9)}V(Q)] + \sum_{Q \in C_h} G_h(P,Q)V(Q) \, . \tag{4.22}$$

Proof: Let $P \in R_h$ and let

$$W(P) = h^2 \textstyle\sum_{Q \in R_h} G_h(P,Q)[-\Delta_h^{(9)}V(Q)] + \sum_{Q \in C_h} G_h(P,Q)V(Q).$$

We calculate

$$\Delta_h^{(9)} W(P) = h^2 \Delta_h G_h(P,P)[\Delta_h^{(9)} V(P)]$$
$$= h^2 \cdot \left(-h^{-2}\right)(-\Delta_h V(P))$$
$$= \Delta_h V(P)\ .$$

Let $P \in C_h$ then $W(P) = G_h(P,P)V(P) = V(P)$.

Lemma 3. $\qquad G_h(P,Q) \geq 0 \ , \quad Q \in R_h + C_h\ .$ (4.23)

Proof: From (4.20), it follows that

$$\Delta_{h,P}^{(9)}(-G_h(P,Q)) = \delta(P,Q)h^{-2} \geq 0 \quad \text{on} \quad R_h\ ,$$
$$-G_h(P,Q) = -\delta(P,Q) \leq 0 \qquad \text{on} \quad C_h\ .$$

By Lemma 1, the function $G_h(P,Q)$ can obtain its maximum on C_h .
Hence

$$-G_h(P,Q) \leq 0\ , \qquad P \in R_h\ ,$$

or

$$G_h(P,Q) \geq 0$$

Lemma 4. If d is the diameter of the smallest circumscribed circle containing R then

$$h^2 \sum_{Q\in R_h} G_h(P,Q) \leq \frac{d^2}{16} \quad , \quad P \in R_h + C_h \quad . \tag{4.24}$$

Proof: Let 0 be the center of the circumscribed circle about R of diameter d, and let for any $P = P(x_1, x_2) \in R_h + C_h$,

$$W(P) = \frac{x_1^2 + x_2^2}{4}.$$

Then

$$\begin{aligned}\Delta_h^{(9)} W = \frac{1}{6h^2}[4((x_1 + h)^2 + x_2^2 + x_1^2 + (x_2 + h)^2 + (x_1 - h)^2 + x_2^2 + x_1^2 \\ + (x_2 - h)^2) + (x_1 + h)^2 + (x_2 + h)^2 + (x_1 - h)^2 \\ + (x_2 + h)^2 + (x_1 - h)^2 + (x_2 - h)^2 + (x_1 + h)^2 \\ + (x_2 - h)^2 - 20(x_1^2 + x_2^2)]\end{aligned}$$

$$\begin{aligned}= \frac{1}{6h^2}\Big[4\big(x_1^2 + 2x_1h + h^2 + x_2^2 + x_1^2 + x_2^2 + 2x_2h + h^2 + x_1^2 - 2x_1h + h^2 \\ + x_2^2 + x_1^2 + x_2^2 - 2x_2h + h^2\big) + x_1^2 + 2x_1h + h^2 + x_2^2 \\ + 2x_2h + h^2 + x_1^2 - 2x_1h + h^2 + x_2^2 + 2x_2h + h^2 + x_1^2 \\ - 2x_1h + h^2 + x_2^2 - 2x_2h + h^2 + x_1^2 + 2x_1h + h^2 + x_2^2 \\ - 2x_2h + h^2 - 20(x_1^2 + x_2^2)\Big]\end{aligned}$$

$$= \frac{1}{6h^2}\left[16x_1^2 + 16x_2^2 + 16h^2 + 4x_1^2 + 4x_2^2 + 8h^2 - 20(x_1^2 + x_2^2)\right]$$

$$= \frac{1}{6h^2}\left[24h^2\right] = 4\,.$$

So that

$$\Delta_h^{(9)} \frac{x_1^2 + x_2^2}{4} = 1\,.$$

Now define the mesh function

$$V(P) = h^2 \sum_{Q \in R_h} G_h(P,Q) \,.$$

We see from (4.20) that

$$\Delta_h^{(9)} V(P) = -1 \;\; , \;\; P \in R_h \quad (\text{as } \Delta_h^{(9)} V(P) = h^2 \textstyle\sum_{Q \in R_h} G_h(P,Q) = 1)$$
$$V(P) = 0 \qquad , \quad P \in C_h$$

Hence

$$[\Delta_h^{(9)} V(P) + \; W] = 0 \quad \text{For} \;\; P \in R_h$$

And

$$V(P) + W(P) \le \frac{d^2}{16} = \frac{r(P)^2}{4} = \frac{(^d/_2)^2}{4} \quad \text{for} \quad P \in C_h \;\; .$$

By the maximum principle, since $W \ge 0$, it follows that

$$V(P) \le \frac{d^2}{16} \quad , \quad P \in R_h + C_h \,.$$

i.e.

$$h^2 \textstyle\sum_{Q \in R_h} G_h(P,Q) \le \frac{d^2}{16} \quad , \quad P \in R_h + C_h$$

Theorem 1. Let $u(x,y)$ be the solution of (3.1) and $U(x,y)$ the solution of (4.19). Then the truncation error $\varepsilon(P) = u(P) - U(P)$ satisfies the inequality

$$|\varepsilon|_M \le \frac{65\, d^2 M_8}{6 \cdot 8!} h^6 \tag{4.25}$$

Proof: Since $\varepsilon(P) = 0$, $P \in C_h$ we see from Lemma 2 that

$$\varepsilon(P) = h^2 \sum_{Q \in R_h} G_h(P,Q)[-\Delta_h^{(9)} \varepsilon(Q)] \, .$$

As

$$\left|-\Delta_h^{(9)} \varepsilon(Q)\right| = |\Delta_h u(Q) - \Delta u(Q)|,$$

since we have

$$\left|\Delta_h^{(9)} u(x,y) - \Delta u(x,y)\right| \le \frac{520 h^6}{3 \cdot 8!} M_8 \, .$$

Therefore,

$$|\varepsilon(P)| = h^2 \sum_{Q \in R_h} G_h(P,Q)\left[-\Delta_h^{(9)} \varepsilon(Q)\right]$$

$$\le \left| h^2 \sum_{Q \in R_h} G_h(P,Q) \right| \frac{520\, h^6 M_8}{3 \cdot 8!}$$

$$\le \frac{d^2}{16}\left(\frac{520\, h^6 M_8}{3 \cdot 8!}\right)$$

$$\le \frac{65\, d^2 M_8}{6 \cdot 8!} h^6.$$

CHAPTER 5

CONCLUSION (RESULTS)

In this thesis, we have discussed the finite-difference approximation of elliptic equations, and we obtained some more estimates of the type suggested by Gershgorin. Here we take the approaches to define a related finite difference Green's function for various finite difference analogues. The analogue of Green's third identity is given in each case and used to obtain estimates for the truncation error.

When the boundary value problem is defined on a rectangular domain by discrete Green's function method to obtain effective error estimations are analyzed.

In the case of problem on domains with curved boundaries by discrete Green's function method, when different type of interpolation formula on the irregular grids are used, the first and the second order error estimations are obtained.

Furthermore, Bramble and Hubbard (1962), by constructing fourth order interpolation in irregular grids and using 9-point approximation on square regular grids by using discrete Green's function method obtained $O(h^4)$ order of estimation.

In this thesis, when solution domain is a rectangle we have used the 9-point approximation on square grid, and by applying Green's function method we obtain $O(h^6)$ order of uniform convergence of the approximate solution.

To extend this result for the problem on the domain with curved boundary in the irregular grids higher order than Bramble and Hubbard's (1962) formula is needed.

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