A COMPREHENSIVE STUDY ON SYSTEM OF q −DIFFENCE EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

LUQMAN YUSUF

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2017

LUQMAN YUSUF, A COMPREHENSIVE STUDY ON SYSTEM OF q − DIFFERENCE EQUATION, 2017

A COMPREHENSIVE STUDY ON SYSTEM OF q −DIFFENCE EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
LUQMAN YUSUF

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2017

LUQMAN YUSUF: A COMPREHENSIVE STUDY ON SYSTEM OF q −DIFFERENCE EQUATION

Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify that, this thesis is satisfactory for the award of the degree of Master of Sciences in Mathematics.

Examining Committee in Charge:

Prof. Dr. Allaberen Ashyralyev, Committee Chairman, Department of Mathematics, Near East University

Assist. Prof. Dr. Mohammad Momenzadeh, Supervisor, Department of Mathematics, Near East University

Assoc. Prof. Dr. Suzan Cival Buranay, Department of Mathematics, Eastern Mediterranean University

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Luqman, Yusuf

Signature:

Date:

ACKNOLEDGMENTS

My profound gratitude goes to Almighty Allah, my creator and sustainer for being my strength and protector throughout the period of my master's program, without him nothing could be possible from my side.

Furthermore, I want to express my sincere appreciation to my able and intellectual supervisor Assist. Prof. Mohammad Momenzadeh for his fatherly advice, assistance, guidance and thorough supervision; your kind attitude, motivations, patience and keenness toward my supervision are what made me to successfully complete this thesis. I say thank you sir and really appreciate you efforts.

However, I cannot close this section without acknowledging my sponsor: Kaduna State Government, Nigeria under the leadership Dr. Muktar Ramalan Yero, I say thank you and Almighty reward you abundantly.

Moreover, to my able and tireless parent, siblings, relatives, friends and course mates, I have no word that I can express my appreciations to you. With deep sincerity I really appreciate your supports, meaningful criticisms throughout my stay here. May almighty Allah bless and grant you all your heart desires.

To my family...

ABSTRACT

The thesis introduced the preliminaries of q −calculus which is the base of q −difference equation. We further considered the Picard's existence and uniqueness theorem for ordinary differential equations for which is the base for q −analogue of this theorem. We therefore define system of q-difference equation and detailed proofs of theorems for first order system of q − difference equation and the Cauchy problem are provided. At the end, we work on a special case of q-Cauchy problem and later extend this problem to the nth order. The second order of this q-difference equation is studied by the several mathematician, we therefore extend this problem to the general form at the last chapter.

Keywords: q-calculus; Jackson Integral; Existence and uniqueness of solutions for differential equation; system of q −difference equation; successive approximation; q −Cauchy problem with boundary values

ÖZET

Tez, q-fark denkleminin temeli olan q-hesabı öncüllerini tanıtmıştır. Ayrıca, Picard'ın bu teoremin q −analogu için temel olan adi diferansiyel denklemler için varoluş ve teklik teorisi encelendi. Bu nedenle, q −fark denkleminin sistemini tanımlıyoruz ve q −fark denkleminin birinci dereceden sistemi için teoremlerin ayrıntılı delilleri ve Cauchy problemi sunuluyor. Sonuçta, özel bir q −Cauchy problemi üzerinde çalışıyoruz ve daha sonra bu sorunu n. Sınıfa kadar genişlettik. Bu q −fark denkleminin ikinci derecesi birkaç matematikçi tarafından incelendiğinden, bu problemi son bölümde genel forma genişletiyoruz.

Anahtar Kelimeker: q −calculus; Jackson Integral; Diferansiyel denklem için özümlerin varlığı ve özgünlüğü; q −fark denklemi sistemi; ardışık yaklaşım; sınır değerli q − Cauchy problemi

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

BVP: Boundary Value Problem

IVP: Initial Value Problem

ODE: Ordinary Differential Equation

LIST OF SYMBOLS

$\mathbb{C}$	Set of Complex Number
$\mathbb{C}^+$	Set of Positive Complex Number
$\boldsymbol{C}[\boldsymbol{a},\boldsymbol{b}]$	Set of continuous function
$\boldsymbol{d_q}$	$q-$ Differential
$\boldsymbol{d_h}$	$h-$ Differential
$\boldsymbol{D_q}$	$q-$ Derivative
$\boldsymbol{D_h}$	$h-$ Derivative
$\mathbb{N}$	Set of Natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\boldsymbol{p}$	Number of functions
$\mathbb{R}$	Set of Real Number
$\mathbb{R}^+$	Set of Positive Real Number
$\mathbb{Z}$	Set of Integers
$\mathbb{Z}^+$	Set of Positive Integers
$\|\cdot\|$	Norm function

CHAPTER 1
INTRODUCTION

Much research on Ordinary Calculus has being carried-out by various scholars in different fields of studies. This result to evolvement of a new concept of calculus called the Quantum Calculus (q - calculus).

As we earlier knew that the Ordinary Calculus encompasses many terminologies and definition(s) of such terminologies, likewise each terminology in Quantum Calculus has its own definition and representation which is called "q- analogue of the term", (Kac and Cheung, 2002). However, we shall discuss the terminologies vis-`a-vis the q-calculus in detail in this thesis before we add some new concepts on the field. On the other hand, to have deep knowledge on the field, one needs to know the following terminologies.

Now, consider the below mathematical expression:

$$\frac{f(x) - f(x_0)}{x - x_0}. \tag{1.1}$$

We knew that the limit of the above expression as x tends to x_0 if it exist give us the ordinary definition of the derivative $\frac{df}{dx}$ of a given function $f(x)$ at $x = x_0$. Now, suppose we substitute $x = qx_0$ or $x = x_0 + h,$ where q is a fixed number other than 1, h be a fixed number distinct from 0, and we do not take the limit, then this lead us to the fascinating world of the Quantum Calculus. However, the corresponding expressions are what we called the definition of q-derivative in relation to q-Calculus and h-derivative in relation h-calculus respectively (Kac and Cheung, 2002).

Been stated above of the two types of Quantum Calculus, that is (the q-Calculus and the h-calculus), in the course of developing the field along with the traditional lines of ordinary calculus some important expressions, equations and results were discovered in the different fields of mathematics. Examples of such of the fields are combinatorics, number theory, and other fields which we shall later discuss the discoveries made and prove some of the results found in detail.

Furthermore, due to some similarities of this field with the ordinary calculus, one need not to disturb himself or herself cogitating on the field. The most important thing for an enthusiastic student of this branch of mathematics is to revise his/ her ordinary calculus.

1.1 q-Derivative and h-Derivative

As we mentioned earlier of the two types of Quantum Calculus, we now begin with the definitions of the terms associated with each type.

1.1.1 Quantum Differentials

Definition 1.1.1. Ernst (2002); suppose $f(x)$ is an arbitrary function defined on the set of real numbers. Then the q- differential of $f(x)$ is defined as:

$$d_q f(x) = f(qx) - f(x) \tag{1.2}$$

And its h-differential is:

$$d_h f(x) = f(x+h) - f(x) \tag{1.3}$$

For instance, suppose $f(x) = x$. Then $d_q x = (q-1)x$ and $d_h x = h$, results from the (1.2) and (1.3) as:

$$d_q x = qx - x = (q-1)x,$$

and

$$d_h x = x + h - x = h.$$

1.1.2 Quantum Differential of Product of Two Functions

Proposition 1.1.1. Kac and Cheung (2002); let $f(x)$ and $g(x)$ be arbitrary functions defined on R. Then the q- differentials of the product of $f(x)$ and $g(x)$ are as:

$$d_q\big(f(x)g(x)\big) = g(qx)d_q f(x) + f(x)d_q g(x) \tag{1.4}$$

$$d_q\big(f(x)g(x)\big) = f(qx)d_q g(x) + g(x)d_q f(x) \tag{1.5}$$

Proof. Kac and Cheung (2002); consider $d_q\big(f(x)g(x)\big) = f(qx)g(qx) - f(x)g(x)$

$$= f(qx)g(qx) - f(x)g(qx) + f(x)g(qx) - f(x)g(x),$$

we have:

$$d_q\big(f(x)g(x)\big) = g(qx)[f(qx) - f(x)] + f(x)[g(qx) - g(x)]$$

From the above equations, it implies;

$$d_q\big(f(x)g(x)\big) = g(qx)d_q f(x) + f(x)d_q g(x) \text{ from (1.2).}$$

Similarly, suppose we expressed $d_q\big(f(x)g(x)\big) = f(qx)g(qx) - f(x)g(x)$ to be equals to:

$$f(qx)g(qx) - f(qx)g(x) + f(qx)g(x) - f(x)g(x),$$

Then, one can easily show that

$$d_q\big(f(x)g(x)\big) = f(qx)g(qx) - f(x)g(x)$$

$$= f(qx)d_q g(x) + g(x)d_q f(x)$$

Furthermore, the h-differentials of product of $f(x)$ and $g(x)$ can also be found in similar way as its counterpart.

Proposition 1.1.2. Kac and Cheung (2002); let $f(x)$ and $g(x)$ be arbitrary functions defined on R. Then the h-differentials of the product of $f(x)$ and $g(x)$ are:

$$d_h\big(f(x)g(x)\big) = g(x+h)d_h f(x) + f(x)d_h g(x) \tag{1.6}$$

$$d_h\big(f(x)g(x)\big) = f(x+h)d_h g(x) + g(x)d_h f(x) \tag{1.7}$$

Proof. Consider $d_h\big(f(x)g(x)\big) = f(x+h)g(x+h) - f(x)g(x)$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x),$$

we have,

$$d_h\big(f(x)g(x)\big) = g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]$$

From the above equations, it implies;

$$d_h\big(f(x)g(x)\big) = g(x+h)d_h f(x) + f(x)d_h g(x) \text{ from equation (1.3).}$$

Similarly, suppose we expressed $d_h\big(f(x)g(x)\big) = f(x+h)g(x+h) - f(x)g(x)$ to be equals to:

$$f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x),$$

then, one can easily show that

$$\begin{aligned} d_h\big(f(x)g(x)\big) &= f(x+h)g(x+h) - f(x)g(x) \\ &= f(x+h)d_h g(x) + g(x)d_h f(x). \end{aligned}$$

From the above proofs of equations (1.4), (1.5), (1.6) and (1.7), one can easily see the lack of symmetry in the differential of the product of two functions in quantum calculus; unlike the ordinary calculus whereby the differential of the product of two functions are symmetric.

However, by considering the definitions of q-differentials and the h-differentials we can now define the corresponding quantum derivative of each as the follows.

Definition 1.1.2. Let $f(x)$ be an arbitrary functions defined on R. Then the q-derivative of $f(x)$ is defined as:

$$D_q f(x) = \begin{cases} \lim_{x\to 0} D_q f(x), & x = 0 \\ \frac{d_q f(x)}{d_q x} = \frac{f(qx)-f(x)}{(q-1)x}, & x \neq 0, q \neq 1 \end{cases} \tag{1.8}$$

Similarly, the h-derivative of $f(x)$ is defined as:

$$D_h f(x) = \frac{d_h f(x)}{d_h x} = \frac{f(x+h)-f(x)}{h}, \qquad h \neq 0 \tag{1.9}$$

Note that:

$$\lim_{q\to 1} D_q f(x) = \lim_{h\to 0} D_h f(x) = \frac{df(x)}{dx}$$

whenever the function $f(x)$ is differentiable. By considering the Leibniz notation $\frac{df(x)}{dx}$ which is a ratio of two ‘‘infinitesimals’’, it is difficult to understand it because the notion of the differential $df(x)$ needs detailed explanation. But on the other hand, the notion of $d_q f(x)$ and $d_h f(x)$ are obvious, and $D_q f(x)$ and $D_h f(x)$ are plain ratios.

Properties of D_q and D_h Operators

Let $f(x)$ be an arbitrary function and a, b be any constants. Then we have:

$$D_q\big(af(x) + bg(x)\big) = aD_q f(x) + bD_q f(x) \tag{1.10}$$

$$D_h\big(af(x) + bgf(x)\big) = aD_h f(x) + bD_h f(x) \tag{1.11}$$

From the above equations (1.10) and (1.11), one can easily see that the two operators are linear operators.

Example: If $f(x) = x^n$, for $0 < n \epsilon Z$, then one can easily compute the q-derivative and h- derivative of the given function using equations (1.8) and (1.9).

That is to say:

$$D_q x^n = \frac{(qx)^n - x^n}{(q-1)x} = \frac{q^n x^n - x^n}{(q-1)x} = \frac{q^n - 1}{q-1} x^{n-1}, \tag{1.12}$$

By letting

$$[n]_q = \frac{(q^n - 1)}{(q-1)} = q^{n-1} + \cdots + 1 \tag{1.13}$$

for $0 < n \epsilon Z$, this is called the *q-anologue of n*. And it implies that equation (1.12) becomes $D_q x^n = [n]_q x^{n-1}$ which looks like the ordinary derivative of x^n. As $q \to 1$, $[n]_q \to 1 + 1 + \cdots + 1 = n$.

Similarly,

$$D_h x^n = \frac{(x+h)^n - x^n}{h}$$

By using binomial expansion on $\frac{(x+h)^n - x^n}{h}$, we can express $D_h x^n$ as follows:

$$D_h x^n = \frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)x^{n-2}h}{2} + \cdots + h^{n-1} \tag{1.14}$$

However, before we proceed to the derivatives of product of two functions, it is important to note that this thesis will mainly focus on q-calculus.

1.1.3 Quantum Derivative of Product of Two Functions

Proposition 1.1.3. Let $f(x)$ and $g(x)$ be two arbitrary functions defined on R. Suppose that $0 < q < 1$. Then from (1.4), (1.5) and (1.8), the q-derivative of the product of $f(x)$ and $g(x)$ are

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x) \tag{1.15}$$

$$D_q\big(f(x)g(x)\big) = f(x)D_q g(x) + g(qx)D_q f(x) \tag{1.16}$$

(Mansour and Annaby, 2012)

Proof. By considering the left hand side of (1.15), it implies

$$D_q\big(f(x)g(x)\big) = \frac{d_q\big(f(x)g(x)\big)}{d_q x} = \frac{f(qx)d_q g(x) + g(x)d_q f(x)}{(q-1)x}, \quad q \neq 1, x \neq 0.$$

and hence,

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x)$$

By symmetry, one can interchange f and g, and obtain (1.16)

$$D_q\big(f(x)g(x)\big) = f(x)D_q g(x) + g(qx)D_q f(x)$$

which is equivalent to (1.15).

However, we can apply (1.15) and (1.16) to derive the q-derivative of $\frac{f(x)}{g(x)}$ using some technics.

1.1.4 Quantum Derivative of Quotient of Two Functions

Proposition 1.1.4. Ernst (2002); let $f(x)$ and $g(x)$ be two arbitrary functions defined on R. Then from (1.15) and (1.16), the q-derivative of the quotient of $f(x)$ and $g(x)$ are:

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}, \quad g(x)g(qx) \neq 0\,, \tag{1.17}$$

and

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}, \quad g(x)g(qx) \neq 0 \tag{1.18}$$

The above proposition can be prove by applying (1.15) and (1.16) to differentiate $f(x) = g(x).\frac{f(x)}{g(x)}$, where $g(x) \neq 0$.

Proof. Kac and Cheung (2002); consider $f(x) = g(x).\frac{f(x)}{g(x)}$, by applying (1.15) we have

$$D_q f(x) = D_q\left(g(x).\frac{f(x)}{g(x)}\right).$$

From (1.15), we have

$$D_q f(x) = D_q\left(g(x).\frac{f(x)}{g(x)}\right) = g(qx)D_q\left(\frac{f(x)}{g(x)}\right) + \left(\frac{f(x)}{g(x)}\right)D_q\big(g(x)\big).$$

It implies

$$D_q f(x) - \left(\frac{f(x)}{g(x)}\right)D_q\big(g(x)\big) = g(qx)D_q\left(\frac{f(x)}{g(x)}\right).$$

By dividing both sides by $g(qx)$ we have

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{D_q f(x)}{g(qx)} - \left(\frac{f(x)}{g(x)g(qx)}\right)D_q(g(x))$$

$$= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}$$

Similarly, using (1.16) we can obtain the (1.18) by considering the same function $f(x) = g(x).\frac{f(x)}{g(x)}$.

This means that; $D_q f(x) = D_q\left(g(x).\frac{f(x)}{g(x)}\right)$

From (1.16), we have;

$$D_q f(x) = D_q\left(g(x).\frac{f(x)}{g(x)}\right) = g(x)D_q\left(\frac{f(x)}{g(x)}\right) + \left(\frac{f(qx)}{g(qx)}\right)D_q(g(x))$$

It implies;

$$D_q f(x) - \left(\frac{f(qx)}{g(qx)}\right)D_q(g(x)) = g(x)D_q\left(\frac{f(x)}{g(x)}\right)$$

By dividing both sides by $g(x)$ we have;

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{D_q f(x)}{g(x)} - \left(\frac{f(qx)}{g(qx)g(x)}\right)D_q(g(x))$$

$$= \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}$$

1.1.5 Quantum Version of the Chain Rule

Not there exists a general chain rule for q- differentiation except for a function that takes the form$f(u(x))$, where $u = u(x) = \alpha x^{\beta}$ with α, β being constants. We can demonstrate how the role applies by considering $D_q[f(u(x))] = D_q[f(\alpha x^{\beta})]$.

One can easily see that $D_q[f(u(x))] = D_q[f(\alpha x^{\beta})] = \frac{f(\alpha q^{\beta} x^{\beta}) - f(\alpha x^{\beta})}{qx - x}$

$$= \frac{f(\alpha q^{\beta} x^{\beta}) - f(\alpha x^{\beta})}{\alpha q^{\beta} x^{\beta} - \alpha x^{\beta}} . \frac{(\alpha q^{\beta} x^{\beta}) - (\alpha x^{\beta})}{qx - x}$$

$$= \frac{f(q^{\beta} u) - f(u)}{q^{\beta} u - u} . \frac{u(qx) - u(x)}{qx - x},$$

and hence,

$$D_{q^{\beta}} f(u(x)) = (D_{q^{\beta}} f)(u(x)). D_q u(x). \tag{1.19}$$

1.2 q-Taylor's Formula for Polynomials

As we knew in ordinary calculus that if a function $f(x)$ is analytic at $x = a$, then it possesses a Taylor's series which is given as:

$$f(x) = \sum_{n=0}^{\infty} f^n(a) \frac{(x-a)^n}{n!}. \tag{1.20}$$

Likewise, in quantum calculus such series exist but in different form. Before we state the q- Taylor's formula for polynomial we firstly begin with the definition of the following important terms as defined in (Kac and Cheung, 2002) and (Momenzadeh and Mahmudov, 2014).

1.2.1 q-analogue of $n!$

Let $n \in Z^+ \cup \{0\}$, the q-analogue of $n!$ is defined as:

$$[n]_q! = \begin{cases} 1 & if\ n = 0 \\ [n]_q \times [n-1]_q \times \cdots \times [1]_q & if\ n = 1,2, \dots . \end{cases} \tag{1.21}$$

1.2.2 q-analogue of $(x-a)^n$ for $n \geq 0$

Let $n \in Z^+ \cup \{0\}$, the q-analogue of $(x-a)^n$ is a polynomial defined as:

$$(x-a)_q^n = \begin{cases} 1 & if\ n = 0 \\ (x-a)(x-qa) \cdots (x-q^{n-1}a) & if\ n \geq 1 \end{cases} \tag{1.22}$$

1.2.3 q-analogue of $(x-a)^{-n}$ for $n \in Z$

Let $n \in Z$, the q-analogue of $(x-a)^{-n}$ is given as:

$$(x-a)_q^{-n} = \frac{1}{(x-q^{-n}a)_q^n} \tag{1.23}$$

Note that definition (1.22) is an extension of definition (1.21), (Kac and Cheung, 2002).

1.2.4 q-analogue of α, for $\alpha \in Z$

Let $\alpha \in \mathbb{C}$, the q-analogue of α is defined as:

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} \tag{1.24}$$

Note that definition (1.23) is an extension of equation (1.13), (Kac and Cheung, 2002).

Proposition 1.2.1. Kac and Cheung, (2002); for any integers $m, n \in Z$ the following properties hold.

I. $(x-a)_q^{m+n} \neq (x-a)_q^m (x-a)_q^n$.

II. $(x-a)_q^{m+n} = (x-a)_q^m (x-q^m a)_q^n.$

III. $D_q (x-a)_q^n = [n]_q (x-a)_q^{n-1}.$

IV. $D_q \left(\frac{1}{(x-a)_q^n}\right) = [-n]_q (x-q^n a)_q^{-n-1}.$

V. $(-1)^n q^{n(n-1)/2} (x-q^{-n+1} a)_q^n = (a-x)_q^n.$

VI. $D_q (a-x)_q^n = [-n]_q (a-qx)_q^{n-1}.$

VII. $D_q \left(\frac{1}{(a-x)_q^n}\right) = \frac{[n]_q}{(a-x)_q^{n+1}}.$

With the above definitions and propositions we have the q-Taylor's formula for polynomials as follows.

Theorem 1.2.1. For any polynomial $f(x)$ of degree N and any number c, we have the following q-Taylor's expansion:

$$f(x) = \sum_{j=0}^{N} (D_q^j f)(c) \frac{(x-c)_q^j}{[j]_q!} \tag{1.25}$$

(Kac and Cheung, 2002).

1.3 The Two Euler's Idendities and Two q- Exponential Functions

Before we state the identities and q- exponential functions, let us consider a definition and some properties associated to both of them. These properties are called the Properties of q-Binomial Coefficients.

1.3.1 q- Binomial Coefficients

Let $n \geq j$, for $0 \leq n \in Z$ the q- Binomial Coefficients is defined as:

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q [n-1]_q \cdots [n-j+1]_q}{[j]_q!} = \frac{[n]_q!}{[j]_q! \, [n-j]_q!}$$

(Kac and Cheung, 2002).

Proposition 1.3.1. Kac and Cheung, (2002); let $1 \leq j \leq n-1$, then the following $q-Pascal\ rules$ hold.

I. $$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tag{1.26}$$

II. $$\begin{bmatrix} n \\ j \end{bmatrix}_q = q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tag{1.27}$$

Proof. Consider the given condition

$$1 \leq j \leq n-1$$

We have:

$$\begin{aligned} [n]_q &= 1 + q + \cdots q^{n-1} \\ &= \left(1 + q + \cdots + q^{j-1}\right) + q^j (1 + q + \cdots + q^{n-j-1}) \\ &= [j]_q + q^j [n-j]_q. \end{aligned}$$

We now consider (1.26)

$$\begin{aligned} \begin{bmatrix} n \\ j \end{bmatrix}_q &= \frac{[n]_q!}{[j]_q![n-j]_q!} = \frac{[n-1]_q![n]_q}{[j]_q![n-j]_q!} \\ &= \frac{[n-1]_q!(\,[j]_q + q^j[n-j]_q)}{[j]_q![n-j]_q!} \\ &= \frac{[n-1]_q!}{[j-1]_q![n-j]_q!} + q^j \frac{[n-1]_q}{[j]_q![n-j-1]_q!} \\ &= \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_q, \end{aligned}$$

also, by consider (1.27), we have

$$\begin{aligned} \begin{bmatrix} n \\ j \end{bmatrix}_q &= \begin{bmatrix} n \\ j-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ n-j-1 \end{bmatrix}_q + q^j \begin{bmatrix} n-1 \\ n-j \end{bmatrix}_q \\ &= \begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \end{aligned}$$

Note that the symmetric property of the q −binomial coefficients "$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q![n-j]_q!} = \begin{bmatrix} n \\ n-j \end{bmatrix}_q$" gives the above second rule.

Corollary 1.3.1. Each q −binomial coefficient is a polynomial in q of degree $j(n-j)$, with 1 as the leading coefficient.

Proof. For any nonnegative integer n,

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1,$$

which is of course a polynomial. Now, using the symmetric property of q −binomial coefficients "$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q![n-j]_q!} = \begin{bmatrix} n \\ n-j \end{bmatrix}_q$" and induction on n, for any $1 \leq j \leq n-1$, $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ is the sum of two polynomials, thus is itself a polynomial (Kac and Cheung, 2002).

Now, by definition (1.26) and (1.13), the explicit expression of a q −binomial coefficient is

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-j+1}-1)}{(q^j-1)(q^{j-1}-1)\cdots(q-1)}. \tag{1.28}$$

Since both the numerator and denominator of (1.28) are polynomials in q with leading coefficient 1, so is their quotient. Finally, the degree of $\begin{bmatrix} n \\ j \end{bmatrix}_q$ in q is the difference of the degree of the numerator and denominator, which is $[n+(n-1)+\cdots+(n-j+1)] - [j+(j-1)+\cdots+1] = (n-1)+(n-j)+\cdots+(n-j) = j(n-j)$.

Another fact can be deduced from the explicit expression (1.28) of the q −binomial coefficient. Knowing that it is a polynomial in q of degree $j(n-j)$, we let

$$a_0 + a_1 q + \cdots + a_{j(n-j)-1} q^{j(n-j)-1} + a_{j(n-j)} q^{j(n-j)}$$

$$= \frac{\left(q^j-1\right)(q^{n-1}-1)\cdots\left(q^{n-j+1}-1\right)}{(q^j-1)(q^{j-1}-1)\cdots(q-1)}.$$

If we replace q by $1/q$ and multiply both sides by $q^{j(n-j)}$, it is easy to check that the right-hand side will be unchanged, while the left-hand side,

$$a_0 q^{j(n-j)} + a_1 q^{j(n-j)-1} + \cdots + a_{j(n-j)-1} q + a_{j(n-j)},$$

has the sequence of coefficients a_i reversed in order. By comparing coefficients, we observe that the coefficients in the polynomial expression of $\begin{bmatrix} n \\ j \end{bmatrix}_q$ are symmetric, $a_i = a_{j(n-j)-i}$.

However, to derive the two Euler's identities and the two q −Exponential functions we also have to consider the Gauss's and Heine's binomial formulas which were derived from the q-Taylor's formula respectively.

Now consider the Gauss's binomial formula

$$(x + a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j x^{n-j}. \tag{1.29}$$

by replacing x and a with 1 and x respectively, we have

$$(1 + x)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} x^j. \tag{1.30}$$

Also, consider the Heine's binomial formula

$$\frac{1}{(1 - x)_q^n} = \sum_{j=0}^{\infty} \frac{[n]_q [n+1]_q \dots [n+j-1]_q}{[j]_q!} x^j \tag{1.31}$$

Suppose we let $n \to \infty$ in (1.30) and (1.31). We knew in ordinary calculus that for $q = 1$ the result will not be very interesting because it is either going to be infinitely large or infinitely small depending on the value of x. But in quantum calculus the result will be totally different because, by considering an example for $|q| < 1$, the expression $(1 + x)_q^\infty$ will be $(1 + x)(1 + qx)(1 + q^2 x) \dots$ from definition (1.21), and so converges to some finite limit. Furthermore, if we let $|q| < 1$, then we have:

$$\lim_{n\to\infty} [n]_q = \lim_{n\to\infty} \frac{1 - q^n}{1 - q} = \frac{1}{1 - q} \quad for\ q \neq 1 \tag{1.32}$$

and

$$\lim_{n\to\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q = \lim_{n\to\infty}\frac{(1-q^n)(1-q^{n-1})\cdots\left(1-q^{n-j+1}\right)}{(1-q)(1-q^2)\dots(1-q^j)}$$
$$= \frac{1}{(1-q)(1-q^2)\dots(1-q^j)} \qquad (1.33)$$

By considering the $q-$anlogue of integer and binomial coefficients behavior when n is large, we can easily see the difference when compared with that of ordinary calculus.

Suppose we apply (1.32) and (1.33) to equations (1.30) and (1.31), then as $n\to\infty$ we have the following two identities of formal power series in x which are called the Euler first and second identities (with the assumption that $|q|<1$).

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{j(j-1)/2}\frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)}\,. \qquad (1.34)$$

$$\frac{1}{(1-x)_q^\infty} = \sum_{j=0}^{\infty}\frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)}. \qquad (1.35)$$

Now consider the second Euler's identity (1.35), by dividing both the numerator and the denominator of it by $(1-q)$ we have

$$\sum_{j=0}^{\infty}\frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)} = \sum_{j=0}^{\infty}\frac{\left(\frac{x}{1-q}\right)^j}{1\left(\frac{1-q^2}{1-q}\right)\dots\left(\frac{1-q^j}{1-q}\right)}$$
$$= \sum_{j=0}^{\infty}\frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!}. \qquad (1.36)$$

Clearly equation (1.36) looks like Taylor's expansion of the classical exponential function

$$e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}. \qquad (1.37)$$

Definition 1.3.2. A $q-$ analogue of the classical exponential function e^x is

$$e_q^x = \sum_{j=0}^{\infty}\frac{x^j}{[j]_q!}, \quad for\ |x| < \frac{1}{|q-1|} \qquad (1.38)$$

(Kac and Cheung, 2002).

Lemma 1.3.1. The interval of convergence of (1.38) is $|x| < \frac{1}{|q-1|}$.

Proof. Using ratio test we have the interval of convergence of (1.38) as,

$$\lim_{j\to\infty}\left|\frac{x^{j+1}/[j+1]_q!}{x^j/[j]_q!}\right| = \lim_{j\to\infty}\frac{|x|}{|[j+1]_q|} = \lim_{j\to\infty}\frac{|x||q-1|}{|q^{j+1}-1|} = |x||q-1| < 1$$

It implies $|x| < \frac{1}{|q-1|}$.

Also consider the first Euler's identity (1.34), by dividing the numerator and the denominator of it by $(1-q)$ we have:

Definition 1.3.3. Another $q-$ analogue of the classical exponential function e^x is

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2}\,\frac{x^j}{[j]_q!} = (1+(1-q)x)_q^{\infty}, \qquad for\ |x| < \infty. \tag{1.39}$$

Lemma 1.3.2. The radius of convergence of (1.39) is infinity.

Proof. Using ratio test we have the interval of convergence of (1.39) as

$$\lim_{j\to\infty}\left|\frac{q^{j(j+1)}/2\ \ x^{j+1}/[j+1]_q!}{q^{j(j-1)}/2\ \ x^j/[j]_q!}\right| = \lim_{j\to\infty}\frac{|q^jx|}{|[j+1]_q|} = |x|\lim_{j\to\infty}\frac{|q^j||q-1|}{|q^{j+1}-1|} = |x|.0 < 1$$

Hence the radius of convergence of (1.39) is infinity since $R = \frac{1}{\alpha} = \frac{1}{0}$.

Proposition 1.3.2. The classical exponential functions (1.38) and (1.39) are unchanged under differential**.**

Proof. Consider the left side of equation (1.38),

This means that

$$D_q\left(e_q^x\right) = \sum_{j=0}^{\infty}\frac{D_q(x^j)}{[j]_q!} = \sum_{j=1}^{\infty}\frac{[j]_q(x^{j-1})}{[j]_q!} = \sum_{j=1}^{\infty}\frac{x^{j-1}}{[j-1]_q!} = \sum_{j=0}^{\infty}\frac{x^j}{[j]_q!},$$

and,

$$D_q(E_q^x) = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{D_q(x^j)}{[j]_q!} = \sum_{j=1}^{\infty} q^{j(j-1)/2} \frac{[j]_q(x^{j-1})}{[j]_q!}$$
$$= \sum_{j=1}^{\infty} q^{(j-1)(j-2)/2} q^{j-1} \frac{x^{j-1}}{[j-1]_q!} = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{q^j x^j}{[j]_q!}.$$

We have

$$D_q(e_q^x) = e_q^x \text{ and } D_q(E_q^x) = E_q^{qx}. \tag{1.40}$$

Note that the derivative of E_q^x is not exactly itself. The results in (1.40) may also be obtained by letting $n \to \infty$ in

$$D_q \frac{1}{(1-(1-q)x)_q^n} = \frac{(1-q)[n]_q}{(1-(1-q)x)_q^{n+1}} \tag{1.41}$$

and

$$D_q(1+(1-q)x)_q^n = (1-q)[n]_q(1+q(1-q)x)_q^{n-1}. \tag{1.42}$$

1.4 q −Antiderivative

Definition 1.4.1. Let $F(x)$ and $f(x)$ be two functions defined on R, then $F(x)$ is called a q −antiderivative of $f(x)$ if $D_q(F(x)) = f(x)$, and it is denoted by

$$F(x) = \int f(x) d_q x \tag{1.43}$$

(Ernst, 2002) and (Ernst, 2012).

1.4.1 Jackson Integral

Definition 1.4.2. Let $f(x)$ be a functions defined on the set of real line R, then the Jackson integral is defined as

$$\int f(x)d_q x = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x). \tag{1.44}$$

However, from (1.44) we can easily derive a more general formula

$$\begin{aligned}\int f(x)D_q g(x)d_q x &= (1-q)x\sum_{j=0}^{\infty} q^j f\left(q^j x\right) D_q g\left(q^j x\right)\\ &= (1-q)x\sum_{j=0}^{\infty} q^j f\left(q^j x\right)\frac{g\left(q^j x\right)-g(q^{j+1}x)}{(1-q)q^j x}\\ &= \sum_{j=0}^{\infty} f\left(q^j x\right)\left(g\left(q^j x\right)-g\left(q^{j+1}x\right)\right). \end{aligned} \tag{1.45}$$

Theorem 1.4.1. Annaby and Mansour, (2012); Suppose $0 < q < 1$. If $f(x)x^{\alpha}$ is bounded on the interval $(0, A]$ for some $0 < \alpha < 1$, then the Jackson integral defined by (1.4.2) converges to a function $F(x)$ on $(0, A]$, which is a q-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$

Proof. Suppose $|f(x)x^{\alpha}| < M$ on $(0, A]$. For any $0 < x \leq A, j \geq 0$, then we can substitute x by $q^j x$ since then $0 < q < 1$, and $0 < q^j x \leq x \leq A$.

This means that

$$\left|f\left(q^j x\right)\right| < M\left(q^j x\right)^{-\alpha}.$$

Thus, for any $0 < x \leq A$, by multiplying both-sides of $\left|f\left(q^j x\right)\right| < M\left(q^j x\right)^{-\alpha}$ by q^j we have

$$\left|q^j f\left(q^j x\right)\right| < Mq^j\left(q^j x\right)^{-\alpha} = Mx^{-\alpha}(q^{1-\alpha})^j. \tag{1.46}$$

Since $1-\alpha > 0$ and $0 < q < 1$, we see that the series is majorized by a convergent geometric series. Hence, the right-hand side of (1.44) convergences pointwise to some function $F(x)$. It follows directly from (1.44) that $F(0) = 0$. The fact that $F(x)$ is continuous at $x = 0$, i.e $F(x)$ tends to zero as $x \to 0$, it is clear if we consider (1.46) by using geometric series, taking summation of both-sides for j starts from 0 to ∞, and multiplying each side with $(1-q)x$ we have

$$\left|\sum_{j=0}^{\infty} q^j f\left(q^j x\right)\right| \leq \sum_{j=0}^{\infty}\left|q^j f\left(q^j x\right)\right| < M \sum_{j=0}^{\infty} x^{-\alpha}\left(q^{1-\alpha}\right)^j = M x^{-\alpha} \frac{1}{1-q^{1-\alpha}}$$

$$= \left|(1-q)x \sum_{j=0}^{\infty} q^j f(q^j x)\right| < \frac{M(1-q)x^{1-\alpha}}{1-q^{1-\alpha}}, \qquad 0 < x \leq A.$$

To verify that $F(x)$ is a q-antiderivative, we q-differentiate it

$$D_q F(x) = \frac{1}{(1-q)x}\left((1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) - (1-q)qx \sum_{j=0}^{\infty} q^j f(q^{j+1} x)\right),$$

$$= \sum_{j=0}^{\infty} q^j f\left(q^j x\right) - \sum_{j=0}^{\infty} q^{j+1} f\left(q^{j+1} x\right),$$

$$= \sum_{j=0}^{\infty} q^j f\left(q^j x\right) - \sum_{j=1}^{\infty} q^j f\left(q^j x\right) = f(x).$$

Note that if $x \in (0, A]$ and $0 < q < 1$, then $qx \in (0, A]$, and the q-differentiation is valid.

Definition 1.4.3. Kac aand Cheung, (2002). Let $0 < a < b$, then the definite q-integral is defined as

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b), \tag{1.47}$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{1.48}$$

As seen before in (1.45), we derived from (1.47) a more general formula:

$$\int_0^b f(x)d_q g(x) = \sum_{j=0}^{\infty} f(q^j b)\left(g(q^j b) - g(q^{j+1}b)\right). \tag{1.49}$$

Definition 1.4.4. The improper q-integral of $f(x)$on $[0, +\infty)$ is defined to be

$$\begin{cases} \int_0^{\infty} f(x)d_q x = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} f(x)d_q x, & \text{if } 0 < q < 1. \\ \int_0^{\infty} f(x)d_q x = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^{j+1}} f(x)d_q x, & \text{if } q > 1. \end{cases} \tag{1.50}$$

1.5 Fundamental Theorem of q −Calculus and Integration by Parts.

Theorem 1.5.1. (Fundamental Theorem of q −Calculus)

If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have

$$\int_a^b f(x)d_q x = F(b) - F(a), \tag{1.51}$$

where $0 \leq a < b \leq \infty$ (Kac and Cheung, 2002).

Proof. Kac and Cheung (2002); Since $F(x)$ is continuous at $x = 0$, $F(x)$ is given by the Jackson formula, up to adding a constant, that is

$$F(x) = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) + F(0).$$

Since by definition,

$$\int_0^a f(x)d_q x = (1-q)a \sum_{j=0}^{\infty} q^j f(q^j a),$$

we have

$$\int_0^a f(x)d_q x = F(a) - F(0).$$

Similarly, we have, for finite b,

$$\int_0^b f(x)d_qx = F(b) - F(0),$$

and thus

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx = F(b) - F(a).$$

Putting $a = q^{j+1}$or q^j and $b = q^j$ (or q^{j+1}), where $0 < q < 1$ $(or\ q > 1)$, and considering the definition of improper q-integral (1.4.8), we see that (1.5.1) is true for $b = \infty$ as well if $\lim_{x\to\infty} F(x)$ exists.

Corollary 1.5.1. Annaby and Mansour (2012). If $f'(x)$ exists in a neighborhood of $x = 0$ and is continuous at $x = 0$, where $f'(x)$ denotes the ordinary derivative of $f(x)$, we have

$$\int_a^b D_qf(x)d_qx = f(b) - f(a) \tag{1.52}$$

Proof. Using L'Hospital's rule, we get

$$\lim_{x\to 0} D_qf(x) = \lim_{x\to 0}\frac{f(qx) - f(x)}{(q-1)x} = \lim_{x\to 0}\frac{qf'^{(qx)} - f'(x)}{(q-1)} = f'(0).$$

Hence $D_qf(x)$ can be made continuous at $x = 0$ if we define $(D_qf)(0) = f'(0)$, and (1.5.2) follows from the theorem.

1.5.1 q-Integration by Part Formula

Let $f(x)$ and $g(x)$ be two arbitrary differentiable functions defined on R. Then

$$\int_a^b f(x)d_qg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)d_qf(x) \tag{1.53}$$

is called the formula of q-integration by parts. Note that b can be equals to infinity as well

Theorem 1.5.2. Suppose $D_q^j f(x)$ is continuous at $x = 0$ for any $j \leq n + 1$. Then, we have a q-analogue of Taylor's formula with the Cauchy remainder:

$$f(b) = \sum_{j=0}^{n} \left(D_q^j f\right)(a) \frac{(b-a)_q^j}{[j]_q!} + \frac{1}{[n]_q!} \int_a^b D_q^{n+1} f(x)(b-qx)_q^n d_q x \tag{1.54}$$

Proof. Since $D_q f(x)$is continuous at $x = 0$, by Theorem (1.51) we have

$$f(b) - f(a) = \int_a^b D_q f(x)\, d_q x = -\int_a^b D_q f(x)\, d_q(b-x),$$

which proved (1.54) in the case where $n = 0$. Assume that (1.54) holds for $n - 1$

$$f(b) = \sum_{j=0}^{n+1} \left(D_q^j f\right)(a) \frac{(b-a)_q^j}{[j]_q!} + \frac{1}{[n-1]_q!} \int_a^b D_q^n f(x)(b-qx)_q^{n-1} d_q x$$

Using $D_q(a-x)_q^n = [-n]_q(a-qx)_q^{n-1}$ and applying q-integral by part (1.53), we obtain

$$\int_a^b D_q^n f(x)(b-qx)_q^{n-1} d_q x = -\frac{1}{[n]_q!} \int_a^b D_q^n f(x) d_q(b-x)_q^n$$

$$= \left(D_q^n f\right)(a) \frac{(b-a)_q^n}{[n]_q!} + \frac{1}{[n]_q!} \int_a^b (b-qx)_q^n D_q^{n+1} f(x) d_q x,$$

and the proof is complete by induction.

CHAPTER 2
EXISTENCE AND UNIQUENESS OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

2.1 Existence and Uniqueness of a Solution of an Ordinary Differential Equation

Before we state the Picard theorem for existence and uniqueness of a solution of a given differential equation we found that the following definitions and theorem are important from (Rudin, 1976), (Kreyszig, 1978), (Kolmogorov and Fomin, 1957), (Ashyralyev, 2013), and (Nagle et al, 2012).

2.1.1 Norm

A complex norm is a function $\|\cdot\| : X \to R$ having the following properties:

I. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$. For all $x \epsilon X$

II. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \epsilon X$ as $\alpha \epsilon C$.

III. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \epsilon X$. (triangular equality)

2.1.2 Normed Space

Let X be a nonempty set. Then, the pair $(X, \|\cdot\|)$ is called a normed space or normed vector space.

2.1.3 Complete Normed Space

A normed space is called complete if every Cauchy sequence contained in it converges to some point in it.

2.1.4 Banach Space

Let $(X, \|\cdot\|)$ be a normed space, then $(X, \|\cdot\|)$ is said to be Banach Space if it is a complete normed space.

2.1.5 Fixed Point of an Operator

A fixed point of an operator or a transformation is an element in the domain that the operator or transformation maps to itself.

2.1.6 Weierstrass M-Test Theorem

Let $\{f_n\}$ be a sequence of functions defined on a set E. Suppose that for all $n \in N$, there exist $M_n \in R$ such that

$$|f_n(x)| \leq M_n \ \forall x \in E$$

Then if $\sum M_n$ converges, then $\sum f_n$ must converges uniformly on E.

2.1.7 Banach Fixed Point Theorem for Operators

Let S be the set of continuous functions on $[a,b]$ that lie within a fixed distance $\alpha > 0$ of a given function $y^t(x) \in C[a,b], i.e. S = \{\, y \in C[a,b] : \|y - y^t\| \leq \alpha\}$. Suppose that G is an operator mapping S into S and it is a contraction on S, that is

$$\exists\ k \in R, 0 \leq k < 1,\ \ni\ \|G[w] - G[z]\| \leq k\|w - z\|\ \forall\ w, z\ \in S.$$

Then the operator G has a unique fixed point solution in S. Moreover, the sequence of successive approximations defined by $y_{n+1} = G[y_n], n = 0,1,2 \ldots$ converges uniformly to this fixed point, for any choice of starting function $y_0 \in S$.

Proof. Choose any starting function $y_0 \in S$. Since y_0 is an element of the domain of G, then $y_1 = G[y_0]$ is defined. Since G maps S to itself, $y_1 \in S$. By induction, $y_n \in S$ and $G[y_n]$ is well-defined, for all $n \geq 0$.

We rewrite

$$y_n = y_0 + (y_1 + y_0) + (y_2 + y_1) + \cdots + (y_n + y_{n-1}),$$

so that

$$y_n(x) = y_0(x) + \sum_{j=0}^{n-1} \left(y_{j+1}(x) - y_j(x)\right) \tag{2.1}$$

We now show that the sequence $\{y_n\}$ converges uniformly to an element in the set S. We can do this by using Theorem (2.1.1) which is an extension of the Comparison Test.

Now we need to find a bound M on the terms of the series (2.1).

Claim: $\|y_{j+1} - y_j\| \leq k^j \|y_1 - y_0\|$.

Then the claim is clearly true for $j = 0$. Suppose that the claim is true for $j = q$, where $q \in N$, $q \geq 0$. Then

$$\|G[y_{q+2}] - G[y_{q+1}]\| = \left\|G\left[G[y_{q+1}]\right] - G\left[G[y_q]\right]\right\|$$
$$\leq k\|G[y_{q+1}] - G[y_q]\| \leq k^{q+1}\|y_1 - y_0\|,$$

proving the claim

By considering equation (2.1) again, from the claim above it is clear that $\max_{x\in[a,b]}|y_{j+1}(x) - y_j(x)| = \|y_{j+1} - y_j\| \leq k^j\|y_1 - y_0\|$. Let $M_j := k^j\|y_1 - y_0\|$. Because

$$\sum_{j=1}^{\infty} M_j = \|y_1 - y_0\| \sum_{j=1}^{\infty} k_j$$

converges (since it is geometric series and also with the assumption that $0 \leq k \leq 1$), then the Weierstrass M-Test shows that $\{y_n\}$ converges uniformly to a continuous function y_∞. Moreover, $y_\infty \in S$ because the assumption that $\|y_\infty - y^t\| > \alpha$ implies that $\|y_n - y^t\| > \alpha$ not for all n, contradicting the fact that $y_n \in S$.

Recall that G is a contraction, this mean that $\|G[y_\infty] - G[y_n]\| \leq k\|y_\infty - y_n\|$ for any n. But we have $\|y_\infty - y_n\| \to 0$ as $n \to \infty$, so $\|y_\infty - y_{n+1}\| \to 0$ as $n \to \infty$. Of course, $G[y_n] = y_{n+1}$.

Thus,

$$\lim_{n\to\infty}\|G[y_\infty] - G[y_n]\| = \lim_{n\to\infty}\|G[y_\infty] - y_{n+1}\| \leq \lim_{n\to\infty} k\|y_\infty - y_{n+1}\| = 0.$$

Finally, $G[y_\infty] - y_\infty = (G[y_\infty] - y_{n+1}) + (y_{n+1} - y_\infty)$,

so that by triangular inequality for norm

$$\|G[y_\infty] - y_\infty\| \leq \|G[y_\infty] - y_{n+1}\| + \|y_{n+1} - y_\infty\|. \tag{2.2}$$

Since both terms on the right side of (2.2) tends to zero as n tends to ∞, it follows that $\|G[y_\infty] - y_\infty\| = 0$, or $G[y_\infty] = y_\infty$.

Thus, y_∞ is a fixed point of G.

Now suppose that $z \in S$ is any fixed point of G, i.e. that z satisfies $G[z] = z$. Then $\|y_\infty - z\| = \|G[y_\infty] - G[z]\| \leq k\, \|y_\infty - z\| < \|y_\infty - z\|$, which is possible if and only if $\|y_\infty - z\| = 0$. In other words, $z = y_\infty$, so that y_∞ is the unique fixed point of G.

2.1.8 Picard's Existence and Uniqueness Theorem

Consider the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

Suppose that $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous functions in some open rectangle $R = \{(x, y): a < x < b,\ c < y < d\}$ that contains the point (x_0, y_0). Then the Initial Value Problem has a unique solution in some closed interval $I = [x_0 - \delta, x_0 + \delta]$, where $\delta > 0$ (Nagle et al, 2012).

Proof. Picard's Theorem is proved by applying the Banach Fixed Point Theorem for Operators to the operator T. We the unique fixed point to be the limit of the Picard's Iterations given by

$$y_{n+1} = T[y_n],\ y_0(x) \equiv y_0$$

Recall that if y is a fixed point of T, then $y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)dt$, which is equivalent to the initial value problem. If such a function, $y(x)$ exists, then it is the unique solution to the initial value problem $y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)dt$.

To apply the Banach Fixed Point Theorem for Operators, we must show that T will map a suitable set S to itself and that T is a contraction. This may not be true for all real x. Also; our information pertains only to the particular intervals for x and y referred to the hypothesis of Picard's Theorem.

First we find an interval $I = [x_0 - \delta,\ x_0 + \delta]$ and $\alpha \in R, \alpha > 0$ such that T maps $S = \{g \in C[I] : \|y - y_0\| \leq \alpha\}$ into itself and T is a contraction. Here, $C[I] = C[x_0 - \delta,\ x_0 + \delta]$ and we adopt the norm

$$\|y\|_I = \max_{x \in I} |y|.$$

Choose δ_1 and α_1 such that

$$R_1 \coloneqq \{(x,y): |x-x_0| \le \delta_1, |y-y_0| \le \alpha_1\} \subseteq R$$

Because f and $\frac{\partial f}{\partial y}$ are continuous on the compact set R_1, it follows that both f and $\frac{\partial f}{\partial y}$ attain their supremum (and infimum) on R_1.

It follows that there exist $M > 0$ and $L > 0$ such that

$$\forall (x,y) \in R_1, |f(x,y)| \le M \text{ and } \left|\frac{\partial f}{\partial y}\right| \le L.$$

Now let g be a continuous function on $I_1 = [x_0 - \delta_1, x_0 + \delta_1]$ satisfying $|g(x) - y_0| \le \alpha_1$ for all $x \in I$, then $T[g](x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)dt$ so that, for all $x \in I$,

$$|T[g](x) - y_0| = \left|\int_{x_0}^{x} f\big(t,y(t)\big)dt\right| \le \int_{x_0}^{x} \left|f\big(t,y(t)\big)\right|dt \le M\left|\int_{x_0}^{x} dt\right| = M|x - x_0|.$$

Now choose δ such that $0 < \delta < \min\left\{\delta_1, \frac{\alpha_1}{M}, \frac{1}{L}\right\}$. Let $\alpha = \alpha_1, I = [x_0 - \delta, x_0 + \delta]$, and $S = \{g \in C(I): \|g - y_0\|_I \le \alpha\}$. Then T maps S into S; moreover, $T[g](x)$ is clearly a continuous function on I since it is differentiable, and we knew that differentiability of a function implies continuity of that function (Rudin, 1976).

For any $g \in S$, we have for any $x \in I$,

$$|T[g](x) - y_0| \le M|x - x_0| \le M\delta < M\left(\frac{\alpha_1}{M}\right) = \alpha_1.$$

In other words, $\|T[g] - y_0\| \le \alpha_1$, so $T[g] \in S$.

Now we show that T is a contraction. Let $u, v \in S$. On R_1, $\left|\frac{\partial f}{\partial y}\right| \le L$, so by the Mean Value Theorem there is a function $z(t)$ between $u(t)$ and $v(t)$ such that

$$|T[u](t) - T[v](t)| = \left|\int_{x_0}^{x}\{f\big(t,u(t)\big) - f\big(t,v(t)\big)\}dt\right| = \left|\frac{\partial f}{\partial y}\big(t,z(t)\big)[u(t) - v(t)]dt\right|$$

$$\le \left|\int_{x_0}^{x}\{f\big(u(t)\big) - f\big(t,v(t)\big)\}dt\right| \le L\|u - v\|_I|x - x_0| \le Lh\|u - v\|_I,$$

for all $x \in I$. Thus $\|T[u](x) - T[v](x)\| \le k\|u - v\|$, where $k = Lh < 1$, so T is a contraction on S.

The Banach Fixed Point Theorem for Operators therefore implies that T has a unique fixed point in S. It follows that the IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ has a unique solution in S. Moreover, this solution is the uniform limit of the Picard iterations.

Now we have found the unique solution to the IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ in S, there is one important point that remains to be resolved. We must show that any solution to the IVP on $I = [x_0 - \delta,\ x_0 + \delta]$ must lie in S.

Suppose that $u(x)$ is a solution to the IVP on $[x_0 - \delta,\ x_0 + \delta]$. Recall that $|f(x,y)| < M$ on the rectangle R_1. Since $u(x_0) = y_0$, the graph of $u(x)$ must lie inR_1 for x close to x_0. For such an x, we have $|f(x,u(x))| \leq M$, which implies that $|u'(x)| = |f(x,u(x))| \leq M$. Therefore, for x close to x_0 , the graph of $u = T[u]$ must lie within the shaded region. Moreover, the graph cannot escape from this region in $[x_0 - \delta,\ x_0 + \delta]$, since if it did, $|u'(x)| = |f(x,u(x))| > M$ at some point of the region, which is clearly impossible. Thus $|u(x) - y_0| \leq \alpha_1$ for all $x \in [x_0 - \delta,\ x_0 + \delta]$, which shows that $u(x) \in S$.

Example 1: consider the initial value problem

$$y' = 3y^{2/3},\ y(2) = 0$$

Then we have

$$f(x,y) = 3y^{2/3}\ and\ \frac{\partial f}{\partial y}(x,y) = 2y^{-1/3},$$

By considering $f(x,y) = 3y^{2/3}$ we can see that when $y = 0, f(x,y)$ is continuous. But at $y = 0$, $\frac{\partial f}{\partial y}$ is not continuous. Therefore the hypothesis of Picard's Theorem does not hold, and neither does the conclusion; the initial value problem has two solutions,

$$y^{1/3} = x - 2 \text{ and } y \equiv 0$$

Example 2: consider the initial value problem

$$y' = 2y\ \ y(0) = 1$$

Then we have

$$f(x, y) = 2y \text{ and } \frac{\partial f}{\partial y}(x, y) = 2,$$

Clearly $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are both continuous at the point $y = 1$.

However, we have the initial value problem to be $y = 1 + \int_0^x 2y dt$, and so the Picard's iterates are $y_0(x) \equiv 1$

$$y_1(x) = 1 + \int_0^x 2y_0(t)dt = 1 + 2x$$

$$y_2(x) = 1 + \int_0^x 2(1 + 2t)dt = 1 + 2x + \frac{(2x)^2}{2!}$$

and so by induction the nth iterate will be,

$$1 + 2x + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} = \sum_{i=0}^{n} \frac{(2x)^i}{i!}$$

which is the nth partial sum of the Maclaurin's series for e^{2x}.

Thus, as $n \to \infty$, $y_n(x) \to e^{2x}$.

2.2 Existence and Uniqueness of a Solution of System of Differential Equation

In the previous section we mainly focused to understand the Lipschitz condition and its connection with existence and uniqueness of solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE). Lipschitz condition guarantees uniform continuity but it does not ensure differentiability of the function (Rudin, 1976). In 2.0 we have shown that continuity is sufficient for existence of solution and locally Lipschitz is a sufficient condition for uniqueness of the solution of a IVP of first order ODE.

We construct the similar theorem for system of differential equation with two equations. Assume the following system of differential equation with the given initial values for two unknown functions call y_1 and y_2

$$\begin{cases} y_1'(\mathrm{x}) &= \mathrm{F}_1(x, y_1(x), y_2(x)) \\ y_2'(\mathrm{x}) &= \mathrm{F}_2(x, y_1(x), y_2(x)) \\ & y_1(x_0) = y_{1,0} \\ & y_2(x_0) = y_{2,0} \end{cases} \tag{2.3}$$

In addition, we assume that F_1 and F_2 are two continuous functions with continuous and bounded $y_1 -$ and $y_2 -$derivatives $\frac{\partial F_i}{\partial y_1}, \frac{\partial F_i}{\partial y_2}$ on the following domains

$$D = \{(x, y_1, y_2): \ x \in [x_0 - a \ \ x_0 + a], \ \ y_1 \& y_2 \in \mathbb{R}\}$$

By the another words, for some positive real value K, we have

$$\left|\frac{\partial F_i}{\partial y_j}(x, y_1, y_2)\right| < K \quad i = 0,1 \quad j = 0,1$$

Theorem 2.2.1. Suppose that F_i satisfies the assumption above. Then there is a unique pair of functions y_1 and y_2 defined on $[x_0 - a \ \ x_0 + a]$, with continuous first derivative, such that the system holds for all $x \in [x_0 - a \ \ x_0 + a]$. (Poria and Dhiman, 2013)

Proof. The procedure of the proof is as the same as 2.0.3 (Picard Theorem), so we just write out the iteration sequences. We assume the following successive approximation, set the recurrence relation as

$$\varphi_{0,1}(x) = y_{1,0}, \quad \varphi_{n,1}(x) = y_{1,0} + \int_{x_0}^{x} F_1\left(t, \varphi_{n-1,1}(t), \varphi_{n-1,2}(t)\right) dt,$$

$$\varphi_{0,2}(x) = y_{2,0}, \quad \varphi_{n,2}(x) = y_{2,0} + \int_{x_0}^{x} F_2\left(t, \varphi_{n-1,1}(t), \varphi_{n-1,2}(t)\right) dt.$$

Under the given assumptions, these two sequences converge to y_1 and y_2 respectively. We will discuss about system of q- difference equations in chapter 4.

Remark: If F_1 and F_2 can be demonstrated as a linear expressions of $y_1$and y_2 then the system is called linear system of differential equations and we can represent it by using matrix. In this case, eigenvalue and eigenvectors of this matrix make an important role.

CHAPTER 3
BRIEF HISTORY OF q −DIFFERENCE EQUATION

Scholarly works on q-difference equations begun at the beginning of the nineteenth century in thorough works especially in papers like (Jackson, 19010), (Carmichael, 1912), (Mason, 1915), (Adams 1915), (Trjitzinsky, 1933) and by other authors such as Poincare, Picard, Ramanujan. Unfortunately, from the thirties up to the beginning of the eighties, there was not significant interest in field (Bangerezako, 2008).

However, at eighties a thorough and somewhat astonishing interest in the subject appeared again in different areas of mathematics and applications comprising mainly new difference calculus, q-combinatorics, orthogonal polynomials, q-arithmetics, q-integrable systems and variational q-calculus (Annaby and Mansour, 2012).

Furthermore, despite of the plenteousness of specialized scientific publications and a relative classicality of the subject, an insufficiency of popularized publications in the form of books that can be accessible to a broad public comprising under and upper graduated students is so sensitive (Bangerezako, 2008).

As we earlier mentioned of the research works that were carried-out by different scholars, the study of q-difference equations have been introduced by Jackson in (Jackson, 1908). The paper (Carmichael, 1913) is the first research of the problem of existence of solutions of linear q-difference equations using the technique established by Birkhoff in his text (Birkhoff, 1941). Furthermore, in Mason's paper (Mason, 1915) he studied the existence of solutions of entire function relevant to homogeneous ($f = 0$) and non-homogeneous linear q-difference equations of nth order of the following form

$$\sum_{j=0}^{n} a_j(x)y\left(q^{n-j}x\right) = f(x) \tag{3.1}$$

such that the coefficients a_j are considered to be entire functions. Then Adams in the papers (Adams, 1925), (Adams, 1928/1929) and (Adams, 1928/1929) thoroughly studied the existence of solutions of the equation (3.1) when the coefficients are analytic or have pole of finite order at the origin. More recently, (Trjitzinsky, 1933) has brought into existence an analytic theory of existence of solutions of homogenous linear q-difference equations and their properties. The existence and uniqueness of solutions of first order linear q-difference equations in the space $C[0,\infty)$ and $L^P(\mathbb{R}^+)$ are disclosed in the paper (Liu, 1995). Apart from this old history of q-difference equations, the field received a significant interest of many mathematicians and from many fields of study in both theoretical and practical aspects (Annaby and Mansour, 2012).

However, we want to establish a theory for q-difference equation in the next chapter similar to that of that of ordinary differential equation in (Eastham, 1970), (Coddington, 1913) and (Nagle et al, 2012). In the course of this, we will study the Cauchy problem of q-difference equation in the neighborhood of a point say a, where $0 \leq a \leq \infty$. Also, we will derive the existence and uniqueness theorem for the cases $a = 0$ and $0 \leq a < \infty$. This will be form by the use of a q $-$analogue of the Picard Lindelöf method of differential equations and equations with deviating arguments, respectively. Furthermore, the validity's ranges of the solutions are examined in individual case, while the existence and uniqueness theorem of the solutions of q $-$difference equations of order n in a neighborhood of zero will also be proved. The situation when the initial conditions are given at a point $a > 0$ is rather complicated. In (Exton, 1982), it is stated that the Cauchy problem is

$$D_q\{K(x)D_q y\} - G(x)y(qx) = 0,\ a \leq x \leq b,\ \ y(c) = \gamma_0,\ \ D_q y(c) = \gamma \qquad (3.2)$$

Where K and G are continuous functions on $[a,b], c$ is an interior point of $[a,b]$ and γ_0, γ are complex numbers, has only one continuous solution with a continuous q-derivative. This is not necessarily true as the following counter example below shows.

Example 3.1. Suppose that $0 < q < 1$. Let $g(t) = -(t - q^2c)(t - qc)$,where $t \in [q^2c, qc]$ and $c \in \mathbb{C}$. Also, let $x \in [q^2c, \infty)$ for $x \geq q^2c$ then for some $t \in [q^2c,\ qc)$ we have $x = tq^{-n}$ or $x = tq^n$;

By another word, there exist $n \in \mathbb{N}$ such that $q^2c \leq xq^n \leq qc$ for $x \geq q^2c$.

Now let $A = \{m \,/\, q^2c \leq xq^m\}$, then at least $0 \in A$. Also, let n be maximum of A. $q^2c \geq xq^n$ and contrary to the assumption $xq^n > qc$ then $q^2c < xq^{n+1}$. It implies that $(n+1) \in A$, but $(n+1) > n$.

Therefore, there exist $n \in \mathbb{N}$ such that

$$q^2c \leq xq^n \leq qc.$$

Now the relation

$$\emptyset(x) = \emptyset(tq^{-n}) \coloneqq g(t), \text{ where } q^2c \leq t \leq qc \leq x,$$

defines a function $\emptyset$ on $[q^2c, \infty)$. Clearly, $\emptyset$ is a continuous since it is defined by $g(t)$ and is continuous function since it a parabola. Moreover, the discontinuity can only be occurred at the endpoints.

Also $\emptyset$ on $[q^2c, \infty)$ is a q-periodic function since

$$D_q\emptyset(x) = \frac{\emptyset(qx) - \emptyset(x)}{qx - x} = \frac{g(t) - g(t)}{qx - x} = 0.$$

Since $D_q\emptyset(x) \equiv 0$, it implies $D_q^2\emptyset(x) \equiv 0$, $D_q\emptyset(c) = 0$ and

$$\emptyset(c) = \emptyset((\, q^2c)\, q^{-2}) = g(\, q^2c) = 0.$$

Hence the q $-$initial value problem

$$D_q^2 y(x) = 0, y(c) = D_q y(c) = 0,$$

has the functions

$$y(x) = \emptyset(x) \text{ and } y(x) = 0.$$

This implies that the problem has no unique solution.

CHAPTER 4
SYSTEM OF q −DIFFERENCE EQUATION

4.1 Existence and Uniqueness of a Solution of System of q −Difference Equation

In this chapter, we will establish the existence and uniqueness of a solution of the first order system of q −difference equation in a neighborhood of point $a,$ such that $0 \leq a \leq \infty$ by the use of a q −analogue of the Picard Lindelöf method of successive approximations. However, before we describe the q −analogue of the Picard Lindelöf method at the point $a = 0$, and $0 < a < \infty$ respectively, we realized the following definition and theorem are important.

Definition 4.1.1. Let r, s and $n_i, i = 0,1, \dots, r,$ be element of $\mathbb{Z}^+$ and let

$$N = (n_0 + 1) + \cdots + (n_r + 1) - 1.$$

$$= n_0 + n_1 + \cdots n_r + r$$

Let $F_j(x, y_0, y_1, \cdots y_N), j = 0, 1, \cdots, s,$ be real or complex-valued functions where x is a real variable lying in some interval I and each y_i is a complex variable lying in some region D_i of the complex plane. That is F_j is equivalent to

$$F_0(x, y_0, y_1, \cdots y_N), \quad x \in I,\ y_0 \in D_0,\ y_1 \in D_1, \cdots y_N \in D_N.$$

$$F_1(x, y_0, y_1, \cdots y_N),$$

$$\vdots$$

$$F_{s-1}(x, y_0, y_1, \cdots y_N),$$

$$F_s(x, y_0, y_1, \cdots y_N).$$

If there is a sub-interval $J(J \subseteq I)$ of I and functions $(\emptyset_i,\ 0 \leq i \leq r)$ defined in J such that

a) $\emptyset_i$ has n_i q −derivatives in J for $i = 0,1, \cdots r$.
b) $D_q^m \emptyset_i$ exists and lies in the region D_i for all $x \in J,\ 0 \leq m \leq n_i,$ and $0 \leq i \leq r,$ for which the left-hand side in (4.1) below is defined.
c) For all $x \in J$ and $0 \leq i \leq s,$ the following equations hold

$$F_j\left(x, \emptyset_0(x), D_q\emptyset_0(x), \cdots, D_q^{n_0}\emptyset_0(x), \cdots, \emptyset_r(x), \cdots, D_q^{n_r}\emptyset_r(x)\right) = 0, \qquad (4.1)$$

then we say that $\{\emptyset_i\}_{i=0}^{r}$ is a solution to the system of the q −difference equations

$$F_0\left(x, y_0(x), D_q y_0(x), \cdots, D_q^{n_0} y_0(x), \cdots, y_r(x), \cdots, D_q^{n_r} y_r(x)\right) = 0, \qquad (4.2)$$

$$F_1\left(x, y_0(x), D_q y_0(x), \cdots, D_q^{n_0} y_0(x), \cdots, y_r(x), \cdots, D_q^{n_r} y_r(x)\right) = 0,$$

$$\vdots$$

$$F_s\left(x, y_0(x), D_q y_0(x), \cdots, D_q^{n_0} y_0(x), \cdots, y_r(x), \cdots, D_q^{n_r} y_r(x)\right) = 0.$$

valid in J, or that the set $\{\emptyset_i\}_{i=0}^{r}$ satisfies (4.2) in J. If there exist such J and functions $\emptyset_i$, we say that the system (4.2) has no solutions. The system (4.2) is said to be of order n, where $n = \max\{n_0, n_1, \cdots, n_p\} = \max_{0\le i\le p} n_i$.

However, we will only consider first order system of (4.2) where $r = s = p$. If the functions F_j are such that (4.2) can be solved for the $D_q^{n_i} y_i(x)$ in the form

$$D_q^{n_i} y_i(x) = f_i\big(x, y_0(x), D_q y_0(x), \cdots, y_1(x), \cdots\big)\, where\ (i = 0,1, \cdots, p) \qquad (4.3)$$

the system (4.3) is called the normal system. The following is example of normal system of first order:

$$D_q y_i(x) = f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right) \text{ where } (i = 0,1, \cdots, p). \qquad (4.4)$$

Or equivalently,

$$D_q y_0(x) = f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right),$$

$$D_q y_1(x) = f_1\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right),$$

$$\vdots$$

$$D_q y_p(x) = f_p\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right).$$

4.1.1 q −Initial Value Problems in a Neighborhood of Zero

Annaby and Mansour (2012); let I be an interval containing zero and E_r be disks of the form

$$E_r \coloneqq \{y \in \mathbb{C} : |y - b_r| < \beta\}, \ \beta > 0, \ b_r \in \mathbb{C}$$

and $r = 0,1,\cdots,p$. Let $f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right)$ where $(i = 0,1,\cdots,p)$ be functions defined on $I \times E_0 \times E_1 \times E_2 \times, \cdots \times E_{r-1} \times E_r$. By a q −initial value problem in a neighborhood of zero we mean the problem of finding functions $\{y_i\}_{i=0}^{p}$ that are continuous at zero, satisfying system (4.4) and the initial conditions

$$y_i(0) = b_i \ where \ (i = 0,1,\cdots,p) \tag{4.5}$$

Lemma 4.1.1. Annaby and Mansour (2012); let $J \subseteq I$ such that $0 \in J$. Let f_n be functions defined in the interval $I, n \in \mathbb{N}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in I$ and f_n tends uniformly to f on J.

Then,

$$\lim_{n\to\infty} \int_0^t f_n(x) d_q x = \int_0^t f(x) d_q x \ , \forall \ t \in I. \tag{4.6}$$

Theorem 4.1.1. Let I be an interval containing zero and E_r be disks of the form

$$E_r \coloneqq \{y \in \mathbb{C} : |y - b_r| < \beta\}, \ \beta > 0, \ b_r \in \mathbb{C}$$

and $r = 0,1,\cdots,p$. Let $f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right)$ where $(i = 0,1,\cdots,p)$ be functions defined on $I \times E_0 \times E_1 \times E_2 \times, \cdots \times E_{r-1} \times E_r$ such that the following conditions hold.

a) For any $y_r \in E_r, 0 \le r \le p,$ the function $f_i\left(x, y_0(x), y_1(x), \cdots, y_{p-1}(x), y_p(x)\right)$ is continuous at $x = 0, \ 0 \le i \le p.$
b) There exist a positive constant A such that for any $x \in I$ and $y_r, \ \tilde{y}_r \in E_r, 0 \le i \le p$ the following Lipschitz condition hold.

$$\left|f_i(x,\tilde{y}_0,\tilde{y}_1,\cdots,\tilde{y}_p)-f_i(x,y_0,y_1,\cdots,y_p)\right|\le A\left(|\tilde{y}_0-y_0|+\cdots+|\tilde{y}_p-y_p|\right) \quad (4.7)$$

Then, if zero is not an end point of I, there exist $h>0$ such that (4.4) has a unique solution which is valid for $|x|<h$. Moreover, if zero is the left or right end point of I, the result holds, except that the interval $[-h,h]$ is substituted by $[0,h]$ or $[-h,0]$ respectively. (Annaby and Mansour, 2012)

Proof. The proof is given in (Annaby and Mansour, 2012) as follows when zero is an interior point of I. Also, the proof when zero is the boundary of I is similar.

Now we define sequence of functions $\{\emptyset_{i,m}\}_{m=1}^{\infty}$, $i=0,1,2,\dots,p$ by the equations

$$\emptyset_{i,m+1}(x)=\begin{cases} b_i & m=0 \\ b_i+\int_0^x f_i\left(t,\emptyset_{0,m}(t),\emptyset_{1,m}(t),\cdots,\emptyset_{p,m}(t)\right)d_qt\,, & m\ge 1\end{cases} \quad (4.8)$$

By applying the Lipschitz condition (4.7), we have

$$\left|f_i(x,y_0,y_1,\cdots,y_p)\right|\le\left|f_p(0,b_0,b_1,\cdots,b_p)\right|$$

$$+\left|f_i(x,y_0,y_1,\cdots,y_p)-f_i(x,b_0,b_1,\cdots,b_p)\right|+\left|f_i(x,b_0,b_1,\cdots,b_p)-f_i(0,b_0,b_1,\cdots,b_p)\right|$$

$$\le A\sum_{j=0}^{p}|y_j-b_j|+\left|f_i(x,b_0,b_1,\cdots,b_p)-f_i(0,b_0,b_1,\cdots,b_p)\right|+\left|f_i(0,b_0,b_1,\cdots,b_p)\right|$$

Since the function $f_i(x,b_0,b_1,\cdots,b_p)$ is continuous at zero from the first condition, then for $\varepsilon=1$ there exist $\gamma>0$ such that

$$|x|\le\gamma \text{ which implies } \left|f_i(x,b_0,b_1,\cdots,b_p)-f_i(0,b_0,b_1,\cdots,b_p)\right|<1.$$

Hence,

$$\left|f_i(x,y_0,y_1,\cdots,y_p)\right|\le A\sum_{j=0}^{p}\beta+1+\max_{0\le i\le p}\left|f_i(0,b_0,b_1,\cdots,b_p)\right|$$

$$=A(p+1)\beta+1+\max\nolimits_{0\le i\le p}\left|f_i(0,b_0,b_1,\cdots,b_p)\right|$$

for all $y_r\in E_r, r=0,1,\dots,p,$ and $|x|\le\gamma$. Define the non-zero constant K,h to be

$$K:=\max_{0\le i\le p}\ \sup_{|x|\le\gamma,|y_j-b_j|\le\beta}\left|f_i(x,y_0,y_1,\cdots,y_p)\right|$$

$$h:=min\left\{\gamma,\frac{\beta}{K},\frac{1}{A(p+1)(1-q)}\right\}.$$

We will establish the existence of the solution $\{\emptyset_i\}_{i=0}^{p}$ of (4.4) and (4.5) on $J = [-h, h]$ using the method of successive approximations. We will consider the sequence defined by (4.8).

Existence: We will prove the existence of the solution in four steps.

1) We show that $\emptyset_{i,m},\ m \in \mathbb{N}$ are well defined. First

$$\emptyset_{i,m}(x) \in E_i \ \ (x \in J: m \in \ \mathbb{N}\ ). \tag{4.9}$$

Then from the definition of $\emptyset_{i,m}$ equation (4.1) we have

$$\emptyset_{i,m} = b_i + \int_0^x f_i\big(t, \emptyset_{0,m-1}(t), \emptyset_{1,m-1}(t), \cdots, \emptyset_{p,m-1}(t)\big) d_q t.$$

It implies

$$\left|\emptyset_{i,m} - b_i\right| \le \int_0^x \left| f_i\left(t, \emptyset_{0,m-1}(t), \emptyset_{1,m-1}(t), \cdots, \emptyset_{p,m-1}(t)\right)\right| d_q t \le \int_0^x K d_q t \tag{4.10}$$

Thus each $\emptyset_{i,m}(x)$ is continuous at zero and (4.8) is well defined.

2) For all $m \in \mathbb{N},\ x \in J$ we can prove by induction on m that

$$\left|\emptyset_{i,m+1}(x) - \emptyset_{i,m}(x)\right| \le K B^{m-1}(1-q)^m \frac{|x|^m}{(q:q)_m}, \tag{4.11}$$

where $B \coloneqq A(p+1)$.

Now, let $m = 1$ then we have

$$\left|\emptyset_{i,2}(x) - \emptyset_{i,1}(x)\right| = \left| b_i + \int_0^x f_i\big(t, \emptyset_{0,1}(t), \emptyset_{1,1}(t), \cdots, \emptyset_{p,1}(t)\big) d_q t - b_i \right|$$

$$\le K|x| = K B^0 (1-q)^1 \frac{|x|^1}{(1-q)^1}.$$

(4.12) Suppose the statement is true for $m = n$. Then we have

$$
\begin{aligned}
&\left|\emptyset_{i,n+1}(x)-\emptyset_{i,n}(x)\right| \\
&\qquad = \left|b_i+\int_0^x f_i\big(t,\emptyset_{0,n}(t),\emptyset_{1,n}(t),\cdots,\emptyset_{p,n}(t)\big)d_qt-b_i\right. \\
&\qquad \left.-\int_0^x f_i\big(t,\emptyset_{0,n-1}(t),\emptyset_{1,n-1}(t),\cdots,\emptyset_{p,n-1}(t)\big)d_qt\right| \\
&\qquad \le KB^{n-1}(1-q)^n\frac{|x|^n}{(q:q)_n}
\end{aligned}
$$

We prove that the statement is true for $m=n+1$.

This means that $\left|\emptyset_{i,m+1}(x)-\emptyset_{i,m}(x)\right|$ becomes

$$\left|\emptyset_{i,n+2}(x)-\emptyset_{i,n+1}(x)\right|,$$

and

$$
\begin{aligned}
&\left|\emptyset_{i,m+1}(x)-\emptyset_{i,m}(x)\right| \\
&\qquad = \left|b_i+\int_0^x f_i\big(t,\emptyset_{0,n+1}(t),\emptyset_{1,n+1}(t),\cdots,\emptyset_{p,n+1}(t)\big)d_qt-b_i\right. \\
&\qquad \left.-\int_0^x f_i\big(t,\emptyset_{0,n}(t),\emptyset_{1,n}(t),\cdots,\emptyset_{p,n}(t)\big)d_qt\right|
\end{aligned}
$$

$$= \left|\int_0^x f_i\left(\big(t,\emptyset_{0,n+1}(t),\emptyset_{1,n+1}(t),\cdots,\emptyset_{p,n+1}(t)\big)-f_i\big(t,\emptyset_{0,n}(t),\emptyset_{1,n}(t),\cdots,\emptyset_{p,n}(t)\big)\right)d_qt\right|$$

It implies from (4.7) we have

$$\left|\int_0^x f_i\left(\big(t,\emptyset_{0,n+1}(t),\emptyset_{1,n+1}(t),\cdots,\emptyset_{p,n+1}(t)\big)-f_i\big(t,\emptyset_{0,n}(t),\emptyset_{1,n}(t),\cdots,\emptyset_{p,n}(t)\big)\right)d_qt\right|$$

$$\le \int_0^x A\left(\left|\emptyset_{0,n+1}(t)-\emptyset_{0,n}(t)\right|+\cdots+\left|\emptyset_{p,n+1}(t)-\emptyset_{p,n}(t)\right|\right)d_qt$$

By induction assumption we have

$$\int_0^x A\left(\left|\emptyset_{0,n+1}(t)-\emptyset_{0,n}(t)\right|+\cdots+\left|\emptyset_{p,n+1}(t)-\emptyset_{p,n}(t)\right|\right)d_qt$$

$$\leq AK\int_0^x \left(A^{n-1}(p+1)^{n-1}(p+1)\frac{|t|^n}{[n]_q!}\right)d_q t$$

$$= KA^n(p+1)^n\int_0^x \left(\frac{|t|^n}{[n]_q!}\right)d_q t$$

This means that

$$KA^n(p+1)^n\int_0^x \left(\frac{|t|^n}{[n]_q!}\right)d_q t = KA^n(p+1)^{n+1}\frac{|x|^{n+1}}{[n]_q!\,[n+1]_q}.$$

It implies

$$KA^n(p+1)^n\frac{|x|^{n+1}}{[n]_q!\,[n+1]_q} = KB^n\frac{|x|^{n+1}}{[n+1]_q!}$$

$$= KB^n(1-q)^{n+1}\frac{|x|^{n+1}}{(q;q)_{n+1}}.$$

However, note that the inequality

$$\left|\int_a^a f(t)d_q t\right| \leq \int_a^a |f(t)|d_q t, \quad (0 \leq a < b < \infty).$$

is not valid always. (Annaby and Mansour, 2012)

3) We show that $\emptyset_{i,m}$ tends to a function $\emptyset_i$ uniformly on J.

Now from (4.1.21) we have

$$\emptyset_{i,m}(x) = \emptyset_{i,1}(x) + \sum_{l=1}^{m-1}\emptyset_{i,l+1}(x) - \emptyset_{i,l}\ (x)$$

$$= \emptyset_{i,1}(x) + \left(\emptyset_{i,2}(x) - \emptyset_{i,1}(x)\right) + \left(\emptyset_{i,3}(x) - \emptyset_{i,2}(x)\right) + \cdots + \left(\emptyset_{i,m}(x) - \emptyset_{i,m-1}(x)\right).$$

It implies that

$$\left|\emptyset_{i,m}(x)\right| \le |b_i| + \sum_{l=1}^{m-1}\left|\emptyset_{i,l+1}(x) - \emptyset_{i,l}\,(x)\right| = |b_i| + \sum_{l=1}^{m-1} KB^{l-1}\frac{|x|^l}{[l]_q!}$$

Now as $m \to \infty$ we have

$$K\sum_{l=1}^{\infty} B^{l-1}\frac{|x|^l}{[l]_q!} = K\sum_{l=1}^{\infty} A^{l-1}\,(p+1)^{l-1}\frac{|x|^l}{[l]_q!} \le K\sum_{l=1}^{\infty} A^{l-1}\,(p+1)^{l-1}\frac{(1-q)^{l-1}h^l}{(q;q)_l}$$

$$\le K\sum_{l=1}^{\infty}\frac{h^l}{h^{l-1}(q,q)_l} = K\sum_{l=1}^{\infty}\frac{h}{(q,q)_l} = Kh\sum_{l=1}^{\infty}\frac{1}{(q,q)_l} < \infty$$

By the Weiestrass m-Test it is uniformly continuous.

4) Now we show that $\{\emptyset_i\}_{i=0}^{p}$ satisfies (4.4) and (4.5). Indeed, from (4.7) we have

$$\left|f_i\left(t,\emptyset_{0,m}(t),\emptyset_{1,m}(t),\cdots,\emptyset_{p,m}(t)\right) - f_i\left(t,\emptyset_0(t),\emptyset_1(t),\cdots,\emptyset_p(t)\right)\right|$$

$$\le A\left(\left|\emptyset_{0,m}(t)-\emptyset_0(t)\right| + \left|\emptyset_{1,m}(t)-\emptyset_1(t)\right| + \cdots + \left|\emptyset_{p,m}(t)-\emptyset_p(t)\right|\right)$$

for all $t \in J$ and for all $m \in \mathbb{N}$. Since the right-hand side of number (3) approaches uniformly to zero on J as $m \to \infty$, it follows that,

$$\lim_{m\to\infty} f_i\left(t,\emptyset_{0,m}(t),\emptyset_{1,m}(t),\cdots,\emptyset_{p,m}(t)\right) = f_i\left(t,\emptyset_0(t),\emptyset_1(t),\cdots,\emptyset_p(t)\right)$$

is uniformly on J.

By letting $m \to \infty$ in (4.8) and using Lemma (4.1.1), we have

$$\emptyset_i = b_i + \int_0^x f_i\big(t,\emptyset_0(t),\emptyset_1(t),\cdots,\emptyset_p(t)\big)d_q t\,,\quad (0 \le i \le p;\ \ x \in J). \qquad (4.13)$$

By the use of conditions (a) and (b) of theorem (4.1.1), and the continuity of the function $\{\emptyset_i\}_{i=0}^{p}$ at zero, one can verify that the functions $f_i\big(t,\emptyset_0(t),\emptyset_1(t),\cdots,\emptyset_p(t)\big)$ are continuous at the point $\left(0,b_0,b_1,\cdots,b_p\right)$. Thus,

$$D_q\emptyset_i = f_i\big(x,\emptyset_0(x),\emptyset_1(x),\cdots,\emptyset_p(x)\big) \quad (0 \le i \le p;\ \ x \in J).$$

Hence, the set $\{\emptyset_i\}_{i=0}^{p}$ is a solution of (4.4), and (4.5) is valid in J since it is satisfied.

Uniqueness: To prove the uniqueness of the solution of the system (4.4) we assume that $\{\varphi_i\}_{i=0}^{p}$ is another solution to (4.4) such that the solution is valid in $|x| \le h_1 \le h$ and satisfies (4.5). However, for $0 \le i \le p, |x| \le h_1$.

$$\varphi_i(x) = \varphi_i(qx) + x(1-q)f_i\left(x, \varphi_0(x), \varphi_1(x), \cdots, \varphi_p(x)\right); \tag{4.14}$$

$$\emptyset_i(x) = \emptyset_i(qx) + x(1-q)f_i\left(x, \emptyset_0(x), \emptyset_1(x), \cdots, \emptyset_p(x)\right). \tag{4.15}$$

Now consider the equations (4.14) and (4.15), by subtracting (4.14) from (4.15) and applying (4.7) we have

$$|\emptyset_i(x) - \varphi_i(x)| \le |\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q|\sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)| \tag{4.16}$$

Now by taking summation of both-sides of (4.16) we have

$$\sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)| \le \sum_{i=0}^{p}|\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q|(P+1)\sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)|$$

Now let $\sigma(x) = \sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)|, |x| \le h_1$. It implies

$$\sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)| \le \sum_{i=0}^{p}|\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q|(P+1)\sum_{i=0}^{p}|\emptyset_i(x) - \varphi_i(x)|,$$

resulting;

$$\sigma(x) \le \sigma(qx) + A(P+1)|1-q||x|\sigma(x).$$

This means that

$$\sigma(x) \le \sigma(qx) + A(P+1)|1-q||x|\sigma(x) = \sigma(x)(1 - A(P+1)|1-q||x|) \le \sigma(qx)$$

Since $B = A(P+1)$, then we have

$$\sigma(x) \le \frac{\sigma(qx)}{(1 - B|1-q||x|)} \tag{4.17}$$

By replacing x with qx. (it is valid for $|x| \le h_1$)

This means that (4.17) becomes

$$\sigma(qx) \le \frac{\sigma(q^2x)}{(1 - B|1-q||qx|)} \tag{4.18}$$

By combining the inequalities (4.17) and (4.18) we have

$$\sigma(x) \le \frac{\sigma(q^2x)}{(1 - B|1 - q||x|)(1 - B|1 - q|q|x|)}$$

In the same manner by induction we have following

$$\sigma(x) \le \frac{\sigma(q^m x)}{\prod_{k=0}^{m-1}(1 - B|1 - q|q^k|x|)} \qquad (|x| \le h_1)$$

By calculating the limit as $m \to \infty$, we have

$$\sigma(x) \le \frac{\sigma(0)}{\prod_{k=0}^{\infty}(1 - B|1 - q|q^k|x|)} \qquad (|x| \le h_1)$$

According to the definition of $\sigma(x)$ we have $\sigma(0) = 0$. So $\sigma(x) = 0$ which implies $\varphi(x) = \emptyset(x)$.

Theorem 4.1.2. (Range of validity). Annaby and Mansour (2012); suppose that all the condition of theorem (4.1.1) hold with $E_r = \mathbb{C}$ for all r; $\ni r = 0,1, \ldots, p$. Then the problem (4.4) with initial condition (4.5) a unique solution which is valid for at least in $I \cap \left(-\frac{1}{A(p+1)}, \frac{1}{A(p+1)(1-q)}\right)$.

Proof. we will prove the theorem by trying to prove the existence and uniqueness of solution of the problem (4.4) with initial condition (4.5) on any subinterval

$$[-h, h] \subseteq I^* \coloneqq I \cap \left(-\frac{1}{A(p+1)}, \frac{1}{A(p+1)(1-q)}\right),$$

for $h > 0$. By considering the strategy used in proving theorem (4.1.1), we can determine a constant $\gamma \le h \ni \emptyset_{i,m}$ approaches uniformly to $\emptyset_i$ on $[-\gamma, \gamma]$, such that $\emptyset_{i,m}$ are defined in equation (4.8). In addition, it is not difficult to verify that $\emptyset_{i,m}$ converges to $\emptyset_i$ pointwise on $[-h, h]$. By the use of Lemma (4.1.1) it can be shown that the solution $\{\emptyset_i\}_{i=0}^{p}$ could be extended throughout the interval $[-h, h]$.

Remark 4.1.1. Theorem (4.1.1) holds for the other Cauchy problem

$$D_q y_i(x) = f_i\left(qx, y_0(qx), y_1(qx), \cdots, y_p(qx)\right),$$

$$y_i(0) = b_i, (i = 0,1, \cdots, p),$$

but the solution is valid only throughout whole interval I whenever the function f_i's satisfy the conditions (a), (b) of the theorem 4.1.1 with $E_r = \mathbb{C}, 0 \leq r, 0 \leq i \leq p$.

The below corollary shows that Theorem (4.1.1) can be used to discuss the existence and uniqueness of the nth order q −initial value problem

$$D_q^n y(x) = f\left(x, y(x), D_q y(x), \cdots, , D_q^{n-1} y(x)\right), \tag{4.19}$$
$$D_q^{i-1} y(0) = b_i, \qquad (b_i \in \mathbb{C};\ \ 1 \leq i \leq n).$$

Corollary 4.1.1. Annaby and Mansour (2012); let $p, I,\ E_r, b_r$ be as in the theorem (4.1.2). Let $f(x, y_0, y_1, \cdots y_p)$ be a function defined on $I \times E_0 \times E_1 \times E_2 \times, \cdots \times E_{p-1} \times E_p$ such that the following conditions hold.

a) For any fixed values of $y_r \in E_r$, the function $f(x, y_0, y_1, \cdots y_p)$ at the point zero is continuous.
b) There exist a constant $A > 0$ such that for all $x \in I$ and $y_r,\ \tilde{y}_r \in E_r, 0 \leq i \leq p$ the following Lipschitz condition hold.

$$\left|f_i(x, \tilde{y}_0, \cdots, \tilde{y}_p) - f_i(x, y_0, \cdots, y_p)\right| \leq A\left(|\tilde{y}_0 - y_0| + \cdots + |\tilde{y}_p - y_p|\right) \tag{4.20}$$

Then, if the point zero is not a boundary point of I, there exist $h > 0$ such that the Cauchy problem (4.19) has a unique solution $\emptyset$ which is valid for $|x| < h$. Moreover, if zero is the left or right end point of I, the result holds, except that the interval $[-h, h]$ is substituted by $[0, h]$ or $[-h, 0]$ respectively.

Proof. Suppose that zero is an interior point of I. Then the Cauchy problem is identical to the first order q −initial value problem

$$D_q y_i(x) = f_i(x, y_0, y_1, \cdots, y_{n-1}), \qquad y_i = \ b_i;\ \ 0 \leq i \leq n-1, \tag{4.21}$$

That is

$$q y_0(x) = f_0(x, y_0(x), y_1(x), \cdots, y_{n-1}(x)),$$
$$D_q y_1(x) = f_1(x, y_0(x), y_1(x), \cdots, y_{n-1}(x)),$$
$$\vdots$$

$$D_q y_{n-1}(x) = f_{n-1}\big(x, y_0(x), y_1(x), \cdots, y_{n-1}(x)\big),$$

whereby $\{\emptyset_i\}_{i=0}^{n-1}$ is a solution to the equation (4.21) if and only $\emptyset_0$ is a solution to (4.20). However,

$$\begin{aligned}&f_0(x, y_0, y_1, \cdots, y_{n-1}),\\ &f_1(x, y_0, y_1, \cdots, y_{n-1}),\\ &f_2(x, y_0, y_1, \cdots, y_{n-1}),\\ &\vdots\\ &f_{n-1}(x, y_0, y_1, \cdots, y_{n-1}),\end{aligned}$$

are the functions

$$f_i(x, y_0, y_1, \cdots, y_{n-1}) = \begin{cases} y_{i+1}, & 0 \le i \le n-2, \\ f(x, y_0, y_1, \cdots, y_{n-1}), & i = n-1. \end{cases}$$

Therefore, by theorem (4.1.1), there exist $h > 0$ such that the system (4.21) has a unique solution which is valid for $|x| \le h$.

Corollary 4.1.2. Annaby and Mansour (2012); consider the q −differential equation (Cauchy differential equation) below

$$a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = b(x) \tag{4.22}$$

$$D_q^i y(0) = b_i,\ (0 \le i \le n-1)$$

Let the $a_j(x)$ (for $0 \le j \le n$) and $b(x)$ be defined on an interval I containing zero such that $a_0(x) \ne 0$ for all $x \in I$. Let the function $a_j(x)$ and $b(x)$ be continuous at the point zero and bounded on I. Then, for all complex numbers b_i, there is a sub-interval J of I where zero is an element of J such that (4.22) has a unique solution.

Proof. Consider the Cauchy's q −differential equation (4.22)

$$a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = b(x),$$

$$D_q^i y(0) = b_i,\ (0 \le i \le n-1).$$

By dividing the equation through by $a_0(x)$ and making $D_q^n y(x)$ the subject of the formula we have

$$D_q^n y(x) = A_1(x)D_q^{n-1} y(x) + \cdots + A_n(x)y(x) + B(x) \tag{4.23}$$

where $A_i(x) = -a_j/a_0$, $1 \leq j \leq n$, and $B(x) = b(x)/a_0(x)$.

By comparing, equation (4.23) is of the form of (4.21), and it implies that;

$$f\left(x, y, \cdots, D_q^{n-1}y\right) = A_1(x)D_q^{n-1}y(x) + \cdots + A_n(x)y(x) + B(x).$$

Since given that all $a_j(x)$ are continuous at zero and bounded on I, it implies that $A_j(x)$ and $B(x)$ are continuous at the point zero and also bounded on I. The function $f\left(x, y, \cdots, D_q^{n-1}y\right)$ satisfies the conditions of Corollary (4.1.1). Hence, there exists a sub-interval J of I where zero is an element of J such that (4.23) has a unique solution that is valid in J.

Remark 4.1.2. From the above equations, it implies that we can use the method of power series to obtain solution of some linear q −difference equation. For instance, if we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ to be the solution of the q −initial value problem

$$D_q y(x) = y(x), \qquad y(0) = 1, \tag{4.24}$$

by considering $f(x)$, that is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we have

$$D_q f(x) = \sum_{n=1}^{\infty} a_n [n]_q x^{n-1}.$$

By substituting $f(x)$ and $D_q f(x)$ in (4.24) we have

$$\sum_{n=1}^{\infty} a_n [n]_q x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0, \qquad y(0) = 1.$$

This means that

$$\sum_{n=1}^{\infty} a_n [n]_q x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0, \qquad y(0) = 1.$$

Furthermore,

$$\sum_{n=0}^{\infty} a_{n+1}[n+1]_q x^n - \sum_{n=0}^{\infty} a_n x^n = 0, \qquad y(0) = 1.$$

by the use of Shift of Index of Summation (Boyce and Diprima,1992).

This means that $a_{n+1}[n+1]_q - a_n = 0$ since $x^n \neq 0$.

It also implies that

$$a_{n+1} = \frac{a_n}{[n+1]_q}.$$

By considering $a_{n+1} = \frac{a_n}{[n+1]_q}$, it implies $a_{n+1} = a_n \frac{1-q}{1-q^{n+1}}$ from definition (1.2.4).

Now, for $n = 0$, we have

$$a_1 = a_0 \frac{1-q}{1-q^1}.$$

For $n = 1$, we have

$$a_2 = a_1 \frac{1-q}{1-q^2} = a_0 \frac{(1-q)(1-q)}{(1-q^1)(1-q^2)}.$$

For $n = 2$, we have

$$a_3 = a_2 \frac{1-q}{1-q^3} = a_0 \frac{(1-q)(1-q)(1-q)}{(1-q^1)(1-q^2)(1-q^3)}.$$

For $n = 3$, we have

$$a_4 = a_3 \frac{1-q}{1-q^4} = a_0 \frac{(1-q)(1-q)(1-q)(1-q)}{(1-q^1)(1-q^2)(1-q^3)(1-q^4)}.$$

$$\vdots$$

Therefore, we have the generalization of the sequence as

$$a_n = a_0 \frac{(1-q)^n}{(1-q^n)} = a_0 \frac{(1-q)^n}{(q;q)_n}, \quad n \in \mathbb{N}.$$

It follows that

$$y(x) = f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Therefore,

$$y(x) = \sum_{n=0}^{\infty} a_0 \frac{(1-q)^n}{(q;q)_n} x^n, \qquad (q;q)_0 = 1\,.$$

By the given condition $y(0) = 1$, we have

$$y(0) = \sum_{n=0}^{\infty} a_0 \frac{(1-q)^n}{(q;q)_n} 0^n = 1, \;\; (q;q)_0 = \; 1.$$

This means that $a_0 = 1$, and hence

$$y(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} x^n, \quad (q;q)_0 = 1.$$

By comparing $y(x)$ with q −exponential function we found $y(x) = e_q^{x(1-q)}$. By the Theorem (4.1.2), the solution $y(x) = e_q^{x(1-q)} = f(x)$ is valid in $|x| < (1-q)^{-1}$. This can achieve using the well-known Ratio test.

Another example is the q −initial value problem

$$D_q y(x) = y(qx), \qquad y(0) = 1. \tag{4.25}$$

The term b_n of the solution $g(x) = \sum_{n=0}^{\infty} b_n x^n$ satisfies the given q-initial value problem (4.25). However, this can be shown by considering the function $g(x)$ and differentiating it.

Now, since

$$g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

This means that

$$D_q g(x) = \sum_{n=1}^{\infty} b_n [n]_q x^{n-1}.$$

By substituting $g(x)$ and $D_q f(x)$ in (4.1.33), we have

$$\sum_{n=1}^{\infty} b_n [n]_q x^{n-1} - \sum_{n=0}^{\infty} b_n (qx)^n = 0, \qquad y(0) = 1.$$

This means that

$$\sum_{n=1}^{\infty} b_n [n]_q x^{n-1} - \sum_{n=0}^{\infty} b_n (qx)^n = 0, \qquad y(0) = 1.$$

Is equals to

$$\sum_{n=0}^{\infty} b_{n+1} [n+1]_q x^n - \sum_{n=0}^{\infty} b_n q^n x^n = 0, \qquad y(0) = 1.$$

by the use of Shift of Index of Summation (Bender and Orszag,1978).

This means that $b_{n+1}[n+1]_q - q^n b_n = 0$ since $x^n \neq 0$..

It also implies that

$$b_{n+1} = q^n \frac{b_n}{[n+1]_q} = q^n b_n \frac{(1-q)}{(1-q^{n+1})}, \quad b_0 = 1.$$

By considering $b_{n+1} = q^n \frac{b_n}{[n+1]_q}$, it implies$b_{n+1} = q^n b_n \frac{(1-q)}{(1-q^{n+1})}$ from definition (1.2.4).

Now, for $n = 0$, we have

$$b_1 = b_0 q^0 \frac{1-q}{1-q^1}.$$

For $n = 1$, we have

$$b_2 = b_1 q^1 \frac{1-q}{1-q^2} = b_0 q^1 \frac{(1-q)(1-q)}{(1-q^1)(1-q^2)}.$$

For $n = 2$, we have

$$b_3 = b_2 \frac{1-q}{1-q^3} = b_0 q^0 q^1 q^2 \frac{(1-q)(1-q)(1-q)}{(1-q^1)(1-q^2)(1-q^3)}.$$

For $n = 3$, we have

$$b_4 = b_3 q^3 \frac{1-q}{1-q^4} = b_0 q^0 q^1 q^2 q^3 \frac{(1-q)(1-q)(1-q)(1-q)}{(1-q^1)(1-q^2)(1-q^3)(1-q^4)}.$$

$$\vdots$$

Therefore, by analyzing the sequence above, we have the generalization of it as

$$b_n = b_0 q^{\frac{n(n-1)}{2}} \frac{(1-q)^n}{(1-q^n)} = b_0 \frac{q^{\frac{n(n-1)}{2}}(1-q)^n}{(q;q)_n}, \quad n \in \mathbb{N},\ (q;q)_0 = 1.$$

It follows that

$$y(x) = b(x) = \sum_{n=0}^{\infty} b_n x^n,$$

yielding

$$y(x) = \sum_{n=0}^{\infty} b_0 \frac{q^{\frac{n(n-1)}{2}}(1-q)^n}{(q;q)_n} x^n, \qquad (q;q)_0 = 1.$$

By the given condition $y(0) = 1$, we have

$$y(0) = \sum_{n=0}^{\infty} b_0 \frac{q^{\frac{n(n-1)}{2}}(1-q)^n}{(q;q)_n} 0^n = 1, \ \ (q;q)_0 = \ 1$$

This means that $b_0 = 1$, and hence

$$y(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}(1-q)^n}{(q;q)_n} x^n, \quad (q;q)_0 = 1$$

By comparing $g(x)$ with the q −exponential function we found $g(x) = E_q^{(x(1-q))}$. By the Lemma (1.3.2), the solution $y(x) = g(x) = E_q^{(x(1-q))}$ is valid throughout $\mathbb{C}$. This can achieve using the well-known Ratio test.

However, the q −initial value problems (4.24) and (4.25) are q −analogues of the initial value problem

$$y' = y(x), \quad y(0) = 1.$$

4.1.2 q −Initial Value Problem in a Neighborhood of Infinity

As we earlier mentioned in the introduction part of this chapter that even if an initial condition is given at a point $a > 0$, the uniqueness of solutions of the q −initial value problem is not guaranteed. However, in this segment we shall examine the problem of existence and uniqueness of q −initial value problem with initial conditions different from zero. Furthermore, in the course of this we will have an initial interval instead of an initial point c and instead of $b_{i's}$we had, we will have initial q −periodic functions. On the other hand, there is need to define the Cauchy problems in this case. The two types of the problems are the forward and backward problem depending on the q −difference equations (Annaby and Mansour, 2012).

Definition 4.1.2. By a Forward value problem at $a > 0$, we mean the problem of finding a solution of $D_q y_i(x) = f_i\left(qx, y_0(qx), y_1(qx), \cdots, y_p(qx)\right), (i = 0,1, \cdots, p)$ in an interval of the form $[a, \infty)$, such that the forward conditions $y_i(x) = g_i(x),\ x \in [qa, a)$ are satisfied. The arbitrary functions g_i are called forward initial functions and the interval $[qa, a)$ is called the initial interval (Annaby and Mansour, 2012).

Definition 4.1.3. By a Backward value problem at $b > 0$, we mean the problem of finding a solution of (4.4) in an interval of the form $(0, b]$, such that the backward conditions $y_i(x) = g_i(x),\ x \in]b, bq^{-1}]$ are satisfied. The arbitrary functions g_i are called backward functions and the interval $]b, bq^{-1}]$ is called the backward interval (Annaby and Mansour, 2012).

Theorem 4.1.3. Annaby and Mansour (2012); let $f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right)$ be functions defined for $x \in [qa, \infty)$, $y_j \in \mathbb{C}$, where $0 \leq j, i \leq p$, and $a > 0$. Let $\{g_i(t)\}_{i=0}^{p}$ be a set of q −periodic functions such that the following conditions hold.

a) $\left|f_i\left(x, g_0(x), g_1(x), \cdots, g_{p-1}(x), g_p(x)\right)\right|$ is bounded on $[qa, \infty)$,

b) There exist a positive constant A such that for any $x \in [qa, \infty)$ and y_r, $\tilde{y}_r \in \mathbb{C}$, the following Lipschitz condition hold for $0 \leq i \leq p$.

$$\left|f_i(x, \tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_p) - f_i(x, y_0, y_1, \cdots, y_p)\right| \leq A\left(|\tilde{y}_0 - y_0| + \cdots + |\tilde{y}_p - y_p|\right) \quad (4.26)$$

Thus, the forward value problem

$$D_q y_i(x) = f_i\left(qx, y_0(qx), y_1(qx), \cdots, y_p(qx)\right) \text{ for } x \in [a, \infty), \quad (4.27)$$

$$\emptyset_i(x) = g_i(x) \qquad (x \in [qa, \infty), i = 0,1,2, \dots, p)$$

has a unique solution $\{\emptyset_{\mathrm{i}}\}_{\mathrm{i}=0}^{\mathrm{p}}$ which is valid in $[\mathrm{a}, \infty)$.

Proof. The proof is given as follows:

From (a), it implies that there exist a positive constant $\mathbb{C}$ such that

$$C \coloneqq \max_{0 \leq i \leq p} \sup_{x \geq qa} \left|f_i\left(x, g_0(x), g_1(x), \cdots, g_{p-1}(x), g_p(x)\right)\right| \quad (4.28)$$

Now we define sequence of functions $\left\{\emptyset_{i,m}(x)\right\}_{m=1}^{\infty}$, $i = 0,1,2, \dots, p$ by the equations

$$\emptyset_{i,m+1}(tq^{-n}) = \begin{cases} c_i(t) & m = 0 \\ c_i(t) + \int_t^{tq^{-n}} f_i\left(qu, \emptyset_{0,m}(qu), \cdots, \emptyset_{p,m}(qu)\right) d_q t\,, & m \geq 1 \end{cases} \quad (4.29)$$

Where $t \in [qa, a)$, $n \in \mathbb{N}$ and $c_i(t)$ are q −periodic functions.

In this setting the successive approximation associated with the problem (4.27), $\emptyset_{i,m+1}(x)$, $i = 0,1,\cdots,p, m \in \mathbb{N}$, is the sequence (4.29). In the equation (4.29) above it is clearly understood from definition (1.4.1) that the integral over $[t, tq^{-k}]$ is

$$\int_t^{tq^{-k}} h(u)d_q u = (1-q)\sum_{j=1}^{k} tq^{-j}h(tq^{-j})$$

Thus, the successive approximation (4.29) is well defined. We now let $t \in [qa, a)$and $k \in \mathbb{N}$ be fixed.

Existence: we will prove the existence of the solution in three steps.

1) For all $m \in \mathbb{N},\ B = A(p+1)$, we can prove by induction on m that

$$\left|\emptyset_{i,m+1}(tq^{-k}) - \emptyset_{i,m}(tq^{-k})\right| \le CB^{m-1}q^{\frac{m(m-1)}{2}}\frac{t^m(1-q)^m}{q^{mk}(q:q)_m} \tag{4.30}$$

for all $t \in [qa, a)$and $k \in \mathbb{N}$.

Now, since the proof is by induction then let $m = 1$.

If $m = 1$, then we have;

$$\left|\emptyset_{i,2}(tq^{-k}) - \emptyset_{i,1}(tq^{-k})\right| = \left|c_i(t) + \int_t^{tq^{-k}} f_i\big(qu, \emptyset_{0,1}(qu), \cdots, \emptyset_{p,1}(qu)\big)d_q u - c_i(t)\right|$$

$$\le \int_t^{tq^{-k}} \left|f_i\big(qu, \emptyset_{0,1}(qu), \cdots, \emptyset_{p,1}(qu)\big)\right|d_q u \ \le \int_t^{tq^{-k}} C d_q u = Ct(q^{-k}-1)$$

$$\le CB^0 q^0 \frac{t^1(1-q)^1}{q^k(q:q)_1}\ = CB^0\frac{t^1}{q^k} \tag{4.31}$$

Suppose that the statement is true for $m = n$. Then we have

$$\left|\emptyset_{i,n+1}(tq^{-k}) - \emptyset_{i,n}(tq^{-k})\right|$$

$$= \left| c_i(t) + \int_t^{tq^{-k}} f_i\left(qu, \emptyset_{0,n}(qu), \cdots, \emptyset_{p,n}(qu)\right) d_q u - c_i\,(t) - \int_t^{tq^{-k}} f_i\left(qu, \emptyset_{0,n-1}(qu), \cdots, \emptyset_{p,n-1}(qu)\right) d_q u \right|$$

$$\leq CB^{n-1} q^{\frac{n(n-1)}{2}} \frac{t^n (1-q)^n}{q^{nk}(q:q)_n}.$$

We now prove that the statement is true for $m = n + 1$

This means that $\left|\emptyset_{i,m+1}(tq^{-k}) - \emptyset_{i,m}(tq^{-k})\right|$ becomes

$$\left|\emptyset_{i,n+2}(tq^{-k}) - \emptyset_{i,n+1}(tq^{-k})\right|$$

And

$$\left|\emptyset_{i,m+1}(tq^{-k}) - \emptyset_{i,m}(tq^{-k})\right| = \left|\emptyset_{i,n+2}(tq^{-k}) - \emptyset_{i,n+1}(tq^{-k})\right|$$

$$= \left| c_i(t) + \int_t^{tq^{-k}} f_i\left(qu, \emptyset_{0,n+1}(qu), \emptyset_{1,n+1}(qu), \cdots, \emptyset_{p,n+1}(qu)\right) d_q u - c_i(t) - \int_t^{tq^{-k}} f_i\left(qu, \emptyset_{0,n}(qu), \emptyset_{1,n}(qu), \cdots, \emptyset_{p,n}(qu)\right) d_q u \right|$$

$$= \left| \int_t^{tq^{-k}} f_i\left(\left(qu, \emptyset_{0,n+1}(qu), \cdots, \emptyset_{p,n+1}(qu)\right) - f_i\left(qu, \emptyset_{0,n}(qu), \cdots, \emptyset_{p,n}(qu)\right)\right) d_q \right|$$

It implies from (4.26) we have

$$\left| \int_t^{tq^{-k}} f_i\left(\left(qu, \emptyset_{0,n+1}(qu), \cdots, \emptyset_{p,n+1}(qu)\right) - f_i\left(qu, \emptyset_{0,n}(qu), \cdots, \emptyset_{p,n}(qu)\right)\right) d_q \right|$$

$$\leq \int_t^{tq^{-k}} A\left(\left|\emptyset_{0,n+1}(qu) - \emptyset_{0,n}(qu)\right| + \cdots + \left|\emptyset_{p,n+1}(qu) - \emptyset_{p,n}(qu)\right|\right) d_q u$$

By induction assumption we have

$$\int_t^{tq^{-k}} A\left(\left|\emptyset_{0,n+1}(qu)-\emptyset_{0,n}(qu)\right|+\cdots+\left|\emptyset_{p,n+1}(qu)-\emptyset_{p,n}(qu)\right|\right)d_qu$$

$$\leq AC\int_t^{tq^{-k}}\left(A^{n-1}(p+1)^{n-1}(p+1)q^{\frac{n(n-1)}{2}}\frac{u^nq^{n(1+k)}(1-q)^n}{q^{nk}(q;q)_n}\right)d_qu$$

$$=CA^n(p+1)^nq^{\frac{n(n-1)}{2}}\int_t^{tq^{-k}}\left(\frac{u^nq^{n(1+k)}(1-q)^n}{q^{nk}(q;q)_n}\right)d_qu$$

This means that

$$CA^n(p+1)^nq^{\frac{n(n-1)}{2}}\int_t^{tq^{-k}}\left(\frac{u^nq^{n(1+k)}(1-q)^n}{q^{nk}(q;q)_n}\right)d_qu$$

$$=CA^n(p+1)^nq^{\frac{n(n-1)}{2}}q^nq^{nk}\left(\frac{(tq^{-k})^{n+1}}{q^{nk}[n+1]_q!}-\frac{t^{n+1}}{q^{nk}[n+1]_q!}\right)$$

It implies

$$CA^n(p+1)^nq^{\frac{n(n-1)+2n}{2}}\quad\left(\frac{(tq^{-k})^{n+1}}{[n+1]_q!}-\frac{t^{n+1}}{[n+1]_q!}\right)$$

$$=CB^nq^{\frac{n(n+n)}{2}}\quad\left(\frac{(tq^{-k})^{n+1}}{[n+1]_q!}-\frac{t^{n+1}}{[n+1]_q!}\right)$$

Since $B=A(P+1)$.

Also,

$$CB^nq^{\frac{n(n+1)}{2}}\left(\frac{(tq^{-k})^{n+1}}{[n+1]_q!}-\frac{t^{n+1}}{[n+1]_q!}\right)$$

$$=CB^nq^{\frac{n(n+1)}{2}}\frac{t^{n+1}(q^{-k(n+1)}-1)}{q^{k(n+1)}[n+1]_q!}$$

$$\leq CB^n q^{\frac{n(n+1)}{2}} \frac{t^{n+1}(1-q^{k(n+1)})}{q^{k(n+1)}[n+1]_q!}$$

$$\leq CB^n q^{\frac{n(n+1)}{2}} \frac{t^{n+1}}{q^{n(k+1)}[n+1]_q!}$$

$$= CB^n q^{\frac{n(n+1)}{2}} \frac{t^{n+1}}{q^{n(k+1)}[n+1]_q!}$$

$$= CB^n q^{\frac{n(n+1)}{2}} \frac{t^{n+1}(1-q)^{n+1}}{q^{k(n+1)}(q;q)_{n+1}}$$

$$\leq CB^n q^{\frac{n(n+1)}{2}} \frac{t^{n+1}(1-q)^{n+1}}{q^{k(n+1)}(q;q)_{n+1}}$$

Note that: *the inequality*

$$\left|\int_a^a f(t) d_q t\right| \leq \int_a^a |f(t)| d_q t, \quad (0 \leq a < b < \infty).$$

is not valid always. (Annaby and Mansour, 2012)

2) We show that $\emptyset_{i,m}$ tends to a function $\emptyset_i, x \in [qa, \infty)$.

Now from (4.31) we have

$$\emptyset_{i,m}(tq^{-k}) = \emptyset_{i,1}(tq^{-k}) + \sum_{l=1}^{m-1} \emptyset_{i,l+1}(tq^{-k}) - \emptyset_{i,l}\,(tq^{-k})$$

$$= \emptyset_{i,m}(tq^{-k}) + \left(\emptyset_{i,2}(tq^{-k}) - \emptyset_{i,1}(tq^{-k})\right) + \cdots + \left(\emptyset_{i,m}(tq^{-k}) - \emptyset_{i,m-1}(tq^{-k})\right)$$

It implies that

$$\left|\emptyset_{i,m}(tq^{-k})\right| \leq |c_i(tq^{-k})| + \sum_{l=1}^{m-1} \left|\emptyset_{i,l+1}(tq^{-k}) - \emptyset_{i,l}\,(tq^{-k})\right|$$

$$= |c_i(tq^{-k})| + \sum_{l=1}^{m-1} CB^l \, q^{\frac{l(l+1)}{2}} \frac{t^{l+1}(1-q)^{l+1}}{q^{lk+k}(q;q)_{l+1}}, \quad l \in \mathbb{N}$$

Now as $m \to \infty$ we have

$$C \sum_{l=1}^{\infty} B^l \, q^{\frac{l(l+1)}{2}} \frac{t^{l+1}(1-q)^{l+1}}{q^{lk+k}(q;q)_{l+1}} = C \sum_{l=1}^{\infty} A^l \, (p+1)^l q^{\frac{l(l+1)}{2}} \frac{t^{l+1}(1-q)^{l+1}}{q^{lk+k}(q;q)_{l+1}}$$

$$\leq C \sum_{l=1}^{\infty} B^{l-1} \, q^{\frac{(l-k)(l+1)}{2}} \frac{t^{l+1}(1-q)^{l+1}}{(q,q)_{l+1}}$$

Now $(l-k)(l+1)$ is a parabola with minimum value at $l = \frac{k-1}{2}$, where k is an arbitrary point. So we have

$$q^{\frac{(l-k)(l+1)}{2}} \leq \; q^{-\frac{(k+1)^2}{8}} = \frac{1}{q^{\frac{(k+1)^2}{8}}}$$

$$\text{Let } h \coloneqq \min\left\{\frac{1}{A(p+1)}, a, \frac{1}{q^{\frac{(k+1)^2}{8}}}\right\}$$

Then we have

$$C \sum_{l=1}^{\infty} B^{l-1} \, q^{\frac{(l-k)(l+1)}{2}} \frac{t^{l+1}(1-q)^{l+1}}{(q,q)_{l+1}} \leq C \sum_{l=1}^{\infty} h^{-l-1} \frac{h^{l+1}(1-q)^{l+1}}{(q,q)_{l+1}}$$

$$C \sum_{l=1}^{\infty} \frac{1}{(q,q)_{l+1}} = C(e_q^1 - 2)$$

By the Weiestrass m-Test it is uniformly continuous.

3) Now we show that $\{\emptyset_i\}_{i=0}^{p}$ is a solution of the initial value problem (4.25) in $[a, \infty)$. Indeed, from (4.26) we have

$$\left| f_i\left(x, \emptyset_{0,m}(x), \emptyset_{1,m}(x), \cdots, \emptyset_{p,m}(x)\right) - f_i\left(x, \emptyset_0(x), \emptyset_1(x), \cdots, \emptyset_p(x)\right) \right|$$

$$\leq A\left(\left|\emptyset_{0,m}(x)-\emptyset_0(x)\right|+\left|\emptyset_{1,m}(x)-\emptyset_1(x)\right|+\cdots+\left|\emptyset_{p,m}(x)-\emptyset_p(x)\right|\right)$$

for all $x \in [qa\ \infty)$ and for all $m \in \mathbb{N}$. Since the right-hand side of number (3) approaches uniformly to zero on$[qa\ \infty)$ as $m \to \infty$, it follows that for all $x \in [qa, \infty)$,

$$\lim_{m\to\infty} f_i\left(x, \emptyset_{0,m}(x), \emptyset_{1,m}(x), \cdots, \emptyset_{p,m}(x)\right) = f_i\left(x, \emptyset_0(x), \emptyset_1(x), \cdots, \emptyset_p(x)\right)$$

is uniformly on $[a, \infty)$.

By letting $m \to \infty$ in (4.29), we have

$$\emptyset_i(tq^{-k}) = g_i(t) + \int_t^{tq^{-k}} f_i\left(qu, \emptyset_0(qu), \cdots, \emptyset_p(qu)\right) d_q u, \qquad (4.32)$$

for all $t \in [qa, a), k \in \mathbb{N}_0$. By substituting $k = 0$ in (4.32), we can see that the initial conditions on (4.27) hold. And that the functions $\{\emptyset_i(x)\}_{i=0}^{p}$ is a solution to the q −initial Value problem (4.27) which valid in $[a, \infty)$.

Uniqueness:- In order to prove the uniqueness of the solution of the system (4.27) we assume that $\{\varphi_i(x)\}_{i=0}^{p}$ is another solution to (4.27) such that it the solution is valid in $[a, b] \subseteq [a, \infty)$ and $\varphi_i(x) = g_i(x)$ for all x in $[qa, a)$, where $0 \leq i \leq p$.

Thus,

$$D_q\varphi_i = f_i\big(qx, \varphi_0(qx), \varphi_1(qx), \cdots, \varphi_p(qx)\big) \quad (0 \leq i \leq p;\ \ x \in [a, b]).$$

Consequently,

$$\varphi_i(x) = \varphi_i(qx) + x(1-q)f_i\left(qx, \varphi_0(qx), \varphi_1(qx), \cdots, \varphi_p(qx)\right); \qquad (4.33)$$

In addition,

$$\emptyset_i(x) = \emptyset_i(qx) + x(1-q)f_i\left(qx, \emptyset_0(qx), \emptyset_1(qx), \cdots, \emptyset_p(qx)\right). \qquad (4.34)$$

Now, consider the equations (4.33) and (4.34), by subtracting (4.33) from (4.34) and applying (4.26) we have

$$|\emptyset_i(x) - \varphi_i(x)| \leq |\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q| \sum_{i=0}^{p} |\emptyset_i(qx) - \varphi_i(qx)| \qquad (4.35)$$

Now by taking summation on both-sides of (4.35) we have

$$\sum_{i=0}^{p} |\emptyset_i(x) - \varphi_i(x)| \leq \sum_{i=0}^{p} |\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q|(P+1) \sum_{i=0}^{p} |\emptyset_i(qx) - \varphi_i(qx)|$$

Now let $\sigma(x) \coloneqq \sum_{i=0}^{p} |\emptyset_i(x) - \varphi_i(x)|\,, x \in [a,b]$. It implies

$$\sum_{i=0}^{p} |\emptyset_i(x) - \varphi_i(x)| \leq \sum_{i=0}^{p} |\emptyset_i(qx) - \varphi_i(qx)| + A|x||1-q|(P+1) \sum_{i=0}^{p} |\emptyset_i(qx) - \varphi_i(qx)|$$

Will be;

$$\sigma(x) \leq \sigma(qx) + A(P+1)|1-q||x|\sigma(qx).$$

This means that

$$\sigma(x) \leq \sigma(qx)(1 + A(P+1)|1-q||x|) \text{ for all } x \in [a,b]$$

But, for any $x \in [a,b]$, there exist $t \in [aq, a)$ and $k \in \mathbb{N}_0$ such that $x = tq^{-k}$. This can be shown by the following proof.

Let $x \in [a,b]$, we acclaim that there exist $t \in [aq, a)$ or $qa \leq t < a$ and $k \in \mathbb{N}_0$ such that $t = xq$ or equivalently $x = tq^{-k}$.

Equivalently, we prove that for some $k \in \mathbb{N}_0$, $qa \leq xq^k < a$ and by the assumption $a \leq x \leq b$.

Now, let $A = \{r \colon xq^r < a\} \subseteq \mathbb{N}_0$.Since $0 < q < 1$, it implies $\lim_{r\to\infty} q^r = 0$ which is less than $''a''$ and the set A is not empty. Let k be $\min A$, this means that $xq^k < a$ which is contrary to the assumption that $qa < xq^k$. This means that $xq^k < qa$ which also implies $xq^{k-1} < a$.

Thus $k - 1 \in A$, contrary to the assumption that k is the minimum element of A. It implies that there exist $k \in \mathbb{N}_0$ such that $qa < xq^k < a$.

Let $xq^k = t$, then there exist $t \in [qa, a)$ such that $t = xq^k$ for some $k \in \mathbb{N}_0$ or $x = tq^{-k}$.

Now, by substituting x with tq^{-k} in $\sigma(x) \leq \sigma(qx)(1 + A(P+1)|1-q||x|)$ for all $x \in [a,b]$

We have,

$$\sigma(tq^{-k}) \leq \sigma(qtq^{-k})(1 + A(P+1)|1-q||tq^{-k}|)$$
$$= \sigma(tq^{1-k})(1 + A(P+1)|1-q||tq^{-k}|)$$

Also, by substituting t with tq^{1-k} in the above equation, the relation is valid for all $t \in [qa, a)$.Thus

$$\sigma(tq^{1-k}) \leq \sigma(tq^{2-k})(1 + A(P+1)|1-q||tq^{1-k}|)$$

But

$$\frac{\sigma(tq^{-k})}{(1 + A(P+1)|1-q||tq^{-k}|)} \leq \sigma(tq^{1-k})$$

This means that

$$\sigma(tq^{1-k}) \leq \sigma(tq^{2-k})(1 + A(P+1)|1-q||tq^{1-k}|)$$

Will become

$$\frac{\sigma(tq^{-k})}{(1 + A(P+1)|1-q||tq^{-k}|)} \leq \sigma(tq^{2-k})(1 + A(P+1)|1-q||tq^{1-k}|)$$

Thus, by transitive property we have

$$\sigma(tq^{-k}) \leq \sigma(tq^{2-k})(1 + A(P+1)|1-q||tq^{-k}|)(1 + A(P+1)|1-q||tq^{1-k}|)$$

In the same manner by induction we have following

$$\sigma(tq^{-k}) \leq \prod_{j=1}^{k} \left(1 + A(1+p)|1-q| \left|q^k \frac{t}{q^j}\right|\right) \sigma(t)$$

But $\sigma(t) = 0$ for $qa \leq t < a$ this means that $\sigma(tq^{-k}) = 0$ for all $t \in [qa, a)$ and $k \in \mathbb{N}_0$ such that $x = tq^{-k} \in [a, b]$.

$$\sigma(x) \leq \prod_{j=1}^{k} \left(1 + A(1+p)|1-q| \left|q^k \frac{t}{q^j}\right|\right) \sigma(0), \qquad x \in [a, b]$$

According to the definition of $\sigma(x)$ we have $\sigma(0) = 0$. So $\sigma(x) = 0$ which implies $\varphi(x) = \emptyset(x)$.

Theorem 4.1.4. Let $f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right)$ be functions defined for $x \in (0, b]$, $y_j \in \mathbb{C}$, where $0 \leq j, i \leq p$, and $b > 0$. Let $\{g_i(t)\}_{i=0}^{p}$ be a set of q −periodic functions such that the following conditions hold.

a) $f_i\left(x, g_{01}(x), g_1(x), \cdots, g_{p-1}(x), g_p(x)\right)$ are bounded on $(0, b]$.

b) There exist a positive constant A such that for any $x \in (0, b]$ and $y_r,\ \tilde{y}_r \in \mathbb{C}$, the following Lipschitz condition hold for $0 \leq i \leq p$.

$$\left|f_i(x, \tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_p) - f_i(x, y_0, y_1, \cdots, y_p)\right| \leq A\left(|\tilde{y}_0 - y_0| + \cdots + \left|\tilde{y}_p - y_p\right|\right) \quad (4.36)$$

Moreover, there exists a point $c \in (0, b)$ such that the following system

$$D_q y_i(x) = f_i\left(x, y_0(x), y_1(x), \cdots, y_p(x)\right) \quad (4.37)$$

$$\emptyset_i(x) = g_i(x) \qquad (x \in (c, cq^{-1}]), i = 0,1,2, \dots, p)$$

has a unique solution $\{\emptyset_\mathrm{i}\}_{\mathrm{i}=0}^{\mathrm{p}}$ which is valid in$(0, \mathrm{c}]$.

Proof. The proof of this theorem follows in similar way to that of the previous theorem (4.26).

CHAPTER 5
CONLUSION

5.1 Conclusion

In this thesis we have introduced the q-difference equations and their properties. Recently, a lot of mathematicians work on this area and they found several forms of q-difference equations. In (Agarwal et al, 2015), they consider the following q-difference equation

$$\begin{cases} D_q^2 u(t) = f\big(t, u(t)\big) + I_q g\big(t, u(t)\big), \ \ t \in I_q = [0, T] \cup \{0\}, \\ u(0) = \eta u(T), \qquad\qquad D_q\big(u(0)\big) = \eta D_q u(T), \end{cases} \tag{5.1}$$

which is the second order Cauchy problem and we discussed about this case in a general form. In this case some boundary value is changed. They solved this q-difference equation by using individual solution in a form of Jackson integral and by using this operator; they showed existence and uniqueness of this equation. In (Ahmad et al, 2012.) and (Ahmad et al, 2016.), they used similar methods to solve following q $-$difference equation as well.

$$\begin{cases} D_q^2 u(t) = f\big(t, u(t)\big), \qquad\qquad\qquad t \in I, \\ u(0) = \eta u(T), \qquad D_q\big(u(0)\big) = \eta D_q u(T). \end{cases} \tag{5.2}$$

These investigations motivate us to solve q-difference equation with new boundary points. In this case, we will discuss about system of q-difference equation with another boundary value such that the above cases can be represented as a special case.

For instance, let us assume the following system of q $-$difference equation:

$$\begin{cases} D_q u(t) = u_1(t), & u(0) = \eta u(T), \\ D_q u_1(t) = D_q^2 u(t) = u_2(t), & u_1(0) = D_q u(0) = \eta D_q u(T), \\ \quad\vdots & \\ D_q u_{n-2}(t) = D_q^{n-1} u(t) = u_{n-1}(t), & u_{n-2}(0) = D_q^{n-2} u(0) = \eta D_q^{n-2} u(T), \\ D_q D_q^{n-1} u(t) = g\left(t, u(t), D_q u(t), \ldots, D_q^{n-1} u(t)\right) & D_q^{n-1} u(0) = \eta D_q^{n-1} u(T), \end{cases}$$

This system of q −difference equations is a general form of (5.1) and (5.2). Actually this system demonstrates the Cauchy problem in an order n. Following proposition states the existence and uniqueness of this system.

Proposition 5.1.1. Let E_n be a neighborhood of $\eta D_q^n u(T)$ for a fixed value of T such that the radius of this neighbor is less than β, also let I be an interval in around zero, In addition let the following statements holds true:

a) For any fix value of y_n in E_n function $g(t, y_1, y_2, \dots, y_n)$ and $D_q^k u(t)$ are continuous at zero for $k = 1, \dots, n-1$.

b) g satisfy Lipschitz condition

$$|g(t, y_1, y_2, \dots, y_n) - g(t, \widetilde{y_1}, \widetilde{y_2}, \dots, \widetilde{y_n})| \le A(|y_1 - \widetilde{y}_1| + \cdots + |y_n - \widetilde{y}_n|)$$

where $t \in I, A > 0, y_k, \widetilde{y_k} \in E_k$.

Then the given system of q-difference equations has a unique solution in a neighborhood of zero.

Proof. Procedure of proof is exactly similar to the main theorem in a previous chapter. We focus on a sequence of functions that reach to the solution. If we apply the same recurrence sequences to find the solution, then we have:

$$\varphi_{i,k}(t) = \begin{cases} \eta u(T), & k = 1, \\ \eta D_q^{k-1} u(T) + \int_0^t f_i\left(t, \varphi_{0,k-1}(t), \dots \varphi_{n,k-1}(t)\right) d_q t & k \ge 2. \end{cases}$$

where $f_n = g$ and $f_i\big(t, u(t), \dots u_n(t)\big) = D_q^i u(t) = u_i(t)$ $i = 1, \dots n-1$. Since the fundamental theorem of calculus for Jackson integral is true, we may rewrite this sequence by taking integral from q-derivative.

Let us focus on the special case of this system when the order of q −Cauchy problem is n, then we have the q − difference equation with initial values as follow:

$$\begin{cases} D_q^n u(t) = f\left(t, u(t), D_q u(t), \dots D_q^{n-1} u(t)\right), & t \in I, \\ u(0) = \eta u(T), \qquad D_q^k u(0) = \eta D_q^k u(T) & k = 1,2, \dots n-1 \end{cases}$$

According to Proposition 5.1.1, this equation has unique solution. Actually, this is the general forms of 5.2.

Lemma 5.1.2. Following relation for interchanging the order of Jackson integral holds true

$$\int_0^t \left(\int_0^v f(s, u(s))\, d_q s\right) d_q v = \int_0^t \left(\int_{qs}^t f(s, u(s))\, d_q v\right) d_q s$$

Proof. We use the definition of Jackson integral to prove this relation. We may write the right side of this equation as follow

$$\begin{aligned} \int_0^t \left(\int_{qs}^t f(s, u(s))\, d_q v\right) d_q s &= \int_0^t \left((t - qs)(1-q) \sum_{i=0}^{\infty} q^i f(s, u(s))\right) d_q s \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} t(1-q)^2 \left(t - q q^j t\right) q^{j+i} f\left(q^j t, u(q^j t)\right) \\ &= \sum_{j=0}^{\infty} t^2 (1-q)(1 - q^{j+1})\, q^j f\left(q^j t, u(q^j t)\right) \\ &= \sum_{j=0}^{\infty} t^2 (1-q)^2 \left(\sum_{i=0}^{\infty} q^{2i}\right) \frac{1-q^2}{1-q} \left(1 - q^{j+1}\right) q^j f\left(q^j t, u(q^j t)\right) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t(1-q)^2\, t\, q^{2i+j} f\left(q^j t, u(q^j t)\right) \\ &= \int_0^t \left(v(1-q) \sum_{j=0}^{\infty} q^j f\left(q^j t, u(q^j t)\right)\right) d_q v \\ &= \int_0^t \left(\int_0^v f(s, u(s))\, d_q s\right) d_q v. \end{aligned}$$

Example 5.1.3. We may solve 5.2 by using the following function. Integrating the equation $D_q^2 u(t) = f(t, u(t))$, we get

$$D_q u(t) = \int_0^t f\big(s, u(s)\big)\, d_q s + b_1. \tag{5.3}$$

Again taking integral from (5.3) lead us to

$$u(t) = \int_0^t \left(\int_0^v f\big(s, u(s)\big)\, d_q s \right) d_q v + b_1 t + b_2. \tag{5.4}$$

If we change the order of integration, we lead to

$$u(t) = \int_0^t (t - qs) f\big(s, u(s)\big)\, d_q s + b_1 t + b_2. \tag{5.5}$$

By substituting in the equation, we can find b_1 and b_2.

Note 5.1.4. Previous example introduced an operator to make a successive approximation. Indeed, the given operator could be fined by the sequences at Proposition 5.1.1.

REFERENCES

Adams, C. R.. (1925). Note on the existence of analytic solutions of non-homogeneous linear $q-$difference equations, ordinary and partial. *Annals of Mathematics, 27*(2), 73–83.

Agarwal, P. R., Ahmad, B., Zhang, L., Hobiny, A., Wang, G. (2015). Existence and nonexistence of solutions for nonlinear second order $q-$integro-difference equations with non-separated boundary condition. *Journal of Nonlinear Science and Applications 8,* 976-985.

Adams, C.R. (1928/1929). On the linear ordinary q-difference equation. *Annals of Mathematics, (2) 30*(1–4), 195–205

Adams, C.R. (1928/1929). Note on the existence of analytic solutions of non-homogeneous linear $q-$difference equations, ordinary and partial. *Annals of Mathematics,(2) 30*(1–4), 626

Annaby, M. H. & Mansour, Z. S. (2012). $q-$Fractional Calculus and Equations, *$q-$Difference Equation* (pp. 41-56). Berlin Heidelberg, BH: Springer-Verlag.

Adam, C. R.. (1928). On the linear ordinary $q-$difference equation, *Annals of Mathematics. 30*(1-4), 195-205

Ashyralyev, A. (2013). Computational Mathematics Textbook. South Kazakhstan State University.

Ahmad, B., Alsaedi, A., Ntouyas, S. K. (2012). A study of second-order q-difference equations with boundary conditions. *Advances in Difference Equation, a Springer Open Journal*.

Bender, C. M & Orszag, S. A (1978). Advanced Mathematical Method for Scientist and Engineers. McGraw-Hill

Boyce, W. E. & Diprima, R.C. (1992). Elementary Differential Equation and Boundary Value Problems, Wiley

Bangerezako, G. (2008). An Introduction to $q-$Difference Equation. Preprint, Bujumbura

Birkhoff, G. D. (1941). Note on a canonical form for the linear q-differences. *Proc. Natl. Acad. Sci.27,* 218-222

Carmichael, R. D. (1912). The general theory of linear $q -$ difference equations. *American Journal of Mathematics 34*, 147-168

Carmichael, R.D. (1911). Linear difference equations and their analytic solutions. *Transaction. Of American Mathematical. Society. 12*(1), 99–134

Chen, B.Q., Chen, X.Z. & Li, S. (2010). Properties on solutions of some $q -$difference equations. *Acta Mathematica Sinica, English Series. 26*(10), 1877–1886

Coddington, E. A & Levinson, N. (1955). Theory of Ordinary Differential Equations. New York, NK: McGraw-Hill.

Carmichael, R.D.(1912).The general theory of linear q-difference equations. *American Journal of Mathematic. 34,* 147-168.

Carmichael, R. D. (1913). On the theory of liner difference equations. *American Journal of mathematics. 35*(2), 163-182

Gupta, D.B. (2011). Mathematical Physics, New Delhi, ND: Vicas.

Ernst, T. (2002). The History of $q -$calculus and new method, *Student's Phd Dissertation 25, Uppsala University, Sweden.*

Ernst, T. (2012). Comprehensive Treatement of $q -$Calculus. Birkhaű̋ser Basel.

Eastham, M. S.P. (1970). Theory of Ordinary Differential Equations. New York, NK: Van Nostrand, Reinhold.

Exton, H. (1983). $q -$Hypergeometric Functions and Application. United Kingdom, UK: Ellis-Horwood, Chichester.

Jackson, F.H. (1909). A q-form of Taylor's theorem; *Messenger Math. 38* ,62–64.

Jackson, F. H. (1910) . q-Difference equations, *American Journal of Mathematics. 32,* 305-314.

Jackson, F.H. (1908). On q-functions and a certain difference operator; *Transaction of Society of Edinburg. 46,* 64–72

Kac, V. & Cheung, P. (2002). Quantum Calculus. New York, NY: Springer-Verlag

Kolmogorov, A.N. & Fomin, S.V. (1957). Introductory Real Analysis. New York, NK: Dover Publications.

Kreyszig, E. (1978). Introductory Functional Analysis with Applications. United States of America, US: John Wiley & Sons Inc.

Liu, D. Q. (1995). The linear q −difference equation $y(x) = ay(qx) + f(x)$. Appl. Math *belt. 8*(1), 15-18

Mahmudov, M. I. & Momenzadeh, M. (2014). On a class of q −Bernoulli, q −Euler and q −Genocchi Polynomials. *Research Gate,* doi: 10.1155/2014/696454. Source: arXiv

Mason, T.E. (1915). On properties of the solutions of linear q −difference equations with entire function coefficients. *American Journal of Mathematics, 37*(4), 439–444

Nagle, R. K., Snider, A. D. & Saff, E. B. (2012). Fundamentals of Differential Equations and Boundary Value Problems. Pearson

Poria, S. & Dhiman, A. (2013). Existence and uniqueness theorem for ODE, an overview. Department of mathematics, university of Calcutta (india).

Rudin, W. (1976). Principle of Mathematical Analysis. McGraw-Hill, Inc.

Richard, C. (2009). On q −functional equations and excursion moments. *Discrete Math. 309*(1), 207–230

Romanenko, E.Yu. (2001). Differential-difference equations reducible to difference and q −difference equations. *Comput. Math. Appl. 42*(3–5), 615–626

Trjitzinsky, W.J. (1933). Analytic theory of linear q −difference equations. *Acta Mathematica. 61*(1), *1–38*