YASMINA F OMAR BADER

FRACTIONAL CALCULUS AND IT'S APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
YASMINA F OMAR BADER

In Partial Fulfilment of the Requirements for The Degree of Master of Science in Mathematics

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Approval of Director of Graduate School of Applied Sciences

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my parents

ABSTRACT

In this thesis fractional calculus and its applications to stability for the fractional Basset equation are studied. Most important properties of fractional order integrals and derivatives are discussed. In applications, methods for the solutions of initial value problem for fractional differential equations are considered. Stability of initial value problem is illustrated with a special type of fractional differential equation.

$$A\,D_t x(t) + B D_t^{\alpha} x(t) + C g(x) = f(t)\,,$$

where $A \neq 0$ and $B, C \in R,\ 0 < \alpha < 1$ which is known as Basset equation.

***Keywords*:** Fractional calculus; Fractional differential equations; Basset equation; Stability; Numerical solution

ÖZET

Bu tezde kesirli kalkülüs ve Basset denklemi için kararlılığa uygulamaları incelenmiştir. Kesirli mertebeden integrallerin ve türevlerin en önemli özellikleri tartışılmştır. Uygulamalarda, kesirli diferansiyel denklemler için başlangıç değer probleminin çözümleri için yöntemler göz önüne alınmıştır. Başlangıç değer probleminin kararlılığı, $A \neq 0$ ve $B, C \in R,\ 0 < \alpha < 1$ olarak üzere Basset denklemi olarak bilinir.

$$A\, D_t x(t) + BD_t^{\alpha} x(t) + Cg(x) = f(t)\,,$$

özel bir kesirli diferansiyel denklem için gösterilmiştir.

Anahtar Kelimeler: Kesirli hesap; Kesirli diferansiyel denklemler; Baset denklemi; Kararlılık; Sayısal çözüm

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CHAPTER 1
INTRODUCTION

The study of fractional calculus achieves a wide range of applications in many areas. Especially in computer engineering it becomes a popular subject. Moreover, fractional derivatives have been successfully applied to problems in system of biology, physics, chemistry and biochemistry [see, e.g, (Liu, Anh, & Turner, 2004; Yuste & Lindenberg, 2001) and the references given therein]. The history of it began with a letter from L'Hospital to Leibniz in which is asked the meaning of the derivative of order 1/2 in 1695. In 1738, Euler did the first attempt with observing the result of evaluation of the non - integer order derivative of a power function x^a has a meaning and right after in 1820, Lacroix repeated the Euler's idea and nearly found the exact formula for the evaluation of the half derivative of the power function x^a. Then, first definition for the derivative of arbitrary positive order suitable for any sufficiently good function, not necessarily a power function was given by Fourier (1822) as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^\alpha \, d\lambda \int_{-\infty}^{\infty} f(\mathrm{t}) \cos(\lambda x - t\lambda + \alpha\pi/2) \, dt \,. \qquad (1.1)$$

Near all of these studies, the first solution of a fractional order equation was made by Abel in 1823 with the formulation of the tautochrone problem as an integral equation

$$\int_a^x \frac{\varphi(t)}{(x-t)^\mu} dt = f(x), \qquad x > a, 0 < \mu < 1. \qquad (1.2)$$

After 1832, applications of the fractional calculus to the solution of some types of linear ordinary differential equations were seen in the papers of Liouville. His initial definition based on the formula for the differentiating an exponential function which may be expanded as the series

$$f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x} \quad \text{is}$$

$$D^{\alpha} f(x) = \sum_{k=0}^{\infty} c_k a_k^{\alpha} e^{a_k x} \ , \quad \text{for any complex } \alpha\,. \tag{1.3}$$

Starting from the definition (1.3), he obtained the formula for the differentiation of a power function and fractional integration which is known as Liouville's first formula

$$D^{-\alpha} f(x) = \frac{1}{(-1)^{\alpha}\Gamma(\alpha)} \int_0^{\infty} \varphi(x+t) t^{\alpha-1}\, dt, \tag{1.4}$$

$$-\infty < x < \infty, \mathrm{Re}\ \alpha > 0\,.$$

Next, Riemann's expression which was done when he was a student in 1847 has become one of the main formula with Liouville's construction. Riemann had lastly arrived the expression:

$$\frac{1}{\Gamma(\alpha)} \int_0^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt\,, \qquad x > 0\,. \tag{1.5}$$

Studies on fractional calculus achieved a significant and suitable level for modern mathematicians after 1880's. Being more applicable and veritable greatly enhanced the power of fractional calculus. Therefore, need of efficient and reliable techniques to solve the problems which are modelled with fractional integral and differential operators occur. Liouville was the first person who tried to solve fractional differential equations as mentioned above. Then, some books written by (Miller & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1998; Samko, Kilbas, & Marichev, 1993) played a considerable role to understand the subject and gave the applications of fractional differential equations and methods for solutions.

In the present study, fractional calculus and it's applications to stability for the fractional Basset equation are considered. Most important properties of fractional order integrals and derivatives are discussed. This material was written on the basis notes that were used in a graduate course at Near East University, Lefkoşa, Cyprus. In applications, methods for the solutions of initial value problem for fractional differential equations are considered. Stability of initial value problem is illustrated with a special type of fractional differential equation

$$A\,D_t x(t) + B D_t^{\alpha} x(t) + C g(x) = f(t)\ ,$$

where $A \neq 0$ and $B, C \in R,\ 0 < \alpha < 1$.

CHAPTER 2
RIEMANN – LIOUVILLE FRACTIONAL INTEGRAL

This chapter contain the definition and some properties of the Riemann-Liouville fractional integrals.

2.1 Auxiliary Lemma

We start this section by the first order integral operator I defined by the following formula

$$If(x) = \int_0^x f(s)ds\,.$$

From that it follows

$$I^2 f(x) = I\big(If(x)\big) = I\left(\int_0^x f(s)\,ds\right) = \int_0^x \int_0^y f(s)\,ds\,dy\,.$$

Therefore, the second order integral operator I^2 defined by the following formula

$$I^2 f(x) = \int_0^x (x-s)\,f(s)\,ds\,.$$

Lemma 2.1. The following formula is true

$$I^n f(x) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)\,!}\ f(s)\ ds \tag{2.1}$$

for any $n \epsilon N$.

Proof. Assume that (2.1) is true for $n = k$. That means

$$I^k f(x) = \int_0^x \frac{(x-s)^{k-1}}{(k-1)\,!}\ f(s)\ ds.$$

Now, we will prove (2.1) for $n = k + 1$.

Applying the definition of the integral of integer order, we get

$$I^{k+1} f(x) = I\Big(I^k\, f(x)\Big) = I\left[\int_0^x \frac{(x-s)^{k-1}}{(k-1)\,!}\ f(s)\, ds\right]$$

$$= \int_0^x \int_0^y \frac{(y-s)^{k-1}}{(k-1)\,!}\ f(s)\, ds\, dy\ .$$

Changing the order of integration and using
$\{0 \le y \le x, 0 \le s \le y\} = \{0 \le s \le x, s \le y \le x\}$, we get

$$I^{k+1} f(x) = \int_0^x \int_s^x \frac{(y-s)^{k-1}}{(k-1)\,!}\ f(s)\, dy\, ds\ = \int_0^x f(s) \int_s^x \frac{(y-s)^{k-1}}{(k-1)\,!}\ dy\, ds$$

$$= \int_0^x f(s)\ \frac{(x-s)^k}{k(k-1)!}\ ds = \int_0^x \frac{(x-s)^k}{k\,!}\ f(s)\, ds\ .$$

So, (2.1) is true for $n = k + 1$. By the induction it is true for any $n \epsilon N$. Lemma 2.1 is proved.

2.2 Riemann - Liouville fractional integral

Let us consider some of the starting points for a discussion of classical fractional calculus. One development begins with a generalization of repeated integration. In the same manner as Lemma 2.1 if f is locally integrable on (a, ∞), then n-fold integrated integral is given by

$$I^n f(x) = \int_a^x ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{n-1}} f(s_n)\, ds_n$$

$$= \frac{1}{(n-1)!} \int_a^x \frac{1}{(x-s)^{1-n}} f(s)\, ds \tag{2.2}$$

for almost all of x with $-\infty \le a < x < \infty$ and $n \in N$. Writing $(n - 1)! = \Gamma(n)$, an immediate generalization is the integral of f of fractional order $\alpha > 0$,

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{(x-s)^{1-\alpha}} f(s) ds \quad \text{(right hand)} \tag{2.3}$$

and similarly for $-\infty < x < b < \infty$

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{1}{(s-x)^{1-\alpha}} f(s) ds \quad \text{(left hand)} \tag{2.4}$$

both being defined for suitable f. When a = $-\infty$ Equation (2.3) is equivalent to Liouville's definition, and when a = 0 we have Riemann's definition. The right and left hand integrals $I_{a+}^{\alpha} f(x)$ and $I_{b-}^{\alpha} f(x)$ are related via Parseval equality (fractional integration by parts) which we give for convenience for a = 0 and $b = \infty$:

$$\int_0^\infty f(x) I_{0+}^{\alpha}\, g(x)\, dx = \int_0^\infty g(x) I_{\infty-}^{\alpha} f(x)\, dx\,. \tag{2.5}$$

Proof. Using the definition of I^{α}, we get

$$\int_0^{\infty} f(x)\, I_{0+}^{\alpha}\, g(x)\, dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x) \int_0^{x} \frac{1}{(x-s)^{1-\alpha}} g(s)\, ds\, dx\, .$$

Changing the order of integration and using
$\{0 \le x < \infty,\ 0 \le s \le x\} = \{0 \le s < \infty,\ s \le x < \infty\}$, we get

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} g(s) \int_s^{\infty} \frac{f(x)}{(x-s)^{1-\alpha}}\, dx\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} g(x) \int_x^{\infty} \frac{f(s)}{(s-x)^{1-\alpha}}\, ds\, dx$$

$$= \int_0^{\infty} g(x) \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(s)}{(s-x)^{1-\alpha}}\, ds\, dx$$

$$= \int_0^{\infty} g(x) I_{\infty -}^{\alpha}\, f(x)\, dx\, .$$

The following properties are stated for right handed fractional integrals (with obvious changes in the case of left handed integrals). We will consider right hand fractional integral when a = 0 we will use the following notation

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x} \frac{1}{(x-s)^{1-\alpha}} f(s) ds \qquad (2.6)$$

for the Riemann-Liouville integral operator I^{α} of order α . We have the following properties of the Riemann - Liouville integral operator I^{α} of order α .

1) The Riemann - Liouville integral operator I^{α} of order α is a linear operator. That means

$$I^{\alpha}\big(a\mathrm{f}(x)+\ bg(x)\big)=a\,I^{\alpha}\mathrm{f}(x)+\ bI^{\alpha}g(x), a,b\in R, \alpha\in R^{+}.$$

Proof. Using the definition of I^{α}, we get

$$I^{\alpha}\big(a\mathrm{f}(x)+\ bg(x)\big)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{a\mathrm{f}(s)+\ bg(s)}{(x-s)^{1-\alpha}}\,ds$$

$$=a\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\mathrm{f(s)}\,ds}{(x-s)^{1-\alpha}}+\ b\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\mathrm{f(s)}\,ds}{(x-s)^{1-\alpha}}$$

$$=a\,I^{\alpha}\mathrm{f}(x)+bI^{\alpha}g(x)\,.$$

2) The following semigroup properties hold

$$I^{\alpha}\left(I^{\beta}\mathrm{f}(x)\right)=\ I^{\alpha+\beta}\big(\mathrm{f}(x)\big), \alpha,\beta\in R^{+}.$$

Proof. Using the definition of fractional integral operator, we get

$$I^{\alpha}\left(I^{\beta}\mathrm{f}(x)\right)=I^{\alpha}\left[\frac{1}{\Gamma(\beta)}\int_{0}^{x}\frac{\mathrm{f}(s)\,ds}{(x-s)^{1-\beta}}\right]$$

$$=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{1}{(x-y)^{1-\alpha}}\frac{1}{\Gamma(\beta)}\int_{0}^{\mathrm{y}}\frac{\mathrm{f}(s)\,ds}{(y-s)^{1-\beta}}\,dy\,.$$

Changing the order of the integration, we get

$$I^{\alpha}\left(I^{\beta}\mathrm{f}(x)\right)=\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_{0}^{x}\int_{0}^{y}\frac{1}{(x-y)^{1-\alpha}}\,\frac{1}{(y-s)^{1-\beta}}\,\mathrm{f}(s)\,ds\,dy$$

$$=\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_{0}^{x}\left[\int_{s}^{x}\frac{1}{(x-\mathrm{y})^{1-\alpha}}\,\frac{dy}{(\mathrm{y}-\mathrm{s})^{1-\beta}}\right]\mathrm{f}(\mathrm{s})\,ds\,.$$

Now, we will obtain the integral

$$A(x,s)=\int_{s}^{x}\frac{1}{(x-y)^{1-\alpha}}\,\frac{dy}{(y-s)^{1-\beta}}\,. \tag{2.7}$$

Putting $y-s=t$, we get $dy=dt$ and

$$A(x,s)=\int_{0}^{x-s}\frac{dt}{(x-s-t)^{1-\alpha}\,t^{1-\beta}}\,.$$

Putting $t=(x-s)u$, we get $dt=(x-s)du$. Then

$$A(x,s)=\int_{0}^{1}\frac{(x-s)du}{(x-s)^{1-\alpha}\,(1-u)^{1-\alpha}\,(x-s)^{1-\beta}\,u^{1-\beta}}$$

$$=\frac{1}{(x-s)^{1-(\alpha+\beta)}}\int_{0}^{1}\frac{du}{(1-u)^{1-\alpha}\,u^{1-\beta}}$$

$$=\frac{1}{(x-s)^{1-(\alpha+\beta)}}\int_{0}^{1}(1-u)^{\alpha-1}\,u^{\beta-1}\,du$$

$$=\frac{1}{(x-s)^{1-(\alpha+\beta)}}\,B(\alpha\,,\beta)$$

$$= \frac{1}{(x-s)^{1-(\alpha+\beta)}} \; \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_s^x \frac{dy}{(x-y)^{1-\alpha}\,(y-s)^{1-\beta}} = \frac{1}{(x-s)^{1-(\alpha+\beta)}} \; \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

Then

$$I^{\alpha}\left(I^{\beta}\mathrm{f}(x)\right) = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^x \frac{1}{(x-\mathrm{s})^{1-(\alpha+\beta)}} \; \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\mathrm{f}(\mathrm{s})\,ds$$

$$= \frac{1}{\Gamma(\alpha+\beta)}\int_0^x \frac{\mathrm{f}(\mathrm{s})\,ds}{(x-s)^{1-(\alpha+\beta)}} \;=\; I^{\alpha+\beta}\big(\mathrm{f}(x)\big)\,.$$

3) The following commutative properties hold

$$I^{\alpha}\left[I^{\beta}\mathrm{f}(x)\right] = I^{\beta}\,[I^{\alpha}\mathrm{f}(x)], \alpha,\beta \in R^{+}\,.$$

Proof. Applying semigroup properties, we get

$$I^{\alpha}\left[I^{\beta}\mathrm{f}(x)\right] = I^{\alpha+\beta}\mathrm{f}(x) = I^{\beta+\alpha}\mathrm{f}(x)$$

$$= \; I^{\beta}[I^{\alpha}\mathrm{f}(x)]\,.$$

4) Introduce the following causal function (Vanishing for $x < 0$)

$$\Phi_{\alpha}(x) = \frac{x_{+}^{\alpha-1}}{\Gamma(\alpha)}\,, \alpha > 0\,.$$

Then, we have that

a) $\phi_\alpha(x) * \phi_\beta(x) = \phi(x)_{\alpha+\beta}\,, \alpha, \beta \in\ R^+$,

b) $I^\alpha\, f(x) = \ \phi_\alpha(x) * f(x),\ \alpha\ \in\ R^+$.

Proof. By the definition of convolution operator , we have that

$$\phi_\alpha(x) * \phi_\beta(x) = \int_0^x \phi_\alpha(s)\ \phi_\beta(x-s)ds$$

$$= \int_0^x \frac{s^{\alpha-1}}{\Gamma(\alpha)} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x s^{\alpha-1}(x-s)^{\beta-1}\, ds = \ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x y^{\alpha-1}(x-y)^{\beta-1}\, dy\,.$$

From 2.7. It follows that

$$\phi_\alpha(x) * \phi_\beta(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\ A(x,0)\,.$$

Therefore

$$\phi_\alpha(x) * \phi_\beta(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{x^{1-(\alpha+\beta)}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,.$$

a) is proved. b) follows from the definition of convolution and definition of fractional integral operator

$$\phi_\alpha(x) * f(x) = \int_0^x \phi_\alpha(x-s)\, f(s)\, ds$$

$$= \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s)\, ds = I^\alpha f(x)\,.$$

b) is proved.

5) For the Laplace transforms of I^α the following formula holds

$$\mathcal{L}\,\{I^\alpha f(x)\} = \frac{1}{s^\alpha}\,\mathcal{L}\,\{f\,(x)\}\,.$$

Proof. Applying the definition of the Laplace transform, we get

$$\mathcal{L}(I^\alpha f(x)) = \int_0^\infty e^{-sx}\, I^\alpha\big(f(x)\big)\, dx = \int_0^\infty e^{-sx}\, \phi_\alpha(x) * f(x)\, dx$$

$$= \mathcal{L}\,\{\phi_\alpha\,(x)\}\,\mathcal{L}\,\{f(x)\}\,.$$

Now, we will prove that

$$\mathcal{L}\,\{\phi_\alpha(x)\} = \int_0^\infty e^{-sx}\, \frac{x^{\alpha-1}}{\Gamma(\alpha)}\, dx = \frac{1}{s^\alpha}\,.$$

Putting $= p$, we get $dx = \frac{dp}{s}$. Then

$$\mathcal{L}\,\{\phi_\alpha\,(x)\} = \int_0^\infty \frac{e^{-p}\left(\frac{p}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\, \frac{dp}{s}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{s^{\alpha}} \int_{0}^{\infty} e^{-p} p^{\alpha-1} dp = \frac{1}{\Gamma(\alpha)} \frac{1}{s^{\alpha}} \Gamma(\alpha) = \frac{1}{s^{\alpha}}.$$

6) Effect on power functions is satisfied.

$$I^{\alpha}(x^{\beta}) = \frac{x^{\beta+\alpha}}{\Gamma(\beta+1+\alpha)} \Gamma(\beta+1) \quad \text{for all } \alpha > 0 \text{ and } \beta > -1, x > 0.$$

Proof. Using the definition of fractional integral of I^{α} and the property of $B(\alpha, \beta)$ function, we get

$$I^{\alpha}(x^{\beta}) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{s^{\beta}}{(x-s)^{1-\alpha}} ds.$$

Putting $s = xp$, we get $ds = x\, dp$. Then

$$I^{\alpha}(x^{\beta}) = \frac{1}{\Gamma(\alpha)} \int_{0}^{1} \frac{x^{\beta} p^{\beta}}{x^{1-\alpha} (1-p)^{1-\alpha}} x\, dp = \frac{x^{\beta+\alpha}}{\Gamma(\alpha)} \int_{0}^{1} p^{\beta} (1-p)^{\alpha-1} dp$$

$$= \frac{x^{\beta+\alpha}}{\Gamma(\alpha)} B(\beta+1, \alpha) = \frac{x^{\beta+\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} = x^{\beta+\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}.$$

Note.

1) $I^{\alpha}(1) = \frac{1}{\Gamma(1+\alpha)} x^{\alpha}$ for all $\alpha > 0$, $x > 0$.

2) Let $f(x)$ be an analytic function, then

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{1}{(x-s)^{1-\alpha}} f(s) ds$$

$$= f(0) \frac{x^{\alpha}}{\Gamma(1+\alpha)} + f'(0) \frac{x^{\alpha+1}}{\Gamma(2+\alpha)} + \dots + f^{(n)}(0) \frac{x^{\alpha+n}}{\Gamma(n+1+\alpha)} + \cdots$$

for all $\alpha > 0$.

Applying this formula, we can obtain the fractional integral of order $\alpha > 0$ from elementary functions, for example, we have that

$$I^{\alpha}(e^{\alpha x}) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{e^{as}}{(x-s)^{1-\alpha}} ds$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} + a \frac{x^{\alpha+1}}{\Gamma(2+\alpha)} + \dots + a^{n} \frac{x^{\alpha+n}}{\Gamma(n+1+\alpha)} + \dots$$

for all $\alpha > 0, x \geq 0$.

CHAPTER 3

CAPUTO FRACTIONAL DIFFERENTIAL OPERATOR

This chapter contain the definition and some properties of the Caputo fractional differential operator.

Definition 3.1. Suppose that $\alpha > 0, x > 0, \alpha, x \in R$. The fractional operator

$$D_*^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \dfrac{f^{(n)}(s)\, ds}{(x-s)^{\alpha+1-n}}, & n-1 < \alpha < n \ \in N, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in N, \end{cases}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α.

Lemma 3.1. Let $n-1 < \alpha < n\,, n \in N\,, \alpha \ \in \ R$ and $f(x)$ be such that $D_*^{\alpha} f(x)$ exists. Then

$$D_*^{\alpha} f(x) = I^{\lceil \alpha \rceil - \alpha} D^{\lceil \alpha \rceil} f(x)\ .$$

This mean that the Caputo fractional operator is equivalent to $(\lceil \alpha \rceil - \alpha)$-fold integration after $\lceil \alpha \rceil$-th order differentiation.

We have the following properties of the Caputo fractional differential operator D_*^{α} of order α.

If $f(x)$ and $g(x)$ are sufficiently smooth function. Then

1) The Caputo fractional differential operator D_*^{α} of order α is a linear operator. That means

$$D_*^{\alpha}\big(af(x)+bg(x)\big)=aD_*^{\alpha}f(x)+bD_*^{\alpha}g(x), a,b\in R, \alpha\in R^+ .$$

Proof. Using the definition of D_*^{α} , we get

$$D_*^{\alpha}\big(af(x)+bg(x)\big)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\,\frac{d^{\lceil\alpha\rceil}}{ds^{\lceil\alpha\rceil}}[af(s)+bg(s)]\,ds$$

$$=a\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\,\frac{d^{\lceil\alpha\rceil}}{ds^{\lceil\alpha\rceil}}f(s)\,ds$$

$$+b\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\,\frac{d^{\lceil\alpha\rceil}}{ds^{\lceil\alpha\rceil}}\,g(s)\,ds$$

$$=aD_*^{\alpha}f(x)+b\,D_*^{\alpha}g(x).$$

2) The following non-semigroup properties hold

$$D_*^{\alpha}\,D_*^{\beta}f(x)\neq D_*^{\alpha+\beta}\,f(x), \alpha,\beta\in R^+.$$

Proof. Let $=1$, $\beta=\frac{1}{2}$, $f(x)=x$. Then applying the definition, we get

$$D_*^{1}D_*^{\frac{1}{2}}\,(x)=D(D_*^{\frac{1}{2}}(x))$$

$$D_*^{\frac{1}{2}}(x)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{1}{(x-s)^{\frac{1}{2}}}\,\frac{d}{ds}(s)\,ds$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{ds}{(x-s)^{\frac{1}{2}}}=\frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$D_*^1(D_*^{\frac{1}{2}}(x)) = D_*^1\left(\frac{2\sqrt{x}}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{\pi\, x}},$$

and

$$D_*^{\frac{3}{2}}(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{1}{(x-s)^{\frac{1}{2}}} \frac{d^{\lceil 2 \rceil}}{ds^{\lceil 2 \rceil}}(s)\, ds$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x \frac{1}{(x-s)^{\frac{1}{2}}} \frac{d}{ds}(1)\, ds = 0 .$$

We see that

$$D_*^1(D_*^{\frac{1}{2}}(x)) = \frac{1}{\sqrt{\pi\, x}} \neq 0 = D_*^{\frac{3}{2}}(x) .$$

3) The following non-commutative properties hold
Suppose that $n-1 < \alpha < n$, $m, n \in N$, $\alpha \in R^+$ and $D_*^{\alpha} f(x)$ exists. Then in general

$$D_*^{\alpha}\, D^m\, f(x) = D_*^{\alpha+m} f(x) \neq D^m D_*^{\alpha} f(x).$$

Proof. Using the definition of D_*^{α} , we get

$$D_*^{\alpha}\, D^m\, f(x) = D_*^{\alpha}(D^m f(x)) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^x \frac{f^{\lceil \alpha \rceil + m}(s)\, ds}{(x-s)^{1-\lceil \alpha \rceil + \alpha}},$$

and

$$D_*^{\alpha+m} f(x) = \frac{1}{\Gamma\big((\lceil \alpha \rceil + m) - (\alpha + m)\big)} \int_0^x \frac{f^{\lceil \alpha \rceil + m}(s)\, ds}{(x-s)^{1-(\lceil \alpha \rceil + m)+(\alpha+m)}}$$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} f^{\lceil\alpha\rceil+m}(s)\, ds\,.$$

Corollary 3.1. Suppose that $n-1 < \alpha < n, \beta = \alpha - (n-1)\,, (0 < \beta < 1), n \in N,$ $\alpha, \beta \in$ R and the function $f(x)$ is such that $D_*^\alpha f(x)$ exists. Then

$$D_*^\alpha f(x) = D_*^\beta D^{n-1} f(x)\,.$$

Proof. Substitute β for α and $n-1$ for m in

$$D_*^\alpha D^m f(x) = D_*^{\alpha+m} f(x) \neq D^m D_*^\alpha f(x)\,.$$

Then

$$D_*^\beta D^{n-1} f(x) = D_*^{\beta+(n-1)} f(x) = D_*^{\alpha-(n-1)+(n-1)} f(x) = D_*^\alpha f(x)\,.$$

This means

$$D_*^\alpha D^m f(x) = D_*^{\alpha+m} f(x) \neq D^m D_*^\alpha f(x)\,.$$

4) For any constant properties hold

$$D_*^\alpha(c) = 0\,.$$

Proof. Using the definition of D_*^α, we get

$$D_*^\alpha(c) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} \frac{d^{\lceil\alpha\rceil}}{ds^{\lceil\alpha\rceil}}(c)\, ds = 0\,.$$

5) For the Laplace transform of D_*^α the following formula holds

$$\mathcal{L}\{D_*^\alpha f(x)\} = s^\alpha \mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) .$$

Proof. Applying the definition of Laplace transform, we get

$$\mathcal{L}\{D_*^\alpha f(x)\} = \int_0^\infty e^{-sx}\left(D_*^\alpha f(x)\right)dx$$

$$= \int_0^\infty e^{-sx} \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} f^{\lceil\alpha\rceil}(p)dp\, dx .$$

Changing the order of integration and using
$\{0 < x < \infty, 0 \leq p \leq x\} = \{0 < p < \infty, p \leq x < \infty\}$, we get

$$\mathcal{L}\{D_*^\alpha f(x)\} = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty f^{\lceil\alpha\rceil}(p) \int_p^\infty \frac{e^{-sx}}{(x-p)^{1-\lceil\alpha\rceil+\alpha}}\, dx\, dp .$$

Putting $x - p = t$, we get $dx = dt$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty e^{-sp} f^{\lceil\alpha\rceil}(p)dp \int_0^\infty \frac{e^{-st}}{t^{1-\lceil\alpha\rceil+\alpha}}\, dt .$$

Now, we will obtain the integral

$$A(0,\infty) = \int_0^\infty \frac{e^{-st}}{t^{1-\lceil\alpha\rceil+\alpha}} dt.$$

Putting $st = y$, we get $dt = \frac{dy}{s}$ and

$$A(0,\infty) = s^{\alpha-\lceil\alpha\rceil} \int_0^\infty e^{-y} y^{\lceil\alpha\rceil-\alpha-1}\, dy = s^{\alpha-\lceil\alpha\rceil}\Gamma(\lceil\alpha\rceil - \alpha)\,.$$

Therefore

$$= \left(\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty e^{-sp}\, f^{\lceil\alpha\rceil}(p) dp\right)\left(s^{\alpha-\lceil\alpha\rceil}\Gamma(\lceil\alpha\rceil - \alpha)\right)$$

$$= s^{\alpha-n}\mathcal{L}\,\{f^n(x)\}$$

$$= s^{\alpha-n}\left\{s^n\mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^{n-k-1}\, f^k(0)\right\}$$

$$= s^{\alpha}\mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, f^k(0).$$

6) The Riemann-Liouville integral operator I^α and the Caputo fractional differential operator D_*^α are inverse operators in the sense that

a) $D_*^\alpha\, I^\alpha f(x) = f(x)\,.$

Proof. Using the definition of D_*^α, we get

$$D_*^\alpha\, I^\alpha f(x) = I^{\lceil\alpha\rceil-\alpha}\, D^{\lceil\alpha\rceil}\, I^{\lceil\alpha\rceil} I^{\alpha-\lceil\alpha\rceil} f(x) = I^{\lceil\alpha\rceil-\alpha}\, D\left(D^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil}\right) I^{\alpha-\lceil\alpha\rceil} f(x)$$

$$= I^{\lceil\alpha\rceil-\alpha} D\, I^{\alpha-\lceil\alpha\rceil} f(x).$$

From that it follows

$$D_*^{\alpha}\, I^{\alpha} f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{D I^{\alpha - \lceil\alpha\rceil} f(y) dy}{(x-y)^{1-\lceil\alpha\rceil+\alpha}}$$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-y)^{1-\lceil\alpha\rceil+\alpha}} D\left(\frac{1}{\Gamma(\alpha - \lceil\alpha\rceil)} \int_0^y \frac{f(s) ds}{(y-s)^{1-\alpha+\lceil\alpha\rceil}}\right) dy$$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-y)^{1-\lceil\alpha\rceil+\alpha}} D\left(\frac{1}{\Gamma(\alpha - \lceil\alpha\rceil + 1)} \int_0^y \frac{f(s) ds}{(y-s)^{\lceil\alpha\rceil-\alpha}}\right) dy.$$

Now, we obtain the formula for

$$D\left(\frac{1}{\Gamma(\alpha - \lceil\alpha\rceil + 1)} \int_0^y \frac{f(s) ds}{(y-s)^{\lceil\alpha\rceil-\alpha}}\right).$$

We have that

$$\frac{1}{\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \int_0^y \frac{f(s) ds}{(y-s)^{\lceil\alpha\rceil-\alpha}}$$

$$= -\frac{1}{\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \int_0^y f(s) \frac{d(y-s)^{-\lceil\alpha\rceil+\alpha+1}}{\alpha + 1 - \lceil\alpha\rceil}$$

$$= \frac{1}{(\alpha + 1 - \lceil\alpha\rceil)\Gamma(\alpha + 1 - \lceil\alpha\rceil)}$$

$$\left[f(0) y^{-\lceil\alpha\rceil+\alpha+1} + \int_0^y f'(s)(y-s)^{-\lceil\alpha\rceil+\alpha+1}\, ds\right]$$

$$= \frac{1}{\Gamma(\alpha + 2 - \lceil\alpha\rceil)} \left[f(0) y^{1+\alpha-\lceil\alpha\rceil} + \int_0^y f'(s)(y-s)^{-\lceil\alpha\rceil+\alpha+1} \; ds \right].$$

Therefore,

$$D\left(\frac{1}{\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \int_0^y \frac{f(s)ds}{(y-s)^{\lceil\alpha\rceil-\alpha}} \right)$$

$$= \frac{1}{\Gamma(\alpha + 2 - \lceil\alpha\rceil)}$$

$$\left[f(0)(\alpha + 1 - \lceil\alpha\rceil) y^{-\lceil\alpha\rceil+\alpha} + \int_0^y f'(s)(1 + \alpha - \lceil\alpha\rceil)(y-s)^{-\lceil\alpha\rceil+\alpha} \; ds \right]$$

$$= \frac{1}{\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \left[f(0) y^{-\lceil\alpha\rceil+\alpha} + \int_0^y f'(s)(y-s)^{-\lceil\alpha\rceil+\alpha} \; ds \right].$$

Applying this formula, we get

$$D_*^\alpha \, I^\alpha f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-y)^{1-\lceil\alpha\rceil+\alpha}}$$

$$\left\{ \frac{1}{\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \left[f(0) y^{-\lceil\alpha\rceil+\alpha} + \int_0^y f'(s)(y-s)^{-\lceil\alpha\rceil+\alpha} \; ds \right] \right\} dy$$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)\Gamma(\alpha + 1 - \lceil\alpha\rceil)} \left\{ f(0) \int_0^x \frac{y^{-\lceil\alpha\rceil+\alpha}}{(x-y)^{1-\lceil\alpha\rceil+\alpha}} \; dy \right\}$$

$$+\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\alpha+1-\lceil\alpha\rceil)}\left\{\begin{array}{c}\displaystyle\int_0^x \frac{1}{(x-y)^{1-\lceil\alpha\rceil+\alpha}} \\ \displaystyle\int_0^y f'(s)(y-s)^{-\lceil\alpha\rceil+\alpha}\ ds\,dy\end{array}\right\}.$$

Now, we will obtain the integral

$$A(0,x)=\int_0^x \frac{f(0)\,y^{\alpha-\lceil\alpha\rceil}}{(x-y)^{1-\lceil\alpha\rceil+\alpha}}\ dy\,.$$

Putting $=ux$, we get $dy=x\,du$

$$=f(0)\int_0^1 \frac{(ux)^{\alpha-\lceil\alpha\rceil}}{(x-ux)^{1-\lceil\alpha\rceil+\alpha}}x\,du=f(0)\,\beta(\alpha-\lceil\alpha\rceil+1,\lceil\alpha\rceil-\alpha)$$

$$A(0,x)=\Gamma(\alpha-\lceil\alpha\rceil+1)\ \Gamma(\lceil\alpha\rceil-\alpha)f(0)\,.$$

Now, we will obtain the integral

$$A\big((0,x)(0,y)\big)=\int_0^x\int_0^y \frac{f'(s)}{(x-y)^{1-\lceil\alpha\rceil+\alpha}\,(y-s)^{\lceil\alpha\rceil-\alpha}}ds\,dy\,.$$

Changing the order of integral and using
$[0\le y\le x, 0\le s\le y]=[0\le s\le x, s\le y\le x]$, we get

$$A\big((0,x)(0,y)\big)=\int_0^x f'(s)\int_s^x \frac{1}{(x-y)^{1-\lceil\alpha\rceil+\alpha}\,(y-s)^{\lceil\alpha\rceil-\alpha}}dy\,ds\,.$$

Putting $y-s=t$, we get $dy=dt$ and

$$A\big((0,x)(0,y)\big) = \int_0^x f'(s) \int_0^{x-s} \frac{dt}{(x-s-t)^{1-\lceil\alpha\rceil+\alpha}\, t^{\lceil\alpha\rceil-\alpha}} ds$$

$$= \int_0^x f'(s) \int_0^{x-s} \frac{dt}{(x-s)^{1-\lceil\alpha\rceil+\alpha}\, (1-\frac{t}{x-s})^{1-\lceil\alpha\rceil+\alpha}\, t^{\lceil\alpha\rceil-\alpha}} ds\,.$$

Putting $t = (x-s)u$, we get $dt = (x-s)du$ and

$$A\big((0,x)(0,y)\big) = \int_0^x f'(s)$$

$$\int_0^1 \frac{(x-s)du}{(x-s)^{1-\lceil\alpha\rceil+\alpha}\, (1-u)^{1-\lceil\alpha\rceil+\alpha}\, [(x-s)u]^{\lceil\alpha\rceil-\alpha}} ds$$

$$= \int_0^x f'(s) \int_0^1 (1-u)^{\lceil\alpha\rceil-\alpha-1}\; u^{\alpha-\lceil\alpha\rceil}\, du\, ds$$

$$= \int_0^x f'(s)\, \beta(\lceil\alpha\rceil-\alpha, \alpha-\lceil\alpha\rceil+1)\, ds$$

$$= \beta(\lceil\alpha\rceil-\alpha, \alpha-\lceil\alpha\rceil+1) \int_0^x f'(s)\; ds$$

$$= \Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\alpha-\lceil\alpha\rceil+1)[f(x)-f(0)]\,.$$

Therefore,

$$D_*^\alpha\, I^\alpha f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)\, \Gamma(\alpha+1-\lceil\alpha\rceil)}$$

$$[\Gamma(\alpha-\lceil\alpha\rceil+1)\Gamma(\lceil\alpha\rceil-\alpha)f(0)+\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\alpha-\lceil\alpha\rceil+1)f(x)$$

$$-\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(\alpha-\lceil\alpha\rceil+1)f(0)] \ = f(x)\,.$$

b) $$I^{\alpha}D_{*}^{\alpha}\, f(x)=f(x)-\sum_{k=0}^{\lceil\alpha\rceil}\frac{x^k}{k!}D^k f(0^+), \alpha\in R^+\,.$$

Proof. It is easy to see that

$$I\,Df(x)=\int_0^x f^{(1)}(t_1)dt_1=f(x)-f(0).$$

From that it follows

$$I^2D^2f(x)=\int_0^x\int_0^{t_1} f^{(2)}\,(t_2)dt_2\,dt_1$$

$$I=\int_0^x(\,f^{(1)}(t_1)-f^{(1)}(0))dt_1=f(x)-f(0)-xf^{(1)}(0)\,,$$

$$I^3D^3f(x)=\int_0^x\int_0^{t_1}\int_0^{t_2} f^{(3)}(t_3)\,dt_3\,dt_2dt_1$$

$$=\int_0^x\left[f^{(1)}(t_1)-f^{(1)}(0)-t_1f^{(2)}(0)\right]\,dt_1$$

$$=f(x)-x\,f^{(1)}(0)-\frac{x^2}{2}\,f^{(2)}(0)-f(0)\,.$$

Suppose it is true for $n = k - 1$. That means

$$I^{k-1}D^{k-1}f(x) = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} f^{(k-1)}(t_{k-1})\ dt_{k-1}\ \ldots$$

$$= f(x) - f(0) - xf^{(1)}(0) - \cdots - \frac{x^{k-1}}{(k-1)!}\, f^{(k-1)}(0).$$

Then

$$I^k D^k f(x) = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-1}} f^{(k)}\,(t_k)dt_k\ dt_{k-1}\ \cdots dt_2 dt_1$$

$$= \int_0^x \left[f^{(1)}(t_1) - f^{(1)}(0) - \frac{t}{1!}f^{(2)}(0) - \frac{t^2}{2!}f^{(3)}(0) - \cdots - \frac{x^{k-1}}{(k-1)!}f^{(k)}(0) \right] dt_1$$

$$= f(x) - f(0) - xf^{(1)}(0) - \frac{x^2}{2!}f^{(2)}(0) - \cdots - \frac{x^k}{k!}f^{(k)}(0)$$

$$= f(x) - \sum_{k=0}^{k} \frac{x^k}{k!} f^{(k)}(0)\,.$$

Therefore,

$$I^n\, D^n f(x) = f(x) - \sum_{k=0}^{n} \frac{x^n}{n!} f^{(n)}(0)$$

is true for any integer n .

Finally, we have the following formula

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha + 1)}$$

$$\left(x^{\lceil\alpha\rceil-\alpha}\, D^{\lceil\alpha\rceil}\, f(0^+) + \int_0^x (x-s)^{\lceil\alpha\rceil-\alpha}\, (D^{1+\lceil\alpha\rceil} f)(s)\, ds \right).$$

Proof. Using the integration by parts

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x D^{\lceil\alpha\rceil} \frac{f(s)}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\, ds$$

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)}$$

$$\left(-\frac{(x-s)^{\lceil\alpha\rceil-\alpha}}{(\lceil\alpha\rceil-\alpha)} D^{\lceil\alpha\rceil} f(s) \Big|_0^x - \int_0^x \frac{-(x-s)^{\lceil\alpha\rceil-\alpha}}{(\lceil\alpha\rceil-\alpha)} \left(D^{1+\lceil\alpha\rceil} f\right)(s)\, ds \right)$$

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha + 1)}$$

$$\left(x^{\lceil\alpha\rceil-\alpha}\, D^{\lceil\alpha\rceil}\, f(0^+) + \int_0^x (x-s)^{\lceil\alpha\rceil-\alpha}\, (D^{1+\lceil\alpha\rceil} f)(s)\, ds \right).$$

Example. Prove that

$$D_*^{\alpha}\left(x^{\beta}\right) = \begin{cases} 0 & \text{if } \beta \in N^0 \text{ and } \beta < \lceil\alpha\rceil, \\ \dfrac{\Gamma(\beta+1)\, x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)} & \text{if } \beta \in N^0 \text{and } \beta \geq \lceil\alpha\rceil \text{ or } \beta \notin N \text{ and } \beta > \lceil\alpha\rceil. \end{cases}$$

Here $N^0 = N \cup \{0\}$.

Solution. If $\beta \in N^0$ and $\beta < \lceil \alpha \rceil$, than $D^{\lceil \alpha \rceil}\left(u^{\beta}\right) = 0$, and using this formula

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^x \frac{1}{(x-u)^{1-\lceil \alpha \rceil + \alpha}} D^{\lceil \alpha \rceil} f(u)\, du \,, \alpha, x \in R^+, \qquad (3.1)$$

we get $D_*^{\alpha} f(x) = 0$.

If $\beta \in N^0$ and $\beta \geq \lceil \alpha \rceil$ or $\beta \notin N$ and $\beta > \lceil \alpha \rceil$, then

$$D^{\lceil \alpha \rceil}\left(u^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} u^{\beta-\alpha}.$$

Using formula (3.1), we get

$$D_*^{\alpha}\left(x^{\beta}\right) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^x \frac{1}{(x-u)^{1-\lceil \alpha \rceil + \alpha}} D^{\lceil \alpha \rceil}\left(u^{\beta}\right) du$$

$$= \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^x \frac{1}{(x-u)^{1-\lceil \alpha \rceil + \alpha}} \; \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \; u^{\beta-\alpha} \; .$$

Putting $u = xp$, we get $du = x\, dp$. Then

$$D_*^{\alpha}\left(x^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\lceil \alpha \rceil - \alpha)\, \Gamma(\beta+1-\alpha)} \int_0^1 \frac{x}{[x\,(1-p)]^{1-\lceil \alpha \rceil + \alpha}} (xp)^{\beta-\lceil \alpha \rceil}\, dp$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\lceil \alpha \rceil - \alpha)\, \Gamma(\beta+1-\alpha)} \int_0^1 p^{\beta-\lceil \alpha \rceil}\, (1-p)^{\lceil \alpha \rceil - \alpha - 1} dp$$

$$= \frac{\Gamma(\beta+1)\, x^{\beta-\alpha}}{\Gamma(\lceil\alpha\rceil-\alpha)\,\Gamma(\beta+1-\alpha)}\beta(\beta-\lceil\alpha\rceil+1,\lceil\alpha\rceil-\alpha)$$

$$= \frac{\Gamma(\beta+1)\, x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)}.$$

CHAPTER 4

RIEMANN-LIOUVILLE FRACTIONAL OPERATOR

This chapter contain the definition and some properties of the Riemann-Liouville fractional operator.

Definition 4.1. Suppose that $\alpha > 0, x > 0,\ \alpha, x \in R$. Then

$$D^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_0^x \dfrac{f(s)ds}{(x-s)^{1-n+\alpha}}, & n-1 < \alpha < n, n \in N, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in N \end{cases}$$

is called the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional operator of order α.

Lemma 4.1. Let $n-1 < \alpha < n, n \in N, \alpha \in R$ and $f(x)$ be such that $D^{\alpha} f(x)$ exists. Then

$$D^{\alpha} f(x) = D^{\lceil \alpha \rceil} I^{\lceil \alpha \rceil - \alpha} f(x) .$$

This means the Riemann-Liouville fractional derivative is equivalent to $(\lceil \alpha \rceil - \alpha)$-fold integration and $\lceil \alpha \rceil$-th order differential.

We have the following properties of the Riemann-Liouville fractional differential operator D^{α} of order α .

1) The Riemann-Liouville fractional differential operator D^{α} of order α is a linear operator. That means

$$D^{\alpha}\big(af(x) + bg(x)\big) = aD^{\alpha} f(x) + bD^{\alpha} g(x), a, b \in R, \alpha \in R^{+} .$$

Proof. Using the definition of D^{α} , we get

$$D^{\alpha}\big(a\, f(x) + b\, g(x)\big) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} \int_{0}^{x} \frac{a\, f(s) + b\, g(s)}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} ds$$

$$= \frac{a}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} \int_{0}^{x} \frac{f(s)ds}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} + \frac{b}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} \int_{0}^{x} \frac{g(s)ds}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}$$

$$= a\, D^{\alpha} f(x) + b\, D^{\alpha}\, g(x)\, .$$

2) The following non-semigroup and non-commutative properties hold

$$D^{\alpha} D^{\beta} f(x) \neq D^{\alpha+\beta} f(x), \alpha, \beta \in R^{+}.$$

Suppose that $n-1 < \alpha < n,\ \ n, m \in N,\ \ \alpha \in R_{+}$. Then in general

$$D^{m} D^{\alpha} f(x) = D^{\alpha+m} f(x) \neq D^{\alpha} D^{m} f(x)\, .$$

Proof. Let $\alpha = \frac{1}{2}, f(x) = 1, m = 1$ using the definition of D^{α}, we get

$$D^{\frac{1}{2}} D^{1}(1) = D^{\frac{1}{2}}\,(0) = 0,$$

$$D^{\frac{3}{2}}(1) = \frac{-1}{2\sqrt{\pi}}\, x^{\frac{-3}{2}},$$

$$D^{\frac{1}{2}} D^{1}(1) = 0 \neq \frac{-1}{2\sqrt{\pi}}\, x^{\frac{-3}{2}} = D^{\frac{3}{2}}(1)\, .$$

That means

$$D^{\frac{1}{2}}D^{1}(1) \neq D^{\frac{3}{2}}(1) \text{ (non-semigroup)}$$

and

$$D^{1}D^{\frac{1}{2}}(1) = D^{1}\left(\frac{1}{\sqrt{\pi}}\, x^{\frac{-1}{2}}\right) = \frac{-1}{2\sqrt{\pi}}\, x^{\frac{-3}{2}}$$

$$D^{\frac{1}{2}}D^{1}(1) = 0 \neq \frac{-1}{2\sqrt{\pi}} x^{\frac{-3}{2}} = D^{1}D^{\frac{1}{2}}(1)\,.$$

That means

$$D^{\frac{1}{2}}D^{1}(1) \neq D^{1}D^{\frac{1}{2}}(1) \text{ (non-commutative)}\,.$$

3) For any constant C, the formulas hold

$$D^{\alpha}(c) = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha}\,.$$

Proof. Using the definition of D^{α}, we get

$$D^{\alpha}(c) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\,\frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}}\int_{0}^{x}\frac{c\,ds}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}$$

$$= \frac{c}{\Gamma(1-\alpha)}\,\frac{d}{dx}\left[-\frac{(x-s)^{1-\alpha}}{(1-\alpha)}\,\Big|_{0}^{x}\right]$$

$$= \frac{c}{\Gamma(1-\alpha)}\,\frac{d}{dx}\,\frac{x^{1-\alpha}}{(1-\alpha)} = \frac{c}{\Gamma(1-\alpha)}\,x^{-\alpha}.$$

4) For the Laplace transform of D^{α} the following formula holds

$$\mathcal{L}\{D^{\alpha}f(x)\} = s^{\alpha}\mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^{k}\,[D^{\alpha-k-1}f(x)]_{x=0}\,.$$

Proof. Applying the definition of Laplace transform, we get

$$\mathcal{L}\{D^{\alpha}f(x)\} = \int_0^{\infty} e^{-sx}\,[D^{\alpha}f(x)]\,dx$$

$$= \int_0^{\infty} e^{-sx}\left\{\frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dx^n}\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\right\}dx$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_0^{\infty} e^{-sx}\left\{\frac{d^n}{dx^n}\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\right\}dx$$

$$= \frac{1}{\Gamma(n-\alpha)}s^n\mathcal{L}\left\{\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\right\} - \frac{1}{\Gamma(n-\alpha)}\left\{s^{n-1}\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\right\}_{x=0}$$

$$- \cdots - \frac{1}{\Gamma(n-\alpha)}\left\{\frac{d^{n-1}}{dx^{n-1}}\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\right\}_{x=0}$$

$$= \frac{s^n}{\Gamma(n-\alpha)}\int_0^{\infty} e^{-sx}\int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\,dx - s^{n-1}I^{n-\alpha}f(x)_{x=0} - \cdots$$

$$-(D^{\alpha-1}f(x))_{x=0}\,.$$

we obtain formula for the integral

$$T(s) = \frac{s^n}{\Gamma(n-\alpha)} \int_0^\infty e^{-sx} \int_0^x \frac{f(p)dp}{(x-p)^{1-n+\alpha}}\, dx\,.$$

Changing the order of integration and using

$[0 \le x < \infty, 0 \le p \le x] = [0 \le p < \infty, p \le x < \infty]$, we get

$$T(s) = \frac{s^n}{\Gamma(n-\alpha)} \int_0^\infty f(p) \int_p^\infty \frac{e^{-sx}}{(x-p)^{1-n+\alpha}}\, dx\, dp\,.$$

Putting $-p = t$, we get $dx = dt$

$$T(s) = \frac{S^n}{\Gamma(n-\alpha)} \int_0^\infty e^{-sp} f(p)\, dp \int_0^\infty \frac{e^{-st}}{t^{1-n+\alpha}}\, dt\,.$$

Now, we will obtain the integral

$$A(0,\infty) = \int_0^\infty \frac{e^{-st}}{t^{1-n+\alpha}}\, dt\,.$$

Putting $st = y$, we get $dt = \frac{dy}{s}$

$$A(0,\infty) = \int_0^\infty \frac{e^{-y}}{\left(\frac{y}{s}\right)^{1-n+\alpha}} \frac{dy}{s}$$

$$= s^{\alpha-n} \int_0^\infty e^{-y}\, y^{n-\alpha-1}\, dy \; = s^{\alpha-n}\Gamma(n-\alpha)\,.$$

Therefore,

$$T(s) = \frac{s^n}{\Gamma(n-\alpha)} \int_0^\infty e^{-sp} f(p) \ \left(s^{\alpha-n}\Gamma(n-\alpha)\right) dp = s^\alpha \int_0^\infty e^{-sp} f(p)\, dp$$

$$= s^\alpha \mathcal{L}\,\{f(x)\}$$

and

$$\mathcal{L}\,\{D^\alpha f(x)\} = s^\alpha \mathcal{L}\,\{f(x)\} - \sum_{k=0}^{n-1} s^k \left[D^{\alpha-k-1} f(x)\right]_{x=0} .$$

5) In general the two operators Riemann - Liouville and Caputo, do not coincide. Actually,

$$D_*^\alpha f(x) = I^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil} f(x) \neq D^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha} f(x) = D^\alpha f(x) .$$

But, we have the following formula

$$D_*^\alpha f(x) = D^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} \ f^{(k)}(0) \right).$$

Proof. The well-known Taylor series expansion about the point 0 is

$$f(x) = f(0) + x f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \cdots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{x^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} ,$$

where, considering also (2.2)

$$R_{n-1} = \int_0^x \frac{f^{(n)}(s)(x-s)^{n-1}}{(n-1)!}\, ds \;= \frac{1}{\Gamma(n)}\int_0^x f^{(n)}(s)(x-s)^{n-1} ds$$

$$= I^n f^{(n)}(x)\,.$$

Now, using the linearity property of the Riemann - Liouville fractional derivative, the Riemann - Liouville fractional derivative of the power function, the properties of the fractional integral and representation formula

$$D_*^\alpha f(x) = I^{n-\alpha} D^n f(x)\,.$$

$$D^\alpha f(x) = D^\alpha \left(\sum_{k=0}^{n-1} \frac{x^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \right)$$

$$= \sum_{k=0}^{n-1} \frac{D^\alpha x^k}{\Gamma(k+1)} f^{(k)}(0) + D^\alpha R_{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + D^\alpha I^n f^{(n)}(x)$$

$$= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + I^{n-\alpha} f^{(n)}(x)$$

$$= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + D_*^\alpha f(x)\,.$$

This means that

$$D_*^{\alpha} f(x) = D^{\alpha}\left(f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0) \right).$$

Note. Suppose that $n-1 < \alpha < n, n \in N$. Let $f(x)$ be an analytic function, Then

1) $D_*^{\alpha} f(x) = f^{(n)}(0) \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} + \cdots,$

2) $D^{\alpha} f(x) = f(0) \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + f^{(1)}(0) \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \cdots + f^{(n)}(0) \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} + \cdots.$

From that it follows that

1) $D_*^{\alpha} e^x = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \cdots + \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} + \cdots$ for all $\alpha \in (0,1)$.

2) $D^{\alpha} e^x = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \cdots + \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} + \cdots$ for all $\alpha \in (0,1)$.

CHAPTER 5

FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

This chapter contain methods for the solutions of initial value problem for fractional differential equations.

First, we consider the Cauchy problem for the fractional differential equation

$$D^{\alpha}\, u(t) = f\big(t, u(t)\big), 0 < \alpha \leq 1, t > 0, u(0) = u_0 \,.$$

Assume that $f\big(t, u(t)\big)$ be a smooth function. Then

$$u(t) = I^{\alpha}\{D^{\alpha}u(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\big(D^{\alpha}u(s)\big)\, ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} f\big(s, u(s)\big)\, ds \,. \qquad (5.1^*)$$

Then, applying the fixed point Theorem, we can write

$$u(t) = \lim_{m\to\infty} u_m(t),$$

where $u_m(t)$ is defined by the formula

$$u_m(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} f\big(s, u_{m-1}(s)\big)\, ds, m \geq 1, \qquad (5.1)$$

$u_0(t)$ is given .

Example 5.1. Solve the Cauchy problem

$$D^{\frac{1}{2}}\, u(t) = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}, t > 0, u(0) = 0\,.$$

Solution. We will use three different methods. First, we consider the Green's function method. Using Green's formula (5.1*), we get

$$u(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left\{\frac{8}{3\sqrt{\pi}} s^{\frac{3}{2}}\right\} ds$$

$$= \frac{\frac{8}{3}}{\sqrt{\pi}\sqrt{\pi}} \int_0^t \frac{s^{\frac{3}{2}}}{(t-s)^{\frac{1}{2}}} ds = \frac{8}{3\pi} \int_0^t (t-s)^{\frac{1}{2}-1}\, s^{\frac{5}{2}-1} ds\,.$$

Putting $s = tp$, we get $ds = tdp$

$$u(t) = \frac{8}{3\pi} \int_0^1 (1-p)^{\frac{1}{2}-1}\; t^{\frac{1}{2}-1}\, t^{\frac{5}{2}-1}\, p^{\frac{5}{2}-1} tdp = t^2\, \frac{8}{3\pi}\; B\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= t^2\; \frac{8}{3\pi}\; \frac{\Gamma\left(\frac{1}{2}\right)\, \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{5}{2}\right)} = t^2\; \frac{8}{3\pi}\; \frac{\sqrt{\pi}\; \frac{3}{2}\; \frac{1}{2}\; \sqrt{\pi}}{\Gamma(3)} = t^2.$$

Then

$$u(t) = t^2.$$

Second, we will obtain the solution of this problem by the power series. Actually,

$$u(t) = \sum_{k=0}^{\infty} c_k\, t^{k\alpha}\,.$$

Taking $= \frac{1}{2}$, we get

$$u(t) = \sum_{k=0}^{\infty} c_k\, t^{k\alpha} = c_0 + c_1 t^{\frac{1}{2}} + c_2 t + c_3 t^{\frac{3}{2}} + c_4 t^2 + c_5 t^{\frac{5}{2}} + \cdots\,.$$

$$u(0) = c_0 = 0\,.$$

Then,

$$D^{\frac{1}{2}}\, u(t) = \sum_{k=1}^{\infty} c_k\, D^{\frac{1}{2}}\, \left\{t^{\frac{k}{2}}\right\} = \sum_{k=1}^{\infty} c_k\, \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}\, t^{\frac{k}{2}-\frac{1}{2}} = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}.$$

So,

$$\sum_{k=1}^{\infty} c_k\, \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}\, t^{\frac{k-1}{2}} = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}.$$

Equating the coefficients of $t^{\frac{k-1}{2}}$, we get

$$c_4 \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} = \frac{8}{3}\frac{1}{\sqrt{\pi}}\,, c_k \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} = 0, k \neq 4\,.$$

From that it follows

$$c_4 = 1, c_k = 0, k \neq 4 .$$

Then

$$u(t) = \sum_{k=0}^{\infty} c_k\, t^{\frac{k}{2}} = c_4 t^{\frac{4}{2}} = t^2.$$

Third, applying the Laplace transform, we get

$$\mathcal{L}\left\{D^{\frac{1}{2}} u(t)\right\} = s^{\frac{1}{2}} \mathcal{L}\{u(t)\} .$$

Then,

$$s^{\frac{1}{2}} \mathcal{L}\{u(t)\} = \int_0^{\infty} e^{-st} \left(\frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right) dt .$$

Putting $y = st$, we get $dy = sdt$

$$s^{\frac{1}{2}} \mathcal{L}\{u(t)\} = \frac{8}{3\sqrt{\pi}} \int_0^{\infty} e^{-y} \left(\frac{y}{s}\right)^{\frac{3}{2}} \frac{dy}{s}$$

$$= \frac{8}{3\sqrt{\pi}} s^{\frac{-5}{2}} \int_0^{\infty} e^{-y}\, y^{\frac{3}{2}}\, dy = \frac{8}{3\sqrt{\pi}} s^{\frac{-5}{2}}\, \Gamma\left(\frac{5}{2}\right) = 2s^{\frac{-5}{2}} .$$

Therefore

$$\mathcal{L}\{u(t)\} = \frac{2}{s^3} = \frac{2!}{s^3} .$$

Then

$$u(t) = \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2.$$

Example 5.2. Solve the Cauchy problem

$$D^{\frac{1}{2}}u(t) = \frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}} - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + \cdots + \frac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)}t^{n-\frac{1}{2}} + \cdots + \frac{1}{2}u(t) - \frac{1}{2}e^{-t},$$

$$t > 0, u(0) = 1\,.$$

Solution.

$$f\big(t, u(t)\big) = \frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}} - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + \cdots + \frac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)}t^{n-\frac{1}{2}} + \cdots + \frac{1}{2}u(t) - \frac{1}{2}e^{-t}$$

$f(t, u)$ is the continuous and

$$|f(t, u_1) - f(t, u_2)| = \frac{1}{2}|u_1 - u_2| < 1\,,$$

where $\alpha = \frac{1}{2} < 1$.

Therefore, there exists

$$u(t) = \lim_{m\to\infty} u_m(t)$$

where $u_m(t)$ is defined by formula

$$u_m(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}$$

$$\left[\frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\,s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots + \frac{1}{2}u_{m-1}(s) - \frac{1}{2}e^{-s}\right]ds, m = 1, \cdots,$$

$u_0(t)$ is given function.

Putting $u_0(t) = e^{-t}$, we get

$$u_1(t) = \frac{1}{\pi}\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\,ds - \frac{2}{\pi}\int_0^t \frac{s^{\frac{1}{2}}}{\sqrt{t-s}}ds + \cdots$$

$$+\frac{1}{\sqrt{\pi}}\frac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^t \frac{s^{n-\frac{1}{2}}}{\sqrt{t-s}}ds + \cdots + \frac{1}{2\sqrt{\pi}}\int_0^t \frac{(u_0(s) - e^{-s})}{\sqrt{t-s}}\,ds$$

$$= \frac{1}{\pi}\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}ds - \frac{2}{\pi}\int_0^t \frac{s^{\frac{1}{2}}}{\sqrt{t-s}}ds + \cdots + \frac{(-1)^n}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\int_0^t \frac{s^{n-\frac{1}{2}}}{\sqrt{t-s}}ds\,.$$

Now, we will obtain the integral

$$I(t) = \int_0^t \frac{s^{k-\frac{1}{2}}}{\sqrt{t-s}}ds\,, k = 0, 1, 2, \dots\,.$$

Putting $s = tp$, we get $ds = t\,dp$

$$= \int_0^1 \frac{t^{k-\frac{1}{2}}\,p^{k-\frac{1}{2}}}{\sqrt{t}\,(1-p)^{\frac{1}{2}}}t\,dp = t^k\int_0^1 p^{k+\frac{1}{2}-1}\,(1-p)^{\frac{1}{2}-1}\,dp$$

$$= t^k B\left(k+\frac{1}{2},\frac{1}{2}\right) = t^k \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)}.$$

Using this formula, we get

$$u_1(t) = t^0 \frac{\Gamma^2\left(\frac{1}{2}\right)}{\pi} - t\frac{\Gamma^2\left(\frac{1}{2}\right)}{\pi} + \cdots + \frac{(-1)^n}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} t^n \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots$$

$$= 1 - t + \frac{t^2}{2!} + \cdots + \frac{(-1)^n\, t^n}{n!} + \cdots = e^{-t}.$$

Assume that

$$u_{m-1}(t) = e^{-t}.$$

Then

$$u_m(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\left[\frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{1}{2}u_{m-1}(s) - \frac{1}{2}e^{-s}\right] ds = e^{-t}.$$

So, by the induction $u_m(t) = e^{-t}$ for any m.

Then, passing limit when $m \to \infty$, we get

$$u(t) = \lim_{m\to\infty} u_m(t) = \lim_{m\to\infty} e^{-t} = e^{-t}.$$

Second, we consider the Cauchy problem for the Basset fractional differential equation

$$Du(t) + D^\alpha u(t) = f\big(t, u(t)\big), 0 < \alpha < 1, t > 0, u(0) = u_0.$$

Assume that $f\big(t, u(t)\big)$ be a smooth function. Then

$$u(t) = u_0 + \int_0^t \left(-D^\alpha u(s) + f\big(s, u(s)\big)\right) ds .$$

Then, applying the fixed point Theorem, we can write

$$u(t) = \lim_{m\to\infty} u_m(t)$$

where

$$u_m(t) = u_0 + \int_0^t [-D^\alpha u_{m-1}(s) + f(s, u_{m-1}(s))]\, ds, m \geq 1, u_0(t) \text{ is given.}$$

Example 5.3. Solve the Cauchy problem

$$Du(t) + D^{\frac{1}{2}} u(t) + u(t) = 2t + t^2 + \frac{8}{3}\frac{1}{\sqrt{\pi}} t^{\frac{3}{2}}, t > 0, u(0) = 0$$

for the Basset fractional differential equation.

Solution. First, we will obtain the solution of this problem by the power series. Actually,

$$u(t) = \sum_{k=0}^{\infty} c_k\, t^{k\alpha} .$$

Taking $\alpha = \frac{1}{2}$ and $u(0) = 0$, we get $c_0 = 0$. Then

$$u(t) = \sum_{k=1}^{\infty} c_k\, t^{\frac{k}{2}} .$$

Since

$$u'(t) = \sum_{k=1}^{\infty} c_k \frac{k}{2} t^{\frac{k}{2}-1} = \sum_{k=1}^{\infty} \frac{c_k k}{2} t^{\frac{k-2}{2}},$$

$$D^{\frac{1}{2}} u(t) = \sum_{k=1}^{\infty} c_k D^{\frac{1}{2}} \left\{ t^{\frac{k}{2}} \right\} = \sum_{k=1}^{\infty} c_k \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} t^{\frac{k-2}{2}},$$

we have that

$$\sum_{k=1}^{\infty} c_k \frac{k}{2} t^{\frac{k-2}{2}} + \sum_{k=1}^{\infty} c_k \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} t^{\frac{k-1}{2}} + \sum_{k=0}^{\infty} c_k t^{\frac{k}{2}} = 2t + t^2 + \frac{8}{3} \frac{1}{\sqrt{\pi}} t^{\frac{3}{2}}.$$

Equating the coefficients of $t^{\frac{k}{2}}$ for $k = 1, 2, \ldots$, we get

$$c_1 = 0, c_2 + c_1 \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(1)} = 0,$$

$$c_3 \frac{3}{2} + c_2 \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} + c_1 = 0,$$

$$c_4 \frac{4}{2} + c_3 \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(2)} + c_2 = 2,$$

$$c_5 \frac{5}{2} + c_4 \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} + c_3 = \frac{8}{3} \frac{1}{\sqrt{\pi}},$$

$$c_6 \frac{6}{2} + c_5 \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma(3)} + c_4 = 1,$$

$$k \geq 7, c_k \frac{k}{2} + c_{k-1} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} + c_{k-2} = 0.$$

It is easy to see that $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_4 = 1$ and $c_k = 0$ for $k \geq 7$. Thus,

$$u(t) = c_4 t^{\frac{4}{2}} = t^2\,.$$

Second, applying the Laplace transform, we get

$$\mathcal{L}\,\{u'(t)\} + \mathcal{L}\,\left\{D^{\frac{1}{2}}u(t)\right\} + \mathcal{L}\,\{u(t)\} = 2\mathcal{L}\,\{t\} + \mathcal{L}\,\{t^2\} + \frac{8}{3\sqrt{\pi}}\mathcal{L}\,\left\{t^{\frac{3}{2}}\right\},$$

$$s\mathcal{L}\,\{u(t)\} + s^{\frac{1}{2}}\,\mathcal{L}\,\{u(t)\} + \mathcal{L}\,\{u(t)\} = \frac{2}{s^2} + \frac{2}{s^3} + \frac{8}{3\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}},$$

$$\left(s + s^{\frac{1}{2}} + 1\right)\mathcal{L}\,\{u(t)\} = \frac{2}{s^2} + \frac{2}{s^3} + \frac{2}{s^{\frac{5}{2}}} = \frac{2}{s^3}\left(s + s^{\frac{1}{2}} + 1\right).$$

Therefore

$$\mathcal{L}\,\{u(t)\} = \frac{2}{s^3} = \frac{2!}{s^3},$$

$$u(t) = \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2.$$

Example 5.4. Solve the Cauchy problem

$$Du(t) + D^{\frac{1}{2}}u(t) + \frac{u(t)}{2} = -\frac{e^{-t}}{2} + \frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}} - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + \cdots + \frac{(-1)^n}{\Gamma\left(n + \frac{1}{2}\right)}t^{n-\frac{1}{2}},$$

$u(0) = 1\,.$

Solution. We have that

$$u(t) = 1 + \int_0^t \left[\begin{array}{c} -D^{\frac{1}{2}}\, u(s) - \dfrac{u(s)}{2} - \dfrac{e^{-s}}{2} \\ + \left\{ \dfrac{1}{\sqrt{\pi}} s^{-\frac{1}{2}} - \dfrac{2}{\sqrt{\pi}} s^{\frac{1}{2}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}} + \cdots \right\} \end{array} \right] ds\,.$$

Therefore

$$u(t) = \lim_{m\to\infty} u_m(t)\,,$$

where $u_m(t)$ is defined by the following formula

$$u_m(t) = 1 + \int_0^t \left[\begin{array}{c} -D^{\frac{1}{2}}\, u_{m-1}(s) - \dfrac{u_{m-1}(s)}{2} - \dfrac{e^{-s}}{2} \\ + \left\{ \dfrac{1}{\sqrt{\pi}} s^{-\frac{1}{2}} - \dfrac{2}{\sqrt{\pi}} s^{\frac{1}{2}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}} + \cdots \right\} \end{array} \right] ds,$$

$m = 1, 2, \ldots\,,$

$u_0(t)$ is given smooth function.

Putting, $u_0(t) = e^{-t}$, we get

$$u_1(t) = 1 + \int_0^t \left[\begin{array}{c} -D^{\frac{1}{2}}\, u_0(s) - \dfrac{u_0(s)}{2} - \dfrac{e^{-s}}{2} \\ + \left\{ \dfrac{1}{\sqrt{\pi}} s^{-\frac{1}{2}} - \dfrac{2}{\sqrt{\pi}} s^{\frac{1}{2}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}} + \cdots \right\} \end{array} \right] ds$$

$$= 1 + \int_0^t \begin{bmatrix} -\left\{\dfrac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \dfrac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}}\right\} - e^{-s} \\ + \left\{\dfrac{1}{\sqrt{\pi}} s^{-\frac{1}{2}} - \dfrac{2}{\sqrt{\pi}} s^{\frac{1}{2}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}} + \cdots\right\} \end{bmatrix} ds$$

$$= 1 - \int_0^t e^{-s}\, ds = 1 + e^{-t} - e^0 = e^{-t}\,.$$

Then

$$u_1(t) = e^{-t}.$$

Assume that

$u_{m-1}(t) = e^{-t}$. Then

$$u_m(t) = 1 + \int_0^t \begin{bmatrix} -D^{\frac{1}{2}} u_{m-1}(s) - \dfrac{u_{m-1}(s)}{2} - \dfrac{e^{-s}}{2} \\ + \left\{\dfrac{1}{\sqrt{\pi}} s^{-\frac{1}{2}} - \dfrac{2}{\sqrt{\pi}} s^{\frac{1}{2}} + \cdots + \dfrac{(-1)^n}{\Gamma\left(n+\frac{1}{2}\right)} s^{n-\frac{1}{2}} + \cdots\right\} \end{bmatrix} ds = e^{-t}.$$

So, by the induction $u_m(t) = e^{-t}$ for any m.

Then, passing limit when $m \to \infty$, we get

$$u(t) = \lim_{m\to\infty} u_m(t) = \lim_{m\to\infty} e^{-t} = e^{-t}\,.$$

Third, we consider the Cauchy problem for the fractional differential equation

$$D^{\alpha}u(t) = f\big(t, u(t), u'(t)\big), 1 < \alpha \leq 2, t > 0, u(0) = u_0, u'(0) = u_0 \,.$$

Assume that $f\big(t, u(t)\big)$ be a smooth function. Then

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} f\big(s, u(s), u'(s)\big)\, ds \,.$$

Then, applying the fixed point Theorem, we can write

$$u(t) = \lim_{m \to \infty} u_m(t)\,,$$

where $u_m(t)$ is defined by the formula

$$u_m(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} f\big(s, u_{m-1}(s), u'_{m-1}(s)\big)\, ds, m = 1, 2, \ldots\,,$$

$u_0(t)$ is given.

Example 5.5. Solve the Cauchy problem

$$D^{\frac{3}{2}}u(t) = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}, t > 0, u(0) = 0, u'(0) = 0.$$

Solution. We will use three different methods. First, we consider the Green's function method. Using Green's formula (5.1*), we get

$$u(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t \frac{1}{(t-s)^{1-\frac{3}{2}}} \frac{4}{\sqrt{\pi}} s^{\frac{1}{2}}\, ds$$

$$= \frac{8}{\pi} \int_0^t \frac{s^{\frac{1}{2}}}{(t-s)^{-\frac{1}{2}}} \, ds \, .$$

Putting $s = tp$, we get $ds = t\, dp$

$$u(t) = \frac{8}{\pi} t^2 B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{8}{\pi} t^2 \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{8\, t^2 \, \left(\frac{1}{2}\right)^2 \left(\sqrt{\pi}\right)^2}{2\pi} = t^2.$$

Then

$$u(t) = t^2.$$

Second, we will obtain the solution of this problem by the power series. Actually,

$$u(t) = \sum_{k=0}^{\infty} c_k \, t^{k\alpha}$$

$$= \sum_{k=0}^{\infty} c_k \, t^{k\alpha} = c_0 + c_1 t^{\frac{1}{2}} + c_2 t + c_3 t^{\frac{3}{2}} + c_u t^2 + \cdots.$$

Applying $u(0) = 0, u'(0) = 0$, we get

$$c_0 = c_1 = c_2 = 0 \, .$$

Then,

$$u(t) = \sum_{k=3}^{\infty} c_k \, t^{\frac{k}{2}}$$

and

$$D^{\frac{3}{2}} u(t) = \sum_{k=3}^{\infty} c_k \; D^{\frac{3}{2}} \left[t^{\frac{k}{2}}\right] = \sum_{k=3}^{\infty} c_k \; \frac{\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2} - \frac{1}{2}\right)} \; t^{\frac{k}{2} - \frac{3}{2}} = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}} \, .$$

Equating the coefficients of $t^{\frac{k}{2}}$ for $k = 3, \ldots$, we get

$$c_4 \frac{4}{\Gamma\left(\frac{1}{2}\right)} = \frac{4}{\sqrt{\pi}} \; , c_k \frac{\Gamma\left(1 + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2} - \frac{1}{2}\right)} = 0, k \neq 4 \, .$$

From that it follows

$$c_4 = 1, c_k = 0, k \neq 4 \, .$$

Then

$$u(t) = \sum_{k=0}^{\infty} c_k \, t^{\frac{k}{2}} = c_4 t^{\frac{4}{2}} = t^2 \, .$$

Third, applying the Laplace transform, we get

$$\mathcal{L}\left\{D^{\frac{3}{2}} u(t)\right\} = \frac{4}{\sqrt{\pi}} \; \mathcal{L}\left\{t^{\frac{1}{2}}\right\} .$$

Then,

$$s^{\frac{3}{2}}\mathcal{L}\{u(t)\} = \frac{4}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}$$

$$= \frac{4}{\sqrt{\pi}}\,\frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} = \frac{2}{s^{\frac{3}{2}}}.$$

Therefore

$$\mathcal{L}\{u(t)\} = \frac{2}{s^3}$$

and

$$u(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2\,.$$

Example 5.6. Solve the Cauchy problem

$$D^{\frac{3}{2}}u(t) + \frac{1}{2}u'(t) + \frac{1}{2}u(t) = \frac{2\,t^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\,t^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)} + \cdots + \frac{t}{2}\ , u(0) = 0,$$

$$u'(0) = 0\,.$$

Solution. We have that

$$u(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}$$

$$\left\{-\frac{1}{2}u'(s) - \frac{1}{2}u(s) + \frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\,s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)} + \cdots + \frac{s}{2}\right\}ds\,.$$

Therefore

$$u(t) = \lim_{m\to\infty} u_m(t) \,,$$

where $u_m(t)$ is defined by the following formula

$$u_m(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}$$

$$\left\{\frac{-1}{2} u'_{m-1}(s) - \frac{1}{2} u_{m-1}(s) + \frac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)} + \cdots + \frac{s}{2}\right\} ds \,,$$

$$m = 1, 2, \ldots \,,$$

$u_0(t)$ is given smooth function.

Putting $u_0(t) = e^{-t} - 1 + t$, we get

$$u_1(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}$$

$$\left\{\frac{-1}{2} u'_0(s) - \frac{1}{2} u_0(s) + \frac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)} + \cdots + \frac{s}{2}\right\} ds$$

$$= \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}$$

$$\left\{\frac{-1}{2}(-e^{-s}+1)-\frac{1}{2}(e^{-s}-1+s)+\frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}}+\cdots+\frac{(-1)^n\,s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)}+\cdots+\frac{s}{2}\right\}ds$$

$$=\frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}\left\{\frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}}+\cdots+\frac{(-1)^n\,s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)}+\cdots\right\}ds$$

$$=\frac{4}{\pi}\int_0^t \frac{s^{\frac{1}{2}}}{(t-s)^{-\frac{1}{2}}}ds+\cdots+\frac{(-1)^n}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}\int_0^t \frac{s^{n-\frac{3}{2}}}{(t-s)^{-\frac{1}{2}}}ds+\cdots.$$

Now, we will obtain the integral

$$I_1(t)=\int_0^t \frac{s^{\frac{1}{2}}}{(t-s)^{-\frac{1}{2}}}ds\,.$$

Putting $s=tp$, we get $ds=t\,dp$

$$I_1(t)=\int_0^1 \frac{(tp)^{\frac{1}{2}}}{(t-tp)^{-\frac{1}{2}}}\;t\,dp=\int_0^1 t^2p^{\frac{1}{2}}\;(1-p)^{\frac{1}{2}}\,dp$$

$$=t^2\int_0^1 p^{\frac{3}{2}-1}\,(1-p)^{\frac{3}{2}-1}dp=t^2\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}=\frac{t^2\,\pi}{8}\,.$$

Now, we will obtain the integral

$$I_2(t)=\int_0^t \frac{s^{n-\frac{3}{2}}}{(t-s)^{-\frac{1}{2}}}\;ds\,.$$

Putting $s=tp$, we get $ds=t\,dp$

$$I_2(t) = \int_0^1 \frac{(tp)^{n-\frac{3}{2}}}{(t-tp)^{-\frac{1}{2}}} t\, dp = t^n \int_0^1 p^{n-\frac{3}{2}}\ (1-p)^{\frac{1}{2}}\, dp$$

$$= t^n \int_0^1 p^{(n-\frac{1}{2})-1}\ (1-p)^{\frac{3}{2}-1}\, dp$$

$$= t^n \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n-\frac{1}{2}+\frac{3}{2}\right)} = t^n \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+1)}.$$

Therefore,

$$u_1(t) = \frac{4}{\pi}\frac{\pi}{8} t^2 + \cdots + \frac{(-1)^n}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+1)}\ t^n + \cdots$$

$$= \frac{t^2}{2} + \cdots + \frac{(-1)^n\ t^n}{n!} + \cdots = e^{-t} - 1 + t\,.$$

Assume that

$u_{m-1}(t) = e^{-t} - 1 + t.$ Then

$$u_m(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t \frac{1}{(t-s)^{-\frac{1}{2}}}$$

$$\left\{ \frac{-1}{2} u'_{m-1}(s) - \frac{1}{2} u_{m-1}(s) + \frac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\ s^{n-\frac{3}{2}}}{\Gamma\left(n-\frac{1}{2}\right)} + \cdots + \frac{s}{2} \right\} ds$$

$$= e^{-t} - 1 + t\,.$$

So, by the induction $u_m(t) = e^{-t} - 1 + t$ for any m.

Then, passing limit when $m \to \infty$, we get

$$u(t) = \lim_{m\to\infty} u_m(t) = \lim_{m\to\infty} e^{-t} - 1 + t = e^{-t} - 1 + t\,.$$

Fourth, we consider the Cauchy problem for the Bagley Torvik fractional differential equation

$$D^2 u(t) + D^\alpha u(t) = f\big(t, u(t)\big), 0 < \alpha < 2, t > 0, u(0) = u_0, u'(0) = u'_0\,.$$

Assume that $f(t, u(t))$ be a smooth function. Then

$$u(t) = u_0 + t\,u'_0 + \int_0^t (t-s)\left(-D^\alpha u(s) + f(s, u(s))\right) ds\,.$$

Then applying the fixed point Theorem, we can write

$$u(t) = \lim_{m\to\infty} u_m(t),$$

where

$$u_m(t) = u_0 + t\,u'_0 + \int_0^t (t-s)\left[-D^\alpha u_{m-1}(s) + f\big(s, u_{m-1}(s)\big)\right] ds,$$

$m \geq 1, u_0(t)$ is given .

Example 5.7. Solve the Cauchy problem

$$D^2 u(t) + D^{\frac{3}{2}} u(t) + u(t) = 6t + t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}}\,, 0 < t, u(0) = 0, u'(0) = 0$$

for the Bagley Torvik fractional differential equation.

Solution. First, we will obtain the solution of this problem by the power series. Actually,

$$u(t)=\sum_{k=0}^{\infty}c_k t^{\frac{k}{2}}.$$

We have that

$$u'(t)=\sum_{k=0}^{\infty}c_k\frac{k}{2}t^{\frac{k}{2}-1}.$$

Applying initial conditions, we get

$$c_0=c_1=c_2=0.$$

Then

$$u(t)=\sum_{k=3}^{\infty}c_k t^{\frac{k}{2}}\ ,u'(t)=\sum_{k=3}^{\infty}c_k\frac{k}{2}t^{\frac{k}{2}-1},u''(t)=\sum_{k=3}^{\infty}\frac{k}{2}\left(\frac{k}{2}-1\right)c_k t^{\frac{k}{2}-2},$$

$$D^{\frac{3}{2}}u(t)=\sum_{k=3}^{\infty}c_k\,D^{\frac{3}{2}}\left(t^{\frac{k}{2}}\right)=\sum_{k=3}^{\infty}c_k\frac{\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}t^{\frac{k}{2}-\frac{3}{2}}.$$

So,

$$\sum_{k=3}^{\infty}\frac{k(k-2)}{4}c_k t^{\frac{k-4}{2}}+\sum_{k=3}^{\infty}c_k\frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)}t^{\frac{k-3}{2}}+\sum_{k=3}^{\infty}c_k t^{\frac{k}{2}}=6t+t^3+\frac{8t^{\frac{3}{2}}}{\sqrt{\pi}}.$$

Equating the coefficients of $t^{\frac{k}{2}}$ for $k=3,\ldots$, we get

$$\frac{3}{4}c_3 = 0, 2c_4 + c_3\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(1)} = 0, \frac{5.3}{4}c_5 + c_4\frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} + 0 = 0,$$

$$\frac{6.4}{4}c_6 + c_5\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma(2)} + 0 = 6, \frac{7.5}{4}c_7 + c_6\frac{\Gamma(4)}{\Gamma\left(\frac{3}{2}\right)} + c_3 = \frac{8}{\sqrt{\pi}}$$

$$\frac{8.6}{4}c_8 + c_7\frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma(3)} + c_4 = 0, \frac{9.7}{4}c_9 + c_8\frac{\Gamma(5)}{\Gamma\left(\frac{7}{2}\right)} + c_5 = 0,$$

$$\frac{10.8}{4}c_{10} + c_9\frac{\Gamma\left(\frac{11}{2}\right)}{\Gamma(4)} + c_6 = 1, k \geq 11;$$

$$\frac{k(k-2)}{4}c_k + c_{k-1}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-2}{2}\right)} + c_{k-4} = 0\,.$$

It is easy to see that $c_3 = c_4 = c_5 = 0, c_6 = 1, c_7 = c_8 = c_9 = c_{10} = 0$ and $c_k = 0$ for $k \geq 11$. Thus,

$$u(t) = c_6 t^{\frac{6}{2}} = t^3.$$

Second, applying the Laplace transform, we get

$$\mathcal{L}\{D^2\,u(t)\} + \mathcal{L}\left\{D^{\frac{3}{2}}\,u(t)\right\} + \mathcal{L}\{u(t)\} = 6\mathcal{L}\{t\} + \mathcal{L}\{t^3\} + \frac{8}{\sqrt{\pi}}\mathcal{L}\left\{t^{\frac{3}{2}}\right\},$$

$$s^2\,\mathcal{L}\{u(t)\} + s^{\frac{3}{2}}\,\mathcal{L}\{u(t)\} + \mathcal{L}\{u(t)\} = \frac{6}{s^2} + \frac{3!}{s^4} + \frac{8}{\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}\,,$$

$$\left(s^2 + s^{\frac{3}{2}} + 1\right)\mathcal{L}\{u(t)\} = \frac{6}{s^4}\left(s^2 + 1 + s^{\frac{3}{2}}\right).$$

Therefore

$$\mathcal{L}\{u(t)\} = \frac{6}{s^4},$$

$$u(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{3!}{s^{3+1}} = t^3.$$

Example 5.8. Solve the Cauchy problem

$$D^2u(t) + D^{\frac{1}{2}}u(t) + \frac{1}{2}u(t) = \frac{3}{2}e^{-t} + \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\,t^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\,t^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots,$$

$$u(0) = 1, u'(0) = -1.$$

Solution. We have that

$$u(t) = 1 - t + \int_0^t (t-s)\left[\begin{array}{c} -D^{\frac{1}{2}}u(s) - \frac{1}{2}u(s) + \frac{3}{2}e^{-s} \\ + \left\{\frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\,s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\,s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots\right\}\end{array}\right]ds.$$

Therefore

$$u(t) = \lim_{m\to\infty} u_m(t),$$

where $u_m(t)$ is defined by the following formula

$$u_m(t) = 1 - t + \int_0^t (t-s)\begin{bmatrix} -D^{\frac{1}{2}}\, u_{m-1}(s) - \frac{1}{2}\, u_{m-1}(s) + \frac{3}{2} e^{-s} \\ + \left\{ \frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots \right\} \end{bmatrix} ds,$$

$m = 1, 2, \ldots\;,$

$u_0(t)$ is given smooth function.

Putting $u_0(t) = e^{-t}$, we get

$$u_1(t) = 1 - t + \int_0^t (t-s)\begin{bmatrix} -D^{\frac{1}{2}}\, u_0(s) - \frac{1}{2}\, u_0(s) + \frac{3}{2} e^{-s} \\ + \left\{ \frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2\, s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots \right\} \end{bmatrix} ds$$

$$= (1-t) + \int_0^t (t-s)\begin{bmatrix} -D^{\frac{1}{2}}(e^{-s}) + e^{-s} \\ + \left\{ \frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots \right\} \end{bmatrix} ds$$

$$= (1-t) + \int_0^t (t-s)\, e^{-s} ds$$

$$+ \int_0^t (t-s)\begin{bmatrix} -D^{\frac{1}{2}}\left(1 - s + \frac{s^2}{2!} + \cdots + \frac{(-1)^n\, s^n}{n!} + \cdots\right) \\ + \left\{ \frac{s^{-\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \frac{(-1)^n\, s^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} + \cdots \right\} \end{bmatrix} ds$$

$$= (1-t) + \int_0^t (t-s)\, e^{-s} ds = 1 + e^{-t} - 1 = e^{-t}\,.$$

So,

$$u_1(t) = e^{-t}.$$

Assume that

$$u_{m-1}(t) = e^{-t} \,.$$

Then

$$u_m(t) = 1 - t + \int_0^t (t-s) \left[\begin{array}{c} -D^{\frac{1}{2}}\, u_{m-1}(s) - \dfrac{1}{2} u_{m-1}(s) + \dfrac{s}{2}\; e^{-s} \\ + \left\{ \dfrac{s^{-\frac{1}{2}}}{\sqrt{\pi}} - \dfrac{2\; s^{\frac{1}{2}}}{\sqrt{\pi}} + \cdots + \dfrac{(-1)^n\; s^{n-\frac{1}{2}}}{\Gamma\left(n + \frac{1}{2}\right)} + \cdots \right\} \end{array} \right] ds$$

$$= e^{-t}.$$

So, by the induction $u_m(t) = e^{-t}$ for any m.

Then, passing limit when $m \to \infty$, we get

$$u(t) = \lim_{m\to\infty} u_m(t) = \lim_{m\to\infty} e^{-t} = e^{-t} \,.$$

CHAPTER 6
STABILITY OF DIFFERENTIAL AND DIFFERENCE PROBLEMS

In this chapter, we use the Basset equation for the solution of the initial value problem and differential scheme for the numerical solution on the stability estimates.

6.1 The stability of the initial-value problem for Basset equation

We consider the initial value problem for Basset equation

$$\begin{cases} D_t u(t) + \frac{1}{2} D_t^{\frac{1}{2}} u(t) + \frac{1}{2} u(t) = f(t), \qquad 0 < t < T , \\ u(0) = 0 . \end{cases} \tag{6.1}$$

Here

$$D_t\, u(t) = u'(t) .$$

Assume that $f(t)$ is the continuous function defined on $[0, T]$.

Theorem 6.1. For the solution of problem (6.1) the following stability estimates hold

$$\max_{0 \le t \le T} |u(t)| + \max_{0 \le t \le T} |u'(t)| + \max_{0 \le t \le T} \left| D_t^{\frac{1}{2}}\, u(t) \right| \le c_1 \max_{0 \le t \le T} |f(t)| ,$$

where c_1 does not depend on f(t).

Proof. From (6.1) it follows the following Cauchy problem

$$\begin{cases} D_t u(t) + \frac{1}{2}\, u(t) = f(t) - \frac{1}{2} D_t^{\frac{1}{2}} u(t), \qquad 0 < t < T, \\ u(0) = 0 . \end{cases} \tag{6.2}$$

It is a linear problem and the following formula holds

$$u(t) = e^{-\frac{1}{2}t}u(0) + \int_0^t e^{-\frac{1}{2}(t-s)} \left[f(s) - \frac{1}{2} D_s^{\frac{1}{2}} u(s)\right] \, ds$$

$$= \int_0^t e^{-\frac{1}{2}(t-s)} \left[f(s) - \frac{1}{2} D_s^{\frac{1}{2}} u(s)\right] \, ds \, . \tag{6.3}$$

Using the last formula, we can write

$$u'(t) = f(t) - \frac{1}{2} D_t^{\frac{1}{2}} u(t) - \frac{1}{2} \int_0^t e^{-\frac{1}{2}(t-s)} \left[f(s) - \frac{1}{2} D_s^{\frac{1}{2}} u(s)\right] \, ds \, . \tag{6.4}$$

Using the definition of fractional derivative and formula (6.3) and (6.4), we get

$$D_t^{\frac{1}{2}} u(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{u'(s)}{(t-s)^{\frac{1}{2}}} \, ds$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left\{ f(s) - \frac{1}{2} \int_0^s e^{-\frac{1}{2}(s-y)} f(y) \, dy \right\} \, ds$$

$$+ \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left\{ -\frac{1}{2} D_s^{\frac{1}{2}} u(s) + \frac{1}{4} \int_0^s e^{-\frac{1}{2}(s-y)} D_y^{\frac{1}{2}} u(y) dy \right\} \, ds$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left\{ f(s) - \frac{1}{2} \int_0^s e^{-\frac{1}{2}(s-y)} f(y) \, dy \right\} ds$$

$$-\frac{1}{2}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} D_s^{\frac{1}{2}} u(s)\, ds$$

$$+\frac{1}{4}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \int_0^s e^{-\frac{1}{2}(s-y)}\, D_y^{\frac{1}{2}} u(y)\, dy\, ds\,.$$

We denote that

$$v(t) = D_t^{\frac{1}{2}} u(t)\,. \tag{6.5}$$

Then, from the last formula it follows that

$$v(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left\{ f(s) - \frac{1}{2}\int_0^s e^{-\frac{1}{2}(s-y)} f(y)\, dy \right\} ds$$

$$-\frac{1}{2}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} v(s)\, ds$$

$$+\frac{1}{4}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \int_0^s e^{-\frac{1}{2}(s-y)}\, v(y)\, dy\, ds\,. \tag{6.6}$$

First, we will consider the integral

$$J(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} v(s)\, ds$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} D_s^{\frac{1}{2}} u(s)\, ds = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}}\int_0^s \frac{1}{(s-y)^{\frac{1}{2}}} u'(y) dy\, ds\,.$$

Changing the order of integration and using

$[0 \le s \le t, 0 \le y \le s] = [0 \le y \le t, y \le s \le t]$, we get

$$= \frac{1}{\pi}\int_0^t \int_0^s \frac{u'(y)\, dy\, ds}{(t-s)^{\frac{1}{2}}\,(s-y)^{\frac{1}{2}}} = \frac{1}{\pi}\int_0^t \int_y^t \frac{ds}{(t-s)^{\frac{1}{2}}\,(s-y)^{\frac{1}{2}}}\, u'(y)\, dy$$

$$= \int_0^t B(t,y)\, u'(y) dy\,,$$

where

$$B(t,y) = \frac{1}{\pi}\int_y^t \frac{ds}{(t-s)^{\frac{1}{2}}\,(s-y)^{\frac{1}{2}}}\,.$$

Putting $p = s - y$, we get $dp = ds$ and

$$B(t,y) = \frac{1}{\pi}\int_0^{t-y} \frac{dp}{(t-y-p)^{\frac{1}{2}}\, p^{\frac{1}{2}}}\,.$$

Putting $p = (t-y)u,$ we get $dp = (t-y)du$ and

$$B(t,y) = \frac{1}{\pi}\int_0^1 \frac{(t-y)\, du}{(1-u)^{\frac{1}{2}}\, u^{\frac{1}{2}}\,(t-y)} = \frac{1}{\pi}\int_0^1 (1-u)^{\frac{1}{2}-1}\, u^{\frac{1}{2}-1}\, du$$

$$= \frac{1}{\pi} B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{\pi}\, \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)} = 1\,.$$

Then

$$J(t) = \int_0^t u'(y)\, dy = u(t)\,.$$

Using formulas (6.3) and (6.5), we obtain

$$J(t) = -\frac{1}{2}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} D_s^{\frac{1}{2}} u(s)\, ds$$

$$= -\frac{1}{2}\int_0^t e^{-\frac{1}{2}(t-s)}\left[f(s) - \frac{1}{2}v(s)\right]\, ds\,. \tag{6.7}$$

Second, we will estimate the double integral above

$$I(t) = \frac{1}{4}\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\int_0^s e^{-\frac{1}{2}(s-y)}\, v(y)\, dy\, ds\,.$$

Changing the order of integration and using
$[0 \le s \le t, 0 \le y \le s] = [0 \le y \le t, y \le s \le t]$, we get

$$I(t) = \frac{1}{4}\frac{1}{\sqrt{\pi}}\int_0^t \int_y^t \frac{e^{-\frac{1}{2}(s-y)}}{(t-s)^{\frac{1}{2}}}\, ds\, v(y) dy = \int_0^t C(t,y)\, v(y)\, dy\,.$$

Were

$$C(t,y) = \frac{1}{4}\frac{1}{\sqrt{\pi}}\int_y^t \frac{e^{-\frac{1}{2}(s-y)}}{(t-s)^{\frac{1}{2}}}\, ds\,.$$

Putting $s - y = p$, we get $ds = dp$ and

$$C(t,y)=\frac{1}{4}\frac{1}{\sqrt{\pi}}\int_{0}^{t-y}\frac{e^{-\frac{1}{2}p}}{(t-y-p)^{\frac{1}{2}}}\,dp\le\frac{1}{4}\frac{1}{\sqrt{\pi}}\int_{0}^{t-y}\frac{dp}{(t-y-p)^{\frac{1}{2}}}$$

$$=\frac{\sqrt{t-y}}{2\sqrt{\pi}}\le\frac{\sqrt{T}}{2\sqrt{\pi}}. \tag{6.8}$$

Applying the triangle inequality, formulas (6.6), (6.7) and estimate (6.8), we get

$$|v(t)|\le\frac{1}{\sqrt{\pi}}\int_{0}^{t}\frac{1}{(t-s)^{\frac{1}{2}}}\left\{|f(s)|+\frac{1}{2}\int_{0}^{s}e^{-\frac{1}{2}(s-y)}|f(y)|\,dy\right\}ds$$

$$+\frac{1}{2}\int_{0}^{t}e^{-\frac{1}{2}(t-s)}\left\{|f(s)|+\frac{1}{2}|v(s)|\right\}ds+\int_{0}^{t}C(t,y)\ |v(y)|\,dy$$

$$\le\left(1+\frac{4\sqrt{T}}{\sqrt{\pi}}\right)\max_{0\le s\le T}|f(s)|+\int_{0}^{t}\left(\frac{1}{4}+\frac{\sqrt{T}}{2\sqrt{\pi}}\right)|v(y)|\,dy\,.$$

Applying the integral inequality, we get

$$|v(t)|\le\left(1+\frac{4\sqrt{T}}{\sqrt{\pi}}\right)\max_{0\le s\le T}|f(s)|\,e^{\left(\frac{1}{4}+\frac{\sqrt{T}}{2\sqrt{\pi}}\right)t}$$

for any $t\in[0,T]$. From that it follows that

$$\max_{0\le t\le T}\left|D_t^{\frac{1}{2}}u(t)\right|\le\left(1+\frac{4\sqrt{T}}{\sqrt{\pi}}\right)e^{\left(\frac{1}{4}+\frac{\sqrt{T}}{2\sqrt{\pi}}\right)T}\max_{0\le t\le T}|f(t)|\,. \tag{6.9}$$

Applying the triangle inequality and estimate (6.9), we get

$$|u'(t)| \leq |f(t)| + \frac{1}{2}\int_0^t e^{-\frac{1}{2}(t-s)} |f(s)|\, ds$$

$$+\frac{1}{2}\left|D_t^{\frac{1}{2}} u(t)\right| + \frac{1}{4}\int_0^t e^{-\frac{1}{2}(t-s)} \left|D_s^{\frac{1}{2}} u(s)\right|\, ds$$

$$\leq 2 \max_{0\leq t\leq T}|f(t)| + \max_{0\leq t\leq T}\left|D_t^{\frac{1}{2}} u(t)\right|$$

$$\leq \left[1 + \left(1 + \frac{4\sqrt{T}}{\sqrt{\pi}}\right) e^{\left(\frac{1}{4}+\frac{\sqrt{T}}{2\sqrt{\pi}}\right)T}\right] \max_{0\leq t\leq T}|f(t)|\,. \tag{6.10}$$

Applying the triangle inequality and estimates (6.9) and (6.10), we get

$$|u(t)| \leq 2|f(t)| + 2|D_t u(t)| + \left|D_t^{\frac{1}{2}}\, u(t)\right| \leq C_2 \max_{0\leq t\leq T}|f(t)| \tag{6.11}$$

Finally, applying estimate (6.9), (6.10) and (6.11), we get

$$\max_{0\leq t\leq T}|u(t)| + \max_{0\leq t\leq T}|u'(t)| + \max_{0\leq t\leq T}\left|D_t^{\frac{1}{2}}\, u(t)\right| \leq C_1 \max_{0\leq t\leq T}|f(t)|\,.$$

Theorem 6.1 is proved.

6.2 The stability of the difference scheme for the Basset equation

Applying the formula

$$D_t^{\frac{1}{2}} u(t_k) \approx \frac{1}{\tau^{\frac{1}{2}}}\, \frac{1}{\sqrt{\pi}} \sum_{n=1}^{k} \frac{1}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}\, e^{-t}\, dt\, (u_k - u_{k-1})$$

$$= D_\tau^{\frac{1}{2}}\, u_k\,, \tag{6.12}$$

and implicit difference scheme, we get the following difference scheme

$$\begin{cases} \dfrac{u_k - u_{k-1}}{\tau} + \dfrac{1}{2} u_k + \dfrac{1}{2} D_\tau^{\frac{1}{2}} u_k = \varphi_k, \varphi_k = f(t_k), \\ \quad t_k = k\tau, 1 \le k \le N, u_0 = 0, N\tau = T \ , \end{cases} \tag{6.13}$$

for the numerical solution of the initial value problem (6.1).

We have that

$$u_k = R^k u_0 + \sum_{i=1}^{k} \tau R^{k-i+1} \left[\varphi_i - \frac{1}{2} D_\tau^{\frac{1}{2}} u_i \right] \tag{6.14}$$

$$= \sum_{i=1}^{k} \tau R^{k-i+1} \left[\varphi_i - \frac{1}{2} D_\tau^{\frac{1}{2}} u_i \right] \ , k \ge 1,$$

where

$$R = \left(1 + \frac{\tau}{2}\right)^{-1}.$$

From formula (6.14) it follows

$$\frac{u_1 - u_0}{\tau} = R \left[\varphi_1 - \frac{1}{2} D_\tau^{\frac{1}{2}} u_1 \right]$$

$$\frac{u_k - u_{k-1}}{\tau} = R \left[\varphi_k - \frac{1}{2} D_\tau^{\frac{1}{2}} u_k \right] - \frac{1}{2} \sum_{i=1}^{k-1} \tau R^{k-i+1} \left[\varphi_i - \frac{1}{2} D_\tau^{\frac{1}{2}} u_i \right],$$

$$k \ge 2 \ . \tag{6.15}$$

Theorem 6.2. For the solution of difference scheme (6.13) the following stability estimates hold

$$\max_{1\le k\le N}|u_k| + \max_{1\le k\le N}\left|\frac{u_k - u_{k-1}}{\tau}\right| + \max_{1\le k\le N}\left|D_\tau^{\frac{1}{2}}u_k\right| \le C_2 \max_{1\le k\le N}|\varphi_k|, \quad (6.16)$$

where C_2 does not depend on τ and φ_k.

Proof. Applying the formulas (6.12), (6.15), we get

$$D_\tau^{\frac{1}{2}}u_k = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt$$

$$\left\{\tau R\left(\varphi_n - \frac{1}{2}D_\tau^{\frac{1}{2}}u_n\right) - \frac{\tau}{2}\sum_{i=1}^{n-1}\tau R^{n-i+1}\left(\varphi_i - \frac{1}{2}D_\tau^{\frac{1}{2}}u_i\right)\right\}$$

$$+\frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\frac{1}{(k-1)!}\int_0^\infty t^{k-1-\frac{1}{2}}\,e^{-t}dt\left\{\tau R\left(\varphi_1 - \frac{1}{2}D_\tau^{\frac{1}{2}}u_1\right)\right\},k\ge 1\,.$$

We denote that

$$v_k = D_\tau^{\frac{1}{2}}u_k\,. \quad (6.17)$$

Then, from the last formula it follows that

$$v_k = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k}\tau\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt$$

$$\left\{R\varphi_n - \frac{1}{2}\sum_{i=1}^{n-1}\tau R^{n-i+1}\varphi_i\right\} + \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt$$

$$\left\{-\frac{1}{2}R\,v_n + \frac{1}{4}\sum_{i=1}^{n-1}\tau R^{n-i+1}v_i\right\}$$

$$= I_k + I_{1,k} + J_k + J_{1,k} + I_{2,k} + J_{2,k}\ , \tag{6.17a}$$

where

$$I_k = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt\,\left\{-\frac{1}{2}\sum_{i=1}^{n-1}\tau R^{n-i+1}\varphi_i\right\},$$

$$J_k = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt\,\left\{\frac{1}{4}\sum_{i=1}^{n-1}\tau R^{n-i+1}v_i\right\},$$

$$I_{1,k} = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt\,\{R\varphi_n\},$$

$$J_{1,k} = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}e^{-t}\,dt\,\left\{-\frac{1}{2}Rv_n\right\},$$

$$I_{2,k} = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\tau\int_0^\infty t^{-\frac{1}{2}}\,e^{-t}\,dt\,\left\{R\,\varphi_k - \frac{1}{2}\sum_{i=1}^{k-1}\tau\,R^{k-i+1}\,\varphi_i\right\},$$

$$J_{2,k} = \frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\tau\int_0^\infty t^{-\frac{1}{2}}\,e^{-t}\,dt\,\left\{-\frac{1}{2}\,R\,v_k + \frac{1}{4}\sum_{i=1}^{k-1}\tau\,R^{k-i+1}\,v_i\right\}.$$

Now, we will estimate $|I_{1,k}|$, $|I_k|$, $|J_{1,k}|$, $|J_k|$, $|I_{2,k}|$ and $|J_{2,k}|$, separately.

Applying the triangle inequality, Holder's inequality, we get

$$|I_{1,k}| \le \frac{1}{\sqrt{\pi}}\sqrt{\tau}\sum_{n=2}^{k-1}\int_0^\infty \frac{t^{k-n-\frac{1}{2}}}{(k-n)!}e^{-t}\,dt\,R|\varphi_n|$$

$$\leq \frac{\sqrt{\tau}}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{1}{\sqrt{k-n}} \int_0^\infty \frac{(t^{k-n})^{\frac{1}{2}}}{\left((k-n)!\right)^{\frac{1}{2}}} e^{\frac{-t}{2}} \frac{(t^{k-n-1})^{\frac{1}{2}} e^{\frac{-t}{2}}}{\left((k-n-1)!\right)^{\frac{1}{2}}} dt \max_{1\leq k\leq N} |\varphi_n|$$

$$\leq \frac{\sqrt{\tau}}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{1}{\sqrt{k-n}} \left(\int_0^\infty \frac{t^{k-n} e^{-t}}{(k-n)!} dt \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{t^{k-n-1} e^{-t}}{(k-n-1)!} dt \right)^{\frac{1}{2}} \max_{1\leq k\leq N} |\varphi_k|$$

$$\leq \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{\sqrt{\tau}}{\sqrt{k-n}} \max_{1\leq k\leq N} |\varphi_k| \leq \frac{1}{\sqrt{\pi}} \int_0^t \frac{ds}{\sqrt{t-s}} \max_{1\leq k\leq N} |\varphi_k|$$

$$\leq \frac{1}{\sqrt{\pi}} 2\sqrt{T} \max_{1\leq k\leq N} |\varphi_k|$$

for any $k, k = 1, \dots, N$.

Applying the triangle inequality, we can obtain

$$|I_k| \leq \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \tau \frac{\Gamma\left(k-n+\frac{1}{2}\right)}{(k-n)!} \left\{ \frac{1}{2} \sum_{i=1}^{n-1} \tau \; R^{n-i+1} |\varphi_i| \right\}$$

$$\leq \max_{1\leq k\leq N} |\varphi_k| \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \tau \frac{\Gamma\left(k-n+\frac{1}{2}\right)}{(k-n)!}$$

$$= \max_{1\leq k\leq N} |\varphi_k| \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{\tau}{\sqrt{k\tau - n\tau}} \frac{\Gamma\left(k-n+\frac{1}{2}\right)}{\sqrt{k-n}\,(k-n-1)!}.$$

Applying Holder's inequality, we get

$$\frac{\Gamma\left(k-n+\frac{1}{2}\right)}{\sqrt{k-n}\,(k-n-1)!} \leq \sqrt{k-n} \;\; \frac{1}{(k-n)!} \int_0^\infty t^{k-n-\frac{1}{2}} e^{-t} dt$$

$$\leq \sqrt{k-n}\ \frac{1}{(k-n)!}\int_0^\infty t^{\frac{k-n}{2}}\ e^{-\frac{t}{2}}\, t^{\frac{k-n-1}{2}}\ e^{-\frac{t}{2}}\, dt$$

$$\leq \sqrt{k-n}\ \frac{1}{(k-n)!}\left(\int_0^\infty t^{k-n}\ e^{-t}\, dt\right)^{\frac{1}{2}}\left(\int_0^\infty t^{k-n-1}\ e^{-t}\, dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{k-n}\ \frac{1}{(k-n)!}(\Gamma(k-n+1))^{\frac{1}{2}}\,(\Gamma(k-n))^{\frac{1}{2}}$$

$$\leq \sqrt{k-n}\ \frac{1}{(k-n)!}\,\big((k-n)!\big)^{\frac{1}{2}}\,\big((k-n-1)!\big)^{\frac{1}{2}} = 1\,. \tag{6.18}$$

Therefore

$$|I_k| \leq \max_{1\leq k\leq N}|\varphi_k|\frac{1}{\sqrt{\pi}}\int_0^{t_k}\frac{ds}{\sqrt{t_k-s}} \leq \frac{2\sqrt{T}}{\sqrt{\pi}}\ \max_{1\leq k\leq N}|\varphi_k|$$

for any $k, k = 1, \dots, N$.

Now, we will estimate $|J_k|$.

Applying the triangle inequality and estimate (6.18), we get

$$|J_k| \leq \frac{1}{\tau^{\frac{1}{2}}}\,\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{\tau}{(k-n)!}\int_0^\infty t^{k-n-\frac{1}{2}}\,e^{-t}\, dt\ \left\{\frac{1}{4}\sum_{i=1}^{n-1}\tau R^{n-i+1}|v_i|\right\}$$

$$\leq \sqrt{\tau}\,\frac{1}{\sqrt{\pi}}\sum_{n=2}^{k-1}\frac{1}{\sqrt{k-n}}\left\{\frac{1}{4}\sum_{i=1}^{n-1}\tau R^{n-i+1}|v_i|\right\}$$

since $[2 \leq n \leq k-1, 1 \leq i \leq n-1] = [1 \leq i \leq k-2, i+1 \leq n \leq k-1]$, we have that

$$= \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{\sqrt{\tau}}{\sqrt{k-n}} \left\{ \frac{1}{4} \sum_{i=1}^{n-1} \tau R^{n-i+1} |v_i| \right\}$$

$$= \frac{1}{4} \frac{\tau}{\sqrt{\pi}} \sum_{i=1}^{k-2} \sqrt{\tau} \sum_{n=i+1}^{k-1} \frac{1}{\sqrt{k-n}} \left\{ R^{n-i+1} |v_i| \right\}.$$

Therefore,

$$|J_k| \le \sqrt{\tau} \frac{1}{\sqrt{\pi}} \frac{1}{4} \sum_{i=1}^{k-2} \tau |v_i| \sum_{n=i+1}^{k-1} \frac{R^{n-i+1}}{\sqrt{k-n}}$$

$$\le \frac{1}{4\sqrt{\pi}} \sum_{i=1}^{k-2} \tau \, |v_i| \sum_{n=i+1}^{k-1} \frac{\tau}{\sqrt{k\tau - n\tau}}$$

$$\le \frac{1}{4\sqrt{\pi}} \sum_{i=1}^{k-2} \tau \, |v_i| \int_{i\tau}^{k\tau} \frac{ds}{\sqrt{k\tau - s}} \le \frac{\sqrt{T}}{2\sqrt{\pi}} \sum_{i=1}^{k-2} \tau \, |v_i|$$

for any $k, k = 1, \dots, N$.

Now, we will estimate $|J_{1,k}|$. By (6.13), we have that

$$|J_{1,k}| = \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \sum_{n=2}^{k-1} \frac{\tau}{(k-n)!} \int_0^\infty t^{k-n-\frac{1}{2}} e^{-t} dt \left(-\frac{1}{2}\right) R \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}}$$

$$\sum_{m=1}^{n} \frac{1}{(n-m)!} \int_0^\infty s^{n-m-\frac{1}{2}} e^{-s} ds \, (u_m - u_{m-1}) \, .$$

It is clear that $[2 \le n \le k-1, 1 \le m \le n] = [1 \le m \le k-1, m \le n \le k-1]$.

Therefore, changing the order of summation, we get

$$\left|J_{1,k}\right| = \sum_{m=1}^{k-1} B(k,m)\ \frac{u_m - u_{m-1}}{\tau}\tau, \tag{6.19}$$

where

$$B(k,m) = \frac{1}{\pi}\left(-\frac{1}{2}\right) R\ \sum_{m=1}^{k-1} \frac{1}{(k-n)!\,(n-m)!} \int_0^{\infty} t^{k-n-\frac{1}{2}}\ e^{-t} dt$$

$$\int_0^{\infty} s^{n-m-\frac{1}{2}}\ e^{-s}\ ds = \frac{1}{\pi}\left(-\frac{1}{2}\right) R \sum_{m=1}^{k-1} \frac{\Gamma\left(k-n+\frac{1}{2}\right)\ \Gamma\left(n-m+\frac{1}{2}\right)}{(k-n)!\,(n-m)!}.$$

Applying (6.18), we get

$$|B(k,m)| \le \frac{R}{2\,\pi} \sum_{m=1}^{k-1} \frac{1}{\sqrt{k-n}\,\sqrt{n-m}}$$

$$\le \frac{R}{2\,\pi} \sum_{m=1}^{k-1} \frac{\tau}{\sqrt{k\tau - n\tau}\,\sqrt{n\tau - m\tau}}$$

$$\le C_1 \int_{m\tau}^{k\tau} \frac{ds}{\sqrt{k\tau - s}\,\sqrt{s - m\tau}} = C_1 \int_0^{(k-m)\tau} \frac{dy}{\sqrt{(k-m)\tau - y}\,\sqrt{y}}.$$

Putting $y = (k-m)\tau t$, we get $dy = (k-m)\tau\ dt$

$$\int_0^{(k-m)\tau} \frac{dy}{\sqrt{(k-m)\tau - y}\,\sqrt{y}} = \int_0^1 \frac{dt}{\sqrt{1-t}\,\sqrt{t}} = \int_0^1 t^{\frac{1}{2}-1}\,(1-t)^{\frac{1}{2}-1}\,dt$$

$$= B\left(\frac{1}{2},\frac{1}{2}\right) = \pi\,.$$

Therefore

$$|B(k,m)| \leq C_1\pi = C_2\,. \tag{6.20}$$

Using the triangle inequality formulas (6.15), (6.19) and estimate (6.20), we get

$$|J_{1,k}| \leq \sum_{m=1}^{k-1} |B(k,m)| \left|\frac{u_m - u_{m-1}}{\tau}\right| \tau$$

$$\leq C_2 \left[\sum_{m=1}^{k-1} R\left\{|\varphi_m| + \frac{1}{2}|\upsilon_m|\right\}\tau + \sum_{m=1}^{k-1}\sum_{i=1}^{m-1} \frac{1}{2}\tau R^{m-i+1}\left\{|\varphi_i| + \frac{1}{2}|\upsilon_i|\right\}\tau\right]$$

$$\leq C_2 \left[k\tau \max_{1\leq k\leq N}|\varphi_k| + \frac{1}{2}\sum_{m=1}^{k-1} R|\upsilon_m|\tau\right]$$

$$+C_2 \left[\sum_{m=1}^{k-1}\sum_{i=1}^{m-1} \frac{1}{2}\tau^2 R^{m-i+1} \max_{1\leq k\leq N}|\varphi_k| + \sum_{m=1}^{k-1}\sum_{i=1}^{m-1} \frac{1}{4}\tau R^{m-i+1}|\upsilon_i|\tau\right]$$

$$\leq C_2 \left[2T \max_{1\leq k\leq N}|\varphi_k| + \frac{1}{2}\sum_{m=1}^{k-1} R|\upsilon_m|\tau + \sum_{m=1}^{k-1}\sum_{i=1}^{m-1} \frac{\tau}{4} R^{m-i+1}|\upsilon_i|\tau\right]$$

since $[1 \leq m \leq k-1, 1 \leq i \leq m-1] = [1 \leq i \leq k-1, i \leq m \leq k-1]$, we have that

$$\sum_{m=1}^{k-1}\sum_{i=1}^{m-1} \frac{\tau}{4} R^{m-i+1}|\upsilon_i|\tau = \sum_{i=1}^{k-1} \tau|\upsilon_i| \sum_{m=i}^{k-1} \frac{\tau}{4} R^{m-i+1} \leq \frac{1}{2}\sum_{i=1}^{k-1} \tau|\upsilon_i|\,.$$

Therefore,

$$\left|J_{1,k}\right| \leq C_2 \sum_{i=1}^{k-1} \tau|v_i| + C_2\ 2T \max_{1\leq k\leq N}|\varphi_k|$$

for any $\mathrm{k}, \mathrm{k} = 1, \dots, \mathrm{N}$.

Now, we will estimate $\left|\mathrm{I}_{2,\mathrm{k}}\right|$.

Appling the triangle inequality, we can obtain

$$\left|I_{2,k}\right| \leq \frac{\sqrt{\tau}}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt\ \left\{|\varphi_k| + \frac{1}{2}\max_{1\leq k\leq N}|\varphi_k|k\tau\right\}$$

$$\leq \sqrt{\tau}\max_{1\leq k\leq N}|\varphi_k|\left\{1+\frac{1}{2}\right\} = \frac{3}{2}\sqrt{\tau}\max_{1\leq k\leq N}|\varphi_k|\,,$$

for any $k, k = 1, \dots, N$.

Finally, we will estimate $\left|J_{2,k}\right|$.

Applying the triangle inequality, we can obtain

$$\left|J_{2,k}\right| \leq \frac{\sqrt{\tau}}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt\ \left\{\frac{1}{2}R|v_k| + \frac{1}{4}\sum_{i=1}^{k-1}\tau\, R^{k-i+1}|v_i|\right\}$$

$$\leq \sqrt{\tau}\left\{\frac{1}{2}|v_k| + \frac{1}{4}\sum_{i=1}^{k-1}\tau\,|v_i|\right\} \leq \frac{\sqrt{\tau}}{2}|v_k| + \frac{\sqrt{\tau}}{4}\sum_{i=1}^{k-1}\tau\,|v_i|\,,$$

for any $\mathrm{k}, \mathrm{k} = 1, \dots, \mathrm{N}$.

Applying formula (6.17a) and estimates for $\left|\mathrm{I}_{1,\mathrm{k}}\right|$, $|\mathrm{I}_\mathrm{k}|$, $\left|\mathrm{J}_{1,\mathrm{k}}\right|$, $|\mathrm{J}_\mathrm{k}|$, $\left|\mathrm{I}_{2,\mathrm{k}}\right|$ and $\left|\mathrm{J}_{2,\mathrm{k}}\right|$, we get

$$|v_k| \leq \frac{2}{\sqrt{\pi}}\sqrt{T}\max_{1\leq k\leq N}|\varphi_k| + \frac{\sqrt{T}}{2\sqrt{\pi}}\sum_{i=1}^{k-2}\tau|v_i| + C_2\sum_{i=1}^{k-1}\tau|v_i|$$

$$+C_2 2\,T \max_{1\le k\le N}|\varphi_k| + \frac{2}{\sqrt{\pi}}\sqrt{T} \max_{1\le k\le N}|\varphi_k| + \frac{3}{2}\sqrt{\tau}\max_{1\le k\le N}|\varphi_k| + \frac{\sqrt{\tau}}{2}|\upsilon_k|$$

$$+\frac{\sqrt{\tau}}{4}\sum_{i=1}^{k-1}\tau\,|\upsilon_i|$$

$$\le \left(2\,C_2\,T + 4\sqrt{\frac{T}{\pi}} + \frac{3}{2}\sqrt{\tau}\right)\max_{1\le k\le N}|\varphi_k|$$

$$+\left(C_2 + \frac{\sqrt{T}}{2\sqrt{\pi}} + \frac{\sqrt{\tau}}{4}\right)\sum_{i=1}^{k-1}\tau|\upsilon_i| + \frac{\sqrt{\tau}}{2}|\upsilon_k|\,.$$

Therefore,

$$\left(1-\frac{\sqrt{\tau}}{2}\right)|\upsilon_k| \;\le \left(2\,C_2\,T + 4\sqrt{\frac{T}{\pi}} + \frac{3}{2}\sqrt{\tau}\right)\max_{1\le k\le N}|\varphi_k|$$

$$+\left(C_2 + \frac{\sqrt{T}}{2\sqrt{\pi}} + \frac{\sqrt{\tau}}{4}\right)\sum_{i=1}^{k-1}\tau|\upsilon_i|\,,$$

$$|\upsilon_k| \le \frac{1}{1-\frac{\sqrt{\tau}}{2}}\left(2\,C_2\,T + 4\sqrt{\frac{T}{\pi}} + \frac{3}{2}\sqrt{\tau}\right)\max_{1\le k\le N}|\varphi_k|$$

$$+\frac{1}{1-\frac{\sqrt{\tau}}{2}}\left(C_2 + \frac{\sqrt{T}}{2\sqrt{\pi}} + \frac{\sqrt{\tau}}{4}\right)\sum_{i=1}^{k-1}\tau|\upsilon_i|\,.$$

for any $k, k = 1, \dots, N$.

Applying the discrete analogue of integral inequality, we get

$$|v_k| \le \frac{1}{1-\frac{\sqrt{\tau}}{2}}\left(2\,C_2\,T + 4\sqrt{\frac{T}{\pi}} + \frac{3}{2}\sqrt{\tau}\right)\max_{1\le k\le N}|\varphi_k|\; e^{\frac{\left(C_2+\frac{\sqrt{T}}{2\sqrt{\pi}}+\frac{\sqrt{\tau}}{4}\right)k\tau}{1-\frac{\sqrt{\tau}}{2}}},$$

for any $k, k = 1, \ldots, N$. From that it follows that

$$\max_{1\le k\le N}\left|D_\tau^{\frac{1}{2}}\,u_k\right| \le \frac{1}{1-\frac{\sqrt{\tau}}{2}}\left(2\,C_2\,T + 4\sqrt{\frac{T}{\pi}} + \frac{3}{2}\sqrt{\tau}\right)$$

$$e^{\frac{\left(C_2+\frac{\sqrt{T}}{2\sqrt{\pi}}+\frac{\sqrt{\tau}}{4}\right)k\tau}{1-\frac{\sqrt{\tau}}{2}}}\max_{1\le k\le N}|\varphi_k|. \qquad (6.21)$$

Applying formula (6.15), the triangle inequality and estimate (6.21), we get

$$\left|\frac{u_1-u_0}{\tau}\right| \le R\left(|\varphi_1| + \frac{1}{2}\left|D_\tau^{\frac{1}{2}}\,u_1\right|\right) \le\; C_2\max_{1\le k\le N}|\varphi_k|\,,$$

$$\left|\frac{u_k-u_{k-1}}{\tau}\right| \le R|\varphi_k| + \frac{1}{2}\sum_{i=i}^{k-1}\tau R^{k-i+1}\,|\varphi_i| + \frac{1}{2}R\left|D_\tau^{\frac{1}{2}}\,u_k\right|$$

$$+\frac{1}{2}\sum_{i=i}^{k-1}\tau R^{k-i+1}\,\frac{1}{2}\left|D_\tau^{\frac{1}{2}}\,u_i\right|$$

$$\le C_2\max_{1\le k\le N}|\varphi_k|\;, k \ge 2\,. \qquad (6.22)$$

Using the triangle inequality and estimates (6.21), (6.22) and (6.13), we get

$$|u_k| \le 2\left|\frac{u_k-u_{k-1}}{\tau}\right| + \left|D_\tau^{\frac{1}{2}}u_k\right| + 2|\varphi_k| \le C_1\max_{1\le k\le N}|\varphi_k| \qquad (6.23)$$

Finally, estimates (6.21), (6.22) and (6.23) we get estimate (6.16). Therefore 6.2 is proved. Now, for support of theoretical results, we consider the numerical solution of the test initial value problem

$$u'(t) + \frac{1}{2} D_t^{\frac{1}{2}} u(t) + \frac{1}{2} u(t) = 4t + t^2 + \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}},$$

$$0 \le t \le 1, u(0) = 0 \qquad (6.24)$$

for the Basset equation. The exact solution of this test example is $u(t) = 2t^2$.

We get the following difference scheme of first order of accuracy for the numerical solution of the initial value problem (6.24)

$$\frac{u_k - u_{k-1}}{\tau} + \frac{1}{2} u_k + \frac{1}{2} \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \sum_{n=1}^{k} \frac{1}{(k-n)!} \Gamma\left(k - n + \frac{1}{2}\right) (u_n - u_{n-1}) = \varphi_k,$$

$$\varphi_k = 4t_k + t_k^2 + \frac{8}{3\sqrt{\pi}} (t_k)^{\frac{3}{2}}, t_k = k\tau, 1 \le k \le N, u_0 = 0,$$

$$N\tau = 1. \qquad (6.25)$$

For solving difference scheme (6.25), we will transform it in following matrix form:

$$A^\tau u^\tau = \varphi^\tau,$$

where

$$A^\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{2,0} & a_{2,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{3,0} & a_{3,1} & a_{3,2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{N,0} & a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,N-3} & a_{N,N-2} & a_{N,N-1} & a_{N,N} \end{bmatrix}$$

where

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & b & c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \frac{a\Gamma\left(\frac{3}{2}\right)}{2!} & b & c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \frac{a\Gamma\left(\frac{5}{2}\right)}{3!} & \frac{a\Gamma\left(\frac{3}{2}\right)}{2!} & b & c & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \frac{a\Gamma\left(N-\frac{5}{2}\right)}{(N-2)!} & \frac{a\Gamma\left(N-\frac{7}{2}\right)}{(N-3)!} & \frac{a\Gamma\left(N-\frac{9}{2}\right)}{(N-4)!} & \cdots & \cdots & \cdots & \frac{a\Gamma\left(\frac{3}{2}\right)}{2!} & b & c & 0 \\ \cdots & \frac{a\Gamma\left(N-\frac{3}{2}\right)}{(N-1)!} & \frac{a\Gamma\left(N-\frac{5}{2}\right)}{(N-2)!} & \frac{a\Gamma\left(N-\frac{7}{2}\right)}{(N-3)!} & \cdots & \cdots & \cdots & \frac{a\Gamma\left(\frac{5}{2}\right)}{3!} & \frac{a\Gamma\left(\frac{3}{2}\right)}{2!} & b & c \end{bmatrix}$$

$$u^\tau = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, \qquad \varphi^\tau = \begin{bmatrix} 0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{bmatrix}$$

are unknown and given grid functions.

Solving it, we get

$$u^\tau = (A^\tau)^{-1}\, \varphi^\tau .$$

We obtain the following table for the error E_N of solution of difference scheme defined by formula

$$E_N = \max_{0 \le k \le N} |u_k - u(t_k)|$$

Difference scheme *N*	30	60	120
(6.25)	0.0473	0.0237	0.0119

As it is seen in this table, we get some numerical results. If N are doubled, the value of errors E_N decrease by a factor of approximately $\frac{1}{2}$ for first order of accuracy difference scheme.

CHAPTER 7
CONCLUSIONS

This work is dedicate to study fractional calculus and its applications for the fractional Basset equation.

The following results are obtained:

- Study properties of fractional integral.
- Study properties of Caputo fractional differential operator.
- Study properties of Riemann - Liouville fractional differential operator.
- Methods for the solutions of initial value problems fractional differential equations are applied.
- The theorem on the stability estimates for the solution of the initial value problem for the fractional Basset equation is established.
- The theorem on the stability estimates for the solution of the first order of accuracy differential scheme for the numerical solution of the initial value problem for the fractional Basset equation is proved.
- The MATLAB implementation of the difference scheme for the numerical solution of the test Basset problem is presented.
- The theoretical expressions for the solutions of the difference scheme are supported by the results of numerical examples.

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APPENDIX

MATLAB implementation of the difference scheme

```
N=120;
 tau=1/N;
a=-1/(4*(tau*pi)^(1/2));
b=-(1/tau)-(1/(2*(tau)^(1/2)))+(1/(4*(tau)^(1/2)));
c=(1/tau)+(1/2)+(1/(2*(tau)^(1/2)));
A=zeros(N+1,N+1);
for i=2:N+1;
for j=2:N+1;
     A(i,i)=c;
     if i>j;
      A(i,j)=b;
     end;
  if i>j+1;
  A(i,j)=(a*(gamma(i-j-0.5))/factorial(i-j));
  end;
end;
end;
A(1,1)=1;
A;
fii=zeros(N+1,1) ;
for k=1:N+1;
t=(k-1)*tau;
fii(k)=4*t+(t^2)+(8/(3*(pi^(1/2))))*t^(3/2);
end;
fii;
G=inv(A);
u=zeros(N+1,1);
 u=G*(fii);
u;
%\%\%\%\%\%'EXACT SOLUTION OF THIS DDE' \%\%\%\%\%\%\%\%
eu=zeros(N+1,1);
for k=1:N+1;
t=(k-1)*tau;
eu(k)=2*(t^2);
end;
eu
% ABSOLUTE DIFFERENCES ;
absdiff=max(abs(eu-u))
```