A HIGH ORDER ACCURATE METHOD FOR SOLUTION AND ITS DERIVATIVES OF THE LAPLACE EQUATION

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
HEDİYE SARIKAYA

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy of Science in Mathematics

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy of Science in Mathematics

Examining Committee in Charge:

Prof.Dr. Tanıl Ergenç	Committee Chairman, Department of Mathematics, ATÜ
Prof.Dr. Agamirza Bashirov	Department of Mathematics, EMU
Prof.Dr. Adıgüzel Dosiyev	Supervisor, Department of Mathematics, NEU
Prof.Dr. Evren Hınçal	Head of the Department of Mathematics, NEU
Assoc.Prof.Dr. Murat Tezer	Department of Mathematics, NEU

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Name, Last name: HEDİYE SARIKAYA

Signature:

Date:

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To my parents …

ABSTRACT

In this thesis, we investigate on the highly accurate finite-difference approximation of the solution of Laplace's equation and its derivatives on a rectangle and on a rectangular parallelepiped.

The approximation of first and second order pure and mixed derivative of the solution of Dirichlet problem on a rectangular parallelepiped, will be examined. It is assume that the $p-th$ order derivatives, $p \in \{4,5\}$ of the functions which are given on the boundary satisfy Hölder condition. On the edges the compatibility conditions hold for continuity, and for second and fourth order derivatives which follow from the Laplace equation. The uniform estimations of the approximate solution and its first order derivative are of order $O(h^{p-1})$ with step size h. Also, it is proved that the obtained approximate values for the second order pure and mixed derivatives of the solution of Laplace equation have estimations with the order of $O(h^{p-2+\lambda})$ and $O(h^{p-2})$, respectively.

The multi stage method is constructed and justified to obtain a high order approximation of the solution and its derivatives of the Dirichlet problem for Laplace's equation on a rectangular domain. For the sufficiently smooth boundary values, it is proved that the constructed functions for the solution, and for the first and second order pure derivatives are convergent of order, $O(h^8)$ uniformly.

In the case of problem for the Laplace equation with the mixed boundary condition on a rectangular domain, it is assumed that the fourth order derivatives of the function given on the boundary satisfy the Hölder condition. On the edges the compatibility conditions hold for the second and fourth order derivatives which follow from the Laplace equation. The solutions of the finite-difference problem and it the first order derivative are of order $O(h^4)$ and $O(h^3)$, respectively.

The numerical experiments are presented to support the obtained theoretical results.

***Keywords*:** Approximation of derivatives; uniform error; finite difference method; Laplace’s equation; mixed boundary condition; Dirichlet problem; error estimation

ÖZET

Bu tezde, Laplace denkleminin dikdörtgensel bölgede ve dikdörtgenler prizması üzerinde çeşitli sınır şartları göz önünde bulundurularak çözümü ve türevleri incelenmiştir.

Laplace denkleminin dikdörtgenler prizması üzerinde Dirichlet probleminin çözümü, birinci mertebeden türevi, ikinci mertebeden saf ve karışık türevlerinin yaklaşımı tartışılır. Prizmanın yüzlerinde verilen sınır fonksiyonlarının p. türevlerinin Hölder şartını sağladığı kabul edilir, burada $p \in \{4,5\}$ olarak kabul edilecektir. Kenarlarda süreklilik şartının yanı sıra ikinci ve dördüncü mertebeden türevleri Laplace denkleminden sonuçlanan uyumluluk koşulunu sağlar. Önerilen fark şemalarının çözümünün küp ızgaralar üzerinde h ızgara uzunluğu olduğunda Laplace denkleminin çözümünün ve birinci türevinin $O(h^{p-1})$ mertebesinden düzgün yakınsadığı, ikinci dereceden püre türevinin $O(h^{p-2+\lambda})$ ve karışık türevinin ise $O(h^{p-2})$ mertebesinden yakınsadığı ispatlanmıştır.

Laplace denkleminin dikdörtgensel bölge üzerinde Dirichlet probleminin çözümü, birinci mertebeden türevi ve ikinci mertebeden saf türevleri için çok aşamalı yöntem oluşturularak kullanıldı. Dikdörtgenin kenarlarında verilen sınır fonksiyonunun yeterince düzgün seçildiğinde, Dirichlet probleminin kare ızgara üzerinde çözümü için ve çözümün birinci mertebeden ve ikinci mertebeden saf türevleri için $O(h^8)$ düzgün yakınsaklığı sade bir fark şeması ile elde edildi.

Aynı zamanda dikdörtgensel bölge üzerinde Laplace denkleminin karışık sınır şartı problemi de incelenmiştir. Dikdörtgenin kenarlarında verilen sınır fonksiyonunun dördüncü türevlerinin Hölder şartını sağladıkları kabul edildi. Köşelerde süreklilik şartının yanı sıra ikinci ve dördüncü türevlerinin de uyumluluk şartlarını sağladığı kabul edildi. Bu şartlar altında karışık sınır probleminin kare ızgara üzerinde çözümü için $O(h^4)$ ve çözümün birinci mertebeden türevi için $O(h^3)$, h adım uzunluğu olmak üzere sağlandığı ispatlandı.

Elde edilen teorik sonuçları desteklemek için sayısal sonuçlar da sunulmuştur.

Anahtar Kelimeler: Türevlerin yaklaşımı; düzgün hata; sonlu fark metodu; Laplace denklemi; karışık sınır şartı; Dirichlet sınır şartı; noktasal hata tahminleri

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CHAPTER 1
INTRODUCTION

Elliptic equations are widely used in many applied sciences to represent equilibrium or steady-state problems. Laplace's equation in particular, which is one of the most encountered elliptic equations, has been used to model many real-life situations such as the steady flow of heat or electricity in homogeneous conductors, the irrotational flow of incompressible fluid, problems arising in magnetism, and so on. In many applied problems most interesting is not only to find the solution itself but also its derivatives as in the problems: (i) in the electrostatics the first derivatives of electrostatic potential function define electric field. Furthermore, for the calculation of ray tracing in electrostatic fields by the interpolation methods require the specification at each mesh point not only the potential Φ but also the gradients $\left\{\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right\}$ and the mixed derivative $\frac{\partial^2 \Phi}{\partial x \partial y}$. The accuracy of interpolation depends on the accuracy of potentials and derivatives which are specified (see, Chmelfk and Barth, 1993); (ii) in the fracture problem the first derivative of the intensity function defines the stress intensity factor which is a fundamental problem of fracture mechanics.

For the numerical solution of this equation, a highly accurate method becomes a powerful tool in reducing the number of unknowns, which is the main problem in the numerical solution of differential equations, to get reasonable results. This becomes more valuable in 3D problems when we are looking for the derivatives of the unknown solution by the finite difference or finite element methods for a small discretization parameter h.

It is known that if we have an approximation of a function $f(x)$ by the function $\varphi(x)$ as

$$f(x) = \varphi(x) + R(x), \tag{1.1}$$

with small residual term $R(x)$ of this approximation, then by differentiating of (1.1) for the $k-th$ order derivatives

$$f^{(k)}(x) = \varphi^{(k)}(x) + R^{(k)}(x), \tag{1.2}$$

the residual term $R^{(k)}(x)$ can be very large (see (Berezin and Zhidkov, 1965)).

Therefore, a highly accurate approximation for the derivatives of the solution of above mentioned problems become important.

As is well known, an accuracy of the Domain Decomposition or Combined Methods for the solution of partial differential equations depends on the accuracy of a numerical solution on the standard subdomains covering the given domain of the exact solution (see (Volkov 1968; 1976), (Dosiyev 1992; 1994; 2002), (Li, 1998)). Therefore, an error analysis of the Finite Difference or Finite Element Methods on standard domains becomes important. It is also known that, to enlarge a class of problems to apply theoretical results, the maximum possible order of accuracy should be obtained by minimum requirements on the functions given in the boundary conditions.

The investigation of approximate derivatives started in (Lebedev, 1960), where it was proved that the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the 2D Laplace equation in any strictly interior subdomain, with the same order h with which the difference solution converges on the given domain. The uniform convergence of the difference derivatives over the whole grid domain to the corresponding derivatives of the solution for the 2D Laplace equation with the order $O(h^2)$ was proved in (Volkov, 1999).

In (Dosiyev and Sadeghi, 2015), for the first and pure second derivatives of the solution of the 2D Laplace equation special finite difference problems were investigated. It was proved that the solution of these problems converge to the exact derivatives with the order $O(h^4)$. In (Volkov, 2005) for the 3D Laplace equation the convergence of order $O(h^2)$ of the difference derivatives to the corresponding first order derivatives of the exact solution was proved. It was assumed that on the faces the boundary functions have third derivatives satisfying the Holder condition. Furthermore, they are continuous on the edges, and their second derivatives satisfy the compatibility condition that is implied by the Laplace equation. Whereas in (Volkov, 2004) when the boundary values on the faces of a parallelepiped are supposed to have the fourth derivatives satisfying the Holder condition, the constructed difference schemes converge with order $O(h^2)$ to the first and pure second derivatives of the exact solution. The mixed second derivative of the solution to the Dirichlet problem is found on a grid with accuracy $O\left(\frac{h^2}{\rho+h}\right)$, where ρ is the distance from the current mesh node to the parallelepiped boundary, by the numerical differentiation of the approximate first derivative. The appearance of the distance function ρ in the error estimation of the approximation of the second order mixed derivatives just because of unboundedness of the fourth order mixed derivatives with respect to odd number of times to each variable. In (Dosiyev and Sadeghi, 2016) it is assumed that the boundary functions on the faces have sixth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Different difference schemes with the use of the 26-point averaging operator are constructed on a cubic grid with mesh size h, to approximate the first and pure second derivatives of the solution of the Dirichlet problem with order $O(h^4)$.

One of the effective methods of increased accuracy with a simplest finite difference approximation by correcting the right hand term using the high order differences of the numerical solution of 2D Laplace's equation without justification was proposed by L. Fox (1947). Some modification of Fox's approach was given by L.C. Woods (1950). A theoretical justification of Fox's method was done by Volkov in (1954; 1965). From the Volkov's results in the case of Dirichlet problem for Poisson's equation on a rectangular domain Π follows that the approximate solution obtained by the $q-th$ correction of the right hand side of the $5-$point scheme, the convergence order in the uniform metric is $O(h^{2q})$, h is the mesh step, when the exact solution u has $(2q+2)-th$ derivatives on $\overline{\Pi}$ satisfying a Hölder condition with exponent $\lambda \in (0,1)$, i.e., $u \in C^{2q+2,\lambda}(\overline{\Pi})$.

In (Volkov, 2009) a two-stage difference method for solving the Dirichlet problem for 3D Laplace's equation on a rectangular parallelepiped was proposed. It was assumed that the given boundary functions on the faces of a parallelepiped are supposed to have the sixth derivatives satisfying the Hölder condition, and on the edges, besides the continuity they satisfy the compatibility condition for second derivatives, which results from the Laplace equation. It was proved that by using a simple $7-$point scheme in two stages the order of uniform error can be improved up to $O(h^4 ln\, h^{-1})$. From the conditions imposed on the boundary functions in (Volkov, 2009) does not follow, as it was declared in (Berikelashvili and Midodashvili, 2015) that the exact solution belongs to $C^{6,\lambda}(\overline{\Pi})$ (see (Volkov, 1969)).

In this thesis, we investigate of the approximation of a solution of the Dirichlet problem for Laplace's equation, and its first and second order derivatives in a rectangular parallelepiped. Furthermore, we construct a three stage ($9-$point, $5-$point, $5-$point) difference method for approximating of the solution and its first and second derivatives of the mixed boundary value problem for Laplace’s equation on a rectangle.

In Chapter 2, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. It is assumed that the boundary values on the faces have $p-th$ order, $p \in \{4,5\}$ derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Four different schemes with the $14-$point averaging operator, are constructed on a cubic grid with mesh size h, whose solutions separately approximate; (i) the solution of the Dirichlet problem with the order $O(h^4\rho^{p-4})$, (ii) approximates its first derivatives with the order $O(h^{p-1})$, (iii) approximates its pure second order derivatives with the order $O(h^{p-2+\lambda})$ and the second order mixed derivatives with the order $O(h^{p-2})$.

In Chapter 3, a new three-stage difference method for the solution, and its first and second order pure derivatives of the Dirichlet problem for Laplace’s equation on a rectangular domain is proposed. At the first stage the $9-$point scheme, and at the second and third stages the $5-$point schemes are used. For the error of the approximate solution a pointwise estimation of order $O(\rho h^8)$ is obtained, where $\rho = \rho(x, y)$ is the distance from the current grid point $(x, y) \in \Pi^h$ to the boundary of the rectangle Π. Then, at the first stage, to approximate of order $O(h^6)$ of the first derivative of the sum of pure fourth derivatives the 9-point scheme is used. At the second stage, approximate values of the first derivative of the sum of the pure eighth derivatives is approximated of order $O(h^2)$ by the $5-$point scheme. At the final third stage, the system of simplest $5-$point difference equations approximating the first derivative of the solution is corrected by introducing the quantities determined at the first and second stages. It is proved that, when the exact solution is from the Hölder classes $C^{10,\lambda}(\overline{\Pi})$ the uniform error of the approximate values of the first derivatives and second order pure derivatives are of order $O(h^8)$.

In Chapter 4, in a rectangular domain, we discuss an approximation of the first order derivatives for the solution of the mixed boundary value problem. The boundary values on the sides of the rectangle are supposed to have the fourth derivatives satisfying the Hölder condition. On the vertices besides the continuity condition, the compatibility conditions which result from the Laplace equation for the second and fourth derivatives of the boundary values given on the adjacent sides are satisfied. Under these conditions for the approximate values of the first derivatives of the solution of the mixed boundary problem on a square grid, as a solution of the constructed difference scheme a uniform error estimation of order $O(h^3)$ (h is the grid size) is obtained.

In Chapter 5, the numerical experiments, to justify the obtained theoretical results in each Chapters are demonstrated.

The results of the dissertation are published in (Dosiyev and Sarıkaya, 2017; 2018; 2018^*; 2019).

CHAPTER 2

14-POINT DIFFERENCE OPERATOR FOR THE APPROXIMATION OF THE SOLUTION, FIRST AND SECOND ORDER DERIVATIVES OF LAPLACE'S EQUATION IN A RECTANGULAR PARALLELEPIPED

In this chapter, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. It is assumed that the boundary values on the faces have $p-th$ order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Four different schemes are constructed on a cubic grid with mesh size h, whose solutions seperately approximate the solution of the Dirichlet problem with the order $O(h^4\rho^{p-4})$, where $\rho=\rho(x_1,x_2,x_3)$ is the distance from the current point $(x_1,x_2,x_3)\in R$ to the boundary of the rectangular parallelepiped R, boundary fuctions on the faces are from $C^{p,\lambda}$, approximates its first derivatives converges uniformly with order $O(h^{p-1})$, where $p\in\{4,5\}$. It is proved that the proposed difference schemes for the approximation of the pure and mixed second derivatives converge uniformly with order $O\left(h^{p-2+\lambda}\right)$, $0<\lambda<1$ and $O(h^{p-2})$, respectively.

2.1 Some Properties of a Solution of the Dirichlet Problem on a Rectangular Parallelepiped

Let $R=\{(x_1,x_2,x_3): 0<x_i<a_i, i=1,2,3\}$ be an open rectangular parallelepiped; Γ_j, $j=1,2,\ldots,6$ be its faces including the edges; Γ_j for $j=1,2,3$ ($j=4,5,6$) belongs to the plane $x_j=0$ ($x_{j-3}=a_{j-3}$), and let $\Gamma=\cup\,\Gamma_j$ be the boundary of R; $\gamma_{\mu\nu}=\Gamma_\mu\cap\Gamma_\nu$ be the edges of the parallelepiped R. $C^{k,\lambda}(E)$ is the class of functions that have continuous $k-th$ derivatives satisfying the Hölder condition with an exponent $\lambda\in(0,1)$.

Consider the boundary value problem

$$\Delta u = 0 \; on \; R, u = \varphi_j \; on \; \Gamma_j, j = 1,2, \dots ,6 \tag{2.1}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{p,\lambda}\left(\Gamma_j\right),\; 0 < \lambda < 1,\;\; j = 1,2,\dots,6,\;\; \mathrm{p} \in \{4,5\} \tag{2.2}$$

$$\varphi_\mu = \varphi_\nu \;\; on \; \gamma_{\mu\nu}, \tag{2.3}$$

$$\left(\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2}\right) + \left(\frac{\partial^2 \varphi_\nu}{\partial t_\nu^2}\right) + \left(\frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2}\right) = 0 \; on \; \gamma_{\mu\nu}, \tag{2.4}$$

$$\left(\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4}\right) + \left(\frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2}\right) = \left(\frac{\partial^4 \varphi_\nu}{\partial t_\nu^4}\right) + \left(\frac{\partial^4 \varphi_\nu}{\partial t_\nu^2 \partial t_{\nu\mu}^2}\right) \; on \; \gamma_{\nu\mu}, \tag{2.5}$$

where $1 \leq \mu < \nu \leq 6, \nu - \mu \neq 3, t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, t_μ and t_ν is element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

Lemma 2.1 Under conditions (2.2)-(2.5) the solution u of the Dirichlet problem (2.1) belong to the Hölder class $C^{p,\lambda}(\bar{R})$, $0 < \lambda < 1, p \in \{4,5\}$.

The proof of Lemma 2.1 follows from Theorem 2.2 in (Volkov, 1969).■

Lemma 2.2 Let $\rho(x_1, x_2, x_3)$ be the distance from the current point of the open parallelepiped R to its boundary and let $\left(\frac{\partial}{\partial l}\right) \equiv \alpha_1 \left(\frac{\partial}{\partial x_1}\right) + \alpha_2 \left(\frac{\partial}{\partial x_2}\right) + \alpha_3 \left(\frac{\partial}{\partial x_3}\right)$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$.

Then the next inequality holds

$$\left|\frac{\partial^6 u(x_1,x_2,x_3)}{\partial l^6}\right| \leq c\rho^{p+\lambda-6}(x_1, x_2, x_3),\;\; (x_1, x_2, x_3) \in R \;\text{ and } p \in \{4,5\} \tag{2.6}$$

where u is the solution of the problem (2.1), c is a constant independent of the direction of derivative $\left(\frac{\partial}{\partial l}\right)$.

Proof. We choose an arbitrary point $(x_{10}, x_{20}, x_{30}) \in R$. Let $\rho_0 = \rho(x_{10}, x_{20}, x_{30})$ and $\bar{\sigma}_0 \subset \bar{R}$ be the closed ball of radius ρ_0 centered at (x_{10}, x_{20}, x_{30}).

Consider the harmonic function on R

$$v(x_1, x_2, x_3) = \frac{\partial^p u(x_1, x_2, x_3)}{\partial l^p} - \frac{\partial^p u(x_{10}, x_{20}, x_{30})}{\partial l^p}.$$

By Lemma 2.2, $u \in C^{p,\lambda}(\bar{R})$ for $0 < \lambda < 1$. Therefore,

$$\max_{(x_1,x_2,x_3)\in\bar{\sigma}_0} |v(x_1, x_2, x_3)| \leq c_1 \rho_0^{\lambda}, \quad (2.7)$$

where c_1 is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$. Using estimate (2.7) and applying Lemma 3 from (Mikhailov, 1978) (see Chapter 4, Section 3), we obtain

$$\left|\left(\frac{\partial^6 u(x_{10}, x_{20}, x_{30})}{\partial l^6}\right)\right| \leq c \rho_0^{p+\lambda-6}, \quad (2.8)$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$.

Since the point $(x_{10}, x_{20}, x_{30}) \in R$ *is* arbitrary, inequality (2.6) holds true.■

2.2 Finite Difference Problem

We introduce a cubic grid with a step $h > 0$ defined by the planes $x_i = 0, h, 2h, \ldots,\ i = 1,2,3$. It is assumed that the edge lengths of R and h are such that $\left(\frac{a_i}{h}\right) \geq 4$ $(i = 1,2,3)$ are integers.

Let D_h be the set of nodes of the grid constructed, $\bar{R}_h = \bar{R} \cap D_h, R_h = R \cap D_h$, $R_h^k \subset R_h$ be the set of nodes of R_h lying at a distance of kh away from the boundary Γ of R, and $\Gamma_h = \Gamma \cap D_h$.

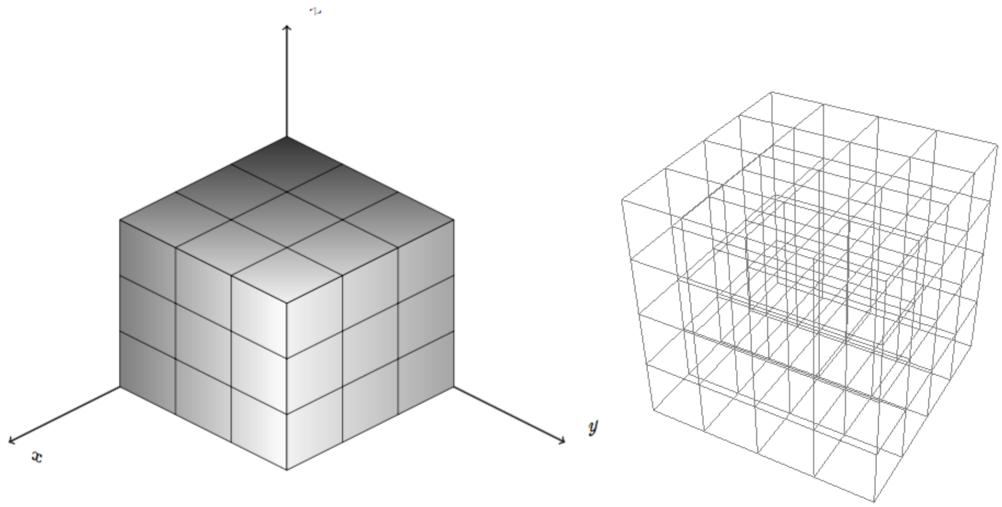

Figure 2.1: $R_h = R \cap D_h$

The 14-point difference operator S on the grid is defined as (Volkov, 2010)

$$Su(x_1, x_2, x_3) = \left(\frac{1}{56}\right)\left(8 \sum_{p=1^{(1)}}^{6} u_p + \sum_{p=6^{(3)}}^{14} u_q\right), (x_1, x_2, x_3) \in R_h, \qquad (2.9)$$

where $\sum_m$ is the sum extending over the nodes lying at a distance of $\sqrt{m}h$ away from the point (x_1, x_2, x_3) and u_p and u_q are the values of u at the corresponding nodes.

Let's give the image of 14-point operator on coordinate axis below.

$$\rho = \begin{cases} Black\ Points, & distance\ between\ h, \\ Blue\ Points, & distance \text{ between } \sqrt{3}h, \end{cases}$$

away from the current point which is green node.

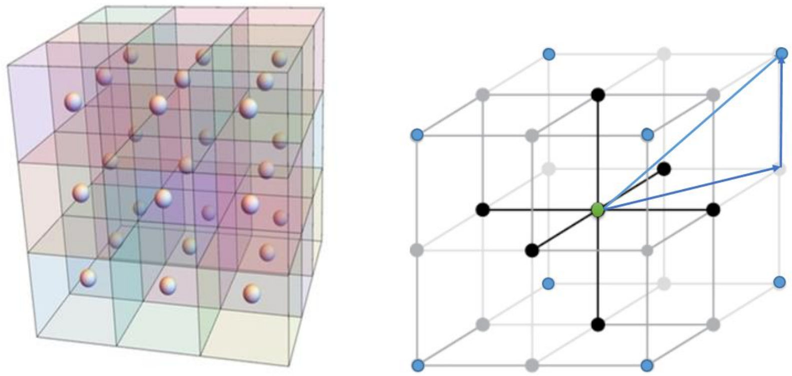

Figure 2.2 : 14-point operator around center using in operator S. Each point has distance of $\sqrt{m}h$ from the point (x_1, x_2, x_3)

On the boundary Γ of R, we define the continuous function φ, on the entire boundary including the edges of R as follows

$$\varphi = \begin{cases} \varphi_1 \; on \; \Gamma_1 \\ \varphi_j \; on \; \Gamma_1 \backslash \left(\bigcup_{i=1}^{j-1} \Gamma_i \right), \quad j = 2, \ldots, 6. \end{cases} \tag{2.10}$$

Obviously,

$$\varphi = \varphi_j \; on \; \Gamma_j, \; j = 1,2, \ldots, 6.$$

We consider the finite difference problem approximating Dirichlet problem (2.1):

$$u_h = S u_h \; on \; R_h, u_h = \varphi \; on \; \Gamma_h, \tag{2.11}$$

where S is the difference operator given by (2.9) and φ is the function defined by (2.10). By maximum principle, the system (2.11) has a unique solution (see Samarskii (2001) in Chap. 4).

In what follows and for simplicity, we denote by $c, c_1, c_2, \ldots$ constants, which are independent of h and the nearest factors, the identical notation will be used for various constants.

Consider two systems of grid equations below

$$v_h = Sv_h + g_h, \ on\ R_h,\ v_h = 0\ on\ \Gamma_h, \tag{2.12}$$

$$\bar{v}_h = S\bar{v}_h + \bar{g}_h, \ on\ R_h,\ \bar{v}_h = 0\ on\ \Gamma_h, \tag{2.13}$$

where g_h and $\bar{g}_h$ are given functions and $|\bar{g}_h| \leq g_h$ on R_h.

Lemma 2.3 The solutions v_h and $\bar{v}_h$of systems (2.12) and (2.13) satisfy the inequality $|\bar{v}_h| \leq v_h\ on\ R_h$.

Proof. The proof of Lemma 2.3 is similar to the comparison theorem in (Samarskii, 2001) (Chap.4, Sec.3).■

Define

$$N(h) = \left[\frac{min\{a_1,a_2,a_3\}}{2h}\right], \tag{2.14}$$

where $[a]$ is the integer part of a.

For a fixed $k, 1 \leq k \leq N(h)$ consider the systems of grid equations,

$$v_h^k = Sv_h^k + g_h^k\ on\ R_h^k,\ v_h^k = 0\ on\ \Gamma_h, \tag{2.15}$$

where

$$g_h^k = \begin{cases} 1, & \rho(x_1,x_2,x_3) = kh, \\ 0, & \rho(x_1,x_2,x_3) \neq kh. \end{cases}$$

Lemma 2.4 The solution v_h^k of the system (2.15) satisfies the following inequality

$$\max_{(x_1,x_2,x_3)\in R^h} v_h^k \leq 5k, \qquad 1 \leq k \leq N(h). \tag{2.16}$$

Proof. Let the function w_h^k be defined on $R_h \cup \Gamma_h$ as follows

$$w_h^k = \begin{cases} 0, & (x_1, x_2, x_3) \in \Gamma^h, \\ 5m, & (x_1, x_2, x_3) \in R_m^h, \quad 1 \le m < k, \\ 5k, & (x_1, x_2, x_3) \in R_l^h, \quad k \le l < N(h). \end{cases} \tag{2.17}$$

It is obvious that

$$\max_{(x_1,x_2,x_3)\in R_h} w_h^k \le 5k. \tag{2.18}$$

We have $w_h^k - Sw_h^k \ge g_h^k$ on R_h, $k = 1,2,\ldots,N(h)$. Consider two cases to show the correctness of inequality (2.18).

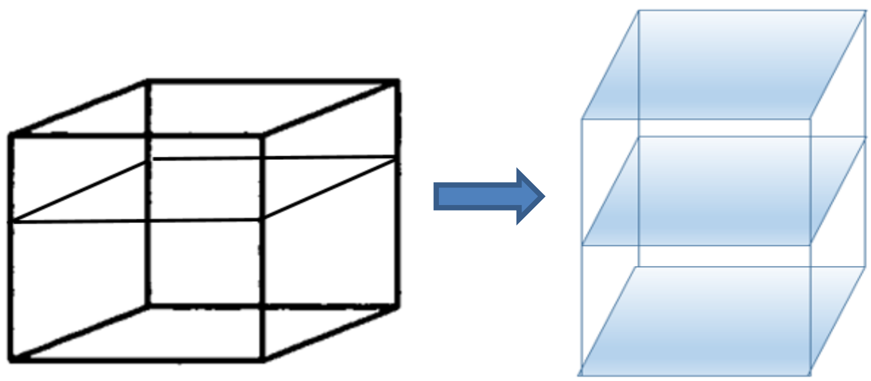

Figure 2.3: The selected plane R_h

Case 1:

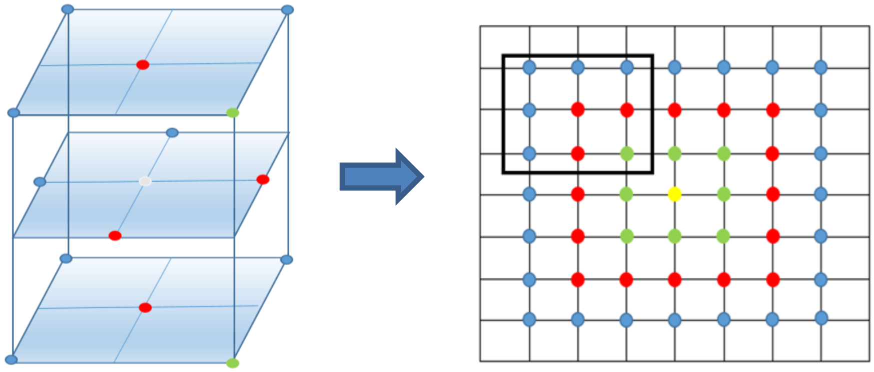

Figure 2.4: The selected plane R_h for Case 1 contains points which are blue ones R_h^1, red ones R_h^2 and grey ones R_h^3

i) If $m = k$ then,

$$Sw_h = \frac{1}{56}\{8(2.5(k-1)+4.(5k))+6.5(k-1)+2(5k))\}$$

$$= \frac{1}{56}(8(30k-10)+40k-30)$$

$$= \frac{280k-110}{56}$$

$$= 5k - \frac{110}{56}.$$

We have,

$$Sw_h^k = w_h^k - \frac{110}{56} \; then \; Sw_h^k - w_h^k = \frac{110}{56} = g_h^k > 1.$$

ii) If $m \neq k$ and $m > k$ then ,

$$Sw_h = \frac{1}{56}\{8(2.(5k) + 4.(5k)) + 6.(5k) + 2(5k))\}$$

$$= \frac{1}{56}(280k)$$

$$= 5k.$$

We have,

$$Sw_h^k = w_h^k \; then \; Sw_h^k - w_h^k = 0 = g_h^k.$$

iii) If $m \neq k$ and $m < k$ then,

$$Sw_h = \frac{1}{56}\{8(2.5(k-1) + 4.(5k)) + 6.5(k-1) + 2.5(k+1))\}$$

$$= \frac{1}{56}(8(30k - 10) + 40k - 20)$$

$$= \frac{280k - 100}{56}$$

$$= 5k - \frac{100}{56}.$$

We have,

$$Sw_h^k = w_h^k - \frac{100}{56} \; then \; Sw_h^k - w_h^k = \frac{100}{56} = g_h^k > 1.$$

Case 2:

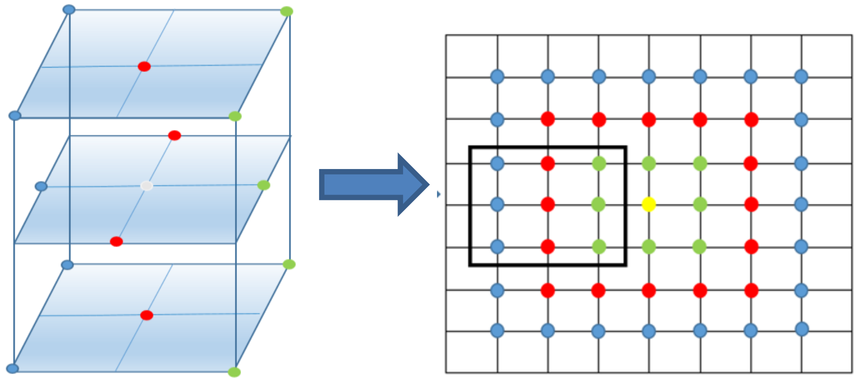

Figure 2.5: The selected plane R_h for Case 2 contains points which are blue ones R_h^1, red ones R_h^2 and grey ones R_h^3

i) If $m = k$ then,

$$Sw_h = \frac{1}{56}\{8(1.5(k-1) + 4.(5k) + 1.(5k))$$

$$+4.5(k-1) + 4(5k))\}$$

$$= \frac{1}{56}(8(30k-5) + 40k - 20)$$

$$= \frac{280k - 60}{56}$$

$$= 5k - \frac{60}{56}.$$

We have,

$$Sw_h^k = w_h^k - \frac{60}{56} \; then \; Sw_h^k - w_h^k = \frac{60}{56} = g_h^k > 1.$$

ii) If $m \neq k$ and $m > k$ then ,

$$Sw_h = \frac{1}{56}\{8(1.(5k) + 4.(5k) + 1.(5k)) + 4.(5k) + 4(5k))\}$$

$$= \frac{1}{56}(280k)$$

$$= 5k.$$

We have,

$$w_h^k = Sw_h^k.$$

iii) If $m \neq k$ and $m < k$ then,

$$Sw_h = \frac{1}{56}\{8(5(k-1) + 4.(5k) + 5(k+1))$$

$$+4.5(k-1) + 4.5(k+1))\}$$

$$= \frac{1}{56}(280k)$$

$$= 5k.$$

We have,

$$Sw_h^k = w_h^k \; then \; Sw_h^k - w_h^k = 0 = g_h^k.$$

Then by Lemma 2.3, and by (2.18), we obtain

$$v_h^k \leq w_h^k \leq 5k \; on \; R_h,$$

which proves the inequality (2.16).■

Let $x_0 = (x_{10}, x_{20}, x_{30})$, for brevity we use Taylor formula to represent the solution of the Dirichlet problem around some point $x_0 \in R_h$:

$$u(x_1, x_2, x_3) = p_5(x_1, x_2, x_3; x_0) + r_6(x_1, x_2, x_3; x_0), \qquad (2.19)$$

where p_5 is fifth-degree Taylor polynomial and r_6 is remainder. Here,

$$p_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_1, x_2, x_3), \qquad r_6(x_{10}, x_{20}, x_{30}; x_0) = 0.$$

Lemma 2.5 It is true that

$$Su(x_{10}, x_{20}, x_{30}) = \quad p_5(x_{10}, x_{20}, x_{30}) + Sr_6(x_{10}, x_{20}, x_{30}) \in R_h,$$

where u solves the Dirichlet problem, r_6 is the remainder in the Taylor formula, and S is the averaging operator defined by (2.9).

Proof. Let $p_5(x_1, x_2, x_3; x_0)$ be a Taylor polynomial and u is a harmonic function and S is linear, by taking into account that,

$$Sp_5(x_{10}, x_{20}, x_{30}; x_0) = \quad \left(\frac{1}{56}\right)\{8[p_5(x_{10} + h, y_{20}, z_{30})$$

$$+p_5(x_{10}, y_{20} + h, z_{30}) + p_5(x_{10}, y_{20}, z_{30} + h) + \; p_5(x_{10} - h, y_{20}, z_{30})$$

$$+p_5(x_{10}, y_{20} - h, z_{30}) + p_5(x_{10}, y_{20}, z_{30} - h)]$$

$$+p_5(x_{10} + h, y_{20} + h, z_{30} + h) + \; p_5(x_{10} - h, y_{20} + h, z_{30} + h)$$

$$+p_5(x_{10} + h, y_{20} - h, z_{30} + h) + \; p_5(x_{10} + h, y_{20} + h, z_{30} - h)$$

$$+p_5(x_{10} + h, y_{20} - h, z_{30} - h) + p_5(x_{10} - h, y_{20} - h, z_{30} + h)$$

$$+p_5(x_{10} - h, y_{20} + h, z_{30} - h) + p_5(x_{10} - h, y_{20} - h, z_{30} - h)\}$$

$$= \quad u(x_{10}, x_{20}, x_{30})$$

$$+\frac{3}{4}h^2\sum_{i=1}^{3}\frac{\partial^2 u(x_{10}, x_{20}, x_{30})}{\partial x_i^2} \quad +\frac{h^4}{56}\sum_{i=1}^{3}\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_i^4}$$

$$+\frac{h^4}{28}\left\{\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_2^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_3^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^2 \partial x_3^2}\right\}.$$

Since u is harmonic function,

$$\sum_{i=1}^{3}\frac{\partial^2 u(x_{10}, x_{20}, x_{30})}{\partial x_i^2} = 0$$

the second term on the right-hand side of this equality vanishes. By taking derivative of the above function twice with respect to x_1, x_2 and x_3, we have

$$\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^4}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_2^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_3^2} = 0,$$

$$\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_2^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^4}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^2 \partial x_3^2} = 0,$$

$$\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_3^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^2 \partial x_3^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_3^4} = 0.$$

The sum of the above three equations gives us the following result

$$\sum_{i=1}^{3}\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_i^4}+2\left\{\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_2^2}+\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^2 \partial x_3^2}\right\} = 0$$

while the third and fourth terms cancel each other.

Thus,

$$Sp_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30}) \quad (2.20)$$

from (2.20) follows

$$Su(x_{10}, x_{20}, x_{30}) = u(x_{10}, x_{20}, x_{30}) + Sr_5(x_{10}, x_{20}, x_{30}; x_0). \quad \blacksquare \quad (2.21)$$

Lemma 2.6 It is true that

$$\max_{(x_1,x_2,x_3)\epsilon R_h^1} |Su - u| \le ch^{p+\lambda}, p \in \{4,5\}, \quad (2.22)$$

where u is the solution of the Dirichlet problem (2.1) , with conditions (2.2)-(2.5) and S is the averaging operator defined by (2.9), R_h^1 is the subset of R_h lying at a distance of h away form the boundary of the parallelepiped R .

Proof. Let (x_{10}, x_{20}, x_{30}) be a node of the grid $R_h^1 \subset R_h$ and let

$$\Theta_0 = \{(x_1, x_2, x_3): |x_i - x_{i0}| < h, i = 1,2,3\} \quad (2.23)$$

be an elementary cube some of whose faces lie on the boundary of R. The nodes of the operator S calculating the averaged value $Su(x_{10}, x_{20}, x_{30})$ of u lie at the vertices of the cube of and the centers of its faces.

Let us estimate the remainder r_6 in (2.19) at the point $(x_{10} + h, x_{20} + h, x_{30} + h)$ which is one of the nodes of S.

Consider the function

$$\tilde{u}(s) = \left(x_{10} + \left(\frac{s}{\sqrt{3}}\right), x_{20} + \left(\frac{s}{\sqrt{3}}\right), x_{30} + \left(\frac{s}{\sqrt{3}}\right)\right), -\sqrt{3}h \le s \le \sqrt{3}h \quad (2.24)$$

of single variable s, which is the arc length along the straight line through the points $(x_{10} - h, x_{20} - h, x_{30} - h)$ and $(x_{10} + h, x_{20} + h, x_{30} + h)$. Regardless of whether or not $(x_{10} + h, x_{20} + h, x_{30} + h)$ lies on the boundary of R, by Lemma 2.2, we have

$$\left|u^{(6)}(s)\right| \leq c\left(\sqrt{3}h - s\right)^{p+\lambda-6}, 0 \leq \lambda \leq 1, 0 \leq s < \sqrt{3}h, \qquad (2.25)$$

where c is a constant independent of the chosen point $(x_{10}, x_{20}, x_{30}) \in R_h^1$.

By using the Taylor formula, function (2.24) around the point $s = 0$ can be represented as

$$\tilde{u}(s) = \tilde{p}_5(s) + \tilde{r}_6(s),$$

where $p_5(s)$ is the fifth-degree Taylor polynomial (in single variable s) and $r_6(s)$ is the remainder. Since

$$\tilde{p}_5(s) \equiv p_5(x_{10} + (\frac{s}{\sqrt{3}}), x_{20} + (\frac{s}{\sqrt{3}}), x_{30} + (\frac{s}{\sqrt{3}}); x_0),$$

where $p_5(x_1, x_2, x_3; x_0)$ is the Taylor polynomial in (2.19), we have

$$r^6\left(x_{10} + \left(\frac{s}{\sqrt{3}}\right), x_{20} + \left(\frac{s}{\sqrt{3}}\right), x_{30} + \left(\frac{s}{\sqrt{3}}\right); x_0\right) = \tilde{r}^6(s), 0 \leq |s| < \sqrt{3}h. \qquad (2.26)$$

Since the remainder r_6 in (2.19) is continuous on the closure of cube (2.23) and $\tilde{r}_6(s)$ is continuous on the interval $[-\sqrt{3}h, \sqrt{3}h]$, it follows from (2.26) that

$$\begin{aligned}|r_6(\sqrt{3}h - \varepsilon)| \leq\ & \left(\frac{1}{5!}\right) \int_0^{\sqrt{3}h-\varepsilon} (\sqrt{3}h - \varepsilon - t)^5 |\tilde{u}^{(6)}(t)| dt \\ \leq\ & c_1 \int_0^{\sqrt{3}h-\varepsilon} \left(\sqrt{3}h - \varepsilon - t\right)^5 \left(\sqrt{3}h - t\right)^{p+\lambda-6} dt \\ \leq\ & c_2 \int_0^{\sqrt{3}h-\varepsilon} \left(\sqrt{3}h - t\right)^{p+\lambda-1} dt \\ \leq\ & c_3 h^{p+\lambda}, \ \ 0 < \varepsilon \leq \left(\frac{\sqrt{3}h}{2}\right). \qquad (2.27)\end{aligned}$$

Thus, combining (2.25)-(2.27) yields

$$|r_6(x_{10}+h, x_{20}+h, x_{30}+h; x_0)| \leq c_4 h^{p+\lambda}, \tag{2.28}$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R'_h$.

Similarly, we can get the same estimates (2.28) of r_6 at the remainder 13 nodes of come (2.27). Then form (2.9), we have

$$|Sr_6(x_{10}, x_{20}, x_{30}; x_0)| \leq c_5 h^{p+\lambda},$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R'_h$. Combining this with equality (2.21) in Lemma 2.5 yields inequality (2.22).■

Lemma 2.7 It is true that

$$\max_{(x_1,x_2,x_3)\epsilon R_h^k} |Su - u| \leq c\left(\frac{h^{p+\lambda}}{k^{6-p-\lambda}}\right), k = 1,2,\ldots, N(h), p \in \{4,5\}, \tag{2.29}$$

where u is the solution of the Dirichlet problem (2.1) and S is the averaging operator defined by (2.19) and $N(h)$ is given by (2.14).

Proof. For $k = 1$, inequality (2.29) holds by Lemma 2.6. Let $x_0 \in R_h^{k_0}$ be an arbitrary point for arbitrary k_0 such that $2 \leq k_0 \leq N(h)$.

Let $r_6(x_1, x_2, x_3; x_0)$ be the Lagrange remainder corresponding to this point in Taylor formula (2.19).

Then $Sr_6(x_{10}, x_{20}, x_{30}; x_0)$ can be expressed linearly in terms of a fixed number of sixth derivatives of u at some points of open cube

$$K_0 = \{(x_1, x_2, x_3): |x_i - x_{i0}| < h, i = 1,2,3\} \tag{2.30}$$

which is a distance of at least $k_0h/2$ away from the boundary of R. The sum of the absolute values of the coefficients multiplying the sixth derivatives does not exceed ch^6 which is independent k_0 $(2 \leq k_0 \leq N(h))$ or the point $x_0 \in R_h^{k_0}$. By Lemma 2, we have

$$|Sr_6(x_{10}, x_{20}, x_{30}; x_0)| \leq \quad c_1\left(\frac{h^6}{(k_0h)^{6-p-\lambda}}\right)$$

$$= \quad c_2\left(\frac{h^{p+\lambda}}{k_0^{6-p-\lambda}}\right), \tag{2.31}$$

where c is a constant independent of k_0 $(2 \leq k_0 \leq N(h))$ and of the point $x_0 \in R_h^{k_0}$. On the basis of (2.21), (2.28), and (2.31) estimation (2.22) holds.■

Theorem 2.1 Assume that the boundary functions φ_j satisfy conditions (2.2)-(2.5). Then at each point $(x_1, x_2, x_3) \in R_h$

$$|u_h - u| \leq ch^4\rho^{p-4}\,, p \in \{4,5\},$$

where u_h is the solution of the finite difference problem (2.11), and u is the exact solution of problem (2.1) and $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R_h$ to the boundary of the rectangular parallelepiped R.

Proof. Let

$$\varepsilon_h = u_h - u \ on \ \bar{R}_h\,. \tag{2.32}$$

By (2.11) and (2.12) the error function satisfies the system of equations

$$\varepsilon_h = S\varepsilon_h + (Su - u) \ on \ R_h, \varepsilon_h = 0 \ on \ \Gamma_h. \tag{2.33}$$

We represent a solution of the system (2.33) as follows

$$\varepsilon_h = \sum_{k=1}^{N(h)} \varepsilon_h^k, \tag{2.34}$$

where $N(h)$ is defined by (2.14) and $\varepsilon_h^k, 1 \le k \le N(h)$, is a solution of the system

$$\varepsilon_h^k = S\varepsilon_h^k + \sigma_h^k \; on \; R_h, \qquad \varepsilon_h^k = 0 \; on \; \Gamma_h, \tag{2.35}$$

where

$$\sigma_h^k = \begin{cases} Su - u \; on \; R_h^k \\ 0 \; on \; R_h | R_h^k \end{cases}$$

Then on the basis of Lemma 2.7 and Lemma 2.6, for the solution of (2.34), we have

$$|\varepsilon_h| \le \sum_{k=1}^{N(h)} |\varepsilon_h^k| \le \sum_{k=1}^{N(h)} 5k. \max_{(x_1,x_2,x_3)\in R_h^k} |Su - u|$$

$$\le 5c_1 h^{p+\lambda} \sum_{k=1}^{\rho/h-1} \frac{k}{k^{6-p-\lambda}} + 5c_1 h^6 \sum_{k=\rho/h}^{N(h)} \frac{\rho/h}{(kh)^{6-p-\lambda}}$$

$$\le 5c_1 h^{p+\lambda} \sum_{k=1}^{\rho/h-1} k^{-5+p+\lambda} + 5c_1 h^{p-1+\lambda}\rho \sum_{k=\rho/h}^{N(h)} k^{-6+p+\lambda}$$

$$\le 5c_1 h^{p+\lambda} \left[1 + \int_1^{\rho/h-1} x^{-5+p+\lambda} dx \right]$$

$$+5c_1 h^{p-1+\lambda}\rho \left[1 + \int_{\rho/h}^{N(h)} x^{-6+p+\lambda} dx \right]$$

$$\le c_2 h^4 \rho^{-4+p+\lambda} + c_3 h^4 \rho$$

$$\le c_4 h^4 \rho^{p-4}, \;\; p = \{4,5\}. \tag{2.36}$$

On the basis of (2.32), (2.34) and (2.36), we obtain

$$|u_h - u| = \max_{(x_1,x_2,x_3 \epsilon R_h^k)} |\varepsilon_h| \le ch^4 \rho^{p-4}. \quad \blacksquare$$

2.3 Approximation of the First Derivative

2.3.1 Boundary function is from $C^{5,\lambda}$

Let the boundary functions $\varphi_j, j = 1,2,\dots,6$, in problem (2.1) on the faces Γ_j be satisfied by the conditions

$$\varphi_j \in C^{5,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\dots,6, \tag{2.37}$$

i.e., $p = 5$ in (2.2). Let u be a solution of the problem (2.1) with the conditions (2.37) and (2.3)-(2.5). We give the following Lemmas and Theorem related to the function u.

Let $v = (\frac{\partial u}{\partial x_1})$ and let $\phi_j = \left(\frac{\partial u}{\partial x_1}\right)$ on $\Gamma_j, j = 1,2,\dots,6$, and consider the boundary value problem:

$$\Delta v = 0 \; on \; R, v = \phi_j \; on \; \Gamma_j, j = 1,2,\dots,6, \tag{2.38}$$

where u is a solution of the boundary value problem (2.1) for $p = 5$.

We define the following operators $\phi_{ph},\ p = 1,2,\dots,6$,

$$\phi_{1h}(u_h) = \frac{1}{12h}[-25\varphi_1(x_2,x_3) + 48u_h(h,x_2,x_3) - 36u_h(2h,x_2,x_3) + 16u_h(3h,x_2,x_3) - 3u_h(4h,x_2,x_3)] \text{ on } \Gamma_1^h, \tag{2.39}$$

$$\phi_{4h}(u_h) = \frac{1}{12h}[25\varphi_4(x_2,x_3) - 48u_h(a_1-h,x_2,x_3) + 36u_h(a_1-2h,x_2,x_3) - 16u_h(a_1-3h,x_2,x_3) + 3u_h(a_1-4h,x_2,x_3)] \text{ on } \Gamma_4^h, \tag{2.40}$$

$$\phi_{ph}(u_h) = \left(\frac{\partial \phi_p}{\partial x_1}\right), \text{ on } \Gamma_p^h, p = 2,3,5,6, \tag{2.41}$$

where u_h is the solution of the finite difference problem (2.11).

Let ν_h be the solution of the following finite difference problem

$$\nu_h = S\nu_h \; on \; R_h, \nu_h = \phi_{jh} \; on \; \Gamma_j^h, j = 1,2,\ldots,6, \tag{2.42}$$

where $\phi_{jh},\ j = 1,2,\ldots,6$, are defined by (2.39)-(2.41)

Lemma 2.8 The following inequality is true

$$|\phi_{kh}(u_h) - \phi_{kh}(u)| \leq ch^4, k = 1,4, \tag{2.43}$$

where u_h is the solution of the finite difference problem (2.11) and u is the solution of problem (2.1).

Proof: It is obvious that $\phi_{ph}(u_h) - \phi_{ph}(u) = 0$ for $p = 2,3,5,6$. For $k = 1$, by (2.39) and Theorem 2.1, we have

$$|\phi_{1h}(u_h) - \phi_{1h}(u)| \leq \left(\frac{1}{12h}\right)\{25|u_h(h, x_2, x_3) - u(h, x_2, x_3)|$$

$$+48|u_h(2h, x_2, x_3) - u(2h, x_2, x_3)| + 16|u_h(3h, x_2, x_3) - u(3h, x_2, x_3)|$$

$$+3|u_h(4h, x_2, x_3) - u(4h, x_2, x_3)|\}$$

$$\leq \left(\frac{1}{12h}\right)[25(ch)h^4 + 48(c2h)h^4 + 16(c3h)h^4 + 3(c4h)h^4]$$

$$\leq ch^4.$$

It is also shown that the same inequality is true when $k = 4$.

$$|\phi_{4h}(u_h) - \phi_{4h}(u)| \leq \left(\frac{1}{12h}\right)\{25|u_h(a_1 - h, x_2, x_3) - u(a_1 - h, x_2, x_3)|$$

$$+48|u_h(a_1 - 2h, x_2, x_3) - u(a_1 - 2h, x_2, x_3)| + 16|u_h(a_1 - 3h, x_2, x_3) - u(a_1 - 3h, x_2, x_3)|$$

$$+3|u_h(a_1 - 4h, x_2, x_3) - u(a_1 - 4h, x_2, x_3)|\}$$

$$\leq \left(\frac{1}{12h}\right)[25(ch)h^4 + 48(c2h)h^4 + 16(c3h)h^4 + 3(c4h)h^4]$$

$$\leq ch^4. \quad \blacksquare$$

Lemma 2.9 The following inequality holds

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_k| \leq ch^4, k = 1,4, \tag{2.44}$$

where $\phi_{kh}, k = 1,4$ are defined by (2.39), (2.40), and $\phi_k = \left(\frac{\partial u}{\partial x_1}\right)$ $on\ \Gamma_k, k = 1,4.$

Proof. From Lemma 2.1 it follows that $u \in C^{5,\lambda}(R)$. Then, at the end points $(0, \nu h, wh) \in \Gamma_1^h$ and $(a_1, \nu h, wh) \in \Gamma_4^h$ of each line segment $\{(x_1, x_2, x_3): 0 \leq x_1 \leq a_1, 0 \leq x_2 = \nu h < a_2, 0 \leq x_3 = wh < a_3\}$, expressions (2.39) and (2.40) give the third order appriximation of $\left(\frac{\partial u}{\partial x_1}\right)$, respectively. From the truncation error formulas it (Burden and Douglas, 2011) follows that

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi(u) - \phi_k| \leq c_1 \left[\frac{h^4}{5}\right] \max_{(x_1,x_2,x_3)\in\Gamma_k^h} \left|\frac{\partial^5 u}{\partial x_1^5}\right|$$

$$\leq c_2 h^4, \quad k = 1,4. \tag{2.45}$$

On the basis of Lemma 2.8 and inequality (2.43), (2.45) follows,

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_k| = \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_{kh}(u) + \phi_{kh}(u) - \phi_k|$$

$$\leq \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_{kh}(u)|$$

$$+ \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u) - \phi_k|$$

$$\leq c_3 h^4 + c_4 h^4$$

$$\leq c_5 h^4 \quad \blacksquare$$

Theorem 2.2 The following estimation is true

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h}\left|\nu_h-\left(\frac{\partial u}{\partial x_1}\right)\right|\leq ch^4, \tag{2.46}$$

where u is the solution of the problem (2.1), ν_h is the solution of the finite difference problem (2.42).

Proof. Let

$$\varepsilon_h=\nu_h-\nu \; on \; R^k, \tag{2.47}$$

where $\nu=\left(\frac{\partial u}{\partial x_1}\right)$. From (2.42) and (2.47), we have

$$\varepsilon_h \; = \; S\varepsilon_h+(S\nu-\nu)\; on \; R^h,$$

$$\varepsilon_h \;\; = \; \phi_{kh}(u_h)-\nu \; on \; \Gamma_k^h, k=1,4, \varepsilon_h=0 \; on \; \Gamma_p^h, p=2,3,5,6.$$

We represent

$$\varepsilon_h=\varepsilon_h^1+\varepsilon_h^2, \tag{2.48}$$

where

$$\varepsilon_h^1 \; = S\varepsilon_h^1 \; on \; R^h, \tag{2.49}$$

$$\varepsilon_h^1 \; = \; \phi_{kh}(u_h)-\nu \; on \; \Gamma_k^h, k=1,4, \;\; \varepsilon_h^1=0 \; on \; \Gamma_p^h, p=2,3,5,6; \tag{2.50}$$

$$\varepsilon_h^2 \; = \; S\varepsilon_h^2+(S\nu-\nu)\; on \; R^h, \varepsilon_h^2=0 \; on \; \Gamma_j^h, j=1,2,\ldots,6. \tag{2.51}$$

By Lemma 2.9 and by the maximum principle, for the solution of system (2.49), (2.50), we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h}|\varepsilon_h^1|\leq\max_{q=1,4}\max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\phi_{qh}(u_h)-\nu\right|\leq c_1h^4. \tag{2.52}$$

The solution ε_h^2 of system (2.51) is the error of the approximate solution obtained by the finite difference method for problem (2.38), when on the boundary nodes Γ_{jh}, the values of the functions ϕ_j in (2.38) are used. It is obvious that $\phi_j, j=1,2,\ldots,6$, satisfy the conditions

$$\phi_j \in C^{4,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \tag{2.53}$$

$$\phi_\mu = \phi_\nu \; on \; \gamma_{\mu\nu}, \tag{2.54}$$

$$\left(\frac{\partial^2 \phi_\mu}{\partial t_\mu^2}\right) + \left(\frac{\partial^2 \phi_\nu}{\partial t_\nu^2}\right) + \left(\frac{\partial^2 \phi_\mu}{\partial t_{\mu\nu}^2}\right) = 0 \; on \; \gamma_{\mu\nu} \, . \tag{2.55}$$

Therefore, for the error ε_h^2 of the finite difference problem for the continuous problem (2.42), on the basis of Theorem 1 in (Volkov, 2010) we have;

$$\max_{(x_1,x_2,x_3)\in \bar{R}^h} |\varepsilon_h^2| \leq c_2 h^4. \tag{2.56}$$

By (2.48), (2.52) and (2.56) inequality (2.46) holds. ■

2.3.2 Boundary function is from $C^{4,\lambda}$

Let the boundary functions $\varphi_j \in C^{4,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6$, in (2.1)-(2.5), i.e., $p = 4$ in (2.1), and let $v = \left(\frac{\partial u}{\partial x_1}\right)$ and let $\phi_j = \left(\frac{\partial u}{\partial x_1}\right) \; on \; \Gamma_j, j = 1,2,\ldots,6$, and consider the boundary value problem:

$$\Delta v = 0 \; on \; R, v = \phi_j \; on \; \Gamma_j, j = 1,2,\ldots,6, \tag{2.58}$$

where u is a solution of the boundary value problem (2.1).

We define the following third order numerical differentiation operators $\phi_{kh}, k = 1,4$

$$\begin{aligned}\phi_{1h}\,(u_h) \;\; = \;\; & \left(\frac{1}{6h}\right)[-11\varphi_1(x_2,x_3) + 18u_h(h,x_2,x_3) \\ & -9u_h(2h,x_2,x_3) + 2u_h(3h,x_2,x_3)] \; on \; \Gamma_1^h \;\; ,\end{aligned} \tag{2.59}$$

$$\phi_{4h}(u_h) = \left(\frac{1}{6h}\right)[11\varphi_4(x_2,x_3) - 18u_h(a_1-h,x_2,x_3)$$

$$+9u_h(a_1-2h,x_2,x_3) - 2u_h(a_1-3h,x_2,x_3)] \ on \ \Gamma_4^h \ ,$$

(2.60)

$$\phi_{ph}(u_h) = \left(\frac{\partial\phi_p}{\partial x_1}\right), on\ \Gamma_p^h, p = 2,3,5,6, \qquad (2.61)$$

where u_h is the solution of the finite difference problem (2.11).

It is obvious that $\phi_j, j = 1,2,\ldots,6$, satisfy the conditions

$$\phi_j \in C^{3,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \qquad (2.62)$$

$$\phi_\mu = \phi_\nu \ on\ \gamma_{\mu\nu}, \qquad (2.63)$$

$$\left(\frac{\partial^2\phi_\mu}{\partial t_\mu^2}\right) + \left(\frac{\partial^2\phi_\nu}{\partial t_\nu^2}\right) + \left(\frac{\partial^2\phi_\mu}{\partial t_{\mu\nu}^2}\right) = 0 \ on\ \gamma_{\mu\nu}. \qquad (2.64)$$

Let v_h be the solution of the following finite difference problem

$$v_h = Sv_h \ on\ R_h, v_h = \phi_{jh} \ on\ \Gamma_j^h, j = 1,2,\ldots,6, \qquad (2.65)$$

where $\phi_{jh}, j = 1,2,\ldots,6$, are defined by (2.59)-(2.61).

Lemma 2.10 The following inequality is true

$$|\phi_{kh}(u_h) - \phi_{kh}(u)| \le ch^3, k = 1,4, \qquad (2.66)$$

where u_h is the solution of the finite difference problem (2.11), u is the solution of problem (2.1).

Proof. It is obvious that $\phi_{ph}(u_h) - \phi_{ph}(u) = 0\ for\ p = 2,3,5,6$. For $k = 1$, by (2.59) and Theorem 2.1, we have

$$
\begin{aligned}
|\phi_{1h}(u_h) - \phi_{1h}(u)| \le\ & \left(\frac{1}{6h}\right)[18|u_h(h, x_2, x_3) - u(h, x_2, x_3)|| \\
& +9|u_h(2h, x_2, x_3) - u(2h, x_2, x_3) \\
& +2|u_h(3h, x_2, x_3) - u(3h, x_2, x_3)|] \\
\le\ & \left(\frac{1}{6h}\right)[29c_1h^4] \\
\le\ & c_2h^3.
\end{aligned}
$$

The same inequality is true when $k = 4$ also, as follows;

$$
\begin{aligned}
|\phi_{4h}(u_h) - \phi_{4h}(u)| \le\ & \left(\frac{1}{6h}\right)[18|u_h(a_1 - h, x_2, x_3) - u(a_1 - h, x_2, x_3)|| \\
& +9|u_h(a_1 - 2h, x_2, x_3) - u(a_1 - 2h, x_2, x_3) \\
& +2|u_h(a_1 - 3h, x_2, x_3) - u(a_1 - 3h, x_2, x_3)|] \\
\le\ & \left(\frac{1}{6h}\right)[29c_3h^3] \\
\le\ & c_4h^3. \qquad \blacksquare
\end{aligned}
$$

Lemma 2.11 The following inequality holds

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_k| \le ch^3, k = 1,4, \tag{2.67}$$

where $\phi_{kh}, k = 1,4$ are defined by (2.59), (2.60), and $\phi_k = \left(\frac{\partial u}{\partial x_1}\right)\ on\ \Gamma_k, k = 1,4$.

Proof. From Lemma 2.1 it follows that $u \in C^{4,\lambda}(R)$. Then, at the end points $(0, \nu h, wh) \in \Gamma_1^h$ and $(a_1, \nu h, wh) \in \Gamma_4^h$ of each line segment $\{(x_1, x_2, x_3): 0 \leq x_1 \leq a_1, 0 \leq x_2 = \nu h < a_2, 0 \leq x_3 = wh < a_3\}$, expressions (2.59) and (2.60) give the third order approximation of $\left(\frac{\partial u}{\partial x_1}\right)$, respectively. From the truncation error formulas it (Burden and Douglas, 2011) follows that

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\ \phi\ (u) - \ \phi_k\ \ | \leq \ \ c_1\left(\frac{h^3}{4}\right) \max_{(x_1,x_2,x_3)\in\Gamma_k^h} \left|\frac{\partial^4 u}{\partial x_1^4}\right|,$$

$$\leq \ \ c_2 h^3, \ \ k = 1,4. \qquad (2.68)$$

On the basis of Lemma 2.10 and estimation (2.68), (2.67) follows,

$$\begin{aligned}\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_k| = & \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_{kh}(u) + \phi_{kh}(u) - \phi_k| \\ \leq & \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u_h) - \phi_{kh}(u)| \\ & + \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\phi_{kh}(u) - \phi_k| \\ \leq & \ c_3 h^3 + c_4 h^3 \ \leq c_5 h^3. \end{aligned}$$

■

Theorem 2.3 The following estimation is true

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} |\nu_h - \left(\frac{\partial u}{\partial x_1}\right)| \leq ch^3, \qquad (2.69)$$

where u is the solution of the problem (2.1), ν_h is the solution of the finite difference problem (2.57).

Proof. Let

$$\varepsilon_h = \nu_h - \nu \ on \ R^k, \qquad (2.70)$$

where $\nu = \left(\frac{\partial u}{\partial x_1}\right)$. From (2.57) and (2.70), we have

$$\varepsilon_h \ = \ S\varepsilon_h + (S\nu - \nu) \ on \ R^h,$$

$$\varepsilon_h = \phi_{kh}(u_h) - \nu \ on \ \Gamma_k^h, k = 1,4, \varepsilon_h = 0 \ on \ \Gamma_p^h, p = 2,3,5,6.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{2.71}$$

where

$$\varepsilon_h^1 = S\varepsilon_h^1 \ on \ R^h, \tag{2.72}$$

$$\varepsilon_h^1 = \phi_{kh}(u_h) - \nu \ on \ \Gamma_k^h, k = 1,4, \varepsilon_h^1 = 0 \ on \ \Gamma_p^h, p = 2,3,5,6; \tag{2.73}$$

$$\varepsilon_h^2 = S\varepsilon_h^2 + (S\nu - \nu) \ on \ R^h, \varepsilon_h^2 = 0 \ on \ \Gamma_j^h, j = 1,2,\ldots,6. \tag{2.74}$$

By Lemma 2.11 and by the maximum principle, for the solution of system (2.72), (2.73), we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} |\varepsilon_h^1| \leq \max_{q=1,4} \max_{(x_1,x_2,x_3)\in\Gamma_q^h} \left|\phi_{qh}(u_h) - \nu\right| \leq c_1 h^3. \tag{2.75}$$

The solution ε_h^2 of system (2.74) is the error of the approximate solution obtained by the finite difference method for problem (2.65), when on the boundary nodes Γ_{jh}, the values of the functions ϕ_j in (2.65) are used. It is obvious that $\phi_j, j = 1,2,\ldots,6$, satisfy the conditions

$$\phi_j \in C^{3,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \tag{2.76}$$

$$\phi_\mu = \phi_\nu \ on \ \gamma_{\mu\nu}, \tag{2.77}$$

$$\left(\frac{\partial^2 \phi_\mu}{\partial t_\mu^2}\right) + \left(\frac{\partial^2 \phi_\nu}{\partial t_\nu^2}\right) + \left(\frac{\partial^2 \phi_\mu}{\partial t_{\mu\nu}^2}\right) = 0 \ on \ \gamma_{\mu\nu}\,. \tag{2.78}$$

Therefore, for the error ε_h^2 of the finite-difference problem for the continuous problem (2.74), on the basis of Theorem 2 in (Volkov, 2010) we have;

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} |\varepsilon_h^2| \leq c_2 h^{3+\lambda} \quad , 0 < \lambda < 1. \tag{2.79}$$

By (2.71), (2.75) and (2.79) inequality (2.69) holds. ■

Remark 2.1: We have investigated the method of high order approximations of the first derivative $\partial u/\partial x_1$. The same results are obtained for the derivatives $\partial u/\partial x_l, l = 2,3,$ by using the same order forward or backward formulae in the corresponding faces of the parallelepiped.

2.4 Approximation of the Pure Second Derivatives

We denote $\omega = \left(\frac{\partial^2 u}{\partial x_1^2}\right)$. The function ω is harmonic on R, on the basis of Lemma 2.1 is continuous on R, and is the solution of the following Dirichlet problem

$$\Delta\omega = 0 \; on \; R, \omega = \chi_j \; on \; \Gamma_j, j = 1,2, \dots,6, \tag{2.80}$$

where

$$\chi_\tau = \left(\frac{\partial^2 \varphi_\tau}{\partial x_1^2}\right), \tau = 2,3,5,6, \tag{2.81}$$

$$\chi_\upsilon = -\left(\left(\frac{\partial^2 \varphi_\upsilon}{\partial x_2^2}\right) + \left(\frac{\partial^2 \varphi_\upsilon}{\partial x_3^2}\right)\right), \upsilon = 1,4. \tag{2.82}$$

Let ω_h be the solution of the finite difference problem

$$\omega_h = S\omega_h \; on \; R_h, \omega_h = \chi_j \; on \; \Gamma_j^h, j = 1,2,\dots,6, \tag{2.83}$$

where $\chi_j, j = 1,2,\dots,6$ are the functions defined by (2.80) and (2.81).

Theorem 2.4 The following estimation holds

$$\max_{\bar{R}_h}|\omega_h - \omega| \le ch^{p-2+\lambda}, \text{ where } p \in \{4,5\}, \tag{2.84}$$

where $\omega = \left(\frac{\partial^2 u}{\partial x_1^2}\right)$, u is the solution of problem (2.1) and ω_h is the solution of the finite difference problem (2.83).

Proof. From the continuity of the function ω on $\bar{R}$, and from (2.81), (2.82) it follows that

$$\chi_j \in C^{q,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \text{where } q \in \{2,3\}, \tag{2.85}$$

$$\chi_\mu = \chi_\nu \ on\ \gamma_{\mu\nu}, \tag{2.86}$$

$$\left(\frac{\partial^2 \chi_\mu}{\partial t_\mu^2}\right) + \left(\frac{\partial^2 \chi_\nu}{\partial t_\nu^2}\right) + \left(\frac{\partial^2 \chi_\mu}{\partial t_{\mu\nu}^2}\right) = 0\ on\ \gamma_{\mu\nu}. \tag{2.87}$$

The boundary functions $\chi_j, j = 1,2,\ldots,6$, in (2.80) on the basis of (2.85)-(2.87) satisfy all conditions of Theorem 2 in (Volkov, 2010) in which follows the proof of the error estimation (2.84). ■

Remark 2.2: If the boundary function $\varphi_j \in C^{6,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6$, the second and fourth order derivatives satisfy some compatibility conditions on the edges then the finite difference solution for the pure second derivatives with order $O(h^4)$.

2.5 Approximation of the Mixed Second Derivatives.

2.5.1 Boundary function from $C^{5,\lambda}$

Let the boundary functions $\varphi_j, j = 1,2,\ldots,6$, in problem (2.1) on the faces Γ_j be satisfied by the condition

$$\varphi_j \in C^{5,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \tag{2.88}$$

i.e., $p = 5$ in (2.2). Let u be a solution of the problem (2.1) with the conditions (2.88) and (2.2)-(2.5).

Let $\varpi = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$ and let $\Psi_j = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)\ on\ \Gamma_j$, j=1,2,...,6, where u is a solution of the boundary value problem (2.1) and v is a solution of the boundary value problem (2.38).

Consider the boundary value problem:

$$\varDelta\varpi = 0 \; on \; R, \varpi = \Psi_j \; on \; \Gamma_j, j = 1,2,\ldots,6. \tag{2.89}$$

We define the sets

$$\Gamma_k^{h+} = \left\{0 \leq x_2 \leq \tfrac{a_2}{2}, \;\; x_1 = t_k\right\} \cap \Gamma_k^h, \quad k = 1,4, \tag{2.90}$$

And

$$\Gamma_k^{h-} = \left\{\tfrac{a_2}{2} + h \leq x_2 \leq a_2, \;\; x_1 = t_k\right\} \cap \Gamma_k^h, \quad k = 1,4, \tag{2.91}$$

where $t_1 = 0 \; and \; t_4 = a_1$.

We define the following operators $\Psi_{qh}, q = 1,2,\ldots,6$,

$$\begin{aligned}\Psi_{kh}(v_h) \;\; &= \;\; \left(\frac{1}{6h}\right)[-11v_h(t_k, x_2, x_3) + 18v_h(t_k, x_2 + h, x_3) \\ &\quad -9v_h(t_k, x_2 + 2h, x_3) + 2v_h(t_k, x_2 + 3h, x_3)] \; on \; \Gamma_k^{h+}, \quad (2.92)\end{aligned}$$

$$\begin{aligned}\Psi_{kh}(v_h) \;\; &= \;\; \left(\frac{1}{6h}\right)[11v_h(t_k, x_2, x_3) - 18v_h(t_k, x_2 - h, x_3) \\ &\quad +9v_h(t_k, x_2 - 2h, x_3) - 2v_h(t_k, x_2 - 3h, x_3)] \; on \; \Gamma_k^{h-}, \quad (2.93)\end{aligned}$$

where $k = 1,4$.

$$\begin{aligned}\Psi_{2h}(v_h) \;\; &= \;\; \left(\frac{1}{6h}\right)[-11\phi_2(x_1, x_3) + 18v_h(x_1, h, x_3) - 9v_h(x_1, 2h, x_3) \\ &\quad +2v_h(x_1, 3h, x_3)] \; on \;\; \Gamma_2^h, \quad (2.94)\end{aligned}$$

$$\begin{aligned}\Psi_{5h}(v_h) \;\; &= \;\; \left(\frac{1}{6h}\right)[11\phi_5(x_1, x_3) - 18v_h(x_1, a_2 - h, x_3) \\ &\quad +9v_h(x_1, a_2 - 2h, x_3) - 2u_h(x_1, a_2 - 3h, x_3)] \;\; on \;\; \Gamma_5^h, \quad (2.95)\end{aligned}$$

$$\Psi_{ph}(v_h) \;=\; \left(\frac{\partial^2\varphi_p}{\partial x_1 \partial x_2}\right), on\ \Gamma_p^h, p = 3,6, \tag{2.96}$$

where $\phi_2\ and\ \phi_5$ are functions defined in (2.38), $\varphi_3\ and\ \varphi_5$ are the given functions in (2.1), v_h is the solution of the finite difference problem (2.42).

Let ϖ_h be the solution of the following finite difference problem

$$\varpi_h = S\varpi_h\ on\ R_h, \varpi_h = \Psi_{jh}\ on\ \Gamma_j^h, j = 1,2,\ldots,6, \tag{2.97}$$

where $\Psi_{jh}, j = 1,2,\ldots,6$, are defined by (2.92)-(2.96).

Lemma 2.12 The following inequality is true

$$|\Psi_{kh}(v_h) - \Psi_{kh}(v)| \le ch^3, k = 1,2,\ldots,6 \tag{2.98}$$

where v_h is the solution of the finite difference problem (2.42), v is the solution of problem (2.38).

Proof. It is obvious that $\Psi_{ph}(v_h) - \Psi_{ph}(v) = 0\ for\ p = 3,6.$

For $t = 1,4$, by using the inequality (2.92) and applying the Theorem 2.2, we have

$$\begin{aligned}
|\Psi_{th}(v_h) - \Psi_{th}(v)| \;\le\; & \left(\frac{1}{6h}\right)[11|v_h(t_k,x_2,x_3) - v(t_k,x_2,x_3)| \\
& +18|v_h(\mathrm{t_k},\mathrm{x_2}+h,\mathrm{x_3}) - v(\mathrm{t_k},\mathrm{x_2}+h,\mathrm{x_3})| \\
& +9|v_h(t_k,x_2+2h,x_3) - v(t_k,x_2+2h,x_3)| \\
& +2|v_h(t_k,x_2+3h,x_3) - v(t_k,x_2+3h,x_3)|] \\
\le\; & \left(\frac{1}{6h}\right)[40c_1h^4] \\
\le\; & c_2h^3 on\ \Gamma_k^{h+}.
\end{aligned}$$

The same inequality is obtained on Γ_k^{h-} by using (2.93). We have

$$
\begin{aligned}
|\Psi_{th}(v_h) - \Psi_{th}(v)| \leq \; & \left(\frac{1}{6h}\right)[11|v_h(t_k, x_2, x_3) - v(t_k, x_2, x_3)| \\
& +18|v_h(\mathrm{t_k}, \mathrm{x_2} - h, \mathrm{x_3}) - v(\mathrm{t_k}, \mathrm{x_2} - h, \mathrm{x_3})| \\
& +9|v_h(t_k, x_2 - 2h, x_3) - v(t_k, x_2 - 2h, x_3)| \\
& +2|v_h(t_k, x_2 - 3h, x_3) - v(t_k, x_2 - 3h, x_3)|] \\
\leq \; & \left(\frac{1}{6h}\right)[40c_3h^4] \\
\leq \; & c_4h^3 on\ \Gamma_k^{h-}.
\end{aligned}
$$

Since $\Gamma_k^h = \Gamma_k^{h+} \cup \Gamma_k^{h-}$, we have,

$$
|\Psi_{th}(v_h) - \Psi_{th}(v)| \leq ch^3,\ \ k = 1,4\ \ on\ \Gamma_k^h. \tag{2.99}
$$

For $k = 2$, by (2.94) and Theorem 2.2, we have

$$
\begin{aligned}
|\Psi_{2h}(v_h) - \Psi_{2h}(v)| \leq \; & \left(\frac{1}{6h}\right)[18|v_h(x_1, h, x_3) - v(x_1, h, x_3)| \\
& +9|v_h(x_1, 2h, x_3) - v(x_1, 2h, x_3)| \\
& +2|v_h(x_1, 3h, x_3) - v(x_1, 3h, x_3)|] \\
\leq \; & \left(\frac{1}{6h}\right)[29c_5h^4] \\
\leq \; & c_6h^3.
\end{aligned} \tag{2.100}
$$

The same inequality is true when $k = 5$,

$$
\begin{aligned}
|\Psi_{5h}(v_h) - \Psi_{5h}(v)| \leq\ & \left(\frac{1}{6h}\right)[18|v_h(x_1, a_2 - h, x_3) - v(x_1, a_2 - h, x_3)| \\
& +9|v_h(x_1, a_2 - 2h, x_3) - v(x_1, a_2 - 2h, x_3)| \\
& +2|v_h(x_1, a_2 - 3h, x_3) - v(x_1, a_2 - 3h, x_3)|] \\
\leq\ & \left(\frac{1}{6h}\right)[29c_7h^4] \\
\leq\ & c_8h^3. \qquad (2.101)
\end{aligned}
$$

The inequality (2.98) follows by virtue of (2.99)-(2.101). ■

Lemma 2.13 The following inequality holds

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h) - \Psi_k| \leq ch^3, k = 1,2,4,5 \qquad (2.102)$$

where $\Psi_{kh}, k = 1,2,4,5$ are defined by (2.92)-(2.96), and $\Psi_k = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$ on $\Gamma_k, k = 1,2,4,5$.

Proof. From Lemma 2.1 it follows that $u \in C^{5,\lambda}(\bar{R})$. From (2.92)-(2.96) follows that $\Psi_{qh}(v), q = 1,2,4,5$ are the third order forward ($q = 1,2$) and backward ($q = 4,5$) formulae for the approximation of $((\partial^2 u)/(\partial x_1 \partial x_2))$. Since the solution u of problem (2.1)-(2.5) is from $C^{5,\lambda}$ (R), from the truncation error formulas (Burden and Douglas, 2011), we have

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi(v) - \Psi_k| \leq c_1h^3, \ \ k = 1,2,4,5. \qquad (2.103)$$

On the basis of Lemma 2.12 and estimation (2.103), (2.102) hold as follows,

$$\begin{aligned}\max_{(x_1,x_2,x_3)\in\Gamma_k^h}|\Psi_{kh}(v_h)-\Psi_k| &= \max_{(x_1,x_2,x_3)\in\Gamma_k^h}|\Psi_{kh}(v_h)-\Psi_{kh}(v)+\Psi_{kh}(v)-\Psi_k| \\ &\le \max_{(x_1,x_2,x_3)\in\Gamma_k^h}|\Psi_{kh}(v_h)-\Psi_{kh}(v)| \\ &\quad + \max_{(x_1,x_2,x_3)\in\Gamma_k^h}|\Psi_{kh}(v)-\Psi_k| \\ &\le c_2h^3,\ \ k=1,2,4,5. \quad \blacksquare \qquad (2.104)\end{aligned}$$

From the conditions (2.2)-(2.5), it follows that the solution of the problem (2.1) belongs to the class $C^{5,1-\varepsilon}(\bar{R})$ for any $\varepsilon \in (0,1)$ (see Theorem 2.1 in (Volkov, 1969)). Then, its any second order derivatives belongs to $C^{3,1-\varepsilon_1}(\bar{R})$, for any $\varepsilon_1 \in (0,1)$. Since any order derivatives of harmonic functions are harmonic, the following Lemma follows from Lemma 1.2 in (Volkov, 1979), (see Lemma 3, Chap. 4, Sec. 3 in (Mikahilov, 1978), also).

Lemma 2.14 Let $\rho(x_1,x_2,x_3)$ be the distance from the current point of the open parallelepiped R to its boundary. Then for any derivative ϖ of the solution of the problem (2.89) of order m, $(m>3)$ with respect to x_1,x_2,x_3 satisfy the inequality

$$|\varpi^m(x_1,x_2,x_3)| \le M_m\rho^{-m+3}, \qquad (2.105)$$

where $M_m>0$ is constant dependent on m only.

Let $\Psi_j(h)$ be the trace of $\varpi = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)$ on Γ_j^h, and let ϖ_h' be the solution of the following problem

$$\varpi_h' = S\varpi_h' \; on \; R, \varpi_h' = \Psi_j^h \; on \; \Gamma_j^h, j=1,2,\ldots,6. \qquad (2.106)$$

Lemma 2.15 The following estimation holds

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h}|\varpi_h' - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)| \le ch^3, \qquad (2.107)$$

where ϖ_h' is a solution of the finite difference problem (2.106).

Proof. From the definition of the boundary grid function $\Psi_j(h)$ and from (2.106) for the error function

$$\bar{\varepsilon}_h = \varpi_h' - \varpi \tag{2.108}$$

we have

$$\bar{\varepsilon}_h = S\bar{\varepsilon}_h + (S\varpi - \varpi) \; on \; R_h, \;\; \bar{\varepsilon}_h = 0 \; on \; \Gamma_j^h, j = 1,2,\ldots,6 \tag{2.109}$$

where ϖ is the solution of problem (2.89). By virtue of (2.105), by analogy with the proof of Lemma 3 (Volkov and Dosiyev, 2012) it follows, that

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |S\varpi - \varpi| \le c_1\left(\frac{h^3}{k^3}\right), \;\; k = 1,2,\ldots,N(h). \tag{2.110}$$

On the basis of Lemma 2.1 and (2.110) for the solution of problem (2.109), we obtain

$$\begin{aligned} |\bar{\varepsilon}_h| &\le c_2 h^3 \sum_{k=1}^{N(h)} \left(\frac{1}{k^3}\right) \\ &\le c_3 h^3. \blacksquare \end{aligned}$$

Theorem 2.5 The following estimation is true

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} \left|\varpi_h - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)\right| \le ch^3 \tag{2.111}$$

where u is the solution of the problem (2.1), ϖ_h is the solution of the finite difference problem (2.97).

Proof. Let

$$\varepsilon_h = \varpi_h - \varpi \; on \; R^k, \tag{2.112}$$

where $\varpi = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$. From (2.97) and (2.112), we have

$$\varepsilon_h = S\varepsilon_h + (S\varpi - \varpi) \; on \; R_h,$$

$$\varepsilon_h = \Psi_{kh}(v_h) - \varpi \; on \; \Gamma_k^h, k = 1,2,4,5 \; \varepsilon_h = 0 \; on \; \Gamma_p^h, p = 3,6.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{2.113}$$

where

$$\varepsilon_h^1 = S\varepsilon_h^1 \; on \; R_h, \tag{2.114}$$

$$\varepsilon_h^1 = \Psi_{kh}(v_h) - \varpi \; on \; \Gamma_k^h, k = 1,2,4,5, \varepsilon_h^1 = 0 \; on \; \Gamma_p^h, p = 3,6; \tag{2.115}$$

$$\varepsilon_h^2 = S\varepsilon_h^2 + (S\varpi - \varpi) \; on \; R_h, \; \varepsilon_h^2 = 0 \; on \; \Gamma_j^h, j = 1,2,\ldots,6. \tag{2.116}$$

By Lemma 2.13 and by the maximum principle, for the solution of system (2.114), (2.115), we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |\varepsilon_h^1| \le \max_{q=1,2,\ldots,6} \max_{(x_1,x_2,x_3)\in\Gamma_q^h} |\Psi_{qh}(v_h) - \varpi| \le c_1 h^3. \tag{2.117}$$

The solution ε_h^2 of system (2.116) is the error of the approximate solution obtained by the finite difference method of problem (2.89), when it is assumed that on the boundary nodes Γ_j^h, the exact values of the functions $\Psi_j, j = 1,2,\ldots,6$ are used.

From Lemma 2.15, it follows that

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |\varepsilon_h^2| \le c_2 h^3. \tag{2.118}$$

By (2.112), (2.13), (2.117) and (2.118) yields the estimation (2.111).■

2.5.2 Boundary function from $C^{4,\lambda}$

Let the boundary functions $\varphi_j, j = 1,2,\ldots,6$, in problem (2.1) on the faces Γ_j be satisfied by the condition

$$\varphi_j \in C^{4,\lambda}(\Gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,6, \tag{2.119}$$

i.e., $p = 4$ in (2.2). Let u be a solution of the problem (2.1) with the conditions (2.119) and (2.3)-(2.5).

Let $\varpi = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$ and let $\Psi_j = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$ $on\ \Gamma_j$, j=1,2,...,6, where u is a solution of the boundary value problem (2.1) and v is a solution of the boundary value problem (2.57).

Consider the boundary value problem:

$$\Delta\varpi = 0\ on\ R, \varpi = \Psi_j\ on\ \Gamma_j, j = 1,2,\ldots,6. \tag{2.120}$$

We define the sets

$$\Gamma_k^{h+} = \left\{0 \le x_2 \le \frac{a_2}{2},\ \ x_1 = t_k\right\} \cap \Gamma_k^h,\ \ k = 1,4, \tag{2.121}$$

and

$$\Gamma_k^{h-} = \left\{\frac{a_2}{2} + h \le x_2 \le a_2,\ \ x_1 = t_k\right\} \cap \Gamma_k^h,\ k = 1,4, \tag{2.122}$$

where $t_1 = 0\ and\ t_4 = a_1$.

We define the following operators $\Psi_{qh}, q = 1,2,\ldots,6$,

$$\Psi_{kh}(v_h) = \left(\frac{1}{2h}\right)[-3v_h(t_k, x_2, x_3) + 4v_h(t_k, x_2 + h, x_3)$$

$$-v_h(t_k, x_2 + 2h, x_3)]\ on\ \Gamma_k^{h+}, \tag{2.123}$$

$$\Psi_{kh}(v_h) = \left(\frac{1}{2h}\right)[3v_h(t_k,x_2,x_3) - 4v_h(t_k,x_2-h,x_3)$$

$$+v_h(t_k,x_2-2h,x_3)] \; on \; \Gamma_k^{h-}, \tag{2.124}$$

where $k = 1,4$.

$$\Psi_{2h}(v_h) = \left(\frac{1}{2h}\right)[-3\phi_2(x_1,x_3) + 4v_h(x_1,h,x_3) - v_h(x_1,2h,x_3) \; on \; \Gamma_2^h, \tag{2.125}$$

$$\Psi_{5h}(v_h) = \left(\frac{1}{2h}\right)[3\phi_5(x_1,x_3) - 4v_h(x_1,a_2-h,x_3)$$

$$+v_h(x_1,a_2-2h,x^3) on \; \Gamma_5^h, \tag{2.126}$$

$$\Psi_{ph}(v_h) = \left(\frac{\partial^2\varphi_p}{\partial x_1 \partial x_2}\right), on \; \Gamma_p^h, p = 3,6, \tag{2.127}$$

where ϕ_2 and ϕ_5 are the functions defined in (2.65), $\varphi_3 \; and \; \varphi_5$ are the given functions in (2.1), v_h is the solution of the finite difference problem (2.65).

Let ϖ_h be the solution of the following finite difference problem,

$$\varpi_h = S\varpi_h \; on \; R_h, \varpi_h = \Psi_{jh} \; on \; \Gamma_j^h, j = 1,2,\ldots,6, \tag{2.128}$$

where $\Psi_{jh}, j = 1,2,\ldots,6$, are defined by (2.123)-(2.127).

Lemma 2.16 The following inequality is true,

$$|\Psi_{kh}(v_h) - \Psi_{kh}(v)| \le ch^2, k = 1,2,\ldots,6 \tag{2.129}$$

where v_h is the solution of the finite difference problem (2.65), v is the solution of problem (2.57).

Proof. It is obvious that $\Psi_{ph}(v_h) - \Psi_{ph}(v) = 0 \; for \; p = 3{,}6$.

For $t = 1{,}4$, by using the inequality (2.123) and applying the Theorem 2.3, we have

$$
\begin{aligned}
|\Psi_{th}(v_h) - \Psi_{th}(v)| \;\leq\; & \left(\frac{1}{2h}\right)[3|v_h(t_k, x_2, x_3) - v(t_k, x_2, x_3)| \\
& +4|v_h(t_k, x_2 + h, x_3) - v(t_k, x_2 + h, x_3)| \\
& +|v_h(t_k, x_2 + 2h, x_3) - v(t_k, x_2 + 2h, x_3)|] \\
\leq\; & \left(\frac{1}{2h}\right)[8c_1 h^3] \\
\leq\; & c_2 h^2 on \; \Gamma_k^{h+}.
\end{aligned}
$$

The same inequality is obtained on Γ_k^{h-} by using (2.124). We have

$$
\begin{aligned}
|\Psi_{th}(v_h) - \Psi_{th}(v)| \;\leq\; & \left(\frac{1}{2h}\right)[3|v_h(t_k, x_2, x_3) - v(t_k, x_2, x_3)| \\
& +4|v_h(t_k, x_2 - h, x_3) - v(t_k, x_2 - h, x_3)| \\
& +|v_h(t_k, x_2 - 2h, x_3) - v(t_k, x_2 - 2h, x_3)|] \\
\leq\; & \left(\frac{1}{2h}\right)[8c_3 h^3] \\
\leq\; & c_4 h^2 on \; \Gamma_k^{h-}.
\end{aligned}
$$

Since $\Gamma_k^h = \Gamma_k^{h+} \cup \Gamma_k^{h-}$, we have,

$$
|\Psi_{th}(v_h) - \Psi_{th}(v)| \leq c_5 h^2, \; k = 1{,}4 \; on \; \Gamma_k^h. \tag{2.130}
$$

For $k = 2$, by (2.125) and Theorem 2.3, we have

$$|\Psi_{2h}(v_h) - \Psi_{2h}(v)| \leq \left(\frac{1}{2h}\right)[4|v_h(x_1, h, x_3) - v(x_1, h, x_3)|$$

$$+|v_h(x_1, 2h, x_3) - v(x_1, 2h, x_3)|]$$

$$\leq \left(\frac{1}{2h}\right)[5c_5h^3] \leq c_6h^2. \qquad (2.131)$$

The same inequality is true when $k = 5$,

$$|\Psi_{5h}(v_h) - \Psi_{5h}(v)| \leq \left(\frac{1}{2h}\right)[4|v_h(x_1, a_2 - h, x_3) - v(x_1, a_2 - h, x_3)|$$

$$+|v_h(x_1, a_2 - 2h, x_3) - v(x_1, a_2 - 2h, x_3)|]$$

$$\leq \left(\frac{1}{2h}\right)[5c_7h^3] \leq c_8h^2. \qquad (2.132)$$

By virtue of (2.130)-(2.132) the inequality (2.129) is obtained. ■

Lemma 2.17 The following inequality holds

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h) - \Psi_k| \leq ch^2, k = 1,2,4,5 \qquad (2.133)$$

where $\Psi_{kh}, k = 1,2,4,5$ are defined in (2.123)-(2.126), and $\Psi_k = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$ on $\Gamma_k, k = 1,2,4,5$.

Proof. From Lemma 2.1 it follows that $u \in C^{4,\lambda}(\bar{R})$. From (2.123)-(2.126) follows that $\Psi_{qh}(v), q = 1,2,4,5$ are the second order forward ($q = 1,2$) and backward ($q = 4,5$) formulae for the approximation of $((\partial^2 u)/(\partial x_1 \partial x_2))$. Since the solution u of problem (2.1)-(2.5) is from $C^{4,\lambda}$ $(\bar{R})$, from the truncation error formulas (Burden and Douglas, 2011), we have

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi(v)-\Psi_k| \le c_1 h^2, \;\; k = 1,2,4,5. \tag{2.134}$$

On the basis of Lemma 2.16 and estimation (2.134), (2.133) follows,

$$\begin{aligned}\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h)-\Psi_k| = &\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h)-\Psi_{kh}(v)+\Psi_{kh}(v)-\Psi_k| \\ \le &\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h)-\Psi_{kh}(v)| \\ &+ \max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v)-\Psi_k| \\ \le &\; c_2 h^2, \;\; k = 1,2,4,5. \quad \blacksquare \end{aligned} \tag{2.135}$$

From the conditions (2.2)-(2.5), it follows that the solution of the problem (2.1) belongs to the class $C^{4,1-\varepsilon}(\bar{R})$ for any $\varepsilon \in (0,1)$ (see Theorem 2.1 in (Volkov, 1969)). Then, its any second order derivatives belongs to $C^{2,1-\varepsilon_1}(\bar{R})$, for any $\varepsilon_1 \in (0,1)$. Since any order derivatives of harmonic functions are harmonic, the following Lemma follows from Lemma 1.2 in (Volkov, 1979) (see Lemma 3, Chap. 4, Sec. 3 in (Mikahilov, 1978), also).

Lemma 2.18 Let $\rho(x_1, x_2, x_3)$ be the distance from the current point of the open parallelepiped R to its boundary. Then for any derivative ϖ of the solution of the problem (2.120) of order m, $(m > 2)$ with respect to x_1, x_2, x_3 satisfy the inequality

$$|\varpi^m(x_1, x_2, x_3)| \le M_m \rho^{-m+2}, \tag{2.136}$$

where $M_m > 0$ is constant dependent on m only.

Let $\Psi_j(h)$ be the trace of $\varpi = ((\partial^2 u)/(\partial x_1 \partial x_2))$ on Γ_j^h, and let ϖ_h' be the solution of the following problem

$$\varpi_h' = S\varpi_h' \; on \; R, \varpi_h' = \Psi_j^h \; on \; \Gamma_j^h, j = 1,2,\ldots,6. \tag{2.137}$$

Lemma 2.19 The following estimation holds

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |\varpi_h' - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)| \leq ch^2, \tag{2.138}$$

where ϖ_h' is a solution of the finite difference problem (2.137).

Proof. From the definition of the boundary grid function $\Psi_j(h)$ and from (2.137) for the error function

$$\bar{\varepsilon}_h = \varpi_h' - \varpi \tag{2.139}$$

we have

$$\bar{\varepsilon}_h = S\bar{\varepsilon}_h + (S\varpi - \varpi) \ on\ R_h,\ \bar{\varepsilon}_h = 0 \ on\ \Gamma_j^h, j = 1,2,\dots,6 \tag{2.140}$$

where ϖ is the solution of problem (2.120). By virtue of (2.136), by analogy with the proof of Lemma 3 (Volkov and Dosiyev, 2012) it follows, that

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |S\varpi - \varpi| \leq c_1\left(\frac{h^3}{k^3}\right),\ k = 1,2,\dots,N(h). \tag{2.141}$$

On the basis of Lemma 2.1 and (2.141) for the solution of problem (2.140), we obtain

$$\begin{aligned} |\bar{\varepsilon}_h| &\leq c_2 h^3 \sum_{k=1}^{N(h)} \left(\frac{1}{k^3}\right) \\ &\leq c_3 h^3. \ \blacksquare \end{aligned}$$

Theorem 2.6 The following estimation is true

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |\varpi_h - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)| \leq ch^2 \tag{2.142}$$

where u is the solution of the problem (2.1), ϖ_h is the solution of the finite difference problem (2.128).

Proof. Let

$$\varepsilon_h = \varpi_h - \varpi \; on \; R^k, \tag{2.143}$$

where $\varpi = \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial v}{\partial x_2}\right)$. From (2.128) and (2.143), we have

$$\varepsilon_h \; = \; S\varepsilon_h + (S\varpi - \varpi) \; on \; R_h,$$

$$\varepsilon_h \; = \; \Psi_{kh}(v_h) - \varpi \; on \; \Gamma_k^h, k = 1,2,4,5 \; \varepsilon_h = 0 \; on \; \Gamma_p^h, p = 3,6.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{2.144}$$

where

$$\varepsilon_h^1 \; = \; S\varepsilon_h^1 \; on \; R_h, \tag{2.145}$$

$$\varepsilon_h^1 = \; \Psi_{kh}(v_h) - \varpi \; on \; \Gamma_k^h, k = 1,2,4,5, \; \varepsilon_h^1 = 0 \; on \; \Gamma_p^h, p = 3,6; \tag{2.146}$$

$$\varepsilon_h^2 = \; S\varepsilon_h^2 + (S\varpi - \varpi) \; on \; R_h, \; \varepsilon_h^2 = 0 \; on \; \Gamma_j^h, j = 1,2,\dots,6. \tag{2.147}$$

By Lemma 2.17 and by the maximum principle, for the solution of system (2.145), (2.146), we have

$$\max_{(x_1,x_2,x_3)\in \bar{R}_h} |\varepsilon_h^1| \leq \max_{q=1,2,\dots,6} \max_{(x_1,x_2,x_3)\in \Gamma_q^h} |\Psi_{qh}(v_h) - \varpi| \leq c_1 h^2. \tag{2.148}$$

The solution ε_h^2 of system (2.147) is the error of the approximate solution obtained by the finite difference method for problem (2.120), when it is assumed that on the boundary nodes Γ_j^h, the exact values of the functions $\Psi_j, j = 1,2,\dots,6$ are used.

From Lemma 2.19, it follows that

$$\max_{(x_1,x_2,x_3)\in \bar{R}_h} |\varepsilon_h^2| \leq c_2 h^2. \tag{2.149}$$

By (2.143), (2.144), (2.148) and (2.149), the estimation (2.142) holds. ■

Remark 2.3: We have investigated the method of high order approximations of the second order mixed derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ and pure derivative $\frac{\partial^2 u}{\partial x_1^2}$. The same results are obtained for the second order mixed derivatives $\partial^2 u/\partial x_m \partial x_n, (m \neq n)$ and pure derivative $\partial^2 u/\partial^2 x_l$, where $m, n = \{1,2,3\}$ and $l = \{2,3\}$, analogously, by using the same order forward and backward formulae in appropriate faces of the parallelepiped.

CHAPTER 3

A HIGHLY ACCURATE DIFFERENCE METHOD FOR APPROXIMATING OF THE FIRST DERIVATIVES OF THE DIRICHLET PROBLEM FOR LAPLACE'S EQUATION ON A RECTANGLE

In this chapter, we concentrate highly accurate difference method for approximation of the first derivatives of the Dirichlet problem for Laplace's equation. $O(h^8\rho)$, where ρ is the distance from the current grid point to the boundary of the rectangle and h is the mesh size, order of accurate three-stage difference method on a square grid for the approximate solution of the Dirichlet problem for Laplace's equation on a rectangle is proposed and justified without taking more than 9 nodes of the grid. At the first stage, by using the 9-point scheme the sum of the pure fourth derivatives of the desired solution is approximated of order $O(h^6\rho)$. At the second stage, approximate values of the sum of the pure eighth derivatives is approximated of order $O(h^2\rho)$ by the 5-point scheme. At the final third stage, the system of simplest 5-point difference equations approximating the Dirichlet problem is corrected by introducing the quantities determined at the first and second stages. By using this error estimation it is proved that the proposed three stage method constructed to find an approximate value of the first and second order pure derivatives of the solution converges uniformly with an order of $O(h^8)$.

3.1 Some Differential Properties of the Solution to the Dirichlet Problem

Let $\Pi = \{(x, y): 0 < x < a, 0 < y < b\}$ be rectangle, a/b be rational, $\gamma_j(\gamma_j')$, j=1,2,3,4, be the sides, including (excluding) the ends, enumerated counterclockwise starting from the left side $(\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5)$ and let $\gamma = \bigcup_{j=1}^{4} \gamma_j$ be the boundary of Π. Denote by s the arclength, measured along γ, and by s_j the value of s at the beginning of γ_j. We say that $f \in C^{k,\lambda}(D)$, if f has k −th derivatives on D satisfying a Hölder condition with exponent $\lambda \in (0,1)$.

We consider the boundary value problem

$$\Delta u = 0 \;\; on\; \Pi, u = \varphi_j(s) \;\; on\; \gamma_j, \qquad j = 1,2,3,4, \tag{3.1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2, \varphi_j$ are given functions of s. Assume that

$$\varphi_j \in C^{12,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \tag{3.2}$$

$$\varphi_j^{2q}(s_j) = (-1)^q \varphi_{j-1}^{2q}(s_j), q = 0,1,2,3,4,5. \tag{3.3}$$

Lemma 3.1 Under the conditions (3.2) and (3.2) the solution u of the Dirichlet problem (3.1) belongs to the class $C^{11,\lambda}(\overline{\Pi}),\; 0 < \lambda < 1$.

Proof. Lemma 3.1 follows from Theorem 3.1 in (Volkov, 1965). ■

Lemma 3.2 The following type of partial derivatives of the solution u to problem (3.1) satisfy the next inequality

$$\max_{0\le p\le 6} \sup_{(x,y)\in\Pi} \left|\frac{\partial^{12}u}{\partial x^{2p}\partial y^{12-2p}}\right| < \infty. \tag{3.4}$$

Proof. From the conditions (3.2) and (3.3) follows that the derivatives $\left(\frac{\partial^{10}u}{\partial x^{10}}\right)$ and $\left(\frac{\partial^{10}u}{\partial y^{10}}\right)$ of the solution u of problem (3.1) are continuous on Π. We put $w = \left(\frac{\partial^{10}u}{\partial x^{10}}\right)$. The function w is harmonic in Π, and is the solution of the problem

$$\Delta w = 0 \;\; on \;\; \Pi, \qquad w = \Psi_j \;\; on\; \gamma_j, j = 1, \dots, 4,$$

where

$$\Psi_\tau = -\left(\frac{\partial^{10}\varphi_\tau}{\partial x^{10}}\right), \qquad \tau = 1,3$$

$$\Psi_\nu = \left(\frac{\partial^{10}\varphi_\nu}{\partial x^{10}}\right), \qquad \nu = 2,4.$$

From conditions (3.2)-(3.3) it follows that

$$\Psi_j \in C^{2,\lambda}(\gamma_j), 0 < \lambda < 1,$$

$$\Psi_j(s_j) = \Psi_{j-1}(s_j), \quad j = 1,\ldots,4.$$

Hence, on the basis of Theorem 4.1 in (Volkov, 1969), we have

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^{12}u}{\partial x^{12}}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^{2}w}{\partial x^{2}}\right| < \infty, \tag{3.5}$$

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^{12}u}{\partial x^{10}\partial y^{2}}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^{2}w}{\partial y^{2}}\right| < \infty. \tag{3.6}$$

Similarly, it is proved that,

$$\sup_{(x,y)\in\Pi}\left\{\left|\frac{\partial^{12}u}{\partial y^{12}}\right|, \left|\frac{\partial^{12}u}{\partial y^{10}\partial x^{2}}\right|\right\} < \infty. \tag{3.7}$$

When $w = \left(\frac{\partial^{10}u}{\partial y^{10}}\right)$, the function w is harmonic on Π, and is the solution of the problem

$$\Delta w = 0 \;\; on \;\; \Pi, \qquad w = \emptyset_j \;\; on\; \gamma_j, j = 1, \ldots, 4,$$

where

$$\emptyset_\tau = -\left(\frac{\partial^{10}\varphi_\tau}{\partial y^{10}}\right), \qquad \tau = 1,3$$

$$\emptyset_\nu = \left(\frac{\partial^{10}\varphi_\nu}{\partial x^{10}}\right), \qquad \nu = 2,4.$$

From conditions (3.2)-(3.3) it follows that,

$$\emptyset_j \in C^{2,\lambda}(\gamma_j), 0 < \lambda < 1,$$

$$\emptyset_j(s_j) = \emptyset_{j-1}(s_j), \quad j = 1,\ldots,4.$$

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^{12}u}{\partial y^{12}}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^{2}w}{\partial y^{2}}\right| < \infty, \tag{3.8}$$

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^{12}u}{\partial y^{10}\partial x^{2}}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^{2}w}{\partial x^{2}}\right| < \infty. \tag{3.9}$$

From (3.5)-(3.9), estimation (3.4) holds.■

3.2 Finite Difference Problem For 3-Stage Method

Let $h > 0$, and $a/h \geq 4, b/h \geq 4$ be integers. We assign Π^h, a square net on Π, with step size h, obtained by the lines $x, y = 0, h, 2h, \ldots$. Let γ_j^h be a set of nodes on the interior of γ_j , and let

$$\gamma^h = \cup \gamma_j^h, \gamma_j = \gamma_{j-1} \cap \gamma_j, \gamma^h = \cup (\gamma_j^h \cup \gamma_j), \Pi^h = \Pi^h \cup \gamma^h.$$

Let the operators A and B be defined as follows:

$$Au(x,y) = [u(x+h,y) + u(x-h,y) + u(x,y+h)$$
$$+u(x,y-h)]/4 \tag{3.10}$$

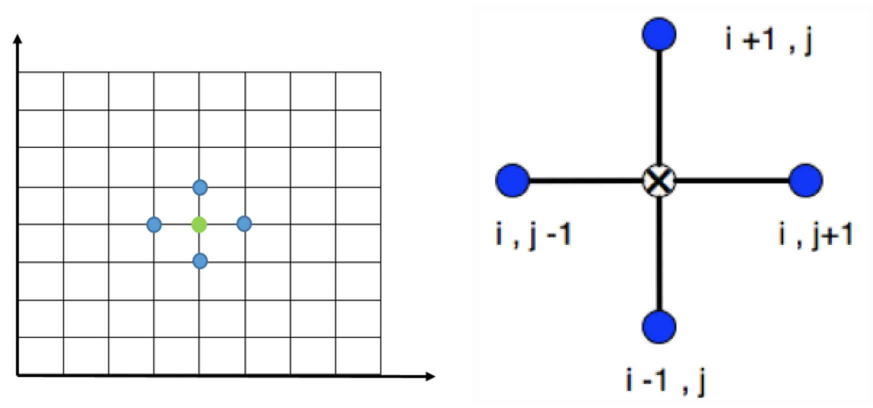

Figure 3.1: four points around center using the operator A. Each point has a distance of h from the point (x, y)

$$Bu(x,y) = \{4[u(x+h,y) + u(x-h,y) + u(x,y+h)$$
$$+u(x,y-h)] + u(x+h,y+h) + u(x-h,y+h)$$
$$+u(x+h,y-h) + u(x-h,y-h)\}/20 \quad (3.11)$$

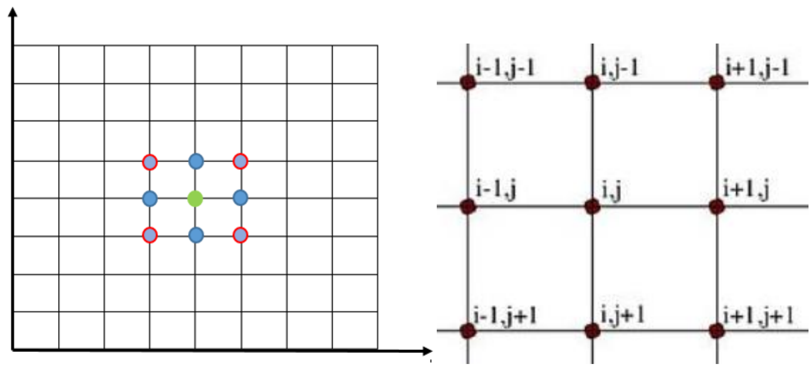

Figure 3.2: eight points around center using the operator B. Blue point has a distance of h and red point has a distance of $\sqrt{2}h$ from the point (x,y)

To simplicity, we will denote by $c, c_1, c_2, \ldots$ as constants which are independent of h and the nearest factor, and identical notation will be used for various constants.

We present two more lemmas. Consider the following systems:

$$q_h = Aq_h + g_h, on\ \Pi^h, q_h = 0\ on\ \gamma_h, \quad (3.12)$$

$$\bar{q}_h = A\bar{q}_h + \bar{g}_h, on\ \Pi^h, \bar{q}_h \geq 0\ on\ \gamma_h, \quad (3.13)$$

and

$$p_h = Bp_h + f_h, on\ \Pi^h, p_h = 0\ on\ \gamma_h, \quad (3.14)$$

$$\bar{p}_h = B\bar{p}_h + \bar{f}_h, on\ \Pi^h, \bar{p}_h \geq 0\ on\ \gamma_h, \quad (3.15)$$

where $g_h, \bar{g}_h, f_h$ and $\bar{f}_h$ are given functions, and $|g_h| \leq \bar{g}_h$ and $|f_h| \leq \bar{f}_h$ on Π^h.

Lemma 3.3 The solutions $q_h, \bar{q}_h, p_h$ and $\bar{p}_h$ of systems (3.12)-(3.15) satisfy the inequality

$$|q_h| \leq \bar{q}_h \ on \ \Pi^h,$$

and

$$|p_h| \leq \bar{p}_h \ on \ \Pi^h.$$

Proof. The proof of Lemma 3.3 follows from the comparison theorem (see Chapter 4 in Samarskii, 2001).■

3.3 $O(h^8)$ Order of Accurate Approximate Solution

Let u be a solution of the following finite difference problem

$$u = Au \ on \ \Pi^h, u_h = \varphi_j \ on \ \Gamma_{jh}, \qquad j = 1,2,3,4,$$

where $\varphi_j, \ j = 1,2,3,4,$ are functions (3.1). A is five point operator defined by (3.10). Taking into account that the function u is harmonic, by exhaustive calculations, we have

$$Au = u + \frac{1}{2}\left\{\frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \frac{h^6}{6!}\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right)\right.$$
$$\left. + \frac{h^8}{8!}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right) + + \frac{h^{10}}{10!}\left(\frac{\partial^{10} u}{\partial x^{10}} + \frac{\partial^{10} u}{\partial y^{10}}\right) + \frac{1}{12!}O(h^{12})\right\}.$$

Since u is harmonic so

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \text{ and } \left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right) = 0 \text{ and } \left(\frac{\partial^{10} u}{\partial x^{10}} + \frac{\partial^{10} u}{\partial y^{10}}\right) = 0.$$

Then we have,

$$Au = u + \frac{1}{2}\left\{\frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \frac{h^8}{8!}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right) + \frac{1}{12!}O(h^{12})\right\}.$$

3.3.1 First stage for the solution

On γ_h, set

$$u^4 = u^4(x,y) = \frac{1}{2}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right), \tag{3.16}$$

where u is the solution of the Dirichlet problem (3.1).

The function (3.16) is the solution of the following boundary value problem

$$\Delta u^4 = 0 \quad on\ \Pi, u^4 = \Psi_j^4\ on\ \gamma_j^h \cup \gamma_j, j = 1,2,3,4 \tag{3.17}$$

where

$$\Psi_j^4 = \begin{cases} \Psi_j^4(x) = \frac{d^4\varphi_j}{dx^4}, & j = 2,4 \\ \Psi_j^4(y) = \frac{d^4\varphi_j}{dy^4}, & j = 1,3 \end{cases}. \tag{3.18}$$

On the basis of the Lemma 3.1 solution u of problem (3.1) belongs to the class $C^{11,\lambda}(\Pi), 0 < \lambda < 1$. Let us show that u^4 is a harmonic

$$\Delta u^4 = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\right)$$

$$\Delta u^4 = \frac{1}{2}\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial y^6}\right).$$

$$\Delta u^4 = \frac{1}{2}\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} + \frac{\partial^4}{\partial x^2 \partial y^2}\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right]\right).$$

Since u harmonic, we have

$$\frac{\partial^6 u}{\partial x^6} = -\frac{\partial^6 u}{\partial x^2 \partial y^4} \quad \text{and} \quad \frac{\partial^6 u}{\partial y^6} = -\frac{\partial^6 u}{\partial x^4 \partial y^2},$$

then

$$\Delta u^4 = \frac{1}{2}\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right) = \frac{1}{2}\frac{\partial^4}{\partial x^2 \partial y^2}\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right] = 0.$$

Let us show that boundary functions of u^4 are

$$u^4 \rfloor_{\gamma^h} = \Psi_j^4 = \frac{1}{2}\left(\frac{\partial^4 \varphi}{\partial x^4} + \frac{\partial^4 \varphi}{\partial y^4}\right) = \begin{cases} \Psi_j^4(x) = \dfrac{d^4 \varphi_j}{dx^4}, & j = 2,4 \\ \Psi_j^4(y) = \dfrac{d^4 \varphi_j}{dy^4}, & j = 1,3 \end{cases}.$$

Furthermore, as follows from Lemma 3.2 the following $8 - th$ order derivatives are bounded on Π:

$$\max_{0 \le p \le 4} \sup_{(x,y) \in \Pi} \left| \frac{\partial^8 u^4}{\partial x^{2p} \partial y^{8-2p}} \right| < \infty.$$

We consider the following system of difference equation for the approximation of problem (3.17) and (3.18):

$$u_h^4 = B u_h^4 \ \ on\ \Pi^h, u_h^4 = \Psi_j^4 \ \ on\ \gamma_j^h \cup \gamma_j, j = 1,2,3,4, \tag{3.19}$$

by the maximum principle, problem (3.19) has the unique solution, where Ψ_j^4 are functions (3.18) and B averaging operator defined by (3.11).

Lemma 3.4 For the solution of the problems

$$p_h \ = \ B p_h + h^8 \ on\ \Pi^h, q_h = 0 \ on\ \gamma_h, \tag{3.20}$$

the following inequality holds:

$$p_h \ \le \ \left(\tfrac{5}{3}\right) \rho d h^6 \ on\ \Pi^h \ ,$$

where $d = max\{a, b\}, \rho = \rho(x, y)$ is the distance from the current point $(x, y) \in \Pi^h$ to the boundary of the rectangle Π^h.

Proof. We consider the functions

$$\bar{p}_h^{(1)}(x,y) = \left(\frac{5}{3}\right) h^6(ax - x^2) \geq 0, \quad \bar{p}_h^{(2)}(x,y) = \left(\frac{5}{3}\right) h^6(bx - x^2) \geq 0 \; on \; \overline{\Pi}.$$

Let $\bar{p}_h(x,y) = \bar{p}_h^{(1)}(x,y)$, then

$$\begin{aligned}
\mathrm{B}\bar{p}_h(x_0', y_0') = \; & \frac{1}{5}(\bar{p}_h(x_0+h,y_0) + \bar{p}_h(x_0-h,y_0) + \bar{p}_h(x_0,y_0+h) \\
& + \bar{p}_h(x_0,y_0-h)) + \frac{1}{20}(\bar{p}_h(x_0+h,y_0+h) + \bar{p}_h(x_0+h,y_0-h) \\
& \bar{p}_h(x_0-h,y_0-h) + \bar{p}_h(x_0-h,y_0+h)) \\
= \; & \frac{h^6}{3}(ax_0 + ah - x_0^2 - 2x_0h - h^2 + ax_0 - ah - x_0^2 \\
& + 2x_0h - h^2 + ax_0 - x_0^2 + ax_0 - x_0^2) + \frac{h^6}{12}(ax_0 + ah - x_0^2 - 2x_0h \\
& - h^2 + ax_0 + ah - x_0^2 - 2x_0h - h^2 + ax_0 - ah - x_0^2 + 2x_0h - h^2 \\
& + ax_0 - ah - x_0^2 + 2x_0h - h^2) \\
= \; & \frac{h^6}{3}(4ax_0 - 4x_0^2 - 2h^2) + \frac{h^6}{12}(4ax_0 - 4x_0^2 - 4h^2) \\
= \; & \frac{h^6}{12}(20ax_0 - 20x_0^2 - 12h^2) \\
= \; & \frac{5}{3}h^6(ax_0 - x_0^2) - h^8 \\
= \; & \bar{p}_h(x_0', y_0') - h^8.
\end{aligned}$$

Similarly let $\bar{p}_h(x,y) = \bar{p}_h^{(2)}(x,y)$, then

$$\begin{aligned}
B\bar{p}_h(x_0', y_0') = \; & \frac{1}{5}(\bar{p}_h(x_0+h,y_0) + \bar{p}_h(x_0-h,y_0) + \bar{p}_h(x_0,y_0+h) \\
& + \bar{p}_h(x_0,y_0-h)) + \frac{1}{20}(\bar{p}_h(x_0+h,y_0+h) + \bar{p}_h(x_0+h,y_0-h) \\
& \bar{p}_h(x_0-h,y_0-h) + \bar{p}_h(x_0-h,y_0+h))
\end{aligned}$$

$$= \frac{h^6}{3}(bx_0 + bh - x_0^2 - 2x_0h - h^2 + bx_0 - bh - x_0^2$$

$$+2x_0h - h^2 + bx_0 - x_0^2 + bx_0 - x_0^2) + \frac{h^6}{12}(bx_0 + bh - x_0^2 - 2x_0h$$

$$-h^2 + bx_0 + bh - x_0^2 - 2x_0h - h^2 + bx_0 - bh - x_0^2 + 2x_0h - h^2$$

$$+bx_0 - bh - x_0^2 + 2x_0h - h^2)$$

$$= \frac{h^6}{3}(4bx_0 - 4x_0^2 - 2h^2) + \frac{h^6}{12}(4bx_0 - 4x_0^2 - 4h^2)$$

$$= \frac{h^6}{12}(20bx_0 - 20x_0^2 - 12h^2)$$

$$= \frac{5}{3}h^6(bx_0 - x_0^2) - h^8$$

$$= \bar{p}_h(x_0', y_0') - h^8,$$

which are solutions of the equation $p_h = Bp_h + h^8 \ on \ \bar{\Pi}^h$. By virtue of Lemma 3.3 we obtain

$$p_h \leq \min_{i=1,2} \bar{p}_h^i(x,y) \leq \tfrac{5}{3}\rho d h^6 \ \ on \ \ \bar{\Pi}^h. \ \blacksquare$$

Let ϖ be a solution of the problem

$$\Delta\varpi = 0 \ \ on \ \Pi, \varpi = \theta_j(s) \ \ on \ \gamma_j, \ \ j = 1,2,3,4, \qquad (3.21)$$

where $\theta_j, j = 1,2,3,4$ are given functions and

$$\theta_j \in C^{8,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \qquad (3.22)$$

$$\theta_j^{(2q)}(s_j) = (-1)^q \theta_{j-1}^{(2q)}(s_j), q = 0,..,3. \qquad (3.23)$$

Lemma 3.5 The estimation holds

$$\max_{(x,y)\in\Pi} |B\varpi - \varpi| \leq ch^8,$$

where ϖ is the of problem (3.21)-(3.23) on Π^h.

Proof.

$$
\begin{aligned}
B\varpi = \;& \{4[\varpi(x+h,y)+\varpi(x-h,y)+\varpi(x,y+h) \\
& +\varpi(x,y-h)]+\varpi(x+h,y+h)+\varpi(x-h,y+h) \\
& +\varpi(x+h,y-h) \;+\varpi(x-h,y-h)\}/20 \\
= \;& \frac{1}{20}\left(20\varpi+\frac{h^4}{6}\left(3\frac{\partial^4\varpi}{\partial x^4}+6\frac{\partial^4\varpi}{\partial x^2\partial y^2}+3\frac{\partial^4\varpi}{\partial x^4}\right)\right. \\
& +\frac{4h^8}{8!}\left(3\frac{\partial^8\varpi}{\partial x^8}+28\frac{\partial^8\varpi}{\partial x^6\partial y^2}+70\frac{\partial^8\varpi}{\partial x^4\partial y^4}+28\frac{\partial^8\varpi}{\partial x^2\partial y^6}\right. \\
& \left.\left.+3\frac{\partial^8\varpi}{\partial x^8}\right)+\dots\right)
\end{aligned}
$$

Since ϖ harmonic so

$$\left(3\frac{\partial^4\varpi}{\partial x^4}+6\frac{\partial^4\varpi}{\partial x^2\partial y^2}+3\frac{\partial^4\varpi}{\partial x^4}\right)=0.$$

We have

$$B\varpi-\varpi=\frac{h^8}{5.8!}\left(3\frac{\partial^8\varpi}{\partial x^8}+28\frac{\partial^8\varpi}{\partial x^6\partial y^2}+70\frac{\partial^8\varpi}{\partial x^4\partial y^4}+28\frac{\partial^8\varpi}{\partial x^2\partial y^6}+3\frac{\partial^8\varpi}{\partial x^8}\right).$$

Since $B\varpi-\varpi$ contains only eighth order derivatives of ϖ of the form

$$\left(\frac{\partial^8\varpi}{\partial x^{2k}\partial y^{8-2k}}\right),\;\; 0\le k\le 4$$

are bounded the proof of Lemma 3.5 becomes true.■

Lemma 3.6 The following estimation holds

$$\max_{(x,y)\in\Pi^h}|u_h^4-u^4|\le ch^6\rho, \tag{3.24}$$

where u_h^4 is the solution of the system (3.19), u^4 is the trace of exact solution of problem (3.17), (3.18) and ρ is the distance from the current grid point to the boundary of the rectangle $on\ \Pi^h$.

Proof. Let

$$\varepsilon_h = u_h^4 - u^4 \ on\ \Pi^h. \qquad (3.25)$$

Then

$$B\varepsilon_h = Bu_h^4 - Bu^4 \ \Longrightarrow Bu_h^4 = B\varepsilon_h + Bu^4.$$

Moreover,

$$u_h^4 = \varepsilon_h + u^4.$$

By considering problem (3.19) it is obvious that

$$\varepsilon_h = B\varepsilon_h + (Bu^4 - u^4)\ on\ \Pi^h,\ \ \varepsilon_h = 0\ on\ \gamma^h. \qquad (3.26)$$

By virtue of Lemma 3.5 for $(Bu^4 - u^4)$ and applying Lemma 3.3 to the problem (3.20) and (3.26), on the basis of Lemma 3.4 we obtain

$$|\varepsilon_h| \leq c\rho h^6. \qquad (3.27)$$

From (3.25) and (3.27) the proof of Lemma 3.6 can be done. ■

3.3.2 Second stage for the solution

On γ_h, set

$$u^8 = u^8(x,y) = \frac{1}{2}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right), \qquad (3.28)$$

where u is the solution of Dirichlet problem (3.1).

The function (3.28) is the solution of the following boundary value problem

$$\Delta u^8 = 0 \quad on\ \Pi, u^8 = \Psi_j^8\ on\ \gamma_j^h \cup \gamma_j, j = 1,2,3,4 \tag{3.29}$$

where

$$\Psi_j^8 = \begin{cases} \Psi_j^8(x) = \frac{d^8\varphi_j}{dx^8}, & j = 2,4 \\ \Psi_j^8(y) = \frac{d^8\varphi_j}{dy^8}, & j = 1,3 \end{cases}. \tag{3.30}$$

On the basis of Lemma 3.1 solution u of the problem (3.1) belongs the class $C^{11,\lambda}(\Pi), 0 < \lambda < 1$. Let us show that u^8 is a harmonic

$$\Delta u^8 = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right)\right)$$

$$\Delta u^8 = \frac{1}{2}\left(\frac{\partial^{10} u}{\partial x^{10}} + \frac{\partial^{10} u}{\partial x^2 \partial y^8} + \frac{\partial^{10} u}{\partial x^8 \partial y^2} + \frac{\partial^{10} u}{\partial y^{10}}\right),$$

$$\Delta u^8 = \frac{1}{2}\left(\frac{\partial^{10} u}{\partial x^{10}} + \frac{\partial^{10} u}{\partial y^{10}} + \frac{\partial^4}{\partial x^2 \partial y^2}\left[\frac{\partial^6 u}{\partial y^6} + \frac{\partial^6 u}{\partial x^6}\right]\right).$$

Since u harmonic, we have

$$\frac{\partial^6 u}{\partial x^6} = \frac{\partial^6 u}{\partial x^2 \partial y^4} \quad \text{and} \quad \frac{\partial^6 u}{\partial y^6} = \frac{\partial^6 u}{\partial x^4 \partial y^2}$$

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = \frac{\partial^4}{\partial x^2 \partial y^2}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial^{10} u}{\partial x^{10}} = -\frac{\partial^{10} u}{\partial x^4 \partial y^6} \quad \text{and} \quad \frac{\partial^{10} u}{\partial y^6} = -\frac{\partial^{10} u}{\partial x^6 \partial y^4}$$

then

$$\Delta u^8 = \frac{1}{2}\left(\frac{\partial^{10} u}{\partial x^{10}} + \frac{\partial^{10} u}{\partial y^{10}}\right) = \frac{1}{2}\left(-\frac{\partial^8}{\partial x^4 \partial y^4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\right) = 0.$$

Let us show that boundary functions of u^8 are

$$u^8|_{\gamma^h} = \Psi_j^8 = \frac{1}{2}\left(\frac{\partial^8\varphi}{\partial x^8} + \frac{\partial^8\varphi}{\partial y^8}\right) = \begin{cases} \Psi_j^8(x) = \dfrac{d^8\varphi_j}{dx^8}, & j = 2,4 \\ \Psi_j^8(y) = \dfrac{d^8\varphi_j}{dy^8}, & j = 1,3 \end{cases}. \quad \blacksquare$$

Furthermore, as follows from Lemma 3.2 the following $4-th$ order derivatives are bounded on Π:

$$\max_{0\le p\le 2} \sup_{(x,y)\in\Pi} \left|\frac{\partial^4 u^8}{\partial x^{2p}\partial y^{4-2p}}\right| < \infty.$$

We consider for the approximation of problems (3.29) and (3.30) the following system of difference equation:

$$u_h^8 = Au_h^8 \ \ on\ \Pi^h, u_h^8 = \Psi_j{}^8 \ \ on\ \gamma_j^h, j = 1,2,3,4, \tag{3.31}$$

by the maximum principle, problem (3.31) has the unique solution, where Ψ_j^8 are functions (3.30) and A averaging operator defined by (3.10).

Lemma 3.7 For the solution of the problems

$$q_h \ = \ Aq_h + h^4 \ on\ \Pi^h, q_h = 0\ on\ \gamma_h, \tag{3.32}$$

the following inequality holds:

$$q_h \ \le \ 2\rho d h^2 \ on\ \Pi^h\ ,$$

where $d = max\{a, b\}, \rho = \rho(x, y)$ is the distance from the current point $(x, y) \in \Pi^h$ to the boundary of the rectangle Π^h.

Proof. We consider the functions

$$\bar{q}_h^{(1)}(x,y) = 2h^2(ax - x^2) \geq 0, \quad \bar{q}_h^{(2)}(x,y) = 2h^2(bx - x^2) \geq 0 \; on \; \bar{\Pi}.$$

Let $\bar{q}_h(x,y) = \bar{q}_h^{(1)}(x,y)$, then

$$\begin{aligned} A\bar{q}_h(x_0', y_0') = & \frac{1}{4}\big(\bar{q}_h(x_0+h, y_0) + \bar{q}_h(x_0-h, y_0) + \bar{q}_h(x_0, y_0+h) \\ & +\bar{q}_h(x_0, y_0-h)\big) \\ = & \; 2h^2(ax_0 + ah - x_0^2 - 2x_0h - h^2 + ax_0 - ah - x_0^2 \\ & +2x_0h - h^2 + ax_0 - x_0^2 + ax_0 - x_0^2) \\ = & \; \frac{h^2}{2}(4ax_0 - 4x_0^2 - 2h^2) \\ = & \; 2h^2(ax_0 - x_0^2) - h^4 \\ = & \; \bar{p}_h(x_0', y_0') - h^4. \end{aligned}$$

Similarly let $\bar{q}_h(x,y) = \bar{q}_h^{(2)}(x,y)$, then

$$\begin{aligned} A\bar{q}_h(x_0', y_0') = & \frac{1}{4}\big(\bar{q}_h(x_0+h, y_0) + \bar{q}_h(x_0-h, y_0) + \bar{q}_h(x_0, y_0+h) \\ & +\bar{q}_h(x_0, y_0-h)\big) \\ = & \; 2h^2(bx_0 + bh - x_0^2 - 2x_0h - h^2 + bx_0 - bh - x_0^2 \\ & +2x_0h - h^2 + bx_0 - x_0^2 + bx_0 - x_0^2) \\ = & \; \frac{h^2}{2}(4bx_0 - 4x_0^2 - 2h^2) \\ = & \; 2h^2(bx_0 - x_0^2) - h^4 \\ = & \; \bar{q}_h(x_0', y_0') - h^4. \end{aligned}$$

which are solutions of the equation $q_h = Aq_h + h^4 \; on \; \bar{\Pi}^h$. By virtue of Lemma 3.3 we obtain

$$q_h \leq \min_{i=1,2} \bar{q}_h^i(x,y) \leq 2\rho d h^2 \; on \; \bar{\Pi}^h. \blacksquare$$

Let ω be a solution of the problem

$$\Delta\omega = 0 \;\; on\, \Pi, \;\; \omega = \Psi_j(s) \;\; on\, \gamma_j, \;\; j = 1,2,3,4, \tag{3.33}$$

where $\Psi_j, j = 1,2,3,4$ are given functions and

$$\Psi_j \in C^{4,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \tag{3.34}$$

$$\Psi_j^{(2q)}(s_j) = (-1)^q \Psi_{j-1}^{(2q)}(s_j), q = 0,1. \tag{3.35}$$

Lemma 3.8 The following estimation holds

$$\max_{(x,y)\in\Pi} |A\omega - \omega| \le ch^4,$$

where ω is the of problem (3.33)-(3.35) on Π^h.

Proof.

$$\begin{aligned} A\omega = \quad & \{\omega(x+h,y) + \omega(x-h,y) + \omega(x,y+h) \\ & +\omega(x,y-h)\}/4 \\ = \quad & \frac{1}{4}\left(4\omega + 2\frac{h^2}{2!}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) + 2\frac{h^4}{4!}\left(\frac{\partial^4\omega}{\partial x^4} + \frac{\partial^4\omega}{\partial y^4}\right) + \cdots\right. \end{aligned}$$

Since ω harmonic so

$$\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) = 0.$$

We have

$$A\omega - \omega = \frac{h^4}{2.4!}\left(\frac{\partial^4\omega}{\partial x^4} + \frac{\partial^4\omega}{\partial y^4}\right).$$

Since $A\omega - \omega$ contains only fourth order derivatives of ω of the form

$$\left(\frac{\partial^4\omega}{\partial x^{2k}\partial y^{4-2k}}\right), \;\; 0 \le k \le 2$$

are bounded the proof of Lemma 3.8 becomes true.■

Lemma 3.9 The following estimation holds

$$\max_{(x,y)\in\Pi^h} |u_h^8 - u^8| \leq ch^2\rho, \tag{3.36}$$

where u_h^8 is the solution of the system (3.31), u^8 is the trace of exact solution of problem (3.29), (3.30) and ρ is the distance from the current grid point to the boundary of the rectangle $on\ \Pi^h$.

Proof. Let

$$\varepsilon_h = u_h^8 - u^8 \ on\ \Pi^h. \tag{3.37}$$

Then

$$A\varepsilon_h = Au_h^8 - Au^8 \implies Au_h^8 = A\varepsilon_h + Au^8.$$

Moreover,

$$u_h^8 = \varepsilon_h + u^8.$$

By considering problem (3.32) it is obvious that

$$\varepsilon_h = A\varepsilon_h + (Au^8 - u^8)\ on\ \Pi^h,\ \varepsilon_h = 0\ on\ \gamma^h. \tag{3.38}$$

By virtue of Lemma 3.8 for $(Au^8 - u^8)$ and applying Lemma 3.3 to the problem (3.32) and (3.38), on the basis of Lemma 3.7 we obtain

$$|\varepsilon_h| \leq c\rho h^6. \tag{3.39}$$

From (3.37) and (3.39) the proof of Lemma 3.9 can be determined. ■

3.3.3 Third stage for the solution

Let u_h^4 and u_h^8 be the solution of the difference problem (3.19) and (3.31) respectively. We approximate the solution of the given Dirichlet problem (3.1) on the grid Π^h as a solution u_h of the following difference problem

$$u_h = Au_h - \frac{h^4}{24}u_h^4 - \frac{h^8}{40320}u_h^8\ on\ \Pi^h, u_h = \varphi_j\ \ on\ \gamma_j^h, j = 1, \dots, 4. \tag{3.40}$$

Lemma 3.10 For the solution of the problems

$$q_h = Aq_h + h^{10} \ on \ \Pi^h, q_h = 0 \ on \ \gamma_h,$$

the following inequality holds:

$$q_h \leq 2\rho d h^8 \ on \ \Pi^h,$$

where $d = max\{a, b\}, \rho = \rho(x, y)$ is the distance from the current point $(x, y) \in \Pi^h$ to the boundary of the rectangle Π^h.

Proof. We consider the functions

$$\bar{q}_h^{(1)}(x, y) = 2h^8(ax - x^2) \geq 0, \quad \bar{q}_h^{(2)}(x, y) = 2h^8(bx - x^2) \geq 0 \ on \ \bar{\Pi}.$$

Let $\bar{q}_h(x, y) = \bar{q}_h^{(1)}(x, y)$, then

$$\begin{aligned} A\bar{q}_h(x_0', y_0') = & \frac{1}{4}(\bar{q}_h(x_0 + h, y_0) + \bar{q}_h(x_0 - h, y_0) + \bar{q}_h(x_0, y_0 + h) \\ & +\bar{q}_h(x_0, y_0 - h)) \\ = & 2h^8(ax_0 + ah - x_0^2 - 2x_0 h - h^2 + ax_0 - ah - x_0^2 \\ & +2x_0 h - h^2 + ax_0 - x_0^2 + ax_0 - x_0^2) \\ = & \frac{h^8}{2}(4ax_0 - 4x_0^2 - 2h^2) \\ = & 2h^8(ax_0 - x_0^2) - h^{10} \\ = & \bar{p}_h(x_0', y_0') - h^8. \end{aligned}$$

Similarly let $\bar{q}_h(x, y) = \bar{q}_h^{(2)}(x, y)$, then

$$\begin{aligned} A\bar{q}_h(x_0', y_0') = & \frac{1}{4}(\bar{q}_h(x_0 + h, y_0) + \bar{q}_h(x_0 - h, y_0) + \bar{q}_h(x_0, y_0 + h) \\ & +\bar{q}_h(x_0, y_0 - h)) \\ = & 2h^8(bx_0 + bh - x_0^2 - 2x_0 h - h^2 + bx_0 - bh - x_0^2 \\ & +2x_0 h - h^2 + bx_0 - x_0^2 + bx_0 - x_0^2) \end{aligned}$$

$$= \frac{h^8}{2}(4bx_0 - 4x_0^2 - 2h^2)$$
$$= 2h^8(bx_0 - x_0^2) - h^{10}$$
$$= \bar{q}_h(x_0', y_0') - h^{10}.$$

which are solutions of the equation $q_h = Aq_h + h^8 \; on \; \bar{\Pi}^h$. By virtue of Lemma 3.5 we obtain

$$q_h \leq \min_{i=1,2} \bar{q}_h^i(x, y) \leq 2\rho d h^8 \; on \; \bar{\Pi}^h. \blacksquare$$

Theorem 3.1 On Π^h, it holds that

$$\max_{(x,y)\in\Pi^h} |u_h - u| \leq ch^8\rho, \tag{3.41}$$

where u is the trace of the exact solution of problem (3.1) on Π^h, u_h is a solution to system (3.40), ρ is the distance from the current grid point to the boundary of the rectangle.

Proof. On the basis of Lemma 3.6 and Lemma 3.9 by using Taylor's formula, we have

$$u(x, y) = Au(x, y) - \left(\frac{h^4}{4!}\right)u^4(x, y) - \left(\frac{h^8}{8!}\right)u^8(x, y) - r(x, y) \tag{3.42}$$

where $(x, y) \in \Pi^h$, u is the solution to Dirichlet problem (3.1), u^4 and u^8 are the functions defined in (3.16) and (3.28),

$$r(x, y) = \left(\frac{h^{12}}{2\times 12!}\right)\left(\left(\frac{\partial^{12}u(x+\theta_1, y)}{\partial x^{12}}\right) + \left(\frac{\partial^{12}u(x, y+\theta_2)}{\partial y^{12}}\right)\right), |\theta_i| < 1, i = 1,2, \tag{3.43}$$

and by Lemma 3.2

$$|r(x, y)| \leq c_1 h^{12}. \tag{3.44}$$

We put

$$\varepsilon_h = u_h - u \ \ on \ \ \Pi^h \tag{3.45}$$

where u is the solution to Dirichlet problem (3.1) and u_h is the solution of the system (3.40).

From (3.40), (3.42) and (3.45), we have

$$\varepsilon_h = \quad A\varepsilon_h + \left(\frac{h^4}{4!}\right)[u_h^4(x,y) - u^4(x,y)] + \left(\frac{h^8}{8!}\right)[u_h^8(x,y) - u^8(x,y)]$$
$$+r \ \ on \ \Pi^h, \varepsilon_h = 0 \ \ on \ \gamma_j^h. \tag{3.46}$$

By virtue of Lemma 3.6 and Lemma 3.9 from (3.44) follows

$$\left(\frac{h^4}{4!}\right)|u^4 - u_h^4| + \left(\frac{h^8}{8!}\right)\left|u^8 - u_h^8\right| + |r| \le c_2 h^{10} \ on \ \Pi^h. \tag{3.47}$$

Now, from (3.46) and (3.47) we have

$$\bar{\varepsilon}_h = A\bar{\varepsilon}_h + c_3 h^{10} \ on \ \Pi^h,$$

$$\bar{\varepsilon}_h = 0 \ on \ \gamma^h.$$

Then, by Lemma 3.3, Lemma 3.6 and Lemma 3.9, we obtain

$$|\varepsilon_h| \le |\bar{\varepsilon}_h| \le 2c_4 d h^8 \rho \qquad \text{or} \qquad |\varepsilon_h| \le c h^8 \rho,$$

where $c = 2c_4 d, d = \max\{a, b\}$, and ρ is the distance from the current point $(x, y)\epsilon\overline{\Pi}^h$ to the boundary of the rectangle Π. ■

3.4 Establish Of The First Derivative Problem

We denote $\chi_j = \left(\frac{\partial u}{\partial x}\right)$ on $\gamma_j, j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v = 0 \ on\ \Pi, v = \chi_j \ on\ \gamma_j, j = 1,2,3,4, \quad (3.48)$$

where u is the solution of the boundary value problem (3.1).

On Π, we introduce the functions $v^m, m = 4,8\ as$

$$v^m = v^m(x,y) = \left(\frac{1}{2}\right)\left[\left(\frac{\partial^m v(x,y)}{\partial x^m}\right) + \left(\frac{\partial^m v(x,y)}{\partial y^m}\right)\right], \quad (3.49)$$

where v is the solution to Dirichlet problem (3.48).

We put $\chi_j^m = v^m(x,y)$ on $\gamma_j, j = 1,2,3,4$. It is easy to check that the functions (3.49) are unique bounded solution of the following boundary value problem

$$\Delta v^m = 0 \ on\ \Pi, \quad (3.50)$$

$$v^m = \chi_j^m \ on\ \gamma_j, \qquad j = 1,2,3,4; \quad m = 4,8. \quad (3.51)$$

From (3.2), (3.3), (3.49)-(3.51), and Theorem 3.1 in (Volkov, 1965) follows that the boundary functions $\chi_j^m, j = 1,2,3,4$ satisfy the conditions

$$\chi_j^m \in C^{11-m,\lambda}(\gamma_j), 0 < \lambda < 1, \quad (3.52)$$

$$\left(\chi_j^m\right)^{2q}(s_j) = (-1)^q\left(\chi_{j-1}^m\right)^{2q}, m = 4,8;\ q = 0,1,\dots,(m/2). \quad (3.53)$$

On the basis of Theorem 3.1 in (E.A. Volkov, 1965) by taking (3.52) and (3.53) into account, it follows that $v^m \in C^{11-m,\lambda}(\Pi), m = 4,8$.

3.5 3 Stage Method for the First Derivative

Let v be a solution of the following finite difference problem

$$v = Av \ on \ \Pi^h, v_h = \chi_j \ on \ \Gamma_{jh}, \qquad j = 1,2,3,4,$$

where $\chi_j, j = 1,2,3,4,$ are functions $(3.48).$ A is five point operator defined by (3.10).

$$Av(x,y) = \frac{1}{4}\big(v(x+h,y) + v(x-h,y) + v(x,y+h) + v(x,y-h)\big)$$

Taking into account that the function v is harmonic, by axhaustive calculations, we have

$$Av = v + \frac{1}{2}\left\{\frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4}\right) + \frac{h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6} + \frac{\partial^6 v}{\partial y^6}\right)\right.$$
$$\left. + \frac{h^8}{8!}\left(\frac{\partial^8 v}{\partial x^8} + \frac{\partial^8 v}{\partial y^8}\right) + \frac{1}{10!}O(h^{10})\right\}.$$

Since u is harmonic so $v = \frac{\partial u}{\partial x}$ is also harmonic, then

$$\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \text{ and } \left(\frac{\partial^6 v}{\partial x^6} + \frac{\partial^6 v}{\partial y^6}\right) = 0.$$

Then we have,

$$Av = v + \frac{1}{2}\left\{\frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4}\right) + \frac{h^8}{8!}\left(\frac{\partial^8 v}{\partial x^8} + \frac{\partial^8 v}{\partial y^8}\right) + \frac{1}{10!}O(h^{10})\right\}.$$

3.5.1 First stage for the first derivative

On γ_h, set

$$v^4 = v^4(x,y) = \frac{1}{2}\left(\frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4}\right), \qquad (3.54)$$

$Let\ v = \left(\frac{\partial u}{\partial x}\right)$ and put equation (3.54), we have

$$v^4 = v^4(x,y) = \frac{1}{2}\left(\frac{\partial^4}{\partial x^4}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^4}{\partial y^4}\left(\frac{\partial u}{\partial x}\right)\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)$$

$$= \frac{\partial}{\partial x} u^4 \qquad (3.55)$$

where u^4 is the solution to Dirichlet problem (3.17).

We denote ${\chi_j}^4 = v^4\ on\ \gamma_j, j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v^4 = 0\ on\ \Pi, v^4 = \chi_j^4 on\ \gamma_j, j = 1,2,3,4, \qquad (3.56)$$

$$\chi_j^4 \in C^{7,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \qquad (3.57)$$

$$\left(\chi_j^4\right)^{2q}(s_j) = (-1)^q\left(\chi_{j-1}^4\right)^{2q}(s_j), q = 0,1,2,3. \qquad (3.58)$$

where v^4 is the solution of the boundary value problem (3.55).

We put

$$\chi_{1h}^4(u_h^4) = \left(\frac{1}{60h}\right)[-147\Psi_1^4(0,y) + 360u_h^4(h,y)$$

$$- 450u_h^4(2h,y) + 400u_h^4(3h,y) - 225u_h^4(4h,y)$$

$$+72u_h^4(5h,y) - 10u_h^4(5h,y)]\ on\ \gamma_1^h, \qquad (3.59)$$

$$\chi_{3h}^4(u_h^4) = \left(\frac{1}{60h}\right)[147\Psi_3^4(a,y) - 360u_h^4(a-h,y)$$

$$+450u_h^4(a-2h,y) - 400u_h^4(a-3h,y)$$

$$+225u_h^4(a-4h,y) - 72u_h^4(a-5h,y)$$

$$+10u_h^4(a-6h,y)]\ \ on\ \gamma_3^h \qquad (3.60)$$

$$\chi_{ph}^4(u_h^4)=\left(\frac{\partial\Psi_p^4}{\partial x}\right)\ on\ \gamma_p^h, p=2,4, \tag{3.61}$$

where u_h^4 is the solution of the finite difference boundary value problem (3.19) and $\Psi_j^4,\ j=1,2,3,4$ are boundary functions of problem (3.20).

Lemma 3.11 The following inequality is true

$$|\chi_{kh}^4(u_h^4)-\chi_{kh}(u^4)|\leq ch^6, k=1,3,$$

where u_h^4 is the solution of the problem (3.19), u^4 is the solution of problem (3.17).

Proof. On the basis of (3.59), (3.60) and Lemma 3.6, if $k=1$, we have

$$|\chi_{1h}^4(u_h^4)-\chi_{1h}^4(u^4)|\ \leq\ \left(\frac{1}{60h}\right)(360|u_h^4(h,y)-u^4(h,y)|$$

$$+450|u_h^4(2h,y)-u^4(2h,y)|+400|u_h^4(3h,y)-u^4(3h,y)|$$

$$+225|u_h^4(4h,y)-u^4(4h,y)|+72|u_h^4(5h,y)-u^4(5h,y)|$$

$$+10|u_h^4(6h,y)-u^4(6h,y)|)$$

$$\leq\ \left(\frac{1}{60h}\right)(360(ch)h^6+450(c2h)h^6+400(c3h)h^6$$

$$+225(c4h)h^6+72(c5h)h^6+10(c6h)h^6)$$

$$\leq\ \left(\frac{1}{60h}\right)(3780c_1h^7)\leq c_2h^6.$$

Similarly if $k=3$,

$$|\chi_{3h}^4(u_h^4)-\chi_{3h}^4(u^4)|\ \leq\ \left(\frac{1}{60h}\right)(360|u_h^4(a-h,y)-u^4(a-h,y)|$$

$$+450|u_h^4(a-2h,y)-u^4(a-2h,y)|+400|u_h^4(a-3h,y)-u^4(a-3h,y)|$$

$+225|u_h^4(a-4h,y)-u^4(a-4h,y)|+72|u_h^4(a-5h,y)-u^4(a-5h,y)|$

$+10|u_h^4(a-6h,y)-u^4(a-6h,y)|)$

$$\leq \left(\frac{1}{60h}\right)(360(ch)h^6+450(c2h)h^6+400(c3h)h^6$$

$$+225(c4h)h^6+72(c5h)h^6+10(c6h)h^6)$$

$$\leq \left(\frac{1}{60h}\right)(3780c_3h^7) \leq c_4h^6.$$

Hence

$$|\chi_{kh}^4(u_h^4)-\chi_{kh}(u^4)| \leq ch^6, k=1,3, \qquad \blacksquare$$

Lemma 3.12 The following inequality holds

$$\max_{(x,y)\in\gamma_k^h} |\chi_{kh}^4(u_h^4)-\chi_{kh}^4| \leq ch^6, k=1,3. \qquad (3.62)$$

Proof. From Lemma 3.1 it follows that $u_h^4 \in C^{7,\lambda}(\overline{\Pi})$. Then, at the and points $(0,vh) \in \gamma_1^h$ and $(a,vh) \in \gamma_3^h$ of each line segment $\{(x,y): 0 \leq x \leq a, 0 \leq y = vh \leq b\}$, (3.59) and (3.60) give the sixth order approximation of $\left(\frac{\partial u_h^4}{\partial x}\right)$, respectively. From the truncation error formulas (Burden and Douglas, 2011) it follows that

$$\frac{\max_{(x,y)\in\gamma_k^h}\left|\frac{\partial^7 v^4}{\partial x^7}\right|}{(n+1)!}\prod_{\substack{k=0\\k\neq j}}^{n}(x_j-x_k)$$

Hence,

$$\max_{(x,y)\in\gamma_k^h}|\varphi_{kh}^4(u^4)-\varphi_k^4| \leq \frac{\max_{(x,y)\in\gamma_k^h}\left|\frac{\partial^7 u^4}{\partial x^7}\right|}{7!}h(2h)(3h)(4h)(5h)(6h)$$

$$= \quad \frac{h^6}{7} \max_{(x,y)\in\Pi} \left|\frac{\partial^7 u^4}{\partial x^7}\right|$$

$$\leq \quad c_1 h^6, \quad k = 1,3. \tag{3.63}$$

On the basis of Lemma 3.11 and estimation (3.63) follows (3.62),

$$\max_{(x,y)\in\gamma_k^h} |\varphi_{kh}^4(u_h^4) - \varphi_k^4| = \max_{(x,y)\in\gamma_k^h} |\varphi_{kh}^4(u_h^4) - \varphi_{kh}^4(u^4) + \varphi_{kh}^4(u^4) - \varphi_k^4|$$

$$\leq \max_{(x,y)\in\gamma_k^h} |\varphi_{kh}^4(u_h^4) - \varphi_{kh}^4(u^4)|$$

$$+ \max_{(x,y)\in\gamma_k^h} |\varphi_{kh}^4(u^4) - \varphi_k^4|$$

$$\leq c_2 h^6 + c_3 h^6$$

$$\leq ch^6. \ \blacksquare$$

We consider the finite difference boundary value problem

$$v_h^4 = Bv_h^4 \ on \ \Pi^h, v_h^4 = \chi_{jh}^4 \ on \ \gamma_j^h, j = 1,2,3,4, \tag{3.64}$$

where $\chi_{jh}^4, j = 1,2,3,4$, are defined by (3.59)-(3.61).

Lemma 3.13 The estimation is true

$$\max_{(x,y)\in\overline{\Pi}^h} \left|v_h^4 - \left(\frac{\partial u^4}{\partial x}\right)\right| \leq ch^6, \tag{3.65}$$

where v^4 is the solution of problem (3.56) and v_h^4 is the solution of the finite difference problem (3.64).

Proof. Let

$$\varepsilon_h = v_h^4 - v^4 on \ \Pi, \tag{3.66}$$

where $v^4 = \left(\frac{\partial u^4}{\partial x}\right)$. From (3.64) and (3.66), we have

$$\varepsilon_h = B\varepsilon_h + (Bv^4 - v^4) \ on \ \Pi^h,$$
$$\varepsilon_h = \varphi_{kh}^4(u^4) - v^4 on \ \gamma_k^h, \ \ k = 1,3,$$
$$\varepsilon_h = 0 \ on \ \gamma_p^h \ , p = 2,4.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{3.67}$$

where

$$\varepsilon_h^1 = B\varepsilon_h^1 \tag{3.68}$$

$$\varepsilon_h^1 = \varphi_{kh}^4(u^4) - v^4 on \ \gamma_k^h, \ k = 1,3, \ \ \varepsilon_h^1 = 0 \ on \ \gamma_p^h, \ p = 2,4, \tag{3.69}$$

$$\varepsilon_h^2 = B\varepsilon_h^2 + (Bv^4 - v^4) on \ \Pi^h, \ \ \varepsilon_h^2 = 0 \ on \ \gamma_j^h \ , j = 1,2,3,4. \tag{3.70}$$

By Lemma 3.12 and by the maximum principle, for the solution of system (3.68), (3.69), we have

$$\max_{(x,y)\in\bar{\Pi}} |\varepsilon_h^1| \le \max_{k=1,3} \max_{(x,y)\in\gamma_q^h} |\varphi_{kh}^4(u_h^4) - v^4| \le ch^6. \tag{3.71}$$

The solution ε_h^2 of system (3.70) is the error of the approximate solution obtained by the finite difference method of problem (3.56), when the boundary values satisfy the conditions below;

$$\varphi_j^4 \in C^{7,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \tag{3.72}$$

$$\left(\varphi_j^4\right)^{(2q)}(s_j) = (-1)^q \left(\varphi_{j-1}^4\right)^{(2q)}(s_j), q = 0,1,2. \tag{3.73}$$

Since the function $v^4 = \left(\frac{\partial u^4}{\partial x}\right)$ is harmonic on Π with the boundary functions $\varphi_j^4, j = 1,2,3,4$, on the basis of (3.72), (3.73), and Theorem 12 in (Dosiyev, 2003), we have

$$\max_{(x,y)\in\bar{\Pi}^h} |\varepsilon_h^2| \le ch^6. \tag{3.74}$$

By (3.67),(3.71) and (3.74) follows the proof of Lemma 3.13. ■

3.5.2 Second stage for the first derivative

On γ_h, set

$$v^8 = v^8(x,y) = \frac{1}{2}\left(\frac{\partial^8 v}{\partial x^8} + \frac{\partial^8 v}{\partial y^8}\right), \tag{3.75}$$

$Let\ v = \left(\frac{\partial u}{\partial x}\right)$ and substitute it to (3.75), we have

$$v^8 = v^8(x,y) = \frac{1}{2}\left(\frac{\partial^8}{\partial x^8}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^8}{\partial y^8}\left(\frac{\partial u}{\partial x}\right)\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^8 u}{\partial x^8} + \frac{\partial^8 u}{\partial y^8}\right) = \frac{\partial}{\partial x}u^8 \tag{3.76}$$

where u^8 is the solution of the Dirichlet problem (3.29).

We denote $\chi_j^8 = v^8\ on\ \gamma_j, j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v^8 = 0\ on\ \Pi, \quad v^8 = \chi_j^8 on\ \gamma_j, \ \ j = 1,2,3,4, \tag{3.77}$$

$$\chi_j^8 \in C^{3,\lambda}(\gamma_j), \qquad 0 < \lambda < 1, \qquad j = 1,2,3,4, \tag{3.78}$$

$$\left(\chi_j^8\right)^{2q}(s_j) = (-1)^q\left(\chi_{j-1}^8\right)^{2q}(s_j),\ q = 0,1. \tag{3.79}$$

where v^8 is the solution of the boundary value problem (3.76).

We put

$$\chi_{1h}^8(u_h^8) = \left(\frac{1}{2h}\right)[-3\Psi_1^8(0,y) + 4u_h^8(h,y) - u_h^8(2h,y)]on\ \gamma_1^h\ , \tag{3.80}$$

$$\chi_{3h}^8(u_h^8) = \left(\frac{1}{2h}\right)[3\Psi_3^8(a,y) - 4u_h^4(a-h,y)$$

$$+u_h^8(a-2h,y)]\ \ on\ \gamma_3^h, \tag{3.81}$$

$$\chi^8_{ph}(u^8_h) = \left(\frac{\partial \Psi^8_p}{\partial x}\right) \text{ on } \gamma^h_p, p = 2,4, \quad (3.82)$$

where u^8_h is the solution of the finite difference boundary value problem (3.31) and $\Psi^8_j,\ j = 1,2,3,4$ are boundary functions of problem (3.30).

Lemma 3.14 The following inequality is true

$$|\chi^8_{kh}(u^8_h) - \chi_{kh}(u^8)| \leq ch^2, k = 1,3,$$

where u^8_h is the solution of the problem (3.31) and u^8 is the solution of problem (3.29).

Proof. On the basis of (3.80), (3.81) and Lemma 3.13, if $k = 1$, we have

$$\begin{aligned}
|\chi^8_{1h}(u^8_h) - \chi^8_{1h}(u^8)| &\leq \left(\frac{1}{2h}\right)(4\left|u^8_h(h,y) - u^8(h,y)\right| \\
&\quad + \left|u^8_h(2h,y) - u^8(2h,y)\right|) \\
&\leq \left(\frac{1}{2h}\right)(4(ch)h^2 + (c2h)h^2) \\
&\leq \left(\frac{1}{2h}\right)(6c_1h^3) \\
&\leq c_2h^2.
\end{aligned}$$

Similarly if $k = 3$,

$$\begin{aligned}
|\chi^8_{3h}(u^8_h) - \chi^8_{3h}(u^8)| &\leq \left(\frac{1}{2h}\right)(4\left|u^8_h(a-h,y) - u^8(a-h,y)\right| \\
&\quad + \left|u^8_h(a-2h,y) - u^8(a-2h,y)\right|) \\
&\leq \left(\frac{1}{2h}\right)(4(ch)h^2 + (c2h)h^2) \\
&\leq \left(\frac{1}{2h}\right)(6c_3h^3) \\
&\leq c_4h^2.
\end{aligned}$$

Hence

$$\left|\chi_{kh}^{8}(u_h^{8}) - \chi_{kh}(u^{8})\right| \leq ch^2, k = 1,3, \qquad \blacksquare$$

Lemma 3.15 The following inequality holds

$$\max_{(x,y)\in\gamma_k^h} |\chi_{kh}^{8}(u_h^{8}) - \chi_{kh}^{8}| \leq ch^2, k = 1,3. \qquad (3.83)$$

Proof. From Lemma 3.1 it follows that $u_h^8 \in C^{3,\lambda}(\overline{\Pi})$. Then, at the and points $(0, vh) \in \gamma_1^h$ and $(a, vh) \in \gamma_3^h$ of each line segment $\{(x, y): 0 \leq x \leq a, 0 \leq y = vh \leq b\}$, (3.80) and (3.81) give the second order approximation of $\left(\frac{\partial u_h^8}{\partial x}\right)$, respectively. From the truncation error formulas (Burden and Douglas, 2011) it follows that

$$\frac{\max_{(x,y)\in\gamma_k^h}\left|\frac{\partial^3 v^8}{\partial x^3}\right|}{(n+1)!}\prod_{\substack{k=0\\k\neq j}}^{n}\left(x_j - x_k\right)$$

Hence,

$$\begin{aligned}\max_{(x,y)\in\gamma_k^h}|\varphi_{kh}^{8}(u^{8}) - \varphi_k^{8}| \leq & \frac{\max_{(x,y)\in\gamma_k^h}\left|\frac{\partial^3 u^8}{\partial x^8}\right|}{3!} h(2h) \\ = & \frac{h^2}{3}\max_{(x,y)\in\Pi}\left|\frac{\partial^3 u^8}{\partial x^3}\right| \\ \leq & c_1 h^2, \quad k = 1,3. \qquad (3.84)\end{aligned}$$

On the basis of Lemma 3.14 and estimation (3.84) follows (3.83),

$$\max_{(x,y)\in\gamma_k^h}\left|\varphi_{kh}^8(u_h^8)-\varphi_k^8\right| = \max_{(x,y)\in\gamma_k^h}\left|\varphi_{kh}^8(u_h^8)-\varphi_{kh}^8(u^8)+\varphi_{kh}^8(u^8)-\varphi_k^8\right|$$

$$\leq \max_{(x,y)\in\gamma_k^h}\left|\varphi_{kh}^8(u_h^8)-\varphi_{kh}^8(u^8)\right|$$

$$+\max_{(x,y)\in\gamma_k^h}\left|\varphi_{kh}^8(u^8)-\varphi_k^8\right|$$

$$\leq c_2h^2+c_3h^2$$

$$\leq ch^2. \blacksquare$$

We consider the finite difference boundary value problem

$$v_h^8 = Av_h^8 \; on \; \Pi^h, v_h^8 = \chi_{jh}^8 \; on \; \gamma_j^h, j = 1,2,3,4, \tag{3.85}$$

where $\chi_{jh}^8, j = 1,2,3,4$, are defined by (3.80)-(3.82).

Lemma 3.16 The following estimation is true

$$\max_{(x,y)\in\overline{\Pi}^h}\left|v_h^8-\left(\frac{\partial u^8}{\partial x}\right)\right| \leq ch^2, \tag{3.86}$$

where v^8 is the solution of problem (3.77) and v_h^8 is the solution of the finite difference problem (3.85).

Proof. Let

$$\varepsilon_h = v_h^8 - v^8 on \; \Pi, \tag{3.87}$$

where $v^8 = \left(\frac{\partial u^8}{\partial x}\right)$. From (3.85) and (3.87), we have

$$\varepsilon_h \; = \; A\varepsilon_h + (Av^8 - v^8) \; on \; \Pi^h,$$

$$\varepsilon_h = \varphi_{kh}^8(u^8) - v^8 on \; \gamma_k^h, \;\; k = 1,3,$$

$$\varepsilon_h = \; 0 \; on \; \gamma_p^h \; , p = 2,4.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \quad (3.88)$$

where

$$\varepsilon_h^1 = A\varepsilon_h^1 \quad (3.89)$$

$$\varepsilon_h^1 = \varphi_{kh}^8(u^8) - v^8 on\, \gamma_k^h,\ k = 1,3,\ \ \varepsilon_h^1 = 0\ on\, \gamma_p^h,\ p = 2,4, \quad (3.90)$$

$$\varepsilon_h^2 = A\varepsilon_h^2 + (Av^8 - v^8) on\, \Pi^h,\ \ \varepsilon_h^2 = 0\ on\, \gamma_j^h, j = 1,2,3,4. \quad (3.91)$$

By Lemma 3.15 and by the maximum principle, for the solution of system (3.89), (3.90), we have

$$\max_{(x,y)\in\overline{\Pi}} |\varepsilon_h^1| \le \max_{k=1,3} \max_{(x,y)\in\gamma_q^h} \left|\varphi_{kh}^8(u_h^8) - v^8\right| \le ch^2. \quad (3.92)$$

The solution ε_h^2 of system (3.91) is the error of the approximate solution obtained by the finite difference method of problem (3.77), when the boundary values satisfy the conditions below

$$\varphi_j^8 \in C^{3,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \quad (3.93)$$

$$\left(\varphi_j^8\right)^{(2q)}\left(s_j\right) = (-1)^q \left(\varphi_{j-1}^8\right)^{(2q)}\left(s_j\right), q = 0,1. \quad (3.94)$$

Since the function $v^8 = \left(\frac{\partial u^8}{\partial x}\right)$ is harmonic on Π with the boundary functions $\varphi_j^8, j = 1,2,3,4$, on the basis of (3.93), (3.94), and Theorem 2.1 in (Volkov, 1976), we have

$$\max_{(x,y)\in\overline{\Pi}^h} |\varepsilon_h^2| \le ch^2. \quad (3.95)$$

By (3.88),(3.92) and (3.95) follows the proof of Lemma 3.16. ■

3.5.3 Third stage for the first derivative

Let the boundary functions $\varphi_j, j = 1,2,\ldots,6$, in problem (3.1) on the sides γ_j be satisfied by the conditions

$$\varphi_j \in C^{12,\lambda}\left(\gamma_j\right), 0 < \lambda < 1, j = 1,2,\ldots,4. \quad (3.96)$$

Let u be a solution of the problem (3.1) with the conditions (3.2) and (3.3). We give the following Lemmas and Theorem related to the function u.

Let $v = (\frac{\partial u}{\partial x_1})$ and let $\chi_j = \left(\frac{\partial u}{\partial x_1}\right)$ on $\gamma_j, j = 1,2,\ldots,4$, and consider the boundary value problem:

$$\Delta v = 0 \ on \ \Pi, v = \chi_j \ on \ \gamma_j, j = 1,2,\ldots,4, \quad (3.97)$$

where u is a solution of the boundary value problem (3.1).

We define the following operators $\chi_{ph}, \ p = 1,2,\ldots,4$,

$$\begin{aligned} \chi_{1h}(u_h) \ = \ & \left(\frac{1}{840h}\right)[-2283\varphi_1(0,y) + 6720u_h(h,y) \\ & -11760u_h(2h,y) + 15680u_h(3h,y) \\ & -14700u_h(4h,y) + 9408u_h(5h,y) \\ & -3920u_h(6h,y) + 960u_h(7h,y) \\ & -105u_h(8h,y)\ ] \ on \ \gamma_1^h, \end{aligned} \quad (3.98)$$

$$\chi_{3h}(u_h) = \left(\frac{1}{840h}\right)[2283\varphi_3(a,y) - 6720u_h(a-h,y)$$

$$+11760u_h(a-2h,y) - 15680u_h(a-3h,y)$$

$$+14700u_h(a-4h,y) - 9408u_h(a-5h,y)$$

$$+3920u_h(a-6h,y) - 960u_h(a-7h,y)$$

$$+105u_h(a-8h,y)\,]\ on\ \gamma_3^h\,, \qquad (3.99)$$

$$\chi_{ph}(u_h) = \left(\frac{\partial\varphi_p}{\partial x}\right)\ on\ \gamma_p^h, p = 2,4, \qquad (3.100)$$

where u_h is the solution of the finite difference problem (3.47).

Lemma 3.17 The following inequality is true

$$|\chi_{kh}(u_h) - \chi_{kh}(u)| \le ch^8,\ \ k = 1,4, \qquad (3.101)$$

where u_h is the solution of the finite difference problem (3.40), u is the solution of problem (3.1).

Proof: It is obvious that $\chi_{ph}(u_h) - \chi_{ph}(u) = 0$ for $p = 2,3$. For $k = 1$, by (3.98) and Theorem 3.1, we have

$$|\chi_{1h}(u_h) - \chi_{1h}(u)| \le \left(\frac{1}{840h}\right)\{6720|u_h(h,y) - u(h,y)|$$

$$+11760|u_h(2h,y) - u(2h,y)| + 15680|u_h(3h,y) - u(3h,y)|$$

$$+14700|u_h(4h,y) - u(4h,y)| + 9408|u_h(5h,y) - u(5h,y)|$$

$$+3920|u_h(6h,y) - u(6h,y)| + 960|u_h(7h,y) - u(7h,y)|$$

$$\leq \left(\frac{1}{840h}\right)[6720(ch)h^8 + 11760(c2h)h^8$$

$$+15680(c3h)h^8 + 14700(c4h)h^8 + 9408(c5h)h^8 + 3920(c6h)h^8$$

$$+960(c7h)h^8 + 105(c8h)h^8]$$

$$\leq \left(\frac{1}{840h}\right).c.214200.h^9$$

$$= c_1 213361h^8$$

$$= c_2 h^8.$$

It is shown the same inequality is true when $k = 3$ also.

$$|\chi_{3h}(u_h) - \chi_{3h}(u)| \leq \left(\frac{1}{840h}\right)\{6720|u_h(a-h,y) - u(a-h,y)|$$

$$+11760|u_h(a-2h,y) - u(a-2h,y)| + 15680|u_h(a-3h,y) - u(a-3h,y)|$$

$$+14700|u_h(a-4h,y) - u(a-4h,y)| + 9408|u_h(a-5h,y) - u(a-5h,y)|$$

$$+3920|u_h(a-6h,y) - u(a-6h,y)| + 960|u_h(a-7h,y) - u(a-7h,y)|$$

$$+105|u_h(a-8h,y) - u(a-8h,y)|\}$$

$$\leq \left(\frac{1}{840h}\right)[6720(ch)h^8 + 11760(c2h)h^8$$

$$+15680(c3h)h^8 + 14700(c4h)h^8 + 9408(c5h)h^8 + 3920(c6h)h^8$$

$$+960(c7h)h^8 + 105(c8h)h^8]$$

$$\leq \left(\frac{1}{840h}\right).c.214200.h^9 = c_2 h^8. \blacksquare$$

Lemma 3.18 The following inequality holds

$$\max_{(x,y)\in\Pi_k^h} |\chi_{kh}(u_h) - \chi_k| \leq ch^8, \; k = 1,3, \tag{3.102}$$

where $\chi_{kh}, k = 1,3$ are defined by (3.98), (3.99), and $\chi_k = \left(\frac{\partial u}{\partial x_1}\right) \; on \; \Pi_k, k = 1,3.$

Proof. From Lemma 3.1 it follows that $u \in C^{11,\lambda}(R)$. Then, at the end points $(0, \nu h) \in \gamma_1^h$ and $(a, \nu h) \in \gamma_3^h$ of each line segment $\{(x, y): 0 \leq x \leq a, 0 \leq y = \nu h < b\}$, expressions (3.98) and (3.99) give the eighth order approximation of $\left(\frac{\partial u}{\partial x_1}\right)$, respectively. From the truncation error formulas it (Burden and Douglas, 2011) follows that

$$\begin{aligned}\max_{(x,y)\in\Pi_k^h} |\chi(u) - \chi_k| &\leq c_1 \left[\frac{h^8}{9}\right] \max_{(x,y)\in\Pi_k^h} \left|\frac{\partial^9 u}{\partial x^9}\right| \\ &\leq c_2 h^8, \; k = 1,3. \end{aligned} \tag{3.103}$$

On the basis of Lemma 3.17 and estimation (3.101), (3.103) holds,

$$\begin{aligned}\max_{(x,y)\in\Pi_k^h} |\chi_{kh}(u_h) - \chi_k| = &\max_{(x,y)\in\Pi_k^h} |\chi_{kh}(u_h) - \chi_{kh}(u) + \chi_{kh}(u) - \chi_k| \\ \leq &\max_{(x,y)\in\Pi_k^h} |\chi_{kh}(u_h) - \chi_{kh}(u)| \\ &+ \max_{(x,y)\in\Pi_k^h} |\chi_{kh}(u) - \chi_k| \\ \leq &\; c_3 h^8 + c_4 h^8 \\ \leq &\; c_5 h^8 \end{aligned}$$

■

Let ν_h be the solution of the following finite difference problem

$$\nu_h = A\nu_h \; on \; \Pi_h, \nu_h = \chi_{jh} \; on \; \Pi_j^h, j = 1,2, \ldots, 4, \tag{3.104}$$

where $\chi_{jh}, \; j = 1,2, \ldots, 4$, are defined by (3.98)-(3.100).

Let v_h^4 and v_h^8 be the solution of the difference problem (3.64) and (3.85) respectively.

We approximate the solution $v = \left(\frac{\partial u}{\partial x}\right)$ of the Dirichlet problem (3.97) on the grid Π^h as a solution v_h of the following difference problem

$$\left.\begin{aligned} v_h &= Av_h - \left(\frac{h^4}{4!}\right)v_h^4 - \left(\frac{h^8}{8!}\right)v_h^8 \ \ on\ \Pi^h, \\ v_h &= \chi_j \ \ on\ \gamma_j^h, j = 1, \dots 4, \end{aligned}\right\} \tag{3.105}$$

where v_h^4 and v_h^8 are the solution of problem (3.64) and (3.85), respectively.

Theorem 3.2 The following estimation is true

$$\max_{(x,y)\in\Pi_k^h}\left|v_h - \left(\frac{\partial u}{\partial x_1}\right)\right| \leq ch^8, \tag{3.106}$$

where u is the solution of the problem (3.1) and v_h is the solution of the finite difference problem (3.105).

Proof. On the basis of (3.10), (3.49) and Taylor's formula, for the solution of problem (3.48) at any grid $(x, y) \in \Pi^h$, we have

$$v(x,y) = Av(x,y) - \left(\frac{h^4}{4!}\right)v^4(x,y) - \left(\frac{h^8}{8!}\right)v^8(x,y) - r(x,y), \tag{3.107}$$

where

$$r(x,y) = \left(\frac{h^{10}}{2 \times 10!}\right)\left(\left(\frac{\partial^{10}v(x+\theta_1,y)}{\partial x^{10}}\right) + \left(\frac{\partial^{10}v(x,y+\theta_2)}{\partial y^{10}}\right)\right),$$

$$|\theta_i| < 1, i = 1,2. \tag{3.108}$$

By Lemma 3.1

$$|r(x,y)| \leq ch^{10}. \tag{3.109}$$

We put

$$\varepsilon_h = v_h - v \ \ on \ \ \Pi^h \tag{3.110}$$

where v is the solution to Dirichlet problem (3.48) and v_h is the solution of the system (3.104).

From (3.105), (3.107) and (3.110), we have

$$\varepsilon_h = A\varepsilon_h + \left(\frac{h^4}{4!}\right)(v_h^4(x,y) - v^4(x,y))$$

$$+\left(\frac{h^8}{8!}\right)(v_h^8(x,y) - v^8(x,y)) + r \ \ on \ \Pi^h, \tag{3.111}$$

$$\varepsilon_h = \varphi_{kh}(u) - v \ \ on \ \gamma_k^h, \qquad k = 1,3, \tag{3.112}$$

$$\varepsilon_h = 0 \ on \ \gamma_l^h, \qquad l = 2,4. \tag{3.113}$$

By virtue of (3.102) and Lemma 3.13 and Lemma 3.16 it follows that

$$\left(\frac{h^4}{4!}\right)|v^4 - v_h^4| + \left(\frac{h^8}{8!}\right)\left|v^8 - v_h^8\right| + |r| \leq c_4 h^{10} \ on \ \Pi^h. \tag{3.114}$$

On the basis of (3.109), estimation (3.114) by analogy with the proof of Theorem 3.1 from (3.111)-(3.113), we obtain

$$\max_{(x,y)\in\overline{\Pi}^h} |\varepsilon_h| \leq ch^8. \quad \blacksquare$$

3.6. Approximation of the Second Order Pure Derivatives

We denote by $w = \left(\frac{\partial^2 u}{\partial x^2}\right)$. The function w is harmonic on Π, on the basis of Lemma 3.1 is continuous on Π, and is the solution of the following Dirichlet problem

$$\Delta w = 0 \; on \; \Pi, w = \vartheta_j \; on \; \gamma_j, j = 1,2,\dots,4. \tag{3.115}$$

On Π, we introduce the functions $w^m, m = 4,8$ as

$$w^m = w^m(x,y) = \left(\frac{1}{2}\right)\left[\left(\frac{\partial^m w(x,y)}{\partial x^m}\right) + \left(\frac{\partial^m w(x,y)}{\partial y^m}\right)\right], \tag{3.116}$$

where w is the solution of Dirichlet problem (3.115).

Lemma 3.19 Functions (3.116) coincide with the unique bounded solution to the boundary value problems

$$\Delta w^m = 0 \; on \; \Pi, \tag{3.117}$$

$$w^m = \vartheta_k^m = \vartheta_k^m(y) \; on \; \gamma_k, \qquad k = 1,3, \tag{3.118}$$

$$w^m = \vartheta_l^m = \vartheta_l^m(x) \; on \; \gamma_l, \qquad l = 2,4, \tag{3.119}$$

where

$$\left.\begin{aligned} \vartheta_k^m(y) &= \left(\frac{d^{m+2}\varphi_k}{dy^{m+2}}\right), k = 1,3, \\ \vartheta_l^m(x) &= \left(\frac{d^{m+2}\varphi_l}{dx^{m+2}}\right), l = 2,4 \end{aligned}\right\}; m = 4,8. \tag{3.120}$$

Proof. It is easy to check that the functions (3.116) are harmonic on Π and satisfy the boundary conditions (3.118) and (3.119). From (3.2), (3.3) and (3.120), it follows that the boundary functions $\vartheta_j^m, j = 1, \ldots, 4$ satisfy the conditions

$$\left.\begin{array}{l} \vartheta_j^4 \in C^{6,\lambda}(\gamma_j), \quad 0 < \lambda < 1, \\ \left(\vartheta_j^4\right)^{2q}(s_j) = (-1)^q \left(\vartheta_{j-1}^4\right)^{2q}, q = 0,1,2,3 \end{array}\right\} \tag{3.121}$$

and

$$\left.\begin{array}{l} \vartheta_j^8 \in C^{2,\lambda}(\gamma_j), \ 0 < \lambda < 1, \\ \left(\vartheta_j^8\right)^{2q}(s_j) = (-1)^q \left(\vartheta_{j-1}^8\right)^{2q}, q = 0,1 \end{array}\right\} \tag{3.122}$$

On the basis of Theorem 3.1 in (Volkov, 1965) by taking (3.121) and (3.122) into account, it follows that $w^4 \in C^{6,\lambda}(\overline{\Pi}), w^8 \in C^{2,\lambda}(\overline{\Pi}), 0 < \lambda < 1$. ■

9-point difference approximation, when $m = 4$,

$$w_h^4 = Bw_h^4 \ \ on \ \Pi^h, \qquad w_h^4 = \vartheta_j^4 \ \ on \ \gamma_j^h \cup \gamma_j, \qquad j = 1,2,3,4, \tag{3.123}$$

and the 5-point difference approximation, when $m = 8$,

$$w_h^8 = Aw_h^8 \ \ on \ \Pi^h, \qquad w_h^8 = \vartheta_j^8 \ \ on \ \gamma_j^h \cup \gamma_j, \qquad j = 1,2,3,4, \tag{3.124}$$

By the maximum principle, problems (3.123) and (3.124) have the unique solution.

Lemma 3.20 The following estimation holds

$$\max_{(x,y)\in\Pi^h} |w_h^4 - w^4| \leq ch^6 \tag{3.125}$$

where w_h^4 is the solution of the system (3.123) and w^4 is the trace of exact solution of problem (3.117)-(3.119) when $m = 4$, on Π^h.

Proof. From (3.2), (3.3) and (3.120) it follows that $\vartheta_j^4 \in C^{6,\lambda}(\gamma_j)$. By virtue of the conjugation conditions (3.121), for $m = 4$, at the vertices and on the basis of Theorem 12 in (Dosiyev, 2013) the inequality (3.125) holds. ■

Lemma 3.21 The following estimation holds

$$\max_{(x,y)\in\Pi^h} \left|w_h^8 - w^8\right| \leq ch^2 \tag{3.126}$$

where w_h^8 is the solution of the system (3.124) and w^8 is the trace of exact solution of problem (3.117)-(3.119) when $m = 8$, on Π^h.

Proof. From (3.2), (3.3) and (3.120) it follows that $\vartheta_j^8 \in C^{2,\lambda}(\gamma_j)$. By virtue of the conjugation conditions (3.122), for $m = 8$, at the vertices, on the basis of Theorem 1.1 in (Volkov, 1976) follows the inequality (3.126). ■

Let w_h^4 and w_h^8 be the solution of the difference problem (3.123) and (3.124) respectively. We approximate the solution of the given Dirichlet problem (3.115) on the grid Π^h as a solution w_h of the following difference problem

$$w_h = Aw_h - \left(\frac{h^4}{4!}\right)w_h^4 - \left(\frac{h^8}{8!}\right)w_h^8 \;\; on\ \Pi^h, w_h = \vartheta_j \;\; on\ \gamma_j^h, j = 1,\ldots,4. \tag{3.127}$$

Theorem 3.3 On Π^h, it holds that

$$\max_{(x,y)\in\Pi^h} |w_h - w| \leq ch^8 \tag{3.128}$$

where w is the trace of the exact solution of problem (3.115) on Π^h, w_h is a solution to system (3.128).

Proof. On the basis of Lemma 3.2 and Lemma 3.19 by using Taylor's formula, we have

$$w(x,y) = Aw(x,y) - \left(\frac{h^4}{4!}\right) w^4(x,y) - \left(\frac{h^8}{8!}\right) w^8(x,y) - r(x,y) \quad (3.129)$$

where $(x,y) \in \Pi^h$, w is the solution to Dirichlet problem (3.115), w^4 and w^8 are the functions defined in (3.116),

$$r(x,y) = \left(\frac{h^{10}}{2\times 10!}\right)\left(\left(\frac{\partial^{10} w(x+\emptyset_1, y)}{\partial x^{10}}\right) + \left(\frac{\partial^{10} w(x, y+\emptyset_2)}{\partial y^{10}}\right)\right),$$

$$|\emptyset_i| < 1, i = 1,2, \quad (3.130)$$

and by Lemma 3.2

$$|r(x,y)| \le c_3 h^{10}. \quad (3.131)$$

We put

$$\varepsilon_h = w_h - w \;\; on \;\; \Pi^h \quad (3.132)$$

where w is the solution to Dirichlet problem (3.115) and w_h is the solution of the system (3.128).

From (3.127), (3.129) and (3.132), we have

$$\varepsilon_h = A\varepsilon_h + \left(\frac{h^4}{4!}\right)[w_h^4(x,y) - w^4(x,y)] + \left(\frac{h^8}{8!}\right)[w_h^8(x,y) - w^8(x,y)]$$

$$+r \;\; on \; \Pi^h, \varepsilon_h = 0 \;\; on \; \gamma_j^h. \quad (3.133)$$

By virtue of Lemma 3.20 and Lemma 3.21 from (3.133) it follows that;

$$\left(\frac{h^4}{4!}\right)|w^4 - w_h^4| + \left(\frac{h^8}{8!}\right)|w^8 - w_h^8| + |r| \le c_4 h^{10} \;\; on \; \Pi^h. \quad (3.134)$$

On the basis of (3.133), (3.134) and comparison theorem (see in (see Chapter 4., Section 3 in (Samarskii, 2001), we obtain

$$\max_{(x,y)\in\pi^h} |\varepsilon_h| \leq ch^8. \qquad \blacksquare$$

CHAPTER 4

A HIGHLY ACCURATE DIFFERENCE METHOD FOR APPROXIMATING OF THE FIRST DERIVATIVES OF THE MIXED BOUNDARY VALUE PROBLEMFOR LAPLACE'S EQUATION ON A RECTANGLE

In a rectangular domain, we discuss about an approximation of the first order derivatives for the solution of the mixed boundary value problem. The boundary values on the sides of the rectangle are supposed to have the fourth derivatives satisfying the Hölder condition. On the vertices, besides the continuity condition, the compatibility conditions, which result from the Laplace equation for the second and fourth derivatives of the boundary values, given on the adjacent sides, are also satisfied. Under these conditions for the approximate values of the first derivatives of the solution of mixed boundary problem on a square grid, as the solution of the constructed difference scheme a uniform error estimation of order $O(h^3)$, (h is the grid size) is obtained.

4.1 Finite Difference Approximation

Let $\Pi = \{(x,y): 0 < x < a, 0 < y < b\}$ be rectangle, a/b be rational, $\gamma_j(\gamma_j'), j = 1,2,3,4$, be the sides, including (excluding) the ends, enumerated counterclockwise starting from the side which is located on the $x-$axis $(\gamma^0 \equiv \gamma^4, \gamma^1 \equiv \gamma^5)$. Denote by s the arclength, measured along γ, and by s_j the value of s at the beginning of γ_j and by $\gamma = \bigcup_{j=1}^{4} \gamma_j$, the boundary of Π, by v_j a parameter taking the values 0 or 1, and $\overline{v_j} = 1 - v_j$.

We consider the boundary value problem

$$\Delta u = 0 \ \ on\ \Pi, \qquad (4.1)$$

$$v_j u + \overline{v_j} u_n^{(1)} = v_j \varphi_j + \overline{v_j} \psi_j \ \ on\ \gamma_j, \qquad j = 1,2,3,4, \qquad (4.2)$$

where $u_n^{(1)}$ is the derivative along the inner normal, φ_j and ψ_j are the given functions at the arclength taken along γ,

$$1 \leq \sum_{j=1}^{4} v_j \leq 4, \quad v_1 = 1. \tag{4.3}$$

Definition 4.1 We say that the solution u of the problem (4.1) and (4.2) belongs to $\tilde{C}_{k,\lambda}(\overline{\Pi})$, if

$$v_j\varphi_j + \bar{v}_j\psi_j \epsilon C_{k,\lambda}(\gamma_j), \ \ 0 < \lambda < 1, \qquad j = 1,2,3,4, \tag{4.4}$$

and at the vertices $A_j = \gamma_j \cap \gamma_{j-1}$ the conjugation conditions

$$\begin{aligned} v_j\varphi_j^{2q+\delta_{\tau-2}} + \bar{v}_j\psi_j^{2q+\delta_\tau} = \ & (-1)^{q+\delta_\tau+\delta_{\tau-1}}(v_{j-1}\varphi_{j-1}^{2q+\delta_{\tau-1}} \\ & +\bar{v}_{j-1}\psi_{j-1}^{2q+\delta_\tau}), \end{aligned} \tag{4.5}$$

are satisfied, except may be the case when $q = \frac{k}{2}$ for $\tau = 3$, where $\tau = v_{j-1} + 2v_j$, $\delta_\omega = 1$ for $\omega = 0$; $\delta_\omega = 0$ for $\omega \neq 0, q = 0,1,\dots,Q, Q = \left[\frac{k-\delta_{\tau-1}-\delta_{\tau-2}}{2}\right] - \delta_\tau$.

Let $h > 0$, and $a/h \geq 2, b/h \geq 2$ be integers. We assign Π^h, a square net on Π, with step size h, obtained by the lines $x, y = 0, h, 2h, \dots . Let\ \gamma_j^h$ be a set of nodes on the interior of γ_j , and let

$$\dot{\gamma_j}^h = \gamma_j \cap \gamma_{j+1}, \qquad \gamma^h = \cup\ (\gamma_j^h \cup \dot{\gamma_j}^h), \qquad \overline{\Pi}^h = \Pi^h \cup \gamma^h.$$

We consider the system of finite difference equation (see Dosiyev, 2003)

$$u_h = \ \ Bu_h \text{ on } \Pi^h, \tag{4,6}$$

$$u_h = \ \ \bar{v}_j B_j u_h + E_{jh}(\varphi_j, \psi_j) \text{ on } \gamma_j^h, \tag{4.7}$$

$$u_h = \ \ \bar{v}_j\bar{v}_{j+1}\dot{B}_j u_h + \dot{E_{jh}}(\varphi_j, \varphi_{j+1}, \psi_j, \psi_{j+1}) \text{ on } \dot{\gamma_j}^h, j = 1,2,3,4 \tag{4.8}$$

where

$$Bu(x,y) \equiv \frac{1}{5}\{u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)\}$$
$$+\frac{1}{20}\{u(x+h,y+h)+u(x+h,y-h)+u(x-h,y+h)$$
$$+u(x-h,y-h)\}, \quad (4.9)$$

the operators $B_j,\ E_{jh},\ \dot{B}_j,\ \dot{E}_{jh}$ in the right coordinate system with the axis x_j, directed along γ_{j+1} and the axis y_j, directed along γ_j have the expressions below:

$$B_j u(0,y_j) \equiv \quad (2u(h,y_j)+u(0,y_j+h)+u(0,y_j-h))/5$$
$$+(u(h,y_j+h)+u(h,y_j-h))/10, \quad (4.10)$$

$$E_{jh}(\varphi_j,\psi_j) \equiv \quad v_j\varphi_j - \bar{v}_j\left(\frac{3h}{5}\psi_j - \frac{2h^5}{5!\,5}\psi_j^{(4)}\right), \quad (4.11)$$

$$\dot{B}_j(0,0) \equiv \quad (2u(h,0)+2u(0,h)+u(h,h))/5, \quad (4.12)$$

$$\dot{E}_{jh}(\varphi_j,\varphi_{j+1},\psi_j,\psi_{j+1}) \equiv v_j\varphi_j + \bar{v}_j v_{j+1}\varphi_{j+1} - \bar{v}_j\bar{v}_{j+1}\left\{\frac{3h}{5}(\psi_j+\psi_{j+1})\right.$$
$$\left.+\frac{h^2}{5}\,\psi_{j+1}^{(1)} - \frac{2h^5}{5!\,5}(\psi_j^{(4)}+\psi_{j+1}^{(4)})\right\}. \quad (4.13)$$

The system of finite difference equations (4.6)-(4.8) which has nonnegative coefficients, with the conditions (4.3) is uniquely solvable.

For simplicity, we will denote constants which are independent of h by c.

Theorem 4.1. Let u be the solution of problem (4.1), (4.2). If $u\epsilon\tilde{C}_{4,\lambda}(\overline{\Pi})$ and the condition (4.3) holds, then

$$\max_{\overline{\Pi}^h}|u_h - u| \le ch^4, \quad (4.14)$$

where u_h is the solution of the system (4.6)-(4.8).

Proof. The proof of Theorem 4.3 follows from the Remark 15 in (Dosiyev, 2003).■

Let us prove a theorem for the proof of the necessary theorems before examining the first derivative.

Let ω be a solution of the problem

$$\Delta\omega = 0 \; on \; \Pi, \qquad \omega = \Phi_{\mathrm{j}} \; on \; \gamma_j, \;\; j = 1,2,3,4, \tag{4.15}$$

where Φ_{j}, $j = 1,2,3,4$, are the given functions, and

$$\Phi_j \in C^{3,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,3,4, \tag{4.16}$$

$$\Phi_j^{2q}(s_j) = (-1)^q \Phi_{j-1}^{2q}(s_j), q = 0,1. \tag{4.17}$$

Consider the following system of grid equations approximating Dirichlet problem (4.15)

$$\omega_h = B\omega_h \;\; on \; \Pi^h, \quad \omega_h = \Phi_j \;\; on \; \gamma_{jh}, \;\; j = 1,2,3,4, \tag{4.18}$$

where B is the averaging operator given by (4.9) and Φ is the function defined by (4.16). By the maximum principle, which obviously holds for this system, system (4.18) has a unique solution.

Lemma 4.1 The solution ω of problem (4.15) is from $C^{3,\lambda}(\bar{\Pi})$.
The proof of Lemma 4.1 follows from Theorem 3.1 by (Volkov, 1969).

Lemma 4.2 Let $\rho(x,y)$ be the distance from the current point of open rectangle Π to its boundary and let $\partial/\partial l \equiv \alpha\,\partial/\partial x + \beta\,\partial/\partial y$, $\alpha^2 + \beta^2 = 1$. Then the inequality below holds

$$\left|\frac{\partial^8\omega(x,y)}{\partial l^8}\right| \le c\rho^{\lambda-5}(x,y), \;\; (x,y) \in \Pi, \tag{4.19}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$, and ω is a solution of problem (4.15).

Proof. We choose an arbitrary point $(x_0,y_0) \in \Pi$. Let $\rho_0 = \rho(x_0,y_0)$, and $\bar{\sigma}_0 \subset \bar{\Pi}$ be the closed circle of radius ρ_0 centered at (x_0,y_0). Consider the harmonic function on Π

$$v(x,y) = \frac{\partial^3\omega(x,y)}{\partial l^3} - \frac{\partial^3\omega(x_0,y_0)}{\partial l^3}. \tag{4.22}$$

By Lemma 4.2, $\omega \in C^{3,\lambda}(\bar{\Pi})$, for $0 < \lambda < 1$. Then for the function (4.22) we have

$$\max_{(x,y)\in\bar{\sigma}_0} |v(x,y)| \le c_0\rho_0^{\lambda}, \tag{4.23}$$

where c_0 is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since ω is harmonic on Π, by using estimation (4.23) and applying Lemma 3 from (Mikhailov, 1978) we have

$$\left|\frac{\partial}{\partial l}\left(\frac{\partial^3 \omega(x,y)}{\partial l^3} - \frac{\partial^3 \omega(x_0,y_0)}{\partial l^3}\right)\right| \leq c_1 \frac{\rho_0^{\lambda}}{\rho_0}.$$

or

$$\left|\frac{\partial^8 \omega(x,y)}{\partial l^8}\right| \leq c_1 \rho_0^{\lambda-5}(x_0, y_0),$$

where $c_1$is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since the point $(x_0, y_0) \in \Pi$ is arbitrary, inequality (4.21) holds true. ■

Let Π^{kh} be the set of nodes of grid Π^h whose distance from γ is kh. It is obvious that $1 \leq k \leq N(h)$, where

$$N(h) = \left[\frac{1}{2h} min\{a, b\}\right], \tag{4.24}$$

$[d]$ is the integer part of d.

We define for $1 \leq k \leq N(h)$ the function

$$f_h^k = \begin{cases} 1, & \rho(x,y) = kh, \\ 0, & \rho(x,y) \neq kh. \end{cases} \tag{4.25}$$

Consider the following systems

$$q_h = Bq_h + g_h \text{ on } \Pi^h, \quad q_h = 0 \text{ on } \gamma^h, \tag{4.26}$$

$$\bar{q}_h = B\bar{q}_h + \bar{g}_h \text{ on } \Pi^h, \quad \bar{q}_h = 0 \text{ on } \gamma^h, \tag{4.27}$$

where g_h and $\bar{g}_h$ are given function, and $|g_h| \leq \bar{g}_h$ on Π^h.

Lemma 4.3 The solution q_h and $\bar{q}_h$ of systems (4.26) and (4.27) satisfy the inequality

$$|q_h| \leq \bar{q}_h \text{ on } \overline{\Pi}^h .$$

The proof of Lemma 4.3 follows from comparison Theorem in Chapter 4 (Samarskii, 2001).

Lemma 4.4 The solution of the system

$$v_h^k = Bv_h^k + f_h^k \text{ on } \Pi^h, \quad v_h^k = 0 \text{ on } \gamma^h \tag{4.28}$$

satisfies the inequality

$$v_h^k(x,y) \leq Q_h^k, \quad 1 \leq k \leq N(h), \tag{4.29}$$

where Q_h^k is defined as follows

$$Q_h^k = Q_h^k(x,y) = \begin{cases} \dfrac{6\rho}{h}, & 0 \leq \rho(x,y) \leq kh\,, \\ 6k, & \rho(x,y) > kh\,. \end{cases} \tag{4.30}$$

Proof. Two different cases should be examined for the roof of Lemma 4.4**.** Each cases contains three different cases as well.

<u>Case 1:</u>

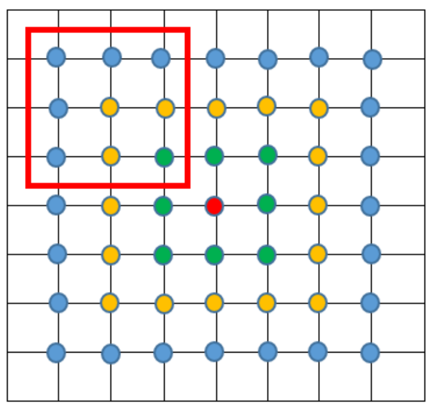

Figure 4.1: For case 1, we selected region in Π

i) By virtue of (4.9) and (4.29) and by considering of Fig. (4.1), for $0 \leq \rho = kh$ we have

$$BQ_h^k = \frac{1}{20}[4(6k + 6(k-1) + 6(k-1) + 6k)$$
$$+6(k-1) + 6k + 6(k-1) + 6(k-1)]$$
$$= 6k - \frac{66}{20},$$

which leads to

$$Q_h^k - BQ_h^k = \frac{66}{20} > 1 = f_h^k.$$

ii) Consider Fig. (4.1), for $\rho > kh$ we have,

$$BQ_h^k = \frac{1}{20}[4(6k + 6k + 6k + 6k) + 6k + 6k + 6k + 6k = 6k,$$

which leads to

$$BQ_h^k = Q_h^k.$$

iii) For $\rho < kh$ we have,

$$BQ_h^k = \frac{1}{20}[4(6(k-1) + 6(k-2) + 6(k-2) + 6(k-1))$$
$$+6(k-2) + 6k + 6(k-2) + 6(k-2)]$$
$$= 6k - 9,$$

which leads to

$$Q_h^k - BQ_h^k = 9 > 1 = f_h^k.$$

<u>Case 2:</u>

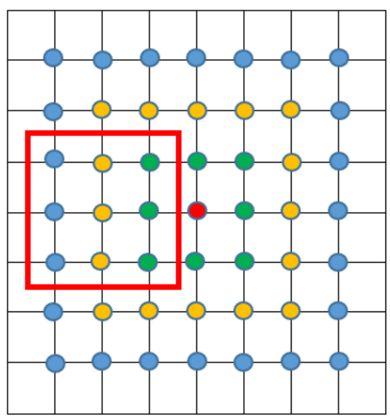

Figure 4.2: For case 2, we selected region in Π

i) Consider Fig. (4.2), for $0 \leq \rho = kh$ we have,

$$BQ_h^k = \frac{1}{20}[4(6k + 6(k-1) + 6k + 6k)$$
$$+6(k-1) + 6k + 6(k-1) + 6k]$$
$$= 6k - \frac{36}{20},$$

which leads to

$$Q_h^k - BQ_h^k = \frac{36}{20} > 1 = f_h^k.$$

ii) Consider Fig. (4.2), for $\rho > kh$ we have

$$BQ_h^k = \frac{1}{20}[4(6k + 6k + 6k + 6k) + 6k \ + 6k + 6k + 6k = 6k,$$

which leads to

$$BQ_h^k = Q_h^k.$$

iii) For $\rho < kh$ we have

$$BQ_h^k = \frac{1}{20}[4(6k + 6(k-2) + 6(k-1) + 6(k-1))$$
$$+6k + 6k + 6(k-2) + 6(k-2)]$$
$$= 6k - 6,$$

which leads to

$$Q_h^k - BQ_h^k = 6 > 1 = f_h^k.$$

From the above calculations we have

$$Q_h^k = BQ_h^k + q_h^k \text{ on } \Pi^h, Q_h^k = 0 \text{ on } \gamma^h, \quad k = 1, \dots, N(h), \qquad (4.31)$$

where $|q_h^k| \geq 1$. On the basis of (4.25), (4.28), (4.31) and the Comparison Theorem (see (Samarskii, 2001), Chap. 4), we obtain

$$|v_h^k| \leq Q_h^k \text{ for all } k, \ \ 1 \leq k \leq N(h). \ \blacksquare$$

Lemma 4.5 It is true that

$$Bp_7(x_0, y_0) = \omega(x_0, y_0)$$

where p_7 is the seventh order Taylor's polynomial at (x_0, y_0), ω is a harmonic function.

Proof. The seventh order Taylor's polynomial at (x_0, y_0) has the form

$$p_7(x,y) = \omega(x,y) + h\left(\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}\right) + \frac{h^2}{2!}\left(\frac{\partial^2\omega}{\partial x^2} + 2\frac{\partial^2\omega}{\partial x\partial y} + \frac{\partial^2\omega}{\partial y^2}\right) + \frac{h^3}{3!}\left(\frac{\partial^3\omega}{\partial x^3}\right.$$

$$+3\frac{\partial^3\omega}{\partial x^2\partial y} + 3\frac{\partial^3\omega}{\partial x\partial y^2} + \left.\frac{\partial^3\omega}{\partial y^3}\right) + \frac{h^4}{4!}\left(\frac{\partial^4\omega}{\partial x^4} + 4\frac{\partial^4\omega}{\partial x^3\partial y} + 6\frac{\partial^4\omega}{\partial x^2\partial y^2} + 4\frac{\partial^4\omega}{\partial x\partial y^3}\right.$$

$$\left.+\frac{\partial^4\omega}{\partial y^4}\right) + \frac{h^5}{5!}\left(\frac{\partial^5\omega}{\partial x^5} + 5\frac{\partial^5\omega}{\partial x^4\partial y} + 20\frac{\partial^5\omega}{\partial x^3\partial y^2} + 20\frac{\partial^5\omega}{\partial x^2\partial y^3} + 5\frac{\partial^5\omega}{\partial x\partial y^4} + \frac{\partial^5\omega}{\partial y^5}\right)$$

$$+\frac{h^6}{6!}\left(\frac{\partial^6\omega}{\partial x^6} + 6\frac{\partial^6\omega}{\partial x^5\partial y} + 15\frac{\partial^6\omega}{\partial x^4\partial y^2} + 20\frac{\partial^6\omega}{\partial x^3\partial y^3} + 15\frac{\partial^6\omega}{\partial x^2\partial y^4} + 6\frac{\partial^6\omega}{\partial x\partial y^5}\right.$$

$$\left.+\frac{\partial^6\omega}{\partial y^6}\right) + \frac{h^7}{7!}\left(\frac{\partial^7\omega}{\partial x^7} + 7\frac{\partial^7\omega}{\partial x^6\partial y} + 21\frac{\partial^7\omega}{\partial x^5\partial y^2} + 35\frac{\partial^7\omega}{\partial x^4\partial y^3} + 35\frac{\partial^7\omega}{\partial x^3\partial y^4}\right.$$

$$\left.+21\frac{\partial^7\omega}{\partial x^2\partial y^5} + 7\frac{\partial^7\omega}{\partial x\partial y^6} + \frac{\partial^7\omega}{\partial y^7}\right). \quad (4.32)$$

Then according to (4.9) and (4.32) we have

$$Bp_7(x_0, y_0) = \frac{1}{20}[4(p_7(x_0 + h, y_0) + p_7(x_0 - h, y_0) + p_7(x_0, y_0 + h)$$

$$+p_7(x_0, y_0 - h)) + p_7(x_0 + h, y_0 + h) + p_7(x_0 + h, y_0 - h)$$

$$+p_7(x_0 - h, y_0 + h) + p_7(x_0 - h, y_0 - h)]$$

$$= \omega(x_0, y_0) + \frac{h^4}{40}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\omega(x_0,y_0)}{\partial x^2} + \frac{\partial^2\omega(x_0,y_0)}{\partial y^2}\right)$$

$$+\frac{h^4}{40}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\omega(x_0,y_0)}{\partial x^2} + \frac{\partial^2\omega(x_0,y_0)}{\partial y^2}\right) + \frac{3h^6}{5\times 6!}\frac{\partial^4}{\partial x^4}\left(\frac{\partial^2\omega(x_0,y_0)}{\partial x^2} + \frac{\partial^2\omega(x_0,y_0)}{\partial y^2}\right)$$

$$+\frac{3h^6}{5\times 6!}\frac{\partial^4}{\partial y^4}\left(\frac{\partial^2\omega(x_0,y_0)}{\partial x^2} + \frac{\partial^2\omega(x_0,y_0)}{\partial y^2}\right) + \frac{2h^6}{5\times 5!}\frac{\partial^4}{\partial x^2\partial y^2}\left(\frac{\partial^2\omega(x_0,y_0)}{\partial x^2} + \frac{\partial^2\omega(x_0,y_0)}{\partial y^2}\right).$$

Since ω is harmonic, we obtain

$$Bp_7(x_0, y_0) = \omega(x_0, y_0) \quad \blacksquare$$

Lemma 4.6 The inequality holds

$$\max_{(x,y)\in\Pi^{kh}}|B\omega - \omega| \le c\frac{h^{3+\lambda}}{k^{5-\lambda}}, \quad k = 1,2,\dots,N(h), \quad (4.33)$$

where ω is a solution of problem (4.15).

Proof. Let (x_0, y_0) be a point of Π^{1h}, and let

$$\Pi_0 = \{(x,y): |x - x_0| < h, |y - y_0| < h\}, \qquad (4.34)$$

be an elementary square, such that some sides of it lie on the boundary of the rectangle Π. On the vertices of Π_0, and on the mid points of its sides lie the nodes of which the function values are used to evaluate $Bu(x_0, y_0)$. We represent a solution of problem (4.15) in some neighborhood of $(x_0, y_0) \in \Pi^{1h}$ by Taylor's formula as;

$$\omega(x,y) = p_7(x,y) + r_8(x,y), \qquad (4.35)$$

where $p_7(x,y)$ is the seventh order Taylor's polynomial, $r_8(x,y)$ is the remainder term, by Lemma 4.4 we have

$$Bp_7(x_0, y_0) = \omega(x_0, y_0). \qquad (4.36)$$

Now, we estimate r_8 at the nodes of the operator B. We take node $(x_0 + h, y_0 + h)$ which is one of the eight nodes of B, and consider the function

$$\widetilde{\omega}(s) = \omega\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \quad -\sqrt{2}h \le s \le \sqrt{2}h \qquad (4.37)$$

of one variable s. By virtue of Lemma 4.2, we have

$$\left|\frac{\partial^8 \widetilde{\omega}(s)}{\partial s^8}\right| \le c_2\left(\sqrt{2}h - s\right)^{\lambda-5}, \quad 0 \le s \le \sqrt{2}h\,. \qquad (4.38)$$

We represent function (4.37) around the point $s = 0$ by Taylor's formula

$$\tilde{u}(s) = \tilde{p}_7(s) + \tilde{r}_8(s), \qquad (4.39)$$

where

$$\tilde{p}_7(s) = p_7\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) \qquad (4.40)$$

is the seventh order Taylor's polynomial of the variable s, and

$$\tilde{r}_8(s) = r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \; 0 \le |s| \le \sqrt{2}h \qquad (4.41)$$

is the remainder term. On the basis of continuity of $\tilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$, it follows from (4.41) that

$$r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) = \lim_{\epsilon \to +0} \tilde{r}_8\left(\sqrt{2}h - \epsilon\right). \qquad (4.42)$$

Applying an integral representation for $\tilde{r}_8$ we have

$$\tilde{r}_8(\sqrt{2}h-\epsilon)=\frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-\epsilon-t\right)^7\tilde{u}^8(t)dt\,,\qquad 0<\epsilon\le\frac{h}{\sqrt{2}}.$$

Using estimation (4.38), we have

$$\begin{aligned}\left|\tilde{r}_8(\sqrt{2}h-\epsilon)\right| &\le c_3\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-\epsilon-t\right)^7\left(\sqrt{2}h-t\right)^{\lambda-5}dt\\ &\le c_4\frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-t\right)^{2+\lambda}dt\\ &\le ch^{\lambda+3},\quad 0<\epsilon\le\frac{h}{\sqrt{2}}.\end{aligned}\tag{4.43}$$

From (4.41)-(4.43) yields

$$|r_8(x_0+h,y_0+h|\le c_1h^{\lambda+3},\tag{4.44}$$

where c_1 is a constant independent of the taken point (x_0,y_0) on Π^{1h}. Proceeding in a similar manner, we can find the same estimates of r_8 at the other vertices of square (4.34) and at the centers of its sides. Since the norm of B in the uniform metric is equal to unity, we have

$$|Br_8(x_0,y_0)|\le c_5h^{\lambda+3}.\tag{4.45}$$

where c_5 is a constant independent of the taken point (x_0,y_0) on Π^{1h}. From (4.35), (4.36), (4.45) and linearity of the operator B, we obtain

$$|B\omega(x_0,y_0)-\omega(x_0,y_0)|\le ch^{\lambda+3},\tag{4.46}$$

for any $(x_0,y_0)\in\Pi^{1h}$.

Now let $(x_0,y_0)\in\Pi^{kh}$, $2\le k\le N(h)$ and $r_8(x,y)$ be the Lagrange remainder corresponding to this point in Taylor's formula (4.35). Then $Br_8(x_0,y_0)$ can be expressed linearly in terms of a fixed number of eighth derivatives of u at some point of the open square Π_0, which is a distance of $kh/2$ away from the boundary of Π. The sum of the coefficients multiplying the eighth derivatives does not exceed ch^8, which is independent of k $(2\le k\le N(h))$. By Lemma 4.2, we have

$$|Br_8(x_0, y_0)| \leq c\frac{h^8}{(kh)^{5-\lambda}} = c\frac{h^{\lambda+3}}{k^{5-\lambda}}, \quad (4.47)$$

where c is a constant independent of k $(2 \leq k \leq N(h))$. On the basis of (4.35), (4.36), (4.46), and (4.47) follows estimation (4.33) at any point $(x_0, y_0) \in \Pi^{kh}$, $1 \leq k \leq N(h)$. ■

Theorem 4.2 The following estimation holds

$$\max_{\overline{\Pi}^h}|\omega_h - \omega| \leq ch^{3+\lambda},$$

where ω is the exact solution of problem (4.15), ω_h is the solution of the finite difference problem (4.20).

Proof. Let

$$\epsilon_h(x, y) = \omega_h(x, y) - \omega(x, y), \quad (x, y) \in \overline{\Pi}^h. \quad (4.48)$$

Putting $\omega_h = \epsilon_h + \omega$ into (4.20), we have

$$\epsilon_h = B\epsilon_h + (B\omega - \omega) \text{ on } \Pi^h, \quad \epsilon_h = 0 \text{ on } \gamma^h. \quad (4.49)$$

We represent a solution of system (4.48) as follows

$$\epsilon_h = \sum_{k=1}^{N(h)} \epsilon_h^k, \quad N(h) = \left[\frac{1}{2h}\min\{a, b\}\right], \quad (4.50)$$

where ϵ_h^k is a solution of the system

$$\epsilon_h^k = B\epsilon_h^k + \sigma_h^k \text{ on } \Pi^h, \quad \epsilon_h = 0 \text{ on } \gamma^h, \quad k = 1,2, \dots, N(h); \quad (4.51)$$

$$\sigma_h^k = \begin{cases} B\omega - \omega & \text{on} \quad \Pi_h^k, \\ 0 & \text{on} \quad \Pi_h/\Pi_h^k. \end{cases} \quad (4.52)$$

By virtue of (4.51), (4.52) and Lemma 4.4, for each k, $1 \leq k \leq N(h)$, we can find the inequality

$$\begin{aligned} \max_{(x,y)\epsilon\Pi^{kh}}\left|\epsilon_h^k(x, y)\right| &\leq Q_h^k(x, y) \max_{(x,y)\epsilon\Pi^{kh}}|(B\omega - \omega)| \text{ on } \overline{\Pi}^h. \\ &\leq c6k\frac{h^{3+\lambda}}{k^{5-\lambda}} \\ &= \frac{ch^{3+\lambda}}{k^{4-\lambda}}, \quad 1 \leq k \leq N(h). \quad (4.53) \end{aligned}$$

According to (4.48)-(4.52),

$$\omega_h - \omega = \varepsilon_h = \varepsilon_h^1 + \cdots + \varepsilon_h^{N(h)}.$$

Comparing this with (4.53) produces

$$\max_{(x,y)\in\Pi^h} |\omega_h - \omega| \leq ch^{3+\lambda} \sum_{k=1}^{N(h)} \frac{1}{k^{4-\lambda}} \leq ch^{3+\lambda}.$$

Theorem 4.2 is proved. ■

4.2 Approximation of the First Derivatives

We will calculate for the approximate values of the first derivatives of the solution of mixed boundary problem on a square grid of the Laplace equation in this section. Denote by v_j a parameter taking the values 0 or 1, and $\overline{v_j} = 1 - v_j$.

Let u be a solution of problem (4.1), (4.2). Let $v = \frac{\partial u}{\partial x}$ and $\phi_j = \frac{\partial u}{\partial x}$ on γ_j, $j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v = 0 \; on \; \Pi, v = \phi_j \; on \; \gamma_j, j = 1,2,3,4. \tag{4.54}$$

$$\gamma_3^{h+} = \left\{0 \leq x \leq \frac{a}{2},\ y = b\right\} \cap \gamma_3^h \tag{4.55}$$

and

$$\gamma_3^{h-} = \left\{\frac{a}{2} + h \leq x \leq a,\ y = b\right\} \cap \gamma_3^h. \tag{4.56}$$

We define the following operator $\phi_{ph},\ p = 1,2,\ldots,4$,

$$\phi_{1h}(u_h) = \frac{\partial u(x,0)}{\partial x} \; on \; \gamma_1^h \tag{4.57}$$

$$\phi_{2h}(u_h) = \bar{v}_2\psi_2 + v_2\frac{1}{6h}[11\varphi_2(a) - 18u_h(a-h,y)$$

$$+9u_h(a-2h,y) - 2(a-3h,y)] \text{ on } \gamma_2^h, \qquad (4.58)$$

$$\phi_{3h}(u_h) = v_3\frac{\partial u(x,b)}{\partial x} + \bar{v}_3\frac{1}{6h}[-11u_h(x,b) + 18u_h(x+h,b)$$

$$-9u_h(x+2h,b) + 2u_h(x+3h,b)] \; on \; \gamma_3^{h+}, \qquad (4.59)$$

$$\phi_{3h}(u_h) = v_3\frac{\partial u(x,b)}{\partial x} + \bar{v}_3\frac{1}{6h}[11u_h(x,b) - 18u_h(x-h,b)$$

$$+9u_h(x-2h,b) - 2u_h(x-3h,b)] \; on \; \gamma_3^{h-}, \qquad (4.60)$$

$$\phi_{4h}(u_h) = \bar{v}_k\psi_4 + v_k\frac{1}{6h}[-11\varphi_4(0) + 18u_h(h,y)$$

$$-9u_h(2h,y) + 2u_h\,(3h,y)] \; on \; \gamma_4^h, \qquad (4.61)$$

u_h is the solution of the finite difference problem (4.6)-(4.8).

Let v_h be the solution of the following finite difference problem

$$v_h = Bv_h \; on \; \Pi_h, v_h = \phi_{jh} \; on \; \gamma_j^h, j = 1,2,\ldots,4, \qquad (4.62)$$

where ϕ_{jh}, $j = 1,2,\ldots,4$, are defined by (4.57)-(4.61).

Lemma 4.7 The following inequality is true

$$|\phi_{th}(u_h) - \phi_{th}(u)| \leq ch^3, \qquad (4.63)$$

where u_h is the solution of the finite difference problem (4.6)-(4.8) and u is the solution of problem (4.1).

Proof: It is obvious that, $\phi_{ph}(u_h) - \phi_{ph}(u) = 0$, if the Dirichlet condition is given on γ_3 or if the Newman condition is given on γ_j for $j = 2,4$, i.e., $v_2 = v_4 = 0$. . Assume that Dirichlet condition is given on γ_2, By (4.58) and Theorem 4.1, we have

$$
\begin{aligned}
|\phi_{2h}(u_h) - \phi_{2h}(u)| \leq\ & \left(\frac{1}{6h}\right)\{18|u_h(a-h,y) - u(a-h,y)| \\
& +9|u_h(a-2h,y) - u(a-2h,y)| \\
& +2|u_h\ (a-3h,y) - u(a-3h,y)|\} \\
\leq\ & \left(\frac{1}{6h}\right)[18ch^4 + 9ch^4 + 2ch^4] \\
\leq\ & c_1 h^3. \blacksquare
\end{aligned}
$$

Lemma 4.8 The following inequality holds

$$\max_{(x,y)\in\Pi_2^h} |\phi_{2h}(u_h) - \phi_2| \leq ch^4, \qquad (4.64)$$

where ϕ_{2h} is defined by (4.62) and $\phi_2 = \frac{\partial u}{\partial x}\ on\ \gamma_2$.

Proof. $u \epsilon C_{4,\lambda}(\overline{\Pi})$. Then, at the end point $(a, \nu h) \in \gamma_2^h$ of each line segment $\{(x,y): 0 \leq x \leq a, 0 \leq y = \nu h < b,\}$, expressions (4.58) give the third order approximation of $\left(\frac{\partial u}{\partial x}\right)$, respectively. From the truncation error formulas it (Burden and Douglas, 2011) follows that

$$
\begin{aligned}
\max_{(x,y)\in\gamma_2^h} |\phi(u) - \phi_2| \leq\ & c_1 \frac{h^3}{4} \max_{(x,y)\in\gamma_2^h} \left|\frac{\partial^4 u}{\partial x^4}\right| \\
\leq\ & c_2 h^3. \qquad (4.65)
\end{aligned}
$$

On the basis of Lemma 4.7 and estimation (4.65) follows (4.64),

$$\begin{aligned}\max_{(x,y)\in\gamma_2^h}|\phi_{2h}(u_h)-\phi_2| = & \max_{(x,y)\in\gamma_2^h}|\phi_{2h}(u_h)-\phi_{2h}(u)+\phi_{2h}(u)-\phi_2| \\ \leq & \max_{(x,y)\in\gamma_2^h}|\phi_{2h}(u_h)-\phi_{2h}(u)| \\ & +\max_{(x,y)\in\gamma_2^h}|\phi_{2h}(u)-\phi_2| \\ \leq & c_3h^3+c_4h^3 \\ \leq & c_5h^3. \ \blacksquare\end{aligned}$$

All the remaining case are solved in the similar way as Lemma 4.7 and Lemma 4.8.

Theorem 4.3 The following estimation is true

$$\max_{(x,y)\in\Pi^h}\left|\nu_h-\frac{\partial u}{\partial x}\right|\leq ch^3, \qquad (4.66)$$

where u is the solution of the problem (4.54) and ν_h is the solution of the finite difference problem (4.62).

Proof. Let

$$\varepsilon_h=\nu_h-\nu \ on \ \pi^k, \qquad (4.67)$$

where $\nu=\frac{\partial u}{\partial x}$. From (4.61) and(4.66), we have

$$\varepsilon_h = B\varepsilon_h+(B\nu-\nu) \ on \ \Pi^h,$$

$$\varepsilon_h = \phi_{kh}(u_h)-\nu \ on \ \gamma_j^h, \qquad j=1,2,3,4.$$

We represent

$$\varepsilon_h=\varepsilon_h^1+\varepsilon_h^2, \qquad (4.68)$$

where

$$\varepsilon_h^1 = B\varepsilon_h^1 \; on \; \Pi^h, \tag{4.69}$$

$$\varepsilon_h^1 = \phi_{kh}(u_h) - \nu \; on \; \gamma_j^h, \qquad j = 1,2,3,4; \tag{4.70}$$

$$\varepsilon_h^2 = B\varepsilon_h^2 + (B\nu - \nu) \; on \; \Pi^h, \varepsilon_h^2 = 0 \; on \; \gamma_j^h, j = 1,2,3,4. \tag{4.71}$$

By Lemma 4.10 and by the maximum principle, for the solution of system (4.69), (4.70) we have

$$\max_{(x,y)\in\Pi^h} |\varepsilon_h^1| \le \max_{q=1,2,3,4} \max_{(x,y)\in\gamma_k^h} \left|\phi_{qh}(u_h) - \nu\right| \le c_1 h^3. \tag{4.72}$$

The solution ε_h^2 of system (4.71) is the error of the approximate solution obtained by the finite difference method for problem (4.62), when on the boundary nodes on γ_j^h, the exact values of the functions ϕ_j in (4.62) are used. It is obvious that $\phi_j, j = 1,2,3,4$, satisfy the conditions

$$\phi_j \in C^{3,\lambda}(\gamma_j), 0 < \lambda < 1, j = 1,2,\ldots,4, \tag{4.73}$$

Since the function $\nu = \frac{\partial u}{\partial x}$ is harmonic on Π, where u is the solution of (4.1), (4.2), the boundary functions $\phi_j = \frac{\partial u}{\partial x} \; on \; \gamma_j$ satisfy the conjunction conditions at the vertices of Π:

$$\phi_j^{2q}(s_j) = (-1)^q \phi_{j-1}^{2q}(s_j), q = 0,1. \tag{4.74}$$

Therefore on the basis of Theorem 4.2, for the error function ε_h^2 as the solution of the finite difference problem (4.71), we have estimation

$$\max_{(x,y)\in\gamma_k^h} |\varepsilon_h^2| \le c_2 h^{3+\lambda}. \tag{4.75}$$

The inequality (4.66) follows by (4.67), (4.72) and (4.75). ■

CHAPTER 5

NUMERICAL EXPERIMENTS

In this chapter we present the numerical results obtained in support of the theoretical part. Our aim is to show the high order accurate approximation of the first, second order pure and mixed derivatives of the Laplace equation on a rectangle and a rectangular parallelepiped.

Numerical Examples

The following part, supports the theoretical part by numerical results are which obtained in a rectangle by using Gauss$-$Seidel iterative method and rectangular parallelepiped by using Discreate Fourier method.

The results on rectangular parallelepiped have four parts:

- The approximate results for the solution of the Dirichlet problem of Laplace’s equation
- The approximate results for the first derivative of the solution
- The approximate results for the pure second derivative of the solution
- The approximate results for the second order mixed derivative of the solution.

The results for the solution of the Dirichlet problem on a rectangle domain have three parts:

- The approximate results for the solution of Laplace’s equation
- The approximate results for the first derivative of the solution
- The approximate results for the pure second derivative of the solution

The results for the solution of the mixed boundary value problem on a rectangle domain have two parts:

- The approximate results for the solution of Laplace’s equation
- The approximate results for the first derivative of the solution

The grid spacing (difference step size) h is defined by $h = \frac{1}{2^n},\ n = 4, \ldots, 7$.

5.1 Domain in the Shape of a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3): 0 < x_i < 1, i = 1,2,3\}$, and let $\Gamma_j, j = 1, \ldots, 6$ be its faces. We consider the following problem:

$$\Delta u = 0 \ \ on\ R, \qquad u = \varphi(x_1, x_2, x_3)\ on\ \Gamma_j, j = 1, \ldots, 6, \qquad (5.1)$$

where φ is the exact solution of this problem.

Let U denote the exact solution and U_n be its approximate values on $\bar{R}^h$ (contains the nodes of the cubic grid formed in R) of the Dirichlet problem of Laplace's equation on the rectangular parallelepiped domain R. We denote

$$\|U_h - U\|_{\bar{R}_h} = \max_{\bar{R}^h} |U_h - U|,\ E_U^m = \frac{\|U - U_{2^{-m}}\|_{R_h}}{\left\|U - U_{2^{-(m+1)}}\right\|_{R_h}}.$$

In the following examples the results are demonstrated in four tables. The first table is related to the approximate of problem (5.1), the second, third and fourth tables are corresponds to the approximate values of $v = \frac{\partial u}{\partial x}, \omega = \frac{\partial^2 u}{\partial x^2}$ and $\varpi = \frac{\partial^2 u}{\partial x \partial y}$, respectively.

These results are obtained for different boundary functions which are given below.

5.1.1 Boundary function from $C^{5,\lambda}$

In the following examples, the forward and backward formulae are used for fourth order accuracy to find a new boundary value on faces when $x_1 = 0$ or $x_1 = 1$ for the first derivative problem. For the find the second order mixed derivative of the solution of the Laplace equation we used the third order forward and backward numerical differentiation formula on faces when $x_1 = 0,1$ or $x_2 = 0,1$.

The results show that the approximate solutions and approximation of the first derivative converge as $O(h^4)$, for the approximation of the second order pure and mixed derivatives converges uniformly with order $O(h^{3+\lambda})$ and $O(h^3)$, respectively.

Example 5.1 : Let $\varphi \epsilon C^{5,\frac{1}{30}}$, on Γ_j, $j = 1,2,\dots,6$, where

$$\varphi(x,y) = \left(x_3 - \frac{1}{2}\right)^2 - \frac{x_1^2 + x_2^2}{2} + (x_1^2 + x_2^2)^{\frac{151}{60}} cos\left(\frac{151}{30}\theta\right)$$

where $\theta = arctan\left(\frac{x_2}{x_1}\right)$.

Table 5.1: The approximate of solution of problem (5.1) when the boundary function is in $C^{5,\frac{1}{30}}$

h	$\lVert u - u_h \rVert$	E_U^m
$\frac{1}{16}$	2,34E-10	32,68
$\frac{1}{32}$	7,16E-12	32,69
$\frac{1}{64}$	2,19E-13	32,78
$\frac{1}{128}$	6,68E-15	

Table 5.2: First derivative approximation results of the solution of problem (5.1) with the fourth-order accurate formula

h	$\|v - v_h\|$	E_U^m
$\frac{1}{16}$	3,39E-04	14,76
$\frac{1}{32}$	2,30E-05	15,42
$\frac{1}{64}$	1,49E-06	15,67
$\frac{1}{128}$	9,51E-08	

Table 5.3: The approximate results for the pure second order derivative of the solution of (5.1)

h	$\|\bar{\omega} - \bar{\omega}_h\|$	E_U^m
$\frac{1}{16}$	3,68E-07	8,20
$\frac{1}{32}$	4,49E-08	8,18
$\frac{1}{64}$	5,49E-09	8,19
$\frac{1}{128}$	6,70E-10	

Table 5.4: The approximation results for the second order mixed derivative with the third-order accurate formula when $\varphi \epsilon C^{5,\frac{1}{30}}$

h	$\|\omega - \omega_h\|$	E_U^m
$\frac{1}{16}$	6,92E-03	7,35
$\frac{1}{32}$	9,41E-04	7,71
$\frac{1}{64}$	1,22E-04	7,92
$\frac{1}{128}$	1,54E-05	

In Table 5.1-Table 5.4 the maximum error are given. Table 5.1 shows that the solution of the Laplace equation converges with order more than 4 which corresponds to the product ρ in Theorem 2.1 when $p = 5$. Table 5.2 justifies approximation of the first derivative when $4 - th$ order accuracy forward backward formula is used in Theorem 2.2, the fourth order converge. Table 5.3 shows that convergence order of the approximations of the pure second derivatives $O(h^3)$ in Theorem 2.4 and Table 5.4 shows that convergence order $O(h^3)$ of the second order mixed derivative when $3 - th$ order accurate forward backward formula is used in Theorem 2.5 of the problem (5.1).

5.1.2 Boundary function from $C^{4,\lambda}$

In the following examples forward and backward formulae is used for third order accuracy to find a new boundary value on faces when $x_1 = 0$ or $x_1 = 1$. For the find the second order mixed derivative of the solution of the Laplace equation we used the second order forward and backward numerical differentiation formula on faces when $x_1 = 0,1$ or $x_2 = 0,1$.

The results show that the approximate solutions converge as $O(h^4)$, and approximation of the first derivative converges uniformly with order $O(h^3)$, for the approximation of the second order pure and mixed derivatives converges uniformly with order $O(h^{2+\lambda})$ and $O(h^2)$, respectively.

Example 5.2 : Let $\varphi \epsilon C^{4,\frac{1}{30}}$, on $\Gamma_j, j = 1,2,\ldots,6$, where

$$\varphi(x,y) = \left(x_3 - \frac{1}{2}\right)^2 - \frac{x_1^2 + x_2^2}{2} + (x_1^2 + x_2^2)^{\frac{121}{60}} cos\left(\frac{121}{30}\theta\right)$$

where $\theta = arctan\left(\frac{x_2}{x_1}\right)$.

Table 5.5: The approximate of solution of problem (5.1) when the boundary when the boundary function is in $C^{4,\frac{1}{30}}$

h	$\|u - u_h\|$	E_u^m
$\frac{1}{16}$	2,15E-09	17,77
$\frac{1}{32}$	1,21E-10	15,09
$\frac{1}{64}$	8,02E-12	16,37
$\frac{1}{128}$	4,90E-13	

Table 5.6: The approximate results for the first derivative when $\varphi \epsilon C^{4,\frac{1}{30}}$, using the third-order accurate formula

h	$\|\|v - v_h\|\|$	E_v^m
$\frac{1}{16}$	1,32E-03	7,29
$\frac{1}{32}$	1,81E-04	7,67
$\frac{1}{64}$	2,36E-05	7,84
$\frac{1}{128}$	3,01E-06	

Table 5.7: The approximate results for the pure second order derivative when $\varphi \epsilon C^{4,\frac{1}{30}}$

h	$\|\|\overline{\omega} - \overline{\omega}_h\|\|$	$E_{\overline{\omega}}^m$
$\frac{1}{16}$	3,11E-07	4,09
$\frac{1}{32}$	7,61E-08	4,09
$\frac{1}{64}$	1,86E-08	4,10
$\frac{1}{128}$	4,54E-09	

Table 5.8: The approximate results for the second order mixed derivative with the second order accurate formula when $\varphi \epsilon C^{4,\frac{1}{30}}$

h	$\|\omega - \omega_h\|$	E_w^m
$\frac{1}{16}$	8,59E-03	3,87
$\frac{1}{32}$	2,22E-03	3,96
$\frac{1}{64}$	5,61E-04	3,98
$\frac{1}{128}$	1,41E-04	

In Table 5.5, the approximate results for the solution of the Dirichlet problem for the Laplace's equation are presented. Table 5.6 shows the maximum errors and convergence order of the first derivative when $3-th$ order accuracy forward bacward formula is used, and in Table 5.7, the maximum errors and the convergence order of the approximations of the pure second derivatives, Table 5.8 shows the maximum errors and convergence order of the first derivative when $second$ order accuracy forward bacward formula is used of the problem (5.1) for different step size h are present.

5.2 Domain in the Shape of a Rectangle

Let $\Pi = \{(x, y): 0 < x < 1,\ 0 < y < 1\}$, and let $\gamma_j, j = 1, \dots, 4$ be the boundary of Π. We consider the following problem:

$$\Delta u = 0 \ \ on\ \Pi, \qquad u = \varphi(x, y) \qquad on\ \ \gamma_j, j = 1, \dots, 4, \tag{5.2}$$

where φ is the exact solution of this problem.

Let U denote the exact solution and U_n be its approximate values on $\overline{\Pi}^h$ (contains the nodes of the cubic grid formed in Π) of the Dirichlet problem of Laplace's equation on the rectangular parallelepiped domain Π. We denote

$$\|U_h - U\|_{\overline{\Pi}_h} = \max_{\overline{\Pi}^h} |U_h - U|,\ \ E_U^m = \frac{\|U - U_{2^{-m}}\|_{\Pi_h}}{\left\|U - U_{2^{-(m+1)}}\right\|_{\Pi_h}}.$$

5.2.1 Three stage method

In the following examples the results are demonstrated in three tables. The first table is related to the approximate values of u, the second and third tables is corresponds to the approximate values of $v = \frac{\partial u}{\partial x}, w = \frac{\partial^2 u}{\partial x^2}$ respectively.

Example 5.3 : Let $\varphi \epsilon C^{12,\frac{1}{30}}$ on $\gamma_j, j = 1,2,3,4$, where

$$\varphi(x, y) = (x^2 + y^2)^{\frac{361}{60}} cos\left(\frac{361}{30} \arctan\left(\frac{y}{x}\right)\right).$$

Table 5.9: The approximate results of u in problem (5.2) by using a three stage difference method when the boundary function is in $C^{12,\frac{1}{30}}$

h	$\|u^4 - u_h^4\|$	$E_{u^4}^m$	$\|u^8 - u_h^8\|$	$E_{u^8}^m$	$\|u - u_h\|$	E_u^m
$\frac{1}{16}$	$7{,}62E-12$	63,98	$2{,}51E-04$	4,00	$2{,}56E-16$	256,12
$\frac{1}{32}$	$1{,}19E-13$	64,03	$6{,}28E-05$	4,00	$1{,}39E-18$	256,46
$\frac{1}{64}$	$1{,}86E-15$	64,14	$1{,}57E-05$	4,01	$5{,}42E-21$	256,87
$\frac{1}{128}$	$2{,}90E-17$		$3{,}93E-06$		$2{,}11E-23$	

In Table 5.9, the values of $E_{\mathrm{u}^4}^{\mathrm{n}}$ and $E_{\mathrm{u}^8}^{\mathrm{n}}$ show that the functions u^4 (the first stage for the solution) and u^8 (the second stage for the solution) are approximated with the order of $O(h^6)$ and $O(h^2)$; respectively. The values of E_{u}^{n} shows that the accuracy of the proposed method (the third stage for the solution) is of order $O(h^8)$.

Table 5.10: The approximate results for the first derivative by using a three stage difference method when the boundary function is in $C^{12,\frac{1}{30}}$

h	$\lVert v^4 - v_h^4 \rVert$	$E_{v^4}^m$	$\lVert v^8 - v_h^8 \rVert$	$E_{v^8}^m$	$\lVert v - v_h \rVert$	E_v^m
$\frac{1}{16}$	$3{,}17E-08$	$54{,}28$	$6{,}49E-03$	$3{,}63$	$1{,}81E-11$	$187{,}37$
$\frac{1}{32}$	$5{,}84E-10$	$59{,}17$	$1{,}79E-03$	$3{,}83$	$9{,}66E-14$	$210{,}49$
$\frac{1}{64}$	$9{,}87E-12$	$61{,}69$	$4{,}67E-04$	$3{,}91$	$4{,}59E-16$	$226{,}11$
$\frac{1}{128}$	$1{,}60E-13$		$1{,}20E-04$		$2{,}03E-18$	

In Table 5.10, the values of $E_{\mathrm{v}^4}^{\mathrm{n}}$ and $E_{\mathrm{v}^8}^{\mathrm{n}}$ show that the functions v^4 (the first stage for the first derivative) using sixth-order accurate numerical differentiation formulae and v^8 (the second stage for the first derivative) using second-order formulae are approximated with the order of $O(h^6)$ and $O(h^2)$; respectively. The values of E_{v}^{n} shows that the accuracy of the proposed method (the third stage for the first derivative) is of order $O(h^8)$.

Table 5.11: The approximate results for the second order pure derivative by using a three stage difference method when the boundary function is in $C^{12,\frac{1}{30}}$

h	$\|w^4 - w_h^4\|$	$E_{w^4}^m$	$\|w^8 - w_h^8\|$	$E_{w^8}^m$	$\|w - w_h\|$	E_w^m
$\frac{1}{16}$	$1{,}35E-10$	63,61	$4{,}40E-03$	3,95	$2{,}27E-15$	257,52
$\frac{1}{32}$	$2{,}12E-12$	63,97	$1{,}12E-03$	3,99	$8{,}82E-18$	255,37
$\frac{1}{64}$	$3{,}31E-14$	64,00	$2{,}80E-04$	4,03	$3{,}45E-20$	256,06
$\frac{1}{128}$	$5{,}17E-16$		6,96E-05		$1{,}35E-22$	

In Table 5.11, the values of $E_{w^4}^n$ and $E_{w^8}^n$ show that the functions w^4 (the first stage for the second order pure derivative) and w^8 (the second stage for the second order pure derivative) are approximated with the order of $O(h^6)$ and $O(h^2)$; respectively. The values of E_w^n shows that the accuracy of the proposed method (the third stage for the second order pure derivative) is of order $O(h^8)$.

5.2.2 Mixed boundary conditions

In the following examples the results are demonstrated in a table. The third column is related to the approximate values of u, the last column is corresponds to the approximate values of $v = \frac{\partial u}{\partial x}$ respectively.

In this section, we solve the problem with a mixed boundary condition on the sides, an approach was developed for the first derivative by using the third order differentiation formula. It is showed as cases which consist all possible problems.

In this section we have examined four different cases:

- <u>Case 1:</u> The Neumann condition on the left side is given;
- <u>Case 2:</u> The Neumann condition on the left and right sides are given;
- <u>Case 3:</u> The Neumann condition on the left and up sides are given;
- <u>Case 4:</u> The Neumann condition on the left, right and up sides are given.

All other possibilities are thought in the same way.

<u>Case 1:</u> The Neumann condition on the left side is given;

Example 5.4 : Let $\Pi = \{(x, y): 0 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem :

$$\Delta u = 0 \; on \; \Pi, u = \varphi_j(x, y) on \; \gamma_j, j = 1,2,3, \;\; u^{(1)} = \frac{\partial u(0,y)}{\partial x} = \Psi_4(y) \; on \; \gamma_4$$

where

$$\varphi_j(x, y) = (x^2 + y^2)^{\frac{181}{90}} cos\left(\frac{181}{45} \arctan\left(\frac{y}{x}\right)\right) \; on \; \gamma_j, \quad j = 1,2,3$$

and

$$\Psi_4\,(y) = \frac{181}{45} y^{\frac{136}{45}} \sin\left(\frac{181\pi}{90}\right) on \; \gamma_4$$

is the exact solution of this problem.

Table 5.12: The approximate results for Case 1 for the solution and first derivative when $\varphi \in C^{4,\frac{1}{45}}$

h	$\|u-u_h\|$	E_u^m	$\|v-v_h\|$	E_v^m
$\frac{1}{16}$	$6{,}08E-06$	$16{,}21$	$1{,}54E-03$	$7{,}99$
$\frac{1}{32}$	$3{,}75E-07$	$16{,}23$	$1{,}93E-04$	$7{,}98$
$\frac{1}{64}$	$2{,}31E-08$	$16{,}27$	$2{,}41E-05$	$8{,}01$
$\frac{1}{128}$	$1{,}42E-09$		$3{,}02E-06$	

Table 5.12 shows that order of the solution of the problem given in Example 5.4 is $O(h^4)$ when the given Neumann condition on the left side and order of the first derivative $O(h^3)$ when used the third order forward backward numerical differentiation formula on the right side.

<u>Case 2:</u> The Neumann condition on the left and right sides are given;

Example 5.5 : Let $\Pi = \{(x,y): 0 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem :

$$\Delta u = 0 \; on \; \Pi, u = \varphi_j(x,y) on \; \gamma_j, j = 1,3, \; u^{(1)} = \frac{\partial u}{\partial x} = \Psi_k(y) \; on \; \gamma_k, k = 2,4$$

where

$$\varphi_j(x,y) = (x^2 + y^2)^{\frac{181}{90}} cos\left(\frac{181}{45} \arctan\left(\frac{y}{x}\right)\right) \; on \; \gamma_j, j = 1,3$$

and

$$\Psi_2(y) = \frac{181}{45}(1+y^2)^{\frac{91}{90}}\left\{ysin\left(\frac{181}{45}\arctan(y)\right) + \cos\left(\frac{181}{45}arctan(y)\right)\right\}$$

$$\Psi_4(y) = \frac{181}{45}y^{\frac{136}{45}}\sin\left(\frac{181\pi}{90}\right) on\ \gamma_4$$

is the exact solution of this problem.

Table 5.13: The approximate results for Case 2 for the solution and first derivative when $\varphi \in C^{4,\frac{1}{45}}$

h	$\|u - u_h\|$	E_u^m	$\|v - v_h\|$	E_v^m
$\frac{1}{16}$	$6{,}08E-06$	16,22	$4{,}97E-08$	8,12
$\frac{1}{32}$	$3{,}75E-07$	16,24	$6{,}12E-09$	8,12
$\frac{1}{64}$	$2{,}31E-08$	16,25	$7{,}54E-10$	8,12
$\frac{1}{128}$	$1{,}42E-09$		$9{,}27E-11$	

Table 5.13 shows that order of the solution of the problem given in Example 5.5 is $O(h^4)$ when the given Neumann condition on the left and right sides and order of the first derivative $O(h^3)$.

<u>Case 3:</u> The Neumann condition on the left and up sides are given;

Example 5.6 : Let $\Pi = \{(x, y): 0 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem :

$$\Delta u = 0 \ on \ \Pi, u = \varphi_j(x, y) \ on \ \gamma_j, j = 1,2,$$

$$u^{(1)} = \frac{\partial u(x, 1)}{\partial y} = \Psi_3(x) \ on \ \gamma_3,$$

$$u^{(1)} = \frac{\partial u(0, y)}{\partial x} = \Psi_4(y) \ on \ \gamma_4,$$

where

$$\varphi_j(x, y) = (x^2 + y^2)^{\frac{181}{90}} cos\left(\frac{181}{45} \arctan\left(\frac{y}{x}\right)\right) \ on \ \gamma_j, \ j = 1,2$$

and

$$\Psi_3\ (x) = \frac{181}{45}(1 + x^2)^{\frac{91}{90}}\left\{cos\left(\frac{181}{45} \arctan\left(\frac{1}{x}\right)\right) - x \sin\left(\frac{181}{45} arctan\left(\frac{1}{x}\right)\right)\right\}$$

$$\Psi_4\ (y) = \frac{181}{45} y^{\frac{136}{45}} \sin\left(\frac{181\pi}{90}\right) on \ \gamma_4$$

is the exact solution of this problem.

Table 5.14: The approximate results for Case 3 for the solution and first derivative when $\varphi \in C^{4,\frac{1}{45}}$

h	$\|\|u-u_h\|\|$	E_u^m	$\|\|v-v_h\|\|$	E_v^m
$\frac{1}{16}$	$6{,}10E-06$	16,25	$1{,}54E-03$	8,00
$\frac{1}{32}$	$3{,}75E-07$	16,25	$1{,}93E-04$	8,00
$\frac{1}{64}$	$2{,}31E-08$	16,25	$2{,}41E-05$	8,00
$\frac{1}{128}$	$1{,}42E-09$		$3{,}02E-06$	

Table 5.14 shows that order of the solution of the problem given in Example 5.6 is $O(h^4)$ when the given Neumann condition on the left and up sides and order of the first derivative $O(h^3)$ when used the third order forward backward numerical differentiation formula on the right and up sides.

<u>Case 4:</u> The Neumann condition on the left, right and up sides are given;

Example 5.7 : Let $\Pi = \{(x,y): 0 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem :

$$\Delta u = 0 \; on \; \Pi, \qquad u = \varphi_1(x,y) \; on \; \gamma_1,$$

$$u^{(1)} = \frac{\partial u(x,1)}{\partial y} = \Psi_3(x) \; on \; \gamma_3,$$

$$u^{(1)} = \frac{\partial u(y)}{\partial x} = \Psi_k(y) \; on \; \gamma_k, \qquad k = 2{,}4$$

where

$$\varphi_1(x) = x^{\frac{181}{45}} \; on \; \gamma_1,$$

and

$$\Psi_2\,(y) = \frac{181}{45}(1+y^2)^{\frac{91}{90}}\left\{ysin\left(\frac{181}{45}\arctan(y)\right) + \cos\left(\frac{181}{30}\,arctan(y)\right)\right\}$$

$$\Psi_3\,(x) = \frac{181}{45}(1+x^2)^{\frac{91}{90}}\left\{cos\left(\frac{181}{45}\arctan\left(\frac{1}{x}\right)\right) - x\sin\left(\frac{181}{45}\,arctan\left(\frac{1}{x}\right)\right)\right\}$$

$$\Psi_4\,(y) = \frac{181}{30}y^{\frac{136}{45}}\sin\left(\frac{181\pi}{90}\right)$$

is the exact solution of this problem.

Table 5.15: The approximate results for Case 4 for the solution and first derivative when $\varphi \in C^{4,\frac{1}{45}}$

$\boldsymbol{h}$	$\lVert \boldsymbol{u}-\boldsymbol{u_h} \rVert$	$\boldsymbol{E_u^m}$	$\lVert \boldsymbol{v}-\boldsymbol{v_h} \rVert$	$\boldsymbol{E_v^m}$
$\frac{1}{16}$	$6{,}11E-06$	16,28	$1{,}54E-03$	7,99
$\frac{1}{32}$	$3{,}76E-07$	16,26	$1{,}93E-04$	8,00
$\frac{1}{64}$	$2{,}31E-08$	16,26	$2{,}41E-05$	8,00
$\frac{1}{128}$	$1{,}42E-09$		$3{,}00E-06$	

Table 5.15 shows that order of the solution of the problem given in Example 5.7 is $O(h^4)$ when the given Neumann condition on the left, ride and up sides and order of the first derivative $O(h^3)$ when used the third order forward backward numerical differentiation formula on the up side.

On the based all this results, by using third-order accurate numerical differentiation formula, we obtain estimation of error the of the derivative of the function which is order of $O(h^3)$ as it is seen in cases.

CHAPTER 6
CONCLUSION

For the approximate solution of the Laplace equation in a rectangular parallelepiped R, a new pointwise error of order $O(h^4\rho^{p-4})$ is obtained, where $p \in \{4,5\}$ and ρ is the distance from the current point to the boundary of R. This estimation shows the additional downturn of the error near the boundary as ρ, which is used to get $O(h^{p-1})$ order of accuracy for the approximate value of the first derivatives of the solution of Laplace's equation. For the approximation of the second order pure and mixed derivatives the obtained rate of convergence are $O\left(h^{p-2+\lambda}\right)$ and $O(h^{p-2})$, respectively.

The multy stage method is constructed and justified to obtain a high order approximation of the solution and its derivatives of the Dirichlet problem for Laplace's equation on a rectangular domain. It is assumed that the boundary functions on the sides are from $C^{12,\lambda}$, $0 < \lambda < 1$, and at the vertices satisfy the compatibility conditions for the even derivatives up to tenth order, which result from the Laplace equation. Under these conditions, the constructed approximate value of the first derivatives by the proposed method converge to the exact derivatives of the solution in the uniform metric of order $O(h^8)$. To obtain this estimation, for the error of the approximate solution of the given problem by the three stage method a new pointwise estimation $O(\rho h^8)$ is proved, where $\rho = \rho(x,y)$ is the distance from the current grid point $(x,y) \in \Pi^h$ to the boundary of the rectangle Π. Then, at the first stage, by using the 9-point scheme the first derivative of the sum of the pure fourth derivatives of the desired solution is approximated of order $O(h^6)$, and at the second stage, approximate values of the first derivative of the sum of the pure eighth derivatives is approximated of order $O(h^2)$ by the 5-point scheme. At the final third stage, the system of simplest 5-point difference equations approximating the Dirichlet problem for the first derivative of the solution is corrected by introducing the quantities determined at the first and second stages which converges at rate of $O(h^8)$.

In a rectangular domain, we investigate an approximation of the first order derivatives for the solution of the mixed boundary value problem. The boundary values on the sides of the rectangle are supposed to have the fourth derivatives satisfying the Hölder condition. On the vertices besides the continuity condition, the compatibility conditions, which result from the Laplace equation for the second and fourth derivatives of the boundary values, given on the adjacent sides are satisfied. For the approximate values of the first derivatives of a solution of the mixed boundary value problem the constructed difference scheme on a square grid a uniform error estimation of order $O(h^3)$ is obtained.

The obtained results can be applied for the approximation of a solution and its derivatives of problems in more complicated domains, when different version of domain decomposition methods are used (see for 2D problem (Dosiyev, 1992), (Dosiyev, 1994), (Dosiyev, 2003), (Dosiyev, 2004), (Dosiyev, 2012), (Dosiyev, 2013), (Dosiyev, 2014) (Volkov, 1976), see for 3D (Smith et at ., 2004), (Volkov, 1979), (Volkov, 2003)). Moreover the constructed difference schemes can be used to get a highly accurate results in many applied problems in electrostatics and fracture mechanics.

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CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name : Sarikaya, Hediye

Nationality : Cyprus

DateandPlace of Birth : 04 July 1989, Lefkoşa

MaritalStatus : Married

EDUCATION

Degree	Institution	Year of Graduation
M.Sc.	NEU, Department of Mathematics	2013
B.Sc.	NEU, Department of Mathematics	2011

WORK EXPERIENCE

Year	Place	Enrollment
March, 2014 -Present	Department of Mathematics	Lecturer

FOREING LANGUAGES

Fluent spoken and written English

HONORS AND AWARDS

- Young Scientist Award, NEU, 2017

PUPLICATIONS IN INTERNATIONAL REFEREED JOURNALS (IN COVERAGE OF SSCI/SCI-ECPANDED AND AHCI):

- Adıgüzel A. Dosiyev and Hediye Sarıkaya, (2018), 14-Point Difference Operator for the Approximation of the First Derivatives of a Solution of Laplace’s Equation in a Rectangular Parallelepiped, FİLOMAT (SCI-E), (https://doi.org/10.2298/FIL1803791D)

- Adıgüzel A. Dosiyev and Hediye Sarıkaya, (Accepted), On the difference method for approximation of second order derivatives of a solution of Laplace's equation in a rectangular parallelepiped, FİLOMAT (SCI-E),

BULLETING PRESENTED IN INTERNATIONAL ACADEMIC MEETINGS AND PUBLISHED IN PROCEEDINGS BOOKS:

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INTERNATIONAL CONFERENCE PRESENTATIONS

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- Adıgüzel Dosiyev and Hediye Sarıkaya, (2017), On the difference method for approximation of second order derivatives of a solution of Laplace's equation in a rectangular parallelepiped, International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM 2017), Kuşadası, Turkey.
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- Adıgüzel Dosiyev and Hediye Sarıkaya, (2018), A highly accurate corrected scheme in solving the Laplace's equation on a rectangle, 3rd International Conference on Computational Mathematics and Engineering Sciences (CMES'2018), Girne, Kıbrıs

THESIS

Master

- Hediye Sarıkaya (2013), Q-Biorthogonal Polynomials, Near East University, Department of Mathematics, Nicosia, Cyprus.

COURSES GIVEN (from 2014-2018)

Undergraduate

- Complex Analysis (English)
- Mathematics for Business and Economics (English)
- İşletme Matematiği (Turkish)
- İstatistik I (Turkish)
- Complex Analysis II (English)
- İstatistik II (Turkish)
- Topology I (English)
- Topology II (English)
- Biyoistatistik (Turkish)
- Mathematıcs I (English)
- Matematik I (Türkish)
- Numerical Analysis (English)
- Sayısal Analiz (Turkish)
- Temel Matematik (Turkish)
- Analiz III (English)
- Analiz IV (English)
- Ağırlama Endüstrisi İçin Matematik (Turkish)
- Matematik Ve Geometri II (Turkish)
- Matematik Ve Geometri I (Turkish)
- Kriptoloji (English)
- Sınıf Öğretmenliği Temel Matematiği (Turkish)

HOBBIES

Reading, Environment and Earth, Music, Travel, Draw a Picture

OTHER INTERESTS

Programming Languages, Databese, Informational Technologies, ProgramingLogic, Information Systems, Algorithm, Kriptology