NUMERICAL SOLUTIONS OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FOR OBSERVING EPIDEMIC MODELS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
AMEERA MANSOUR AMER YOUNES

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

Ameera Mansour Amer Younes

Numerical solutions of the system of partial differential equations for observing epidemic models

NEU 2019

NUMERICAL SOLUTIONS OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FOR OBSERVING EPIDEMIC MODELS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

AMEERA MANSOUR AMER YOUNES

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

Ameera Mansour Amer Younes: NUMERICAL SOLUTIONS OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FOR OBSERVING EPIDEMIC MODELS

Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics Department

Examining Committee in Charge:

Prof.Dr. Evren Hınçal	Committee Chairman Depart of Mathematics, NEU.
Prof.Dr. Allaberen Ashyralyev	Supervisor, Department of Mathematics, NEU.
Assoc.Prof.Dr. Deniz Agirseven	Department of Mathematics Trakya University.

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, last name: Ameera Younes

Signature:

Date:

ACKNOWLEDGMENTS

Firstly, I would like to express my best gratitude and appreciation to my family especially my parent that who have been supported and encourage me in all steps of life including my education journey and my husband that he have been help me to finish this study .In particular, I would like to extend many thanks to Prof. Dr. Allaberen Ashyralyev my supervisor at Near East University, for continuous provision of many useful suggestions and constructive feedback, which enabled me to complete this study. Although, I would like to express many thanks to the Near East University staff and organizations that were involved in providing me with the opportunity to study for a master of science.

To my parents...

ABSTRACT

In the present study, a system of partial differential equations for observing epidemic models is investigated. Using tools of classical approach we are enabled to obtain the solution of the several system of partial differential equations for observing epidemic models. Furthermore, difference schemes for the numerical solution of the system of partial differential equations for observing epidemic models are presented. Then, these difference schemes are tested on an example and some numerical results are presented.

***Keywords*:** System of partial differential equations; Fourier series method; Laplace transform method; Fourier transform method; difference schemes; epidemic models

ÖZET

Bu çalışmada, epidemik modelleri gözlemlemek için bir kısmi diferansiyel denklem sistemi araştırılmıştır. Klasik yaklaşım araçlarını kullanarak epidemik modelleri gözlemlemek için birkaç kısmi diferansiyel denklem sisteminin çözümünü elde etmeyi başardık. Ayrıca, epidemik modelleri gözlemlemek için kısmi diferansiyel denklemler sisteminin sayısal çözümü için fark şemaları sunulmuştur. Daha sonra, bu fark şemaları bir örnek üzerinde test edilipve bazı nümerik sonuçlar Verilmiştir.

***Anahtar Kelimeler*:** Kısmi diferansiyel denklem sistemleri; Fourier serileri yöntemi; Laplace dönüşümü yöntemi; Fourier dönüşümü yöntemi; fark şemaları; salgın modeller

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ABBREVIATIONS

HIV: Human immunodeficiency virus

CHAPTER 1
INTRODUCTION

System of partial differential equations take an important place in applied sciences and engineering applications and have been studied by many authors.

Direct and inverse boundary value problems for system of partial differential equations for observing epidemic models have been a major research area in many branches of science and engineering particularly in applied mathematics.

The mechanism of transmission is usually qualitatively known for most diseases from epidemiological point of view. For modeling the spread process of infectious diseases mathematically and quantitatively, many classical epidemic models have been proposed and studied, such as SIR, SIS, SEIR, and SIRS models (Li & liu, 2014; Samarskii, 2001; Lotfi et al., 2014; Chalub & Souza, 2011; Elkadry, 2013). Modeling infectious diseases can be classified as some basic deterministic models, simple stochastic models and spatial models. An important role of modelling is that they can inform us to the disadvantages in our present consideration of the epidemiology of different infectious diseases, and advise compelling questions for research and data that need to be collected. The rate at which susceptible individuals become infected is called the transmission rate. It is important to know this rate in order to study the spread and the effect of an infectious disease in a population. This study aims at providing an understanding of estimating the transmission rate from mathematical models representing the population dynamics of an infectious diseases using solution of these models.

An important advantage of using models is that the mathematical representation of biological processes enables transparency and accuracy regarding the epidemiological assumptions, thus enabling us to test our understanding of the disease epidemiology by comparing model results and observed patterns (Jun-Jie et al., 2010). A model can also assist in decision-making by making projections regarding important issues such as intervention-induced changes in the spread of disease. A point that deserves emphasis is that transmission models are based on the current understanding of the natural history of infection and immunity. In cases where such knowledge is lacking, assumptions can be made regarding these processes. However, in such cases there can be several possible mechanisms, and therefore several different models, which can lead to similar observed patterns, so that it is not always possible to learn about underlying mechanisms by comparing model outcomes. One must then be very cautious regarding model predictions, because different models that lead to similar outcomes in one context may fail to do so in another. In such instances, it is best to conduct further epidemiological and experimental studies in order to discriminate among the different possible mechanisms. Thus, an important role of modelling enterprises is that they can alert us to the deficiencies in our current understanding of the epidemiology of various infectious diseases, and suggest crucial questions for investigation and data that need to be collected. Therefore, when models fail to predict, this failure can provide us with important clues for further research. Our aim is first to understand the causes of a biological problem or epidemics, then to predict its course, and finally to develop ways of controlling it, including comparisons of different possible approaches. The first step is obtaining and analyzing observed data (Lotfi et al, 2014; Elkadry, 2013).

Various initial-boundary-value problems for the system of partial differential equations can be reduced to the initial-value problem for the system of ordinary differential equations

$$\begin{cases} \frac{du^1(t)}{dt} + \alpha u^1(t) + Au^1(t) = f^1(t), \\ \frac{du^2(t)}{dt} + \beta u^2(t) - \beta_1 u^1(t) + cAu^2(t) = f^2(t), \\ \frac{du^3(t)}{dt} + \gamma u^3(t) - \gamma_1 u^1(t) + eAu^3(t) = f^3(t), \\ \frac{du^4(t)}{dt} + du^4(t) - d_1 u^3(t) - d_2 u^2(t) + lAu^4(t) = f^4(t), \\ 0 < t < T, u^m(0) = \varphi^m, m = 1,2,3,4 \end{cases} \tag{1.1}$$

In a Hilbert space H with a self-adjoint positive definite operator A. In the paper Ashyralyev et al, (2018) stability of initial-boundary value problem (1.1) for the system of partial differential equations for observing HIV mother to child transmission epidemic models is studied. Applying operator approach, theorems on stability of this problem and of difference schemes for approximate solutions of this problem are established. The generality of the approach considered in this paper, however, allows for treating a wider class of multidimensional problems. Numerical results are provided.

In the present thesis, we will consider the application of classical methods of solution of problem (1.1) and of difference scheme for the approximate solution of problem (1.1).

This thesis is organized as follows. Chapter 1 is introduction. In chapter 2, the solution of system of partial differential equations for observing epidemic models is obtained by using tools of classical approach. In chapter3, numerical results are provided by using finite difference method for the solution of system of partial differential equations. In appendix matlab programming that is used for finding numerical results is given.

CHAPTER 2
METHODS FOR SOLUTION OF SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

It is known that system of partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

2.1. Fourier Series Method

Example 1: Obtain the Fourier series solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = \alpha\, e^{-4t} \sin 2x, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (\beta - \beta_1) e^{-4t} \sin 2x, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (\delta - \delta_1) e^{-4t} \sin 2x, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (d - d_1 - d_2) e^{-4t} \sin 2x, \\ 0 < t < T, 0 < x < \pi, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = \sin 2x, 0 \le x \le \pi, \\ u(t,0) = v(t,0) = w(t,0) = z(t,0) = 0, 0 \le t \le T, \\ u(t,\pi) = v(t,\pi) = w(t,\pi) = z(t,\pi) = 0,\ 0 \le t \le T \end{cases} \tag{2.1}$$

for the system of parabolic equations.

Solution: In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x)-\lambda u(x)=0, 0<x<\pi, u(0)=u(\pi)=0$$

Generated by the space operator of problem (2.1). It is easy to see that the solution of this Sturm- Liouville problem is

$$\lambda_k=-k^2, u_k(x)=\sin kx, k=1,2,\dots.$$

Then, we will obtain the Fourier series solution of problem (2.1) by formula

$$\begin{cases} u(t,x)=\sum_{k=1}^{\infty} A_k(t)\sin kx, \\ v(t,x)=\sum_{k=1}^{\infty} B_k(t)\sin kx, \\ w(t,x)=\sum_{k=1}^{\infty} C_k(t)\sin kx, \\ z(t,x)=\sum_{k=1}^{\infty} D_k(t)\sin kx. \end{cases} \tag{2.2}$$

Here $A_k(t)$, $B_k(t), C_k(t)$ and $D_k(t)$ are unknown functions. Applying these formula to the system of equations and initial conditions, we get

$$
\begin{cases}
\sum_{k=1}^{\infty} A'_k(t)\sin kx + \alpha \sum_{k=1}^{\infty} A_k(t)\sin kx + \sum_{k=1}^{\infty} k^2 A_k(t)\sin kx = \alpha\, e^{-4t}\sin 2x, \\
\sum_{k=1}^{\infty} B'_k(t)\sin kx + \beta \sum_{k=1}^{\infty} B_k(t)\sin kx - \beta_1 \sum_{k=1}^{\infty} A_k(t)\sin kx \\
+\sum_{k=1}^{\infty} k^2 B_k(t)\sin kx = (\beta - \beta_1)e^{-4t}\sin 2x, \\
\sum_{k=1}^{\infty} C'_k(t)\sin kx + \delta \sum_{k=1}^{\infty} C_k(t)\sin kx - \delta_1 \sum_{k=1}^{\infty} A_k(t)\sin kx \\
+\sum_{k=1}^{\infty} k^2 C_k(t)\sin kx = (\delta - \delta_1)e^{-4t}\sin 2x, \\
\sum_{k=1}^{\infty} D'_k(t)\sin kx + d \sum_{k=1}^{\infty} D_k(t)\sin kx - d_1 \sum_{k=1}^{\infty} C_k(t)\sin kx \\
-d_2 \sum_{k=1}^{\infty} B_k(t)\sin kx + \sum_{k=1}^{\infty} k^2 D_k(t)\sin kx = (d - d_1 - d_2)e^{-4t}\sin 2x, \\
0 < t < T, 0 < x < \pi,
\end{cases}
$$

$$
\begin{cases}
u(0,x) = \sum_{k=1}^{\infty} A_k(0)\sin kx = \sin 2x, \\
v(0,x) = \sum_{k=1}^{\infty} B_k(0)\sin kx = \sin 2x, \\
w(0,x) = \sum_{k=1}^{\infty} C_k(0)\sin kx = \sin 2x, \\
z(0,x) = \sum_{k=1}^{\infty} D_k(0)\sin kx = \sin 2x, \\
0 \le x \le \pi.
\end{cases}
$$

Equating coefficients $\sin kx, k = 1,...$ to zero, we get

$$\begin{cases} A_2'(t) + \alpha A_2(t) + 4A_2(t) = \alpha e^{-4t}, \\ B_2'(t) + \beta B_2(t) - \beta_1 A_2(t) + 4B_2(t) = (\beta - \beta_1)e^{-4t}, \\ C_2'(t) + \delta C_2(t) - \delta_1 A_2(t) + 4C_2(t) = (\delta - \delta_1)e^{-4t}, \\ D_2'(t) + dD_2(t) - d_1 C_2(t) - d_2 B_2(t) + 4D_2(t) = (d - d_1 - d_2)e^{-4t}, \\ 0 < t < T, A_2(0) = B_2(0) = C_2(0) = D_2(0) = 1 \end{cases}$$

And for k≠2

$$\begin{cases} A_k'(t) + \alpha A_k(t) + k^2 A_k(t) = 0, \\ B_k'(t) + \beta B_k(t) - \beta_1 A_k(t) + k^2 B_k(t) = 0, \\ C_k'(t) + \delta C_k(t) - \delta_1 A_k(t) + k^2 C_k(t) = 0, \\ D_k'(t) + d\,D_k(t) - d_1 C_k(t) - d_2 B_k(t) + k^2 D_k(t) = 0, \\ 0 < t < T, \\ A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0. \end{cases}$$

We will obtain $A_k(t)$, $B_k(t)$, $C_k(t)$ and $D_k(t)$ for $k \neq 2$. Firstly, we consider the problem

$$A_k'(t) + (\alpha + k^2)A_k(t) = 0,\ 0 < t < T, A_k(0) = 0.$$

We have that

$$A_k(t) = e^{-(\alpha + k^2)t} A_k(0) = 0.$$

Secondly, applying $A_k(t) = 0,$ we get the following problem

$$B_k'(t) + (\beta + k^2)B_k(t) = 0,\ 0 < t < T, B_k(0) = 0.$$

Therefore,

$$B_k(t) = e^{-\left(\beta+k^2\right)t}B_k(0) = 0.$$

Thirdly, applying $A_k(t) = 0,$ we get the following problem

$$C_k'(t) + (\delta + k^2)C_k(t) = 0,\ 0 < t < T, C_k(0) = 0.$$

Therefore,

$$C_k(t) = e^{-\left(\delta+k^2\right)t}C_k(0) = 0.$$

Fourthly, using $B_k(t) = 0$ and $C_k(t) = 0,$ we get the following problem

$$D_k'(t) + (d + k^2)D_k(t) = 0,\ 0 < t < T, D_k(0) = 0.$$

Therefore,

$$D_k(t) = e^{-\left(d+k^2\right)t}D_k(0) = 0.$$

Thus, $A_k(t) = B_k(t) = C_k(t) = D_k(t) = 0$ for any $t \in [0,T].$

Now, we obtain $A_2(t),$ $B_2(t),$ $C_2(t)$ and $D_2(t)$. Firstly, we consider the problem

$$A_2'(t) + (\alpha + 4)A_2(t) = \alpha e^{-4t},\ 0 < t < T,\ A_2(0) = 1.$$

We have that

$$A_2(t) = e^{-(\alpha+4)t}A_2(0) + \int_0^t e^{-(\alpha+4)(t-s)}\alpha e^{-4s}ds$$

$$= e^{-(\alpha+4)t} + e^{-(\alpha+4)t}\int_0^t \alpha e^{\alpha s}ds = e^{-(\alpha+4)t} + e^{-(\alpha+4)t}(e^{\alpha t} - 1) = e^{-4t}.$$

Secondly, applying $A_2(t) = e^{-4t},$ we get the following problem

$$B_2'(t) + (\beta + 4)B_2(t) = \beta e^{-4t}, 0 < t < T, B_2(0) = 1.$$

We have that

$$B_2(t) = e^{-(\beta+4)t}B_2(0) + \int_0^t e^{-(\beta+4)(t-s)}\beta e^{-4s}ds$$

$$= e^{-(\beta+4)t} + e^{-(\beta+4)t}\int_0^t \beta e^{\beta s}ds = e^{-(\beta+4)t} + e^{-(\beta+4)t}(e^{\beta t} - 1) = e^{-4t}.$$

Thirdly, applying $A_2(t) = e^{-4t}$, we get the following problem

$$C_2'(t) + (\delta + 4)C_2(t) = \delta e^{-4t},\ 0 < t < T, C_2(0) = 1.$$

We have that

$$C_2(t) = e^{-(\delta+4)t}C_2(0) + \int_0^t e^{-(\delta+4)(t-s)}\delta e^{-4s}ds$$

$$= e^{-(\delta+4)t} + e^{-(\delta+4)t}\left(e^{\delta t} - 1\right) = e^{-4t}.$$

Fourthly, using $B_2(t) = e^{-4t}$ and $C_2(t) = e^{-4t}$, we get

$$D_2'(t) + (d + 4)D_2(t) = de^{-4t},\ 0 < t < T, D_2(0) = 1.$$

We have that

$$D_2(t) = e^{-(d+4)t}D_2(0) + \int_0^t e^{-(d+4)(t-s)}de^{-4s}ds$$

$$= e^{-(d+4)t} + e^{-(d+4)t}(e^{dt} - 1) = e^{-4t}.$$

Thus, $A_2(t) = B_2(t) = C_2(t) = D_2(t) = e^{-4t}$ for any $t \in [0, T]$. Applying formulas obtained for $A_k(t)$, $B_k(t)$, $C_k(t)$ and $D_k(t), k = 1,...,$ we can obtain the exact solution of problem (2.1) by formulas

$$\begin{cases} u(t,x) = A_2(t)\sin 2x = e^{-4t}\sin 2x, \\ v(t,x) = B_2(t)\sin 2x = e^{-4t}\sin 2x, \\ w(t,x) = C_2(t)\sin 2x = e^{-4t}\sin 2x, \\ z(t,x) = D_2(t)\sin 2x = e^{-4t}\sin 2x. \end{cases}$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = \alpha f_1(t,x), \\
\frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 v(t,x)}{\partial x_r^2} = (\beta - \beta_1) f_2(t,x), \\
\frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 w(t,x)}{\partial x_r^2} = (\delta - \delta_1) f_3(t,x), \\
\frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\ w(t,x) - d_2\ v(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 z(t,x)}{\partial x_r^2} \\
= (d - d_1 - d_2) f_4(t,x), \\
x = (x_1, \ldots, x_n) \in \overline{\Omega},\ 0 < t < T, \\
u(0,x) = \varphi(x), v(0,x) = \psi(x), w(0,x) = \xi(x), z(0,x) = \lambda(x), \\
x = (x_1, \ldots, x_n) \in \overline{\Omega}, \\
u(t,x) = v(t,x) = w(t,x) = z(t,x) = 0, x \in S, 0 \le t \le T
\end{cases} \tag{2.3}
$$

For the multidimensional system of partial differential equations. Assume that $\alpha_r > \alpha > 0$ and $f_k(t,x), k = 1,2,3,4 \left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(x), \lambda(x)$ $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$ dimensional Euclidean space $R^n \left(0 < x_k < 1, 1 \le k \le n\right)$ with the boundary $S, \overline{\Omega} = \Omega \cup S.$

However Fourier series method described in solving (2.3) can be used only in the case when (2.3) has constant coefficients.

Example 2: Obtain the Fourier series solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = \alpha\, e^{-t} \cos x, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (\beta - \beta_1) e^{-t} \cos x, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (\delta - \delta_1) e^{-t} \cos x, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (d - d_1 - d_2) e^{-t} \cos x, \\ 0 < t < T, 0 < x < \pi, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = \cos x, 0 \le x \le \pi, \\ u_x(t,0) = v_x(t,0) = w_x(t,0) = z_x(t,0) = 0, 0 \le t \le T, \\ u_x(t,\pi) = v_x(t,\pi) = w_x(t,\pi) = z_x(t,\pi) = 0,\ 0 \le t \le T \end{cases} \tag{2.4}$$

for the system of parabolic equations.

Solution: In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi, u_x(0) = u_x(\pi) = 0$$

Generated by the space operator of problem (2.4). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = -k^2, u_k(x) = \cos kx, k = 0, 1, \ldots.$$

Then, we will obtain the Fourier series solution of problem (2.4) by formula

$$\begin{cases} u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos kx, \\ v(t,x) = \sum_{k=0}^{\infty} B_k(t)\cos kx, \\ w(t,x) = \sum_{k=0}^{\infty} C_k(t)\cos kx, \\ z(t,x) = \sum_{k=0}^{\infty} D_k(t)\cos kx. \end{cases}$$

Here $A_k(t)$, $B_k(t), C_k(t)$ and $D_k(t)$ are unknown functions. Applying these formula to the system of equations and initial conditions, we get

$$
\begin{cases}
\sum_{k=0}^{\infty} A_k'(t)\cos kx + \alpha \sum_{k=0}^{\infty} A_k(t)\cos kx + \sum_{k=0}^{\infty} k^2 A_k(t)\cos kx = \alpha\, e^{-t}\cos x, \\
\sum_{k=0}^{\infty} B_k'(t)\cos kx + \beta \sum_{k=0}^{\infty} B_k(t)\cos kx - \beta_1 \sum_{k=0}^{\infty} A_k(t)\cos kx \\
+\sum_{k=0}^{\infty} k^2 B_k(t)\cos kx = (\beta - \beta_1)e^{-t}\cos x, \\
\sum_{k=0}^{\infty} C_k'(t)\cos kx + \delta \sum_{k=0}^{\infty} C_k(t)\cos kx - \delta_1 \sum_{k=0}^{\infty} A_k(t)\cos kx \\
+\sum_{k=0}^{\infty} k^2 C_k(t)\cos kx = (\delta - \delta_1)e^{-t}\cos x, \\
\sum_{k=0}^{\infty} D_k'(t)\cos kx + d \sum_{k=0}^{\infty} D_k(t)\cos kx - d_1 \sum_{k=0}^{\infty} C_k(t)\cos kx \\
-d_2 \sum_{k=0}^{\infty} B_k(t)\cos kx + \sum_{k=0}^{\infty} k^2 D_k(t)\cos kx = (d - d_1 - d_2)e^{-t}\cos x, \\
0 < t < T, 0 < x < \pi,
\end{cases}
$$

$$
\begin{cases}
u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos kx = \cos x, \\
v(0,x) = \sum_{k=0}^{\infty} B_k(0)\cos kx = \cos x, \\
w(0,x) = \sum_{k=0}^{\infty} C_k(0)\cos kx = \cos x, \\
z(0,x) = \sum_{k=0}^{\infty} D_k(0)\cos kx = \cos x, \\
0 \le x \le \pi.
\end{cases}
$$

Equating coefficients $\cos kx, k = 0,...$ to zero, we get

$$
\begin{cases}
A_1'(t) + \alpha A_1(t) + A_1(t) = \alpha e^{-t}, \\
B_1'(t) + \beta B_1(t) - \beta_1 A_1(t) + B_1(t) = (\beta - \beta_1)e^{-t}, \\
C_1'(t) + \delta C_1(t) - \delta_1 A_1(t) + C_1(t) = (\delta - \delta_1)e^{-t}, \\
D_1'(t) + dD_1(t) - d_1 C_1(t) - d_2 B_1(t) + D_1(t) = (d - d_1 - d_2)e^{-t}, \\
0 < t < T, A_1(0) = B_1(0) = C_1(0) = D_1(0) = 1
\end{cases}
$$

And for k≠1

$$
\begin{cases}
A_k'(t) + \alpha A_k(t) + k^2 A_k(t) = 0, \\
B_k'(t) + \beta B_k(t) - \beta_1 A_k(t) + k^2 B_k(t) = 0, \\
C_k'(t) + \delta C_k(t) - \delta_1 A_k(t) + k^2 C_k(t) = 0, \\
D_k'(t) + d\, D_k(t) - d_1 C_k(t) - d_2 B_k(t) + k^2 D_k(t) = 0, \\
0 < t < T, \\
A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0.
\end{cases}
$$

We will obtain $A_k(t)$, $B_k(t)$, $C_k(t)$ and $D_k(t)$ for $k \neq 1$. Firstly, we consider the problem

$$A_k'(t) + (\alpha + k^2) A_k(t) = 0,\ 0 < t < T, A_k(0) = 0.$$

We have that

$$A_k(t) = e^{-\left(\alpha + k^2\right)t} A_k(0) = 0.$$

Secondly, applying $A_k(t) = 0,$ we get the following problem

$$B_k'(t) + (\beta + k^2) B_k(t) = 0,\ 0 < t < T, B_k(0) = 0.$$

Therefore,

$$B_k(t) = e^{-\left(\beta + k^2\right)t} B_k(0) = 0.$$

Thirdly, applying $A_k(t) = 0,$ we get the following problem

$$C_k'(t) + (\delta + k^2) C_k(t) = 0,\ 0 < t < T, C_k(0) = 0.$$

Therefore,

$$C_k(t) = e^{-\left(\delta + k^2\right)t} C_k(0) = 0.$$

Fourthly, using $B_k(t)=0$ and $C_k(t)=0,$ we get the following problem

$$D_k'(t)+(d+k^2)D_k(t)=0,\ 0<t<T, D_k(0)=0.$$

Therefore,

$$D_k(t)=e^{-\left(d+k^2\right)t}D_k(0)=0.$$

Thus, $A_k(t)=B_k(t)=C_k(t)=D_k(t)=0$ for any $t\in[0,T]$.

Now, we obtain $A_1(t),\quad B_1(t),\quad C_1(t)$ and $D_1(t)$. Firstly, we consider the problem

$$A_1'(t)+(\alpha+1)A_1(t)=\alpha e^{-t},\ 0<t<T,\ A_1(0)=1.$$

We have that

$$A_1(t)=e^{-(\alpha+1)t}A_1(0)+\int_0^t e^{-(\alpha+1)(t-s)}\alpha e^{-s}ds$$

$$=e^{-(\alpha+1)t}+e^{-(\alpha+1)t}\int_0^t \alpha e^{\alpha s}ds=e^{-(\alpha+1)t}+e^{-(\alpha+1)t}(e^{\alpha t}-1)=e^{-t}.$$

Secondly, applying $A_1(t)=e^{-t}$, we get the following problem

$$B_1'(t)+(\beta+1)B_1(t)=\beta e^{-t}, 0<t<T, B_1(0)=1.$$

We have that

$$B_1(t)=e^{-(\beta+1)t}B_1(0)+\int_0^t e^{-(\beta+1)(t-s)}\beta e^{-s}ds$$

$$=e^{-(\beta+1)t}+e^{-(\beta+1)t}\int_0^t \beta e^{\beta s}ds=e^{-(\beta+1)t}+e^{-(\beta+1)t}(e^{\beta t}-1)=e^{-t}.$$

Thirdly, applying $A_1(t)=e^{-t}$, we get the following problem

$$C_1'(t)+(\delta+1)C_1(t)=\delta e^{-t},\ 0<t<T, C_1(0)=1.$$

We have that

$$C_1(t) = e^{-(\delta+1)t}C_1(0) + \int_0^t e^{-(\delta+1)(t-s)}\delta e^{-s}ds$$

$$= e^{-(\delta+1)t} + e^{-(\delta+1)t}(e^{\delta t} - 1) = e^{-t}.$$

Fourthly, using $B_1(t) = e^{-t}$ and $C_1(t) = e^{-t}$, we get

$$D_1'(t) + (d+1)D_1(t) = de^{-t},\ 0 < t < T, D_1(0) = 1.$$

We have that

$$D_1(t) = e^{-(d+1)t}D_1(0) + \int_0^t e^{-(d+1)(t-s)}de^{-s}ds$$

$$= e^{-(d+1)t} + e^{-(d+1)t}(e^{dt} - 1) = e^{-t}.$$

Thus, $A_1(t) = B_1(t) = C_1(t) = D_1(t) = e^{-t}$ for any $t \in [0,T]$.

Applying formulas obtained for $A_k(t)$, $B_k(t)$, $C_k(t)$ and $D_k(t), k = 0,1,...,$ we can obtain the exact solution of problem (2.4) by formulas

$$\begin{cases} u(t,x) = A_1(t)\cos x = e^{-t}\cos x, \\ v(t,x) = B_1(t)\cos x = e^{-t}\cos x, \\ w(t,x) = C_1(t)\cos x = e^{-t}\cos x, \\ z(t,x) = D_1(t)\cos x = e^{-t}\cos x. \end{cases}$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = \alpha f_1(t,x), \\
\frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 v(t,x)}{\partial x_r^2} = (\beta - \beta_1) f_2(t,x), \\
\frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 w(t,x)}{\partial x_r^2} = (\delta - \delta_1) f_3(t,x), \\
\frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 z(t,x)}{\partial x_r^2} \\
= (d - d_1 - d_2) f_4(t,x), \\
x = (x_1, \dots, x_n) \in \overline{\Omega},\ 0 < t < T, \\
u(0,x) = \varphi(x), v(0,x) = \psi(x), w(0,x) = \xi(x), z(0,x) = \lambda(x), \\
x = (x_1, \dots, x_n) \in \overline{\Omega}, \\
\frac{\partial u(t,x)}{\partial \overline{m}} = \frac{\partial v(t,x)}{\partial \overline{m}} = \frac{\partial w(t,x)}{\partial \overline{m}} = \frac{\partial z(t,x)}{\partial \overline{m}} = 0, x \in S, 0 \le t \le T
\end{cases}
\tag{2.5}
$$

For the multidimensional system of partial differential equations. Assume that $\alpha_r > \alpha > 0$ and $f_k(t,x), k = 1,2,3,4 \left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(x), \lambda(x)$ $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to S.

However Fourier series method described in solving (2.5) can be used only in the case when (2.5) has constant coefficients.

Example 3: Obtain the Fourier series solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = (-1+\alpha)e^{-t}, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (-1+\beta-\beta_1)e^{-t}, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (-1+\delta-\delta_1)e^{-t}, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (-1+d-d_1-d_2)e^{-t}, \\ 0 < t < T, 0 < x < \pi, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = 1, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), 0 \le t \le T, \\ v(t,0) = v(t,\pi), v_x(t,0) = v_x(t,\pi), 0 \le t \le T, \\ w(t,0) = w(t,\pi), w_x(t,0) = w_x(t,\pi), 0 \le t \le T, \\ z(t,0) = z(t,\pi), z_x(t,0) = z_x(t,\pi), 0 \le t \le T \end{cases} \tag{2.6}$$

for the system of parabolic equations.

Solution: In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi), u_x(0) = u_x(\pi)$$

Generated by the space operator of problem (2.6). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = -4k^2, u_k(x) = \sin 2kx, k = 1, \ldots, u_k(x) = \cos 2kx, k = 0, 1, \ldots.$$

Then, we will obtain the Fourier series solution of problem (2.6) by formula

$$\begin{cases} u(t,x) = \sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx, \\ v(t,x) = \sum\limits_{k=1}^{\infty} C_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} D_k(t)\cos 2kx, \\ w(t,x) = \sum\limits_{k=1}^{\infty} E_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} F_k(t)\cos 2kx, \\ z(t,x) = \sum\limits_{k=1}^{\infty} M_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} N_k(t)\cos 2kx. \end{cases} \tag{2.7}$$

Here $A_k(t)$, $B_k(t), C_k(t), D_k(t)$, $E_k(t)$, $F_k(t), M_k(t)$ and $N_k(t)$ are unknown functions. Applying these formulas to the system of equations and initial conditions, we get

$$\begin{cases} \sum\limits_{k=1}^{\infty} A_k'(t)\sin 2kx + \alpha\sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx + \sum\limits_{k=1}^{\infty} 4k^2 A_k(t)\sin 2kx \\ \sum\limits_{k=0}^{\infty} B_k'(t)\cos 2kx + \alpha\sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx + \sum\limits_{k=0}^{\infty} 4k^2 B_k(t)\cos 2kx \\ = (-1+\alpha)e^{-t}, \\ \sum\limits_{k=1}^{\infty} C_k'(t)\sin 2kx + \beta\sum\limits_{k=1}^{\infty} C_k(t)\sin 2kx - \beta_1\sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx \\ +\sum\limits_{k=1}^{\infty} 4k^2 C_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} D_k'(t)\cos 2kx + \beta\sum\limits_{k=0}^{\infty} D_k(t)\cos 2kx \\ -\beta_1\sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx + \sum\limits_{k=0}^{\infty} 4k^2 D_k(t)\cos 2kx \\ = (-1+\beta-\beta_1)e^{-t}, \end{cases}$$

$$\left\{\begin{array}{l}
\sum_{k=1}^{\infty} E_k'(t)\sin 2kx + \delta \sum_{k=1}^{\infty} E_k(t)\sin 2kx - \delta_1 \sum_{k=1}^{\infty} A_k(t)\sin 2kx \\
+\sum_{k=1}^{\infty} 4k^2 E_k(t)\sin 2kx + \sum_{k=0}^{\infty} F_k'(t)\cos 2kx + \delta \sum_{k=0}^{\infty} F_k(t)\cos 2kx \\
-\delta_1 \sum_{k=0}^{\infty} B_k(t)\cos 2kx + \sum_{k=0}^{\infty} 4k^2 F_k(t)\cos 2kx \\
= (-1+\delta-\delta_1)e^{-t}, \\
\sum_{k=1}^{\infty} M_k'(t)\sin 2kx + d\sum_{k=1}^{\infty} M_k(t)\sin 2kx - d_1 \sum_{k=1}^{\infty} E_k(t)\sin 2kx \\
-d_2 \sum_{k=1}^{\infty} C_k(t)\sin 2kx + \sum_{k=1}^{\infty} 4k^2 M_k(t)\sin 2kx + \sum_{k=0}^{\infty} N_k'(t)\cos 2kx \\
+d\sum_{k=0}^{\infty} N_k(t)\cos 2kx - d_1 \sum_{k=0}^{\infty} F_k(t)\cos 2kx - d_2 \sum_{k=0}^{\infty} D_k(t)\cos 2kx \\
+\sum_{k=0}^{\infty} 4k^2 N_k(t)\cos 2kx = (-1+d-d_1-d_2)e^{-t}, \\
0 < t < T, 0 < x < \pi,
\end{array}\right.$$

$$\begin{cases} u(0,x) = \sum_{k=1}^{\infty} A_k(0)\sin 2kx + \sum_{k=0}^{\infty} B_k(0)\cos 2kx = 1, \\ v(0,x) = \sum_{k=1}^{\infty} C_k(0)\sin 2kx + \sum_{k=0}^{\infty} D_k(0)\cos 2kx = 1, \\ w(0,x) = \sum_{k=1}^{\infty} E_k(0)\sin 2kx + \sum_{k=0}^{\infty} F_k(0)\cos 2kx = 1, \\ z(0,x) = \sum_{k=1}^{\infty} M_k(0)\sin 2kx + \sum_{k=0}^{\infty} N_k(0)\cos 2kx = 1, \\ 0 \le x \le \pi. \end{cases}$$

Equating coefficients $\sin 2kx, k = 1,...$ and $\cos 2kx, k = 0,1,...$ to zero, we get

$$\begin{cases} B_0'(t) + \alpha B_0(t) = (-1+\alpha)e^{-t}, \\ D_0'(t) + \beta D_0(t) - \beta_1 B_0(t) = (-1+\beta-\beta_1)e^{-t}, \\ F_0'(t) + \delta F_0(t) - \delta_1 B_0(t) = (-1+\delta-\delta_1)e^{-t}, \\ N_0'(t) + dN_0(t) - d_1 F_0(t) - d_2 D_0(t) = (-1+d-d_1-d_2)e^{-t}, \\ 0 < t < T, B_0(0) = D_0(0) = F_0(0) = N_0(0) = 1 \end{cases}$$

and

$$\begin{cases} A_k'(t) + \alpha A_k(t) + 4k^2 A_k(t) = 0, \\ C_k'(t) + \beta C_k(t) - \beta_1 A_k(t) + 4k^2 C_k(t) = 0, \\ E_k'(t) + \delta E_k(t) - \delta_1 A_k(t) + 4k^2 E_k(t) = 0, \\ M_k'(t) + d\,M_k(t) - d_1 E_k(t) - d_2 C_k(t) + 4k^2 M_k(t) = 0, \\ 0 < t < T, \\ A_k(0) = C_k(0) = E_k(0) = M_k(0) = 0, \end{cases}$$

$$\begin{cases} B_k'(t) + \alpha B_k(t) + 4k^2 B_k(t) = 0, \\ D_k'(t) + \beta D_k(t) - \beta_1 B_k(t) + 4k^2 D_k(t) = 0, \\ F_k'(t) + \delta F_k(t) - \delta_1 B_k(t) + 4k^2 F_k(t) = 0, \\ N_k'(t) + d\,N_k(t) - d_1 F(t) - d_2 D_k(t) + 4k^2 N_k(t) = 0, \\ 0 < t < T, \\ B_k(0) = D_k(0) = F_k(0) = N_k(0) = 0. \end{cases}$$

for k=1,2,… .

We will obtain $A_k(t), B_k(t), C_k(t), D_k(t), E_k(t), F_k(t), M_k(t)$ and $N_k(t)$

Firstly, we consider the problem

$$A_k'(t) + (\alpha + 4k^2)A_k(t) = 0, 0 < t < T, A_k(0) = 0.$$

We have that

$$A_k(t) = e^{-\left(\alpha+4k^2\right)t}A_k(0) = 0 \tag{2.8}$$

for any $0 \le t \le T$.

Secondly, we consider the problem

$$B_k'(t) + (\alpha + 4k^2)B_k(t) = 0, 0 < t < T, B_k(0) = 0.$$

We have that

$$B_k(t) = e^{-\left(\alpha+4k^2\right)t}B_k(0) = 0 \tag{2.9}$$

for any $0 \le t \le T$.

Thirdly, we consider the problem

$$C_k'(t) + (\beta + 4k^2)C_k(t) - \beta_1 A_k(t) = 0, 0 < t < T, C_k(0) = 0.$$

Using (2.8), we get

$$C_k'(t) + (\beta + 4k^2)C_k(t) = 0, 0 < t < T, C_k(0) = 0.$$

We have that

$$C_k(t) = e^{-\left(\beta+4k^2\right)t}C_k(0) = 0 \tag{2.10}$$

for any $0 \le t \le T$.

Fourthly, we consider the problem

$$D_k'(t) + (\beta + 4k^2)D_k(t) - \beta_1 B_k(t) = 0, 0 < t < T, D_k(0) = 0.$$

Using (2.9), we get

$$D_k'(t) + (\beta + 4k^2)D_k(t) = 0, 0 < t < T, D_k(0) = 0.$$

We have that

$$D_k(t) = e^{-\left(\beta+4k^2\right)t}D_k(0) = 0 \tag{2.11}$$

for any $0 \le t \le T$.

Fifthly, we consider the problem

$$E_k'(t) + (\delta + 4k^2)E_k(t) - \delta_1 A_k(t) = 0, 0 < t < T, E_k(0) = 0.$$

Using (2.8), we get

$$E_k'(t) + (\delta + 4k^2)E_k(t) = 0, 0 < t < T, E_k(0) = 0.$$

We have that

$$E_k(t) = e^{-\left(\delta+4k^2\right)t}E_k(0) = 0 \tag{2.12}$$

for any $0 \le t \le T.$

Sixthly, we consider the problem

$$F_k'(t) + (\delta + 4k^2)F_k(t) - \delta_1 B_k(t) = 0, 0 < t < T, F_k(0) = 0.$$

Using (2.9), we get

$$F_k'(t) + (\delta + 4k^2)F_k(t) = 0, 0 < t < T, F_k(0) = 0.$$

We have that

$$F_k(t) = e^{-\left(\delta+4k^2\right)t}F_k(0) = 0 \tag{2.13}$$

for any $0 \le t \le T.$

Seventhly, we consider the problem

$$M_k'(t) + (d + 4k^2)M_k(t) - d_1 E_k(t) - d_2 C_k(t) = 0, 0 < t < T, M_k(0) = 0.$$

Using (2.10) and (2.12), we get

$$M_k'(t) + (d + 4k^2)M_k(t) = 0, 0 < t < T, M_k(0) = 0.$$

We have that

$$M_k(t) = e^{-\left(d+4k^2\right)t}M_k(0) = 0 \tag{2.14}$$

for any $0 \le t \le T.$

Eighthly, we consider the problem

$$N_k'(t) + (d + 4k^2)N_k(t) - d_1 F_k(t) - d_2 D_k(t) = 0, 0 < t < T, N_k(0) = 0.$$

Using (2.11) and (2.13), we get

$$N_k'(t) + (d + 4k^2)N_k(t) = 0, 0 < t < T, N_k(0) = 0.$$

We have that

$$N_k(t) = e^{-\left(d+4k^2\right)t} N_k(0) = 0$$

for any $0 \le t \le T$. Therefore,

$A_k(t) = B_k(t) = C_k(t) = D_k(t) = E_k(t) = F_k(t) = M_k(t) = N_k(t) = 0$ for any $0 \le t \le T.$

Now, we obtain $B_0(t)$, $D_0(t)$, $F_0(t)$ and $N_0(t)$. Firstly, we consider the problem

$$B_0'(t) + \alpha B_0(t) = (-1 + \alpha)e^{-t}, 0 < t < T, B_0(0) = 1.$$

We have that

$$B_0(t) = e^{-\alpha t} B_0(0) + \int_0^t e^{-\alpha(t-s)}(-1 + \alpha)e^{-s} ds$$

$$= e^{-\alpha t} + e^{-\alpha t} \int_0^t (-1 + \alpha)e^{(-1+\alpha)s} ds = e^{-t}.$$

Therefore,

$$B_0(t) = e^{-t}. \tag{2.15}$$

Secondly, we consider the problem

$$D_0'(t) + \beta D_0(t) - \beta_1 B_0(t) = (-1 + \beta - \beta_1)e^{-t}, 0 < t < T, D_0(0) = 1$$

Using (2.15), we get

$$D_0'(t) + \beta D_0(t) = (-1 + \beta)e^{-t}, 0 < t < T, D_0(0) = 1$$

We have that

$$D_0(t) = e^{-\beta t}D_0(0) + \int_0^t e^{-\beta(t-s)}(-1+\beta)e^{-s}ds$$

$$= e^{-\beta t} + e^{-\beta t}\int_0^t (-1+\beta)e^{(-1+\beta)s}ds = e^{-t}.$$

Therefore,

$$D_0(t) = e^{-t}. \tag{2.16}$$

Thirdly, we consider the problem

$$F_0'(t) + \delta F_0(t) - \delta_1 B_0(t) = (-1+\delta-\delta_1)e^{-t}, 0 < t < T, F_0(0) = 1.$$

Using (2.15), we get

$$F_0'(t) + \delta F_0(t) = (-1+\delta)e^{-t}, 0 < t < T, F_0(0) = 1.$$

We have that

$$F_0(t) = e^{-\delta t}F_0(0) + \int_0^t e^{-\delta(t-s)}(-1+\delta)e^{-s}ds$$

$$= e^{-\delta t} + e^{-\delta t}\int_0^t (-1+\delta)e^{(-1+\delta)s}ds = e^{-t}.$$

Therefore,

$$F_0(t) = e^{-t}. \tag{2.17}$$

Fourthly, we consider the problem

$$N_0'(t) + dN_0(t) - d_1F_0(t) - d_2D_0(t) = (-1+d-d_1-d_2)e^{-t}, 0 < t < T, N_0(0) = 1.$$

Using (2.16) and (2.17), we get

$$N_0'(t) + dN_0(t) = (-1+d)e^{-t}, 0 < t < T, N_0(0) = 1.$$

We have that

$$N_0(t) = e^{-dt}N_0(0) + \int_0^t e^{-d(t-s)}(-1+d)e^{-s}ds$$

$$= e^{-dt} + e^{-dt}\int_0^t (-1+d)e^{(-1+d)s}ds = e^{-t}.$$

Therefore,

$$N_0(t) = e^{-t}. \tag{2.18}$$

Applying formulas (2.8)-(2.18) and (2.7), we get

$$u(t,x) = B_0(t) = e^{-t},$$

$$v(t,x) = D_0(t) = e^{-t},$$

$$w(t,x) = F_0(t) = e^{-t},$$

$$z(t,x) = N_0(t) = e^{-t}.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$
\left\{
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = \alpha f_1(t,x),\\
&\frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 v(t,x)}{\partial x_r^2} = (\beta - \beta_1) f_2(t,x),\\
&\frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 w(t,x)}{\partial x_r^2} = (\delta - \delta_1) f_3(t,x),\\
&\frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 z(t,x)}{\partial x_r^2}\\
&= (d - d_1 - d_2) f_4(t,x),\\
&x = (x_1,\ldots,x_n) \in \overline{\Omega},\ 0 < t < T,\\
&u(0,x) = \varphi(x), v(0,x) = \psi(x), w(0,x) = \xi(x), z(0,x) = \lambda(x),\\
&x = (x_1,\ldots,x_n) \in \overline{\Omega},\\
&u(t,x)|_{S_1} = u(t,x)|_{S_2}, \frac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_2}, 0 \le t \le T,\\
&v(t,x)|_{S_1} = v(t,x)|_{S_2}, \frac{\partial v(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial v(t,x)}{\partial \overline{m}}\Big|_{S_2}, 0 \le t \le T,\\
&w(t,x)|_{S_1} = w(t,x)|_{S_2}, \frac{\partial w(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial w(t,x)}{\partial \overline{m}}\Big|_{S_2}, 0 \le t \le T,\\
&z(t,x)|_{S_1} = z(t,x)|_{S_2}, \frac{\partial z(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial z(t,x)}{\partial \overline{m}}\Big|_{S_2}, 0 \le t \le T
\end{aligned}
\right. \tag{2.19}
$$

for the multidimensional system of partial differential equations. Assume that $\alpha_r > \alpha > 0$ and $f_k(t,x), k=1,2,3,4\left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(x), \lambda(x) \quad \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, ? = S_1 \cap S_2$.

However Fourier series method described in solving (2.19) can be used only in the case when (2.19) has constant coefficients.

2.2. Laplace Transform Method

Now, we consider Laplace transform solution of problems for the system of partial differential equations.

Example 1: Obtain the Laplace transform solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = (-2+\alpha)e^{-t-x}, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (-2+\beta-\beta_1)e^{-t-x}, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (-2+\delta-\delta_1)e^{-t-x}, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (-2+d-d_1-d_2)e^{-t-x}, \\ 0 < t < T, 0 < x < \infty, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = e^{-x}, 0 \le x < \infty, \\ u(t,0) = v(t,0) = w(t,0) = z(t,0) = e^{-t}, 0 \le t \le T, \\ u_x(t,0) = v_x(t,0) = w_x(t,0) = z_x(t,0) = -e^{-t}, 0 \le t \le T \end{cases} \tag{2.20}$$

for the system of parabolic equations.

Solution: Here and in future we denote

$$\begin{cases} \mathcal{L}\{u(t,x)\} = u(t,s), \\ \mathcal{L}\{v(t,x)\} = v(t,s), \\ \mathcal{L}\{w(t,x)\} = w(t,s), \\ \mathcal{L}\{z(t,x)\} = z(t,s). \end{cases}$$

Using formula

$$\mathcal{L}\{e^{-x}\} = \frac{1}{s+1} \tag{2.21}$$

and taking the Laplace transform of both sides of the system of partial differential equations and conditions

$u(t,0) = v(t,0) = w(t,0) = z(t,0) = e^{-t}, u_x(t,0) = v_x(t,0) = w_x(t,0) = z_x(t,0) = -e^{-t}$,

we can write

$$
\begin{cases}
\mathcal{L}\left\{\frac{\partial u(t,x)}{\partial t}\right\} + \alpha\, \mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} = \mathcal{L}\{(-2+\alpha)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial v(t,x)}{\partial t}\right\} + \beta\, \mathcal{L}\{v(t,x)\} - \beta_1 \mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 v(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+\beta-\beta_1)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial w(t,x)}{\partial t}\right\} + \delta \mathcal{L}\ \{w(t,x)\} - \delta_1 \mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 w(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+\delta-\delta_1)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial z(t,x)}{\partial t}\right\} + d\mathcal{L}\ \{z(t,x)\} - d_1 \mathcal{L}\left\{\ w(t,x)\right\} - d_2\ \mathcal{L}\{v(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 z(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+d-d_1-d_2)e^{-t-x}\}, 0 < t < T, \\
\mathcal{L}\{u(0,x)\} = \mathcal{L}\{v(0,x)\} = \mathcal{L}\{w(0,x)\} = \mathcal{L}\{z(0,x)\} = \mathcal{L}\{e^{-x}\}
\end{cases}
$$

or

$$
\begin{cases}
u_t(t,s) + \alpha u(t,s) - \{s^2 u(t,s) - se^{-t} + e^{-t}\} = (-2+\alpha)e^{-t}\frac{1}{s+1}, \\
v_t(t,s) + \beta v(t,s) - \beta_1 u(t,s) - \{s^2 v(t,s) - se^{-t} + e^{-t}\} \\
= (-2+\beta-\beta_1)e^{-t}\frac{1}{s+1}, \\
w_t(t,s) + \delta w(t,s) - \delta_1 u(t,s) - \{s^2 w(t,s) - se^{-t} + e^{-t}\} \\
= (-2+\delta-\delta_1)e^{-t}\frac{1}{s+1}, \\
z_t(t,s) + dz(t,s) - d_1 w(t,s) - d_2 v(t,s) - \{s^2 z(t,s) - se^{-t} + e^{-t}\} \\
= (-2+d-d_1-d_2)e^{-t}\frac{1}{s+1}, 0 < t < T, \\
u(0,s) = v(0,s) = w(0,s) = z(0,s) = \frac{1}{s+1}
\end{cases}
$$

Now, we taking the Laplace transform with respect to t , we get

$$\begin{cases} \mu u(\mu,s) - \frac{1}{s+1} + (\alpha - s^2)u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\alpha-1}{s+1}\right), \\ \mu v(\mu,s) - \frac{1}{s+1} + (\beta - s^2)v(\mu,s) - \beta_1 u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\beta-\beta_1-1}{s+1}\right), \\ \mu w(\mu,s) - \frac{1}{s+1} + (\delta - s^2)w(\mu,s) - \delta_1 u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\delta-\delta_1-1}{s+1}\right), \\ \mu z(\mu,s) - \frac{1}{s+1} + (d - s^2)z(\mu,s) - d_1 w(\mu,s) - d_2 v(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+d-d_1-d_2-1}{s+1}\right) \end{cases}$$

Firstly, applying equation

$$\mu u(\mu,s) - \frac{1}{s+1} + (\alpha - s^2)u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\alpha-1}{s+1}\right),$$

we get

$$(\alpha - s^2 + \mu)u(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(\frac{-s^2+\alpha-1}{s+1}\right)$$

or

$$u(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.22}$$

Secondly, applying formula (2.22) and equation

$$\mu v(\mu,s) - \frac{1}{s+1} + (\beta - s^2)v(\mu,s) - \beta_1 u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\beta-\beta_1-1}{s+1}\right),$$

we get

$$(\beta - s^2 + \mu)v(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(\frac{-s^2+\beta-\beta_1-1}{s+1}\right) + \frac{\beta_1}{(\mu+1)(s+1)}$$

or

$$v(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.23}$$

Thirdly, applying formula (2.22) and equation

$$\mu w(\mu,s) - \frac{1}{s+1} + (\delta - s^2)w(\mu,s) - \delta_1 u(\mu,s) = \frac{1}{\mu+1}\left(\frac{-s^2+\delta-\delta_1-1}{s+1}\right)$$

we get

$$(\delta - s^2 + \mu)w(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(\frac{-s^2+\delta-\delta_1-1}{s+1}\right) + \frac{\delta_1}{(\mu+1)(s+1)}$$

or

$$w(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.24}$$

Fourthly, applying formulas (2.23), (2.24) and equation

$$\mu z(\mu,s) - \frac{1}{s+1} + (\delta - s^2)z(\mu,s) - d_1 w(\mu,s) - d_2 v(\mu,s)$$

$$= \frac{1}{\mu+1}\left(\frac{-s^2+d-d_1-d_2-1}{s+1}\right),$$

we get

$$\left(d - s^2 + \mu\right)z(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(\frac{-s^2+d-d_1-d_2-1}{s+1}\right) + \frac{d_1+d_2}{(\mu+1)(s+1)}$$

or

$$z(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.25}$$

Applying formulas (2.22) - (2.25) and taking the inverse Laplace transforms with respect to t and x, we obtain

$$u(t,x) = v(t,x) = w(t,x) = z(t,x) = e^{-t-x}.$$

Example 2: Obtain the Laplace transform solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = (-2+\alpha)e^{-t-x}, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (-2+\beta-\beta_1)e^{-t-x}, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (-2+\delta-\delta_1)e^{-t-x}, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (-2+d-d_1-d_2)e^{-t-x}, \\ 0 < t < T, 0 < x < \infty, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = e^{-x}, 0 \le x < \infty, \\ u(t,0) = v(t,0) = w(t,0) = z(t,0) = e^{-t}, 0 \le t \le T, \\ u(t,\infty) = v(t,\infty) = w(t,\infty) = z(t,\infty) = 0, 0 \le t \le T \end{cases} \tag{2.26}$$

for the system of parabolic equations.

Solution: Applying formula (2.21) and taking the Laplace transform of both sides of the system of partial differential equations and conditions $u(t,0) = v(t,0) = w(t,0) = z(t,0) = e^{-t}$, we can write

$$
\begin{cases}
\mathcal{L}\left\{\frac{\partial u(t,x)}{\partial t}\right\} + \alpha\,\mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} = \mathcal{L}\{(-2+\alpha)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial v(t,x)}{\partial t}\right\} + \beta\,\mathcal{L}\{v(t,x)\} - \beta_1\mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 v(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+\beta-\beta_1)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial w(t,x)}{\partial t}\right\} + \delta\mathcal{L}\,\{w(t,x)\} - \delta_1\mathcal{L}\{u(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 w(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+\delta-\delta_1)e^{-t-x}\}, \\
\mathcal{L}\left\{\frac{\partial z(t,x)}{\partial t}\right\} + d\mathcal{L}\,\{z(t,x)\} - d_1\mathcal{L}\left\{\,w(t,x)\right\} - d_2\,\mathcal{L}\{v(t,x)\} - \mathcal{L}\left\{\frac{\partial^2 z(t,x)}{\partial x^2}\right\} \\
= \mathcal{L}\{(-2+d-d_1-d_2)e^{-t-x}\}, 0 < t < T, \\
\mathcal{L}\{u(0,x)\} = \mathcal{L}\{v(0,x)\} = \mathcal{L}\{w(0,x)\} = \mathcal{L}\{z(0,x)\} = \mathcal{L}\{e^{-x}\}
\end{cases}
$$

or

$$
\begin{cases}
u_t(t,s) + \alpha u(t,s) - s^2 u(t,s) + se^{-t} + \gamma_1(t) = (-2+\alpha)e^{-t}\frac{1}{s+1}, \\
v_t(t,s) + \beta v(t,s) - \beta_1 u(t,s) - s^2 v(t,s) + se^{-t} + \gamma_2(t) \\
= (-2+\beta-\beta_1)e^{-t}\frac{1}{s+1}, \\
w_t(t,s) + \delta w(t,s) - \delta_1 u(t,s) - s^2 w(t,s) + se^{-t} + \gamma_3(t) \\
= (-2+\delta-\delta_1)e^{-t}\frac{1}{s+1}, \\
z_t(t,s) + dz(t,s) - d_1 w(t,s) - d_2 v(t,s) - s^2 z(t,s) + se^{-t} + \gamma_4(t) \\
= (-2+d-d_1-d_2)e^{-t}\frac{1}{s+1}, 0 < t < T, \\
u(0,s) = v(0,s) = w(0,s) = z(0,s) = \frac{1}{s+1}.
\end{cases}
$$

Here

$$
\begin{cases}
\gamma_1(t) = u_x(t,0), \\
\gamma_2(t) = v_x(t,0), \\
\gamma_3(t) = w_x(t,0), \\
\gamma_4(t) = z_x(t,0).
\end{cases}
$$

Now, taking the Laplace transform with respect to t , we get

$$\begin{cases} \mu u(\mu,s) - \frac{1}{s+1} + (\alpha - s^2)u(\mu,s) = \frac{1}{\mu+1}\left(-s + \frac{-2+\alpha}{s+1}\right) - \gamma_1(\mu), \\ \mu v(\mu,s) - \frac{1}{s+1} + (\beta - s^2)v(\mu,s) - \beta_1 u(\mu,s) = \frac{1}{\mu+1}\left(-s + \frac{-2+\beta-\beta_1}{s+1}\right) - \gamma_2(\mu), \\ \mu w(\mu,s) - \frac{1}{s+1} + (\delta - s^2)w(\mu,s) - \delta_1 u(\mu,s) = \frac{1}{\mu+1}\left(-s + \frac{-2+\delta-\delta_1}{s+1}\right) - \gamma_3(\mu), \\ \mu z(\mu,s) - \frac{1}{s+1} + (d - s^2)z(\mu,s) - d_1 w(\mu,s) - d_2 v(\mu,s) = \frac{1}{\mu+1}\left(-s + \frac{-2+d-d_1-d_2}{s+1}\right) - \gamma_4(\mu) \end{cases}$$

or

$$\begin{cases} (\mu + \alpha - s^2)u(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+\alpha}{s+1}\right) - \gamma_1(\mu), \\ (\mu + \beta - s^2)v(\mu,s) - \beta_1 u(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+\beta-\beta_1}{s+1}\right) - \gamma_2(\mu), \\ (\mu + \delta - s^2)w(\mu,s) - \delta_1 u(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+\delta-\delta_1}{s+1}\right) - \gamma_3(\mu), \\ (\mu + d - s^2)z(\mu,s) - d_1 w(\mu,s) - d_2 v(\mu,s) = \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+d-d_1-d_2}{s+1}\right) - \gamma_4(\mu). \end{cases}$$

Moreover, taking the Laplace transform from conditions

$u(t,\infty) = v(t,\infty) = w(t,\infty) = z(t,\infty) = 0,$ we get

$$u(\mu,\infty) = 0, v(\mu,\infty) = 0, w(\mu,\infty) = 0, z(\mu,\infty) = 0. \tag{2.27}$$

Firstly, applying equation

$$(\mu + \alpha - s^2)u(\mu,s) = \frac{\mu + \alpha - s^2}{(\mu+1)(s+1)} - \gamma_1(\mu) - \frac{1}{\mu+1},$$

we get

$$u(\mu,s) = \frac{1}{(\mu+1)(s+1)} - \left(\gamma_1(\mu) + \frac{1}{\mu+1}\right)\frac{1}{\mu + \alpha - s^2}.$$

Using the formula

$$\frac{1}{\mu+\alpha-s^2} = \left(\frac{1}{\sqrt{\mu+\alpha}-s}+\frac{1}{\sqrt{\mu+\alpha}+s}\right)\frac{1}{2\sqrt{\mu+\alpha}},$$

we get

$$u(\mu,s) = \frac{1}{(\mu+1)(s+1)}$$

$$-\left(\gamma_1(\mu)-\frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\alpha}}\left(\frac{1}{s+\sqrt{\mu+\alpha}}+\frac{1}{s-\sqrt{\mu+\alpha}}\right). \tag{2.28}$$

Taking the inverse Laplace transform with respect to x , we get

$$u(\mu,x) = \frac{1}{(\mu+1)}e^{-x}$$

$$+\left(\gamma_1(\mu)+\frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\alpha}}\left(e^{-\sqrt{\mu+\alpha}\,x}-e^{\sqrt{\mu+\alpha}\,x}\right). \tag{2.29}$$

Passing to limit in (2.29) when $x\to\infty$ and using (2.27), we get

$$u(\mu,\infty) = \left(\gamma_1(\mu)+\frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\alpha}}\lim_{x\to\infty}e^{\sqrt{\mu+\alpha}\,x} = 0.$$

From that it follows

$$\gamma_1(\mu) = -\frac{1}{\mu+1}. \tag{2.30}$$

Applying (2.28), (2.29) and (2.30), we get

$$(\mu+\alpha-s^2)u(\mu,s) = \frac{\mu+\alpha-s^2}{(\mu+1)(s+1)}$$

or

$$u(\mu,x) = \frac{1}{(\mu+1)}e^{-x}, u(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.31}$$

Secondly, applying (2.31) and equation

$$(\mu+\beta-s^2)v(\mu,s)-\beta_1 u(\mu,s)$$

$$=\frac{1}{s+1}+\frac{1}{\mu+1}\left(-s+\frac{-2+\beta-\beta_1}{s+1}\right)-\gamma_2(\mu),$$

we get

$$(\mu+\beta-s^2)v(\mu,s)=\frac{\mu+\beta-s^2}{(\mu+1)(s+1)}-\gamma_2(\mu)-\frac{1}{\mu+1}$$

or

$$v(\mu,s)=\frac{1}{(\mu+1)(s+1)}-\left(\gamma_2(\mu)+\frac{1}{\mu+1}\right)\frac{1}{\mu+\beta-s^2}.$$

Applying the formula

$$\frac{1}{\mu+\beta-s^2}=\left(\frac{1}{\sqrt{\mu+\beta}-s}+\frac{1}{\sqrt{\mu+\beta}+s}\right)\frac{1}{2\sqrt{\mu+\beta}},$$

we get

$$v(\mu,s)=\frac{1}{(\mu+1)(s+1)}$$

$$-\left(\gamma_2(\mu)-\frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\beta}}\left(\frac{1}{s+\sqrt{\mu+\beta}}+\frac{1}{s-\sqrt{\mu+\beta}}\right) \tag{2.32}$$

Taking the inverse Laplace transform with respect to x, we get

$$v(\mu,x)=\frac{1}{(\mu+1)}e^{-x}$$

$$+\left(\gamma_2(\mu)+\frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\beta}}\left(e^{-\sqrt{\mu+\beta}\,x}-e^{\sqrt{\mu+\beta}\,x}\right). \tag{2.33}$$

Passing the limit in (2.33) when $x\to\infty$ and using (2.27), we get

$$v(\mu,\infty) = \left(\gamma_2(\mu) + \frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\beta}}\lim_{x\to\infty} e^{\sqrt{\mu+\beta}\,x} = 0.$$

From that it follows

$$\gamma_2(\mu) = -\frac{1}{\mu+1}. \tag{2.34}$$

Applying (2.32), (2.33) and (2.34), we get

$$(\mu+\beta-s^2)v(\mu,s) = \frac{\mu+\beta-s^2}{(\mu+1)(s+1)}$$

or

$$v(\mu,x) = \frac{1}{(\mu+1)}e^{-x}, v(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.35}$$

Thirdly, applying (2.31) and equation

$$(\mu+\delta-s^2)w(\mu,s) - \delta_1 u(\mu,s)$$

$$= \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+\delta-\delta_1}{s+1}\right) - \gamma_3(\mu),$$

we get

$$(\mu+\delta-s^2)w(\mu,s) = \frac{\mu+\delta-s^2}{(\mu+1)(s+1)} - \gamma_3(\mu) - \frac{1}{\mu+1}$$

or

$$w(\mu,s) = \frac{1}{(\mu+1)(s+1)} - \left(\gamma_3(\mu) + \frac{1}{\mu+1}\right)\frac{1}{\mu+\delta-s^2}.$$

Applying the formula

$$\frac{1}{\mu+\delta-s^2} = \left(\frac{1}{\sqrt{\mu+\delta}-s} + \frac{1}{\sqrt{\mu+\delta}+s}\right)\frac{1}{2\sqrt{\mu+\delta}},$$

we get

$$w(\mu,s) = \frac{1}{(\mu+1)(s+1)}$$

$$-\left(\gamma_3(\mu) - \frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\delta}}\left(\frac{1}{s+\sqrt{\mu+\delta}} + \frac{1}{s-\sqrt{\mu+\delta}}\right). \quad (2.36)$$

Taking the inverse Laplace transform with respect to x , we get

$$w(\mu,x) = \frac{1}{(\mu+1)}e^{-x}$$

$$+\left(\gamma_3(\mu) + \frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\delta}}\left(e^{-\sqrt{\mu+\delta}\,x} - e^{\sqrt{\mu+\delta}\,x}\right). \quad (2.37)$$

Passing the limit in (2.37) when $x \to \infty$ and using (2.27), we get

$$w(\mu,\infty) = \left(\gamma_3(\mu) + \frac{1}{\mu+1}\right)\frac{1}{2\sqrt{\mu+\delta}}\lim_{x\to\infty} e^{\sqrt{\mu+\delta}\,x} = 0.$$

From that it follows

$$\gamma_3(\mu) = \frac{-1}{\mu+1}. \quad (2.38)$$

Applying (2.36), (2.37) and (2.38), we get

$$(\mu+\delta-s^2)w(\mu,s) = \frac{\mu+\delta-s^2}{(\mu+1)(s+1)}$$

or

$$w(\mu,x) = \frac{1}{(\mu+1)}e^{-x}, w(\mu,s) = \frac{1}{(\mu+1)(s+1)}. \quad (2.39)$$

Fourthly, applying (2.35), (2.39) and equation

$$(\mu+d-s^2)z(\mu,s) - d_1 w(\mu,s) - d_2 v(\mu,s)$$

$$= \frac{1}{s+1} + \frac{1}{\mu+1}\left(-s + \frac{-2+d-d_1-d_2}{s+1}\right) - \gamma_4(\mu),$$

we get

$$(\mu + d - s^2) z(\mu, s) = \frac{\mu + d - s^2}{(\mu + 1)(s + 1)} - \gamma_4(\mu) - \frac{1}{\mu + 1}$$

or

$$z(\mu, s) = \frac{1}{(\mu + 1)(s + 1)} - \left(\gamma_4(\mu) + \frac{1}{\mu + 1}\right)\frac{1}{\mu + d - s^2}.$$

Applying the formula

$$\frac{1}{\mu + d - s^2} = \left(\frac{1}{\sqrt{\mu + d} - s} + \frac{1}{\sqrt{\mu + d} + s}\right)\frac{1}{2\sqrt{\mu + d}},$$

we get

$$z(\mu, s) = \frac{1}{(\mu + 1)(s + 1)}$$

$$-\left(\gamma_4(\mu) - \frac{1}{\mu + 1}\right)\frac{1}{2\sqrt{\mu + d}}\left(\frac{1}{s + \sqrt{\mu + d}} + \frac{1}{s - \sqrt{\mu + d}}\right). \tag{2.40}$$

Taking the inverse Laplace transform with respect to x, we get

$$z(\mu, x) = \frac{1}{(\mu + 1)} e^{-x}$$

$$+\left(\gamma_4(\mu) + \frac{1}{\mu + 1}\right)\frac{1}{2\sqrt{\mu + d}}\left(e^{-\sqrt{\mu+d}\,x} - e^{\sqrt{\mu+d}\,x}\right). \tag{2.41}$$

Passing the limit in (2.41) when $x \to \infty$ and using (2.27), we get

$$z(\mu, \infty) = \left(\gamma_4(\mu) + \frac{1}{\mu + 1}\right)\frac{1}{2\sqrt{\mu + d}} \lim_{x \to \infty} e^{\sqrt{\mu+d}\,x} = 0.$$

From that it follows

$$\gamma_4(\mu) = -\frac{1}{\mu + 1}. \tag{2.42}$$

Applying (2.40), (2.41) and (2.42), we get

$$(\mu + d - s^2)z(\mu, s) = \frac{\mu + d - s^2}{(\mu+1)(s+1)}$$

or

$$z(\mu, x) = \frac{1}{(\mu+1)}e^{-x}, z(\mu, s) = \frac{1}{(\mu+1)(s+1)}. \tag{2.43}$$

Applying formulas (2.31), (2.35), (2.39), (2.43) and taking the inverse Laplace transform with respect to t, we obtain

$$u(t, x) = v(t, x) = w(t, x) = z(t, x) = e^{-t-x}.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = \alpha f_1(t,x), \\
\frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 v(t,x)}{\partial x_r^2} = (\beta - \beta_1) f_2(t,x), \\
\frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 w(t,x)}{\partial x_r^2} = (\delta - \delta_1) f_3(t,x), \\
\frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 z(t,x)}{\partial x_r^2} \\
= (d - d_1 - d_2) f_4(t,x), \\
x = (x_1, \ldots, x_n) \in \overline{\Omega}^+,\ 0 < t < T,, \\
u(0,x) = \varphi(x), v(0,x) = \psi(x), w(0,x) = \xi(x), z(0,x) = \lambda(x), \\
x = (x_1, \ldots, x_n) \in \overline{\Omega}^+, \\
u(t,x) = \alpha_1(t,x), \quad u_{x_r}(t,x) = \beta_1(t,x), \\
v(t,x) = \alpha_2(t,x), \quad v_{x_r}(t,x) = \beta_2(t,x), \\
w(t,x) = \alpha_3(t,x), \quad w_{x_r}(t,x) = \beta_3(t,x), \\
z(t,x) = \alpha_4(t,x), \quad u_{x_r}(t,x) = \beta_4(t,x), \\
1 \le r \le n, 0 \le t \le T, x \in S^+
\end{cases}
\tag{2.44}
$$

for the multidimensional system of partial differential equations. Assume that $\alpha_r > \alpha > 0$ and $f_k(t,x), k = 1,2,3,4 \left(t \in (0,T), x \in \overline{\Omega}^+\right), \varphi(x), \psi(x), \xi(x), \lambda(x)$ $\left(x \in \overline{\Omega}^+\right)$, $\alpha_k(t,x), \beta_k(t,x)$,

$k = 1, 2, 3, 4$ $\left(t \in [0, T], x \in S^{+}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, \varnothing = S_1 \cap S_2$. Here and in future Ω^{+} is the open cube in the n -dimensional Euclidean space $R^n\left(0 < x_k < \infty, 1 \le k \le n\right)$ with the boundary S^{+} and $\overline{\Omega}^{+} = \Omega^{+} \cup S^{+}$.

However Laplace transform method described in solving (2.44) can be used only in the case when (2.44) has $a_r(x)$ polynomials coefficients.

2.3. Fourier Transform Method

Now, we consider the Fourier transform solution of the initial value problem for the system of partial differential equations.

Example 1: Obtain the Fourier transform solution of the initial-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = \left(-4x^2 + 1 + \alpha\right) e^{-t-x^2}, \\ \frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = \left(-4x^2 + 1 + \beta - \beta_1\right) e^{-t-x^2}, \\ \frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (-4x^2 + 1 + \delta - \delta_1) e^{-t-x^2}, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (-4x^2 + 1 + d - d_1 - d_2) e^{-t-x^2}, \\ 0 < t < T, -\infty < x < \infty, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = e^{-x^2}, -\infty < x < \infty \end{cases} \tag{2.45}$$

for the system of parabolic equations.

Solution: We denote

$$F\{u(t,x)\} = u(t,s), F\left\{e^{-x^2}\right\} = q(s).$$

Then, we have that

$$F\left\{\frac{\partial}{\partial t}u(t,x)\right\} = u_t(t,s),$$

$$F\left\{\frac{\partial^2}{\partial x^2}u(t,x)\right\} = -s^2u(t,s).$$

Taking the Fourier transform of both sides of the system equation and using initial conditions, we get

$$\begin{cases} F\left\{\frac{\partial u(t,x)}{\partial t}\right\} + \alpha F\,\{u(t,x)\} - F\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} \\ = F\left\{\left(-4x^2 + 1 + \alpha\,\right)e^{-t-x^2}\right\}, \\ F\left\{\frac{\partial v(t,x)}{\partial t}\right\} + \beta F\,\{v(t,x)\} - \beta_1 F\{u(t,x)\} - F\left\{\frac{\partial^2 v(t,x)}{\partial x^2}\right\} \\ = F\left\{\left(-4x^2 + 1 + \beta - \beta_1\,\right)e^{-t-x^2}\right\}, \\ F\left\{\frac{\partial w(t,x)}{\partial t}\right\} + \delta F\left\{\,w(t,x)\right\} - \delta_1 F\{u(t,x)\} - F\left\{\frac{\partial^2 w(t,x)}{\partial x^2}\right\} \\ = F\left\{(-4x^2 + 1 + \delta - \delta_1)e^{-t-x^2}\right\}, \\ F\left\{\frac{\partial z(t,x)}{\partial t}\right\} + dF\,\{z(t,x)\} - d_1\,F\{w(t,x)\} - d_2 F\,\{v(t,x)\} - F\left\{\frac{\partial^2 z(t,x)}{\partial x^2}\right\} \\ = F\left\{(-4x^2 + 1 + d - d_1 - d_2)e^{-t-x^2}\right\}, \\ 0 < t < T, \\ u(0,s) = v(0,s) = w(0,s) = z(0,s) = q(s) \end{cases}$$

or

$$\begin{cases} u_t(t,s) + (\alpha + s^2)u(t,s) = (1 + \alpha + s^2)e^{-t}q(s), \\ v_t(t,s) + (\beta + s^2)v(t,s) - \beta_1 u(t,s) = (1 + \beta - \beta_1 + s^2)e^{-t}q(s), \\ w_t(t,s) + (\delta + s^2)w(t,s) - \delta_1 u(t,s) = (1 + \delta - \delta_1 + s^2)e^{-t}q(s), \\ z_t(t,s) + (d + s^2)z(t,s) - d_1 w(t,s) - d_2 v(t,s) \\ = (1 + d - d_1 - d_2 + s^2)e^{-t}q(s), 0 < t < T, \end{cases}$$

Firstly, we consider the problem

$$u_t(t,s) + (\alpha + s^2)u(t,s)$$

$$= (1 + \alpha + s^2)e^{-t}q(s), 0 < t < T, u(0,s) = q(s).$$

We have that

$$u(t,s) = e^{-\left(\alpha+s^2\right)t}u(0,s) + \int_0^t e^{-\left(\alpha+s^2\right)(t-y)}(1 + \alpha + s^2)e^{-y}q(s)dy$$

$$= e^{-\left(\alpha+s^2\right)t}q(s) + e^{-\left(\alpha+s^2\right)t}q(s)\int_0^t e^{\left(-1+\alpha+s^2\right)y}(1 + \alpha + s^2)dy$$

$$= e^{-\left(\alpha+s^2\right)t}q(s) + e^{-\left(\alpha+s^2\right)t}q(s)\left(e^{\left(-1+\alpha+s^2\right)t} - 1\right) = e^{-t}q(s).$$

Therefore,

$$u(t,s) = e^{-t}F\left\{e^{-x^2}\right\}$$

and

$$u(t,x) = F^{-1}\left\{e^{-t}F\left\{e^{-x^2}\right\}\right\} = e^{-t-x^2}.$$

Secondly, we consider the problem

$$v_t(t,s)+(\beta+s^2)v(t,s)$$

$$=(1+\beta-\beta_1+s^2)e^{-t}q(s)+\beta_1 u(t,s), 0<t<T, v(0,s)=q(s).$$

Applying $u(t,s)=e^{-t}q(s),$ we get

$$v_t(t,s)+(\beta+s^2)v(t,s)=(1+\beta+s^2)e^{-t}q(s), 0<t<T, v(0,s)=q(s).$$

We have that

$$v(t,s)=e^{-\left(\beta+s^2\right)t}v(0,s)+\int_0^t e^{-\left(\beta+s^2\right)(t-y)}(1+\beta+s^2)e^{-y}q(s)dy$$

$$=e^{-\left(\beta+s^2\right)t}q(s)+e^{-\left(\beta+s^2\right)t}q(s)\left(e^{\left(-1+\beta+s^2\right)t}-1\right)=e^{-t}q(s).$$

Therefore,

$$v(t,s)=e^{-t}F\left\{e^{-x^2}\right\}$$

and

$$v(t,x)=F^{-1}\left\{e^{-t}F\left\{e^{-x^2}\right\}\right\}=e^{-t-x^2}.$$

Thirdly, we consider the problem

$$w_t(t,s)+(\delta+s^2)w(t,s)$$

$$=(1+\delta-\delta_1+s^2)e^{-t}q(s)+\delta_1 u(t,s), 0<t<T, w(0,s)=q(s).$$

Applying $u(t,s)=e^{-t}q(s),$ we get

$$w_t(t,s)+(\delta+s^2)w(t,s)=(1+\delta+s^2)e^{-t}q(s), 0<t<T, w(0,s)=q(s).$$

We have that

$$w(t,s)=e^{-\left(\delta+s^2\right)t}w(0,s)+\int_0^t e^{-\left(\delta+s^2\right)(t-y)}(1+\delta+s^2)e^{-y}q(s)dy$$

$$= e^{-\left(\delta+s^2\right)t}q(s) + e^{-\left(\delta+s^2\right)t}q(s)\left(e^{\left(-1+\delta+s^2\right)t} - 1\right) = e^{-t}q(s).$$

Therefore,

$$w(t,s) = e^{-t}q(s) = e^{-t}F\left\{e^{-x^2}\right\}$$

and

$$w(t,x) = F^{-1}\left\{e^{-t}F\left\{e^{-x^2}\right\}\right\} = e^{-t-x^2}.$$

Fourthly, we consider the problem

$$z_t(t,s) + (d+s^2)z(t,s)$$

$$= (1+d-d_1-d_2+s^2)e^{-t}q(s) + d_1w(t,s) + d_2v(t,s), 0 < t < T, w(0,s) = q(s).$$

Applying $w(t,s) = e^{-t}q(s), v(t,s) = e^{-t}q(s),$ we get

$$z_t(t,s) + (d+s^2)z(t,s) = (1+d+s^2)e^{-t}q(s), 0 < t < T, w(0,s) = q(s).$$

We have that

$$z(t,s) = e^{-\left(d+s^2\right)t}z(0,s) + \int_0^t e^{-\left(d+s^2\right)(t-y)}(1+d+s^2)e^{-y}q(s)dy$$

$$= e^{-\left(d+s^2\right)t}q(s) + e^{-\left(d+s^2\right)t}q(s)\left(e^{\left(-1+d+s^2\right)t} - 1\right) = e^{-t}q(s).$$

Therefore,

$$z(t,s) = e^{-t}q(s) = e^{-t}F\left\{e^{-x^2}\right\}$$

and

$$z(t,x) = F^{-1}\left\{e^{-t}F\left\{e^{-x^2}\right\}\right\} = e^{-t-x^2}.$$

Thus,

$$u(t,x) = v(t,x) = w(t,x) = z(t,x) = e^{-t-x^2}.$$

Note that using similar procedure one can obtain the solution of the following initial value problem

$$
\begin{cases}
\frac{\partial u(t,x)}{\partial t} + \alpha\, u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = \alpha f_1(t,x), \\
\frac{\partial v(t,x)}{\partial t} + \beta\, v(t,x) - \beta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 v(t,x)}{\partial x_r^2} = (\beta - \beta_1) f_2(t,x), \\
\frac{\partial w(t,x)}{\partial t} + \delta\, w(t,x) - \delta_1 u(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 w(t,x)}{\partial x_r^2} = (\delta - \delta_1) f_3(t,x), \\
\frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 z(t,x)}{\partial x_r^2} \\
= (d - d_1 - d_2) f_4(t,x), \\
x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ 0 < t < T, \\
u(0,x) = \varphi(x), v(0,x) = \psi(x), w(0,x) = \xi(x), z(0,x) = \lambda(x), \\
x = (x_1, \ldots, x_n) \in \mathbb{R}^n
\end{cases}
\qquad (2.46)
$$

for the multidimensional system of partial differential equations. Assume that $\alpha_r > \alpha > 0$ and $f_k(t,x), k = 1,2,3,4 (t \in (0,T), x \in \mathbb{R}^n), \varphi(x), \psi(x), \xi(x), \lambda(x) \quad (x \in \mathbb{R}^n)$ are given smooth functions. However Fourier transform method described in solving (2.46) can be used only in the case when (2.46) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the system of differential equations has constant coefficients or polynomial coefficients. It is well-known that the most general method for solving system of partial differential equations with dependent in t and in the space variables is finite difference method.

In final section, we consider the initial-boundary value problem for the one-dimensional system of partial differential equations. The first order of accuracy difference scheme for the numerical solution of this problem is presented. Numerical analysis and discussions are presented.

CHAPTER 3

FINITE DIFFERENCE METHOD FOR THE SOLUTION OF SYSTEM OF PARTIAL DIFFERENTIAL EQUATION

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of the local and nonlocal problems for the system of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature. In this section, we study the numerical solution of the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a\, u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = a\, e^{-t}\cos x, \\ \frac{\partial v(t,x)}{\partial t} + b\, v(t,x) - b_1 u(t,x) - \frac{\partial^2 v(t,x)}{\partial x^2} = (b-b_1)e^{-t}\cos x, \\ \frac{\partial w(t,x)}{\partial t} + c\, w(t,x) - c_1 u(t,x) - \frac{\partial^2 w(t,x)}{\partial x^2} = (c-c_1)e^{-t}\cos x, \\ \frac{\partial z(t,x)}{\partial t} + d\, z(t,x) - d_1\, w(t,x) - d_2\, v(t,x) - \frac{\partial^2 z(t,x)}{\partial x^2} \\ = (d-d_1-d_2)e^{-t}\cos x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = v(0,x) = w(0,x) = z(0,x) = \cos x, 0 \le x \le \pi, \\ u_x(t,0) = v_x(t,0) = w_x(t,0) = z_x(t,0) = 0, 0 \le t \le 1, \\ u_x(t,\pi) = v_x(t,\pi) = w_x(t,\pi) = z_x(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{3.1}$$

for the system of parabolic equations. The exact solution of this problem is $u(t,x) = v(t,x) = w(t,x) = z(t,x) = e^{-t}\cos x$.

For the numerical solution of the problem (3.1), we present first order of accuracy difference scheme. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first order of accuracy difference schemes is given.

For the numerical solution of the problem (3.1), we present the following first order of accuracy difference scheme Algorithm

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}+au_n^k-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=ae^{-t_k}\cos x_n, \\ \frac{v_n^k-v_n^{k-1}}{\tau}+bv_n^k-b_1u_n^k-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}=(b-b_1)e^{-t_k}\cos x_n, \\ \frac{w_n^k-w_n^{k-1}}{\tau}+cw_n^k-c_1u_n^k-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}=(c-c_1)e^{-t_k}\cos x_n, \\ \frac{z_n^k-z_n^{k-1}}{\tau}+dz_n^k-d_1w_n^k-d_2v_n^k-\frac{z_{n+1}^k-2z_n^k+z_{n-1}^k}{h^2} \\ =(d-d_1-d_2)e^{-t_k}\cos x_n, \\ t_k=k\tau, x_n=nh, 1\le k\le N, 1\le n\le M-1, N\tau=1, Mh=\pi, \\ u_n^0=v_n^0=w_n^0=z_n^0=\cos x_n, 0\le n\le M, \\ u_1^k-u_0^k=v_1^k-v_0^k=w_1^k-w_0^k=z_1^k-z_0^k=0, \\ u_M^k-u_{M-1}^k=v_M^k-v_{M-1}^k=w_M^k-w_{M-1}^k=z_M^k-z_{M-1}^k=0, \\ 0\le k\le N. \end{cases} \tag{3.2}$$

for obtaining the solution of difference scheme (3.2)

$$\left\{\{u_n^k\}_{k=0}^N\right\}_{n=0}^M, \left\{\{v_n^k\}_{k=0}^N\right\}_{n=0}^M, \left\{\{w_n^k\}_{k=0}^N\right\}_{n=0}^M, \left\{\{z_n^k\}_{k=0}^N\right\}_{n=0}^M$$

contains four stages. In the first stage, we consider the problem

$$\frac{u_n^k-u_n^{k-1}}{\tau}+au_n^k-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=ae^{-t_k}\cos x_n,$$

$$1\le k\le N, 1\le n\le M-1,$$

$$u_n^0=\cos x_n, 0\le n\le M,$$

$$u_1^k-u_0^k=u_M^k-u_{M-1}^k=0, 0\le k\le N.$$

We will write it in the following boundary value problem for the second order difference equation with respect to n

$$\begin{cases} A_1u_{n+1}+B_1u_n+C_1u_{n-1}=\phi_n, 1\le n\le M-1, \\ \\ u_0=u_1, u_M=u_{M-1}. \end{cases} \tag{3.3}$$

Here, A_1, B_1, C_1 are $(N+1)\times(N+1)$ square matrices and $u_s, s=n, n\pm1, \phi_n$ are $(N+1)\times1$ column matrices and

$$A_1=C_1=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \breve{a} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \breve{a} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \breve{a} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \breve{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \breve{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \breve{a} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \breve{a} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{c} & \tilde{b} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \tilde{c} & \tilde{b} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{c} & \tilde{b} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \tilde{c} & \tilde{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \tilde{c} & \tilde{b} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{c} & \tilde{b} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n = \begin{bmatrix} \cos x_n \\ ae^{-t_1}\cos x_n \\ . \\ ae^{-t_{N-1}}\cos x_n \\ ae^{-t_N}\cos x_n \end{bmatrix}_{(N+1)\times 1}, \quad u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ . \\ u_s^{N-1} \\ u_s^{N-1} \end{bmatrix}_{(N+1)\times 1}$$

for $s = n, n \pm 1,$ where $\tilde{a} = -\frac{1}{h^2}, \tilde{b} = \frac{1}{\tau} + a + \frac{2}{h^2}, c = -\frac{1}{\tau}$. For obtaining $\left\{\left\{u_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ we have the following algorithm

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 0, u_M = (I - \alpha_M)^{-1}\beta_M, \tag{3.4}$$

$$\alpha_{n+1} = -(B_1 + C_1\alpha_n)^{-1} A_1, \alpha_1 = I,$$

$$\beta_{n+1} = (B_1 + C_1\alpha_n)^{-1}(\phi_n - C_1\beta_n), \beta_1 = 0, n = 1, \ldots, M-1.$$

In the second stage, we consider the problem

$$\frac{v_n^k - v_n^{k-1}}{\tau} + bv_n^k - \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} = (b - b_1)e^{-t_k}\cos x_n + b_1 u_n^k,$$

$$1 \le k \le N, 1 \le n \le M - 1,$$

$$v_n^0 = \cos x_n, 0 \le n \le M,$$

$$v_1^k - v_0^k = v_M^k - v_{M-1}^k = 0, 0 \le k \le N.$$

We will write it in the following boundary value problem for the second order difference equation with respect to n

$$\begin{cases} A_2 v_{n+1} + B_2 v_n + C_2 v_{n-1} = \psi_n, 1 \le n \le M - 1, \\ v_0 = v_1, v_M = v_{M-1}. \end{cases} \tag{3.5}$$

Here, A_2, B_2, C_2 are $(N+1)\times(N+1)$ square matrices and $v_s, s = n, n \pm 1, \psi_n$ are $(N+1)\times 1$ column matrices and

$$A_2 = C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a' & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a' & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a' & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a' & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a' \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c' & b' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c' & b' & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c' & b' & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c' & b' & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c' & b' & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c' & b' \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\psi_n = \begin{bmatrix} \cos x_n \\ (b-b_1)e^{-t_1}\cos x_n + b_1 u_n^1 \\ . \\ (b-b_1)e^{-t_{N-1}}\cos x_n + b_1 u_n^{N-1} \\ (b-b_1)e^{-t_N}\cos x_n + b_1 u_n^N \end{bmatrix}_{(N+1)\times 1}, \quad v_s = \begin{bmatrix} v_s^0 \\ v_s^1 \\ . \\ v_s^{N-1} \\ v_s^{N-1} \end{bmatrix}_{(N+1)\times 1}$$

for $s = n, n \pm 1$, where $a' = -\frac{1}{h^2}, b' = \frac{1}{\tau} + b + \frac{2}{h^2}, c' = -\frac{1}{\tau}$. We have the following algorithm for obtaining $\left\{ \{v_n^k\}_{k=0}^{N} \right\}_{n=0}^{M}$

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 0, v_M = (I-\alpha_M)^{-1}\beta_M, \tag{3.6}$$

$$\alpha_{n+1} = -(B_2 + C_2\alpha_n)^{-1} A_2, \alpha_1 = I,$$

$$\beta_{n+1} = (B_2 + C_2\alpha_n)^{-1}(\psi_n - C_2\beta_n), \beta_1 = 0, n = 1, \ldots, M-1.$$

In the third stage, we consider the problem

$$\frac{w_n^k-w_n^{k-1}}{\tau}+cv_n^k-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}=(c-c_1)e^{-t_k}\cos x_n+c_1u_n^k,$$

$$1\le k\le N, 1\le n\le M-1,$$

$$w_n^0=\cos x_n, 0\le n\le M,$$

$$w_1^k-w_0^k=w_M^k-w_{M-1}^k=0, 0\le k\le N.$$

We will write it in the following boundary value problem for the second order difference equation with respect to n

$$\begin{cases} A_3w_{n+1}+B_3w_n+C_3w_{n-1}=\psi_n, 1\le n\le M-1, \\ w_0=w_1, w_M=w_{M-1}. \end{cases} \tag{3.7}$$

Here, A_3, B_3, C_3 are $(N+1)\times(N+1)$ square matrices and $w_s, s=n, n\pm1, \rho_n$ are $(N+1)\times 1$ column matrices and

$$A_3=C_3=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a^* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a^* & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^* & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^* \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c^* & b^* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c^* & b^* & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c^* & b^* & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c^* & b^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c^* & b^* & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c^* & b^* \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\rho_n = \begin{bmatrix} \cos x_n \\ (c-c_1)e^{-t_1}\cos x_n + c_1 u_n^1 \\ . \\ (c-c_1)e^{-t_{N-1}}\cos x_n + c_1 u_n^{N-1} \\ (c-c_1)e^{-t_N}\cos x_n + c_1 u_n^N \end{bmatrix}_{(N+1)\times 1}, \quad w_s = \begin{bmatrix} w_s^0 \\ w_s^1 \\ . \\ w_s^{N-1} \\ w_s^{N-1} \end{bmatrix}_{(N+1)\times 1}$$

for $s = n, n \pm 1$, where $a^* = -\frac{1}{h^2}, b^* = \frac{1}{\tau} + c + \frac{2}{h^2}, c^* = -\frac{1}{\tau}$. For obtaining $\left\{ \{w_n^k\}_{k=0}^N \right\}_{n=0}^M$ We have the following algorithm

$$w_n = \alpha_{n+1} w_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 0, w_M = (I - \alpha_M)^{-1}\beta_M, \quad (3.8)$$

$$\alpha_{n+1} = -(B_3 + C_3\alpha_n)^{-1} A_3, \alpha_1 = I,$$

$$\beta_{n+1} = (B_3 + C_3\alpha_n)^{-1}(\rho_n - C_3\beta_n), \beta_1 = 0, n = 1, \ldots, M-1.$$

In the fourth stage, we consider the problem

$$\frac{z_n^k - z_n^{k-1}}{\tau} + dz_n^k - \frac{z_{n+1}^k - 2z_n^k + z_{n-1}^k}{h^2} = (d - d_1 - d_2)e^{-t_k}\cos x_n + d_1 w_n^k + d_2 v_n^k,$$

$$1 \le k \le N, 1 \le n \le M-1,$$

$$z_n^0 = \cos x_n, 0 \le n \le M,$$

$$z_1^k - z_0^k = z_M^k - z_{M-1}^k = 0, 0 \le k \le N.$$

We will write it in the following boundary value problem for the second order difference equation with respect to n

$$\begin{cases} A_4 z_{n+1} + B_4 z_n + C_4 z_{n-1} = \epsilon_n, 1 \le n \le M-1, \\ z_0 = z_1, z_M = z_{M-1}. \end{cases} \tag{3.9}$$

Here, A_4, B_4, C_4 are $(N+1)\times(N+1)$ square matrices and $z_s, s = n, n \pm 1, \varepsilon_n$ are $(N+1)\times 1$ column matrices and

$$A_4 = C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a^- & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a^- & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^- & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^- & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^- & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^- \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c^- & b^- & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c^- & b^- & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c^- & b^- & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c^- & b^- & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c^- & b^- & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c^- & b^- \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\epsilon_n = \begin{bmatrix} \cos x_n \\ (d-d_1-d_2)e^{-t_1}\cos x_n + d_1 w_n^1 + d_2 v_n^1 \\ . \\ (d-d_1-d_2)e^{-t_{N-1}}\cos x_n + d_1 w_n^{N-1} + d_2 v_n^{N-1} \\ (d-d_1-d_2)e^{-t_N}\cos x_n + d_1 w_n^{N} + d_2 v_n^{N} \end{bmatrix}_{(N+1)\times 1}, \quad z_s = \begin{bmatrix} z_s^0 \\ z_s^1 \\ . \\ z_s^{N-1} \\ z_s^{N-1} \end{bmatrix}_{(N+1)\times 1}$$

for $s = n, n \pm 1$, where $a^- = -\frac{1}{h^2}, b^- = \frac{1}{\tau} + d + \frac{2}{h^2}, c^- = -\frac{1}{\tau}$. for obtaining $\left\{ \{z_n^k\}_{k=0}^{N} \right\}_{n=0}^{M}$ we have the following algorithm

$$z_n = \alpha_{n+1} z_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 0, z_M = (I-\alpha_M)^{-1}\beta_M, \tag{3.10}$$

$$\alpha_{n+1} = -(B_4 + C_4\alpha_n)^{-1}A_4, \alpha_1 = I,$$

$$\beta_{n+1} = (B_4 + C_4\alpha_n)^{-1}(\epsilon_n - C_4\beta_n), \beta_1 = 0, n = 1, \ldots, M-1.$$

The exact solution of problem (3.1) is $u(t,x) = v(t,x) = w(t,x) = z(t,x) = e^{-t}\cos x$. The errors of the numerical solution are computed by

$$(Eu)_M^N = \max_{1\le k\le N, 1\le n\le M-1} \left| u(t_k, x_n) - u_n^k \right| \tag{3.11}$$

Where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in the following table.

Table 3.1: Error analysis

difference scheme $E\cdot_M^N / N, M$	20,20	40,40	80,80	160,160
Eu_M^N	0.0349	0.0167	0.0082	0.0041
Ev_M^N	0.0504	0.0255	0.0128	0.0064
Ew_M^N	0.0504	0.0255	0.0128	0.0064
Ez_M^N	0.0880	0.0449	0.0227	0.0114

As it is seen in Table 3.1, we get some numerical results. If N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately 1/2 for first order difference scheme.

CHAPTER 4
CONCLUSION

In the present study, a system of partial differential equations is studied.

Fourier series, Laplace transform and Fourier transform methods are used for the solution of several system of partial differential equations.

Difference scheme is presented for the numerical solution of the initial-boundary value problem for the system of one dimensional partial differential equations. Numerical results are provided.

REFERENCES

Ashyralyev, A., & Sobolevskii, P, E. (2004). *New Difference Schemes for Partial Differential Equations*, Operator Theory Advances and Applications, Birkhäuser Verlag, Basel, Boston, Berlin.

Ashyralyev, A. (2007). Fractional spaces generated by the positivite differential and difference operator in a Banach space, In: Tas, K, Tenreiro Machado, JA, Baleanu, D, (eds.) Proceedings of the Conference *"Mathematical Methods and Engineering*", Springer, Netherlands, 13-22.

Ashyralyev, A., & Sarsenbi, A. M. (2015). Well-posedness of an elliptic equation with involution. *Electronic Journal of Differential Equations*, 284, 1-8, https://ejde.math.txstate.edu.

Ashyralyev, A., Hincal, E., & Kaymakamzade, B. (2018). Numerical solutions of the system of partial differential equations for observing epidemic models. *AIP Conference Proceedings. 1997,* No. 1, p. 020050. AIP Publishing.

Chalub, F. A., & Souza, M. O. (2011). The SIR epidemic model from a PDE point of view. *Mathematical and Computer Modelling, 53*(7-8), 1568-1574.

Elkadry, A. (2013). Transmission Rate in Partial Differential Equation in Epidemic Models. *Master Thesis, Marshall University*, 53 pages.

Huppert, A., & Katriel, G. (2013). Mathematical modelling and prediction in infectious disease epidemiology. *Clinical Microbiology and Infection*, 19(11), 999-1005.

Jun-Jie, W., Reilly, K. H., Hua, H., Zhi-Hang, P., & Ning, W. (2010). Dynamic characteristic analysis of HIV mother to child transmission in China. *Biomed Environ Sci, 23*(5), 402-408.

Krein, S. G. (1966). *Linear Differential Equations in Banach space*, Nauka: Moscow. (Russian).

Li, M., & liu, X. (2014). An SIR Epidemic Model with Time Delay and General Nonlinear Incidence Rate. *In Abstract and Applied Analysis. 2014*, p. 7. Hindawi.

Lotfi, E. M., Maziane, M., Hattaf, K., & Yousfi, N. (2014). Partial differential equations of an epidemic model with spatial diffusion. *International Journal of Partial Differential Equations, 2014*.

Samarskii, A. A. (2001). *The Theory of Difference Schemes* (1 ed., Vol. 240). CRC Press.

Sobolevskii, P.E. (1975). *Difference Methods for the Approximate Solution of Differential Equations*, Izdat. Voronezh. Gosud. Univ., Voronezh. (Russian).

APPENDICES

APPENDIX 1
MATLAB PROGRAMMING

Matlab programs are presented for the first order of approximation two-step difference scheme for M=N.

```
Clear all; clc; close all;delete '*.asv';
N=80; M= N;
aa=1; bb=3; bb1=2; cc=3; cc1=2; dd=4; dd1=2; dd2=1;
h=pi/M;
tau=1/N;
%%%%%%%%%%%%%%%% solution for u(t,x)%%%%%%%%%%%%%%%
c1=-1/(tau);
a1=-1/(h^2);
b1=(1/tau)+aa+(2/(h^2));
for k=2:N;
A1(k,k)=a1;
A1(N+1,N+1)=a1;
end;A1;
for k=2:N;
B1(k,k)=b1; B1(k,k-1)=c1;
B1(N+1,N+1)=b1; B1(1,1)=1;
B1(N+1,N)=c1;
end;B1;C1=A1;C1;
for j=1:M-1;
for k=2:N+1;
t1=(k-1)*tau; x1=(j)*h;
phy1(k,j:j)=aa*exp(-t1)*cos(x1);
end;
```

```
for j=1:M-1;
x1=(j)*h;
phy1(1,j:j)=cos(x1);
end;phy1;
for i=1:N+1;
D(i,i)=1;
end; D;D;
I=eye(N+1,N+1);
alpha1{1}=eye(N+1,N+1);
betha1{1}=zeros(N+1,1);
for j=1:M-1;
alpha1{j+1}=inv(B1+C1*alpha1{j})*(-A1);
betha1{j+1}=inv(B1+C1*alpha1{j})*(I*phy1(:,j:j)-C1*betha1{j});
end;
U{M}=inv(I-alpha1{M})*betha1{M};
for Z=M-1:-1:1;
U{Z}=alpha1{Z+1}*U{Z+1}+betha1{Z+1};
end;
for Z=1:M;
p1(:,Z+1)=U{Z};
end;
p1(:, 1)=U{1};
for j=1:M+1;
for k=1:N+1;
t1=(k-1)*tau;
x1=(j-1)*h;
es1(k,j:j)=exp(-t1)*cos(x1);
end;
```

```
end;
abs(es1-p1);
maxes1=max(max(es1)) ;
maxapp1=max(max(p1)) ;
maxerror1=max(max(abs(es1-p1)))
%%%%%%%%%%%%%%%% solution for v(t,x)%%%%%%%%%%%%%%%%%
a2=-1/(h^2);
b2=(1/tau)+bb+(2/(h^2));
c2=-1/(tau);
for k=2:N;
A2(k,k)=a2;
A2(N+1,N+1)=a2;
end;A2;
for k=2:N;
B2(k,k)=b2; B2(k,k-1)=c2;
B2(N+1,N+1)=b2; B2(1,1)=1;
B2(N+1,N)=c2;
end;B2;
C2=A2;C2;
for j=1:M-1;
for k=2:N+1;
t2=(k-1)*tau; x2=(j)*h;
phy2(k,j:j)=(bb-bb1)*exp(-t2)*cos(x2)+bb1*p1(k-1,j);
end;
end;
for j=1:M-1;
x2=(j)*h;
phy2(1,j:j)=cos(x2);
```

```
end;phy2;
alpha2{1}=eye(N+1,N+1);
betha2{1}=zeros(N+1,1);
for j=1:M-1;
alpha2{j+1}=inv(B2+C2*alpha2{j})*(-A2);
betha2{j+1}=inv(B2+C2*alpha2{j})*(I*phy2(:,j:j)-C2*betha2{j});
end;
V{M}=inv(I-alpha2{M})*betha2{M};
for Z=M-1:-1:1;
V{Z}=alpha2{Z+1}*V{Z+1}+betha2{Z+1};
end;
for Z=1:M;
p2(:,Z+1)=V{Z};
end;
p2(:, 1)=V{1};
for j=1:M+1;
for k=1:N+1;
t2=(k-1)*tau;
x2=(j-1)*h;
es2(k,j:j)=exp(-t2)*cos(x2);
end;
end;
abs(es2-p2);
maxes2=max(max(es2)) ;
maxapp2=max(max(p2)) ;
maxerror2=max(max(abs(es2-p2)))
%%%%%%%%%%%%%% solution for w(t,x)%%%%%%%%%%%%%%%
a3=-1/(h^2);
```

```
b3=(1/tau)+cc+(2/(h^2));
c3=-1/(tau);
for k=2:N;
A3(k,k)=a3;
A3(N+1,N+1)=a3;
end;A3;
for k=2:N;
B3(k,k)=b3; B3(k,k-1)=c3;
B3(N+1,N+1)=b3; B3(1,1)=1;
B3(N+1,N)=c3;
end;B3;
C3=A3;C3;
for j=1:M-1;
for k=2:N+1;
t3=(k-1)*tau; x3=(j)*h;
phy3(k,j:j)=(cc-cc1)*exp(-t3)*cos(x3)+cc1*p1(k-1,j);
end;
end;
for j=1:M-1;
x3=(j)*h;
phy3(1,j:j)=cos(x3);
end;phy3;
alpha3{1}=eye(N+1,N+1);
betha3{1}=zeros(N+1,1);
for j=1:M-1;
alpha3{j+1}=inv(B3+C3*alpha3{j})*(-A3);
betha3{j+1}=inv(B3+C3*alpha3{j})*(I*phy3(:,j:j)-C3*betha3{j});
end;
```

```
w{M}=inv(I-alpha3{M})*betha3{M};
for Z=M-1:-1:1;
w{Z}=alpha3{Z+1}*w{Z+1}+betha3{Z+1};
end;
for Z=1:M;
p3(:,Z+1)=w{Z};
end;
p3(:, 1)=w{1};
for j=1:M+1;
for k=1:N+1;
t3=(k-1)*tau;
x3=(j-1)*h;
es3(k,j:j)=exp(-t3)*cos(x3);
end;
end;
abs(es3-p3);
maxes3=max(max(es3)) ;
maxapp3=max(max(p3)) ;
maxerror3=max(max(abs(es3-p3)))
%%%%%%%%%%%%%% solution for z(t,x)%%%%%%%%%%%%%%%
a4=-1/(h^2);
b4=(1/tau)+dd+(2/(h^2));
c4=-1/(tau);
for k=2:N;
A4(k,k)=a4;
A4(N+1,N+1)=a4;
end;A4;
for k=2:N;
```

```
B4(k,k)=b4; B4(k,k-1)=c4;
B4(N+1,N+1)=b4; B4(1,1)=1;
B4(N+1,N)=c4;
end;B4; C4=A4;C4;
for j=1:M-1;
for k=2:N+1;
t4=(k-1)*tau; x4=(j)*h;
phy4(k,j:j)=(dd-dd1-dd2)*exp(-t4)*cos(x4)+dd1*p3(k-1,j)+dd2*p2(k-1,j);
end;
end;
for j=1:M-1;
x4=(j)*h;
phy4(1,j:j)=cos(x4);
end;phy4;
alpha4{1}=eye(N+1,N+1);
betha4{1}=zeros(N+1,1);
I=eye(N+1,N+1);
for j=1:M-1;
alpha4{j+1}=inv(B4+C4*alpha4{j})*(-A4);
betha4{j+1}=inv(B4+C4*alpha4{j})*(I*phy4(:,j:j)-C4*betha4{j});
end;
z{M}=inv(I-alpha4{M})*betha4{M};
for Z=M-1:-1:1;
z{Z}=alpha4{Z+1}*z{Z+1}+betha4{Z+1};
end;
for Z=1:M;
p4(:,Z+1)=z{Z};
end;
```

```
p4(:, 1)=z{1};
for j=1:M+1;
for k=1:N+1;
t4=(k-1)*tau;
x4=(j-1)*h;
es4(k,j:j)=exp(-t4)*cos(x4);
end;
end;
abs(es4-p4);
maxes4=max(max(es4)) ;
maxapp4=max(max(p4)) ;
maxerror4=max(max(abs(es4-p4)))
```