AHLAM MUFTAH ABDUSSALAM

INVESTIGATION OF THE FINITE-DIFFERENCE METHOD FOR APPROXIMATING THE SOLUTION AND ITS DERIVATIVES OF THE DIRICHLET PROBLEM FOR 2D AND 3D LAPLACE'S EQUATION

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A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
AHLAM MUFTAH ABDUSSALAM

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy of Science in Mathematics

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Approval of Director of Graduate School of Applied Sciences

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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Last but not least this dissertation is dedicated to my late father who has been my constant source of inspiration.

To my parents and family…

ABSTRACT

The Dirichlet problem for the Laplace equation is carefully weighed in a rectangle and a rectangular parallelepiped. In the case of a rectangle domain, the boundary functions of the Dirichlet problem are supposed to have seventh derivatives satisfying the Hölder condition on the sides of the rectangle Π. Moreover, it is assumed that on the vertices the continuity conditions as well as compatibility conditions, which result from Laplace equation, for even order derivatives up to sixth order are satisfied. Under these conditions the error $u - u_h$ of the 9-point solution u_h at each grid point (x, y) a pointwise estimation $O(\rho h^6)$ is obtained, where $\rho = \rho(x, y)$ is the distance from the current grid point to the boundary of rectangle Π, u is the exact solution and h is the grid step. The solution of difference problems constructed for the approximate values of the first derivatives converge with orders $O(h^6)$ and for pure second derivatives converge with orders $O\left(h^{5+\lambda}\right), 0 < \lambda < 1$. In a rectangular parallelepiped domain, the seventh derivatives for the boundary functions of the Dirichlet problem on the faces of the parallelepiped R are supposed to satisfy the Hölder condition. While, their even order derivatives up to sixth satisfy the compatibility conditions on the edges. For the error $u - u_h$ of the 27-point solution u_h at each grid point (x_1, x_2, x_3), a pointwise estimation $O(\rho h^6)$ is obtained, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point to the boundary of the parallelepiped R. The solution of the constructed 27- point difference problems for the approximate values of the first converge with orders $O(h^6 \ln h)$ and for pure second derivatives converge with orders $O\left(h^{5+\lambda}\right)$. In the constructed three-stage difference method for solving Dirichlet problem for Laplace's equation on a rectangular parallelepiped under some smoothness conditions for the boundary functions the difference solution obtained by 15+7+7- scheme converges uniformly as $O(h^6)$, as the 27-point scheme.

Keywords: Approximation of the derivatives; pointwise error estimations; finite difference method; uniform error estimations; 2D and 3D Laplace's equation; numerical solution of the Laplace equation

ÖZET

Bir dikdörtgen içindeki Laplace denklemi ve dikdörtgenler prizması için Dirichlet problem düşünülmüştür. Tanım bölgesinin dikdörtgen olduğu durumda dikdörtgenin Π kenarlarında verilen sınır fonksiyonlarının yedinci türevlerinin Hölder şartını sağladığı Kabul edildi. Köşelerde süreklilik şartının yanında Laplace denkleminden sonuçlanan köşelerin komşu kenarlarında verilen sınır değer fonksiyonlarının ikinci, dördüncüve altıncı türevleri için uyumluluk şartları da sağlandı. Bu şartlar altında Dirichlet probleminin kare ızgara üzerinde çözümü için $u - u_h$ yaklaşımı $O(\rho h^6)$ olarak düzgün yakınsadığı bulunmuştur, burada u_h 9-nokta yaklaşımı kullanıldığında elde elinen yaklaşık çözüm, $problem$ sağlayan kesin çözüm, $\rho = \rho(x, y)$, dikdörtgen sınırını işaret eden mevcut ızgara uzunlığu ve h, ızgara adımıdır. Çözümün birinci ve pür türevleri için oluşturulan fark problemlerinin sırasıyla çözümleri, $O(h^6)$ ve $O\left(h^{5+\lambda}\right)$, $0 < \lambda < 1$ mertebesi ile yakınsar. Tanım bölgesinin dikdörtgenler prizması olduğu durumda prizmanın yüzeylerinde verilensınır fonksiyonlarının yedinci türevlerinin Hölder şartını sağladığı Kabul edildi. Köşelerde süreklilik şartının yanında Laplace denkleminden sonuçlanan kenarlarının komşu kenarlarında verilen sınır değer fonksiyonlarının ikinci, dördüncü ve altıncı türevleri için uyumluluk şartlarınıda sağlar. Bu şartlar altında Dirichlet probleminin küp ızgaralar üzerindeki çözümü için $u - u_h$ yaklaşımı $O(\rho h^6)$ olarak bulunmuştur, burada u_h 27-nokta yaklaşımı kullanıldığında elde edilen yaklaşık çözüm, problem sağlayan kesin çözüm ve $\rho = \rho(x_1, x_2, x_3)$ prizmanın sınırını işaret edenmevcut ızgara ve h, ızgara adımıdır. Çözümün birinci ve pür türevleri için oluşturulan fark problemlerinin sırasıyla çözümleri $O(h^6 \ln h)$, ve $O(h^6)$, $0 < \lambda < 1$ mertebeleri ile yakınsar. Laplace denkleminin sınır fonksiyonları için bazı pürüzsüzlük koşulları altında Dirichlet problemi çözmek için inşa edilmiş üç aşamalı fark yöntemi, 15 + 7 + 7 - şeması ile elde edilen fark çözümü $O(h^6)$ olarak yakınsar 27 noktalı şema yöntemiyle elde edilen sonuç gibi.

Anahtar Kelimeler: Sonlu fark metodu; noktasal hata tahminleri; türev yaklaşımları; düzgün hata tahminleri; 2D ve 3D Laplace denklemleri; Laplace denkleminin sayısal çözümü

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

The partial differential equations are highly used in many topics of applied sciences in order to solve equilibrium or steady state problem. Laplace equation is one of the most important elliptic equations, which has been used, to model many problems in real life situations.

Further it can be used in the formulation of problems relevant to theory of electrostatics, gravitation and problems arising in the field of interest to mathematical physics. In addition, it is applied in engineering, when dealing with many problems such as analysis of steady heat condition in solid bodies, the irrotational flow of incompressible fluid, and so on.

In many applied problems not only the calculation of the solution of the differential equation but also the calculation of the derivatives are very important to provide information about some physical phenomenas. For example, by the theory of Saint-Venant, the problem of the torsion of any prismatic body whose section is the region D bounded by the contour L reduces to the following boundary- value problem: to find, the solution of the Poisson equation

$$\Delta u = -2,$$

that reduces to zero on the contour L:

$$u = 0 \text{ on } L.$$

Here the basis quantities required from the calculation are expressed in terms of the function u the components of the tangential stress

$$\tau_{zx} = \mathrm{G}\vartheta \frac{\partial u}{\partial y}, \qquad \tau_{zy} = -\mathrm{G}\vartheta \frac{\partial u}{\partial x},$$

and the torsional moment

$$M = \mathrm{G}\vartheta \iint_D u dx dy.$$

Here ϑ is the angle of twist per unit length, and G is the modulus of shear.

The construction and justification of highly accurate approximate methods for the solution and its derivatives of PDEs in a rectangle or in a rectangular parallelepiped are important not only for the development of theory of these methods, but also to improve some version of domain decomposition methods for more complicated domains, (Smith et al., 2004; Kantorovich and Krylov, 1958; Volkov, 1976; Volkov, 1979; Volkov, 2003; Volkov, 2006).

Since the operation of differentiation is ill-conditioned, to find a highly accurate approximation for the derivatives of the solution of a differential equation becomes problematic, especially when smoothness is restricted.

It is obvious that the accuracy of the approximate derivatives depends on the accuracy of the approximate solution. As is proved in (Lebedev, 1960), the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the 2D Laplace equation in any strictly interior subdomain with the same order h, (h is the grid step), with which the difference solution converges on the given domain. In (Volkov, 1999), $O(h^2)$ order difference derivatives uniform convergence of the solution of the difference equation, and its first and pure second difference derivatives over the whole grid domain to the solution, and corresponding derivatives of solution for the 2D Laplace equation was proved. In (Dosiyev and Sadeghi, 2015) three difference schemes were constructed to approximate the solution and its first and pure second derivatives of 2D Laplace's equation with order of $O(h^4)$, when the sixth derivatives of the boundary functions on the sides of a rectangle satisfy the Hölder condition, and on the vertices their second and fourth derivatives satisfy the compatibility condition that is implied by the Laplace equation.

In (Volkov, 2004), for the 3D Laplace equation in a rectangular parallelepiped the constructed difference schemes converge with order of $O(h^2)$ to the first and pure second derivatives of the exact solution of the Dirichlet problem. It is assumed that the fourth derivatives of the boundary functions on the faces of a parallelepiped satisfy the Hölder condition, and on the edges their second derivatives satisfy the compatibility condition that is implied by the Laplace equation. Whereas in (Volkov, 2005), the convergence with order $O(h^2)$ of the difference derivatives to the corresponding first order derivatives was proved, when the third derivatives of the boundary functions on the faces satisfy the Hölder condition. Further, in (Dosiyev and Sadeghi, 2016) by assuming that the boundary functions on the faces have the sixth order derivatives satisfying the Hölder condition, and the second and fourth derivatives satisfy the compatibility conditions on the edges, for the uniform error of the approximate solution $O(h^6|\ln h|)$ order, and for the first and pure second derivatives $O(h^4)$ order was obtained.

We mention one more problem when we use the high order accurate finite- difference schemes for the approximation of the solution and its derivatives Since in the finite- difference approximations the obtained system of difference equations, in general, are banded matrices. To get a highly accurate results in the most of approximations, difference operators with the high number of pattern are used which increase the number of bandwidth of the difference equations. It is obvious that the complexity of the realization methods for the difference equations increases depending on the number of bandwidth of the matrices of these equations. As it was shown in (Tarjan, 1976) that in case of Gaussian elimination method the bandwidth elimination for $n \times n$ matrices with the bandwidth b the computational cost is of order $O(b^2 n)$. Therefore, the construction of multistage finite difference methods with the use of low number of bandwidth matrix in each of stages becomes important.

In (Volkov, 2009) a new two-stage difference method for solving the Dirichlet problem for Laplace's equation on a rectangular parallelepiped was proposed. It was assumed that the given boundary values are six times differentiable at the faces of the parallelepiped, those derivatives satisfy a Hölder condition, and the boundary values are continuous at the edges and their second derivatives satisfy a compatibility condition implied by the Laplace equation. Under these conditions it was proved that by using the 7-point scheme on a cubic grid in each stage the order of uniform error is improved from $O(h^2)$ up to $O(h^4 \ln h^{-1})$, where h is the mesh size. It is known that, to get $O(h^4)$ order of accurate results by the existing one-stage methods for the approximation $3D$ Laplace's equation we have to use at least 15-point scheme, (Volkov, 2010).

In this thesis, a highly accurate schemes for the solution and its the first and pure second derivatives of the Laplace equation on a rectangle and on a rectangular parallelepiped are constructed and justified. Two-dimensional case (Chapter 2) consider the classical 9-point, and in three-dimensional case (Chapter 3) the 27-point finite- difference approximation of Laplace equation are used. In Chapter 4, in the three-stage difference method at the first stage, the difference equations are formulated using the 14-point averaging operator, and the difference equations at the second and third stages are formulated using the simplest six point averaging operator.

The numerical experiments to justify the obtained theoretical results are presented in Chapter 5.

Now, we formulated all results more explicitly.

In Chapter 2, we consider the Dirichlet problem for the Laplace equation on a rectangle, when the boundary values belong to $C^{7,\lambda},\ 0 < \lambda < 1$, on the sides of the rectangle, and as whole are continuous on the vertices. Also, the $2q,\ q = 1,2,3$, order derivatives satisfy the compatibility conditions on the vertices which result from the Laplace equation. Under these conditions, we present and justify difference schemes on a square grid for obtaining the solution of the Dirichlet problem, its first and pure second derivatives. For the approximate solution a pointwise estimation for the error of order $O(\rho h^6)$, where $\rho = \rho(x, y)$ is the distance from the current grid point (x, y) to the boundary of the rectangle, is obtained. This estimation is used to approximate the first derivatives with uniform error of order $O(h^6)$. The approximation of the pure second derivatives are obtained with uniform accuracy $O\left(h^{5+\lambda}\right)$, $0 < \lambda < 1$.

In Chapter 3, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. The boundary functions on the faces of a parallelepiped are supposed to have the seventh order derivatives satisfying the Hölder condition, and on the edges the second, fourth and sixth order derivatives satisfy the compatibility conditions. We present and justify difference schemes on a cubic grid for obtaining the solution of the Dirichlet problem, its first and pure second derivatives. For the approximate solution a pointwise estimation for the error of order $O(\rho h^6)$ with the weight function ρ, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point (x_1, x_2, x_3) to the boundary of the parallelepiped, is obtained. This estimation gives an additional accuracy of the finite difference solution near the boundary of the parallelepiped, which is used to approximate the first derivatives with uniform error of order $O(h^6 \ln h)$. The approximation of the pure second derivatives are obtained with uniform accuracy $O\left(h^{5+\lambda}\right)$, $0 < \lambda < 1$.

In Chapter 4, A three-stage difference method is proposed for solving the Dirichlet problem for the Laplace equation on a rectangular parallelepiped, at the first stage, approximate values of the sum of the pure fourth derivatives of the desired solution are sought on a cubic grid. At the second stage, approximate values of the sum of the pure sixth derivatives of the desired solution are sought on a cubic grid. At the third stage, the system of difference equations approximating the Dirichlet problem corrected by introducing the quantities determined at the first and second stages. The difference equations at the first stage is formulated using the 14-point averaging operator, and the difference equations at the second and third stages are formulated using the simplest six-point averaging operator. Under the assumptions that the given boundary functions on the faces of a parallelepiped have the eighth derivatives satisfying the Hölder condition, and on the edges the second, fourth, and sixth order derivatives satisfy the compatibility conditions, it is proved that the difference solution to the Dirichlet problem converges uniformly as $O(h^6)$.
In Chapter 5, the numerical experiments to justify the theoretical results obtained in each Chapters are demonstrated.

CHAPTER 2

ON THE HIGH ORDER CONVERGENCE OF THE DIFFERENCE SOLUTION OF LAPLACE'S EQUATION IN A RECTANGLE

In this Chapter we consider the Dirichlet problem for the Laplace equation on a rectangle, when the boundary values on the sides of the rectangle are supposed to have the seventh derivatives satisfying the Hölder condition. On the vertices besides the continuity condition, the compatibility conditions, which result from the Laplace equation for the second, fourth and sixth derivatives of the boundary values, given on the adjacent sides are also satisfied. Under these conditions, we present and justify difference schemes on a square grid for obtaining the solution of the Dirichlet problem, its first and pure second derivatives. For the approximate solution a pointwise estimation for the error of order $O(\rho h^6)$ with the weight function ρ, where $\rho = \rho(x, y)$ is the distance from the current grid point (x, y) to the boundary of the rectangle is obtained. This estimation gives an additional accuracy of the finite difference solution near the boundary of the rectangle, which is used to approximate the first derivatives with uniform error of order $O(h^6)$. The approximation of the pure second derivatives are obtained with uniform accuracy $O\left(h^{5+\lambda}\right),\ 0 < \lambda < 1$.

2.1 The Dirichlet Problem for Laplace's Equation on Rectangle and Some Differential Properties of Its Solution

Let $\Pi = \{(x, y): 0 < x < a\,, 0 < y < b\}$ be an open rectangle and a/b is a rational number. The sides are denoted by γ_j, $j = 1,2,3,4$, including the ends. These sides are enumerated counterclockwise where γ_1 is the left side of Π, $(\gamma_0 \equiv \gamma_4\,, \gamma_5 \equiv \gamma_1)$. Also let the boundary of Π be defined by $\gamma = \bigcup_{j=1}^{4} \gamma_j$.

The arclength along γ is denoted by s, and s_j is the value of s at the beginning of γ_j. We denote by $f \in C^{k,\lambda}(D)$ if f has $k - th$ deravatives on D satisfying Hölder condition, where exponent $\lambda \in (0,1)$.

We consider the following boundary value problem

$$\Delta u = 0 \ \text{ on } \Pi, \ u = \varphi_j(s) \ \text{ on } \gamma_j, \quad j = 1,2,3,4 \tag{2.1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j are given functions of s.
Assume that

$$\varphi_j \in C^{7,\lambda}(\gamma_j), \quad 0 < \lambda < 1, \quad j = 1,2,3,4, \tag{2.2}$$

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \quad q = 0,1,2,3. \tag{2.3}$$

Lemma 2.1 The solution u of problem (2.1) is from $C^{7,\lambda}(\overline{\Pi})$.
The proof of Lemma 2.1 follows from Theorem 3.1 by (Volkov, 1969).

Lemma 2.2 Let $\rho(x,y)$ be the distance from the current point of open rectangle Π to its boundary and let $\partial/\partial l \equiv \alpha\,\partial/\partial x + \beta\,\partial/\partial y$, $\alpha^2 + \beta^2 = 1$. Then the next inequality holds

$$\left|\frac{\partial^8 u(x,y)}{\partial l^8}\right| \le c\rho^{\lambda-1}(x,y), \ \ (x,y) \in \Pi, \tag{2.4}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$, and u is a solution of problem (2.1).

Proof. We choose an arbitrary point $(x_0, y_0) \in \Pi$. Let $\rho_0 = \rho(x_0, y_0)$, and $\bar{\sigma}_0 \subset \overline{\Pi}$ be the closed circle of radius ρ_0 centered at (x_0, y_0). Consider the harmonic function on Π

$$v(x,y) = \frac{\partial^7 u(x,y)}{\partial l^7} - \frac{\partial^7 u(x_0,y_0)}{\partial l^7}. \tag{2.5}$$

By Lemma 2.1, $u \in C^{7,\lambda}(\overline{\Pi})$, for $0 < \lambda < 1$. Then for the function (2.5) we have

$$\max_{(x,y)\in\overline{\sigma}_0} |v(x,y)| \le c_0\rho_0^{\lambda}, \tag{2.6}$$

where c_0 is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since u is harmonic in Π, by using estimation (2.6) and applying Lemma 3 from (Mikhailov, 1978) we have

$$\left|\frac{\partial}{\partial l}\left(\frac{\partial^7 u(x,y)}{\partial l^7} - \frac{\partial^7 u(x_0,y_0)}{\partial l^7}\right)\right| \le c_1\frac{\rho_0^{\lambda}}{\rho_0}.$$

or

$$\left|\frac{\partial^8 u(x,y)}{\partial l^8}\right| \le c_1\rho_0^{\lambda-1}(x_0,y_0),$$

where $c_1$is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since the point $(x_0, y_0) \in \Pi$ is arbitrary, inequality (2.4) holds true. ■

2.2 Difference Equations for the Dirichlet Problem and a Pointwise Estimation

Let $h > 0$, and $min\{a/h\,, b/h\} \ge 6$ where a/h and b/h are integers. A square net on Π is assigned by Π^h, with step h, obtained by the lines $x, y = 0, h, 2h, \dots$.

The set of nodes on $\gamma_j$is denoted by γ_j^h, and let

$$\gamma^h = \cup_{j=1}^{4}\gamma_j^h,\ \overline{\Pi}_h = \Pi^h \cup \gamma^h.$$

Let the averaging operator B be defined as following

$$Bu(x,y) = \frac{1}{20}[4(u(x+h,y) + u(x-h,y) + u(x,y+h)$$
$$+u(x,y-h)) + u(x+h,y+h) + u(x+h,y-h)$$
$$+u(x-h,y+h) + u(x-h,y-h)]. \tag{2.7}$$

Let $c, c_0, c_1, \ldots$ be constants which are independent of h and the nearest factor, and for simplicity identical notation will be used for various constants.

Consider the finite difference approximation of problem (2.1) as follows:

$$u_h = Bu_h \text{ on } \Pi^h, u_h = \varphi_j \text{ on } \gamma_j^h, \quad j = 1,2,3,4. \tag{2.8}$$

By the maximum principle system (2.8) has a unique solution (Samarskii, 2001).

Let Π^{kh} be the set of nodes of grid Π^h whose distance from γ is kh. It is obvious that $1 \le k \le N(h)$, where

$$N(h) = \left[\frac{1}{2h} min\{a,b\}\right], \tag{2.9}$$

$[d]$ is the integer part of d.

We define for $1 \le k \le N(h)$ the function

$$f_h^k = \begin{cases} 1, & \rho(x,y) = kh, \\ 0, & \rho(x,y) \ne kh. \end{cases} \tag{2.10}$$

Consider the following systems

$$q_h = Bq_h + g_h \text{ on } \Pi^h, \quad q_h = 0 \text{ on } \gamma^h, \tag{2.11}$$
$$\bar{q}_h = B\bar{q}_h + \bar{g}_h \text{ on } \Pi^h, \quad \bar{q}_h = 0 \text{ on } \gamma^h, \tag{2.12}$$

where g_h and $\bar{g}_h$ are given function, and $|g_h| \le \bar{g}_h$ on Π^h.

Lemma 2.3 The solution q_h and $\bar{q}_h$ of systems (2.11) and (2.12) satisfy the inequality

$$|q_h| \leq \bar{q}_h \text{ on } \bar{\Pi}^h .$$

The proof of Lemma 2.3 follows from comparison Theorem see Chapter 4 (Samarskii, 2001).

Lemma 2.4 The solution of the system

$$v_h^k = Bv_h^k + f_h^k \text{ on } \Pi^h, \quad v_h^k = 0 \text{ on } \gamma^h \tag{2.13}$$

satisfies the inequality

$$v_h^k(x,y) \leq Q_h^k, \quad 1 \leq k \leq N(h), \tag{2.14}$$

where Q_h^k is defined as follows

$$Q_h^k = Q_h^k(x,y) = \begin{cases} \dfrac{6\rho}{h}, & 0 \leq \rho(x,y) \leq kh \, , \\ 6k, & \rho(x,y) > kh \, . \end{cases} \tag{2.15}$$

Proof. By virtue of (2.7) and (2.15) and in consider of Fig. (2.1), we have for $0 \leq \rho = kh$,

$$BQ_h^k = \frac{1}{20}[4(6k + 6(k-1) + 6(k-1) + 6k)$$
$$+6(k-1) + 6k + 6(k-1) + 6(k-1)]$$
$$= 6k - \frac{66}{20},$$

which leads to

$$Q_h^k - BQ_h^k = \frac{66}{20} > 1 = f_h^k.$$

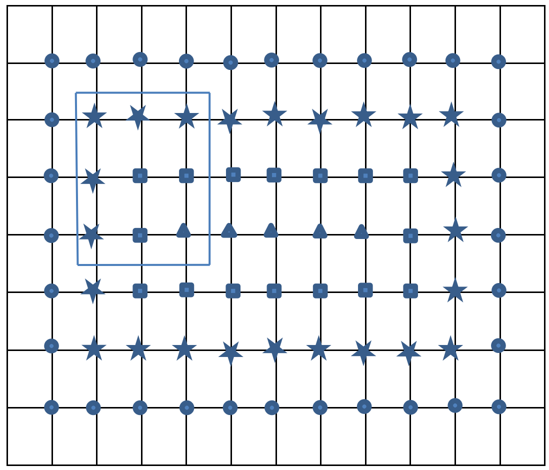


Figure 2.1: The selected region in Π is $\rho = kh$

In consider of Fig. (2.2) for $\rho > kh$, then

$$BQ_h^k = \frac{1}{20}[4(6k + 6k + 6k + 6k) + 6k$$
$$+6k + 6k + 6k = 6k,$$

which leads to

$$BQ_h^k = Q_h^k.$$

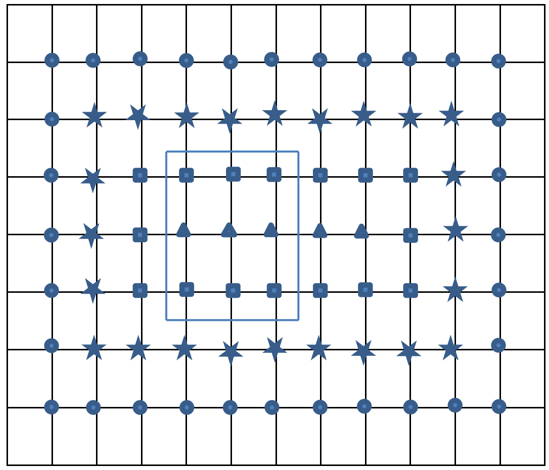


Figure 2.2: The selected region in Π is $\rho > kh$

In consider of Fig. (2.3) for $\rho < kh$, then

$$BQ_h^k = \frac{1}{20}[4(6k + 6(k-2) + 6(k-1) + 6(k-1))$$
$$+6(k-1) + 6k + 6(k-2) + 6(k-2)]$$
$$= 6k - \frac{126}{20},$$

which leads to

$$Q_h^k - BQ_h^k = \frac{126}{20} > 1 = f_h^k.$$

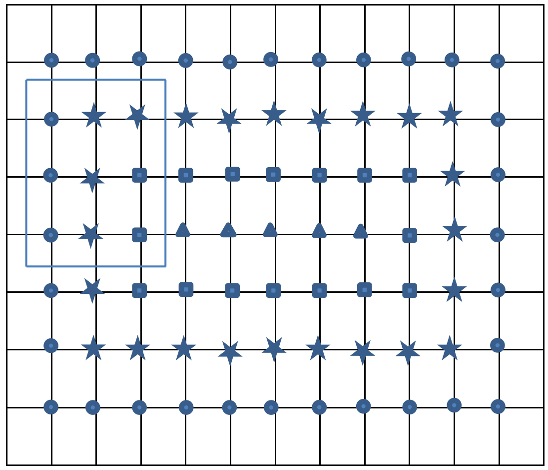


Figure 2.3: The selected region in Π is $\rho < kh$

In consider of Fig. (2.1) for $\rho < kh$, then

$$BQ_h^k = \frac{1}{20} = [4(6(k-1) + 6(k-1) + 6(k-2) + 6k)$$
$$+6k + 6k + 6(k-2) + 6(k-2)]$$
$$= 6k - 6 = 6(k-1),$$

which leads to

$$Q_h^k = BQ_h^k.$$

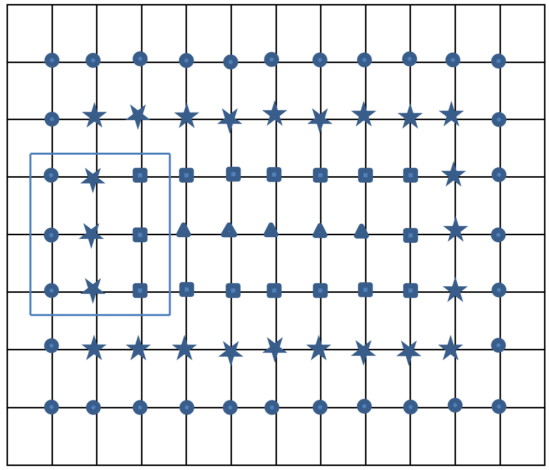


Figure 2.4: The selected region in Π is $\rho < kh$

From the above calculations we have

$$Q_h^k = BQ_h^k + q_h^k \text{ on } \Pi^h, Q_h^k = 0 \text{ on } \gamma^h, \quad k = 1, \dots, N(h), \tag{2.16}$$

where $|q_h^k| \geq 1$. On the basis of (2.10), (2.13), (2.16) and the Comparison Theorem from (Samarskii, 2001), we obtain

$$|v_h^k| \leq Q_h^k \text{ for all } k, \;\; 1 \leq k \leq N(h). \quad \blacksquare$$

Lemma 2.5 The following equality is true

$$Bp_7(x_0, y_0) = u(x_0, y_0)$$

where p_7 is the seventh order Taylor's polynomial at (x_0, y_0), u is a harmonic function.

Proof. The seventh order Taylor's polynomial at (x_0, y_0) has the form

$$p_7(x,y) = u(x,y) + h\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right.$$

$$+3\frac{\partial^3 u}{\partial x^2 \partial y} + 3\frac{\partial^3 u}{\partial x \partial y^2} + \left.\frac{\partial^3 u}{\partial y^3}\right) + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^4 u}{\partial x^3 \partial y} + 6\frac{\partial^4 u}{\partial x^2 \partial y^2}\right.$$

$$+4\frac{\partial^4 u}{\partial x \partial y^3} + \left.\frac{\partial^4 u}{\partial y^4}\right) + \frac{h^5}{5!}\left(\frac{\partial^5 u}{\partial x^5} + 5\frac{\partial^5 u}{\partial x^4 \partial y} + 20\frac{\partial^5 u}{\partial x^3 \partial y^2} + 20\frac{\partial^5 u}{\partial x^2 \partial y^3}\right.$$

$$+5\frac{\partial^5 u}{\partial x \partial y^4} + \left.\frac{\partial^5 u}{\partial y^5}\right) + \frac{h^6}{6!}\left(\frac{\partial^6 u}{\partial x^6} + 6\frac{\partial^6 u}{\partial x^5 \partial y} + 15\frac{\partial^6 u}{\partial x^4 \partial y^2} + 20\frac{\partial^6 u}{\partial x^3 \partial y^3}\right.$$

$$+15\frac{\partial^6 u}{\partial x^2 \partial y^4} + 6\frac{\partial^6 u}{\partial x \partial y^5} + \left.\frac{\partial^6 u}{\partial y^6}\right) + \frac{h^7}{7!}\left(\frac{\partial^7 u}{\partial x^7} + 7\frac{\partial^7 u}{\partial x^6 \partial y} + 21\frac{\partial^7 u}{\partial x^5 \partial y^2}\right.$$

$$\left.+35\frac{\partial^7 u}{\partial x^4 \partial y^3} + 35\frac{\partial^7 u}{\partial x^3 \partial y^4} + 21\frac{\partial^7 u}{\partial x^2 \partial y^5} + 7\frac{\partial^7 u}{\partial x \partial y^6} + \frac{\partial^7 u}{\partial y^7}\right). \tag{2.17}$$

Then according to (2.7) and (2.17) we have

$$Bp_7(x_0, y_0) = \frac{1}{20}[4(p_7(x_0 + h, y_0) + p_7(x_0 - h, y_0) + p_7(x_0, y_0 + h)$$

$$+p_7(x_0, y_0 - h)) + p_7(x_0 + h, y_0 + h) + p_7(x_0 + h, y_0 - h)$$

$$+p_7(x_0 - h, y_0 + h) + p_7(x_0 - h, y_0 - h)]$$

$$= u(x_0, y_0) + \frac{h^4}{40}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x_0, y_0)}{\partial x^2} + \frac{\partial^2 u(x_0, y_0)}{\partial y^2}\right)$$

$$+\frac{h^4}{40}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 u(x_0, y_0)}{\partial x^2} + \frac{\partial^2 u(x_0, y_0)}{\partial y^2}\right) + \frac{3h^6}{5 \times 6!}\frac{\partial^4}{\partial x^4}\left(\frac{\partial^2 u(x_0, y_0)}{\partial x^2} + \frac{\partial^2 u(x_0, y_0)}{\partial y^2}\right)$$

$$+\frac{3h^6}{5 \times 6!}\frac{\partial^4}{\partial y^4}\left(\frac{\partial^2 u(x_0, y_0)}{\partial x^2} + \frac{\partial^2 u(x_0, y_0)}{\partial y^2}\right) + \frac{2h^6}{5 \times 5!}\frac{\partial^4}{\partial x^2 \partial y^2}\left(\frac{\partial^2 u(x_0, y_0)}{\partial x^2} + \frac{\partial^2 u(x_0, y_0)}{\partial y^2}\right).$$

Since u is harmonic, we obtain

$$BP_7(x_0, y_0) = u(x_0, y_0) \quad \blacksquare$$

Lemma 2.6 The inequality holds

$$\max_{(x,y)\in\Pi^{kh}} |Bu - u| \le c\frac{h^{7+\lambda}}{k^{1-\lambda}}, \quad k = 1,2, \dots, N(h), \tag{2.18}$$

where u is a solution of problem (2.1).

Proof. Let (x_0, y_0) be a point of Π^{1h}, and let

$$\Pi_0 = \{(x, y): |x - x_0| < h, |y - y_0| < h\}, \tag{2.19}$$

be an elementary square, some sides of which lie on the boundary of the rectangle Π. On the vertices of Π_0, and on the mid points of its sides lie the nodes of which the function values are used to evaluate $Bu(x_0, y_0)$. We represent a solution of problem (2.1) in some neighborhood of $(x_0, y_0) \in \Pi^{1h}$ by Taylor's formula

$$u(x, y) = p_7(x, y) + r_8(x, y), \tag{2.20}$$

where $p_7(x, y)$ is the seventh order Taylor's polynomial, $r_8(x, y)$ is the remainder term, by Lemma 2.5 we have

$$Bp_7(x_0, y_0) = u(x_0, y_0). \tag{2.21}$$

Now, we estimate r_8 at the nodes of the operator B. We take node $(x_0 + h, y_0 + h)$ which is one of the eight nodes of B, and consider the function

$$\tilde{u}(s) = u\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \quad -\sqrt{2}h \leq s \leq \sqrt{2}h \tag{2.22}$$

of one variable s. By virtue of Lemma 2.2, we have

$$\left|\frac{\partial^8 \tilde{u}(s)}{\partial s^8}\right| \leq c_2\left(\sqrt{2}h - s\right)^{\lambda-1}, \quad 0 \leq s \leq \sqrt{2}h\,. \tag{2.23}$$

We represent function (2.22) around the point $s = 0$ by Taylor's formula

$$\tilde{u}(s) = \tilde{p}_7(s) + \tilde{r}_8(s), \tag{2.24}$$

where

$$\tilde{p}_7(s) = p_7\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) \tag{2.25}$$

is the seventh order Taylor's polynomial of the variable s, and

$$\tilde{r}_8(s) = r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \;\; 0 \leq |s| \leq \sqrt{2}h \tag{2.26}$$

is the remainder term. On the basis of continuity of $\tilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$, it follows from (2.26) that

$$r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) = \lim_{\epsilon \to +0} \tilde{r}_8\left(\sqrt{2}h - \epsilon\right). \tag{2.27}$$

Applying an integral representation for $\tilde{r}_8$ we have

$$\tilde{r}_8(\sqrt{2}h-\epsilon)=\frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-\epsilon-t\right)^7\tilde{u}^8(t)dt\,,\qquad 0<\epsilon\leq\frac{h}{\sqrt{2}}.$$

Using estimation (2.23), we have

$$\left|\tilde{r}_8(\sqrt{2}h-\epsilon)\right|\leq c_3\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-\epsilon-t\right)^7\left(\sqrt{2}h-t\right)^{\lambda-1}dt$$

$$\leq c_4\frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon}\left(\sqrt{2}h-t\right)^{6+\lambda}dt$$

$$\leq ch^{\lambda+7},\quad 0<\epsilon\leq\frac{h}{\sqrt{2}}. \tag{2.28}$$

From (2.26)-(2.28) yields

$$|r_8(x_0+h,y_0+h|\leq c_1h^{\lambda+7}, \tag{2.29}$$

where c_1 is a constant independent of the taken point (x_0,y_0) on Π^{1h}. Proceeding in a similar manner, we can find the same estimates of r_8 at the other vertices of square (2.19) and at the centers of its sides. Since the norm of B in the uniform metric is equal to unity, we have

$$|Br_8(x_0,y_0)|\leq c_5h^{\lambda+7}. \tag{2.30}$$

where c_5 is a constant independent of the taken point (x_0,y_0) on Π^{1h}. From (2.20), (2.21), (2.30) and linearity of the operator B, we obtain

$$|Bu(x_0,y_0)-u(x_0,y_0)|\leq ch^{\lambda+7}, \tag{2.31}$$

for any $(x_0,y_0)\in\Pi^{1h}$.

Now let $(x_0, y_0) \in \Pi^{kh}$, $2 \leq k \leq N(h)$ and $r_8(x, y)$ be the Lagrange remainder corresponding to this point in Taylor's formula (2.20). Then $Br_8(x_0, y_0)$ can be expressed linearly in terms of a fixed number of eighth derivatives of u at some point of the open square Π_0, which is a distance of $kh/2$ away from the boundary of Π. The sum of the coefficients multiplying the eighth derivatives does not exceed ch^8, which is independent of k ($2 \leq k \leq N(h)$). By using Lemma 2.2, we have

$$|Br_8(x_0, y_0)| \leq c\frac{h^8}{(kh)^{1-\lambda}} = c\frac{h^{\lambda+7}}{k^{1-\lambda}}, \qquad (2.32)$$

where c is a constant independent of k ($2 \leq k \leq N(h)$). On the basis of (2.20), (2.21), (2.31), and (2.32) follows estimation (2.18) at any point $(x_0, y_0) \in \Pi^{kh}$, $1 \leq k \leq N(h)$. ■

Theorem 2.1 The next estimation holds

$$|u_h - u| \leq c\rho h^6,$$

where c is a constant independent of ρ and h, u is the exact solution of problem (2.1), u_h is the solution of the finite difference problem (2.8) and $\rho = \rho(x, y)$ is the distance from the current point $(x, y) \in \Pi^h$ to the boundary of rectangle Π.

Proof. Let

$$\epsilon_h(x, y) = u_h(x, y) - u(x, y), \quad (x, y) \in \overline{\Pi}^h. \qquad (2.33)$$

Putting $u_h = \epsilon_h + u$ into (2.8), we have

$$\epsilon_h = B\epsilon_h + (Bu - u) \text{ on } \Pi^h, \quad \epsilon_h = 0 \text{ on } \gamma^h. \qquad (2.34)$$

We represent a solution of system (2.34) as follows

$$\epsilon_h = \sum_{k=1}^{N(h)} \epsilon_h^k, \quad N(h) = \left[\frac{1}{2h}\min\{a, b\}\right], \tag{2.35}$$

where ϵ_h^k is a solution of the system

$$\epsilon_h^k = B\epsilon_h^k + \sigma_h^k \text{ on } \Pi^h, \quad \epsilon_h = 0 \text{ on } \gamma^h, \quad k = 1,2, \dots, N(h); \tag{2.36}$$

$$\sigma_h^k = \begin{cases} Bu - u & \text{on } \quad \Pi_h^k, \\ 0 & \text{on } \quad \Pi_h/\Pi_h^k. \end{cases} \tag{2.37}$$

By virtue of (2.36), (2.37) and Lemma 2.4, for each k, $1 \le k \le N(h)$, follows the inequality

$$\left|\epsilon_h^k(x, y)\right| \le Q_h^k(x, y) \max_{(x,y)\in\Pi^{kh}} |(Bu - u)| \text{ on } \overline{\Pi}^h. \tag{2.38}$$

On the basis of (2.33), (2.35) and (2.38) we have

$$|\epsilon_h| \le \sum_{k=1}^{N(h)} \left|\epsilon_h^k\right| \le \sum_{k=1}^{N(h)} Q_h^k(x, y) \max_{(x,y)\in\Pi^{kh}} |(Bu - u)|$$
$$= \sum_{k=1}^{\frac{\rho}{h}-1} Q_h^k(x, y) \max_{(x,y)\in\Pi^{kh}} |(Bu - u)|$$
$$+ \sum_{k=\frac{\rho}{h}}^{N(h)} Q_h^k(x, y) \max_{(x,y)\in\Pi^{kh}} |(Bu - u)|, \ (x, y) \in \Pi^{kh}.$$

By definition (2.15) of the function Q_h^k, we have

$$\sum_{k=1}^{\frac{\rho}{h}-1} Q_h^k(x,y) \max_{(x,y)\in\Pi^{kh}} |(Bu-u)| \le 6ch^8 \sum_{k=1}^{\frac{\rho}{h}-1} \frac{k}{(kh)^{1-\lambda}}, \tag{2.39}$$

$$\sum_{k=\frac{\rho}{h}}^{N(h)} Q_h^k(x,y) \max_{(x,y)\in\Pi^{kh}} |(Bu-u)| 6ch^8 \sum_{k=\frac{\rho}{h}}^{N(h)} \frac{k}{(kh)^{1-\lambda}}. \tag{2.40}$$

Then from (2.39)-(2.40) we have

$$|\epsilon_h(x,y)| \le 6ch^{7+\lambda} \sum_{k=1}^{\frac{\rho}{h}-1} k^\lambda + 6ch^{6+\lambda}\rho \sum_{k=\frac{\rho}{h}}^{N(h)} \frac{1}{k^{1-\lambda}}$$

$$\le 6ch^{7+\lambda}\left(1 + \int_1^{\frac{\rho}{h}-1} x^\lambda\, dx\right) + 6ch^{6+\lambda}\rho\left(\left(\frac{\rho}{h}\right)^{\lambda-1} + \int_{\frac{\rho}{h}}^{N(h)} x^{\lambda-1}\, dx\right)$$

$$\le ch^{7+\lambda} + ch^{7+\lambda}\left(\frac{\left(\frac{\rho}{h}-1\right)^{\lambda+1}}{\lambda+1} - \frac{1}{\lambda+1}\right) + ch^7\rho^\lambda + ch^{6+\lambda}\rho\left(\frac{(\frac{a}{h})^\lambda}{\lambda} - \frac{\frac{\rho}{h}}{\lambda}\right)$$

$$\le ch^{7+\lambda} + \frac{c}{\lambda+1}h^{7+\lambda}\left(\frac{\rho}{h}-1\right)^{\lambda+1} - \frac{c}{\lambda+1}h^{7+\lambda} + ch^7\rho^\lambda$$

$$+\frac{c}{\lambda}h^{6+\lambda}\rho\frac{\left(\frac{a}{h}\right)^\lambda}{\lambda} - \frac{c}{\lambda}\left(\frac{\rho}{h}\right)^\lambda h^{6+\lambda}\rho$$

$$\le ch^{7+\lambda} + c_1h^{7+\lambda}\left(\frac{\rho}{h}-1\right)^{\lambda+1} - c_1h^{7+\lambda} + ch^7\rho^\lambda + c_2h^6\rho a^\lambda - c_2h^6\rho^{\lambda+1}$$

$$= ch^6\rho(x,y), \qquad (x,y) \in \overline{\Pi}^h.$$

Theorem 2.1 is proved. ■

2.3 Approximation of the First Derivatives

Let u be a solution of the boundary value problem (2.1). We put $v = \frac{\partial u}{\partial x}$. It is obvious that the function v is a solution of boundary value problem

$$\Delta v = 0 \text{ on } \Pi,\ v = \psi_j \text{ on } \gamma_j, \quad j = 1,2,3,4, \tag{2.41}$$

where $\psi_j = \frac{\partial u}{\partial x}$ on γ_j, $j = 1,2,3,4$.

Let u_h be a solution of finite difference problem (2.8). We define the following operators ψ_{vh}, $v = 1,2,3,4$,

$$\psi_{1h}(u_h) = \frac{1}{60h}[-147\varphi_1(y) + 360u_h(h,y) - 450u_h(2h,y) \\ +400u_h(3h,y) - 225u_h(4h,y) + 72u_h(5h,y) \\ -10u_h(6h,y)] \text{ on } \gamma_1^h, \tag{2.42}$$

$$\psi_{3h}(u_h) = \frac{1}{60h}[147\varphi_3(y) - 360u_h(a-h,y) \\ +450u_h(a-2h,y) - 400u_h(a-3h,y) + 225u_h(a-4h,y) \\ -72u_h(a-5h,y) + 10u_h(a-6h,y)] \text{ on } \gamma_3^h, \tag{2.43}$$

$$\psi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x} \text{ on } \gamma_p^h, \quad p = 2,4. \tag{2.44}$$

Lemma 2.7 The next inequality holds

$$|\psi_{kh}(u_h) - \psi_{kh}(u)| \leq ch^6, \quad k = 1,3,$$

where u_h is the solution of problem (2.8), and u is the solution of problem (2.1).

Proof. It is obvious that $\psi_{ph}(u_h) - \psi_{ph}(u) = 0$ for $p = 2,4$.

For $k = 1$, by (2.42) and Theorem 2.1 we have

$$|\psi_{1h}(u_h) - \psi_{1h}(u)| = \left|\frac{1}{60h}[(-147\varphi_1(y) + 360u_h(h,y)\right.$$
$$-450u_h(2h,y) + 400u_h(3h,y) - 225u_h(4h,y) + 72u_h(5h,y)$$
$$-10u_h(6h,y)) - (-147\varphi_1(y) + 360u(h,y) - 450u(2h,y)$$
$$\left.+400u(3h,y) - 225u(4h,y) + 72u(5h,y)10u(6h,y))]\right|$$
$$\leq \frac{1}{60h}[|u_h(h,y) - u(h,y)| + 450|u_h(2h,y) - u(2h,y)|$$
$$+400|u_h(3h,y) - u(3h,y)| + 225|u_h(4h,y) - u(4h,y)|$$
$$+72|u_h(5h,y) - u(5h,y)| + 10|u_h(6h,y) - u(6h,y)|]$$
$$\leq \frac{1}{60h}[360(ch)h^6 + 450(2ch)h^6 + 400(3ch)h^6 + 225(4ch)h^6$$
$$+72(5ch)h^6 + 10(6ch)h^6].$$
$$\leq c_7h^6.$$

The same inequality is true for $k = 3$. ■

Lemma 2.8 The inequality holds

$$\max_{(x,y)\in\gamma_k^h}|\psi_{kh}(u_h) - \psi_k| \leq c_8h^6, \quad k = 1,3,$$

where ψ_{kh}, $k = 1,3$ are the functions defined by (2.42) and (2.43), $\psi_k = \frac{\partial u}{\partial x}$ on γ_k, $k = 1,3$.

Proof. From Lemma 2.1 follows that $u \in C^{7,0}(\overline{\Pi})$. Then at the end points $(0,\upsilon h) \in \gamma_1^h$ and $(a,\upsilon h) \in \gamma_3^h$ of each line segment $\{(x,y): 0 \leq x \leq a, 0 < y = \upsilon h < b\}$ expressions (2.42) and (2.43) give the sixth order approximation of $\frac{\partial u}{\partial x}$ respectively.

From the truncation error formulae follows that

$$\max_{(x,y)\in\gamma_k^h}|\psi_{kh}(u)-\psi_k| \le \max_{(x,y)\in\overline{\Pi}} \frac{1}{7!}\left|\frac{\partial^7 u}{\partial x^7}\right| h(2h)(3h)(4h)(5h)(6h)$$
$$\le c_9 h^6, \quad \text{k} = 1,3. \tag{2.45}$$

On the basis of Lemma 2.7 and estimation (2.45)

$$\max_{(x,y)\in\gamma_k^h}|\psi_{kh}(u_h)-\psi_k| \le \max_{(x,y)\in\gamma_k^h}|\psi_{kh}(u_h)-\psi_{kh}(u)| + \max_{(x,y)\in\gamma_k^h}|\psi_{kh}(u)-\psi_k|$$

$$\le c_7 h^6 + c_9 h^6 = c_8 h^6, \qquad k = 1,3. \blacksquare$$

Consider the finite difference problem

$$v_h = Bv_h \ \text{ on } \Pi^h, \quad v = \psi_{jh} \ \text{ on } \gamma_j^h, \ \ j = 1,2,3,4, \tag{2.46}$$

where $\psi_{jh},\ j = 1,2,3,4$ are defined by the formulas (2.42)-(2.44).
Since the boundary values ψ_{jh} for $j = 1,3$ are defined by the solution of finite-difference problem (2.46) which assumed to be known and $\psi_{jh} = \frac{\partial \varphi_j}{\partial x}$, $j = 2,4$ are calculated by the boundary functions φ_j , $j = 2,4$ the existence and uniqueness follows from the discrete maximum principle.

To estimate the convergence order of problem (2.41) we consider the problem

$$\Delta V = 0 \ \text{ on } \Pi, \quad V = \Phi_j \ \text{ on } \gamma_j, \ \ j = 1,2,3,4,$$

where Φ_j in the given function, which satisfy the following conditions

$$\Phi_j \in C^{6,\lambda}(\gamma_j), \ \ 0 < \lambda < 1, \tag{2.47}$$

$$\Phi_j^{2q}(s_j) = (-1)^q \Phi_{j-1}^{2q}(s_j), \;\; q = 0,1,2. \tag{2.48}$$

Let V_h be a solution of the finite–difference problem

$$V_h = BV_h \;\text{ on } \Pi^h, \; V_h = \Phi_{jh} \;\text{ on } \gamma_j^h, \quad j = 1,2,3,4, \tag{2.49}$$

where ψ_{jh} is the value of ψ_j on γ_j^h, $j = 1,2,3,4$. It is clear that the error function $\epsilon_h = V_h - V$ is a solution of the boundary value problem

$$\epsilon_h = B\epsilon_h + (BV_h - V) \;\text{ on } \Pi^h, \;\; \epsilon_h = 0 \;\text{ on } \gamma_j^h, \quad j = 1,2,3,4. \tag{2.50}$$

As follows from Theorem 12 in (Dosiyev, 2003) the following estimation

$$\max_{(x,y)\in\overline{\Pi}^h} |V_h - V| \le ch^6, \tag{2.51}$$

is true.

Theorem 2.2 The following estimation holds

$$\max_{(x,y)\in\overline{\Pi}} \left| v_h - \frac{\partial u}{\partial x} \right| \le c_0 h^6,$$

where u is the solution of problem (2.1), and v_h is the solution of finite difference problem (2.46).

Proof. Let

$$\epsilon_h = v_h - v \;\text{ on } \overline{\Pi}^h, \tag{2.52}$$

where $v = \frac{\partial u}{\partial x}$. From (2.46) and (2.52), we have

$$\epsilon_h = B\epsilon_h + (Bv - v) \text{ on } \Pi^h, \quad \epsilon_h = \psi_{kh}(u_h) - v \text{ on } \gamma_k^h,$$
$$k = 1,3, \quad \epsilon_h = 0 \text{ on } \gamma_p^h, \quad p = 2,4. \tag{2.53}$$

We represent

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2 \tag{2.54}$$

where

$$\epsilon_h^1 = B\epsilon_h^1 \text{ on } \Pi^h, \epsilon_h^1 = \psi_{kh}(u_h) - v \text{ on } \gamma_k^h, k = 1,3,$$
$$\epsilon_h^1 = 0 \text{ on } \gamma_p^h, \quad p = 2,4; \tag{2.55}$$
$$\epsilon_h^2 = B\epsilon_h^2 + (Bv - v) \text{ on } \Pi^h, \ \epsilon_h^2 = 0 \text{ on } \gamma_j^h, \quad j = 1,2,3,4. \tag{2.56}$$

By Lemma 2.8 and by maximum principle, for the solution of system (2.55), we have

$$\max_{(x,y)\in\Pi^h} |\epsilon_h^1| \leq \max_{q=1,3} \max_{(x,y)\in\gamma_q^h} \left|\psi_{qh}(u_h) - v\right| \leq c_1 h^6. \tag{2.57}$$

From (2.50) follows that ϵ_h^2 in (2.56) is a solution of problem (2.50) with the boundary value of v satisfy the equations (2.47), (2.48). By the estimation (2.51), we obtain

$$\max_{(x,y)\in\overline{\Pi}^h} |\epsilon_h^2| \leq c_2 h^6 \tag{2.58}$$

By virtue of (2.54), (2.57) and (2.58) the proof is completed. ■

2.4 Approximation of the Pure Second Derivatives

We denote $\omega = \frac{\partial^2 u}{\partial x^2}$. The function ω is harmonic on Π, on the basis of Lemma 2.1 ω is continuous on $\overline{\Pi}$, and it is a solution of the following Dirichlet problem

$$\Delta\omega = 0 \text{ on } \Pi, \quad \omega = \chi_j \text{ on } \gamma_j, \quad j = 1,2,3,4 \tag{2.59}$$

where

$$\chi_\tau = \frac{\partial^2 \varphi_\tau}{\partial x^2}, \qquad \tau = 2,4, \tag{2.60}$$

$$\chi_\nu = -\frac{\partial^2 \varphi_\nu}{\partial y^2}, \qquad \nu = 1,3. \tag{2.61}$$

From the continuity of the function ω on $\overline{\Pi}$, and from (2.2), (2.3) and (2.60), (2.61) it follows that

$$\chi_j \in C^{5,\lambda}(\gamma_j), \ \ 0 < \lambda < 1, \quad j = 1,2,3,4. \tag{2.62}$$

$$\chi_j^{(2q)}(s_j) = (-1)^q \chi_{j-1}^{2q}(s_j), \ q = 0,1,2\,, \qquad j = 1,2,3,4. \tag{2.63}$$

Let ω_h be a solution of the finite difference problem

$$\omega_h = B\omega_h \text{ on } \Pi^h, \ \omega_h = \chi_j \text{ on } \gamma_j^h, \ \ j = 1,2,3,4, \tag{2.64}$$

where χ_j are functions determined by (2.60) and (2.61).

Lemma 2.9 The next inequality holds true

$$\left|\frac{\partial^8 \omega(x,y)}{\partial l^8}\right| \leq c_1 \rho^{\lambda-3}(x,y), \quad (x,y) \in \Pi \tag{2.65}$$

where $c_1$is a constant independent of the direction of differentiation $\partial/\partial l$.

Proof. We choose an arbitrary point $(x_0, y_0) \in \Pi$. Let $\rho_0 = \rho(x_0, y_0)$ and $\bar{\sigma} \subset \bar{\Pi}$ be the closed circle of radius ρ_0 centered at (x_0, y_0), consider the harmonic function on

$$v(x,y) = \frac{\partial^5 \omega(x,y)}{\partial l^5} - \frac{\partial^5 \omega(x_0,y_0)}{\partial l^5}. \tag{2.66}$$

By (2.62), $\omega = \frac{\partial^2 u}{\partial x_1^2} \in C^{5,\lambda}(\bar{\Pi})$, for, $0 < \lambda < 1$. Then for the function (2.66) we have

$$\max_{(x,y)\in\bar{\sigma}_0} |v(x,y)| \leq c_0 \rho_0^{\lambda}, \tag{2.67}$$

where c_0 is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since u is harmonic in Π, by using estimation (2.67) and applying Lemma 3 from (Mikhailov, 1978) we have

$$\left|\frac{\partial^3}{\partial l^3}\left(\frac{\partial^5 \omega(x,y)}{\partial l^5} - \frac{\partial^5 \omega(x_0,y_0)}{\partial l^5}\right)\right| \leq c\,\frac{\rho_0^{\lambda}}{\rho_0^{3}},$$

or

$$\left|\frac{\partial^8 \omega(x,y)}{\partial l^8}\right| \leq c \rho_0^{\lambda-3}(x_0,y_0),$$

where c is a constant independent of the point $(x_0, y_0) \in \Pi$ or the direction of $\partial/\partial l$. Since the point $(x_0, y_0) \in \Pi$ is arbitrary, inequality (2.65) holds true. ■

Lemma 2.10 The inequality holds

$$\max_{(x,y)\in\Pi^{kh}} |B\omega - \omega| \le c\frac{h^{5+\lambda}}{k^{3-\lambda}}, \qquad k = 1,2,\dots,N(h), \tag{2.68}$$

where u is a solution of problem (2.1).

Proof. Let (x_0, y_0) be a point of $\Pi^{1h} \subset \Pi^h$, and let

$$\Pi_0 = \{(x,y): |x - x_0| < h\,, |y - y_0| < h\,\}, \tag{2.69}$$

be an elementary square, some sides of which lie on the boundary of the rectangle Π. On the vertices of Π_0, and on the mid points of its sides lie the nodes of which the function values are used to evaluate $B\omega(x_0, y_0)$. We represent a solution of problem (2.59) in some neighborhood of $(x_0, y_0) \in \Pi^{1h}$ by Taylor's formula

$$\omega(x,y) = p_7(x,y) + r_8(x,y), \tag{2.70}$$

where $p_7(x,y)$ is the seventh order Taylor's polynomial, $r_8(x,y)$ is the remainder term, and

$$Bp_7(x_0, y_0) = \omega(x_0, y_0). \tag{2.71}$$

Now, we estimate r_8 at the nodes of the operator B. We take node $(x_0 + h, y_0 + h)$ which is one of the eight nodes of B, and consider the function

$$\tilde{\omega}(s) = \omega\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \ \ -\sqrt{2}h \le s \le \sqrt{2}h \tag{2.72}$$

of one variable s . By virtue of Lemma 2.9, we have

$$\left|\frac{\partial^8\tilde{\omega}(s)}{\partial s^8}\right| \le c_2\left(\sqrt{2}h - s\right)^{\lambda-3}, \ \ 0 \le s \le \sqrt{2}h \tag{2.73}$$

we represent function (2.72) around the point $s = 0$ by Taylor's formula as

$$\tilde{\omega}(s) = \tilde{p}_7(s) + \tilde{r}_8(s), \tag{2.74}$$

where

$$\tilde{p}_7(s) = p_7\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) \tag{2.75}$$

is the seventh order Taylor's polynomial of the variable s, and

$$\tilde{r}_8(s) = r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \ \ 0 \le |s| \le \sqrt{2}h, \tag{2.76}$$

is the remainder term. On the basis of continuity of $\tilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$, it follows from (2.76) that

$$r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right) = \lim_{\epsilon\to+0} \tilde{r}_8(\sqrt{2}h - \epsilon). \tag{2.77}$$

Applying an integral representation for $\tilde{r}_8$ we have

$$\tilde{r}_8\left(\sqrt{2}h - \epsilon\right) = \frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon} \left(\sqrt{2}h - \epsilon - t\right)^7 \tilde{u}^8(t)dt\,, \qquad 0 < \epsilon \le \frac{h}{\sqrt{2}}. \tag{2.78}$$

Using estimation (2.73), we have

$$\left|\tilde{r}_8\left(\sqrt{2}h-\epsilon\right)\right| \le c_3 \int_0^{\sqrt{2}h-\epsilon} \left(\sqrt{2}h-\epsilon-t\right)^7 \left(\sqrt{2}h-t\right)^{\lambda-3} dt$$

$$\le c_4 \frac{1}{7!} \int_0^{\sqrt{2}h-\epsilon} \left(\sqrt{2}h-t\right)^{4+\lambda} dt$$

$$\le ch^{\lambda+5}, \quad 0 < \epsilon \le \frac{h}{\sqrt{2}}. \tag{2.79}$$

From (2.76)-(2.79) yields

$$|r_8(x_0+h, y_0+h| \le c_1 h^{\lambda+5},$$

where c_1 is a constant independent of the taken point (x_0, y_0) on Π^{1h}. Proceeding in a similar manner, we can find the same estimates of r_8 at the other vertices of square (2.69) and at the centers of its sides. Since the norm of B in the uniform metric is equal to unity, we have

$$|Br_8(x_0, y_0)| \le c_5 h^{\lambda+5}, \tag{2.80}$$

where c_5 is a constant independent of the taken point (x_0, y_0) on Π^{1h}. From (2.70), (2.71), (2.80) and linearity of the operator B, we obtain

$$|B\omega(x_0, y_0) - \omega(x_0, y_0)| \le ch^{\lambda+5}, \tag{2.81}$$

for any $(x_0, y_0) \in \Pi^{1h}$.

Now let $(x_0, y_0) \in \Pi^{kh}$, $2 \le k \le N(h)$ and $r_8(x, y)$ be the Lagrange remainder corresponding to this point in Taylor's formula (2.70). Then $Br_8(x_0, y_0)$ can be expressed linearly in terms of a fixed number of eighth derivatives of$u$ at some point of the open square Π_0, which is a distance of $kh/2$ away from the boundary of Π. The sum of the coefficients multiplying the eighth derivatives does not exceed ch^8, which is independent of k, $(2 \leq k \leq N(h))$. By Lemma 2.9, we have

$$|Br_8(x_0, y_0)| \leq c\,\frac{h^8}{(kh)^{3-\lambda}} = c\,\frac{h^{\lambda+5}}{k^{3-\lambda}}, \tag{2.82}$$

where c is a constant independent of $k, (2 \leq k \leq N(h))$. On the basis of (2.70), (2.71), (2.81) and (2.82) follows estimation (2.68) at any point $(x_0, y_0) \in \Pi^{kh}$, $1 \leq k \leq N(h)$. ■

Theorem 2.3 The estimation holds

$$\max_{(x_0,y_0)\in\overline{\Pi}^h} |\omega_h - \omega| \leq c_{12}\, h^{\lambda+5}, \tag{2.83}$$

where $\omega = \frac{\partial^2 u}{\partial x^2}$, u is the solution of problem (2.1), and ω_h is the solution of the finite difference problem (2.64).

Proof. Let

$$\epsilon_h = \omega_h - \omega, \tag{2.84}$$

where ω_h, and ω is a solution of problem (2.64) and (2.59) respectively. Then for ϵ_h, we have

$$\epsilon_h = B\epsilon_h + (B\omega - \omega) \text{ on } \Pi^h, \quad \epsilon_h = 0 \quad \text{on } \gamma^h. \tag{2.85}$$

We represent a solution of system (2.85) as follows

$$\epsilon_h = \sum_{k=1}^{N(h)} \epsilon_{h,}^k \quad N(h) = \left[\frac{1}{2h}\min\{a, b\}\right], \tag{2.86}$$

where ϵ_h^k is a solution of the system

$$\epsilon_h^k = B\epsilon_h^k + f_h^k \text{ on } \Pi^h, \quad \epsilon_h^k = 0 \text{ on } \gamma^h, \quad k = 1,2,\dots,N(h) \tag{2.87}$$

$$f_h^k = \begin{cases} B\omega - \omega & \text{on} \quad \Pi_h^k, \\ 0 & \text{on} \quad \Pi_h/\Pi_h^k. \end{cases} \tag{2.88}$$

By virtue of (2.87), (2.88) and Lemma 2.4 for each k , $1 \le k \le N(h)$, follows the inequality

$$\left|\epsilon_h^k(x,y)\right| \le Q_h^k(x,y) \max_{(x,y)\in\Pi^h} |(B\omega - \omega)|, \text{ on } \overline{\Pi}^h. \tag{2.89}$$

On the basis of (2.84), (2.86) and (2.89) we have

$$\begin{aligned} &\max_{(x,y)\in\Pi^h} |\epsilon_h| \le \sum_{k=1}^{N(h)} 6k \max_{(x,y)\in\Pi^h} |(B\omega - \omega)| \\ &\le \sum_{k=1}^{N(h)} 6k c_{13} \frac{h^{5+\lambda}}{k^{3-\lambda}} \le 6ch^{5+\lambda} \left[1 + \int_1^{N(h)} x^{\lambda-2}\, dx\right] \\ &\le 6c_{14}h^{5+\lambda} \left[1 + \frac{x^{\lambda-1}}{\lambda-1}\Big|_1^{\frac{a}{h}}\right] \\ &= 6c_{14}h^{5+\lambda} \left[1 + \left(\frac{a}{h}\right)^{\lambda-1} \frac{1}{\lambda-1} - \frac{1}{\lambda-1}\right] \\ &= 6c_{14}h^{5+\lambda} + 6c_1h^6a^{\lambda-1} - 6c_1h^{5+\lambda} \\ &\le c_4h^{5+\lambda}. \end{aligned}$$

Theorem 2.3 is proved. ■

CHAPTER 3
ON THE HIGH ORDER CONVERGENCE OF THE DIFFERENCE SOLUTION OF LAPLACE'S EQUATION IN A RECTANGULAR PARALLELEPIPED

In this Chapter, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. The boundary functions on the faces of a parallelepiped are supposed to have the seventh order derivatives satisfying the Hölder condition, and on the edges the second, fourth and sixth order derivatives satisfy the compatibility conditions. We present and justify difference schemes on a cubic grid for obtaining the solution of the Dirichlet problem, its first and pure second derivatives. For the approximate solution a pointwise estimation for the error of order $O(\rho h^6)$ with the weight function ρ, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point (x_1, x_2, x_3) to the boundary of the parallelepiped, is obtained. This estimation gives an additional accuracy of the finite difference solution near the boundary of the parallelepiped, which is used to approximate the first derivatives with uniform error of order $O(h^6|\ln h|)$. The approximation of the pure second derivatives are obtained with uniform accuracy $O(h^{5+\lambda})$, $0 < \lambda < 1$.

3.1 The Dirichlet Problem in a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3): 0 < x_i < a_i\,, i = 1,2,3\}$ be an open rectangular parallelepiped; Γ_j $(j = 1,2,\dots,6)$ be its faces including the edges; Γ_j for $j = 1,2,3$ (for $j = 4,5,6$) belongs to the plane $x_j = 0$ (to the plane $x_{j-3} = a_{j-3}$), let $\Gamma = \bigcup_{\mathrm{j}=1}^{6} \Gamma_{\mathrm{j}}$ be the boundary of the parallelepiped, let γ be the union of the edges of R, and let $\Gamma_{\mu\nu} = \Gamma_\mu \cup \Gamma_\nu$. We say that $f \in C^{k,\lambda}(D)$, if f has continuous $k - th$ deravatives on D satisfying Hölder condition with exponent $\lambda \in (0,1)$.

We consider the following boundary value problem

$$\Delta u = 0 \quad \text{on } R,\ \ u = \varphi_j \quad \text{on } \Gamma_j, \quad j = 1,2,\dots,6, \tag{3.1}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{7,\lambda}(\Gamma_j),\ \ 0 < \lambda < 1, \quad j = 1,2, \dots ,6, \tag{3.2}$$

$$\varphi_\mu = \varphi_\nu \ \text{ on } \gamma_{\mu\nu}, \tag{3.3}$$

$$\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \quad \text{ on } \gamma_{\mu\nu}, \tag{3.4}$$

$$\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \ \text{ on } \gamma_{\mu\nu}, \tag{3.5}$$

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_{\mu\nu}^2} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_\nu^2} = \frac{\partial^6 \varphi_\mu}{\partial t_\mu^2 \partial t_\nu^4} + \frac{\partial^6 \varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\nu^4 \partial t_{\mu\nu}^2} \ \text{ on } \gamma_{\mu\nu}. \tag{3.6}$$

Where $1 \le \mu < \nu \le 6$, $\nu - \mu \ne 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively. The boundary function as hole are continuous on the edges and satisfy second, fourth, and sixth compatibility conditions which result from Laplace equations. Indeed, (3.3) is differentiated twice with respect to t_μ. Then, it is differentiated twice with respect to t_ν. We have

$$\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} = \frac{\partial^2 \varphi_\nu}{\partial t_\mu^2}$$

$$\frac{\partial^2}{\partial t_\nu^2}\left(\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2}\right) = \frac{\partial^2}{\partial t_\nu^2}\left(\frac{\partial^2 \varphi_\nu}{\partial t_\mu^2}\right)$$

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^2\partial t_\nu^2}=\frac{\partial^4\varphi_\nu}{\partial t_\mu^2\partial t_\nu^2}. \tag{3.7}$$

(3.4) is differentiated twice with respect to t_μ

$$\frac{\partial^2}{\partial t_\mu^2}\left(\frac{\partial^2\varphi_\mu}{\partial t_\mu^2}+\frac{\partial^2\varphi_\nu}{\partial t_\nu^2}+\frac{\partial^2\varphi_\mu}{\partial t_{\mu\nu}^2}\right)=0.$$

We have

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^4}+\frac{\partial^4\varphi_\nu}{\partial t_\mu^2\partial t_\nu^2}+\frac{\partial^4\varphi_\mu}{\partial t_{\mu\nu}^2\partial t_\mu^2}=0. \tag{3.8}$$

(3.4) is differentiated twice with respect to t_ν

$$\frac{\partial^2}{\partial t_\nu^2}\left(\frac{\partial^2\varphi_\mu}{\partial t_\mu^2}+\frac{\partial^2\varphi_\nu}{\partial t_\nu^2}+\frac{\partial^2\varphi_\mu}{\partial t_{\mu\nu}^2}\right)=0.$$

We have

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^2\partial t_\nu^2}+\frac{\partial^4\varphi_\nu}{\partial t_\nu^4}+\frac{\partial^4\varphi_\mu}{\partial t_{\mu\nu}^2\partial t_\nu^2}=0. \tag{3.9}$$

From (3.7), (3.8) and (3.9) follows

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^4}+\frac{\partial^4\varphi_\mu}{\partial t_\mu^2\partial t_{\mu\nu}^2}=\frac{\partial^4\varphi_\nu}{\partial t_\nu^4}+\frac{\partial^4\varphi_\mu}{\partial t_\nu^2\partial t_{\nu\mu}^2}.$$

(3.7) is differentiated twice with respect to $t_{\mu\nu}$

$$\frac{\partial^2}{\partial t_{\mu\nu}^2}\left(\frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2}\right) = \frac{\partial^2}{\partial t_{\mu\nu}^2}\left(\frac{\partial^4 \varphi_\nu}{\partial t_\mu^2 \partial t_\nu^2}\right)$$

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^2 \partial t_\nu^2 \partial t_{\mu\nu}^2} = \frac{\partial^6 \varphi_\nu}{\partial t_\mu^2 \partial t_\nu^2 \partial t_{\mu\nu}^2}. \tag{3.10}$$

(3.5) is differentiated twice with respect to t_μ follows

$$\frac{\partial^2}{\partial t_\mu^2}\left(\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2}\right) = \frac{\partial^2}{\partial t_\mu^2}\left(\frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2}\right),$$

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_{\nu\mu}^2} = \frac{\partial^6 \varphi_\nu}{\partial t_\mu^2 \partial t_\nu^4} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^2 \partial t_\nu^2 \partial t_{\nu\mu}^2}. \tag{3.11}$$

(3.5) is differentiated twice with respect to t_ν follows

$$\frac{\partial^2}{\partial t_\nu^2}\left(\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2}\right) = \frac{\partial^2}{\partial t_\nu^2}\left(\frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2}\right)$$

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_\nu^2} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^2 \partial t_\nu^2 \partial t_{\nu\mu}^2} = \frac{\partial^6 \varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\nu^4 \partial t_{\nu\mu}^2}. \tag{3.12}$$

From (3.10), (3.11) and (3.12) follows

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_{\mu\nu}^2} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_\nu^2} = \frac{\partial^6 \varphi_\mu}{\partial t_\mu^2 \partial t_\nu^4} + \frac{\partial^6 \varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\nu^4 \partial t_{\mu\nu}^2}.$$

Lemma 3.1 The solution u of the problem (3.1) is from $C^{7,\lambda}(\bar{R})$,

The proof of Lemma 3.1 follows from Theorem 2.1 (Volkov, 1969).

Lemma 3.2 Let $\rho = (x_1, x_2, x_3)$ be the distance from the current point of the open parallelepiped R to its boundary and let $\partial/\partial l \equiv \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Then the next inequality holds

$$\left| \frac{\partial^8 u(x_1, x_2, x_3)}{\partial l^8} \right| \leq c\rho^{\lambda-1}(x_1, x_2, x_3), \qquad (x_1, x_2, x_3) \in R, \tag{3.13}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$, u is a solution of problem (3.1).

Proof. We choose an arbitrary point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}$. Let $\rho_0 = \rho(x_{10}, x_{20}, x_{30})$, and $\bar{\sigma}_0 \subset \bar{\mathrm{R}}$ be the closed ball of radius ρ_0 centred at (x_{10}, x_{20}, x_{30}). Consider the harmonic function on R

$$v(x_1, x_2, x_3) = \frac{\partial^7 u(x_1, x_2, x_3)}{\partial l^7} - \frac{\partial^7 u(x_{10}, x_{20}, x_{30})}{\partial l^7}. \tag{3.14}$$

As it follows from Theorem 2.1 in (Volkov, 1969) the solution u of problem (3.1) which satisfies the conditions (3.3)-(3.6) belongs to the class $C^{7,\lambda}(\bar{\mathrm{R}})$, for $0 < \lambda < 1$. Then for the function (3.14) we have

$$\max_{(x_1, x_2, x_3) \in \bar{\sigma}_0} |v(x_1, x_2, x_3)| \leq c_0 \rho_0^{\lambda}, \tag{3.15}$$

where c_0 is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$. By using estimation (3.15) and applying Lemma 3 from (Mikeladze, 1978) we have

$$\left| \frac{\partial}{\partial l} \left(\frac{\partial^7 u(x_1, x_2, x_3)}{\partial l^7} - \frac{\partial^7 u(x_{10}, x_{20}, x_{30})}{\partial l^7} \right) \right| \leq c \frac{\rho_0^{\lambda}}{\rho_0},$$

or

$$\left|\frac{\partial^8 u(x_1,x_2,x_3)}{\partial l^8}\right| \leq c\rho_0^{\lambda-1}(x_1,x_2,x_3),$$

where c is a constant independent of the point $(x_{10},x_{20},x_{30}) \in R$ or the direction of $\partial/\partial l$. Since the point $(x_{10},x_{20},x_{30}) \in R$ is arbitrary, inequality (3.13) holds true. ■

Let v be a solution of the problem

$$\Delta v = 0 \text{ on } R,\ \ v = \Psi_j \text{ on } \Gamma_j, \quad j = 1,2,\dots,6, \tag{3.16}$$

where Ψ_j, $j = 1,2,\dots,6$ are given functions and

$$\Psi_j \in C^{5,\lambda}(\Gamma_j), \quad 0 < \lambda < 1, \quad j = 1,2,\dots,6, \quad \Psi_\mu = \Psi_\upsilon \text{ on } \gamma_{\mu\nu}, \tag{3.17}$$

$$\frac{\partial^2\Psi_\mu}{\partial t_\mu^2} + \frac{\partial^2\Psi_\upsilon}{\partial t_\upsilon^2} + \frac{\partial^2\Psi_\mu}{\partial t_{\mu\upsilon}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{3.18}$$

$$\frac{\partial^4\Psi_\mu}{\partial t_\mu^4} + \frac{\partial^4\Psi_\mu}{\partial t_\mu^2\partial t_{\mu\upsilon}^2} = \frac{\partial^4\Psi_\upsilon}{\partial t_\upsilon^4} + \frac{\partial^4\Psi_\mu}{\partial t_\upsilon^2\partial t_{\upsilon\mu}^2} \text{ on } \gamma_{\mu\nu}. \tag{3.19}$$

Lemma 3.3 The next inequality is true

$$\left|\frac{\partial^8 v(x_1,x_2,x_3)}{\partial l^8}\right| \leq c_1\rho^{\lambda-3}(x_1,x_2,x_3), \qquad (x_1,x_2,x_3) \in R, \tag{3.20}$$

where c_1 is a constant independent of the direction of differentiation $\partial/\partial l$.

Proof. We choose an arbitrary point $(x_{10},x_{20},x_{30}) \in$ R. Let $\rho_0 = \rho(x_{10},x_{20},x_{30})$ and $\bar{\sigma} \subset \overline{\mathrm{R}}$ be the closed ball of radius ρ_0 centred at (x_{10},x_{20},x_{30}). Consider the harmonic function on R

$$v(x_1, x_2, x_3) = \frac{\partial^5 v(x_1, x_2, x_3)}{\partial l^5} - \frac{\partial^5 v(x_{10}, x_{20}, x_{30})}{\partial l^5}. \tag{3.21}$$

As it follows from Theorem 2.1 in (Volkov, 1969) the solution u of problem (3.1), which satisfies the conditions (3.17)-(3.19) belongs to the class $C^{5,\lambda}(\overline{\mathrm{R}}),\ 0 < \lambda < 1$. Then for the function (3.21) we have

$$\max_{(x_1,x_2,x_3)\in\overline{\sigma}_0} |v(x_1, x_2, x_3)| \leq c_0 \rho_0^{\lambda}, \tag{3.22}$$

where c_0 is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}$ or the direction of $\partial/\partial l$. Since u is harmonic in R, by using estimation (3.22) and applying Lemma 3 in (Mikeladze, 1978) we have

$$\left|\frac{\partial^3}{\partial l^3}\left(\frac{\partial^5 v(x_1, x_2, x_3)}{\partial l^5} - \frac{\partial^5 v(x_{10}, x_{20}, x_{30})}{\partial l^5}\right)\right| \leq c\,\frac{\rho_0^{\lambda}}{\rho_0^{3}},$$

or

$$\left|\frac{\partial^8 v(x_1, x_2, x_3)}{\partial l^8}\right| \leq c \rho_0^{\lambda-3}(x_1, x_2, x_3),$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}$ or the direction of $\partial/\partial l$. Since the point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}$ is arbitrary, inequality (3.20) holds true. ■

3.2 27-Point Finite Difference Method for the Dirichlet Problem

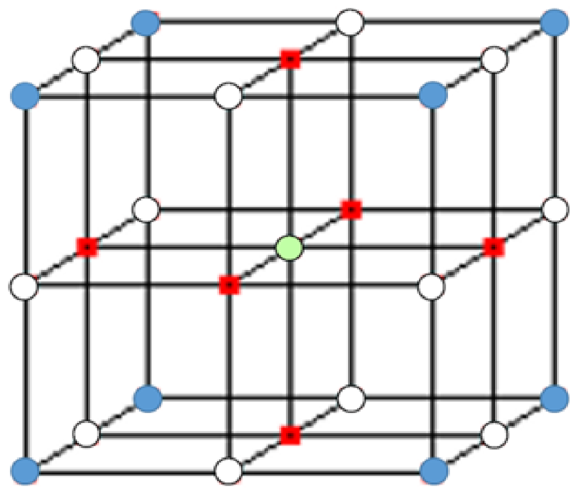
Let $h > 0$, and $a_i/h \geq 6$ where $i = 1,2,3,\ldots,6$ integers. We assign R^h a cubic grid on R, with step h, obtained by the planes $x_i = 0, h, 2h, \ldots, i = 1,2,3$. Let D^h be a set of nodes of this grid, $\mathrm{R}^h = R \cap D^h$, $\Gamma_j^h = \Gamma_j \cap D^h$, and $\Gamma^h = \Gamma_1^h \cup \Gamma_2^h \cup \ldots \cup \Gamma_6^h$.

Let the operator $\mathfrak{R}$ be defined as follows by (Mikeladze, 1938).

$$\Re u(x_1, x_2, x_3) = \frac{1}{128}\left(14 \sum_{p=1^{(1)}}^{6} u_p + 3 \sum_{q=7^{(2)}}^{18} u_q + \sum_{r=19^{(3)}}^{26} u_r\right),$$

$$(x_1, x_2, x_3) \in R, \tag{3.23}$$

where the sum $\sum_{(k)}$ is taken over the grid nodes that are at a distance of $\sqrt{k}h$ from the point (x_1, x_2, x_3), and u_p, u_q and u_r are the values of u at the corresponding grid points.



The red points have distance h from the center point, the white points have distance $\sqrt{2}h$ from the center point and the blue points have distance $\sqrt{3}h$ from center point.

Figure 3.1: 26 points around center point using operator $\Re$

Let $c, c_0, c_1, \ldots$ be constants which are independent of h and the nearest factor, and for simplicity identical notation will be used for various constants.

We consider the finite difference approximations of problem (3.1):

$$u_h = \Re u_h \text{ on } \mathrm{R}^h,\; u_h = \varphi_j \text{ on } \Gamma_j^h, \quad j = 1,2, \ldots, 6. \tag{3.24}$$

By the maximum principle system (3.24) has a unique solution.

Let R^{kh} be the set of the grid nodes R^h whose distance from Γ is kh. It is obvious that $1 \le k \le N(h)$, where

$$N(h) = \left[\frac{1}{2h} min\{a_1, a_2, a_3\}\right], \tag{3.25}$$

$[d]$ is the integer part of d.

We define for $1 \le k \le N(h)$ the function

$$f_h^k = \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \ne kh. \end{cases} \tag{3.26}$$

Consider two systems of grid equations

$$v_h = Av_h + g_h, \quad \text{on } R_h, \qquad v_h = 0 \text{ on } \Gamma_h, \tag{3.27}$$

$$\bar{v}_h = A\bar{v}_h + \bar{g}_h, \quad \text{on } R_h, \qquad \bar{v}_h = 0 \text{ on } \Gamma_h, \tag{3.28}$$

where g_h and $\bar{g}_h$ are given functions and $|g_h| \le \bar{g}_h$ on R_h.

Lemma 3.4 The solution v_h and $\bar{v}_h$ to systems (3.23) and (3.31) satisfy the inequality

$$|v_h| \le \bar{v}_h \text{ on } R_h.$$

Proof. The proof of Lemma 3.4 follows from the comparison theorem (Samarskii, 2001).

Lemma 3.5 The solution of the system

$$v_h^k = \Re v_h^k + f_h^k \text{ on } \mathrm{R}^h, \;\; v_h^k = 0 \text{ on } \Gamma^h \tag{3.29}$$

satisfies the inequality

$$v_h^k(x_1, x_2, x_3) \leq Q_h^k, \;\; 1 \leq k \leq N(h), \tag{3.30}$$

where Q_h^k is defined as follows

$$Q_h^k = Q_h^k(x_1, x_2, x_3) = \begin{cases} \frac{6\rho}{h}, & 0 \leq \rho(x_1, x_2, x_3) \leq kh\,, \\ 6k, & \rho(x_1, x_2, x_3) > kh\,. \end{cases} \tag{3.31}$$

Proof. By virtue of (3.23) and (3.31), and in consider of Fig 3.2, we have for $0 \leq \rho = kh$

$$\Re Q_h^k = \frac{1}{128}[14(2 \times 6(k-1) + 4 \times 6k) + 3(7 \times 6(k-1) + 5 \times 6k)$$

$$+6 \times 6(k-1) + 2 \times 6k] = \frac{1}{128}[504k - 168$$

$$+216k - 126 + 48k - 36] = \frac{1}{128}[768k - 330]$$

$$= 6k - \frac{330}{128},$$

which leads to

$$Q_h^k - \Re Q_h^k = \frac{330}{128} > 1 = f_h^k.$$

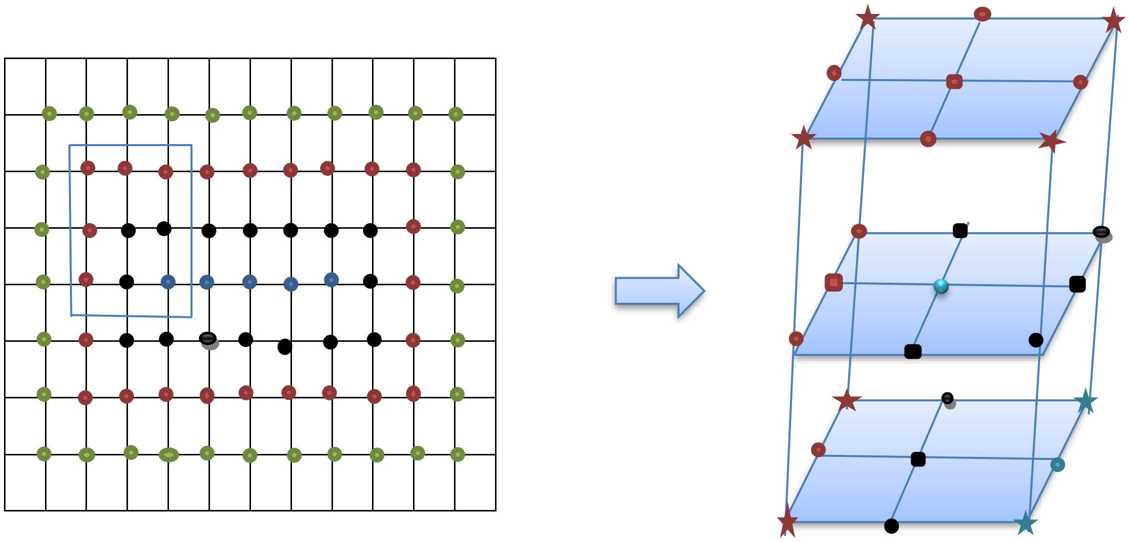


Figure 3.2: The selected region in R is $\rho = kh$

In consider of Fig 3.3 for $\rho > kh$, then

$$\Re Q_h^k = \frac{1}{128}[14(6 \times 6k) + 3(12 \times 6k) + 8 \times 6k] = \frac{768}{128}k = 6k,$$

which leads to

$$Q_h^k - \Re Q_h^k = 0.$$

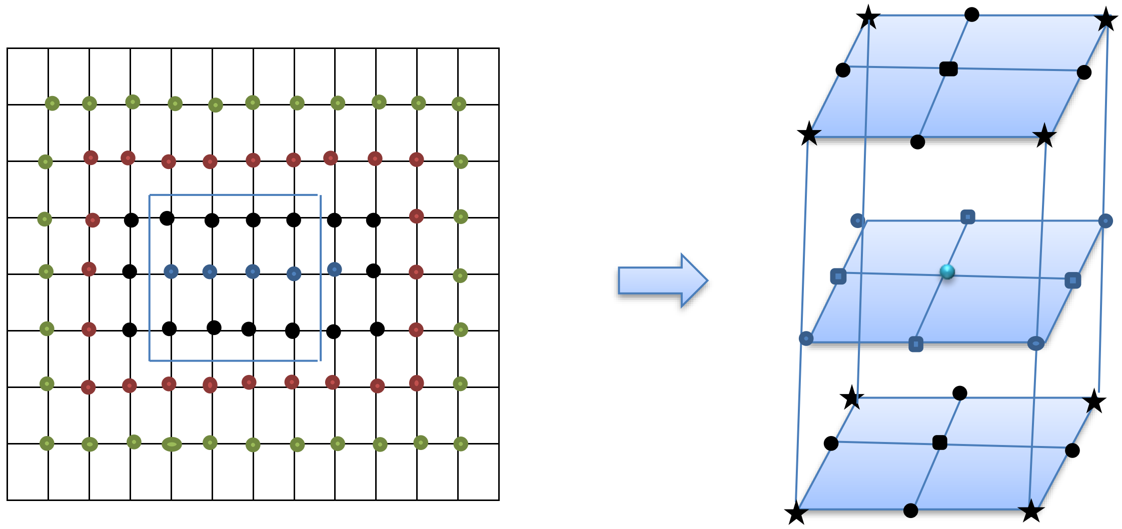


Figure 3.3: The selected region in R is $\rho > kh$

In consider of Fig 3.4 for $\rho < kh$, then

$$\Re Q_h^k = \frac{1}{128}[14(6(k-2) + 4 \times 6(k-1) + 6k) + 3(4 \times 6(k-2)$$
$$+4 \times 6(k-1) + 4 \times 6k) + 4 \times 6(k-2) + 4 \times 6k]$$
$$= 504k - 504 + 216k - 216 + 48k - 48$$
$$= \frac{768}{128}k - \frac{768}{128} = 6k - 6 = 6(k-1),$$

which leads to

$$Q_h^k - \Re Q_h^k = 0.$$

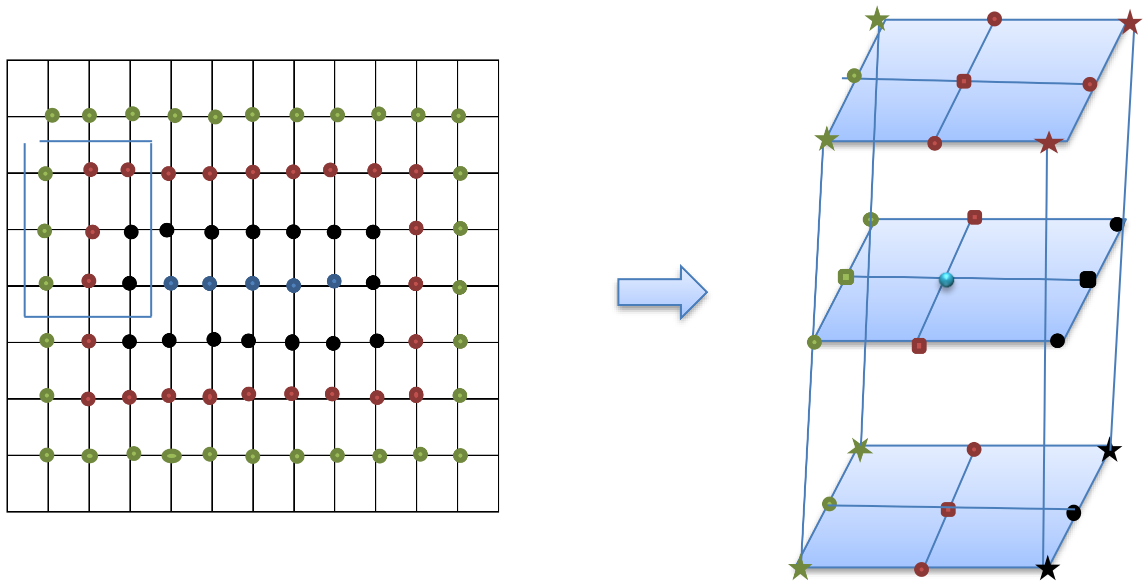


Figure 3.4: The selected region in R is $\rho < kh$

From the above calculations we have

$$Q_h^k = \Re Q_h^k + q_h^k \text{ on } \mathrm{R}^h, \; Q_h^k = 0 \text{ on } \Gamma^h, \; k = 1, \dots, N(h), \tag{3.32}$$

where $|q_h^k| \geq 1$.

On the basis of (3.26), (3.31), (3.32) and Lemma 3.4, we obtain

$$v_h^k \leq Q_h^k \text{ for all } k, \; 1 \leq k \leq N(h).$$

■

Introducing the notation $x_0 = (x_{10}, x_{20}, x_{30})$, we used Taylor's formulate to represent the solution of Dirichlet problem around some point $x_0 \in R_h$

$$u(x_1, x_2, x_3) = p_7(x_1, x_2, x_3; x_0) + r_8(x_1, x_2, x_3; x_0), \tag{3.33}$$

where p_7 is the seventh order Taylor's polynomial and $r_8(x_1, x_2, x_3; x_0)$ is the remainder term.

Here,

$$p_7(x_{10},x_{20},x_{30};x_0) = u(x_{10},x_{20},x_{30};x_0) + r_8(x_{10},x_{20},x_{30};x_0) = 0.$$

Lemma 3.6 It is true that

$$\Re u(x_{10},x_{20},x_{30}) = u(x_{10},x_{20},x_{30}) + \Re r_8(x_{10},x_{20},x_{30};x_0), (x_{10},x_{20},x_{30}) \in R_h,$$

Proof. Let $p_7(x_{10},x_{20},x_{30};x_0)$ be a Taylor's polynomial.

By Direct calculations we have

$$\begin{aligned}
\Re p_7(x_{10},x_{20},x_{30}) &= u(x_{10},x_{20},x_{30}) + \frac{1}{128}\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right.\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + \frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right)\right] + \frac{1}{1536}\left[\frac{\partial^4}{\partial x_1^4}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right.\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + \frac{\partial^4}{\partial x_2^4}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + \frac{\partial^4}{\partial x_3^4}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + 4\frac{\partial^4}{\partial x_1^2 \partial x_2^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + 4\frac{\partial^4}{\partial x_1^2 \partial x_3^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right) + 4\frac{\partial^4}{\partial x_2^2 \partial x_3^2}\left(\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_1^2}\right. \\
&+ \left.\left.\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_2^2} + \frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_3^2}\right)\right] \\
&= u(x_{10},x_{20},x_{30}).
\end{aligned}$$

Since u is harmonic function, all terms on the right-hand side of this equality vanished except the first term. Thus

$$\Re p_7(x_{10}, x_{20}, x_{30}) = u(x_{10}, x_{20}, x_{30}).$$

Combining this with (3.33) and recalling the linearity of $\Re$, Lemma 3.6. is proved. ■

Lemma 3.7 Let u be a solution of problem (3.1) The inequality holds

$$\max_{(x_1,x_2,x_3)\in R^{kh}} |\Re u - u| \leq c\frac{h^{7+\lambda}}{k^{1-\lambda}}, \quad k = 1, \dots, N(h). \tag{3.34}$$

Proof. Let (x_{01}, x_{02}, x_{03}) be a point of R^{1h}, and let

$$R_0 = \{(x_1, x_2, x_3) : |x_i - x_{i0}| < h,\ i = 1,2,3\} \tag{3.35}$$

be an elementary cube, some faces of which lie on the boundary of the rectangular parallelepiped R.

On the vertices of R_0, and on the center of its faces and edges lie the nodes of which the function values are used to evaluate $\Re(x_{10}, x_{20}, x_{30})$. We represent a solution of problem (3.1) in some neighborhood of $x_0 = (x_{10}, x_{20}, x_{30}) \in R^{1h}$ by Taylor's formula

$$u(x_1, x_2, x_3) = p_7(x_1, x_2, x_3; x_0) + r_8(x_1, x_2, x_3; x_0), \tag{3.36}$$

where $p_7(x_1, x_2, x_3; x_0)$ is the seventh order Taylor's polynomial, $r_8(x_1, x_2, x_3; x_0)$ is the remainder term. Taking into account the function u is harmonic, hence by Lemma 3.6 we have

$$\Re p_7(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30}). \tag{3.37}$$

Now, we estimate r_8 at the nodes of the operator $\Re$. We take node $(x_{10} + h, x_{20}, x_{30} + h; x_0)$ which is one of the twenty six nodes of $\Re$, and consider the function

$$\tilde{u}(s) = u\left(x_{10} + \frac{s}{\sqrt{2}}, x_{20}, x_{30} + \frac{s}{\sqrt{2}}\right),$$
$$-\sqrt{2}h \leq s \leq \sqrt{2}h \tag{3.38}$$

of a single variable s, which is the arclength along the straight line through the points $(x_{10} - h, x_{20}, x_{30} - h)$ and $(x_{10} + h, x_{20}, x_{30} + h)$. By virtue of Lemma 3.2 we have

$$\left|\frac{\partial^8 \tilde{u}(s)}{\partial s^8}\right| \leq c_2\left(\sqrt{2}h - s\right)^{\lambda - 1}, \qquad 0 \leq s \leq \sqrt{2}h \tag{3.39}$$

we represent function (3.38) around the point $s = 0$ by Taylor's formula

$$\tilde{u}(s) = \tilde{p}_7(s) + \tilde{r}_8(s), \tag{3.40}$$

where

$$\tilde{p}_7(s) = p_7\left(x_{10} + \frac{s}{\sqrt{2}}, x_{20}, x_{30} + \frac{s}{\sqrt{2}}\right) \tag{3.41}$$

is the seventh order Taylor's polynomial of the variable s, and

$$\tilde{r}_8(s) = r_8\left(x_{10} + \frac{s}{\sqrt{2}}, x_{20}, x_{30} + \frac{s}{\sqrt{2}}; x_0\right), \qquad |s| \leq \sqrt{2}h \tag{3.42}$$

is the remainder term.

On the basis of continuity of $\tilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$ and estimation (3.42), we obtain

$$r_8(x_{10} + h, x_{20}, x_{30} + h) = \lim_{\epsilon \to +0} \tilde{r}_8(\sqrt{2}h - \epsilon). \tag{3.43}$$

Applying an integral representation for $\tilde{r}_8$ we have

$$\tilde{r}_8(\sqrt{2}h-\epsilon)=\frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon}(\sqrt{2}h-\epsilon-t)^7(\sqrt{2}h-t)^{\lambda-1}dt$$

$$\leq ch^{\lambda+7},\quad 0<\epsilon\leq\frac{h}{\sqrt{2}}. \tag{3.44}$$

From (3.42)-(3.44) yields

$$|r_8(x_{10}+h,x_{20},x_{30}+h)|\leq ch^{\lambda+7}, \tag{3.45}$$

where c is a constant independent of the taken point $(x_{10},x_{20},x_{30})\in \mathrm{R}^{1h}$. Proceeding in a similar manner, we can find the same estimates of r_8 at the other sides of cube (3.35) and at the centers of its faces.

Since the norm of $\mathfrak{R}$ in the uniform metric is equal to unity, we have

$$|\mathfrak{R}r_8(x_{10}+h,x_{20},x_{30}+h)|\leq c_1h^{\lambda+7},\ \ 0<\lambda<1. \tag{3.46}$$

where c_1 is a constant independent of the taken point $(x_{10},x_{20},x_{30})\in \mathrm{R}^{1h}$. From (3.36), (3.37), (3.46) and linearity of the operator $\mathfrak{R}$, we obtain

$$|\mathfrak{R}u(x_{10},x_{20},x_{30})-u(x_{10},x_{20},x_{30})|\leq ch^{\lambda+7}, \tag{3.47}$$

for any $(x_{10},x_{20},x_{30})\in \mathrm{R}^{1h}$.

Now let (x_{10},x_{20},x_{30}), be a point of R^{kh}, $2\leq k\leq N(h)$, and $r_8(x_1,x_2,x_3)$ be the Lagrange remainder corresponding to this point in Taylor's formula (3.36). Then $r_8(x_{10},x_{20},x_{30})$ can be expressed linearly in terms of a fixed number of the twenty sixth derivatives of u at some point of the open cube R_0, which is a distance of $kh/2$ away from the boundary of R. The sum of the coefficients multiplying the twenty sixth derivatives does not exceed ch^8, which is independent of $k,(2\leq k\leq N(h))$. By Lemma 3.2, we have

$$|\Re r_8(x_{10}, x_{20}, x_{30})| \leq c\frac{h^8}{(kh)^{1-\lambda}} = c\frac{h^{\lambda+7}}{k^{1-\lambda}}, \quad (3.48)$$

where c is a constant independent of $k, (2 \leq k \leq N(h))$. On the basis of (3.36), (3.37), (3.47), and (3.48) follows estimation (3.34) at any point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{kh}$, $1 \leq k \leq N(h)$.

Theorem 3.1 Assume that the boundary function φ_j satisfy the conditions (3.2)-(3.6).Then at each point$(x_1, x_2, x_3) \in R^h$

$$|u_h - u| \leq c\rho h^6,$$

where c is a constant independent of ρ and h, $u_h$is the solution of the finite difference problem (3.24), u is the exact solution of problem (3.1), and $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R^h$ to the boundary of rectangular parallelepiped R.

Proof. Let

$$\epsilon_h(x_1, x_2, x_3) = u_h(x_1, x_2, x_3) - u(x_1, x_2, x_3), (x_1, x_2, x_3) \text{ on } \bar{R}^h. \quad (3.49)$$

By (3.24) and (3.49) the error function ϵ_h satisfies the system of equations

$$\epsilon_h = \Re\epsilon_h + (\Re u - u) \text{ on } R^h, \quad \epsilon = 0 \text{ on } \Gamma^h. \quad (3.50)$$

We represent a solution of system (3.50) as follows

$$\epsilon_h = \sum_{k=1}^{N(h)} \epsilon_h^k, \qquad N(h) = \left[\frac{1}{2h} min\{a_1, a_2, a_3\}\right], \quad (3.51)$$

where ϵ_h^k is a solution of the system

$$\epsilon_h^k = \Re\epsilon_h^k + r_h^k \text{ on } \mathrm{R}^h, \ \epsilon_h = 0 \text{ on } \Gamma^h, \quad k = 1,2, \dots, N(h); \quad (3.52)$$

$$r_h^k = \begin{cases} \Re u - u & \text{on} \quad (x_1, x_2, x_3) \in R^{kh}, \\ 0 & \text{on} \ (x_1, x_2, x_3) \in R^h / R^{kh}. \end{cases} \tag{3.53}$$

By virtue of (3.52), (3.53) for each $k,\ 1 \le k \le N(h)$, follows the inequality

$$\left|\epsilon_h^k(x_1, x_2, x_3)\right| \le Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)| \ \text{ on } \bar{R}^h. \tag{3.54}$$

On the basis of (3.49), (3.51) and (3.54), we have

$$|\epsilon_h| \le \sum_{k=1}^{N(h)} \left|\epsilon_h^k\right| \le \sum_{k=1}^{N(h)} Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)|$$

$$= \sum_{k=1}^{\frac{\rho}{h}-1} Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)|$$

$$+ \sum_{k=\frac{\rho}{h}}^{N(h)} Q_h^k\,(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)|, \quad (x_1, x_2, x_3) \in \Pi^{kh}. \tag{3.55}$$

By definition (3.31) of the function Q_h^k, we have

$$\sum_{k=1}^{\frac{\rho}{h}-1} Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)|$$

$$\le 6ch^8 \sum_{k=1}^{\frac{\rho}{h}-1} \frac{k}{(kh)^{1-\lambda}}, \tag{3.56}$$

$$\sum_{k=\frac{\rho}{h}}^{N(h)} Q_h^k(x_1,x_2,x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re u - u)|$$

$$\leq 6ch^8 \sum_{k=\frac{\rho}{h}}^{N(h)} \frac{k}{(kh)^{1-\lambda}}. \tag{3.57}$$

Then from (3.55)-(3.57), we obtain

$$\left|\epsilon_h^k(x_1,x_2,x_3)\right| \leq 6ch^{7+\lambda} \sum_{k=1}^{\frac{\rho}{h}-1} k^\lambda + 6ch^{6+\lambda}\rho \sum_{k=\frac{\rho}{h}}^{N(h)} \frac{1}{k^{1-\lambda}}$$

$$\leq 6ch^{7+\lambda}\left[1+\int_1^{\frac{\rho}{h}-1} x^\lambda\, dx\right] + 6ch^{6+\lambda}\rho\left[\left(\frac{\rho}{h}\right)^{\lambda-1} + \int_{\frac{\rho}{h}}^{N(h)} x^{\lambda-1}\, dx\right]$$

$$= ch^{7+\lambda} + c\frac{h^{7+\lambda}}{\lambda+1}\left(\frac{\rho}{h}-1\right)^{\lambda+1} - c\frac{h^{7+\lambda}}{\lambda+1} + ch^{6+\lambda}\rho\left(\frac{\rho}{h}\right)^{\lambda-1}\left(\frac{a}{2h}\right)$$

$$+ch^{6+\lambda}\frac{\rho}{h}\left(\frac{a}{h}\right)^\lambda - ch^{6+\lambda}\frac{\rho}{h}\left(\frac{\rho}{h}\right)^\lambda$$

$$= ch^{7+\lambda} + c_1h^{7+\lambda}\left(\frac{\rho}{h}-1\right)^{\lambda+1} - c_1h^{7+\lambda} + ch^7\rho^\lambda + c_2h^6\rho a^\lambda$$

$$-c_2h^6\rho^{\lambda+1} \leq ch^6\rho$$

$$= ch^{7+\lambda} + c_1h^{7+\lambda}\left(\frac{\rho}{h}-1\right)^{\lambda+1} - c_1h^{7+\lambda} + ch^7\rho^\lambda + c_2h^6\rho a^\lambda$$

$$-c_2h^6\rho^{\lambda+1} \leq ch^6\rho.$$

■

3.3 Approximation of the First Derivatives

Let u be a solution of the boundary value problem (3.1). We put $v = \frac{\partial u}{\partial x_1}$. It is obvious that the function v is a solution of boundary value problem

$$\Delta v = 0 \quad \text{on } R, v = \mathcal{F}_j \ \text{ on } \Gamma_j, \quad j = 1,2,3,4. \tag{3.58}$$

Where $\mathcal{F}_j = \frac{\partial u}{\partial x_1}$ on Γ_j, $\quad j = 1,2, \dots ,6.$

Let u_h be a solution of finite difference problem (3.24). We define the following operators $\mathcal{F}_{\nu h}$, $\nu = 1,2, \dots ,6$, as follows

$$\mathcal{F}_{1h}(u_h) = \frac{1}{60h}[-147\varphi_1(x_2, x_3) + 360u_h(h, x_2, x_3)$$
$$-450u_h(2h, x_2, x_3) + 400u_h(3h, x_2, x_3)$$
$$-225u_h(4h, x_2, x_3) + 72u_h(5h, x_2, x_3)$$
$$-10u_h(6h, x_2, x_3)] \ \text{ on } \Gamma_1^h, \tag{3.59}$$

$$\mathcal{F}_{4h}(u_h) = \frac{1}{60h}[147\varphi_3(x_2, x_3) - 360u_h(a_1 - h, x_2, x_3)$$
$$+450u_h(a_1 - 2h, x_2, x_3) - 400u_h(a_1 - 3h, x_2, x_3)$$
$$+225u_h(a_1 - 4h, x_2, x_3) - 72u_h(a_1 - 5h, x_2, x_3)$$
$$+10u_h(a_1 - 6h, x_2, x_3)] \ \text{ on } \ \Gamma_4^h, \tag{3.60}$$

$$\mathcal{F}_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x_1} \ \text{ on } \Gamma_p^h, \quad p = 2,3,5,6 \ . \tag{3.61}$$

Consider the finite difference boundary value problem

$$v_h = \Re v_h \ \text{ on } R^h, v_h = \mathcal{F}_{jh} \ \text{ on } \Gamma_j^h, \ j = 1,2, \dots ,6, \tag{3.62}$$

where $\mathcal{F}_{jh}$, $j = 1,2, \dots ,6$ are defined by (3.59)–(3.61).

Lemma 3.8 The inequality is true

$$\left|\mathcal{F}_{qh}(u_h) - \mathcal{F}_{qh}(u)\right| \leq ch^6, \quad q = 1,4,$$

where u_h is the solution of problem (3.24), and u is the solutionof problem (3.1).

Proof. It is obvious that $\mathcal{F}_{ph}(u_h) - \mathcal{F}_{ph}(u) = 0$ for $p = 2,3,5,6$.

For $q = 1$, by (3.59) and Theorem 3.1, we have

$$|\mathcal{F}_{1h}(u_h) - \mathcal{F}_{1h}(u)| \leq \frac{1}{60h}[360|u_h(h, x_2, x_3) - u(h, x_2, x_3)|$$
$$+450|u_h(2h, x_2, x_3) - u(2h, x_2, x_3)| + 400|u_h(3h, x_2, x_3)$$
$$-u(3h, x_2, x_3)| + 225|u_h(4h, x_2, x_3) - u(4h, x_2, x_3)|$$
$$+72|u_h(5h, x_2, x_3) - u(5h, x_2, x_3)| + 10|u_h(6h, x_2, x_3)$$
$$-u(6h, x_2, x_3)|] \leq \frac{1}{60h}[360(ch)h^6 + 450(2ch)h^6 + 400(3ch)h^6$$
$$+225(4ch)h^6 + 72(5ch)h^6 + 10(6ch)h^6] \leq c_8 h^6.$$

The same inequality is true for $q = 4$. ■

Lemma 3.9 The inequality holds

$$\max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u_h) - \mathcal{F}_q\right| \leq c_9 h^6, \quad q = 1,4$$

where $\mathcal{F}_{qh}$, $q = 1,4$ are defined by (3.59), (3.60), and $\mathcal{F}_q = \frac{\partial u}{\partial x_1}$ on Γ_q, $q = 1,4$.

Proof. Since $u \in C^{7,\lambda}(\bar{R})$ it follows that $u \in C^{7,0}(\bar{\mathrm{R}})$. Then at the end points $(0, \nu h, wh) \in \Gamma_1^h$ and $(a_1, \nu h, wh) \in \Gamma_4^h$ of each line segment $\{(x_1, x_2, x_3): 0 \leq x_1 \leq a_1, 0 < x_2 = \nu h \leq a_2, 0 <$

$x_3 = wh \le a_3\}$ expressions (3.59) and (3.60) give the sixth order approximation of $\frac{\partial u}{\partial x_1}$, respectively.

From the truncation error formulas (Richard; Dougla, 2011) it follows that

$$\max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u_h) - \mathcal{F}_q\right| \le c_{10}h^6, \;\; q = 1,4. \tag{3.63}$$

On the basis of Lemma 3.8 and estimation (3.63) follows,

$$\max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u_h) - \mathcal{F}_q\right|$$
$$\le \max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u_h) - \mathcal{F}_{qh}(u)\right| + \max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u) - \mathcal{F}_q\right|$$
$$\le c_{11}h^6, \;\; \mathrm{q} = 1,4.$$

■

Theorem 3.2 The estimation is true

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h}\left|v_h - \frac{\partial u}{\partial x_1}\right| \le ch^6(1 + |\ln h|), \tag{3.64}$$

where v_h is the solution of finite difference problem (3.62), and u is the solution of problem (3.1).

Proof. Let

$$\epsilon_h = v_h - v \;\; \text{on } \overline{\mathrm{R}}^h, \tag{3.65}$$

where $v = \frac{\partial u}{\partial x_1}$.

From (3.62) and (3.65), we have

$$\epsilon_h = \Re\epsilon_h + (\Re v - v) \text{ on } \mathrm{R}^h,$$

$$\epsilon_h = \mathcal{F}_{kh}(u_h) - v \text{ on } \Gamma_k^h,\, k = 1{,}4,\ \epsilon_h = 0 \text{ on } \Gamma_p^h,\ p = 2{,}3{,}5{,}6.$$

We represent

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2, \tag{3.66}$$

where

$$\epsilon_h^1 = \Re\epsilon_h^1 \text{ on } \mathrm{R}^h, \tag{3.67}$$

$$\epsilon_h^1 = \mathcal{F}_{qh}(u_h) - v \text{ on } \Gamma_q^h,\, q = 1{,}4,\, \epsilon_h^1 = 0 \text{ on } \Gamma_p^h,\ p = 2{,}3{,}5{,}6, \tag{3.68}$$

$$\epsilon_h^2 = \Re\epsilon_h^2 + (\Re v - v) \text{ on } \mathrm{R}^h,\ \epsilon_h^2 = 0 \text{ on } \Gamma_j^h,\ j = 1{,}2{,}\ldots, \tag{3.69}$$

By Lemma 3.9 and by the maximum principle, for the solution of system (3.67), (3.68), we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h}|\epsilon_h^1| \le \max_{q=1,4}\ \max_{(x_1,x_2,x_3)\in\Gamma_q^h}\left|\mathcal{F}_{qh}(u_h) - v\right| \le ch^6 \tag{3.70}$$

The solution ϵ_h^2 of system (3.69) is the error of the approximate solution obtained by the finite difference method for problem (3.58), when on the boundary nodes Γ_{jh} approximate values are defined as the exact values of the functions $\mathcal{F}_j$ in (3.58). It is obvious that $\mathcal{F}_j$, $j = 1,2,\ldots,6$, satisfy the following conditions

$$\mathcal{F}_j \in C^{6,\lambda}(\Gamma_j),\ \ 0 < \lambda < 1,\ \ j = 1,2,\ldots,6, \tag{3.71}$$

$$\mathcal{F}_\mu = \mathcal{F}_v \text{ on } \gamma_{\mu v}, \tag{3.72}$$

$$\frac{\partial^2 \mathcal{F}_\mu}{\partial t_\mu^2} + \frac{\partial^2 \mathcal{F}_\upsilon}{\partial t_\upsilon^2} + \frac{\partial^2 \mathcal{F}_\mu}{\partial t_{\mu\upsilon}^2} = 0 \text{ on } \gamma_{\mu\upsilon}, \tag{3.73}$$

$$\frac{\partial^4 \mathcal{F}_\mu}{\partial t_\mu^4} + \frac{\partial^4 \mathcal{F}_\mu}{\partial t_\mu^2 \partial t_{\mu\upsilon}^2} = \frac{\partial^4 \mathcal{F}_\upsilon}{\partial t_\upsilon^4} + \frac{\partial^4 \mathcal{F}_\mu}{\partial t_\upsilon^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\upsilon.} \tag{3.74}$$

Since the function $v = \frac{\partial u}{\partial x_1}$, is harmonic on R with the boundary values of $\mathcal{F}_j$, $j = 1,2,\ldots,6$, on the basis of (3.71)-(3.74) and by lemma 2.6 in (Dosiyev and Sadeghi, 2016), we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} |\epsilon_h^2| \leq c_4 h^6 (1 + |\ln h|). \tag{3.75}$$

By virtue of (3.66), (3.70) and (3.75) we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} |\epsilon_h| \leq c h^6 (1 + |\ln h|).$$

Hence, (3.64) follows. ■

3.4 Approximation of the Pure Second Derivatives

Let u be a solution of the value problem (3.1), we put $\omega = \frac{\partial^2 u}{\partial x_1^2}$. It is obvious that the function ω is a solution of boundary value problem

$$\Delta\omega = 0 \text{ on } R, \omega = \Psi_j \text{ on } \Gamma_j, \quad j = 1,2,\ldots,6, \tag{3.76}$$

where

$$\Psi_p = \frac{\partial^2 \varphi_p}{\partial x_1^2},\ p = 2,3,5,6;\ \Psi_q = -\frac{\partial^2 \varphi_q}{\partial x_2^2} - \frac{\partial^2 \varphi_q}{\partial x_3^2},\ \ q = 1,4. \tag{3.77}$$

We consider the finite difference problem

$$\omega_h = \Re\omega_h \ \text{ on } R^h,\ \ \omega_h = \Psi_j \ \text{ on } \Gamma_j^h, \quad j = 1,2,\ldots,6, \tag{3.78}$$

where the boundary functions Ψ_j are functions defined by (3.76).

Lemma 3.10 Let u be a solution of problem (3.1) and ω be solution of problem (3.76). The inequality is true

$$\max_{(x_1,x_2,x_3)\in R^{kh}} |\Re\omega - \omega| \leq c\frac{h^{5+\lambda}}{k^{3-\lambda}}, \quad k = 1,2,\ldots,N(h). \tag{3.79}$$

Proof. Let (x_{01}, x_{02}, x_{03}) be a point of R^{1h}, and let

$$R_0 = \{(x_1,x_2,x_3): |x_i - x_{i0}| < h,\ i = 1,2,3\}, \tag{3.80}$$

be an elementary cube, some faces of which lie on the boundary of the rectangular parallelepiped R.

On the vertices of R_0, and on the center of its faces and edges lie the nodes of which the function values are used to evaluate $\Re\omega(x_{10}, x_{20}, x_{30})$. We represent a solution of problem (3.76) in some neighborhood of $x_0 = (x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{1h}$ by Taylor's formula

$$(x_1,x_2,x_3) = p_7(x_1,x_2,x_3;x_0) + r_8(x_1,x_2,x_3;x_0), \tag{3.81}$$

where $p_7(x_1,x_2,x_3;x_0)$ is the seventh order Taylor's polynomial, $r_8(x_1,x_2,x_3;x_0)$ is the remainder term. Taking into account the function ω is harmonic, hence by Lemma 3.6 we have

$$\Re p_7(x_{10},x_{20},x_{30};x_0) = \omega(x_{10},x_{20},x_{30}). \tag{3.82}$$

Now, we estimate r_8 at the nodes of the operator $\Re$. We take node $(x_{10}+h, x_{20}, x_{30}+h; x_0)$ which is one of the twenty six nodes of $\Re$, and consider the function

$$\tilde{\omega}(s) = \omega\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}\right),$$
$$-\sqrt{2}h \leq s \leq \sqrt{2}h \tag{3.83}$$

of a single variable s, which is the arclength along the straight line through the points $(x_{10}-h, x_{20}, x_{30}-h)$ and $(x_{10}+h, x_{20}, x_{30}+h)$. By virtue of Lemma 3.3 we have

$$\left|\frac{\partial^8 \tilde{\omega}(s)}{\partial s^8}\right| \leq c_2(\sqrt{2}h - s)^{\lambda-3}, \qquad 0 \leq s \leq \sqrt{2}h \tag{3.84}$$

we represent function (3.83) around the point $s=0$ by Taylor's formula

$$\tilde{u}(s) = \tilde{p}_7(s) + \tilde{r}_8(s), \tag{3.85}$$

where

$$\tilde{p}_7(s) = p_7\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}\right) \tag{3.86}$$

is the seventh order Taylor's polynomial of the variable s, and

$$\tilde{r}_8(s) = r_8\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}; x_0\right), \quad |s| \leq \sqrt{2}h \tag{3.87}$$

is the remainder term.

On the basis of continuity of $\tilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$ and estimation (3.87), we obtain

$$r_8(x_{10}+h, x_{20}, x_{30}+h) = \lim_{\epsilon\to+0} \tilde{r}_8(\sqrt{2}h-\epsilon). \quad (3.88)$$

Applying an integral representation for $\tilde{r}_8$ we have

$$\tilde{r}_8(\sqrt{2}h-\epsilon) = \frac{1}{7!}\int_0^{\sqrt{2}h-\epsilon} (\sqrt{2}h-\epsilon-t)^7 (\sqrt{2}h-t)^{\lambda-3} dt$$

$$\le ch^{\lambda+5}, \quad 0<\epsilon\le\frac{h}{\sqrt{2}}. \quad (3.89)$$

From (3.87)-(3.89) yields

$$|r_8(x_{10}+h, x_{20}, x_{30}+h)| \le ch^{\lambda+5}, \quad (3.90)$$

where c is a constant independent of the taken point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{1h}$. Proceeding in a similar manner, we can find the same estimates of r_8 at the other sides of cube (3.80) and at the centers of its faces.

Since the norm of $\mathfrak{R}$ in the uniform metric is equal to unity, we have

$$|\mathfrak{R}r_8(x_{10}+h, x_{20}, x_{30}+h)| \le c_1 h^{\lambda+5}, \quad 0<\lambda<1, \quad (3.91)$$

where c_1 is a constant independent of the taken point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{1h}$. From (3.81), (3.82), (3.91) and linearity of the operator $\mathfrak{R}$, we obtain

$$|\mathfrak{R}\omega(x_{10}, x_{20}, x_{30}) - \omega(x_{10}, x_{20}, x_{30})| \le ch^{\lambda+5}, \quad (3.92)$$

for any $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{1h}$.

Now let (x_{10}, x_{20}, x_{30}), be a point of R^{kh}, $2 \le k \le N(h)$, and $r_8(x_1, x_2, x_3)$ be the Lagrange remainder corresponding to this point in Taylor's formula (3.81). Then $r_8(x_{10}, x_{20}, x_{30})$ can be expressed linearly in terms of a fixed number of the twenty sixth derivatives of u at some point of the open cube R_0, which is a distance of $kh/2$ away from the boundary of R. The sum of the coefficients multiplying the twenty sixth derivatives does not exceed ch^8, which is independent of $k, (2 \leq k \leq N(h))$. By Lemma 3.3, we have

$$|\Re r_8(x_{10}, x_{20}, x_{30})| \leq c\frac{h^8}{(kh)^{3-\lambda}} = c\frac{h^{\lambda+5}}{k^{3-\lambda}}, \tag{3.93}$$

where c is a constant independent of $k,\ (2 \leq k \leq N(h))$. On the basis of (3.81), (3.82), and (3.93) follows estimation (3.79) at any point $(x_{10}, x_{20}, x_{30}) \in \mathrm{R}^{kh}$, $1 \leq k \leq N(h)$.

Theorem 3.3 The estimation holds

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h}\left|\omega_h - \frac{\partial^2 u}{\partial x_1^2}\right| \leq c_2 h^{5+\lambda}, \quad 0 < \lambda < 1, \tag{3.94}$$

where ω_h is a solution of finite difference problem (3.78), u is a solution of problem (3.1).

Proof. Let

$$\epsilon_h = \omega_h - \omega \ \text{ on } \bar{R}. \tag{3.95}$$

By (3.78) and (3.95) the error function ϵ_h satisfies the system of equations

$$\epsilon_h = \Re\epsilon_h + (\Re\omega - \omega) \ \text{ on } \mathrm{R}^h,\ \epsilon_h = 0 \ \text{ on } \Gamma^h. \tag{3.96}$$

We represent a solution of system (3.96) as follows

$$\epsilon_h = \sum_{k=1}^{N(h)} \epsilon_h^k, \tag{3.97}$$

where $\epsilon_h^k,\ 1 \leq k \leq N(h),\ \ N(h)$ defined by (3.25) is a solution of the system

$$\epsilon_h^k = \Re\epsilon_h^k + r_h^k \text{ on } \mathrm{R}^h,\ \epsilon_h = 0 \text{ on } \Gamma^h \tag{3.98}$$

and

$$r_h^k = \begin{cases} \Re\omega - \omega & \text{on} \quad (x_1, x_2, x_3) \in R^{kh}, \\ 0 & \text{on} \quad (x_1, x_2, x_3) \in R^h / R^{kh}. \end{cases} \tag{3.99}$$

By virtue of (3.98), (3.99) and Lemma 3.5 for each $k,\ 1 \le k \le N(h)$, follows the inequality

$$\left|\epsilon_h^k(x_1, x_2, x_3)\right| \le Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re\omega - \omega)| \text{ on } \bar{R}^h. \tag{3.100}$$

On the basis of (3.95), (3.97), (3.100), and Lemma 3.10, we obtain

$$\begin{aligned}
&\max_{(x_1,x_2,x_3)\in \mathrm{R}^h} |\epsilon_h| \le \sum_{k=1}^{N(h)} \left|\epsilon_h^k\right| \\
&\le \sum_{k=1}^{N(h)} Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in \mathrm{R}^{kh}} |(\Re\omega - \omega)|\,, \\
&\le \sum_{k=1}^{N(h)} 6kc_3 \frac{h^{5+\lambda}}{k^{3-\lambda}} \le 6ch^{5+\lambda} \sum_{k=1}^{N(h)} k^{\lambda-2} \le 6ch^{5+\lambda} \left[1 + \int_1^{N(h)} x^{\lambda-2}\, dx\right] \\
&= 6ch^{5+\lambda} \left[1 + \frac{x^{\lambda-1}}{\lambda-1}\bigg|_1^{\frac{a}{h}}\right] = 6ch^{5+\lambda} \left[1 + \left(\frac{a}{h}\right)^{\lambda-1} \frac{1}{\lambda-1} - \frac{1}{\lambda-1}\right] \\
&= 6ch^{5+\lambda} + 6c_1 h^{5+\lambda-\lambda+1} a^{\lambda-1} - 6c_1 h^{5+\lambda} = c_2 h^{5+\lambda} + c_3 h^6 a^{\lambda-1} - c_3 h^{5+\lambda} \\
&\le c_2 h^{5+\lambda}.
\end{aligned}$$

■

CHAPTER 4

A THREE STAGE DIFFERENCE METHOD FOR SOLVING DIRICHLET PROBLEM FOR LAPLACE'S EQUATION

In this Chapter, we use a three-stage difference method to solve the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. Under the assumptions that the boundary functions on the faces have the eighth derivatives satisfying the Hölder condition, and on the edges the second, fourth, and sixth order derivatives satisfy the compatibility conditions. By using in the first stage the 14-point averaging operator and in the second and third stages 6-point averaging operator, we get $O(h^6)$ order of accurate approximation of the solution.

4.1 The Dirichlet Problem on Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3): 0 < x_i < a_i\ ,\ i = 1,2,3\}$ be an open rectangular parallelepiped; Γ_j $(j = 1,2,\ldots,6)$ be its faces including the edges such that Γ_j for $j = 1,2,3$ (for $j = 4,5,6$) belongs to the plane $x_j = 0$ (to the plane $x_{j-3} = a_{j-3}$), let $\Gamma = \bigcup_{\mathrm{j}=1}^{6} \Gamma_{\mathrm{j}}$ be the boundary of the parallelepiped, let γ be the union of the edges of R, and let $\gamma_{\mu\nu} = \Gamma_\mu \cup \Gamma_\nu$. We say that $f \in C^{k,\lambda}(D)$, if f has continuous $k - th$ deravatives on D satisfying Hölder condition with exponent $\lambda \in (0,1)$.

We consider the following boundary value problem

$$\Delta u = 0 \text{ on } R,\ u = \varphi_j \text{ on.}\Gamma_j,\quad j = 1,2,\ldots,6, \qquad (4.1)$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{8,\lambda}(\Gamma_j),\ 0 < \lambda < 1,\quad j = 1,2,\ldots,6, \qquad (4.2)$$

$$\varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu} \tag{4.3}$$

$$\frac{\partial^2\varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2\varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2\varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu} \tag{4.4}$$

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4\varphi_\mu}{\partial t_\mu^2\partial t_{\mu\nu}^2} = \frac{\partial^4\varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4\varphi_\mu}{\partial t_\nu^2\partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu} \tag{4.5}$$

$$\frac{\partial^6\varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6\varphi_\mu}{\partial t_\mu^4\partial t_{\mu\nu}^2} + \frac{\partial^6\varphi_\mu}{\partial t_\mu^4\partial t_\nu^2} = \frac{\partial^6\varphi_\mu}{\partial t_\mu^4\partial t_\nu^2} + \frac{\partial^6\varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6\varphi_\mu}{\partial t_\nu^4\partial t_{\mu\nu}^2} \text{ on } \gamma_{\mu\nu} \tag{4.6}$$

where $1 \le \mu < \nu \le 6$, $\nu - \mu \neq 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

Let $\sigma(j) = 3\left\{\frac{j}{3}\right\} + 1$, where $\{a\}$ is the fractional part of a.

If $j = 1$. Then $\sigma(1) = 3\left\{\frac{1}{3}\right\} + 1 = 2$,

if $j = 2$. Then $\sigma(2) = 3\left\{\frac{2}{3}\right\} + 1 = 3$,

if $j = 3$. Then $\sigma(3) = 3\left\{\frac{3}{3}\right\} + 1 = 3\left\{\frac{0}{3}\right\} + 1 = 1$,

if $j = 4$. Then $\sigma(4) = 3\left\{\frac{4}{3}\right\} + 1 = 3\left\{\frac{1}{3}\right\} + 1 = 2$,

if $j = 5$. Then $\sigma(5) = 3\left\{\frac{5}{3}\right\} + 1 = 3\left\{\frac{2}{3}\right\} + 1 = 3$,

if $j = 6$. Then $\sigma(6) = 3\left\{\frac{6}{3}\right\} + 1 = 3\left\{\frac{0}{3}\right\} + 1 = 1$.

Lemma 4.1 In the open parallelepiped R, it holds that

$$\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_j^4} = \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j+1)}^4} + 2\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^2}, \qquad (4.7)$$

where u is the solution to Dirichlet problem (4.1).

Proof. The proof from the Laplace equation as follows:

In case $j = 1$

$$\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_1^4} = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}\right)$$

$$= \frac{\partial^2}{\partial x_1^2}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} - \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$

$$= -\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_1^2 \partial x_3^2}$$

$$= -\frac{\partial^2}{\partial x_2^2}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} - \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$

$$-\frac{\partial^2}{\partial x_3^2}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} - \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$

$$= \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_3^2 \partial x_2^2} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_3^4}$$

$$= \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_3^4} + 2\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_3^2}$$

Similarly, for $j = 2,3$. This completes proof (4.7). ■

On $\bar{R}$, we define the function

$$v = v(x_1, x_2, x_3) = \frac{1}{2}\sum_{j=1}^{3} \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_j^4}, \tag{4.8}$$

where u is the solution to Dirichlet problem (4.1).

Lemma 4.2 The function (4.8) coincides with the unique continuous on $\bar{R}$ solution to the boundary value problem

$$\Delta v = 0 \text{ on } R, \; v = \psi_j \text{ on } \Gamma_j, \quad j = 1,2, \dots, 6, \tag{4.9}$$

where

$$\psi_j = \psi_j(x_{\sigma(j)}, x_{\sigma(j+1)}) = \frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^4} + \frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j+1)}^4}$$
$$+ \frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^2}. \tag{4.10}$$

Proof. On the basis of (4.2)-(4.6), Theorem 2.1 by (Volkov,1969) and Theorem 3.1 by (Volkov, 1965) it follows that a solution u of problem (4.1) belongs to the class $C^{7,\lambda}(\bar{R}),\; 0 < \lambda < 1$. Let us show that v is a solution of problem (4.9).

$$\Delta v = \frac{1}{2}\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_1^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_2^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_3^4}\right)\right.$$
$$+ \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_1^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_2^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_3^4}\right)$$
$$\left.+ \frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_1^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_2^4} + \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_3^4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^6} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^2 \partial x_2^4} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^2 \partial x_3^4} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_1^4} \right.$$

$$+ \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_3^4} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^4 \partial x_3^2} + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^4 \partial x_3^2}$$

$$\left. + \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^4}{\partial x_1^4}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)\right.$$

$$+ \frac{\partial^4}{\partial x_2^4}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$

$$\left. + \frac{\partial^4}{\partial x_3^4}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2} + \frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)\right]$$

$$= 0.$$

We show that $v = \psi_j$ on Γ_j, $j = 1,2,\ldots,6$. Since

$$\frac{1}{2}\sum_{j=1}^{3}\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_j^4} = \frac{1}{2}\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_k^4} + \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{3}\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_j^4}$$

$$= \frac{1}{2}\left(\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k+1)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^2 \partial x_{\sigma(k+1)}^2}\right)$$

$$+ \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{3}\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_j^4}$$

$$= \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k+1)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^2 \partial x_{\sigma(k+1)}^2},$$

we have

$$v=\frac{\partial^4\varphi_j\left(x_{\sigma(j)},x_{\sigma(j+1)}\right)}{\partial x_{\sigma(j)}^4}+\frac{\partial^4\varphi_j\left(x_{\sigma(j)},x_{\sigma(j+1)}\right)}{\partial x_{\sigma(j+1)}^4}+\frac{\partial^4\varphi_j\left(x_{\sigma(j)},x_{\sigma(j+1)}\right)}{\partial x_{\sigma(j)}^2\partial x_{\sigma(j+1)}^2}$$

on Γ_j, $j=1,2,\ldots,6.$■

Lemma 4.3 In the open parallelepiped R, it holds that

$$\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6}=-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^6}-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j+1)}^6}$$
$$-3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^4\partial x_{\sigma(j+1)}^2}-3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^2\partial x_{\sigma(j+1)}^4}, \qquad (4.11)$$

where u is the solution to Dirichlet problem (4.1).

Proof. The proof from the Laplace equation as follows:

In case $j=1$

$$\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^6}=\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_1^4}\right)$$
$$=\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^4}+\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_3^4}+2\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_2^2\partial x_3^2}\right)$$
$$=\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^2\partial x_2^4}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^2\partial x_3^4}+2\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^2\partial x_2^2\partial x_3^2}$$
$$=\frac{\partial^4}{\partial x_2^4}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$
$$+\frac{\partial^4}{\partial x_3^4}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$
$$+2\frac{\partial^6}{\partial x_2^2\partial x_3^2}\left(-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}-\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)$$
$$=-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6}-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^4\partial x_3^2}-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^2\partial x_3^4}-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6}$$

$$-2\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^4 \partial x_3^2} - 2\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_3^4}$$
$$= -\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6} - \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6} - 3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^4 \partial x_3^2} - 3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^2 \partial x_3^4}.$$

Similarly, for $j = 2,3$. This completes proof (4.11). ■

On $\bar{R}$, we define the function

$$\omega = \omega(x_1,x_2,x_3) = \frac{1}{3}\sum_{j=1}^{3} \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6}, \quad (4.12)$$

where u is the solution to Dirichlet problem (4.1).

Lemma 4.4 The function (4.12) coincides with the unique continuous on $\bar{R}$ solution to the boundary value problem

$$\Delta\omega = 0 \ \text{ on } R, \quad \omega = \kappa_j \ \text{ on } \Gamma_j, \quad j = 1,2,\ldots,6, \quad (4.13)$$

where

$$\kappa_j = \kappa_j\left(x_{\sigma(j)}, x_{\sigma(j+1)}\right)$$
$$= -\frac{\partial^6 \varphi_j\left(x_{\sigma(j)}, x_{\sigma(j+1)}\right)}{\partial x_{\sigma(j)}^4 \partial x_{\sigma(j+1)}^2} - \frac{\partial^6 \varphi_j\left(x_{\sigma(j)}, x_{\sigma(j+1)}\right)}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^4} \quad (4.14)$$

Proof. On the basis of (4.2)-(4.6), Theorem 2.1 by (Volkov, 1969) and Theorem 3.1 by (Volkov, 1965) it follows that a solution u of problem (4.1) belongs to the class $C^{7,\lambda}(\bar{R})$, $0 < \lambda < 1$. Let us show that ω is a solution of problem (4.13).

$$
\begin{aligned}
\Delta\omega &= \frac{1}{3}\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6}\right)\right.\\
&+\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6}\right)\\
&\left.+\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_1^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_2^6}+\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_3^6}\right)\right]\\
&= \frac{1}{3}\left[\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_1^8}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_1^2\partial x_2^6}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_1^2\partial x_3^6}\right.\\
&+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_1^6\partial x_2^2}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_2^8}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_2^2\partial x_3^6}\\
&\left.+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_1^6\partial x_3^2}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_2^6\partial x_3^2}+\frac{\partial^8 u(x_1,x_2,x_3)}{\partial x_3^8}\right]\\
&= \frac{1}{3}\left[\frac{\partial^6}{\partial x_1^6}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)\right.\\
&+\frac{\partial^6}{\partial x_2^6}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)\\
&\left.+\frac{\partial^6}{\partial x_3^6}\left(\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_1^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_2^2}+\frac{\partial^2 u(x_1,x_2,x_3)}{\partial x_3^2}\right)\right]\\
&= 0.
\end{aligned}
$$

We show that $\omega = \kappa_j$ on Γ_j, $j = 1,2,\dots,6$. Since

$$
\begin{aligned}
\frac{1}{3}\sum_{j=1}^{3}\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6} &= \frac{1}{3}\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_k^6}+\frac{1}{3}\sum_{\substack{j=1\\ j\neq k}}^{3}\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6}\\
&= \frac{1}{3}\left(-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^6}-\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(k+1)}^6}-3\,\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^4\partial x_{\sigma(k+1)}^2}\right.
\end{aligned}
$$

$$-3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(k)}^2 \partial x_{\sigma(k+1)}^4}\Bigg) + \frac{1}{3}\sum_{\substack{j=1\\ j\neq k}}^{3} \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6}$$

we have

$$\omega = -\frac{\partial^6 \varphi_j\big(x_{\sigma(j)},x_{\sigma(j+1)}\big)}{\partial x_{\sigma(j)}^4 \partial x_{\sigma(j+1)}^2} - \frac{\partial^6 \varphi_j\big(x_{\sigma(j)},x_{\sigma(j+1)}\big)}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^4} \text{ on } \Gamma_j,\ \ j = 1,2,\dots,6. \ \blacksquare$$

Lemma 4.5 Even order derivatives for each variable in form

$$\frac{\partial^8 u}{\partial x^{2p}\partial x^{2q}\partial x^{8-2p-2q}}, \qquad 0 \le p \le 4, \qquad 0 \le q \le 4-p\,, \tag{4.15}$$

are continuous and bounded on $\bar{R}\backslash\gamma$.

Proof. Let $\omega = \frac{\partial^6 u}{\partial x_1^6}$. We have

$$\Delta\omega = 0 \text{ on } R, \omega = \Phi_j \text{ on } \Gamma_j, \ \ j = 1,2,\dots,6, \tag{4.16}$$

where

$$\Phi_j = \frac{\partial^6 \varphi_j}{\partial x_1^6},\ \ j = 2,3,5,6, \tag{4.17}$$

$$\Phi_j = -\frac{\partial^6 \varphi_j}{\partial x_1^6} - 3\frac{\partial^6 \varphi_j}{\partial x_2^4 \partial x_3^2} - 3\frac{\partial^6 \varphi_j}{\partial x_2^2 \partial x_3^4} - \frac{\partial^6 \varphi_j}{\partial x_3^6},\ \ j = 1,4. \tag{4.18}$$

From (4.1)-(4.6) follows that the boundary functions Φ_j, $j = 1,2,\dots,6$ defined by (4.17) and (4.18) satisfy the conditions

$$\Phi_j \in C^{2,\lambda}\big(\Gamma_j\big),\ \ \Phi_\mu = \Phi_\nu \text{ on } \gamma_{\mu\nu}.$$

Then, on the basis of Theorem 3.1 by (Volkov, 1969) the pure second order derivatives of the function ω are bounded in $\bar{R}\backslash\gamma$. Then

$$\sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^8 u}{\partial x_1^8}\right| = \sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^2\omega}{\partial x_1^2}\right| < \infty$$

$$\sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^8 u}{\partial x_1^6\partial x_2^2}\right| = \sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^2\omega}{\partial x_2^2}\right| < \infty$$

$$\sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^8 u}{\partial x_1^6\partial x_3^2}\right| = \sup_{(x_1,x_2,x_3)\in\bar{R}\backslash\gamma}\left|\frac{\partial^2\omega}{\partial x_3^2}\right| < \infty$$

Similarly, by taking $\omega = \frac{\partial^6 u}{\partial x_2^6}$, and $\omega = \frac{\partial^6 u}{\partial x_3^6}$ the boundedness of the remainder even order derivatives in (4.15) are proved. ■

4.2 A Sixth Order Accurate Approximate Solution

Consider a cubic mesh with the mesh size $h > 0$ formed by the planes $x_i = 0, h, 2h, \ldots (i = 1,2,3)$. Assume that $a_i/h \geq 6$ $(i = 1,2,3)$ are integers. Let D_h be the set of mesh nodes, $R_h = R \cap D_h$, $\Gamma_h = \Gamma \cap D_h$, $\Gamma'_{jh} = \Gamma'_j \cap D_h$, and $\Gamma'_h = \Gamma'_{1h} \cup \ldots \cup \Gamma'_{6h}$.

For the grid functions on R_h, we consider the 6-point difference operator A as

$$Au(x_1,x_2,x_3) = \frac{1}{6}\sum_{p=1^{(1)}}^{6} u_p, \tag{4.19}$$

and the 14-point difference operator S as

$$Su(x_1, x_2, x_3) = \frac{1}{56}\left(8\sum_{p=1^{(1)}}^{6} u_p + \sum_{q=7^{(3)}}^{14} u_q\right), \tag{4.20}$$

where the sum $\sum_{(k)}$ is taken over the grid nodes that are at a distance of $\sqrt{k}h$ from the point (x_1, x_2, x_3), u_p and u_q are the values of u at the corresponding grid points.

4.3 The First Stage

Let v be a solution of the following finite difference problem

$$v_h = Sv_h \text{ on } R_h, \quad v_h = \psi_j \text{ on } \Gamma_{jh}, \quad j = 1,2, \dots, 6. \tag{4.21}$$

where $\psi_j, j = 1,2, \dots, 6$ are functions (4.10).

Lemma 4.6 The following estimation holds

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |v_h - v| \leq c_1 h^4, \tag{4.22}$$

where v is the function (4.8), c_1 is a constant independent of h.

Proof. By Lemma 4.2,

$$\Delta v = 0 \text{ on } R, \; v = \psi_j \text{ on } \Gamma_j, \quad j = 1,2, \dots, 6, \tag{4.23}$$

where functions ψ_j defined by (4.10), and from (4.2)-(4.6) it follows that

$$\psi_j \in C^{4,\lambda}(\Gamma_j), \; 0 < \lambda < 1, \quad j = 1,2, \dots, 6, \tag{4.24}$$

$$\psi_\mu = \psi_\nu \text{ on } \gamma_{\mu\nu}, \tag{4.25}$$

$$\frac{\partial^2\psi_\mu}{\partial t_\mu^2} + \frac{\partial^2\psi_\nu}{\partial t_\nu^2} + \frac{\partial^2\psi_\mu}{\partial t_{\mu\nu}^2} = 0 \ \text{ on } \gamma_{\mu\nu}. \tag{4.26}$$

By (4.24)-(4.26) the boundary functions ψ_j, $j = 1,2,\dots,6$, satisfy the conditions of Theorem 4 by (Volkov, 2010) of which follows the estimation (4.22). ■

4.4 The Second Stage

Let ω be a solution of the following finite difference problem

$$\omega_h = A\omega_h \ \text{ on } R_h, \quad \omega_h = \kappa_j \ \text{ on } \Gamma'_{jh}, \quad j = 1,2,\dots,6. \tag{4.27}$$

where κ_j, $j = 1,2,\dots,6$ are functions defined by (4.14).

Lemma 4.7 On R_h, it holds that,

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |\omega_h - \omega| \le c_2 h^2, \tag{4.28}$$

where ω is the function (4.12), ω_h is a solution to system (4.27), and c_2 is a constant independent of h.

Proof. By Lemma 4.4, we have

$$\Delta\omega = 0 \ \text{ on } R, \quad \omega = \kappa_j \ \text{ on } \Gamma_j, \quad j = 1,2,\dots,6, \tag{4.29}$$

where functions κ_j defined boundary values in (4.14), and from (4.2)-(4.6) follows

$$\kappa_j \in C^{2,\lambda}(\Gamma_j), \ 0 < \lambda < 1, \quad j = 1,2,\dots,6, \tag{4.30}$$

$$\kappa_\mu = \kappa_\nu \ \text{ on } \gamma_{\mu\nu}. \tag{4.31}$$

On the basis of (4.30)-(4.31) that satisfy the conditions of Theorem 1.1 by (Volkov, 2001). This completes the proof. ■

4.5 The Third Stage

Let v_h and ω_h be the solution of the difference problem (4.21) and (4.27) respectively. We approximate the solution of the given Dirichlet problem (4.1) on the grid $\bar{R}_h$ as a solution u_h of the following difference problem

$$u_h = Au_h - \frac{h^4}{36} v_h - \frac{h^6}{720} \omega_h \text{ on } R_h \tag{4.32}$$

$$u_h = \varphi_j \text{ on } \Gamma'_{jh}, \qquad j = 1,2, \dots, 6 \tag{4.33}$$

Theorem 4.1 Under the conditions (4.2)-(4.6), the estimation

$$\max_{(x_1,x_2,x_3)\in\bar{R}_h} |u_h - u| \leq c_3 h^6, \tag{4.34}$$

is valid, where u is the solution of the Dirichlet problem (4.1), u_h is the solution of system (4.32)-(4.33), and $c_3$is a constant independent of h.

Proof. Under the smoothness properties of the boundary values specified in (4.2)-(4.6), the solution u of the Dirichlet problem (4.1) has eighth-order partial derivatives that continuous on R, and by using Taylor formula, for each $(x_1, x_2, x_3) \in R_h$, we obtain

$$u(x_1, x_2, x_3) = Au(x_1, x_2, x_3) - \frac{h^4}{36} v - \frac{h^6}{720} \omega - r(x_1, x_2, x_3) \tag{4.35}$$

where v and ω are the functions defined by (4.8) and (4.12) respectively,

$$r = r(x_1, x_2, x_3) = \frac{h^8}{120960} \left[\frac{\partial^8 u(x_1 + \theta_1 h, x_2, x_3)}{\partial x_1^8} \right.$$

$$+\frac{\partial^8 u(x_1, x_2+\theta_2 h, x_3)}{\partial x_2^8}+\frac{\partial^8 u(x_1, x_2, x_3+\theta_3 h)}{\partial x_3^8}\Bigg],$$

$$|\theta_\tau|<1, \qquad \tau = 1,2,3. \tag{4.36}$$

We put

$$\epsilon_h = u_h - u \ \text{ on } \bar{R}_h, \tag{4.37}$$

where u_h is the solution of the finite difference problem (4.32), (4.33).

From (4.32), and (4.33) and taking into account that $u_h = u = \varphi_j$ on Γ_{jh}, we obtain the following system of difference equations for the error ϵ_h:

$$\epsilon_h = A\epsilon_h + \frac{h^4}{36}(v - v_h) + \frac{h^6}{720}(\omega - \omega_h) + r \ \text{ on } R_h$$

$$\epsilon_h = 0 \ \text{ on } \Gamma'_h \tag{4.38}$$

By Lemma 4.5, we have

$$\max_{(x_1,x_2,x_3)\in\bar{R}\setminus\gamma} |(x_1, x_2, x_3)| \le c_4 h^8, \tag{4.39}$$

where c_4 is a constant independent of h. On the basis of Lemma 4.6, and Lemma 4.7, and (4.39), we obtain

$$\left|\frac{h^4}{36}(v - v_h) + \frac{h^6}{720}(\omega - \omega_h) + r\right| \le c_5 h^8, \tag{4.40}$$

where $c_5 = \max\{c_1/36, c_2/720, c_4\}$.

Define function $\bar{\epsilon}_h = c_5 h^6 (l^2 - r^2) \geq 0$ on R_h, where $l = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is easy to check that the function $\bar{\epsilon}_h$ is a solution of problem

$$\bar{\epsilon}_h = A\bar{\epsilon}_h + c_5 h^8 \text{ on } R_h,\ \bar{\epsilon}_h \geq 0 \text{ on } \Gamma_h'. \tag{4.41}$$

Since,

$$A\bar{\epsilon}_h = \frac{c_5}{6} h^6 [a_1^2 + a_2^2 + a_3^2 - (x_1 + h)^2 - x_2^2 - x_3^2 + a_1^2 + a_2^2 + a_3^2$$
$$-(x_1 - h)^2 - x_2^2 - x_3^2 + a_1^2 + a_2^2 + a_3^2 - x_1 - (x_2 + h)^2 - x_3$$
$$+a_1^2 + a_2^2 + a_3^2 - x_1 - (x_2 - h)^2 - x_3^2 + a_1^2 + a_2^2 + a_3^2 - x_1 - x_2$$
$$-(x_3 + h)^2 + a_1^2 + a_2^2 + a_3^2 - x_1 - x_2 - (x_3 - h)^2]$$
$$= \frac{c_5}{6} h^6 [6(a_1^2 + a_2^2 + a_3^2) - (x_1^2 + 2x_1 h + h^2) - x_2^2 - x_3^2 - (x_1^2 - 2x_1 h + h^2)$$
$$-x_2^2 - x_3^2 - x_1^2 - (x_2^2 + 2x_2 h + h^2) - x_3 - x_1^2 - (x_2^2 - 2x_2 h + h^2) - x_3$$
$$-x_1 - x_2 - (x_3^2 + 2x_3 h + h^2) - x_1 - x_2 - (x_3^2 - 2x_3 h + h^2)]$$
$$= \frac{c_5}{6} h^6 [6l^2 - 6(x_1^2 + x_2^2 + x_3^2) - 6h^2]$$
$$= c_5 h^6 [l^2 - r^2 - h^2] = c_5 h^6 l^2 - c_5 h^6 r^2 - c_5 h^8 = \bar{\epsilon}_h - c_5 h^8.$$

Therefore,

$$\bar{\epsilon}_h = A\bar{\epsilon}_h + c_5 h^8 \text{ on } R_h.$$

Furthermore, from Lemma 3.4 for the solution ϵ_h of problem (4.38) with the solution of problem (4.41) follows that

$$|\epsilon_h| \leq \bar{\epsilon}_h. \tag{4.42}$$

Since,

$$|\epsilon_h| \le \bar{\epsilon}_h \le c_5 \max_{(x_1,x_2,x_3)\in\bar{R}_h} |l^2 - r^2| h^6$$
$$\le c_5(a_1^2 + a_2^2 + a_3^2)h^6 \le c_3 h^6,$$

Therefore, by (4.38) Theorem 4.1 is proved. ■

CHAPTER 5
NUMERICAL EXAMPLES

In this chapter we present the numerical results to support the theoretical results in chapters 2, 3 and 4.

5.1 Domain in the Shape of a Rectangle

$\mathbf{\Pi} = \{(x, y): 0 < x, y < 1\}$, and let γ be the boundary of Π. We consider the following problem:

$$\Delta u = 0 \text{ on } \Pi, \; u = \varphi(x, y) \text{ on } \gamma_j, \quad j = 1,2,3,4, \tag{5.1}$$

where φ is the exact solution of this problem.

Let U denote the exact solution and U_h be its approximate values on $\overline{\Pi}^h$ (contains the nodes of the square grid formed in Π) of the Dirichlet problem for Laplace's equation on the rectangle domainΠ. We denote by

$$\|U - U_h\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h}|U - U_h|, \qquad \Re_U^m = \frac{\|U - U_{2-m}\|_{\overline{\Pi}^h}}{\left\|U - U_{2-(m+1)}\right\|_{\overline{\Pi}^h}}.$$

The results in the following example are demonstrated in two Tables. The first Table is the approximate result for the solution of problem (5.1), the second Table shows the approximate result for the first derivative of the solution $v = \frac{\partial u}{\partial x}$, which convergence as $O(h^6)$.

The difference step size h is defined by $h = \frac{1}{2^n}, n = 4,5,6,7$.

Example 5.1 Let $\varphi \in C^{7,\frac{1}{30}}$ on γ_j, $j = 1,2,3,4$, where

$$\varphi = \left[x^2 + \left(y - \frac{1}{2}\right)^2\right]^{\frac{211}{60}} \sin\left(\frac{211}{30} \arctan \frac{y - \frac{1}{2}}{x}\right), \tag{5.2}$$

where φ is the exact solution of this problem.

We solve the system (2.1) and (2.46) to find the approximate sixth order solution u_h for u and approximate first derivative v_h for $v = \frac{\partial u}{\partial x}$ respectively.

In Table 5.1 the convergence order more than 6 which corresponding to ρ in Theorem 2.1. Table 5.2 justified estimation in Theorem 2.2, i.e., the sixth order convergence.

Table 5.1: The approximate of solution in problem (5.1)

$\boldsymbol{h}$	$\|\boldsymbol{u} - \boldsymbol{u_h}\|$	$\mathfrak{R}_U^m$
$\frac{1}{16}$	$1.47E - 10$	63.91
$\frac{1}{32}$	$2.30E - 12$	63.89
$\frac{1}{64}$	$3.60E - 14$	64.19
$\frac{1}{128}$	$5.61E - 16$	

Table 5.2: First derivative approximation results with the sixth-order accurate formulate for problem (5.1)

h	$\|\boldsymbol{v}-\boldsymbol{v_h}\|$	$\mathfrak{R}_U^m$
$\frac{1}{16}$	$1.17E-06$	44.15
$\frac{1}{32}$	$2.65E-08$	56.14
$\frac{1}{64}$	$4.72E-10$	61.22
$\frac{1}{128}$	$7.71E-12$	

5.2 Domain in the Shape of a Rectangular Parallelepiped

Let $R=\{(x_1,x_2,x_3): 0<x_i<1,\ i=1,2,3\}$, and let Γ be the boundary of R. We consider the following boundary value problem

$$\Delta u=0 \text{ on } R,\ u=\varphi(x_1,x_2,x_3) \text{ on } \Gamma_j, \quad j=1,2,\dots,6, \tag{5.3}$$

where φ is the exact solution of problem (5.3).

Let U denote the exact solution and U_h be its approximate values on $\bar{R}^h$ (contains the nodes of the cubic grid formed in R) of the Dirichlet problem for Laplace's equation on the rectangular parallelepiped domain R. We denote by

$$\|U-U_h\|_{\bar{R}^h}=\max_{\bar{R}^h}|U-U_h|, \qquad \mathfrak{R}_U^m=\frac{\|U-U_{2-m}\|_{\bar{R}^h}}{\left\|U-U_{2-(m+1)}\right\|_{\bar{R}^h}}.$$

The approximate results for the solution of the Dirichlet problem are presented in Table (5.3). Table (5.4), shows the maximum errors and the convergence order of the approximations of the first derivatives when sixth order accuracy forward backward formula is used, and in Table (5.5) the maximum errors and the convergence order of the approximations of the pure second derivatives of problem (5.3) for different step size h are presented.

The difference step size h is defined by $h = \frac{1}{2^n}, n = 4,5,6,7$.

Example 5.2 Let $\varphi \in C^{7,\frac{1}{30}}$ on Γ_j, $j = 1,2,\dots,6$, where

$$\varphi = \left(x_3 - \frac{1}{2}\right)^2 - \left(\frac{x_1^2 + x_2^2}{2}\right) + (x_1^2 + x_2^2)^{\frac{211}{60}} \cos\left(\frac{211}{30}\Theta\right) \quad (5.4)$$

where $\Theta = \arctan\left(\frac{x_2}{x_1}\right)$ and φ is solution of this problem.

In Table 5.3 the convergence order is sixth order accurate. Table 5.4 justified estimation in Theorem 3.2, the sixth order convergence. Table 5.5 shows the convergence order more than 5 which corresponding to λ in Theorem 3.3.

Table 5.3: The approximate of solution in problem (5.3)

$\boldsymbol{h}$	$\|\boldsymbol{u} - \boldsymbol{u_h}\|$	$\boldsymbol{\Re_U^m}$
$\frac{1}{16}$	$2.32E - 10$	63.74
$\frac{1}{32}$	$3.64E - 12$	63.97
$\frac{1}{64}$	$5.69E - 14$	64
$\frac{1}{128}$	$8.89E - 16$	

Table 5.4: First derivative approximation results with the sixth-order accurate formulae for problem (5.3)

h	$\|\boldsymbol{v}-\boldsymbol{v}_h\|$	$\mathfrak{R}_U^m$
$\frac{1}{16}$	$4.00E-05$	58.91
$\frac{1}{32}$	$6.79E-07$	61.73
$\frac{1}{64}$	$1.10E-08$	62.50
$\frac{1}{128}$	$1.76E-10$	

Table 5.5: The approximate results for the pure second derivative

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$	$\mathfrak{R}_U^m$
$\frac{1}{16}$	$9.93E-09$	32.66
$\frac{1}{32}$	$3.04E-10$	32.72
$\frac{1}{64}$	$9.29E-12$	32.83
$\frac{1}{128}$	$2.83E-13$	

The difference step size h of the following Example is defined by $h=\frac{1}{2^n}, n=2,3,4,5$.

Example 5.3 Let $R=\{(x_1,x_2,x_3): 0<x_i<1,\ i=1,2,3\}$, and let Γ_j, $j=1,2,\ldots,6$, be its faces. We consider the following problem:

$$\Delta u=0 \text{ on } R, \quad u=\varphi(x_1,x_2,x_3) \text{ on } \Gamma_j, j=1,2,\ldots,6, \tag{5.5}$$

where

$$\varphi(x_1, x_2, x_3) = e^{3x_1} \cosh(4x_2) \cos(5x_3) \quad (5.6)$$

Table 5.6: The approximate results by using a three stage difference method

h	$\|\boldsymbol{\nu} - \boldsymbol{\nu_h}\|_{\bar{R}^h}$	$\boldsymbol{E_{\nu}^m}$	$\|\boldsymbol{\omega} - \boldsymbol{\omega_h}\|_{\bar{R}^h}$	$\boldsymbol{E_{\omega}^m}$	$\|\boldsymbol{u} - \boldsymbol{u_h}\|_{\bar{R}^h}$	$\boldsymbol{E_u^m}$
$\frac{1}{4}$	$1.71E-05$	13.15	$7.84E-04$	2.96	$4.37E-09$	52.84
$\frac{1}{8}$	$1.30E-06$	15.24	$2.65E-04$	3.71	$8.27E-11$	63.62
$\frac{1}{16}$	$8.53E-08$	15.94	$7.15E-05$	3.93	$1.30E-12$	64.01
$\frac{1}{32}$	$5.35E-09$		$1.82E-05$		$2.03E-14$	

In Table 5.6 we have used the following notations:

$$\|U - U_h\|_{\Omega_h} = \max_{\Omega_h} \|U - U_h\| \text{ and } E_U^m = \frac{\|U - U_{2-m}\|_{\Omega_h}}{\left\|U - U_{2-(m+1)}\right\|_{\Omega_h}}$$

where U is the trace of the exact solution of the continuous problem Ω_h, and U_h is its approximate values.

The values of E_v^m and E_ω^m show that the function v (the first stage) and ω (the second stage) are approximated with the order $O(h^4)$ and $O(h^2)$, respectively.

The values of E_u^m show that the accuracy of the proposed method (the third stage) is of order $O(h^6)$, where

$$v = v(x_1, x_2, x_3) = \frac{1}{2} \sum_{j=1}^{3} \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_j^4},$$

$$\omega = \omega(x_1, x_2, x_3) = \frac{1}{3} \sum_{j=1}^{3} \frac{\partial^6 u(x_1, x_2, x_3)}{\partial x_j^6},$$

and u defined in (4.1).

CHAPTER 6
CONCLUSION

For the approximate solution of the *2D* Laplace equation in a rectangle Π and the *3D* Laplace equation in a rectangular parallelepiped R, a new pointwise error of order $O(\rho h^6)$ is obtained, where ρ is the distance from the current point to the boundary of Π or R. This estimation shows the additional downturn of the error near the boundary as ρ, which is used to get $O(h^6)$ order of accuracy for the approximate value of the first derivatives in *2D* and $O(h^6 \ln h)$ order of accuracy for the approximate value of the first derivatives in *3D* Laplace equation. For the approximation of the pure second derivatives $O\left(h^{5+\lambda}\right)$, order is obtained for both cases *2D* and *3D*.

The obtained results can be applied for the approximation of a solution and its derivatives of problems in more complicated domains when different version of domain decomposition methods are used (see for 2D problem (Dosiyev, 1992), (Dosiyev, 1994), (Dosiyev, 2003), (Dosiyev, 2004), (Dosiyev, 2012), (Dosiyev, 2013), (Dosiyev, 2014) (Volkov, 1976), see for 3D (Smith et at ., 2004), (Volkov, 1979), (Volkov, 2003)).

Whereas a new three-stage difference method with an accuracy of order $O(h^6)$, where h is mesh size, is proposed and justified by using one fourth order and two second order schemes for the approximate solution of the $3D$ Laplace's equation. The proposed method can be used to highly approximate the derivatives of the unknown solution of Laplace's equation.

REFERENCES

Andreev, V. (1977). The mixed problem for Laplace's mesh equation in a half-plane. *Doklady Akademii Nauk SSSR 234*, 997-1000.

Bahvalov, N. (1959). Numerical solution of the Dirichlet problem for Laplace's equation. *Vestnik Moskov University Series Matematka. Mechanika. Astronomiya, Fizika,Himiya, 3*(5), 171-195.

Batchelor, G. (1967). Introduction to Fluid Dynamics. Cambridge University Press.

Bear, J. (1972). Dynamics of Fluids in Porous Media. New York.

Bergman, S , & Schiffer, M. (1953). Kernal functions and Elliptic differential equations in mathematical physics. New York, Academic Press.

Buranay, S , & Farinola, L. (2018). Six point implicit methods for the pure second derivatives of the solution of first type boundary problem for one dimensional heat equation. *AIP Conference Proceedings 1997* 2018.

Chorlton, F. (1967). Textbook of fluid dynamics. London, Van Nostrand.

Collins, R. (1961). Flow of fluids through porous media. New York, Reinhold.

Dosiyev, A. (2018). A method of harmonic extension for computing the generalized stress intensity factors for Laplace's equation with singularities. *Computers & Mathematics with Applications* 75(5), 1767-1777.

Dosiyev, A. (1994). A block-grid method of increased accuracy for solving Dirichlet's problem for Laplace's equation on polygons. *Compute Math. and Math Physics.* 34(5), 591-604.

Dosiyev, A. (2003). On the maximum error in the solution of Laplace equation by finite difference method. *International Journal of Pure and Applied Mathematics. 7*, 229-241.

Dosiyev, A. (2004). The high accurate block-grid method for solving Laplace's boundary value problem with singularities. *SIAM Journal on Numerical Analysis*. 42(1), 153-178.

Dosiyev, A. (2014). The block-grid method for the approximation of the pure second order derivatives for the solution of Laplace's equation on a staircase polygon. *Journal of Computational and Applied Mathematics.* 259, 14-23.

Dosiyev, A , & Sadeghi, H. (2015). A fourth order accurate approximation of the first and pure second derivatives of the Laplace equation on a rectangle. *Advances in Difference Equations*, 2015:67.

Dosiyev, A., & Sadeghi, H. (2016). On a highly accurate approximation of the first and pure second derivatives of the Laplace's equation in a rectangular parallelepiped. *Advances in Difference Equations* 2016:145.

Dosiyev, A , & Sarıkaya, H. (2017). A highly accurate difference method for solving the Dirichlet problem for Laplace's equation on a rectangle. *AIP Conference Proceedings* 2017:1880.

Dosiyev, A , & Abdussalam, A. (2018). On high order convergence of the difference solution of Laplace's equation in a rectangular parallelepiped. *Filomat* 2018:145, 893-901.

Dosiyev, A , & Sarıkaya, H. (2018). 14-Point Difference Operator for the Approximation of the First Derivatives of a Solution of Laplace's Equation in a Rectangular Parallelepiped. *Filomat*. 2018:32.

Fichera, G. (1965). Linear elliptic differential systems and eigen-value problems. Berlin, Lecture Notes in Mathematics, Springer-Verlag, 8.

Gilbarg, D , & Trudinger, N. (1977). Elliptic partial differential Equations of the second order. Berlin, Springer- Verlag.

Greenkorn, R. (1983). Flow Phenomena in Porous Media. New York, Marcel-Dekker.

Kantorovich, L., & Krylov, V. (1958). Approximate methods of higher analysis.Noordhoff Ltd, Leiden.

Lebedev, V. I. (1960). Evaluation of the error involved in the grid method for Newmann's two dimensional problem. *Soviet Mathematics Dokl*ady, 1, 703-705.

Mikeladze, Sh. (1938). On the numerical solution of Laplace's and Poisson's differential equations. *Izvestiia Academii Nauk SSSR, Ser. Mat.*, (2), 271-292.

Mikhailov, V. (1978). Partial Differential Equations. Mir, Moscow.

Peyson, Y , & Shoucri, M. (1998). An approximate factorization procedure for solving nine-point elliptic difference equations Application for a fast 2-D relativistic Fokker-Planck solver. *Computer Physics Communication*, 109(55).

Richard, L., & Dougla,, J. (2011). Numerical Analysis. Brooks/Cole, Cengage Learning.

Samarski, A. (2001). The Theory of difference schemes. Marcel Deker, Inc.

Smith, G. (1985). Numerical solution of partial differential equations: Finite difference methods. Oxford University Press.

Volkov, E. (1963). On the solution by the grid method of the inner Dirichlet problem for the Laplace equation. *American Mathematical Society Translations*, 24(2), 279-307.

Volkov, E. (1965). On differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle. *Proceedings of the steklov Institue of Mathematics Issue 77*, 101-126.

Volkov, E. (1966). Effective estimates of the errors in solutions by method of nets of boundary-value problems for Laplace's and Poisson's equations on a rectangle and certain triangles. *Proceedings of the steklov Institue of Mathematics SSSR*, 74, 57-90.

Volkov, E. (1969). On differential properties of solutions of the Laplace and Poisson equations on a parallelepiped and efficient error estimates of the method of nets. *Proceedings of the steklov Institue of Mathematics.*. 105, 54-78.

Volkov, E. (1994). Block method for solving the Laplace equation and constructing conformal mappings. USA, CRC Press.

Volkov, E. (1999). On convergence in C_2 of a difference solution of the Laplace equation on a rectangle. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 14(3), 291-298.

Volkov, E. (2004). On the grid method for approximating the derivatives of the solution of the Dirichlet problem for the Laplace equation on the rectangular parallelepiped. *Russian Journal of Numerical Analysis and Mathematical Modelling*. 19(3), 269-278.

Volkov, E. (2005). On the convergence in C_h^1 of the difference solution to the Laplace equation in a rectangular parallelepiped. *Computational Mathematics and Mathematical Physics*, 45(9), 1531-1537.

Volkov, E. & Dosiyev, A. (2007). A high accurate composite grid method for solving Laplace's boundary value problems with singularities. *Russian Journal of Numerical Analysis and Mathematical Modelling*, 22(3), 291-307.

Volkov, E. (2007). Grid approximation of the first derivatives of the solution to the Dirichlet problem for the Laplace equation on a polygon. *Proceedings of the Steklov Institute of Math*ematics, 225, 92-107.

Volkov, E. (2007). On a combined grid method for solving the Dirichlet problem for the Laplace equation in a rectangular parallelepiped. *Computational Mathematics and Mathematical physics*, 47(4), 638-643.

Volkov, E , & Dosiyev, A. (2012). A highly accurate homogeneous scheme for solving the Laplace equation on a rectangular parallelepiped with boundary values in $\boldsymbol{C^{k,1}}$. *Computational Mathematics and Mathematical physics*, 52(6), 879-886.

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International Conference Presentations

1. **A. A. Dosiyev** and Ahlam. Abdussalam, A three-stage difference method for solving the Dirichlet problem for Laplace's equation on a rectangular parallelepiped, **International conference on applied analysis and mathematical modeling,** July 2017, Istanbul.

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