# SCHRÖDINGER TYPE INVOLUTORY PARTIAL DIFFERNETIAL EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE
SCHOOL OF APPLIED SCIENCES
OF
NEAR EAST UNIVERSITY

By
TWANA ABBAS HIDAYAT

In Partial Fulfillment of the Requirements for
the Degree of Master of Science
in
Mathematics

NICOSIA, 2019

Twana Abbas Hidayat SCHRÖDINGER TYPE INVOLUTORY PARTIAL DIFFERETIAL EQUATIONS NEU 2019

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**Twana Abbas Hidayat: SCHRÖDINGER TYPE INVOLUTORY PARTIAL DIFFERETIAL EQUATIONS**

**Approval of Director of Graduate School of Applied Sciences**

**Prof. Dr. Nadire ÇAVUŞ**

**We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics Department**

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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**To my family...**

ABSTRACT

In the present study, a Schrödinger type involutory differential equation is investigated. Using tools of classical approach we are enabled to obtain the solution of the Schrödinger type involutory differential equations. Furthermore, the first order of accuracy difference scheme for the numerical solution of the Schrödinger type involutory differential equations is presented. Then, this difference scheme is tested on an example and some numerical results are presented.

***Keywords*:** Involutory differential equations; Fourier series method; Laplace transform solution; Fourier transform solution; Difference scheme; Modified Gauss elimination method

## ÖZET

Bu çalışmada Schrödinger tipi involüsyon diferansiyel denklemi incelenmiştir. Klasik yaklaşım araçlarını kullanmak Schrödinger tipi involüsyon diferansiyel denklemlerin çözümünü elde etmemize olanak tanır. Ayrıca, Schrödinger tipi involüsyon diferansiyel denklemlerin nümerik çözümü için birinci basamaktan doğruluklu fark şeması sunulmuştur. Daha sonra, bu fark şeması bir örnek üzerinde test edilip bazı sayısal sonuçlar verilmiştir.

***Anahtar Kelimeler***: Involüsyon diferansiyel denklemler; Fourier serisi yöntemi; Laplace dönüşümü çözümü; Fourier dönüşümü çözümü; Fark şeması; Modifiye Gauss eleminasyon yöntemi

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## LIST OF ABBREVIATIONS

**DDE:** Delay Differential Equation

**FDE:** Functional Differential Equation

**IDE:** Involutory Differential Equation

# CHAPTER 1
# INTRODUCTION

Time delay is a universal phenomenon existing in almost every practical engineering systems (Bhalekar and Patade 2016; Kuralay, 2017; Vlasov and Rautian 2016; Sriram and Gopinathan 2004; Srividhya and Gopinathan 2006). In an experiment measuring the population growth of a species of water fleas, Nesbit (1997), used a DDE model in his study. In simplified form his population equation was

$$N'(t) = aN(t-d) + bN(t).$$

He got into a difficulty with this model because he did not have a reasonable history function to carry out the solution of this equation. To overcome this roadblock he proposed to solve a ”time reversal” problem in which he sought the solution to an FDE that is neither a DDE, nor a FDE. He used a ”time reversal” equation to get the juvenile population prior to the beginning time $t = 0$. The time reversal problem is a special case of a type of equation called an involutory differential equation. These are defined as equations of the form

$$y'(t) = f(t; y(t); y(u(t))), y(t_0) = y_0. \tag{1.1}$$

Here $u(t)$ is involutory function, that is $u(u(t)) = t$, and $t_0$ is a fixed point of $u$. For the ”time reversal” problem, we have the simplest IDE, one in which the deviating argument is $u(t) = -t$. This function is involutory since

$$u(u(t)) = u(-t) = -(-t) = t.$$

We consider the simplest IDE, one in which the deviating argument is $u(t) = d-t$.This function is involutory since $u(u(t)) = u(d-t)$,which is $d-(d-t) = t$. Note $d-t$ is not the ”delay” function as $t-d$.

The theory and applications of delay Schrödinger differential equations have been studied in various papers ( Agirseven, 2018; Guo and Yang, 2010; Gordeziani and Avalishvili, 2005; Han and Xu, 2016; Chen and Zhou, 2010; Guo and Shao, 2005; Sun and Wang, 2018;

Nicaise and Rebiai, 2011; Zhao and Ge, 2011; Kun and Cui-Zhen, 2013; and the references given therein).

The discussions of time delay issues are significant due to the presence of delay that normally makes systems less effective and less stable. Especially, for hyperbolic systems, only a small time delay may cause the energy of the controlled systems increasing exponentially. The stabilization problem of one dimensional Schrödinger equation subject to boundary control is concerned in the paper of Gordeziani and Avalishvili, 2005 .

The control input is suffered from time delay. A partial state predictor is designed for the system and undelayed system is deduced. Based on the undelayed system, a feedback control strategy is designed to stabilize the original system. The exact observability of the dual one of the undelayed system is checked. Then it is shown that the system can be stabilized exponentially under the feedback control.

It is known that various problems in physics lead to the Schrödinger equation. Methods of solutions of the problems for Schrödinger equation without delay have been studied extensively by many researchers (Antoine and Mouysset, 2004; Ashyralyev and Hicdurmaz, 2011; Ashyralyev and Hicdurmaz, 2012; Ashyralyev and Sirma, 2008;Ashyralyev and Sirma, 2009; Eskin and Ralston, 1995; Gordeziani and Avalishvili, 2005; Han and Wu, 2005; Mayfield 1989-Serov and Päivärinta, 2006; Smagin and Shepilova, 2008, and the references given therein).

In this study, Schrödinger type involutory partial differential equations is studied. Using tools of the classical approach we are enabled to obtain the solution of the Schrödinger type involutory differential problem. Furthermore, the first order of accuracy difference scheme for the numerical solution of the initial boundary value problem for Schrödinger type involutory partial differential equations is presented. Then, this difference scheme is tested on an example and some numerical results are presented.

The thesis is organized as follows. Chapter 1 is introduction. In Chapter 2, a Schrödinger type involutory ordinary differential equations is studied and Schrödinger type involutory partial differential equations are investigated. Using tools of the classical approach we are enabled to obtain the solution of the several Schrödinger type involutory differential problems. In Chapter 3, numerical analysis and discussions are presented. Finally, Chapter 4 is conclusion.

# CHAPTER 2

# METHODS OF SOLUTION FOR SCHRÖDINGER TYPE INVOLUTORY PARTIAL DIFFERETIAL EQUATIONS

## 2.1 Schrödinger Type Involutory Ordinary Differential Equations

In this section we consider the Schrödinger type involutory ordinary differential equations

$$iy'(t) = f(t; y(t); y(u(t))), y(t_0) = y_0. \tag{2.1}$$

Here u(t) is involutory function, that is u(u(t)) = t, and $t_0$ is a fixed point of u.

**Example 2.1.1.** Solve the problem

$$iy'(t) = 5y(\pi - t) + 4y(t) \quad \text{on} \quad I = (-\infty, \infty), \;\; y(\frac{\pi}{2}) = 0.$$

**Solution.** We will obtain the initial value problem for the second order differential equation equivalent to given problem. Differentiating this equation, we get

$$iy''(t) = -5y'(\pi - t) + 4y'(t).$$

Substituting $\pi - t$ for t into this equation, we get

$$iy'(\pi - t) = 5y(t) + 4y(\pi - t).$$

Using these equations, we can eliminate the terms of $y(\pi - t)$ and $y'(\pi - t)$. Really, using formulas

$$y'(\pi - t) = \frac{1}{i}\{5y(t) + 4y(\pi - t)\},$$

$$y(\pi - t) = \frac{1}{5}(iy'(t) - 4y(t)),$$

we get

$$iy''(t) = \left\{\frac{1}{i}25y(t) + \frac{4}{i}\left(iy'(t) - 4y(t)\right)\right\}(-1) + 4y(t)$$

or

$$y''(t) = 9y(t).$$

Using initial condition $y(\frac{\pi}{2}) = 0$ and equation, we get

$$iy'(\frac{\pi}{2}) = 5y(\frac{\pi}{2}) + 4y(\frac{\pi}{2}) = 0$$

or

$$y'(\frac{\pi}{2}) = 0.$$

Therefore, we have the following initial value problem for the second order differential equation

$$y''(t) - 9y(t) = 0,\ t \in I, y(\frac{\pi}{2}) = 0, y'(\frac{\pi}{2}) = 0.$$

The auxiliary equation is

$$m^2 - 9 = 0.$$

There are two roots $m_1 = 3$ and $m_2 = -3$. Therefore, the general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{-3t}.$$

Differentiating this equation, we get

$$y'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}$$

Using initial conditions $y(\frac{\pi}{2}) = 0$ and $y'(\frac{\pi}{2}) = 0$, we get

$$c_1 e^{\frac{3\pi}{2}} + c_2 e^{-\frac{3\pi}{2}} = 0,$$

$$3c_1 e^{\frac{3\pi}{2}} - 3c_2 e^{-\frac{3\pi}{2}} = 0.$$

Since

$$\Delta = \begin{vmatrix} e^{\frac{3\pi}{2}} & e^{-\frac{3\pi}{2}} \\ 3e^{\frac{3\pi}{2}} & -3e^{-\frac{3\pi}{2}} \end{vmatrix} = -3 - 3 = -6 \neq 0,$$

we have that $c_1 = c_2 = 0$. Therefore, the exact solution of this problem is

$$y(t) = 0.$$

**Example 2.1.2.** Obtain the solution of the problem

$$iy'(t) = by(\pi - t) + ay(t) + f(t) \quad \text{on } I = (-\infty, \infty), y(\frac{\pi}{2}) = 1. \tag{2.2}$$

**Solution.** We will obtain the initial value(2.2) problem for the second order differential equation equivalent to given problem. Differentiating this equation, we get

$$iy''(t) = -by'(\pi - t) + ay'(t) + f'(t). \tag{2.3}$$

Substituting $\pi - t$ for t into equation (2.2), we get

$$iy'(\pi - t) = by(t) + ay(\pi - t) + f(\pi - t).$$

Using these equations, we can eliminate the $y(\pi - t)$ and $y'(\pi - t)$ terms. Really, using formulas

$$y'(\pi - t) = \frac{1}{i}\{by(t) + ay(\pi - t) + f(\pi - t)\},$$

$$y(\pi - t) = \frac{iy'(t) - ay(t) - f(t)}{b},$$

we get

$$iy''(t) = bi\left\{by(t) + a\left[\frac{iy'(t) - ay(t) - f(t)}{b}\right] + f(\pi - t)\right\} + ay'(t) + f'(t)$$

or

$$iy''(t) = ib^2y(t) - ay'(t) - a^2iy(t) - aif(t) + bif(\pi - t) + ay'(t) + f'(t).$$

From that it follows

$$y''(t) - (b^2 - a^2)y(t) = -af(t) + bf(\pi - t) - if'(t). \tag{2.4}$$

Putting initial condition $y(\frac{\pi}{2}) = 1$ into equation (2.2), we get

$$iy'(\frac{\pi}{2}) = a + b + f(\frac{\pi}{2})$$

or

$$y'(\frac{\pi}{2}) = -i\left\{a + b + f(\frac{\pi}{2})\right\}.$$

We denote

$$F(t) = -af(t) + bf(\pi - t) - if'(t). \tag{2.5}$$

Then, we have the following initial value problem for the second order ordinary differential equation

$$y''(t) - (b^2 - a^2)y(t) = F(t), t \in I, y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = -i\left\{a + b + f(\frac{\pi}{2})\right\}. \tag{2.6a}$$

Now, we obtain the solution of equation (2.6a). There are three cases: $a^2 - b^2 > 0, a^2 - b^2 = 0, a^2 - b^2 < 0$.

In the first case $a^2 - b^2 = m^2 > 0$. Substituting $m^2$ for $a^2 - b^2$ into equation (2.6a), we get

$$y''(t) + m^2 y(t) = F(t).$$

We will obtain Laplace transform solution of equation (2.6a), we get

$$s^2 y(s) - sy(0) - y'(0) + m^2 y(s) = F(s)$$

or

$$(s^2 + m^2) y(s) = sy(0) + y'(0) + F(s).$$

Here and in future

$$F(s) = L\{F(t)\}.$$

Then,

$$y(s) = \frac{s}{s^2 + m^2} y(0) + \frac{1}{s^2 + m^2} y'(0) + \frac{1}{s^2 + m^2} F(s). \tag{2.7}$$

Applying formulas

$$\frac{s}{s^2 + m^2} = \frac{1}{2}\left[\frac{1}{s + im} + \frac{1}{s - im}\right], \frac{1}{s^2 + m^2} = \frac{1}{2im}\left[\frac{1}{s - im} - \frac{1}{s + im}\right],$$

we get

$$y(s) = \frac{1}{2}\left[\frac{1}{s + im} + \frac{1}{s - im}\right] y(0) + \frac{1}{2im}\left[\frac{1}{s - im} - \frac{1}{s + im}\right] y'(0)$$
$$+ \frac{1}{2im}\left[\frac{1}{s - im} - \frac{1}{s + im}\right] F(s).$$

Applying formulas

$$L\left\{e^{\pm imt}\right\} = \frac{1}{s \mp im}, L\left\{\int_0^t e^{\pm im(t-y)} F(y) dy\right\} = \frac{1}{s \mp im} F(s)$$

and taking the inverse Laplace transform, we get

$$y(t) = \frac{1}{2}\left[e^{-imt} + e^{imt}\right] y(0) + \frac{1}{2im}\left[e^{imt} - e^{-imt}\right] y'(0)$$

$$+ \frac{1}{2im}\int_0^t \left[e^{im(t-y)} - e^{-im(t-y)}\right] F(y) dy.$$

Using formulas

$$\cos(mt) = \frac{1}{2}\left[e^{-imt} + e^{imt}\right], \sin(mt) = \frac{1}{2i}\left[e^{imt} - e^{-imt}\right],$$

we get

$$y(t) = \cos(mt)\, y(0) + \frac{1}{m}\sin(mt)\, y'(0) + \frac{1}{m}\int_0^t \sin(m(t-y))F(y)dy.$$

Now, we obtain $y(0)$ and $y'(0)$. Taking the derivative, we get

$$y'(t) = -m\sin(mt)\, y(0) + \cos(mt)\; y'(0) + \int_0^t \cos(m(t-y))F(y)dy.$$

Putting $F(y) = -af(y) + bf(\pi - y) - if'(y)$, we get

$$y(t) = \cos(mt)\, y(0) + \frac{1}{m}\sin(mt)\, y'(0)$$

$$+\frac{1}{m}\int_0^t \sin(m(t-y))\left[-af(y) + bf(\pi - y) - if'(y)\right]dy, \tag{2.8}$$

$$y'(t) = -m\sin(mt)\; y(0) + \cos(mt)\; y'(0)$$

$$+\int_0^t \cos(m(t-y))\left[-af(y) + bf(\pi - y) - if'(y)\right]dy. \tag{2.9}$$

Substituting $\frac{\pi}{2}$ for t into equations (2.8)and (2.9) gives us

$$y(\frac{\pi}{2}) = \cos m\frac{\pi}{2}\; y(0) + \frac{1}{m}\sin m\frac{\pi}{2}\; y'(0)$$

$$+\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin m(\frac{\pi}{2} - y)\left[-af(y) + bf(\pi - y) - if'(y)\right]dy,$$

$$y'(\frac{\pi}{2}) = -m\sin m\frac{\pi}{2}\; y(0) + \cos m\frac{\pi}{2}\; y'(0)$$

$$+\int_0^{\frac{\pi}{2}} \cos m(\frac{\pi}{2} - y)\left[-af(y) + bf(\pi - y) - if'(y)\right]dy.$$

Applying initial conditions $y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = -i\left\{a + b + f(\frac{\pi}{2})\right\}$, we obtain

$$\begin{cases} \cos\left(\frac{m\pi}{2}\right)\; y(0) + \frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\; y'(0) = 1 - \alpha_1, \\ \\ -m\sin\left(\frac{m\pi}{2}\right)\; y(0) + \cos\left(\frac{m\pi}{2}\right)\; y'(0) = -i\left\{a + b + f(\frac{\pi}{2})\right\} - \alpha_2. \end{cases}$$

Here

$$\alpha_1 = \frac{1}{m}\int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy,$$

$$\alpha_2 = \int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy.$$

Since

$$\Delta = \begin{vmatrix} \cos\left(\frac{m\pi}{2}\right) & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ -m\sin\left(\frac{m\pi}{2}\right) & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix} = \cos^2 m\frac{\pi}{2} + \sin^2 m\frac{\pi}{2} = 1 \neq 0,$$

we have that

$$y(0) = \frac{\Delta_0}{\Delta} = \begin{vmatrix} 1-\alpha_1 & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ -i\{a+b+f(\frac{\pi}{2})\}-\alpha_2 & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix}$$

$$y(0) = \cos\left(\frac{m\pi}{2}\right)\left[1-\alpha_1\right] + \frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\left[-i\left\{a+b+f(\frac{\pi}{2})\right\}-\alpha_2\right],$$

$$y'(0) = \frac{\Delta_1}{\Delta} = \begin{vmatrix} \cos\left(\frac{m\pi}{2}\right) & 1-\alpha_1 \\ -m\sin\left(\frac{m\pi}{2}\right) & -i\left\{a+b+f(\frac{\pi}{2})\right\}-\alpha_2 \end{vmatrix}$$

$$y'(0) = -\cos\left(\frac{m\pi}{2}\right)\left[i\left\{a+b+f(\frac{\pi}{2})\right\}+\alpha_2\right] + m\sin\left(\frac{m\pi}{2}\right)\left[1-\alpha_1\right].$$

Putting $y(0)$ and $y'(0)$ into equation (2.8), we get

$$y(t) = \cos(mt)\left\{\cos\left(\frac{m\pi}{2}\right)\right.$$

$$\left\{1-\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}$$

$$+\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\left\{-i\left\{a+b+f(\frac{\pi}{2})\right\}\right.$$

$$\left.\left.-\int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}\right\}$$

$$-\frac{1}{m}\sin(mt)\left\{\cos\left(\frac{m\pi}{2}\right)\left[i\left\{a+b+f(\frac{\pi}{2})\right\}\right.\right.$$

$$\left.+\int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right]$$

$$+m\sin\left(\frac{m\pi}{2}\right)\left\{1-\frac{1}{m}\int_0^{\frac{\pi}{2}}\sin\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}\Bigg\}$$

$$+\frac{1}{m}\int_0^{t}\sin(m(t-y))\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$=\cos mt\cos\frac{m\pi}{2}+\frac{1}{m}\cos mt\sin\frac{m\pi}{2}\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\sin mt\cos\frac{m\pi}{2}\left[i\{a+b+f(\frac{\pi}{2})\right]+\sin mt\sin\frac{m\pi}{2}$$

$$-\cos mt\cos\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sin\frac{m\pi}{2}(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\cos mt\sin\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cos m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\frac{1}{m}\sin mt\cos\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cos m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\sin mt\sin\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sin m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\int_0^{t}\sin m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$=\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\cos m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\sin m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\sin m(\frac{\pi}{2}-t)\int_0^{\frac{\pi}{2}}\cos m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\int_0^{t}\sin m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$= \cos m(t-\frac{\pi}{2}) + \frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_{0}^{\frac{\pi}{2}} \sin m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right] dy$$

$$+\frac{1}{m}\int_{0}^{t} \sin m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right] dy.$$

Therefore, the exact solution of this problem is

$$y(t) = \cos m(t-\frac{\pi}{2}) + \frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_{t}^{\frac{\pi}{2}} \sin m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right] dy. \tag{2.10}$$

In the second case $a^2-b^2=0$. Then,

$$y''(t) = F(t). \tag{2.11}$$

Applying the Laplace transform, we get

$$s^2 y(s) - sy(0) - y'(0) = F(s).$$

Then

$$y(s) = \frac{1}{s}y(0) + \frac{1}{s^2}y'(0) + \frac{1}{s^2}F(s).$$

$$y(s) = y(0)L\{1\} + y'(0)L\{t\} + L\{t\}F(s)$$

Taking the inverse Laplace transform, we get

$$y(t) = y(0) + ty'(0) + \int_{0}^{t} (t-y)F(y)\,dy. \tag{2.12}$$

From that it follows

$$y'(t) = y'(0) + \int_{0}^{t} F(y)\,dy.$$

Applying initial conditions $y(\frac{\pi}{2})=1, y'(\frac{\pi}{2})=-i\left\{a+b+f(\frac{\pi}{2})\right\}$, we obtain

$$1 = y(\frac{\pi}{2}) = y(0) + \frac{\pi}{2}\,y'(0) + \int_{0}^{\frac{\pi}{2}} \left(\frac{\pi}{2}-y\right)F(y)\,dy,$$

$$-i\left\{a+b+f(\frac{\pi}{2})\right\}=y'(\frac{\pi}{2})=y'(0)+\int_0^{\frac{\pi}{2}}F(y)\,dy.$$

Therefore,

$$y'(0)=-i\left\{a+b+f(\frac{\pi}{2})\right\}-\int_0^{\frac{\pi}{2}}F(y)\,dy,$$

$$y(0)=1-\frac{\pi}{2}\left\{-i\left\{a+b+f(\frac{\pi}{2})\right\}-\int_0^{\frac{\pi}{2}}F(y)\,dy\right\}-\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-y\right)F(y)\,dy$$

$$=1+\frac{\pi}{2}i\left\{a+b+f(\frac{\pi}{2})\right\}+\int_0^{\frac{\pi}{2}}yF(y)\,dy.$$

Putting y(0) and $y'(0)$ into equation (2.12), we get

$$y(t)=1+\frac{\pi}{2}i\left\{a+b+f(\frac{\pi}{2})\right\}+\int_0^{\frac{\pi}{2}}yF(y)\,dy$$

$$+t\left\{-i\left\{a+b+f(\frac{\pi}{2})\right\}-\int_0^{\frac{\pi}{2}}F(y)\,dy\right\}+\int_0^{t}(t-y)F(y)\,dy$$

$$=1+\left(\frac{\pi}{2}-t\right)i\left\{a+b+f(\frac{\pi}{2})\right\}-\int_0^{\frac{\pi}{2}}(t-y)F(y)\,dy+\int_0^{t}(t-y)F(y)\,dy$$

$$=1+\left(\frac{\pi}{2}-t\right)i\left\{a+b+f(\frac{\pi}{2})\right\}-\int_t^{\frac{\pi}{2}}(t-y)F(y)\,dy.$$

In the third case $a^2-b^2=m^2<0$. Substituting $-m^2$ for $a^2-b^2$ into equation (2.6a), we get

$$y''(t)-m^2y(t)=F(t).$$

Applying Laplace transform, we get

$$s^2y(s)-sy(0)-y'(0)-m^2y(s)=F(s)$$

or

$$y(s)=\frac{s}{s^2-m^2}y(0)+\frac{1}{s^2-m^2}y'(0)+\frac{1}{s^2-m^2}F(s).$$

Applying formulas

$$\frac{s}{s^2-m^2}=\frac{1}{2}\left[\frac{1}{s+m}+\frac{1}{s-m}\right], \frac{1}{s^2-m^2}$$
$$=\frac{1}{2m}\left[\frac{1}{s-m}-\frac{1}{s+m}\right],$$

we get

$$y(s)=\frac{1}{2}\left[\frac{1}{s+m}+\frac{1}{s-m}\right]y(0)$$
$$+\frac{1}{2m}\left[\frac{1}{s-m}-\frac{1}{s+m}\right]y'(0)+\frac{1}{2m}\left[\frac{1}{s-m}-\frac{1}{s+m}\right]F(s).$$

Applying formulas

$$L\left\{e^{\pm mt}\right\}=\frac{1}{s\mp m}, L\left\{\int_0^t e^{\pm m(t-y)}F(y)dy\right\}=\frac{1}{s\mp m}F(s)$$

and taking the inverse Laplace transform, we get

$$y(t)=\frac{1}{2}\left[e^{-mt}+e^{mt}\right]y(0)+\frac{1}{2im}\left[e^{mt}-e^{-mt}\right]y'(0)$$
$$+\frac{1}{2m}\int_0^t\left[e^{m(t-y)}-e^{-m(t-y)}\right]F(y)dy.$$

Using formulas

$$\cosh(mt)=\frac{1}{2}\left[e^{-mt}+e^{mt}\right],\ \sinh(mt)=\frac{1}{2i}\left[e^{mt}-e^{-mt}\right],$$

we get

$$y(t)=\cosh(mt)y(0)+\frac{1}{m}\sinh(mt)y'(0)+\frac{1}{m}\int_0^t\sinh(m(t-y))F(y)dy.$$

Now, we obtain $y(0)$ and $y'(0)$. Taking the derivative, we get

$$y'(t)=-m\sinh(mt)y(0)+\cosh(mt)\ y'(0)+\int_0^t\cosh(m(t-y))F(y)dy.$$

Putting $F(y)=-af(y)+bf(\pi-y)-if'(y)$, we get

$$y(t)=\cosh(mt)y(0)+\frac{1}{m}\sinh(mt)y'(0)$$

$$+\frac{1}{m}\int_0^t \sinh(m(t-y))\left[-af(y)+bf(\pi-y)-if'(y)\right]dy, \tag{2.13}$$

$$y'(t)=m\sinh(mt)\;y(0)+\cosh(mt)\;y'(0)$$

$$+\int_0^t \cosh(m(t-y))\left[-af(y)+bf(\pi-y)-if'(y)\right]dy. \tag{2.14}$$

Substituting $\frac{\pi}{2}$ for t into equations (2.13)and (2.14), we get

$$y(\frac{\pi}{2})=\cosh m\frac{\pi}{2}\;y(0)+\frac{1}{m}\sinh m\frac{\pi}{2}\;y'(0)$$

$$+\frac{1}{m}\int_0^{\frac{\pi}{2}} \sinh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy,$$

$$y'(\frac{\pi}{2})=m\sinh m\frac{\pi}{2}\;y(0)+\cosh m\frac{\pi}{2}\;y'(0)$$

$$+\int_0^{\frac{\pi}{2}} \cosh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy.$$

Applying initial conditions $y(\frac{\pi}{2})=1, y'(\frac{\pi}{2})=-i\left\{a+b+f(\frac{\pi}{2})\right\}$, we obtain

$$\begin{cases} \cosh\left(\frac{m\pi}{2}\right)\;y(0)+\frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)\;y'(0)=1-\alpha_1, \\ \\ m\sinh\left(\frac{m\pi}{2}\right)\;y(0)+\cosh\left(\frac{m\pi}{2}\right)\;y'(0)=-i\left\{a+b+f(\frac{\pi}{2})\right\}-\alpha_2. \end{cases}$$

Here

$$\alpha_1=\frac{1}{m}\int_0^{\frac{\pi}{2}} \sinh\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy,$$

$$\alpha_2=\int_0^{\frac{\pi}{2}} \cosh\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy.$$

Since

$$\Delta=\begin{vmatrix} \cosh\left(\frac{m\pi}{2}\right) & \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right) \\ m\sinh\left(\frac{m\pi}{2}\right) & \cosh\left(\frac{m\pi}{2}\right) \end{vmatrix}=\cosh^2 m\frac{\pi}{2}-\sinh^2 m\frac{\pi}{2}=1\neq 0,$$

we have that

$$y(0)=\frac{\Delta_0}{\Delta}=\begin{vmatrix} 1-\alpha_1 & \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right) \\ -i\{a+b+f(\frac{\pi}{2})\}-\alpha_2 & \cosh\left(\frac{m\pi}{2}\right) \end{vmatrix}$$

$$y(0)=\cosh\left(\frac{m\pi}{2}\right)\left[1-\alpha_1\right]+\frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)\left[-i\left\{a+b+f(\frac{\pi}{2})\right\}-\alpha_2\right],$$

$$y'(0)=\frac{\Delta_1}{\Delta}=\begin{vmatrix}\cosh\left(\frac{m\pi}{2}\right) & 1-\alpha_1\\ m\sinh\left(\frac{m\pi}{2}\right) & -i\left\{a+b+f(\frac{\pi}{2})\right\}-\alpha_2\end{vmatrix}$$

$$y'(0)=-\cosh\left(\frac{m\pi}{2}\right)\left[i\left\{a+b+f(\frac{\pi}{2})\right\}+\alpha_2\right]-m\sinh\left(\frac{m\pi}{2}\right)\left[1-\alpha_1\right].$$

Putting $y(0)$ and $y'(0)$ into equation (2.13), we get

$$y(t)=\cosh(mt)\left\{\cosh\left(\frac{m\pi}{2}\right)\right.$$

$$\left\{1-\frac{1}{m}\int_0^{\frac{\pi}{2}}\sinh\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}$$

$$+\frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)\left\{-i\left\{a+b+f(\frac{\pi}{2})\right\}\right.$$

$$\left.\left.-\int_0^{\frac{\pi}{2}}\cosh\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}\right\}$$

$$+\frac{1}{m}\sinh(mt)\left\{-\cosh\left(\frac{m\pi}{2}\right)\left\{i\left\{a+b+f(\frac{\pi}{2})\right\}\right.\right.$$

$$\left.+\int_0^{\frac{\pi}{2}}\cosh\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}$$

$$\left.\left.-m\sinh\left(\frac{m\pi}{2}\right)\left\{1-\frac{1}{m}\int_0^{\frac{\pi}{2}}\sin\left(m(\frac{\pi}{2}-y)\right)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy\right\}\right\}\right\}$$

$$+\frac{1}{m}\int_0^t\sin(m(t-y))\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$=\cosh mt\cosh\frac{m\pi}{2}-\frac{1}{m}\cosh mt\sinh\frac{m\pi}{2}\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\sinh mt\cosh\frac{m\pi}{2}\left[i\{a+b+f(\frac{\pi}{2})\right]-\sinh mt\sinh\frac{m\pi}{2}$$

$$-\cosh mt\cosh\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sinh\frac{m\pi}{2}(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\frac{1}{m}\cosh mt\sinh\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cosh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\frac{1}{m}\sinh mt\cosh\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cosh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\frac{1}{m}\sinh mt\sinh\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sinh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\int_0^{t}\sinh m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\cosh m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\sinh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\int_0^{\frac{\pi}{2}}\cosh m(\frac{\pi}{2}-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\int_0^{t}\sinh m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_0^{\frac{\pi}{2}}\sinh m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy$$

$$+\frac{1}{m}\int_0^{t}\sinh m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy.$$

Therefore, the exact solution of this problem is

$$y(t)=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[i\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_t^{\frac{\pi}{2}}\sinh m(t-y)\left[-af(y)+bf(\pi-y)-if'(y)\right]dy.$$

## 2.2 Schrödinger Type Involutory Partial Differential Equations

It is known that initial value problems for Schrödinger type involutory partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

First, we consider Fourier series method for solution of problems for Schrödinger type involutory partial differential equations.

**Example 2.2.1.** Obtain the Fourier series solution of the initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(\pi - t, x) = (-1+a)e^{it}\sin(x) - be^{-it}\sin(x), \\ x \in (0,\pi), -\infty < t < \infty, \\ u(\frac{\pi}{2}, x) = i\sin(x),\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, t \in (-\infty,\infty) \end{cases} \tag{2.15}$$

for one dimensional idempotent Schrödinger's equation.

**Solution.** In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0,\ 0 < x < \pi, u(0) = u(\pi) = 0.$$

generated by the space operator of problem (2.15). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = k^2,\ u_k(x) = \sin kx,\ k = 1,2,....$$

Then, we will obtain the Fourier series solution of problem (2.15) by formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t)\sin kx,$$

Here $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$i\sum_{k=1}^{\infty} A'_k(t)\sin kx + a\sum_{k=1}^{\infty} k^2 A_k(t)\sin kx + b\sum_{k=1}^{\infty} k^2 A_k(\pi - t)\sin kx$$

$$= (-1+a)\, e^{it}\sin(x) - be^{-it}\sin(x).$$

$$\sum_{k=1}^{\infty} A_k\left(\frac{\pi}{2}\right)\sin kx = i\sin(x), x\in[0,\pi],$$

$$u(\frac{\pi}{2}, x) = i\sin(x),\ \ x\in[0,\pi].$$

Equating coefficients $\sin kx, k = 1, 2, ...$ to zero, we get

$$\begin{cases} iA'_1(t) + aA_1(t) + bA_1(\pi - t) = (-1+a)\, e^{it} - be^{-it}, \\ \\ A_1\left(\frac{\pi}{2}\right) = i, \end{cases} \tag{2.16}$$

$$\begin{cases} iA'_k(t) - ak^2 A_k(t) - bk^2 A_k(\pi - t) = 0, k \neq 1, \\ \\ A_k\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{2.17}$$

We will obtain $A_1(t)$. Taking the derivative (2.16), we get

$$iA''_1(t) + aA'_1(t) - bA'_1(\pi - t) = i(-1+a)\, e^{it} + ibe^{-it}. \tag{2.18}$$

Putting $\pi - t$ instead of t, we get

$$iA'_1(\pi - t) + aA_1(\pi - t) + bA_1(t) = (-1+a)\, e^{i(\pi-t)} - be^{-i(\pi-t)}. \tag{2.19}$$

Multiplying equation (2.18) by i and equation (2.19) by b, we get

$$-A''_1(t) + aiA'_1(t) - biA'_1(\pi - t) = -(-1+a)\, e^{it} - be^{-it},$$

$$ibA'_1(\pi - t) + abA_1(\pi - t) + b^2 A_1(t) = b(-1+a)\, e^{i(\pi-t)} - b^2 e^{-i(\pi-t)}.$$

Adding last two equations, we get

$$-A''_1(t) + aiA'_1(t) + abA_1(\pi - t) + b^2 A_1(t)$$

$$= -(-1+a)\, e^{it} - be^{-it} + b(-1+a)\, e^{i(\pi-t)} - b^2 e^{-i(\pi-t)}.$$

Applying formulas

$$e^{-i(\pi-t)} = e^{-i\pi}e^{it} = (\cos(-\pi) + i\sin(-\pi))\, e^{it} = -e^{it},$$

$$e^{i(\pi-t)} = e^{i\pi}e^{-it} = (\cos(\pi) + i\sin(\pi))\, e^{-it} = -e^{-it},$$

$$e^{+i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i,$$

$$e^{-i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = -i,$$

we get

$$-A_1''(t) + aiA_1'(t) + abA_1(\pi - t) + b^2A_1(t) = (1 - a + b^2)e^{it} - abe^{-it}.$$

Multiplying equation (2.16) by (–a), we get

$$-aiA_1'(t) - a^2A_1(t) - abA_1(\pi - t) = \left(a - a^2\right) e^{it} + abe^{-it}.$$

Then, adding these equations, we get

$$-A_1''(t) + \left(b^2 - a^2\right) A_1(t) = \left(b^2 - a^2 + 1\right) e^{it}$$

or

$$A_1''(t) + \left(a^2 - b^2\right) A_1(t) = \left(a^2 - b^2 - 1\right) e^{it}. \tag{2.20}$$

Substituting $\frac{\pi}{2}$ for t into equation (2.16), we get

$$iA_1'\left(\frac{\pi}{2}\right) + aA_1\left(\frac{\pi}{2}\right) + bA_1\left(\pi - \frac{\pi}{2}\right) = (-1 + a)\, e^{i\left(\frac{\pi}{2}\right)} - be^{-i\left(\frac{\pi}{2}\right)}$$

or

$$iA_1'\left(\frac{\pi}{2}\right) + ai + bi = (-1 + a)\, i + bi.$$

Then

$$A_1'\left(\frac{\pi}{2}\right) = -1.$$

Therefore, we get the following problem

$$A_1''(t) + \left(a^2 - b^2\right) A_1(t) = \left(a^2 - b^2 - 1\right) e^{it}, \quad A_1\left(\frac{\pi}{2}\right) = i, A_1'\left(\frac{\pi}{2}\right) = -1.$$

There are three cases : $\quad a^2 - b^2 > 0, \quad a^2 - b^2 = 0, \quad a^2 - b^2 < 0.$

In the first case $a^2-b^2=m^2>0$. Substituting $m^2$ for $a^2-b^2$ into equation (2.20), we get

$$A_1''(t)+m^2A_1(t)=\left(a^2-b^2-1\right)e^{it}. \tag{2.21}$$

We will obtain Laplace transform solution of problem (2.21), we get

$$s^2A_1(s)-sA_1(0)-A_1'(0)+m^2A_1(s)=\left(a^2-b^2-1\right)e^{is}.$$

or

$$(s^2+m^2)A_1(s)=sA_1(0)+A_1'(0)+\left(a^2-b^2-1\right)e^{is}.$$

Then,

$$A_1(s)=\frac{s}{s^2+m^2}A_1(0)+\frac{1}{s^2+m^2}A_1'(0)+\frac{1}{s^2+m^2}\left(a^2-b^2-1\right)e^{is}.$$

Applying formulas

$$\frac{s}{s^2+m^2}=\frac{1}{2}\left[\frac{1}{s+im}+\frac{1}{s-im}\right],\frac{1}{s^2+m^2}=\frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right],$$

we get

$$A_1(s)=\frac{1}{2}\left[\frac{1}{s+im}+\frac{1}{s-im}\right]A_1(0)+\frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right]A_1'(0)$$
$$+\frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right]\left(a^2-b^2-1\right)e^{is}.$$

Taking the inverse Laplace transform, we get

$$A_1(t)=\frac{1}{2}\left[e^{-imt}+e^{imt}\right]A_1(0)+\frac{1}{2im}\left[e^{imt}-e^{-imt}\right]A_1'(0)$$
$$+\frac{\left(a^2-b^2-1\right)}{2im}\int_0^t\left[e^{im(t-y)}-e^{-im(t-y)}\right]e^{iy}dy.$$

Applying formulas

$$\cos(mt)=\frac{1}{2}\left[e^{-imt}+e^{imt}\right],\sin(mt)=\frac{1}{2i}\left[e^{imt}-e^{-imt}\right],$$

we get

$$A_1(t)=\cos(mt)A_1(0)+\frac{1}{m}\sin(mt)A_1'(0)$$
$$+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t\sin(m(t-y))e^{iy}dy. \tag{2.22}$$

Now, we obtain $A_1(0)$ and $A_1'(0)$. Taking the derivative, we get

$$A_1'(t) = -m\sin(mt)\, A_1(0)$$

$$+\cos(mt)\ A_1'(0) + \left(a^2-b^2-1\right)\int_0^t \cos(m(t-y))e^{iy}dy. \tag{2.23}$$

Substituting $\frac{\pi}{2}$ for t into equations (2.22)and (2.23), we get

$$A_1(\frac{\pi}{2}) = \cos\left(\frac{m\pi}{2}\right)A_1(0)$$

$$+\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)A_1'(0) + \frac{\left(a^2-b^2-1\right)}{m}\int_0^{\frac{\pi}{2}} \sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy,$$

$$A_1'(\frac{\pi}{2}) = -m\sin\left(\frac{m\pi}{2}\right)A_1(0) + \cos\left(\frac{m\pi}{2}\right)\ A_1'(0)$$

$$+\left(a^2-b^2-1\right)\int_0^{\frac{\pi}{2}} \cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy.$$

Applying initial conditions $A_1\left(\frac{\pi}{2}\right) = i, A_1'\left(\frac{\pi}{2}\right) = -1$, we obtain

$$\begin{cases} \cos\left(\frac{m\pi}{2}\right)A_1(0) + \frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\ A_1'(0) = i - \alpha_1, \\ \\ -m\sin\left(\frac{m\pi}{2}\right)\ A_1(0) + \cos\left(\frac{m\pi}{2}\right)A_1'(0) = -1-\alpha_2. \end{cases}$$

Here

$$\alpha_1 = \frac{\left(a^2-b^2-1\right)}{m}\int_0^{\frac{\pi}{2}} \sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy,$$

$$\alpha_2 = \left(a^2-b^2-1\right)\int_0^{\frac{\pi}{2}} \cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy.$$

Since

$$\Delta = \begin{vmatrix} \cos\left(\frac{m\pi}{2}\right) & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ -m\sin\left(\frac{m\pi}{2}\right) & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix} = \cos^2 m\frac{\pi}{2} + \sin^2 m\frac{\pi}{2} = 1 \neq 0,$$

we have that

$$A_1(0) = \frac{\Delta_0}{\Delta} = \begin{vmatrix} i-\alpha_1 & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ -1-\alpha_2 & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix}$$

$$A_1(0)=\cos\left(\frac{m\pi}{2}\right)\left[i-\alpha_1\right]+\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\left[1+\alpha_2\right],$$

$$A_1'(0)=\frac{\Delta_1}{\Delta}=\begin{vmatrix}\cos\left(\frac{m\pi}{2}\right) & i-\alpha_1\\ -m\sin\left(\frac{m\pi}{2}\right) & -1-\alpha_2\end{vmatrix}$$

$$A_1'(0)=-\cos\left(\frac{m\pi}{2}\right)\left[1+\alpha_2\right]+m\sin\left(\frac{m\pi}{2}\right)\left[i-\alpha_1\right].$$

Putting $A_1(0)$and $A_1'(0)$ into equation (2.22), we get

$$A_1(t)=\cos(mt)\left\{\cos\left(\frac{m\pi}{2}\right)\left\{i-\frac{\left(a^2-b^2-1\right)}{m}\int_0^{\frac{\pi}{2}}\sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy\right\}\right.$$

$$\left.+\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\left\{1+\left(a^2-b^2-1\right)\int_0^{\frac{\pi}{2}}\cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy\right\}\right\}$$

$$+\frac{1}{m}\sin(mt)\left\{-\cos\left(\frac{m\pi}{2}\right)\left[1+\left(a^2-b^2-1\right)\int_0^{\frac{\pi}{2}}\cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy\right]\right.$$

$$\left.+m\sin\left(\frac{m\pi}{2}\right)\left\{i-\frac{\left(a^2-b^2-1\right)}{m}\int_0^{\frac{\pi}{2}}\sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy\right\}\right\}$$

$$+\frac{\left(a^2-b^2-1\right)}{m}\int_0^{t}\sin(m(t-y))e^{iy}dy$$

$$=i\cos mt\cos\frac{m\pi}{2}-\frac{\left(a^2-b^2-1\right)}{m}\cos mt\cos\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy$$

$$+\frac{1}{m}\cos mt\sin\frac{m\pi}{2}+\frac{\left(a^2-b^2-1\right)}{m}\cos mt\sin\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy$$

$$-\sin mt\cos\frac{m\pi}{2}-\frac{\left(a^2-b^2-1\right)}{m}\sin mt\cos\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy$$

$$-i\sin mt\sin\frac{m\pi}{2}-\left(a^2-b^2-1\right)\sin mt\sin\frac{m\pi}{2}\int_0^{\frac{\pi}{2}}\sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy$$

$$+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t \sin(m(t-y))e^{iy}dy$$

$$=i\cos m(t-\frac{\pi}{2})-\frac{1}{m}\sin m(t-\frac{\pi}{2})-\frac{\left(a^2-b^2-1\right)}{m}\cos m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\sin\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy$$

$$-\frac{\left(a^2-b^2-1\right)}{m}\sin m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\cos\left(m\left(\frac{\pi}{2}-y\right)\right)e^{iy}dy+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t \sin(m(t-y))e^{iy}dy$$

$$=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)-\frac{\left(a^2-b^2-1\right)}{m}\int_0^{\frac{\pi}{2}}\sin(m(t-y))e^{iy}dy$$

$$+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t \sin(m(t-y))e^{iy}dy$$

$$=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)-\frac{\left(a^2-b^2-1\right)}{m}\int_t^{\frac{\pi}{2}}\sin(m(t-y))e^{iy}dy.$$

Therefore, the exact solution of this problem is

$$A_1(t)=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)-\frac{\left(a^2-b^2-1\right)}{m}\int_t^{\frac{\pi}{2}}\sin(m(t-y))e^{iy}dy. \tag{2.24}$$

It is easy to see that

$$A_1(t)=i\cos\left(\pm(t-\frac{\pi}{2})\right)+\frac{1}{\pm 1}\sin\left(\pm(\frac{\pi}{2}-t)\right)=i\cos\left(\frac{\pi}{2}-t\right)-\sin\left(\frac{\pi}{2}-t\right)$$

$$=i\frac{e^{i\left(\frac{\pi}{2}-t\right)}+e^{-i\left(\frac{\pi}{2}-t\right)}}{2}+\frac{e^{i\left(\frac{\pi}{2}-t\right)}-e^{-i\left(\frac{\pi}{2}-t\right)}}{2i}=ie^{-i\left(\frac{\pi}{2}-t\right)}=e^{it}$$

for $m^2=1$. Now, we obtain $A_1(t)$ for $m^2\neq 1$. We denote

$$I=\int \sin(m(t-y))e^{iy}dy.$$

We have that

$$I=\frac{1}{i}\sin(m(t-y))e^{iy}+\frac{m}{i}\int\cos(m(t-y))e^{iy}dy$$

$$=\frac{1}{i}\sin(m(t-y))e^{iy}-m\cos(m(t-y))e^{iy}+m^2\int\sin(m(t-y))e^{iy}dy.$$

Therefore,

$$I\left(1-m^2\right)=\frac{1}{i}\sin(m(t-y))e^{iy}-m\cos(m(t-y))e^{iy}$$

or

$$I=\frac{1}{1-m^2}\left\{\frac{1}{i}\sin(m(t-y))e^{iy}-m\cos(m(t-y))e^{iy}\right\}. \tag{2.25}$$

Therefore,

$$\int_t^{\frac{\pi}{2}}\sin(m(t-y))e^{iy}dy=\frac{1}{1-m^2}\left[\frac{1}{i}\sin\left(m\left(t-\frac{\pi}{2}\right)\right)e^{i\frac{\pi}{2}}-m\cos\left(m\left(t-\frac{\pi}{2}\right)\right)e^{i\frac{\pi}{2}}+me^{it}\right]$$

$$=\frac{1}{1-m^2}\left[\frac{1}{i}i\sin\left(m\left(t-\frac{\pi}{2}\right)\right)-mi\cos\left(m\left(t-\frac{\pi}{2}\right)\right)+me^{it}\right] \tag{2.26}$$

Putting (2.26) into equation (2.24), we get

$$A_1(t)=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)-$$

$$\frac{\left(a^2-b^2-1\right)}{m}\left\{\frac{1}{1-m^2}\left[\frac{1}{i}i\sin\left(m\left(t-\frac{\pi}{2}\right)\right)-mi\cos\left(m\left(t-\frac{\pi}{2}\right)\right)+me^{it}\right]\right\}$$

$$=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)$$

$$-\frac{\left(m^2-1\right)}{m}\left\{\frac{1}{1-m^2}\left[\sin\left(m\left(t-\frac{\pi}{2}\right)\right)-mi\cos\left(m\left(t-\frac{\pi}{2}\right)\right)+me^{it}\right]\right\}$$

$$=i\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)$$

$$+\frac{1}{m}\sin\left(m\left(t-\frac{\pi}{2}\right)\right)-i\cos\left(m\left(t-\frac{\pi}{2}\right)\right)+e^{it}=e^{it}.$$

Therefore

$$A_1(t)=e^{it}.$$

It is easy to see that $A_1(t)=e^{it}$ for $a^2-b^2=0$ and $a^2-b^2<0$.

Now, we will obtain $A_k(t)$ for $k\neq 1$. We consider the problem (2.17). Taking the derivative (2.17), we get

$$iA_k''(t)+ak^2A_k'(t)-bk^2A_k'(\pi-t)=0. \tag{2.27}$$

Putting $\pi - t$ instead of t, we get

$$iA_k'(\pi - t) + aA_k(\pi - t) + bA_k(t) = 0. \tag{2.28}$$

Multiplying equation (2.27) by i and equation (2.28) by $bk^2$, we get

$$-A_k''(t) + aik^2 A_k'(t) - ibk^2 A_k'(\pi - t) = 0,$$

$$ibk^2 A_k'(\pi - t) + abk^2 A_k(\pi - t) + b^2 k^2 A_k(t) = 0.$$

Adding last two equations, we get

$$-A_k''(t) + aik^2 A_k'(t) + abk^4 A_k(\pi - t) + b^2 k^4 A_k(t) = 0.$$

Multiplying equation (2.17) by $(-ak^2)$, we get

$$-aik^2 A_k'(t) - a^2 k^4 A_k(t) - abk^4 A_k(\pi - t) = 0.$$

Then adding these equations, we get

$$-A_k''(t) - \left(a^2 - b^2\right) k^4 A_k(t) = 0$$

or

$$A_k''(t) + \left(a^2 - b^2\right) k^4 A_k(t) = 0. \tag{2.29}$$

Substituting $\frac{\pi}{2}$ for t into equation (2.17), we get

$$iA_k'\left(\frac{\pi}{2}\right) + ak^2 A_k\left(\frac{\pi}{2}\right) + bk^2 A_k\left(\pi - \frac{\pi}{2}\right) = 0.$$

Then

$$A_k'\left(\frac{\pi}{2}\right) = 0.$$

So, we have the following problem

$$A_k''(t) + \left(a^2 - b^2\right) k^4 A_k(t) = 0, \quad A_k\left(\frac{\pi}{2}\right) = 0, A_k'\left(\frac{\pi}{2}\right) = 0.$$

There are three cases : $a^2 - b^2 > 0$, $a^2 - b^2 = 0$, $a^2 - b^2 < 0$. In the first case $a^2 - b^2 = m^2 > 0$. Substituting $m^2$ for $a^2 - b^2$ into equation (2.29), we get

$$A_k''(t) + m^2 k^4 A_k(t) = 0. \tag{2.30}$$

We will obtain Laplace transform solution of problem (2.30), we get

$$s^2 A_k(s) - sA_k(0) - A'_k(0) + m^2k^4 A_k(s) = 0.$$

or

$$(s^2 + m^2k^4)A_k(s) = sA_k(0) + A'_k(0).$$

Then,

$$A_k(s) = \frac{s}{s^2 + m^2k^4} A_k(0) + \frac{1}{s^2 + m^2k^4} A'_k(0).$$

Applying formulas

$$\frac{s}{s^2 + m^2k^4} = \frac{1}{2}\left[\frac{1}{s + imk^2} + \frac{1}{s - imk^2}\right], \frac{1}{s^2 + m^2k^4}$$
$$= \frac{1}{2imk^2}\left[\frac{1}{s - imk^2} - \frac{1}{s + imk^2}\right],$$

we get

$$A_k(s) = \frac{1}{2}\left[\frac{1}{s + imk^2} + \frac{1}{s - imk^2}\right] A_k(0) + \frac{1}{2im}\left[\frac{1}{s - imk^2} - \frac{1}{s + imk^2}\right] A'_k(0).$$

Taking the inverse Laplace transform, we get

$$A_k(t) = \frac{1}{2}\left[e^{-imk^2t} + e^{imk^2t}\right] A_k(0) + \frac{1}{2imk^2}\left[e^{imk^2t} - e^{-imk^2t}\right] A'_k(0).$$

Applying formulas

$$\cos\left(mk^2t\right) = \frac{1}{2}\left[e^{-imk^2t} + e^{imk^2t}\right], \sin\left(mk^2t\right) = \frac{1}{2i}\left[e^{imk^2t} - e^{-imk^2t}\right],$$

we get

$$A_k(t) = \cos\left(mk^2t\right) A_k(0) + \frac{1}{mk^2}\sin\left(mk^2t\right) A'_k(0). \tag{2.31}$$

Now, we obtain $A_k(0)$ and $A'_k(0)$. Taking the derivative, we get

$$A'_k(t) = -mk^2 \sin\left(mk^2t\right) A_k(0) + \cos\left(mk^2t\right) A'_k(0). \tag{2.32}$$

Substituting $\frac{\pi}{2}$ for t into equations (2.31) and (2.32), we get

$$A_k(\frac{\pi}{2}) = \cos\left(\frac{mk^2\pi}{2}\right) A_k(0) + \frac{1}{mk^2}\sin\left(\frac{mk^2\pi}{2}\right) A'_k(0),$$

$$A_k'(\frac{\pi}{2}) = -mk^2 \sin\left(\frac{mk^2\pi}{2}\right) A_k(0) + \cos\left(\frac{mk^2\pi}{2}\right) A_k'(0).$$

Applying initial conditions $A_k\left(\frac{\pi}{2}\right) = 0, A_k'\left(\frac{\pi}{2}\right) = 0$, we obtain

$$\begin{cases} \cos\left(\frac{mk^2\pi}{2}\right) A_k(0) + \frac{1}{m}\sin\left(\frac{mk^2\pi}{2}\right) A_k'(0) = 0, \\ \\ -m\sin\left(\frac{mk^2\pi}{2}\right) A_k(0) + \cos\left(\frac{mk^2\pi}{2}\right) A_k'(0) = 0. \end{cases}$$

Since

$$\Delta = \begin{vmatrix} \cos\left(\frac{mk^2\pi}{2}\right) & \frac{1}{m}\sin\left(\frac{mk^2\pi}{2}\right) \\ -m\sin\left(\frac{mk^2\pi}{2}\right) & \cos\left(\frac{mk^2\pi}{2}\right) \end{vmatrix} = \cos^2 mk^2\frac{\pi}{2} + \sin^2 mk^2\frac{\pi}{2} = 1 \neq 0,$$

we have that

$$A_k(0) = \frac{\Delta_0}{\Delta} = \begin{vmatrix} 0 & \frac{1}{mk^2}\sin\left(\frac{mk^2\pi}{2}\right) \\ 0 & \cos\left(\frac{mk^2\pi}{2}\right) \end{vmatrix} = 0,$$

$$A_k'(0) = \frac{\Delta_1}{\Delta} = \begin{vmatrix} \cos\left(\frac{mk^2\pi}{2}\right) & 0 \\ -mk^2\sin\left(\frac{mk^2\pi}{2}\right) & 0 \end{vmatrix} = 0.$$

Putting $A_k(0)$and $A_k'(0)$ into equation (2.31), we get

$$A_k(t) = \cos\left(mk^2t\right)(0) + \frac{1}{mk^2}\sin\left(mk^2t\right)(0) = 0.$$

It is easy to see that $A_k(t) = 0, k \neq 1$ for $a^2 - b^2 = 0$ and $a^2 - b^2 < 0$.

Therefore,

$$u(t,x) = A_1(t)\sin x = e^{it}\sin x$$

is the exact solution of problem (2.15).

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t}-a\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}-b\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2}=f(t,x), \\ x=(x_1,...,x_n)\in\overline{\Omega},\ -\infty<t<\infty, \\ u(\frac{d}{2},x)=\varphi(x), x\in\overline{\Omega}, d\geq 0, \\ u(t,x)=0, x\in S, t\in(-\infty,\infty) \end{cases} \tag{2.33}$$

for the multidimensional involutory Schrödinger type equation. Assume that $\alpha_r>\alpha>0$ and $f(t,x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$, $\varphi(x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$ are given smooth functions. Here and in future $\Omega$ is the unit open cube in the n–dimensional Euclidean space $\mathbb{R}^n\,(0<x_k<1, 1\leq k\leq n)$ with the boundary

$$S, \overline{\Omega}=\Omega\cup S.$$

However Fourier series method described in solving (2.33) can be used only in the case when (2.33) has constant coefficients.

**Example 2.2.2.** Obtain the Fourier series solution of the initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t}-au_{xx}(t,x)-bu_{xx}(-t,x)=(-1+a)e^{it}\cos(x)+be^{-it}\cos(x), \\ x\in(0,\pi), -\infty<t<\infty, \\ u(0,x)=\cos(x),\ x\in[0,\pi], \\ u_x(t,0)=u_x(t,\pi)=0, t\in(-\infty,\infty) \end{cases} \tag{2.34}$$

for one dimensional involutory Schrödinger's equation.

**Solution.** In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x)-\lambda u(x)=0,\ \ 0<x<\pi,\ u_x(0)=u_x(\pi)=0$$

generated by the space operator of problem (2.34). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = k^2,\ u_k(x) = \cos kx,\ k = 0, 1, 2, \ldots.$$

Then, we will obtain the Fourier series solution of problem (2.34) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos kx,$$

Here $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$i\sum_{k=1}^{\infty} A_k'(t)\cos kx + a\sum_{k=1}^{\infty} k^2 A_k(t)\cos kx + b\sum_{k=1}^{\infty} k^2 A_k(-t)\cos kx$$

$$= (-1+a)\,e^{it}\cos(x) + be^{-it}\cos(x),$$

$$\sum_{k=1}^{\infty} A_k(0)\cos kx = \cos(x), x \in [0, \pi].$$

Equating coefficients $\cos kx, k = 0, 1, 2, \ldots$to zero, we get

$$\begin{cases} iA_1'(t) + aA_1(t) + bA_1(-t) = (-1+a)\,e^{it} + be^{-it}, \\ \\ A_1(0) = 1, \end{cases} \tag{2.35}$$

$$\begin{cases} iA_k'(t) + ak^2A_k(t) + bk^2A_k(-t) = 0, k \neq 1, \\ \\ A_k(0) = 0. \end{cases} \tag{2.36}$$

We will obtain $A_1(t)$. Taking the derivative of (2.35), we get

$$iA_1''(t) + aA_1'(t) - bA_1'(-t) = i\,(-1+a)\,e^{it} - ibe^{-it}. \tag{2.37}$$

Putting $-t$ instead of t, we get

$$iA_1'(-t) + aA_1(-t) + bA_1(t) = (-1+a)\,e^{i(-t)} - be^{-i(-t)}. \tag{2.38}$$

Multiplying equation (2.37) by i and equation (2.38) by b, we get

$$-A_1''(t) + aiA_1'(t) - biA_1'(-t) = -(-1+a)\,e^{it} + be^{-it},$$

$$ibA_1'(-t) + abA_1(-t) + b^2A_1(t) = b(-1+a)e^{-it} + b^2e^{it}.$$

Adding last two equations, we get

$$-A_1''(t) + aiA_1'(t) + abA_1(-t) + b^2A_1(t) = (1-a+b^2)e^{it} + abe^{-it}.$$

Multiplying equation (2.35) by (–a), we get

$$-aiA_1'(t) - a^2A_1(t) - abA_1(-t) = \left(a - a^2\right)e^{it} - abe^{-it}.$$

Then from these equations, we get

$$-A_1''(t) + \left(b^2 - a^2\right)A_1(t) = \left(b^2 - a^2 + 1\right)e^{it}$$

or

$$A_1''(t) + \left(a^2 - b^2\right)A_1(t) = \left(a^2 - b^2 - 1\right)e^{it}. \tag{2.39}$$

Substituting (0) for t into (2.16) equation, we get

$$iA_1'(0) + aA_1(0) + bA_1(0) = (-1+a)e^{i(0)} + be^{-i(0)}$$

$$iA_1'(0) + a + b = (-1+a) + b.$$

or

$$A_1'(0) = i.$$

So, we have the following problem

$$A_1''(t) + \left(a^2 - b^2\right)A_1(t) = \left(a^2 - b^2 - 1\right)e^{it}, \quad A_1(0) = 1, A_1'(0) = i.$$

There are three cases : $\quad a^2 - b^2 > 0, \quad a^2 - b^2 = 0, \quad a^2 - b^2 < 0.$

In the first case $a^2 - b^2 = m^2 > 0$. Substituting $m^2$ for $a^2 - b^2$ into equation (2.39), we get

$$A_1''(t) + m^2A_1(t) = \left(a^2 - b^2 - 1\right)e^{it}. \tag{2.40}$$

We will obtain Laplace transform solution of problem (2.40). We have that

$$s^2A_1(s) - sA_1(0) - A_1'(0) + m^2A_1(s) = \left(a^2 - b^2 - 1\right)e^{is}.$$

or

$$(s^2+m^2)A(s) = sA_1(0)+A_1'(0)+\left(a^2-b^2-1\right)e^{is}.$$

Then,

$$A(s) = \frac{s}{s^2+m^2}A_1(0)+\frac{1}{s^2+m^2}A_1'(0)+\frac{1}{s^2+m^2}\left(a^2-b^2-1\right)e^{is}.$$

Applying formulas

$$\frac{s}{s^2+m^2} = \frac{1}{2}\left[\frac{1}{s+im}+\frac{1}{s-im}\right], \frac{1}{s^2+m^2} = \frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right],$$

we get

$$A(s) = \frac{1}{2}\left[\frac{1}{s+im}+\frac{1}{s-im}\right]A_1(0)+\frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right]A_1'(0)$$
$$+\frac{1}{2im}\left[\frac{1}{s-im}-\frac{1}{s+im}\right]\left(a^2-b^2-1\right)e^{is}.$$

Applying formulas

$$L\left\{e^{\pm imt}\right\} = \frac{1}{s\mp im}, L\left\{\int_0^t e^{\pm im(t-y)}F(y)dy\right\} = \frac{1}{s\mp im}F(s)$$

Taking the inverse Laplace transform, we get

$$A_1(t) = \frac{1}{2}\left[e^{-imt}+e^{imt}\right]A_1(0)+\frac{1}{2im}\left[e^{imt}-e^{=imt}\right]A_1'(0)$$
$$+\frac{\left(a^2-b^2-1\right)}{2im}\int_0^t\left[e^{im(t-y)}-e^{=im(t-y)}\right]e^{iy}dy.$$

Applying formulas

$$\cos(mt) = \frac{1}{2}\left[e^{-imt}+e^{imt}\right], \sin(mt) = \frac{1}{2i}\left[e^{imt}-e^{=imt}\right],$$

we get

$$A_1(t) = \cos(mt)A_1(0)+\frac{1}{m}\sin(mt)A_1'(0)+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t\sin(m(t-y))e^{iy}dy. \qquad (2.41)$$

Applying initial conditions $A_1(0) = 1, A_1'(0) = i$, we obtain

$$A_1(t) = \cos(mt)+i\frac{1}{m}\sin(mt)+\frac{\left(a^2-b^2-1\right)}{m}\int_0^t\sin(m(t-y))e^{iy}dy.$$

It is easy to see that

$$A_1(t) = \cos(\pm t) + i\frac{1}{\pm 1}\sin(\pm t) = \cos(t) + i\sin(t)$$

$$= \frac{e^{it} + e^{-it}}{2} + i\frac{e^{it} - e^{-it}}{2i} = e^{it}$$

for $m^2 = 1$. Now, we obtain $A_1(t)$ for $m^2 \neq 1$. Applying (2.25), we get

$$A_1(t) = \cos(mt) + i\frac{1}{m}\sin(mt) - \cos(mt) - i\frac{1}{m}\sin(mt) + e^{it} = e^{it}.$$

So, $A_1(t) = e^{it}$. It is easy to see that $A_1(t) = e^{it}$ for $a^2 - b^2 = 0, \quad a^2 - b^2 < 0$.

Now, we will obtain $A_k(t)$ for $k \neq 1$. We consider the problem (2.36), we get

$$iA_k''(t) + ak^2A_k'(t) - bk^2A_k'(-t) = 0. \tag{2.42}$$

Putting $-t$ instead of $t$, we get

$$iA_k'(-t) + ak^2A_k(-t) + bk^2A_k(t) = 0. \tag{2.43}$$

Multiplying equation (2.42) by $i$ and equation (2.43) by $bk^2$, we get

$$-A_k''(t) + aik^2A_k'(t) - bik^2A_k'(-t) = 0,$$

$$ibk^2A_k'(-t) + abk^4A_k(-t) + b^2k^4A_k(t) = 0.$$

Adding last two equations, we get

$$-A_k''(t) + aik^2A_k'(t) + abk^4A_k(-t) + b^2k^4A_k(t) = 0.$$

Multiplying equation (2.36) by $(-ak^2)$, we get

$$-aik^2A_k'(t) - a^2k^4A_k(t) - abk^4A_k(-t) = 0.$$

Then from these equations, we get

$$-A_k''(t) - \left(a^2 - b^2\right)A_k(t) = 0$$

or

$$A_k''(t) + \left(a^2 - b^2\right)A_k(t) = 0. \tag{2.44}$$

Substituting (0) for t into equation (2.36), we get

$$iA_k'(0) + ak^2 A_k(0) + bk^2 A_k(\pi - 0) = 0$$

or

$$A_k'(0) = 0.$$

We have the following problem

$$A_k''(t) + \left(a^2 - b^2\right) k^2 A_k(t) = 0, \quad A_k(0) = 0, A_k'(0) = 0.$$

From that it follows $A_k(t) = 0, k \neq 1$. In the same manner $A_k(t) = 0, k \neq 1$ for $a^2 - b^2 = 0$ and $a^2 - b^2 < 0$.

Therefore,

$$u(t,x) = A_1(t)\cos x = e^{it}\cos x$$

is the exact solution of problem (2.34).

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \overline{\Omega}, d \geq 0, \\ \frac{\partial u(t,x)}{\partial \overline{m}} = 0, x \in S, t \in (-\infty, \infty) \end{cases} \tag{2.45}$$

for the multidimensional involutory Schrödinger type equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$, $\varphi(x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to S. However Fourier series method described in solving (2.45) can be used only in the case when (2.45) has constant coefficients.

**Example 2.2.3.** Obtain the Fourier series solution of the initial-boundary value problem

$$\begin{cases} iu_t(t,x) - au_{xx}(t,x) - bu_{xx}(-t,x) = -e^{-it}, -\infty < t < \infty, 0 < x < \pi, \\ u(0,x) = 1,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), \quad u_x(t,0) = u_x(t,\pi),\ \ t \in I \end{cases} \tag{2.46}$$

for one dimensional involutory Schrödinger's equation.

**Solution.** In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0,\ \ 0 < x < \pi,\ u(0) = u(\pi),\ u_x(0) = u_x(\pi)$$

generated by the space operator of problem (2.46). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = 4k^2,\ u_k(x) = \cos 2kx,\ k = 0,1,2,...,\ u_k(x) = \sin 2kx,\ k = 1,2,....$$

Then, we will obtain the Fourier series solution of problem (2.46) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx, \tag{2.47}$$

where $A_k(t)$, $k = 0,1,2,....,$ and $B_k(t)$, $k = 1,2,....$ are unknown functions. Putting formula (2.46) into the main problem and using given initial condition, we obtain

$$i\sum_{k=0}^{\infty} A_k'(t)\cos 2kx + i\sum_{k=1}^{\infty} B_k'(t)\sin 2kx - a\sum_{k=0}^{\infty} 4k^2 A_k(t)\cos 2kx$$

$$-a\sum_{k=1}^{\infty} 4k^2 B_k(t)\sin 2kx - b\sum_{k=0}^{\infty} 4k^2 A_k(-t)\cos 2kx - b\sum_{k=1}^{\infty} 4k^2 B_k(-t)\sin 2kx$$

$$= -e^{-it}, t\varepsilon I,\ x\varepsilon(0,\pi),$$

$$\sum_{k=0}^{\infty} A_k(0)\cos 2kx + \sum_{k=1}^{\infty} B_k(0)\sin 2kx = 1,\ 0 \le x \le \pi,$$

Equating the coefficients of $\cos kx$, $k = 0,1,2,...,$ and $\sin kx$, $k = 1,2,...$ to zero, we get

$$\begin{cases} iB_k'(t) - 4ak^2 B_k(t) - 4bk^2 B_k(-t) = 0,\ t\ \varepsilon I, \\ B_k(0) = 0,\ \ k = 1,2,..., \end{cases} \tag{2.48}$$

$$\begin{cases} iA_0'(0) = -e^{-it}, \quad t\,\varepsilon I, \\ \\ A_0(0) = 1, \end{cases} \tag{2.49}$$

$$\begin{cases} iA_k'(t) - 4ak^2A_k(t) - 4bk^2A_k(-t) = 0, \; t\varepsilon I, \\ \\ A_k(0) = 0, \quad k = 1, 2, .... \end{cases} \tag{2.50}$$

First, we obtain $A_0(t)$. Using (2.49), we get

$$A_0'(t) = ie^{it}.$$

Taking the integral, we get

$$A_0(t) = A_0(0) + e^{it} - 1.$$

From that it follows

$$A_0(t) = e^{it}.$$

Second, we obtain $A_k(t)$ for $k \neq 0$. Using (2.50), we get

$$iA_k'(-t) - 4ak^2A_k(-t) - 4bk^2A_k(t) = 0,$$

$$iA_k''(t) - 4ak^2A_k'(t) + 4bk^2A_k'(-t) = 0.$$

From first equation it follows that

$$-4k^2bA_k'(-t) - 16k^4iabA_k(-t) - 16k^4biA_k(t) = 0.$$

Therefore,

$$iA_k''(t) - 4ak^2A_k'(t) - 4bk^2A_k'(-t) + 4bk^2A_k'(-t) - 16k^4iabA_k(-t) - 16k^4biA_k(t) = 0.$$

or

$$iA_k''(t) - 4ak^2A_k'(t) - 16k^4iabA_k(-t) - 16k^4biA_k(t) = 0.$$

Using (2.50), we get

$$4ak^2A_k'(t) + 16k^4iabA_k(-t) + 16k^4biA_k(t) = 0.$$

From last two equations it follows

$$iA_k''(t) + 16k^4(a^2 - b^2)iA_k(t) = 0, t\varepsilon I.$$

Applying equation (2.50) and initial condition, we get

$$A_k'(0) = 0.$$

Therefore, we get the following Cauchy problem

$$A_k''(t) + 16k^4(a^2 - b^2)A_k(t) = 0, t \in I, A_k(0) = 0,\ A_k'(0) = 0.$$

The auxiliary equation is

$$q^2 + 16k^4(a^2 - b^2) = 0.$$

There are three cases: $a^2 - b^2 > 0$, $a^2 - b^2 = 0$, $a^2 - b^2 < 0$. In the first case $16k^4(a^2 - b^2) = m^2$. Then

$$A_k(t) = \cos(mt)\, A_k(0) + \frac{1}{m}\sin(mt)\, A_k'(0) = 0.$$

In the second case $16k^4(a^2 - b^2) = 0$. Then

$$A_k(t) = A_k(0) + A_k'(0)t = 0.$$

In the third case $16k^4(a^2 - b^2) = -m^2$. Then

$$A_k(t) = \cosh(mt)\, A_k(0) + \frac{1}{m}\sinh(mt)\, A_k'(0) = 0.$$

So, $A_k(t) = 0$ for any $t \in I$. In the same manner, we can obtain $B_k(t) = 0$ for any $t \in I$. Therefore,

$$u(t, x) = A_0(t) = e^{it}.$$

is the exact solution of problem (2.46).

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1,...,x_n)\in\overline{\Omega},\ -\infty < t < \infty, \\ u(\frac{d}{2},x) = \varphi(x), x\in\overline{\Omega}, d\geq 0, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \left.\frac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_1} = \left.\frac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_2}, t\in(-\infty,\infty) \end{cases} \tag{2.51}$$

for the multidimensional involutory Schrödinger type equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$, $\varphi(x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$ are given smooth functions. Here $S = S_1\cup S_2, \varnothing = S_1\cap S_2$. However Fourier series method described in solving (2.51) can be used only in the case when (2.51) has constant coefficients.

Second, we consider Laplace transform solution of problems for Schrödinger type involutory partial differential equations.

**Example 2.2.4.** Obtain the Laplace transform solution of the initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = (-1-a)e^{it}e^{-x} - be^{-it}e^{-x}, \\ x\in(0,\infty), -\infty < t < \infty, \\ u(0,x) = e^{-x},\ x\in[0,\infty), \\ u(t,0) = e^{it}, u_x(t,0) = -e^{it}, t\in(-\infty,\infty) \end{cases} \tag{2.52}$$

for one dimensional involutory Schrödinger's equation.

**Solution.** We will obtain Laplace transform solution of problem (2.52). We denote that

$$u(t,s) = L\{u(t,x)\}.$$

Taking the Laplace transform, we get

$$iu_t(t,s)-a\left\{s^2u(t,s)-se^{it}-\left(-e^{it}\right)\right\}-b\left\{s^2u(-t,s)-se^{-it}-\left(-e^{-it}\right)\right\}$$

$$=-(1+a)e^{it}\frac{1}{1+s}-be^{-it}\frac{1}{1+s},u(0,s)=\frac{1}{1+s}$$

or

$$iu_t(t,s)-as^2u(t,s)-bs^2u(-t,s)=a(s)e^{it}-b(s)e^{-it},u(0,s)=\frac{1}{1+s}. \tag{2.53}$$

Here

$$a(s)=-\frac{as^2+1}{1+s},b(s)=\frac{bs^2}{1+s}. \tag{2.54}$$

Taking the derivative (2.53), we get

$$iu_{tt}(t,s)-as^2u_t(t,s)+bs^2u_t(-t,s)=ia(s)e^{it}+ib(s)e^{-it}. \tag{2.55}$$

Putting –t instead of t into equation (2.53), we get

$$iu_t(-t,s)-as^2u(-t,s)-bs^2u(t,s)=a(s)e^{-it}-b(s)e^{it}. \tag{2.56}$$

Multiplying equation (2.55) by (–i) and equation (2.56) by $\left(bs^2\right)$, we get

$$u_{tt}(t,s)+ias^2u_t(t,s)-ibs^2u_t(-t,s)=a(s)e^{it}+b(s)e^{-it},$$

$$ibs^2u_t(-t,s)-abs^4u(-t,s)-b^2s^4u(t,s)=bs^2a(s)e^{-it}-bs^2b(s)e^{it}.$$

Adding last two equations, we get

$$u_{tt}(t,s)+ias^2u_t(t,s)-abs^4u(-t,s)-b^2s^4u(t,s)$$

$$=\left(a(s)-bs^2b(s)\right)e^{it}+\left(b(s)+bs^2a(s)\right)e^{-it}.$$

Multiplying equation (2.53) by $\left(-as^2\right)$, we get

$$-as^2iu_t(t,s)+a^2s^4u(t,s)+abs^4u(-t,s)=-as^2a(s)e^{it}+as^2b(s)e^{-it}.$$

Adding last two equations, we get

$$u_{tt}(t,s)-b^2s^4u(t,s)+a^2s^4u(t,s)$$

$$= \left(a(s) - as^2 a(s) - bs^2 b(s)\right) e^{it} + \left(b(s) + as^2 b(s) + bs^2 a(s)\right) e^{-it}.$$

Using notations (2.54), we get

$$u_{tt}(t,s) + \left(a^2 s^4 - b^2 s^4\right) u(t,s) = \left(-\frac{as^2+1}{1+s} + as^2 \frac{as^2+1}{1+s} - bs^2 \frac{bs^2}{1+s}\right) e^{it}$$

$$+ \left(\frac{bs^2}{1+s} + as^2 \frac{bs^2}{1+s} - bs^2 \frac{as^2+1}{1+s}\right) e^{-it}$$

or

$$u_{tt}(t,s) + \left(a^2 s^4 - b^2 s^4\right) u(t,s) = \frac{\left(a^2-b^2\right) s^4 - 1}{1+s} e^{it}.$$

Using $u(0,s) = \frac{1}{1+s}$ and equation (2.56), we get

$$u_t(0,s) = -\frac{i}{1+s}.$$

Then, we have the following initial value problem for the second order ordinary differential equation

$$\begin{cases} u_{tt}(t,s) + \left(a^2-b^2\right) s^4 u(t,s) = \frac{\left(a^2-b^2\right)s^4-1}{1+s} e^{it}, t \in I, \\ \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = -\frac{i}{1+s}. \end{cases} \tag{2.57}$$

Now, we obtain the solution of problem (2.57).There are three cases: $a^2 - b^2 > 0, a^2 - b^2 = 0, a^2 - b^2 < 0$.

In the first case $a^2 - b^2 > 0$. Substituting $m^2$ for $\left(a^2-b^2\right) s^4$ into equation (2.57), we get

$$u_{tt}(t,s) + m^2 u(t,s) = \frac{m^2-1}{1+s} e^{it}, t \in I, u(0,s) = \frac{1}{1+s}, u_t(0,s) = -\frac{i}{1+s}.$$

We have that

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

$$u_p(t,s) = w(s)e^{it}.$$

where $u_c(t,s)$ is general solution of equation

$$u_{tt}(t,s) + m^2 u(t,s) = 0$$

and $u_p(t,s)$ is partucally solution of given equation. Then,

$$u_p(t,s) = w(s)e^{it}.$$

It is easy to see that

$$w(s) = \frac{1}{1+s}$$

and

$$u_c(t,s) = c_1 e^t + c_2 e^{-t}.$$

Therefore,

$$u(t,s) = c_1 e^t + c_2 e^{-t} + \frac{1}{1+s}e^{it}.$$

Using initial conditions, we get

$$u(0,s) = \frac{1}{1+s} = c_1 + c_2 + \frac{1}{1+s},$$

$$u_t(0,s) = -\frac{i}{1+s} = c_1 - c_2 - \frac{i}{1+s}.$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = \frac{1}{1+s}e^{it}.$$

In the same manner $u(t,s) = \frac{1}{1+s}e^{it}$ for $a^2 - b^2 = 0$ and $a^2 - b^2 < 0$.Therefore, taking the inverse Laplace transform, we get

$$u(t,x) = e^{it}L^{-1}\left\{\frac{1}{1+s}\right\},$$

$$u(t,x) = e^{it}e^{-x}.$$

is the exact solution of problem (2.52).

**Example 2.2.5.** Obtain the Laplace transform solution of the initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = (-b-a)e^{-x}, \\ x \in (0,\infty), -\infty < t < \infty, \\ u(0,x) = e^{-x},\ x \in [0,\infty), \\ u(t,0) = 1, u(t,\infty) = 0, t \in (-\infty,\infty) \end{cases} \tag{2.58}$$

for one dimensional involutory Schrödinger's equation.

**Solution.** We will obtain Laplace transform solution of problem (2.58). Taking the Laplace transform, we get

$$\begin{cases} iu_t(t,s) - a\left[s^2u(t,s) - s - \beta(t)\right] - b\left[s^2u(-t,s) - s - \beta(-t)\right] \\ = -\frac{a+b}{1+s}, -\infty < t < \infty, \\ u(0,s) = \frac{1}{1+s} \end{cases}$$

or

$$\begin{cases} iu_t(t,s) - as^2u(t,s) - bs^2u(-t,s) \\ = -(a+b)s - a\beta(t) - b\beta(-t) - \frac{a+b}{1+s}, -\infty < t < \infty, \\ u(0,s) = \frac{1}{1+s}. \end{cases} \tag{2.59}$$

From (2.59) it follows that

$$iu_t(0,s) = \frac{(a+b)s^2}{1+s} - (a+b)s - (a+b)\beta(0) - \frac{a+b}{1+s}$$

or

$$u_t(0,s) = i(a+b)\left[1+\beta(0)\right].$$

Taking the derivative (2.59), we get

$$iu_{tt}(t,s)-as^2u_t(t,s)+bs^2u_t(-t,s)=-a\beta'(t)+b\beta'(-t). \tag{2.60}$$

Putting –t instead of t into equation (2.59), we get

$$iu_t(-t,s)-as^2u(-t,s)-bs^2u(t,s)=-(a+b)s-a\beta(-t)-b\beta(t)-\frac{a+b}{1+s}. \tag{2.61}$$

Multiplying equation (2.60) by (–i) and equation (2.61) by $\left(bs^2\right)$, we get

$$u_{tt}(t,s)+ias^2u_t(t,s)-ibs^2u_t(-t,s)=ia\beta'(t)-ib\beta'(-t).$$

$$ibs^2u_t(-t,s)-abs^4u(-t,s)+b^2s^4u(t,s)$$
$$=-(a+b)bs^3-abs^2\beta(-t)+b^2s^2\beta(t)-\left(\frac{a+b}{1+s}\right)bs^2.$$

Adding last two equations, we get

$$u_{tt}(t,s)+ias^2u_t(t,s)-abs^4u(-t,s)-b^2s^4u(t,s)$$

$$=ia\beta'(t)-ib\beta'(-t)-(a+b)bs^3-abs^2\beta(-t)+b^2s^2\beta(t)-\left(\frac{a+b}{1+s}\right)bs^2.$$

Multiplying equation (2.59) by $\left(-as^2\right)$, we get

$$-ias^2u_t(t,s)+a^2s^4u(t,s)+abs^4u(-t,s)$$

$$=(a+b)as^3+a^2s^2\beta(t)+as^2b\beta(-t)+as^2\frac{a+b}{1+s}.$$

Adding last two equations, we get

$$u_{tt}(t,s)+\left(a^2-b^2\right)u(t,s)=ia\beta'(t)-ib\beta'(-t)-(a+b)bs^3+b^2s^2\beta(t)$$

$$+(a+b)as^3+a^2s^2\beta(t)+as^2\frac{a+b}{1+s}-\left(\frac{a+b}{1+s}\right)bs^2$$

or

$$u_{tt}(t,s)+\left(a^2-b^2\right)u(t,s)=ia\beta'(t)-ib\beta'(-t)$$

$$+\left(a^2-b^2\right)s^3+\frac{\left(a^2-b^2\right)}{1+s}s^2+\left(a^2-b^2\right)s^2\beta(t).$$

There are three cases: $a^2-b^2=0$, $a^2-b^2>0$, $a^2-b^2<0$. In the first case $a^2-b^2=0$. Then

$$\begin{cases} u_{tt}(t,s) = ia\beta'(t) - ib\beta'(-t) \\ \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = i(a+b)\left[1+\beta(0)\right]. \end{cases}$$

Applying formula

$$y(t) = y_0 + t\, y_0' + \int_0^t (t-s)\, y''(s) ds,$$

we get

$$u(t,s) = \frac{1}{1+s} + t\,\left\{i(a+b)\left[1+\beta(0)\right]\right\} + \int_0^t (t-y)\left\{ia\beta'(y) - ib\beta'(-y)\right\} dy$$

$$= \frac{1}{1+s} + t\,\left\{i(a+b)\left[1+\beta(0)\right]\right\} + i(t-y)$$

$$\left[a\beta(t) - b\beta(-t)\right]_0^t + \int_0^t i\left(a\beta(y) - ib\beta(-y)\right) dy$$

$$= \frac{1}{1+s} + t\,\left\{i(a+b)\left[1+\beta(0)\right]\right\} - it\,(a+b)\beta(0) + \int_0^t i\left(a\beta(y) - ib\beta(-y)\right) dy$$

$$= \frac{1}{1+s} + it(a+b) + i\int_0^t \left(a\beta(y) - ib\beta(-y)\right) dy.$$

Therefore,

$$u(t,s) - \frac{1}{1+s} = it(a+b) + i\int_0^t \left(a\beta(y) - ib\beta(-y)\right) dy.$$

Putting

$$A(t) = it(a+b) + i\int_0^t \left(a\beta(y) - ib\beta(-y)\right) dy,$$

we get

$$u(t,s) - \frac{1}{1+s} = A(t).$$

Taking inverse the Laplace transform, we get

$$u(t,x) - e^{-x} = L^{-1}\left\{A(t)\right\}. \tag{2.62}$$

Applying $x \to \infty$, we get

$$0 = L^{-1}\{A(t)\}.$$

Then,

$$LL^{-1}\{A(t)\} = 0$$

or

$$A(t) = 0.$$

Putting A(t) into (2.62), we get

$$u(t,x) - e^{-x} = 0.$$

$$u(t,x) = e^{-x}.$$

In the same manner we can obtain

$$u(t,x) = e^{-x}$$

for $a^2 - b^2 > 0$ and $a^2 - b^2 < 0$.Therefore,

$$u(t,x) = e^{-x}$$

is the exact solution of problem (2.58).

Note that using similar procedure one can obtain the solution of the following problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1,...,x_n) \in \overline{\Omega}^+,\ -\infty < t < \infty, \\ u(\frac{d}{2},x) = \varphi(x), x \in \overline{\Omega}^+, \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta_r(t,x), 1 \le r \le n, t \in I, x \in S^+ \end{cases} \tag{2.63}$$

for the multidimensional Schrödinger type involutory partial differential equations. Assume that $a_r > a > 0$ and $f(t,x)$ $\left(t \in I, x \in \overline{\Omega}^+\right)$,

$\varphi(x)\left(x\in\overline{\Omega}^{+}\right)$, $\alpha(t,x),\beta_r(t,x)\left(t\in I, x\in S^{+}\right)$ are given smooth functions. Here and in future $\Omega^{+}$ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n\,(0<x_k<\infty, 1\le k\le n)$ with the boundary $S^{+}$ and $\overline{\Omega}^{+}=\Omega^{+}\cup S^{+}$.

However Laplace transform method described in solving (2.63) can be used only in the case when (2.63) has constant coefficients.

Third, we consider Fourier transform solution of the problem for Schrödinger type involutory partial differential equations.

**Example 2.2.6.** Obtain the Fourier transform solution of the initial value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t}-au_{xx}(t,x)-bu_{xx}(-t,x) \\ =\left(-1-a(4x^2-2)\right)e^{it}e^{-x^2}-b(4x^2-2)e^{-it}e^{-x^2}, \\ x\in(0,\infty), -\infty<t<\infty, \\ u(0,x)=e^{-x^2},\ x\in[0,\infty) \end{cases} \tag{2.64}$$

for one dimensional involutory Schrödinger's equation.

**Solution.**We will obtain Fourier transform solution of problem (2.64). Taking the Fourier transform, we get

$$\begin{cases} iu_t(t,s)+as^2u(t,s)+bs^2u(-t,s) \\ =-e^{it}q(s)+as^2e^{it}q(s)+bs^2e^{-it}q(s), \\ u(0,s)=q(s). \end{cases} \tag{2.65}$$

Here

$$u(t,s)=F\{u(t,x)\}, q(s)=F\left\{e^{-x^2}\right\}.$$

From (2.65) it follows that

$$iu_t(0,s)=-as^2q(s)-bs^2q(s)-q(s)+asq(s)+bs^2q(s)=-q(s)$$

or

$$u_t(0,s) = iq(s).$$

Taking the derivative (2.65), we get

$$iu_{tt}(t,s) + s^2(au_t(t,s) - bu_t(-t,s)) = -ie^{it}q(s) + s^2\left(iae^{it} - ibe^{-it}\right)q(s). \tag{2.66}$$

Putting $-t$ instead of $t$ into equation (2.65), we get

$$iu_t(-t,s) + s^2(au(-t,s) + bu(t,s)) = -e^{-it}q(s) + s^2\left(ae^{-it} + be^{it}\right)q(s). \tag{2.67}$$

Multiplying equation (2.66) by $(-i)$ and equation (2.67) by $\left(-bs^2\right)$, we get

$$u_{tt}(t,s) - s^2aiu_t(t,s) + is^2bu_t(-t,s) = -e^{it}q(s) + s^2ae^{it}q(s) - s^2be^{-it}q(s).$$

$$-is^2bu_t(-t,s) - s^4b(au(-t,s) + bu(t,s)) = s^2be^{-it}q(s) + s^4b\left(ae^{-it} + be^{it}\right)q(s).$$

Adding last two equations, we get

$$u_{tt}(t,s) - s^2aiu_t(t,s) - s^4bau(-t,s) - s^4b^2u(t,s)$$

$$= -e^{it}q(s) + s^2ae^{it}q(s) + bas^4e^{-it}q(s) - s^4b^2e^{it}q(s).$$

Multiplying equation (2.65) by $\left(as^2\right)$, we get

$$ias^2u_t(t,s) + a^2s^4u(t,s) + abs^4u(-t,s) = -as^2e^{it}q(s) + a^2s^4e^{it}q(s) + abs^4e^{-it}q(s).$$

Adding last two equations, we get the following problem

$$\begin{cases} u_{tt}(t,s) + \left(a^2 - b^2\right)s^4u(t,s) = e^{it}q(s)\left\{-1 + \left(a^2 - b^2\right)s^4\right\}, \\ \\ u(0,s) = q(s), \quad u_t(0,s) = iq(s). \end{cases}$$

We have that

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

where $u_c(t,s)$ is the general solution of homogenous equation

$$u_{tt}(t,s) + \left(a^2 - b^2\right)s^4u(t,s) = 0$$

and $u_p(t,s)$ is the particular solution of nonhomogenous equation. The auxillary equation is

$$p^2+\left(a^2-b^2\right)s^4=0.$$

There are three cases: $a^2-b^2>0$, $a^2-b^2=0$, $a^2-b^2<0$. In the first case $a^2-b^2=0$. Then $p_{1,2}=0,0.$

$$u_c(t,s)=c_1+c_2t.$$

In the second case $p_{1,2}=\pm i\sqrt{a^2-b^2}s^2$. Then

$$u_c(t,s)=c_1\cos\sqrt{a^2-b^2}s^2t+c_2\sin\sqrt{a^2-b^2}s^2t.$$

In the third case $p_{1,2}=\pm\sqrt{b^2-a^2}s^2$. Then

$$u_c(t,s)=c_1e^{\sqrt{b^2-a^2}s^2t}+c_2e^{-\sqrt{b^2-a^2}s^2t}.$$

Now, we will obtain the particular solution $u_p(t,s)$ by formula

$$u_p(t,s)=A(s)e^{it}.$$

Putting it into nonhomogenous equation, we get

$$-A(s)e^{it}+\left(a^2-b^2\right)s^4A(s)e^{it}=e^{it}q(s)\left\{-1+\left(a^2-b^2\right)s^4\right\}$$

or

$$\left\{-1+\left(a^2-b^2\right)s^4\right\}A(s)=q\left\{-1+\left(a^2-b^2\right)s^4\right\}.$$

Therefore

$$A(s)=q(s)$$

and

$$u_p(t,s)=q(s)e^{it}.$$

In the first case, we have

$$u(t,s)=c_1+c_2t+q(s)e^{it}.$$

Applying initial conditions, we get

$$u(0,s)=c_1+q(s)=q(s),$$

$$u_t(0,s) = c_2 + iq(s) = iq(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{it}.$$

In the second case, we have

$$u(t,s) = c_1 \cos\sqrt{a^2 - b^2}s^2 t + c_2 \sin\sqrt{a^2 - b^2}s^2 t + q(s)e^{it}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + q(s) = q(s),$$

$$u_t(0,s) = c_2\sqrt{a^2 - b^2} + iq(s) = iq(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{it}.$$

In the third case, we have

$$u(t,s) = c_1 e^{\sqrt{b^2 - a^2}s^2 t} + c_2 e^{-\sqrt{b^2 - a^2}s^2 t} + q(s)e^{it}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + c_2 + q(s) = q(s),$$

$$u_t(0,s) = \sqrt{b^2 - a^2}(c_1 - c_2) + iq(s) = iq(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{it}.$$

Therefore,

$$u(t,s) = q(s)e^{it} = e^{it}F\left\{e^{-x^2}\right\}$$

and

$$u(t,x) = F^{-1}\left\{e^{it}F\left\{e^{-x^2}\right\}\right\} = e^{it}e^{-x^2}.$$

Therefore,

$$u(t,x) = e^{it}e^{-x^2}.$$

is the exact solution of problem (2.64).

Note that using similar procedure one can obtain the solution of the following problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \mathbb{R}^n, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \mathbb{R}^n \end{cases} \tag{2.68}$$

for the multidimensional Schrödinger type involutory partial differential equations. Assume that $a_r > a > 0$ and $f(t,x)$ $\left(t \in I, x \in \mathbb{R}^n\right)$, $\varphi(x)\left(x \in \mathbb{R}^n\right)$ are given smooth functions. However Fourier transform method described in solving (2.68) can be used only in the case when (2.68) has constant coefficients.

# CHAPTER 3

# DIFFERENCE METHOD FOR THE SOLUTION OF SCHRÖDINGER TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of the local and nonlocal problems for the Schrödinger type involutory partial differential equations play an important role in applied mathematics. In this chapter, we study the numerical solution of the initial boundary value problem

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - u_{xx}(t,x) - u_{xx}(-t,x) = e^{-it}\sin x, \\ 0 < x < \pi, \quad -\pi < t < \pi, \\ u(0,x) = \sin x, \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, -\pi \le t \le \pi \end{cases} \tag{3.1}$$

for the Schrödinger type involutory partial differential equation. The exact solution of this problem is $u(t,x) = e^{it}\sin x$. For the numerical solution of the problem (3.1), we present first order of accuracy difference scheme. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first order of accuracy difference scheme is given.

For the numerical solution of the problem (3.1), we present the following first order of accuracy difference scheme

$$
\begin{cases}
\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\frac{u_{n+1}^{-k}-2u_n^{-k}+u_{n-1}^{-k}}{h^2} \\
=e^{-it_k}\sin x_n,\ t_k=k\tau, x_n=nh, N\tau=\pi, Mh=\pi, \\
-N+1\le k\le N,\ \ 1\le n\le M-1, \\
u_n^0=\sin x_n,\ 0\le n\le M, \\
u_0^k=u_M^k=0,\ -N\le k\le N.
\end{cases} \tag{3.2}
$$

We will write it in the following boundary value problem for the second order difference equation with respect to n

$$
\begin{cases}
Au_{n-1}+Bu_n+Cu_{n+1}=\varphi_n,\ 1\le n\le M-1, \\
u_0=0,\ \ u_M=0.
\end{cases} \tag{3.3}
$$

Here, $A, B, C$ are $(2N+1)\times(2N+1)$ square matrices and $u_s, s=n, n\pm 1, \varphi_n$ are $(2N+1)\times 1$ column matrices and

$$
A=C=\begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & a & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & a & 0 \\
0 & 0 & a & \cdot & 0 & 0 & 0 & \cdot & a & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & a & 0 & a & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 2a & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & a & 0 & a & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & a & \cdot & 0 & 0 & 0 & \cdot & a & 0 & 0 \\
0 & a & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & a & 0 \\
a & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & a
\end{bmatrix}_{(2N+1)\times(2N+1)},
$$

$$
B = \begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
b & d & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & c & 0 \\
0 & b & d & \cdot & 0 & 0 & 0 & \cdot & c & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & d & 0 & c & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & b & d+c & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & c & b & d & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & c & \cdot & 0 & 0 & 0 & \cdot & d & 0 & 0 \\
0 & c & 0 & \cdot & 0 & 0 & 0 & \cdot & b & d & 0 \\
c & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & b & d
\end{bmatrix}_{(2N+1)\times(2N+1)},
$$

$$
\varphi_n = \begin{bmatrix}
\sin x_n \\
e^{-it_{-N+1}} \sin x_n \\
\cdot \\
e^{-it_{N-1}} \sin x_n \\
e^{-it_N} \sin x_n
\end{bmatrix}_{(2N+1)\times 1}, \quad
u_s = \begin{bmatrix}
u_s^{-N} \\
u_s^{-N+1} \\
\cdot \\
u_s^{N-1} \\
u_s^{N}
\end{bmatrix}_{(2N+1)\times 1},
$$

where $a = -\frac{1}{h^2}$, $b = -\frac{i}{\tau}$, $c = \frac{2}{h^2}$ and $d = \frac{2}{h^2} + \frac{i}{\tau}$. For obtainig $\{u_n\}_{n=0}^{M}$we have the following algorithm

$$
u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 0, u_M = 0, \tag{3.4}
$$

$$
\alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \alpha_1 = 0,
$$

$$
\beta_{n+1} = (B + C\alpha_n)^{-1} \left(\varphi_n - C\beta_n\right), \; \beta_1 = 0, n = 1, ..., M-1.
$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and $u_n^k$ represents the numerical solution at $(t_k, x_n)$ and the results are given in the following table

**Table 3.1:** Error analysis

| Difference Scheme | 20, 20 | 40, 40 | 80, 80 | 160, 160 |
|---|---|---|---|---|
| $Eu_M^N$ | 0.5088 | 0.2578 | 0.1293 | 0.0647 |

As it is seen in Table 3.1, we get some numerical results. If N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately 1/2 for first order difference scheme.

# CHAPTER 4
# CONCLUSION

This thesis is devoted to initial boundary value problem for Schrödinger type involutory differential equations:The following results are obtained:

- The history of involutory differential equations is given.
- Fourier series, Laplace transform and Fourier transform methods are applied for the solution of six Schrödinger type involutory partial differential equations.
- The first order of accuracy difference scheme is considered for the approximate solution of the one dimensional Schrödinger type involutory partial differential equation with Dirichlet condition.
- The Matlab implementation of the numerical solution is presented.

## REFERENCES


Agirseven, D. (2018). On the Stability of the Schrödinger Equation with Time Delay. *Filomat*, *32*(3).

Antoine, X., Besse, C., & Mouysset, V. (2004). Numerical schemes for the simulation of the two-dimensional Schrödinger equation using non-reflecting boundary conditions. *Mathematics of computation*, *73*(248), 1779-1799.

Ashyralyev, A., & Agirseven, D. (2018). Bounded solutions of nonlinear hyperbolic equations with time delay. *Electronic Journal of Differential Equations*.

Ashyralyev, A., & Agirseven, D. (2013). On convergence of difference schemes for delay parabolic equations. *Computers & Mathematics with Applications*, *66*(7), 1232-1244.

Ashyralyev, A., & Hicdurmaz, B. (2011).A note on the fractional Schrödinger differential equations. *Kybernetes*, *40*(5/6), 736-750.

Ashyralyev, A., & Hicdurmaz, B. (2012). On the numerical solution of fractional Schrödinger differential equations with the Dirichlet condition. *International Journal of Computer Mathematics*, *89*(13-14), 1927-1936.

Ashyralyev, A., & Hicdurmaz, B. (2018). a stable second order of accuracy difference scheme for a fractional schrodinger differential equation. applied and computational mathematics, 17(1), 10-21.

Ashyralyev, A., & Sarsenbi, A. M. (2015). Well-posedness of an elliptic equation with involution. *Electronic Journal of Differential Equations*, *284*, 1-8.

Ashyralyev, A., & Sirma, A. (2008). Nonlocal boundary value problems for the Schrödinger equation. *Computers & Mathematics with Applications*, *55*(3), 392-407.

Ashyralyev, A., & Sirma, A. (2009). A note on the numerical solution of the semilinear Schrödinger equation. *Nonlinear Analysis: Theory, Methods & Applications*, *71*(12), e2507-e2516.

Bhalekar, S., & Patade, J. (2016). Analytical solutions of nonlinear equations with proportional delays. *Appl Comput Math*, *15*, 331-445.

Cui, H. Y., Han, Z. J., & Xu, G. Q. (2016). Stabilization for Schrödinger equation with a time delay in the boundary input. *Applicable Analysis*, *95*(5), 963-977.

Chen, T., & Zhou, S. F. (2010). Attractors for discrete nonlinear Schrödinger equation with delay. *Acta Mathematicae Applicatae Sinica, English Series*, *26*(4), 633-642.

Eskin, G., & Ralston, J. (1995). Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy. *Communications in mathematical physics*, *173*(1), 199-224.

Gordeziani, D. G., & Avalishvili, G. A. (2005). Time-nonlocal problems for Schrodinger type equations: I. Problems in abstract spaces. *Differential Equations*, *41*(5), 703-711.

Gordeziani, D. G., & Avalishvili, G. A. (2005). Time-nonlocal problems for Schrodinger type equations: II. Results for specific problems. *Differential Equations*, *41*(6), 852-859.

Guo, B. Z., & Yang, K. Y. (2010). Output feedback stabilization of a one-dimensional Schrödinger equation by boundary observation with time delay. *IEEE Transactions on Automatic Control*, *55*(5), 1226-1232.

Guo, B. Z., & Shao, Z. C. (2005). Regularity of a Schrödinger equation with Dirichlet control and colocated observation. *Systems & Control Letters*, *54*(11), 1135-1142.

Han, H., Jin, J., & Wu, X. (2005). A finite-difference method for the one-dimensional time-dependent Schrödinger equation on unbounded domain. *Computers & Mathematics with Applications*, *50*(8-9), 1345-1362.

Kang, W., & Fridman, E. (2018). Boundary Constrained Control of Delayed Nonlinear Schrodinger Equation. *IEEE Transactions on Automatic Control*.

Kun-Yi, Y. & Cui-Zhen, Y. (2013), Stabilization of one-dimensional Schrödinger equation with variable coefficient under delayed boundary output feedback, Asian Journal of Control, 15 (5), 1531-1537.

Kuralay, G. (2017). *Design of first order controllers for a flexible robot arm with time delay* (Doctoral dissertation, bilkent university).

Mayfield, M. E. (1989). *Nonreflective boundary conditions for Schroedinger's equation*. Rhode Island Univ., Kingston, RI (United States).

Nakatsuji, H. (2002). Inverse Schrödinger equation and the exact wave function. *Physical Review A*, *65*(5), 052122.

Nicaise, S., & Rebiai, S. E. (2011). Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback. *Portugaliae Mathematica*, *68*(1), 19.

Serov, V., & Päivärinta, L. (2006). Inverse scattering problem for two-dimensional Schrödinger operator. *Journal of Inverse and Ill-posed, Problems jiip*, *14*(3), 295-305.

Skubachevskii, A. L. (1994, March). On the problem of attainment of equilibrium for control-system with delay.In *Doklady Akademii Nauk* (Vol. 335, No. 2, pp. 157-160).

Smagin, V. V., & Shepilova, E. V. (2008). On the solution of a Schrödinger type equation by a projection-difference method with an implicit Euler scheme with respect to time. *Differential Equations*, *44*(4), 580-592.

Sobolevskii, P. E. (1975). Difference Methods for the Approximate Solution of Differential Equations, Izdat. Voronezh. Gosud. Univ., Voronezh,.(Russian).

Sriram, K., & Gopinathan, M. S. (2004). A two variable delay model for the circadian rhythm of Neurospora crassa. *Journal of theoretical biology*, *231*(1), 23-38.

Srividhya, J., & Gopinathan, M. S. (2006). A simple time delay model for eukaryotic cell cycle. *Journal of theoretical biology*, *241*(3), 617-627.

Sun, J., Kou, L., Guo, G., Zhao, G., & Wang, Y. (2018). Existence of weak solutions of stochastic delay differential systems with Schrödinger–Brownian motions. *Journal of inequalities and applications*, *2018*(1), 100.

Vlasov, V.V. & Rautian, N.A. (2016). Spectral Analysis of Functional Differential Equations:Monograph,M.:MAKS Press, 488p.

Zhao, Z., & Ge, W. (2011). Traveling wave solutions for Schrödinger equation with distributed delay. *Applied Mathematical Modelling*, *35*(2), 675-687.

## APPENDIX

## MATLAB PROGRAMMING

Matlab programs are presented for the first order of approximation two-step difference scheme for M=N.

```
function mmmmm(N,M)
if nargin;1;end;
close;close;
%first order
N=20;
M=20;
tau=pi/N;
h=pi/M;
a=-1/(h^2);
b=-i/tau;
c=2/h^2;
d=i/tau+2/h^2;
A=zeros(2*N+1,2*N+1);
for k=2:N;
A(N+1,N+1)=2*a;
A(k,k)=a;
A(k,2*N+2-k)=a;
end;
for k=N+2:2*N+1;
```

```
A(k,k)=a;
A(k,2*N+2-k)=a;
end;
A;
C=A;
B=zeros(2*N+1,2*N+1);
B(1,N+1)=1;
for k=2:N;
B(k,k-1)=b;
B(N+1,N+1)=c+d;
B(N+1,N)=b;
B(k,k)=d;
B(k,2*N+2-k)=c;
end;
for k=N+2:2*N+1;
B(k,k)=d;
B(k,2*N+2-k)=c;
B(k,k-1)=b;
end;
B;
D=eye(2*N+1,2*N+1);
for j=1:M+1;
for k=2:2*N+1;
fii(k,j)=exp(-(k-1-N)*tau*i)*sin((j-1)*h);
```

```
end;
fii(1,j)=sin((j-1)*h);
end;
alpha{1}=zeros(2*N+1,2*N+1);
betha{1}=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(2*N+1,M+1);
for j=M:-1:1;
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
es(k,j)=exp((k-1-N)*tau*i)*sin((j-1)*h);
end;
end;
%.ERROR ANALYSIS.;
maxes=max(max(abs(es)));
maxerror=max(max(abs(es-U)))
relativeerror=maxerror/maxes;
```

```
cevap1=[maxerror,relativeerror] ;
%figure;
%m(1,1)=min(min(abs(U)))-0.01;
%m(2,2)=nan;
%surf(m);
%hold;
%surf(es);rotate3d;axis tight;
%title('Exact Solution');
%figure;
%surf(m);
%hold;
%surf(U);rotate3d;axis tight;
%title('FIRST ORDER');
```