ON A STUDY OF q-FRACTIONAL DIFFERENCE EQUATIONS WITH THREE BOUNDARY VALUES

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

MEZEER SADEEQ IBRAHIM

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

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First and foremost, my utmost gratitude goes to my parents who are always there for me. Your prayers and affection always give me courage in all that I do; my appreciation can never be overemphasized. Thank you.

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To my parents…

ABSTRACT

Recent application of q-fractional difference equations motivates a lot of mathematicians to work in this field. We study q-fractional difference equations with three boundary values which can be reduced to the several forms of these equations. We proved the existence and uniqueness of solution in the aid of application of special kind of operator in Banach space. Some q-analogues of familiar formulae are studied; q-Hypergeometric functions and Hine's transform theorem are applied to make an approach to study q-fractional difference equations. Main formula for q-fractional difference equation is investigated and in the aid of this definition we prove the uniqueness and existence of solution.

***Keywords*:** q -calculus; q-Hypergeometric functions; Hine's transform; q-fractional difference equations; Concave operator on Banach spaces.

ÖZET

q-kesirli fark denklemlerinin son zamanlarda uygulanması, bu alanda çalışmak için birçok matematikçiyi motive etmektedir. q-kesirli fark denklemlerini üç Sinir değeriye birlikte, bu çeşitli şekillere indirgenebilen olarak inceliyoruz. Banach uzayında özel bir operatör uygulaması için çözümün varlığını ve benzersizliğini kanıtladık. Tanıdık formüllerin bazı q-analogları incelenmiştir; q-Hipergeometrik fonksiyonlar ve Hine dönüşüm teoremi, q-kesirli fark denklemlerini incelemek için bir yaklaşım uygulamak için uygulanır. q-kesirli fark denkleminin ana formülü incelenmiştir ve bu tanım yardımı ile çözümün benzersizliğini ve varlığını kanıtlıyoruz.

***Anahtar Kelimeler*:** q-hesabı; q-Hipergeometrik fonksiyonlar; Hine dönüşümü; q-kesirli fark denklemleri; Banach uzaylarında içbükey operator.

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CHAPTER 1

INTRODUCTION

1.1 Some Definitions and Concepts

In this section we present some preliminary concepts and definitions. These concepts will be used in next chapters directly and indirectly. We start by introducing a few definitions that are available in any preliminary books of differential equations and linear algebra. (Ross & Shepley, 1989)

Definition 1.1: We start by defining the most basic concept. The differential equation is defined as an equation which included the ordinary or partial derivatives of one or more than one dependent variables which are regarding to independent variables. In regard with this basic definition, we exclude those equation that are actually derivative identities in the class of differential equations. For instance, we do not include such expressions as (Ross & Shepley, 1989).

$$\frac{d}{dt}e^{nt} = ne^{nt} \tag{1.1}$$

$$\frac{d}{dt}sw^{-1} = \frac{w\, {}^{ds}/_{dt} - s\, {}^{dw}/_{dt}}{w^2}. \tag{1.2}$$

Example 1.2: We can assume the following differential equations (Ross & Shepley, 1989):

$$\left(\frac{dy}{dx}\right)^3 + \frac{dy^4}{dx^4} + y = 2\,sin(x) + cos^3(x)\,, \tag{1.3}$$

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0, \tag{1.4}$$

$$yy' = 0, \tag{1.5}$$

Definition 1.3: We can define the order of the differential equation by knowing the highest ordered of derivative that is available in a given differential equation. (Chen, 2006)

Definition 1.4: A vector space (over the scalar field $\mathbb{R}$) is a collection says V along with two operators "addition" and "multiplication" such that following conditions or axioms hold for any vectors $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathrm{V}$ and scaler $\mathrm{r}, \mathrm{s} \in \mathbb{R}$. (Chen, 2006)

1. Commutativity:
   $A + B = B + A,$
2. Addition operator over the vectors should be associative:
   $(A + B)C = A(B + C),$
3. Addition operator has the identity element, means for any arbitrary vector A, There exist 0 vector, such that
   $0 + A = A + 0 = A,$
4. Addition operator has inverse property, means for arbitrary vector A, There exist $-A$ vector, such that
   $A + (-A) = 0,$
5. The scalar multiplication is associative, means for any constant r, s and vector A:
   $r(sA) = (rs)A$,
6. Scalar Summation has distribution property, means for any constant r, s and vector A:
   $(r + s)A = rA + sA,$
7. Vector summation has distribution property, means for any constant r and vectorsA, B: $r(A + B) = rA + rB,$
8. At the end, unit scalar multiplication make the same vector, means for any vector A:
   $1A = A.$

Definition 1.5: Assume that V as a vector space is given and k is the scalar field and $S \subseteq V$. Then S is linearly dependent if we can find vectors $v_1, v_2, \ldots, v_n$ with $v_i \neq v_j$, $i \neq j$ and $v_i \neq 0$ and scalar $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ with

$$\alpha_i \neq 0 \;\; for\ some \;\; i = 1,2,3, \ldots .$$

Otherwise S is said to be independent. (Hoffman & Kunze, 1971)

Definition 1.6: Let X and Y are a vectors space over a field K. A linear map or linear transformation is a function or mapping $T: X \to Y$ that satisfies the following conditions: (Hoffman & Kunze, 1971)

$$1. T(a+b) = T(a) + T(b), \qquad \forall\ a, b \in X \tag{1.6}$$

$$2. T(\alpha b) = \alpha T(b), \qquad \forall\ \alpha \in K, \qquad \forall\ b \in X \tag{1.7}$$

Or equivalently, we can combine these properties as;

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \qquad x, y \in X, \qquad \alpha, \beta \in K \tag{1.8}$$

Definition 1.7: Let the set of vector $v_1, v_2, \dots, v_n$ and scaler $s_1, s_2, \dots, s_n$, are given, then the vector V determine by the following: (Hoffman & Kunze, 1971)

$$V = s_1 v_1 + s_2 v_2 + \dots + s_n v_n \tag{1.9}$$

is called linear combination of vector $v_1, v_2, \dots, v_n$.

Definition 1.8: Let $G = \{ v_1, v_2, \dots, v_n\}$ be a set of vectors which is a subset of the vector space V then we say that G spans V, if any vector S of V can be expressed as a linear combination of the elements of G

$$S = s_1 v_1 + s_2 v_2 + \dots + s_n v_n.$$

(Joyce, 2015)

Definition 1.9: A set of vector $\{ v_1, v_2, \dots, v_n\}$ is said to form a basis for a vector space if (Chen, 2006), if the following holds:

1- The vectors $v_1, v_2, \dots, v_n$ span the vector space.
2- The vectors $v_1, v_2, \dots, v_n$ are linear independent.

Definition 1.10: V as a vector space is a finite-dimensional space if we can found the finite span set for V (Kandasamy & Vasantha, 2003). The dimension of v say Dim V, represents the number of vectors in a set which is the basis for V. For example we can see:

$$Dim\mathbb{R}^0 = 0, \qquad Dim\mathbb{R}^n = n, \qquad Dim\mathbb{p}_n = n + 1$$

Definition 1.11: A vector V would be in the kernel of a matrix A if and only if $AV = 0$.

Therefore, the span of all these vectors would be the kernel. (Gover, 1988)

1.2 q-Calculus and q-Analogues of Some Expressions

In this section, we present some concepts and definitions related to the q-calculus and history of arising this branch of mathematics and the source of this calculus. In fact, the world of q-calculus was started when we evaluated the derivative of functions without using the limit. In following, we study the basic concepts and related notations of q-calculus. In fact, any expression or function can be interpreted by q-calculus. Here and in the rest of thesis, we assume that q is real or complex constant such that $0 < |q| < 1$.

Definition 1.12: Assume that $s(\tau)$ is an arbitrary complex valued function, then the q-derivative of $w(t)$ can be defined as (Kac & Cheung, 2002):

$$d_q w(\tau) = w(q\tau) - w(\tau), \quad D_q w(\tau) = \frac{d_q w(\tau)}{d_q \tau} \quad \tau \neq 0, \tag{1.10}$$

Remark 1.13: We notice that (Kac & Cheung, 2002)

$$\lim_{q \to 1^-} D_q s(\tau) = \frac{ds(\tau)}{d\tau}. \tag{1.11}$$

Property 1.14: The q- derivative is linear in the means of (Kac & Cheung, 2002)

$$D_q\big(\delta s(\tau) + \varepsilon w(\tau)\big) = \delta D_q s(\tau) + \varepsilon D_q w(\tau) \qquad \delta\,, \varepsilon \in \mathbb{R} \tag{1.12}$$

Definition 1.15: The q-analogue of product rule for derivative, can be expressed as following (Kac & Cheung, 2002):

$$\begin{aligned} d_q\big(s(\vartheta)w(\vartheta)\big) &= s(q\vartheta)w(q\vartheta) - s(\vartheta)w(\vartheta) \\ &= s(q\vartheta)w(q\vartheta) - s(q\vartheta)w(\vartheta) + s(q\vartheta)w(\vartheta) - s(\vartheta)w(\vartheta), \end{aligned}$$

We get

$$d_q\big(s(\vartheta)w(\vartheta)\big) = s(q\vartheta)d_q w(\vartheta) + w(\vartheta)d_q s(\vartheta). \tag{1.13}$$

Definition 1.16: We can describe the quotient rules for q- derivative. There are two q-analogues for this rule. Both of them are true and useful, we have two forms as following (Kac & Cheung, 2002):

$$D_q\left(\frac{u(\tau)}{w(\tau)}\right)=\frac{w(\tau)D_qu(\tau)-u(\tau)D_qw(\tau)}{w(\tau)w(q\tau)}, \tag{1.14}$$

$$D_q\left(\frac{u(\tau)}{w(\tau)}\right)=\frac{w(q\tau)D_qu(\tau)-u(q\tau)D_qw(\tau)}{w(\tau)w(q\tau)}. \tag{1.15}$$

Definition 1.17: The q- number is defined by the following expression (Kac & Cheung, 2002)

$$[u]_q=\frac{q^u-1}{q-1}, \qquad 0\le|q|<1 \;, \qquad u\in ¢. \tag{1.16}$$

Definition 1.18: The q-analogue of u-exponent of $(\mathrm{x}-\alpha)$, for any $\mathrm{x},\alpha,\mathrm{q}\in ¢, |\mathrm{q}|<1$ is defined by (Kac & Cheung, 2002)

$$(x-\alpha)_q^u=\begin{cases}1 & if \quad u=0\\ (x-\alpha)(x-q\alpha)\dots(x-q^{u-1}\alpha) & if \quad u\ge 1\end{cases}. \tag{1.17}$$

Remark 1.19: The q-shifted factorial or q-analogue of u-exponent which is defined in (Definition 1.18) can be expressed by (Kac & Cheung, 2002)

$$(\mu;q)_0=1,\;\; (\mu;q)_u=\prod_{\tau=0}^{u-1}(1-q^\tau\mu), \qquad u\in N, \tag{1.18}$$

$$(\mu;q)_\infty=\prod_{\tau=0}^{\infty}(1-\mu q^\tau), \qquad |q|<1, \tag{1.19}$$

In addition, if α be any real number, then we can define q- shifted factorial as

$$(\mu;q)_\alpha=\frac{(\mu;q)_\infty}{(\mu q^\alpha;q)_\infty} \qquad (\alpha\in\mathbb{R}), \tag{1.20}$$

This expression has the following properties:

$$(\mu;q)_n=(q^{1-n}/\mu\,;q)_n(-1)^n\mu^n q^{\binom{n}{2}}, \tag{1.21}$$

$$\frac{(\mu q^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/\mu\,;q)_n}{(q/b\,;q)_n}\left(\frac{\mu}{b}\right)^n, \tag{1.22}$$

Definition 1.20: We have the following q- analogue of u! (Kac & Cheung, 2002):

$$[u]_q! = \begin{cases} 1 & if \quad u = 0, \\ [u]_q[u-1]_q \dots [1]_q. & if \quad u = 1,2,\dots \end{cases} \tag{1.23}$$

Proposition 1.21: The following formula is q-derivative of q-shifted expression which is defined at (1.18) (Kac & Cheung, 2002)

$$D_q(x-\ \alpha)_q^u = [u](x-\ \alpha)_q^{u-1}. \tag{1.24}$$

Definition 1.22: The q- analogue of combination of two numbers which is denoted by $\binom{t}{z}_q$ can be expressed as (Kac & Cheung, 2002)

$$\binom{t}{t-z}_q = \frac{[t]_q!}{[z]_q!\,[t-z]_q!} = \binom{t}{z}_q. \tag{1.25}$$

Definition 1.23: We have several q-analogues of familiar functions. Two different classical q- exponentials function were defined by Euler in 19 century as the following by (Kac & Cheung, 2002)

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{n=0}^{\infty} \frac{1}{1-(1-q)q^n\ z},$$

$$\text{if} \quad 0 < |q| \ < 1, \quad |z| < \frac{1}{|1-q|}\,. \tag{1.26}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{[n]_q!} = \prod_{n=0}^{\infty} (1+(1-q)q^n z),$$

$$\text{if}\ 0 < |q| < 1, \ \ z \in ¢. \tag{1.27}$$

Definition 1.24: Suppose that U(x) and u(x) be two functions on R, if the written series is convergent, then the Jackson integral is determined as follow (T.Ernst, 2000)

$$\int u(x)d_qx = (1-q)x\sum_{n=0}^{\infty} q^n u(q^n x) \quad , \tag{1.28}$$

Nonetheless, from above formula we can easily derive a more general formula:

$$\int u(x)D_q v(x)d_qx = (1-q)x\sum_{n=0}^{\infty} q^n u(q^n x)\, D_q v(q^n x)$$

$$= (1-q)x\sum_{n=0}^{\infty} q^n u(q^n x)\frac{v(q^n x) - v(q^{n+1}x)}{(1-q)q^n x}$$

$$= \sum_{n=0}^{\infty} u(q^n x)\big(v(q^n x) - v(q^{n+1}x)\big), \tag{1.29}$$

Actually, since the chain rule is not true generally for q-derivative, we cannot use the substitution method to solve the Jackson integral.

Theorem 1.25: Let we using two positive real numbers α and β be such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

In addition, let f and g are two complex function such that they have Riemann integral, then

$$\left|\int_a^b u(\tau)w(\tau)d\tau\right| \le \left(\int_a^b |u(\tau)|^\alpha d\tau\right)^{1/\alpha} \left(\int_a^b |w(\tau)|^\beta d\tau\right)^{1/\beta}$$

The term $\left(\int_a^b |u(\tau)|^\alpha d\tau\right)^{1/\alpha}$ is the norm for $u(\tau)$ and denoted by $\|u\|_p$, in addition when $\alpha = \beta = 2$ the inequality called Schwarz inequality (Rudin, 1976).

Definition 1.26: Let $\mathbb{R}^k$ be the Euclidian space including all k-tuple vectors. A set $E \subseteq \mathbb{R}^k$ is convex if $\lambda x + (1-\lambda)y \in E$ for $x, y \in E$ and $0 < \lambda < 1$. That means the set E has not any holes and all point between x and y are included in the set E. The real function $u: E \to \mathbb{R}$, is convex if

$$u(\lambda x + (1-\lambda)y) \le \lambda u(x) + (1-\lambda)u(y).$$

The definition can be extended to any ordered set instead of real numbers. For instance, if the function is real valued, then second derivative of u should be positive (Rudin, 1976).

1.3 Gamma Function and Related q-Analogue

In this section, we present the Gamma function which is the natural extension of factorial symbol. Some properties of this function will be studied and we will introduce the q-analogue of this function as well. Actually, we will apply these definitions and properties later to define the fractional differential equation and q-analogue of them.

Definition 1.27: For any positive real number x such that $x \in \mathbb{R}^+ = \{x | x > 0, x \in \mathbb{R}^+\}$ we define Gamma function by the following integral (Rudin, 1976).

$$\Gamma(\mathrm{x}) = \int_0^\infty t^{x-1} e^{-t} dt \, , \tag{1.30}$$

Proposition 1.28: The Gamma function can be determined by the following properties. (Rudin, 1976)

$$(a)\ \Gamma(x+1) = x\Gamma(x) \ , \quad \text{for } x \in \mathbb{R}^+, \tag{1.31}$$

$$(b)\ \Gamma(n+1) = n! \qquad , \quad \text{for } n \in \mathbb{N}, \tag{1.32}$$

$$(c)\ loq\ \Gamma \text{ is convex on } (0,\infty) \ , \ n \in \mathbb{N}, \tag{1.33}$$

Proof: we prove this theorem in the following steps:

(a) we apply integral by part for $\Gamma(x)$ then

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \lim_{b\to\infty} -t^{x-1} e^{-t}|_0^b + \int_0^\infty (x-1)t^{x-2} e^{-t} dt$$

$$= \lim_{b\to\infty} \frac{-b^{x-1}}{e^b} + 0^{(x-1)} \cdot e^0 + (x-1)\int_0^\infty t^{x-2} e^{-t} dt$$

$$= (x-1)\Gamma(x-2)$$

(b) Use the induction for n, means for $n = 1$ we have;

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \lim_{b\to\infty} -e^{-t}|_0^b = \lim_{b\to\infty} \frac{1}{e^b} + e^0 = 1$$

Assume that $\Gamma(n) = (n-1)!$ Then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

(c) We apply the holder inequality for any $\alpha, \beta > 0$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then

$$\Gamma\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) = \int_0^\infty t^{\frac{x}{\alpha}+\frac{y}{\beta}-1} e^{-t}dt = \int_0^\infty t^{\frac{x}{\alpha}+\frac{y}{\beta}-\frac{1}{\alpha}-\frac{1}{\beta}} \; e^{-\frac{t}{\alpha}-\frac{t}{\beta}} dt$$

$$= \int_0^\infty \left(t^{\frac{x-1}{\alpha}} \cdot e^{-\frac{t}{\alpha}}\right)\left(t^{\frac{y-1}{\beta}} \cdot e^{-\frac{t}{\beta}}\right) dt$$

$$\leq \left(\int_0^\infty \left|t^{\frac{x-1}{\alpha}} \cdot e^{-\frac{t}{\alpha}}\right|^\alpha\right)^{1/\alpha} \cdot \left(\int_0^\infty \left|t^{\frac{y-1}{\beta}} \cdot e^{-\frac{t}{\beta}}\right|^\beta\right)^{1/\beta}$$

$$= \Gamma(x)^{1/\alpha} \cdot \Gamma(y)^{1/\beta}$$

Now take the log for the last expression;

$$\log \Gamma\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq \frac{1}{\alpha} \log \Gamma(x) + \frac{1}{\beta} \log \Gamma(y)$$

Now assume that $\frac{1}{\alpha} = \lambda > 0$ then $1 - \lambda = \frac{1}{\alpha} > 0$ and we have

$$\log \Gamma(\lambda x + (1-\lambda)y) \leq \lambda \log \Gamma(x) + (1-\lambda) \log \Gamma(y)$$

Therefore log Γ is a convex function.

Theorem 1.29: If on positive real number we have positive mapping denoted by s, such that the same properties like (1.31), (1.32), (1.33) hold true, that means (Rudin, 1976):

(a) $s(\omega + 1) = \omega\, s(\omega)$, (1.34)

(b) $s(1) = 1$, (1.35)

(c) $\log s$ is convex, (1.36)

Then $s(\omega) = \Gamma(\omega)$. That means Gamma function is determined only by these properties.

Proof: Let $\omega \in \mathbb{R}^+$ and $\lfloor\omega\rfloor$ is a less integer than ω. We can write ω as $\lfloor\omega\rfloor + \rho$, where $0 < \rho < 1$. Then according to the first property we should prove it only for the case that $0 < \omega < 1$. Because if $\omega = \psi + \rho$ and $0 < \rho < 1$ then

$$\begin{aligned} s(k+\rho) &= (\rho+k-1)(p+k-2)\ldots\big(\rho+k-(k-1)\big)s(\rho) \\ &= (\rho+\mathrm{k}-1)(\rho+\mathrm{k}-2)\ldots\big(\rho+\mathrm{k}-(\mathrm{k}-1)\big)\Gamma(\rho) \\ &= \Gamma(\rho+\mathrm{k}) \qquad (1.37) \end{aligned}$$

Put $\psi(\omega) = log\, s(\omega)$. Then

$$\begin{aligned} \psi(\omega+1) &= log\, s(\omega+1) = log\big(\omega\, s(\omega)\big) = log\,\omega + log\, s(\omega) \\ &= \psi(\omega) + log\omega \qquad (1.38) \end{aligned}$$

And $\psi(1) = log\, s(1) = log\, 1 = 0$ and ψ is convex function. Now let $\mu \in \mathbb{N}$. Then

$$\begin{aligned} \psi(\mu+1) &= \psi(\mu) + log\,\mu = \psi(\mu-1) + log(\mu-1) + log\,\mu \\ &= \psi(1) + log(2) + log(3) + \cdots + log(\mu) = log(\mu!) \end{aligned}$$

Now consider three different intervals as $[\mu, \mu+1], [\mu+1, \mu+1+\omega], [\mu+1, \mu+2]$. Since ψ is convex we have

$$\begin{aligned} log\,\mu = log\frac{\mu!}{(\mu-1)!} &= \frac{log\,\mu! - log(\mu-1)!}{1} = \frac{\psi(\mu+1)-\psi(\mu)}{1} \\ &\leq \frac{\psi(\mu+1+\omega)-\psi(\mu+1)}{\omega} \\ &\leq log(\mu+1). \qquad (1.39) \end{aligned}$$

On the another hand $\psi(\omega+1) = \psi(\omega) + log\,\omega$, then by induction we have

$$\begin{aligned} \psi(\omega+\mu) &= \psi(\omega+\mu-1) + log(\omega+\mu-1) \\ &= \psi(\omega+\mu-2) + log(\omega+\mu-2) + log(\omega+\mu-1) \\ &= \psi(\omega) + log[\omega(\omega+1)\ldots(\omega+\mu-1)] \qquad (1.40) \end{aligned}$$

From (1.37) and (1.38) we have

$$log\,\mu \leq \frac{\psi(\omega) + log[\omega(\omega+1)\ldots(\omega+\mu-1)] - log\,\mu!}{\omega} \leq log\,\mu + 1$$

Multiply the inequality by ω and then subtract $log\,\mu$

$$0 \leq (\omega) + log[\omega(\omega+1)\ldots(\omega+\mu-1)] - log\,\mu! \leq \ \omega\, log(1+\frac{1}{\mu})$$

Now, if we tend $\mu \to \infty, \omega\, log(1+\frac{1}{\mu}) \to 0$, thus

$$\psi(\omega) = \lim_{\mu\to\infty} log \frac{\mu!\,\mu^{\omega}}{\omega(\omega+1)\ \ldots\ (\omega+\mu)} = log\, s(\omega),$$

Therefore,

$$\Gamma(\omega) = s(\omega) = \lim_{\mu\to\infty} \frac{\mu!\,\mu^{\omega}}{\omega(\omega+1)\ \ldots\ (\omega+\mu)}\ .$$

Theorem 1.30: The Beta function is defined by (Rudin, 1976)

$$\beta(\omega,\gamma) = \int_0^1 \tau^{\omega-1}(1-\tau)^{\gamma-1}d\tau\,. \tag{1.41}$$

This function has the following relation by Gamma function,

$$\beta(\omega,\gamma) = \frac{\Gamma(\omega)\Gamma(\gamma)}{\Gamma(\omega+\gamma)}\,. \tag{1.42}$$

Proof: It’s only needed to verify the properties of (theorem 1.29) for a fixed point y

$$\beta(\omega,\gamma) = \frac{\Gamma(\omega)\Gamma(\gamma)}{\Gamma(\omega+\gamma)} = u(\omega) \tag{1.43}$$

Details are available at (Ferreira, 2011).Then

$$\Gamma(\omega) = \frac{\beta(\omega,\gamma)\Gamma(\omega+\gamma)}{\Gamma(\gamma)}. \tag{1.44}$$

Example 1.31: Let us calculate the interesting value of $\Gamma\left({}^{1}/_{2}\right)$ or equivalently the interpretation of $\left(-\frac{1}{2}\right)$!. For this, substitute $\tau = sin^2\theta$ at Beta function to have:

$$\beta(\omega,\gamma) = \int_0^{\pi/2} sin^{2(\omega-1)}\theta \ \cdot cos^{2(\gamma-1)}\theta \ \cdot 2sin\theta cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} sin^{2\omega-1}\theta \ \cdot cos^{2y-1}\theta \ d\theta = \frac{\Gamma(\omega)\Gamma(\gamma)}{\Gamma(\omega+\gamma)}$$

Now if $\omega = \gamma = {}^{1}/_{2}$ gives

$$\Gamma(\omega) = \Gamma\left({}^{1}/_{2}\right) = \sqrt{\pi}$$

Definition 1.32: We defined the Jackson integral at (Definition 1.24), now let $0 < a < b$, then the definite q-integral with these boundaries is defined as (Kac & Cheung, 2002):

$$\int_0^b u(\omega)d_q\omega = (1-q)b\sum_{j=0}^{\infty} q^j u(q^j b) \tag{1.45}$$

And

$$\int_a^b u(\omega)d_q\omega = \int_0^b u(\omega)d_q\omega - \int_0^a u(\omega)d_q\omega \tag{1.46}$$

As seen before in (1.23), we derived from (1.44) a more general formula:

$$\int_0^b u(\omega)d_q v(\omega) = \sum_{j=0}^{\infty} u(q^j b)\left(v(q^j b) - v(q^{j+1}b)\right). \tag{1.47}$$

Definition 1.33: The improper q-integral of $u(x)$ on $[0,+\infty)$ is defined to be (Kac & Cheung, 2002):

$$\int_0^{\infty} u(\omega)d_q\omega = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} u(\omega)d_q\omega\,. \tag{1.48}$$

If $0 < q < 1$, or

$$\int_0^{\infty} \mathrm{u}(\omega) \mathrm{d}_{\mathrm{q}}\omega = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^{j+1}} \mathrm{u}(\omega) \mathrm{d}_{\mathrm{q}}\omega.$$

If $q > 1$.

Definition 1.34: For any positive real number $\tau > 0$ and $0 < |q| < 1$, q-Gamma function

$$\Gamma_q(\tau) = \int_0^{\infty} \omega^{\tau-1} E_q^{-q\omega} \mathrm{d}_{\mathrm{q}}\omega \tag{1.49}$$

Is the q-Gamma function and actually we can interpret the q-analogue of gamma function by using this expression (Kac & Cheung, 2002).

Lemma 1.35: The q-Gamma function has the following properties (Kac & Cheung, 2002)

(a) $\Gamma_q(\tau+1) = [\tau]_q \Gamma_q(\tau) \qquad for \quad \tau \in \mathbb{R}^+$

(b) $\Gamma_q(\mu) = [\mu-1]_q! \qquad for \quad \mu \in \mathbb{N}$

Proof: (a) First we should mention that

$$\begin{aligned}
\mathrm{D_q}(\mathrm{E_q^{-\omega}}) &= \mathrm{D_q}\left(\sum_{\mu=0}^{\infty} \frac{\omega^{\mu}(-1)^{\mu}\, \mathrm{q}^{\frac{\mu(\mu-1)}{2}}}{[\mu]_{\mathrm{q}}!}\right) \\
&= -\sum_{\mu=1}^{\infty} \frac{\omega^{\mu-1}(-1)^{\mu-1}\, \mathrm{q}^{\frac{\mu(\mu-1)}{2}}}{[\mu-1]_{\mathrm{q}}!} \\
&= -\sum_{\mu=0}^{\infty} \frac{(-\omega)^{\mu}\, \mathrm{q}^{\frac{\mu(\mu-1)}{2}}\, \mathrm{q}^{\mu}}{[\mu]_{\mathrm{q}}!} \\
&= -\mathrm{E_q^{-q\omega}} \qquad (1.50)
\end{aligned}$$

In addition, by using series expansion of E_q^{ω} we can see that $\mathrm{E_q^0} = 1$. Also,

$$\lim_{\omega\to\infty} \mathrm{E_q^{-\omega}} = \lim_{\omega\to\infty} \frac{1}{\mathrm{e_q^{\omega}}} = 0.$$

Now, like the normal case we will apply the integral by part. If we apply q-integral by part, we have;

$$\Gamma_q(\tau) = \int_0^\infty \omega^{\tau-1}\, E_q^{-q\omega} d_q\omega = -\int_0^\infty \omega^{\tau-1}\, d_q(E_q^{-\omega})$$

$$= \lim_{b\to\infty} \frac{-\omega^b}{e_q^b} + 0 \cdot E_q^0 + \int_0^\infty [\tau-1]_q\, \omega^{\tau-2} \cdot E_q^{-q\omega}\, d_q\omega$$

$$= [\tau-1]_q \Gamma_q(\tau-1).$$

(b)This equation can be reach by induction. For $\tau = 1$,

$$\Gamma_q(1) = \int_0^\infty E_q^{-q\omega}\, d_q\omega = \lim_{b\to\infty} -E_q^{\omega}\Big|_0^b = E_q^0 - \lim_{b\to\infty} \frac{1}{e_q^b} = 1 - 0 = 1$$

For any $\tau = \eta$ assume that the relation is true. For $\tau = \eta + 1$

$$\Gamma_q(\eta+1) = [\eta]_q \Gamma_q(\eta) = [\eta]_q [\eta+1]_q! = [\eta]_q!$$

Definition 1.36: For any positive real value constants t and s, the q-analogue of Beta function is (Kac & Cheung, 2002)

$$\beta_q(\tau, s) = \int_0^1 \omega^{\tau-1}\, (1-q\omega)_q^{s-1}\, d_q\omega. \tag{1.51}$$

Remark 1.37: If we tend s to infinity at (1.51) then

$$\Gamma_q(\tau) = \frac{1}{(1-q)^\tau} \lim_{s\to\infty} \beta(\tau, s) \tag{1.52}$$

Proof: First we should mention that according to the (definition 1.33) the improper q-integral of $u(\omega)$ on $[0, \infty)$ is defined by

$$\int_0^\infty u(\omega) d_q\omega = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^{j+1}} u(\omega) d_q\omega\,. \qquad 0 < q < 1$$

In addition, integral of the terms in the summation can be written as

$$\int_{q^j}^{q^{j+1}} u(\omega)d_q\omega = \int_0^{q^{j+1}} u(\omega)d_q\omega - \int_0^{q^j} u(\omega)d_q\omega$$

$$= (1-q)\sum_{\eta=0}^{\infty} q^{j+\eta+1}\, u\left(q^{j+\eta+1}\right) - (1-q)\sum_{\eta=0}^{\infty} q^{j+\eta}\, u\left(q^{j+\eta}\right)$$

$$= -(1-q)q^j u\left(q^j\right)$$

These two relations together show that:

$$\int_0^{\infty} u(\omega)d_q\omega = \sum_{j=-\infty}^{+\infty} (1-q)q^j u\left(q^j\right)$$

Now, consider $(1+q^j)_q^\infty$ where $j \in \mathbb{Z}$ and j is negative. Since $(1+q^j)_q^\infty = \prod_{\eta=1}^{\infty}(1-$ qj+η) then for some η, j+η=0 and (1−qj)q∞=0. Therefore

$$(1-q)\sum_{j=-\infty}^{0} q^j (q^j)^{t-1}\left(1-q^{j+1}\right)_q^\infty = 0$$

Thus, we have;

$$\lim_{s\to\infty} \beta_q(\tau,s) = \lim_{s\to\infty}\int_0^1 \omega^{\tau-1}\,(1-q\omega)_q^{s-1}\,d_q\omega = (1-q)\sum_{j=0}^{\infty} q^j (q^j)^{\tau-1}\left(1-q^{j+1}\right)_q^\infty$$

$$= (1-q)\sum_{j=-\infty}^{\infty} q^j (q^j)^{\tau-1}\left(1-q^{j+1}\right)_q^\infty = \int_0^\infty \omega^{\tau-1}\,(1-q\omega)_q^\infty\, d_q x$$

$$= \int_0^\infty \omega^{\tau-1}\,(1-q\omega)_q^{\frac{-q\omega}{1-q}}\, d_q\omega = (1-q)^\tau \int_0^\infty \omega^{\tau-1}\, E_q^{-q\omega}\, d_q\omega$$

$$= (1-q)^\tau \Gamma_q(\tau)$$

At the last line, we substitute ω by $(1-q)\omega$. This substitution is valid because of the definition of q-exponential function.

Proposition 1.38: The q-Gamma function has the following properties:

$$\text{(a) } \beta_q(\tau,s)=\frac{[\tau-1]_q}{[s]_q}\beta(\tau-1,s+1) \qquad \tau>0, s>0. \tag{1.53}$$

$$\text{(b) } \beta_q(\tau,\mu+1)=\beta_q(\tau,\mu)-\ q^{\mu}\beta_q(\tau+1,\mu) \quad \tau>1,\ \ \mu\in\mathbb{N}. \tag{1.54}$$

$$\text{(c) } \beta_q(\tau,\mu)=\frac{(1-q)(1-q)_q^{\mu+1}}{(1-q^t)_q^{\mu}} \qquad \tau>0,\ \ \mu\in\mathbb{N}. \tag{1.55}$$

$$\text{(d) } \beta_q(\tau,s)=\frac{(1-q)(1-q)_q^{\infty}(1-q^{\tau+s})_q^{\infty}}{(1-q^{\tau})_q^{\infty}\,(1-q^{s})_q^{\infty}} \qquad \tau\,,s>0. \tag{1.56}$$

Proof: (a) First, we should mention that

$$D_q(1-\omega)_q^s=-[s]_q(1-q\omega)_q^{s-1}$$

Now, apply the q-integral by part on a definition of q-Beta function:

$$\begin{aligned}\beta_q(\tau,s)&=\int_0^1\omega^{\tau-1}\,(1-q\omega)_q^{s-1}\,d_q\omega=-\frac{1}{[s]_q}\int_0^1\omega^{\tau-1}\,d_q\big((1-\omega)_q^s\big)\\&=-\frac{1}{[s]_q}\big((1-\omega)_q^s-0\big)+\frac{[\tau-1]_q}{[s]_q}\int_0^1\omega^{\tau-2}\,(1-q\omega)_q^s\,d_q\omega\end{aligned}$$

(b) This property is the direct consequence of definition of q-shifted factorial (1.15), because $(1-\omega q)_q^{\mu}=(1-\omega q)_q^{\mu-1}\cdot(1-\omega q^{\mu})$. Therefore

$$\begin{aligned}\beta_q(\tau,\mu+1)&=\int_0^1\omega^{\tau-1}\,(1-q\omega)_q^{\mu-1}(1-\omega q^{\mu})\,d_q\omega\\&=\int_0^1\omega^{\tau-1}\,(1-q\omega)_q^{\mu-1}-q^{\mu}\int_0^1\omega^{\tau}\,(1-q\omega)_q^{\mu-1}\,d_q\omega\end{aligned}$$

(c) Using part (a) and (b) together to reach

$$\beta_q(\tau,n+1)=\beta_q(\tau,n)-q^n\frac{[\tau]_q}{[n]_q}\beta_q(\tau,n+1)$$

Factorize $\beta_q(\tau,n+1)$ at left to have:

$$\beta_q(\tau,\mu+1)=\frac{1-q^{\mu}}{1-q^{\tau+\mu}}\beta_q(\tau,\mu). \tag{1.57}$$

Repeatedly, use (1.57) to reduce $\mu+1$ to 1 and use the fact that

$$\beta_q(\tau,1)=\int_0^1 \omega^{\tau-1}\ d_q\omega=\frac{1}{[\tau]_q}=\frac{1-q}{(1-q^{\tau})}$$

Thus,

$$\beta_q(\tau,\mu)=\frac{(1-q^{\mu-1})(1-q^{\mu-2})\dots(1-q)(1-q)}{(1-q^{\tau+\mu-1})(1-q^{\tau+\mu-2})\dots(1-q^{\tau+1})(1-q^{\tau})}$$
$$=\frac{(1-q)(1-q)_q^{\mu-1}}{(1-q^{\tau})_q^{\mu}}$$

(d) Since

$$(1-q)_q^{\mu-1}=\frac{(1-q)_q^{\infty}}{(1-q^{\mu})_q^{\infty}}\quad and\ \frac{1}{(1-q^{\tau})_q^{\mu}}=\ \frac{(1-q^{\tau+\mu})_q^{\infty}}{(1-q^{\mu})_q^{\infty}}$$

The equality holds true.

Remark 1.39: The following relation holds true:

$$\Gamma_q(r)=\frac{(1-q)_q^{\infty}}{(1-q)^{r-1}(1-q^{r})_q^{\infty}} \tag{1.58}$$

Proof: From the (Remark 1.37) and (1.55), we have:

$$\Gamma_q(r)=\frac{1}{(1-q)^{\tau}}\lim_{s\to\infty}\beta(\tau,s)=\frac{1}{(1-q)^{\tau}}\cdot\frac{(1-q)(1-q)_q^{\infty}}{(1-q^{\tau})_q^{\infty}}$$
$$=\frac{1}{(1-q)^{\tau-1}}\cdot\frac{(1-q)_q^{\infty}}{(1-q^{\tau})_q^{\infty}}, \tag{1.59}$$

Definition 1.40: In the power function $(\mathrm{a}-\mathrm{b})^{(r)}$ the q-analogue of with $r\in\mathbb{R}$ can be defined (Kac & Cheung, 2002)

$$(\mathrm{a}-\mathrm{b})^{(r)}=a^{r}\prod_{\mu=0}^{\infty}\frac{1-({}^{b}/{}_{a})q^{\mu}}{1-({}^{b}/{}_{a})q^{\mu+r}}\quad ,\qquad a,b\in\mathbb{R} \tag{1.60}$$

Note 1.41: According to the last definition, we may write the q-Gamma function with the new notation as;

$$\Gamma_q(r) = \frac{1}{(1-\mathrm{q})^{r-1}} \cdot \frac{(1-\mathrm{q})_{\mathrm{q}}^{\infty}}{(1-\mathrm{q}^{\mathrm{r}})_{\mathrm{q}}^{\infty}} = \frac{1}{(1-\mathrm{q})^{r-1}} \prod_{\mu=0}^{\infty} \frac{1-q^{r+1}}{1-q^{r+\mu+1}}$$

$$= \frac{(1-\mathrm{q})^{(r-1)}}{(1-\mathrm{q})^{r-1}} \qquad (1.61)$$

1.4 q-Hypergeometric Function and Hine's Transform Formula

In the next step, we introduce the q-analogue of Taylor expansion and in the aid of this theorem; we lead to the famous formula of q-binomial and Gauss summation formula. In addition, q-hypergeometric functions and Hine's transformation will be introduced. In the aid of these definitions, we will prove the most applicable lemma for q-difference equations.

Theorem 1.42: The following expression describe q-analogue of Taylor's theorem for all polynomial $g(x)$ of degree μ and any constant a,

$$g(x) = \sum_{m=0}^{n} (D_q^m u)(a) \frac{(x-a)_q^m}{[m]_q!} \qquad (1.62)$$

(Kac & Cheung, 2002)

Note 1.43: In q-Taylor formula put $g(x) = (x+a)_q^n$ and doing some calculations lead to the Gauss's binomial formula (Kac & Cheung, 2002)

$$(x+a)_q^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-1)/2} a^m \omega^{n-m} \qquad (1.63)$$

Now put $g(x) = \frac{1}{(1-x)_q^m}$ to have the following formula

$$\frac{1}{(1-x)_q^n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q x^n \qquad (1.64)$$

Let m & $n \to \infty$ in the above expression to lead Euler and Gauss summation formulae

$$(1-z)_q^{\infty} = (z;q)_{\infty} = \sum_{m=0}^{n} \frac{(-z)^m}{(q;q)_m} q^{m(m-1)/2} \qquad (1.65)$$

$$\frac{1}{(1-z)_q^{\infty}} = \frac{1}{(z;q)_\infty} = \sum_{m=0}^{n} \frac{z^m}{(q;q)_m} \tag{1.66}$$

We can define q-binomial coefficients with

$$\begin{bmatrix}\alpha\\ m\end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(m+1)\Gamma_q(\alpha-m+1)} = \frac{(q^{m+1};q)_\infty (q^{\alpha-m+1};q)_\infty}{(q;q)_\infty (q^{\alpha+1};q)_\infty},$$
$$\alpha, m, \alpha-m \in \mathbb{R}\backslash\{-1,-2,-3,\dots\}. \tag{1.67}$$

Especially,

$$\begin{bmatrix}\alpha\\ m\end{bmatrix}_q = \frac{(q^{-\alpha};q)_m}{(q;q)_m}(-1)^m q^{\alpha m} q^{-\binom{m}{2}} \quad (m \in \mathbb{N}). \tag{1.68}$$

Remark 1.44: Assume the geometric series as $\sum_{\eta=0}^{\mu} z^{\eta} = \frac{1}{1-z}$ which is convergent, where $|z| < 1$. Now taking normal derivative from two sides leads to the classical binomial theorem (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007)

$${}_1F_0\left(\begin{matrix}a\\ -\end{matrix}|z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \frac{1}{(1-z)^a}, \tag{1.69}$$

The same techniques can applied to have q-analogue of this expression as

$${}_1\varphi_0\left(\begin{matrix}a\\ -\end{matrix}|q;z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \tag{1.70}$$

If we put $a = 0$ then Gauss summation formula is given. Replacing z by $\frac{z}{a}$ and let $a \to \infty$ gives Euler formula. Actually we could directly leads to above expression by using q-Taylor formula for $\frac{(az;q)_\infty}{(z;q)_\infty}$. The notations ${}_1F_0\left(\begin{matrix}a\\ -\end{matrix}|z\right)$ and ${}_1\varphi_0\left(\begin{matrix}a\\ -\end{matrix}|q;z\right)$ are hypergeometric and q-hypergeometric functions which are defined at 1.48 and 1.49.

Lemma 1.45: We may change the order of the integration in Jackson double q-integral. The special case of this, which is too applicable, can be expressed as follow (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007):

$$\int_0^{\omega}\int_0^{v} u(s)d_q s\, d_q v = \int_0^{\omega}\int_{qs}^{\omega} u(s)d_q v\, d_q s \tag{1.71}$$

Proof: We can written easily R.H.S of this identity as follows:

$$\begin{aligned}\int_0^{\omega} u(s)\int_{qs}^{\omega} d_q v\, d_q s &= \int_0^{\omega} u(s)(\omega - qs)\, d_q s\\ &= \omega(1-q)\sum_{i=0}^{\infty} q^i u(\omega q^i)(\omega - \omega q^{i+1})\\ &= \omega^2(1-q)^2\sum_{i=0}^{\infty} q^i u(\omega q^i)\left(\frac{1-q^{i+1}}{1-q}\right)\\ &= \omega^2(1-q)^2\sum_{i=0}^{\infty} q^i u(\omega q^i)\left(\sum_{j=0}^{i} q^j\right)\\ &= \omega^2(1-q)^2\sum_{i=0}^{\infty}\sum_{j=0}^{i} q^{j+i}\, u(\omega q^i)\end{aligned}$$

In addition the left hand side, can be written in the aid of q-integral definition as

$$\begin{aligned}&\int_0^{\omega}\int_0^{v} u(s)d_q s\, d_q v\\ &\qquad= \omega(1-q)\sum_{i=0}^{\infty} q^i \int_0^{\omega q^i} u(s)d_q s\\ &\qquad= \omega^2(1-q)^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q^{j+2i}\, u(\omega q^{i+j})\end{aligned}$$

Let $i + j = t$ and apply the Cauchy product to see that both integrals are as the same.

Note 1.46: The previous lemma is the q-analogue of changing the order of iterated integral. In fact the double integral is evaluated on the right triangular region. Calculus techniques can be used to evaluate the boundary of double integral when we change the order of integration.

Theorem 1.47: Since we will use the Cauchy product temporary in this study, let us state the theorem to support the convergent of production for two series, suppose

(a) $\sum_{\mu=0}^{\infty} a_\mu$ converges absolutely

(b) $\sum_{\mu=0}^{\infty} a_\mu = A,$

(c) $\sum_{\mu=0}^{\infty} b_\mu = B,$

(d) $c_\mu \sum_{\eta=0}^{\infty} a_\eta\, b_{\mu-\eta} \qquad (\mu = 0,1,2, \dots)$

Then $\sum_{\mu=0}^{\infty} c_\mu = AB$

That is product of two convergent series converges, and to the right one of the two series converges completely. (Rudin, 1976)

1.5 Discussion about Hypergeometric Function

In this section, we introduce some concepts of special functions. We will apply them later on chapter 3 to prove semi-group properties of q-fractional integral operator. First let us define Pochhammer symbols to lead the hypergeometric functions

Definition 1.48: $(q)_\mu$is the (rising) Pochhammer symbol, which is defined by: (Andrews & Askey, 1999):

$$(q)_\mu = \begin{cases} 1 & \mu = 0 \\ q(q+1)\dots(q+\mu-1) & \mu > 0 \end{cases}, \tag{1.72}$$

The hypergeometric function can be expressed as power series

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix}; z\right] &= {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; z\right) \\ &= \sum_{\mu=0}^{\infty} \frac{(a_1)_\mu \dots \left(a_p\right)_\mu}{(b_1)_\mu \dots \left(b_q\right)_\mu} \frac{z^\mu}{\mu!}, \end{aligned} \tag{1.73}$$

Lemma 1.49: The q-hypergeometric function is defined as (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007)

$$ {}_2\varphi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\middle| q;\omega\right) = {}_2\varphi_1(a,b;c;q,x) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_\mu}{(c;q)_n (q;q)_\mu} \omega^\mu , \quad (1.74) $$

Where $(a;q)_n$ is q-shifted factorial and

$$ (a;q)_\mu = \prod_{i=0}^{\mu-1} \left(1 - aq^i\right) $$

Theorem 1.50: The following expressions for q-Hyper geometric functions hold (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007)

$$ \begin{aligned} {}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| q;z\right) &= \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/b & z \\ & az \end{matrix}\middle| q;b\right) \\ &= \frac{\left(c/b\, ;q\right)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} abz/c & b \\ & bz \end{matrix}\middle| q; c/b\right) \\ &= \frac{\left(abz/c\, ;q\right)_\infty}{(z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/a & c/b \\ & c \end{matrix}\middle| q; abz/c\right) \end{aligned} $$

Proof: According to the definition of q-hypergeometric function, we have

$$ {}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| q;z\right) = \sum_{\eta=0}^{\infty} \frac{(a;q)_\eta (b;q)_\eta}{(c;q)_\eta (q;q)_\eta} z^\eta $$

Now at the expression in the summation we can substitute the following parts:

$$ \frac{(b;q)_\eta}{(c;q)_\eta} = \frac{(b;q)_\infty}{(bq^\eta;q)_\infty} \frac{(cq^\eta;q)_\infty}{(c;q)_\infty} = \frac{(b;q)_\infty}{(c;q)_\infty} \times \frac{(cq^\eta;q)_\infty}{(bq^\eta;q)_\infty} $$

The last part of above expression can be rewritten by using q-binomial theorem (1.72)

$$\frac{(cq^{\eta};q)_{\infty}}{(bq^{\eta};q)_{\infty}} = \frac{\left({}^{c}/_{b}\, bq^{\eta};q\right)_{\infty}}{(bq^{\eta};q)_{\infty}} = {}_{1}\varphi_{0}\left({}^{c}\!\underline{/}b|q;bq^{\eta}\right)$$

$$= \sum_{\mu=0}^{\infty} \frac{\left({}^{c}/_{b}\, q^{\eta};q\right)_{\mu}}{(q;q)_{\mu}} (bq^{\eta})^{\mu} \Rightarrow {}_{2}\varphi_{1}\left({}^{a}{}_{c}{}^{b}|q;z\right)$$

$$= \frac{(b;q)_{\infty}}{(c;q)_{\infty}} \sum_{\eta=0}^{\infty} \frac{(a;q)_{\eta}}{(q;q)_{\eta}} \left(\sum_{\mu=0}^{\infty} \frac{\left({}^{c}/_{b}\, q^{\eta};q\right)_{\mu}}{(q;q)_{\mu}} (bq^{\mu})^{\mu} \right) z^{\eta}$$

$$= \frac{(b;q)_{\infty}}{(c;q)_{\infty}} \sum_{\mu=0}^{\infty} \frac{\left({}^{c}/_{b}\, q^{\eta};q\right)_{\mu}}{(q;q)_{\mu}} b^{\mu} \sum_{\eta=0}^{\infty} \frac{(a;q)_{\eta}}{(q;q)_{\eta}} (zq^{\mu})^{\eta}$$

Now we can apply q-binomial theorem again in the last part which is

$$\sum_{\eta=0}^{\infty} \frac{(a;q)_{\eta}}{(q;q)_{\eta}} (zq^{\mu})^{\eta} = {}_{1}\varphi_{0}\left({}^{a}_{\_}|q;zq^{\mu}\right) = \frac{(azq^{\mu};q)_{\infty}}{(zq^{\mu};q)_{\infty}} = \frac{(az;q)_{\infty}/(az;q)_{\mu}}{(z;q)_{\infty}/(z;q)_{\mu}}$$

If we put all of these together, we have

$${}_{2}\varphi_{1}\left({}^{a}{}_{c}{}^{b}|q;z\right) = \frac{(b;q)_{\infty}}{(c;q)_{\infty}} \frac{(az;q)_{\infty}}{(z;q)_{\infty}} \sum_{\mu=0}^{\infty} \frac{\left({}^{c}/_{b}\, q^{\eta};q\right)_{\mu} (z;q)_{\mu}}{(q;q)_{\mu} (az;q)_{\mu}} b^{\mu}$$

This is the first part of this theorem. For the second transformation, we can apply the first transformation again, means

$$
{}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| q; z\right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/b & z \\ & az \end{matrix}\middle| q; b\right)
$$

$$
= \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} z & c/b \\ & az \end{matrix}\middle| q; b\right)
$$

$$
= \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \frac{\left(c/b\,;q\right)_\infty (bz;q)_\infty}{(az;q)_\infty (b;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} azb/c & b \\ & zb \end{matrix}\middle| q; c/b\right)
$$

$$
= \frac{\left(c/b\,;q\right)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} azb/c & b \\ & zb \end{matrix}\middle| q; c/b\right)
$$

In fact, we can change the order of the terms in the q-hypergeometric functions. This is the second transformation. For the last transformation, we apply the first transformation again on the last expression to have:

$$
{}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| q; z\right) = \frac{\left(c/b\,;q\right)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} b & abz/c \\ & bz \end{matrix}\middle| q; c/b\right)
$$

$$
= \frac{\left(c/b\,;q\right)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty} \frac{\left(abz/c\,;q\right)_\infty (c;q)_\infty}{(bz;q)_\infty \left(c/b\,;q\right)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/a & c/b \\ & c \end{matrix}\middle| q; b\right)
$$

This is the last transformation.

We apply this transformation in the next lemma, which describe the identity for q-expression. Before that, let us define the notation

Lemma 1.51: For $\omega, y, r \in \mathbb{R}^+$ and $s, \tau \in \mathbb{N}$ in the following properties we have (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007):

$$
(\omega - yq^s)^{(r)} = \omega^r (1 - q^s\, y/\omega)^{(r)} \tag{1.75}
$$

$$
\frac{(\omega - yq^s)^{(r)}}{(\omega - y)^{(r)}} = \frac{(q^r\, y/\omega\,; q)_s}{(y/\omega\,; q)_s} \tag{1.76}
$$

$$
(q^\tau - q^s)^{(r)} = 0 \qquad (s \le \tau) \tag{1.77}
$$

Remark 1.52: According to the (Definition 1.40), we have (Kac & Cheung, 2002)

$$(a-b)^{(r)} = a^r \prod_{\eta=0}^{\infty} \frac{1-\frac{b}{a}q^{\eta}}{1-\frac{b}{a}q^{\eta+r}} = a^r \frac{\left(\frac{b}{a};q\right)_{\infty}}{\left(\frac{b}{a}q^r;q\right)_{\infty}} = a^r \left(\frac{b}{a};q\right)_r = a^r \left(1-\frac{b}{a}\right)_q^r$$

In the aid of (Note1.43) we have

$$(a-b)^{(r)} = a^r \left(1-\frac{b}{a}\right)_q^r = \sum_{k=0}^{\mu} \begin{bmatrix}\mu\\k\end{bmatrix}_q q^{\eta(\eta-1)/2} \left(\frac{-b}{a}\right)^{\eta}$$

In addition, for q-combination term we have

$$\begin{bmatrix}\mu\\\eta\end{bmatrix}_q = \frac{(1-q^{\mu})\dots(1-q)}{(1-q^{\eta})\dots(1-q)(1-q^{\mu-\eta})\dots(1-q)} = \frac{(1-q^{\mu})\dots(1-q^{\mu-\eta+1})}{(q;q)_{\eta}}$$

$$= \frac{(-1)^{\eta}q^{\mu}(1-q^{-n})\dots q^{\mu-\eta+1}(1-q^{\eta-\mu-1})}{(q;q)_{\eta}}$$

$$= \frac{(-1)^{\eta}(q^{-\mu};q)_{\eta}q^{\mu}\dots q^{\mu-\eta+1}}{(q;q)_{\eta}} = \frac{(q^{-\mu};q)_{\eta}}{(q;q)_{\eta}}(-1)^{\eta}q^{\mu\eta}q^{-\frac{\eta(\eta-1)}{2}}$$

Lemma 1.53: For real and positive constants τ, r and η , we have (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007):

$$\sum_{t=0}^{\infty} \frac{(1-\delta q^{1-\tau})^{(r-1)}(1-q^{1+\tau})^{(\eta-1)}}{(1-q)^{(r-1)}(1-q)^{(\eta-1)}}(q^r)^{\tau}$$

$$= \frac{(1-\delta q)^{(r+\eta-1)}}{(1-q)^{(r+\eta-1)}}, \qquad (1.78)$$

Proof: According to formula (1.20), (1.22) and (1.60) we have

$$(1-\delta q^{1-\tau})^{(r-1)} = \frac{(\delta q^{1-\tau};q)_{\infty}}{(\delta q^{r-\tau};q)_{\infty}} = \frac{(\delta q^{1-\tau};q)_{\tau}}{(\delta q^{r-\tau};q)_{\tau}}\frac{(\delta q;q)_{\infty}}{(\delta q^{r};q)_{\infty}}$$

$$= \frac{(\delta q^{1-\tau};q)_{\tau}}{(\delta q^{r-\tau};q)_{\tau}}(1-\delta q)^{(r-1)}$$

$$= \frac{(1-\delta q^{1-\tau})(1-\delta q^{2-\tau})\dots(1-\delta q^{-1})(1-\delta)}{(1-\delta q^{r-\tau})(1-\delta q^{r-\tau+1})\dots(1-\delta q^{r-2})(1-\delta q^{r-1})}(1-\delta q)^{(r-1)}$$

Now write the expression at fraction from right to left and factorize $(-\delta)^{\tau-1}$ to have:

$$(1-\delta q^{1-\tau})^{(r-1)} = \frac{(-\delta)^{\tau-1}\,(1-\delta^{-1})(q^{-1}-\delta^{-1})\dots(q^{-\tau}-\delta^{-1})(q^{1-\tau}-\delta^{-1})}{(-\delta)^{\tau-1}(q^{r-1}-\delta^{-1})(q^{r-2}-\delta^{-1})\dots(q^{r-\tau+1}-\delta^{-1})(q^{r-\tau}-\delta^{-1})}\ (1-\delta q)^{(r-1)}$$

Now factorize q-terms from each brackets, means factorize $q^{-1}.q^{-2}\dots q^{1-\tau}=q^{-\tau(\tau-1)/2}$ from the numerator and $q^{r-1}.q^{r-2}\dots q^{r-\tau}=q^{\tau r-\tau(\tau+1)/2}$ from denominator.

$$(1-\delta q^{1-\tau})^{(r-1)} = \frac{(1-\delta^{-1})(1-q\delta^{-1})\dots(1-\delta^{-1}q^{\tau})(1-\delta^{-1}q^{\tau-1})q^{-\tau(\tau-1)/2}}{(1-\delta^{-1}q^{1-r})(1-\delta^{-1}q^{2-r})\dots(1-\delta^{-1}q^{\tau-1-r})(1-\delta^{-1}q^{\tau-r})q^{\tau r-\tau(\tau+1)/2}}(1-\delta q)^{(r-1)} = \frac{(\delta^{-1};q)_\tau q^{-\tau(\tau-1)/2}q^{-\tau r+\tau(\tau+1)/2}}{(\delta^{-1}q^{1-r};q)_\tau}(1-\delta q)^{(r-1)}$$

$$= \frac{(\delta^{-1};q)_\tau q^{\tau(1-r)}}{(\delta^{-1}q^{1-r};q)_\tau}(1$$

In addition, for another fraction of summation, we can write it as

$$\frac{(1-q^{1+\tau})^{(\eta-1)}}{(1-q)^{(\eta-1)}} = \frac{(1-q^{1+\tau})(1-q^{2+\tau})\dots(1-q^{\eta})(1-q^{\eta+1})\dots}{(1-q^{\eta+\tau})(1-q^{\eta+\tau+1})\dots(1-q)(1-q^2)\dots}$$

After cancelling the common terms, we have

$$\frac{(1-q^{1+\tau})^{(\eta-1)}}{(1-q)^{(\eta-1)}} = \frac{(1-q^{\eta})(1-q^{\eta+1})\dots(1-q^{\eta+\tau-1})}{(1-q)(1-q^2)\dots(1-q^{\tau})}$$

$$= \frac{(q^{\eta};q)_\tau}{(q;q)_\tau} \qquad (1.80)$$

In the aid of (1.79) and (1.80), the left part of (1.78) can be write as

$$\sum_{t=0}^{\infty}\frac{(1-\delta q^{1-\tau})^{(r-1)}(1-q^{1+\tau})^{(\eta-1)}}{(1-q)^{(r-1)}(1-q)^{(\eta-1)}}(q^r)^\tau$$

$$=\frac{(1-\delta q)^{(r-1)}}{(1-q)^{(r-1)}}\sum_{t=0}^{\infty}\frac{(\delta^{-1};q)_\tau}{(\delta^{-1}q^{1-r};q)_\tau}\frac{(q^\eta;q)_\tau}{(q;q)_\tau}(q^{\alpha r})^\tau q^{\tau(1-r)}$$

$$=\frac{(1-\delta q)^{(r-1)}}{(1-q)^{(r-1)}}\,{}_2\varphi_1\left(\begin{matrix}\delta^{-1} & q^\eta\\ \delta^{-1}q^{1-\alpha} & \end{matrix}\middle| q;q\right)$$

Now, use Heine′s q − Euler Transformation to have

$$L.H.S=\frac{(1-\delta q)^{(r-1)}}{(1-q)^{(r-1)}}\frac{(q^{\eta+r};q)_\infty}{(q;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}q^{1-r} & \delta^{-1}q^{1-\alpha-\eta}\\ & \delta^{-1}q^{1-\alpha}\end{matrix}\middle| q;q^{\eta+r}\right)$$

$$=\frac{(1-\delta q)^{(r-1)}}{(1-q)^{(r-1)}}\frac{1}{(1-q)^{(r+\eta-1)}}\sum_{n=0}^{\infty}\frac{(q^{1-r};q)_n(\delta^{-1}q^{1-r-\eta};q)_n}{(q;q)_n(\delta^{-1}q^{1-r};q)_n}(q^{r+\eta})^\mu$$

Let us take a look at the expression inside of the summation, we have

$$\frac{(\delta^{-1}q^{1-r-\eta};q)_\mu}{(\delta^{-1}q^{1-r};q)_\mu}=\frac{(1-\delta^{-1}q^{1-r-\eta})(1-\delta^{-1}q^{2-r-\eta})\ldots(1-\delta^{-1}q^{n-r-\eta})}{(1-\delta^{-1}q^{1-r})(1-\delta^{-1}q^{2-r})\ldots(1-\delta^{-1}q^{n-r})}$$

$$=\frac{(-\delta)^{-\mu}(q^{\mu-r-\eta}-\mu)(q^{\mu-r-\eta+1}-\mu)\ldots(q^{1-r-\eta}-\mu)}{(-\delta)^{-\mu}(q^{\mu-r}-\mu)(q^{\mu-r-1}-\mu)\ldots(q^{1-r}-\mu)}$$

$$=\frac{q^{n-r-\eta}(1-\delta q^{r+\eta-\mu})q^{\mu-r-\eta+1}(1-\delta q^{r+\eta-\mu+1})\ldots q^{1-r-\eta}(1-\delta q^{r+\eta-1})}{q^{\mu-r}(1-\delta q^{r-\mu})q^{\mu-\alpha-1}(1-\delta q^{r-\mu+1})\ldots q^{1-r}(1-\delta q^{r-1})}$$

$$=\frac{(\delta q^{r+\eta-\mu};q)_\mu}{(\delta q^{r-\mu};q)_\mu}q^{\mu^2-\mu\alpha-\mu\eta+{}^{\mu(\mu-1)}/_2-\mu^2-\mu r-{}^{\mu(\mu-1)}/_2}=\frac{(\delta q^{r+\eta-\mu};q)_\mu}{(\delta q^{r-\mu};q)_\mu}q^{-\mu\eta}$$

$$=\frac{(\delta q^{r+\eta-\mu};q)_\infty}{(\delta q^{r-\mu};q)_\infty}\frac{(\delta q^r;q)_\infty}{(\delta q^{r+\eta};q)_\infty}q^{-\mu\eta}$$

Therefore, we can substitute this in the expression of L.H.S and

$$L.H.S=\frac{(1-\delta q)^{(r-1)}}{(1-q)^{(r-1)}}\frac{1}{(1-q)^{(r+\eta-1)}}\frac{(\delta q^r;q)_\infty}{(\delta q^{r+\eta};q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{1-r};q)_n(1-\delta q^{r+\eta-\mu})^{(-\eta)}}{(q;q)_n}q^{\alpha n}$$

$$=\frac{(1-\delta q)^{(r+\eta-1)}}{(1-q)^{(r-1)}}\frac{1}{(1-q)^{(r+\eta-1)}}\sum_{n=0}^{\infty}\frac{(q^{1-r};q)_n(1-\delta q^{r+\eta-\mu})^{(-\eta)}}{(q;q)_n}q^{r\mu}$$

Now apply (Remark 1.52) for the last summation to have

$$\sum_{n=0}^{\infty}\frac{(q^{1-r};q)_n(1-\delta q^{r+\eta-\mu})^{(-\eta)}}{(q;q)_\mu}q^{\alpha\mu}$$
$$=\sum_{n=0}^{\infty}\binom{\alpha-1}{\mu}_q(-1)^\mu q^{-\mu(r-1)+{}^{\mu(\mu-1)}/_2}\,q^{r\mu}(1-\delta q^{r+\eta-\mu})^{(-\eta)}$$
$$=\sum_{n=0}^{\infty}\binom{r-1}{\mu}_q(-1)^{n\mu}q^{-\mu(r-1)+{}^{\mu(\mu-1)}/_2}\,q^{\alpha\mu}\sum_{k=0}^{\infty}\binom{-\eta}{k}_q q^{k(k-1)/2}(\delta q^{\alpha+\eta-\mu})^k(-1)^k$$

Change the order of summation to reach

$$\sum_{t=0}^{\infty}(-1)^t\binom{-\eta}{k}_q q^{\tau(\tau-1)/2}(\delta q^{r+\eta})^\tau\sum_{n=0}^{\infty}(r)_q(-1)^{n\mu}\,q^{\mu(1-\tau)}q^{{}^{\mu(\mu-1)}/_2}$$

Again apply (Remark 1.52) for the last part of summation to have

$$\sum_{t=0}^{\infty}(-1)^t\binom{-\eta}{t}_q q^{\tau(\tau-1)/2}(\delta q^{r+\eta})^t(1-q^{1-\tau})^{(r-1)}$$

But $(1-q^{1-\tau})^{(r-1)}=0$ when $\tau\neq 0$. So this expression has non-zero value for only $\tau=0$, which is $(1-q)^{(r-1)}$. Finally we will write in the left part

$$\frac{(1-\delta q)^{(r+\eta-1)}}{(1-q)^{(r-1)}}\frac{1}{(1-q)^{(r+\eta-1)}}(1-q)^{(r-1)}=\frac{(1-\delta q)^{(r+\eta-1)}}{(1-q)^{(r+\eta-1)}}.$$

Lemma 1.54: For real positive α and $\lambda>-1$ the following identity holds (Rajkovi´c, Marinkovi´c, & Stankovi´, 2007):

$$\int_a^{\omega}(\omega-q\tau)^{(r-1)}\,(t-a)^{(\lambda)}d_q\tau$$
$$=\frac{\Gamma_q(r)\Gamma_q(\lambda+1)}{\Gamma_q(r+\lambda+1)}(\omega-a)^{(\lambda+r)}\qquad 0<a<\omega<b$$

Proof: we can expand the q-integral of the left side as

$$\int_0^{\omega}(\omega-q\tau)^{(r-1)}\,(t-a)^{(\lambda)}d_q\tau-\int_0^{a}(\omega-qt)^{(r-1)}\,(t-a)^{(\lambda)}d_q\tau$$

The second part of the integral is zero, we can show it by taking q-derivative and find that the q-derivative is zero, or we can expand it by the definition of q-integral as follow

$$\int_0^{a}(\omega-q\tau)^{(r-1)}\,(\tau-a)^{(\lambda)}d_q\tau$$

$$=a(1-q)\sum_{i=0}^{\infty}q^i\left(\omega-aq^{i+1}\right)^{(r-1)}\left(aq^i-a\right)^{(\lambda)}=0$$

Now apply (Lemma 1.53) when we put λ instead of $r-1$ and r instaed of β and μ is $\frac{a}{qx}$ for the first integral to have

$$\int_0^{\omega}(\omega-qt)^{(r-1)}\,(t-a)^{(\lambda)}d_qt$$

$$=\omega(1-q)\sum_{i=0}^{\infty}q^i\omega^{r-1}\left(1-q^{i+1}\right)^{(r-1)}\left(\omega q^i-a\right)^{(\lambda)}$$

$$=\omega^{\lambda+r}(1-q)\sum_{i=0}^{\infty}q^i\left(1-q^{i+1}\right)^{(r-1)}q^{\lambda i}\left(1-\frac{a}{q\omega}q^{1-i}\right)^{(\lambda)}$$

$$=\omega^{\lambda+r}(1-q)\frac{\left(1-\frac{a}{q\omega}q\right)^{(\lambda+r)}}{(1-q)^{(\lambda+r)}}(1-q)^{(\lambda)}(1-q)^{(r-1)}$$

$$=(1-q)\frac{(\omega-a)^{(\lambda+r)}}{(1-q)^{(\lambda+r)}}(1-q)^{(\lambda)}(1-q)^{(r-1)}$$

$$=\frac{(1-q)^{\lambda+r}(1-q)^{(\lambda)}(1-q)^{(r-1)}}{(1-q)^{\lambda}(1-q)^{r-1}(1-q)^{(\lambda+r)}}(\omega-a)^{(\lambda+r)}$$

$$=\frac{\Gamma_q(r)\Gamma_q(\lambda+1)}{\Gamma_q(r+\lambda+1)}(\omega-a)^{(\lambda+r)}.$$

We should mention that, this identity is essentially useful when we want to prove the semi-group properties of q-fractional difference equations. In the aid of this lemma we will prove this property in the Chapter 4.

CHAPTER 2

FIXED POINT THEOREM AND CONCAVE OPERATOR

In this chapter, we study the fixed point theorem in Banach space. A fixed point theorem is a result that says a given function which satisfies some conditions will have at least one fixed point i.e. a point x for which $u(x) = x$. Generally, conditions for operator $T: A \to A$ were studied and this result can be used to verify uniqueness and existence of solution for differential equations. We start by introducing some definitions.

2.1 Some Concepts and Definitions

Definition 2.1: The metric space is a space X that equipped with a nonnegative function $d: X \to \mathbb{R}^+$ such that for any $x, y, z \in X$, function d has the following properties (Rudin, 1976):

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.2: The sequence $\{a_n\}$ is Cauchy, if for given $\varepsilon > 0$ there exist $M > 0$ such that for any n and m greater than M we have $d(a_n, a_m) < \varepsilon$. It is obvious that any convergent sequence is Cauchy sequence but the inverse is only true when the space is called complete for example Euclidian spaces $\mathbb{R}^n$ are complete spaces. (Rudin, 1976)

Definition 2.3: A vector space X is said to be a normed space, if for any vector $x \in X$, there is associated a nonnegative real number, say $\|x\|$ or norm of x, such that for $x, y \in X$ and scalar α:

1. $\|x + y\| \leq \|x\| + \|y\|$
2. $\|\alpha x\| = |\alpha| \|x\|$
3. $\|x\| > 0 \quad if \quad x \neq 0$

The word "norm" sometimes used to denote nonnegative function which maps x to $\|x\|$. In the aid of norm definition, we may define corresponding metric space by defining $d(x, y) = \|x - y\|$. Banach space is a normed space which is complete in the metric that is defined by its norm. Many of familiar function spaces are Banach spaces. For example, we can mention Hilbert spaces, space of continuous function on compact spaces, certain spaces of differentiable functions and spaces that contain all continuous linear functions from one Banach space into another one (Rudin, 1976).

Definition 2.4: An order in a set A is denoted by $<$ and is a relation which has the following properties (Rudin, 1976):

1. For any elements $a, b \in A$ we have $a < b$ or $b < a$ or $a = b$,
2. For any elements $a, b, c \in A$, if $a < b$ and $b < c$ then $a < c$.

In this case A is called ordered set. For example rational numbers is an order set with definition of the order as $x < y$ if $y - x$ belongs to positive rational numbers. We use the notation $x \leq y$ when $x < y$ or $x = y$. The partial order $\leq$ on a set A is a relation which has following properties:

1. For any $a \in A$, we have $a \leq a$. (Reflexivity)
2. For any elements $a, b \in A$, if $a \leq b$ and $b \leq a$ then $a = b$.(Antisymmetric)
3. For any elements $a, b, c \in A$, if $a \leq b$ and $b \leq c$ then $a \leq c$.(Transitivity)

The size of the longest chain is called the partial order length. The famous example of partially ordered set is assuming Hasse diagram which assume being subset with symbol of $\subseteq$ as a partial order on the given finite set.

Definition 2.5: Let $(E, \|.\|)$ be a real Banach space, then nonempty closed convex set $P \subseteq E$ is called a Cone, if the following conditions hold true (Krasnoselskii, 1964):

1. For $x \in P$ and $r \geq 0$ as scalar, $rx \in P$.
2. If $x \in P$ and $-x \in P$ then x should be the zero element of $(E, \|.\|)$ which we denote it by θ. By another words, nonzero elements of E has not any additional inverse in P.

This cone induces the partial order on $(E, \|.\|)$. In this case, $x \leq y$ if $y - x \in P$. P is called normal, if there is a constant $M > 0$ such that for all $x, y \in E$ where $\theta \leq x \leq y$ implies that $\|x\| \leq M\|y\|$. Here, M is infimum of such constants and is called the normality constant of cone P. We can define increasing operator like the increasing function, the operator $T: E \rightarrow E$ is increasing if and only if $x \leq y$ implies $T(x) \leq T(y)$. We define the equivalence relation $\sim$ on $(E, \|.\|)$ as $x \sim y$ if and only if there exists two positive constants $\tau, \mu > 0$ such that $\tau x \leq y \leq \mu x$. If we assume $E = \mathbb{R}$ then this relation is true for all real number x and y. In fact, Archimedes properties of real numbers guarantee that for some $N \in \mathbb{N}$, $\frac{1}{N}x \leq y \leq Nx$. This equivalence relation makes the partition on $(E, \|.\|)$. Let h be a nonzero element of $(E, \|.\|)$ means $h \neq \theta$ and let $h \geq \theta$ (in the means of partial order on $(E, \|.\|)$), then the class of h is denoted by P_h which is defined as $P_h = \{x \in E | x \sim h\}$.

Remark 2.6: According to the definition 2.5, we can show that $P_h \subseteq P$. For this, Assume that $x \in P_h$ then $x \sim h$ means there exist $\mu > 0$ such that $h \leq \mu x$, but $\theta \leq h$, transitivity of partial order guarantees that $\theta \leq \mu x$, it means $\mu x - \theta = \mu x \in P$. Since $\frac{1}{\mu} > 0$, According to the second property of cone, we have $\frac{1}{\mu}\mu x = x \in P$. (Krasnoselskii, 1964)

Remark 2.7: In addition, P_h is a convex set. For seeing this, let us assume that $x, y \in P_h$ and let $0 \leq t \leq 1$ then there exist $\alpha, \beta, \gamma, \delta > 0$ such that $\alpha x \leq h \leq \beta x$ and $\gamma y \leq h \leq \delta y$. Since P is a cone $a \leq b$ implies that $b - a \in P$ then $t(b - a) \in P$ means $ta \leq a$. Now multiply all inequalities by t and $1 - t$ to have $\alpha tx \leq th \leq \beta tx$ and $\gamma(1 - t)y \leq (1 - t)h \leq \delta(1 - t)y$. This part specify that $tP_h = P_h$. Furthermore, the addition of these inequalities confirm the convexity of P_h. It is obvious that in the partial ordered and in the general case, we cannot add two inequalities. Here, $\alpha tx \leq th$ means $th - \alpha tx \in P$ and $\gamma(1 - t)y \leq (1 - t)h$ means $(1 - t)h - \gamma(1 - t)y \in P$ and implies that $th - \alpha tx + (1 - t)h - \gamma(1 - t)y = h - \alpha tx - \gamma(1 - t)y \in P$. The last expression is true because of equivalency of $\sim$ and convexity and closed properties of P. (Krasnoselskii, 1964)

Note 2.8: Let us list some assumption on the operator $T: P \to P$. These assumptions will be used temporary in this chapter:

A_1)The operator $T: P \to P$ is increasing.

A_2) For any point $x \in P$ and $0 < t < 1$, there exists $t < \varphi(t) \leq 1$ such that $T(tx) \geq \varphi(t)T(x)$, here $\varphi(t)$ is a function of t.

A_3) For any point $x \in P$ and $0 < t < 1$, there exists $t < \varphi(t, x) \leq 1$ such that $T(tx) \geq \varphi(t, x)T(x)$. Where $\varphi(t, x)$ is a function of t and x and decreasing in x while t is fixed.

A_4) Let θ be the zero of $(E, \|.\|)$, then there exists $h > \theta$ (means $h \geq \theta$ and $h \neq \theta$) and $0 < t_0 < 1$ such that $t_0 h \leq T(h) \leq {t_0}^{-1}\varphi(t_0, {t_0}^{-1}h)h$, where $\varphi(t, x)$ is given as A_3.

2.2. Existence and Uniqueness of Positive Solutions for Nonlinear Operators

In this section, in the aid of definitions of the last section, we will study the existence and uniqueness of $x = Ax + x_0$ on real Banach space $(E, \|.\|)$ which is partially ordered by the cone P. This kind of fixed point theorem is not related to the continuity of operator and compactness of the space. We start it by introducing a lemma.

Lemma 2.9: Assume that for operator T which satisfies A_1) and A_2) $Th \in P_h$. Then there exists $u_o, v_0 \in P_h$ and $0 < r < 1$ such that $rv_0 \leq u_0 < v_0$ and $u_0 \leq T(u_0) \leq T(v_0) < v_0$. (Zhai, Yang, & Zhan, 2010)

Proof: From assumption, $h > \theta$ or $h - \theta = h \in P$. On the other hand, the inequality $\alpha < \beta$, implies that $\alpha h \leq \beta h$ (because $0 < \beta - \alpha$ and $h \in P$). Now according to the assumption of lemma, $Th \in P_h$, means , $Th \sim h$ or there exist $\alpha, \beta > 0$ such that $\alpha h \leq T(h) \leq \beta h$. Let $\beta = \frac{1}{t_0}$ for some $0 < t_0 < 1$. If it is not between 0 and 1, we can choose t_0 such that $\beta < \frac{1}{t_0}$ and the procedure will be the same. Since $\alpha < \beta$, we have $t_0 h \leq \alpha h$ and

$$t_0 h \leq T(h) \leq \frac{1}{t_0} h \qquad (2.1)$$

According to the A_2) condition, T is generalized concave and for $\varphi(t_0)$ we have $T(t_0x) \geq \varphi(t_0)T(x)$. In addition $t_0 < \varphi(t_0) \leq 1$ implies that $1 < \frac{\varphi(t_0)}{t_0}$, so there exists $0 < k$ such that

$$\left(\frac{\varphi(t_0)}{t_0}\right)^k \geq \frac{1}{t_0} \qquad (2.2)$$

In this step, let $u_0 = {t_0}^k h$ and $v_0 = \frac{1}{{t_0}^k} h$. Since u_0 and v_0 are the coefficients of $h, u_0, v_0 \in P_h$. In addition, $u_0 = {t_0}^k h = {t_0}^{2k} \frac{1}{{t_0}^k} h = {t_0}^{2k} v_0 < v_0$. If we choose r such that $0 < r \leq {t_0}^{2k} < 1$, then $rv_0 = r\frac{1}{{t_0}^k} h \leq {t_0}^k h = u_0$. According to the A_1) condition, T is increasing, in the addition of A_2) and in the aid of (2.1) and (2.2), we have

$$\begin{aligned} Tu_0 = T({t_0}^k h\,) = T(t_0{t_0}^{k-1} h\,) \geq \varphi(t_0)T({t_0}^{k-1}h) \ldots \geq (\varphi(t_0))^k T(h) \\ \geq (\varphi(t_0))^k t_0 h \geq {t_0}^k h = u_0 \end{aligned}$$

Now we can apply the second condition for $\frac{1}{t}x$ instead of x, means $T(x) = T(t\frac{1}{t}x) \geq \varphi(t)T(\frac{1}{t}x)$ or $T(\frac{1}{t}x) \leq \frac{1}{\varphi(t)}T(x)$. In a same procedure, we can see that

$$\begin{aligned} Tv_0 = T\left(\frac{1}{{t_0}^k} h\right) = T\left(\frac{1}{t_0}\frac{1}{{t_0}^{k-1}} h\right) \leq \frac{1}{\varphi(t_0)} T\left(\frac{1}{{t_0}^{k-1}} h\right) \ldots \\ \leq \left(\frac{1}{\varphi(t_0)}\right)^k T(h) \leq \left(\frac{1}{\varphi(t_0)}\right)^k \frac{1}{t_0} h \leq \frac{1}{{t_0}^k} h = v_0 \end{aligned}$$

Theorem 2.10: Assume that the cone P is normal in the means of definition 2.5, operator $T: P \to P$ is satisfied A_1) and A_2). In addition, $x_0 \in P$ implies that $T(h) + x_0 \in P_h$, then operator equation $x = T(x) + x_0$ has a unique solution in P_h. (Zhai, Yang, & Zhan, 2010)

Proof: First of all, let us define another operator which has the same properties. Let $A(x)$ be the operator that is defined for all $x \in P$ as $A(x) = T(x) + x_0$. Since $x_0 \in P$ and P is convex set, $T(x) + x_0$ should belongs to P. Thus $A: P \to P$ and because of increasing of T, this operator is increasing as well. In addition, If $0 < \lambda < 1$ and $x \in P$, then in the aid of A_2) we have:

$$A(\lambda x) = T(\lambda x) + x_0 \geq \varphi(\lambda)T(x) + x_0 \geq \varphi(\lambda)(T(x) + x_0) = \varphi(\lambda)A(x)$$

Therefore, operator A has both A_1) and A_2) conditions. Now apply (Lemma 2.9) to verify that for $u_o, v_0 \in P_h$ and $0 < r < 1$ we have

$$rv_0 \leq u_0 < v_0 , \qquad u_0 \leq A(u_0) \leq A(v_0) \leq v_0 \tag{2.3}$$

In this step, we construct the recurrence sequence by using operator A temporary on the elements $u_o, v_0 \in P_h$, means

$$u_k =: A(u_{k-1}) \quad and \quad v_k =: A(v_{k-1}) \quad for\ k = 1,2,3, \dots$$

As we mentioned it before, A is increasing operator and from (2.3) we know $u_0 < v_0$ so $u_1 = A(u_0) < A(v_0) = v_1$. By using same logic, for any $k = 1,2, \dots$ we have $u_k < v_k$. In addition, if we apply A on the inequality (2.3) temporary then we have

$$u_0 \leq u_1 \leq \cdots \leq u_k \leq \cdots \leq v_k \leq \cdots \leq v_1 \leq v_0 \tag{2.4}$$

According to (2.3), we know that $rv_0 \leq u_0$, in the aid of (2.4) $rv_k \leq rv_0 \leq u_0 \leq u_k$. Let τ_k be the $Sup\{\tau > 0 | \tau v_k \leq u_k\}$. Therefore, $\tau_k v_k \leq u_k$ and then

$$\tau_k v_{k+1} \leq \tau_k v_k \leq u_k \leq u_{k+1} \tag{2.5}$$

But $\tau_{k+1} = Sup\{\tau > 0 | \tau v_{k+1} \leq u_{k+1}\}$ and (2.5) shows that τ_k is also have the property that $\tau_k v_{k+1} \leq u_{k+1}$, so $\tau_k \leq \tau_{k+1}$. This means $\{\tau_k\}$ is increasing real bounded sequence. Therefore, $\tau_k \to \tau$ when $k \to \infty$. There is two possibilities that $\tau = 1$ or $\tau \in (0,1)$.

We consider two cases for $\tau \in (0,1)$. First, assume that there exists $N > 0$ such that $\tau_k = \tau$ for all $k \geq N$. So for $k \geq N$, we have

$$u_{k+1} = A(u_k) \geq A(\tau_k v_k) = A(\tau v_k) \geq \varphi(\tau)A(v_k) = \varphi(\tau)v_{k+1}$$

But we define $\tau_{k+1} = Sup\{\tau > 0 | \tau v_{k+1} \leq u_{k+1}\}$ then $\tau = \tau_{k+1} \geq \varphi(\tau)$. At condition A_2) we assume that $\varphi(\tau) > \tau$. Therefore $\tau > \tau$, this is the contradiction.

Second, assume that for all $k > 0$, we have $\tau_k < \tau$. Then we obtain

$$u_{k+1} = A(u_k) \geq A(\tau_k v_k) = A\left(\frac{\tau_k}{\tau}\tau v_k\right) \geq \varphi\left(\frac{\tau_k}{\tau}\right) A(\tau v_k) \geq \frac{\tau_k}{\tau}\varphi(\tau) v_{k+1}$$

Again mention the definition of $\tau_{k+1} = Sup\{\tau > 0 | \tau v_{k+1} \leq u_{k+1}\}$, then $\tau_{k+1} \geq \frac{\tau_k}{\tau}\varphi(\tau)$. If we let $k \to \infty$, then in the aid of $\varphi(\tau) > \tau_k$we reach to contradiction of $\tau > \tau$.

Therefore, when $k \to \infty$, $\tau_k \to 1$. Let $n \in \mathbb{N}$, then we have

$$\theta \leq u_{k+n} - u_k \leq v_k - u_k \leq v_k - \tau_k v_k = (1 - \tau_k) v_k \leq (1 - \tau_k) v_0$$

$$\theta \leq v_k - v_{k+n} \leq v_k - u_k \leq (1 - \tau_k) v_0.$$

Given cone is normal in the means of definition (2.5), so there exists $M > 0$ (normality constant) such that

$$\|u_{k+n} - u_k\| \leq M|1 - \tau_k|\|v_0\|$$

$$\|v_k - v_{k+n}\| \leq M|1 - \tau_k|\|v_0\|$$

When we tend k to infinity, both of these expressions tend to zero. These mean that $\{u_k\}$ and $\{v_k\}$ are Cauchy sequence in Banach space and they are convergent to let say $\tilde{u}$ and $\tilde{v}$ respectively. In the aid of (2.4) we can see $u_k \leq \tilde{u} \leq \tilde{v} \leq v_k$, this implies that $\theta \leq \tilde{v} - \tilde{u} \leq v_k - u_k \leq (1 - \tau_k) v_0$. By another words,

$$\|\tilde{v} - \tilde{u}\| \leq \|v_k - u_k\| \leq M|1 - \tau_k|\|v_0\|$$

Therefore $\tilde{u} = \tilde{v}$ and we can say that $\tilde{u} = \tilde{v} = x^*$. We can rewrite the inequality for x^* again and

$$u_{k+1} = A(u_k) \leq A(x^*) \leq A(v_k) = v_{k+1}$$

Now tend k to infinity and apply squeeze theorem to see that $A(x^*) = x^*$. That means A has the fixed point value in P_h. It remains to show that this fixed point is unique. Let x^* and $\hat{x}$ be two fixed points of A means $A(x^*) = x^*$, and $A(\hat{x}) = \hat{x}$. Since x^* and $\hat{x}$ belong to P_h, we can find $\delta, \mu, \rho, \sigma > 0$ such that $\delta h \leq x^* \leq \mu h$ and $\rho h \leq \hat{x} \leq \sigma h$. Now combine these inequalities to have $\frac{\rho}{\mu} x^* \leq \frac{\rho}{\mu}\mu h \leq \rho h \leq \hat{x}$ so $t x^*$ is bounded above and we can define $\sigma = Sup\{\vartheta > 0 |\, \vartheta x^* \leq \hat{x}\}$. In fact, σ should be the positive number. Let $0 < \sigma < 1$, then

$$\hat{x} = A(\hat{x}) \geq A(\sigma x^*) \geq \varphi(\sigma) A(x^*) = \varphi(\sigma) x^*$$

Therefore $\varphi(\sigma)$ is a kind of positive number such that $\vartheta x^* \leq \hat{x}$ and $\sigma = Sup\{\vartheta > 0 |\, \vartheta x^* \leq \hat{x}\}$, then σ should be the upper bound and $\varphi(\sigma) \leq \sigma$. But $0 < \sigma < 1$ and according to the condition A_2) we have $\varphi(\sigma) > \sigma$ which is the contradiction. Thus $\sigma \geq 1$ and $x^* \leq \sigma x^* \leq \hat{x}$. Similarly, same discussion shows that $\hat{x} \leq x^*$ means that $\hat{x} = x^*$. At the end, A or equivalently T has a unique solution in P_h.

Remark 2.11: Similar discussion with different conditions is available in (Zhai, Yang, & Zhan, 2010). Actually, we can apply the different conditions on given nonlinear operator to make a unique fixed point. For example, if the conditions A_1) and A_3) are true and there exists $0 < \tau < 1$ such that

$$\tau h \leq T(h) + x_0 \leq \tau^{-1} \varphi(\tau, \tau^{-1} h) h$$

Then the operator has a unique solution in P_h. (Cheng-Bo Zhai & Xiao-Min Cao, 2010)

2.3. Existence and Uniqueness in the Aid of τ– ϕ-Concave Operators

In this section, we will extend the idea of the last section by using another concept for the operator. Actually, the definition of $\psi - (h, r) -$concave operator is the extension of condition A_2) this is coming later. First, let us extend the idea for equivalent class P_h on the given cone.

Definition 2.12: Let P be the cone similar to (definition 2.5) and $r \in P$ with $\theta \leq r \leq h$, then the set $P_{h,r}$ is defined as

$$P_{h,r} = \{x | x \in E, x + r \in P_h\} = \{x \in E |\, x + r \sim h\}$$
$$= \{x \in E |\, \exists \alpha(x, r, h), \beta(x, r, h) > 0 \text{ such that } \alpha h \leq x + r \leq \beta h\}$$

Obviously we can see $P_{h,\theta} = P_h$ where θ is the zero of Banach space $(E, \|.\|)$. (Cheng-Bo Zhai & Xiao-Min Cao, 2010)

Definition 2.13: Assume that $A: P_{h,r} \to E$ be an operator over the set which is defined at (definition 2.12), also for any $x \in P_{h,r}$ and $0 < t < 1$ we have

$$A(tx + (1-t)r) \geq \varphi(t)A(x) + \big(1 - \varphi(t)\big)r$$

Then A is called a $\varphi - (h,r)$ −concave operator. (Cheng-Bo Zhai & Xiao-Min Cao, 2010)

Note 2.14: In the aid of (definition 2.12 and definition 2.13) we can state similar fixed point theorem which is coming next. Proof of this theorem is as the same as (2.10) and can found in (Cheng-Bo Zhai & Xiao-Min Cao, 2010). The importance of this theorem and (2.10) is the part that itteration successive method for approximation is introduced. Actually, the sequence $\{u_k\}$ and $\{v_k\}$ and the processes of making these sequence to reach the fixed point is important. In chapter 5, we will study the application of these fixed point theorem and we will reduce the conditions for having q-fractional differential equation with the unique solution.

Theorem 2.15: Assume that $(E, \|.\|)$ be the Banach space and P is the normal cone that is defined at (2.5) and let $T: P \to P$ be the increasing and $\tau - \varphi -$concave operator. Moreover, assume that there exist $h > \theta$ such that $T(h) \in P_h$. Then there exist $u_0, v_0 \in P_h$ and $0 < r < 1$ such that $rv_0 \leq u_0 < v_0$ and $u_0 \leq T(u_0) \leq T(v_0) \leq v_0$. $T(h)$ has a unique fixed point $x^* \in [u_0, v_0]$. For any starting point $x_0 \in P_h$, constructing successively the sequence $x_{n+1} = T(x_n)$ tends to that fixed point when $n \to \infty$. (Cheng-Bo Zhai & Xiao-Min Cao, 2010)

CHAPTER 3

CLASSIC FRACTIONAL DIFFERENTIAL EQUATION

In this chapter we review the classical fraction calculus, then we investigate the q-analogue of these operators.

The concept of fractional differentiation can be traced back for the first time in a letter between L'Hopital and Leibniz in 1695.

Fractional calculus is a branch of analysis that consider the different possibilities of defining real or complex powers of differentiation operator.

The classical differentiation which is defined by newton and Leibniz is known as (Annaby & Mansour, 2012)

$$Du(x) = \frac{d}{dx}u(x)$$

And also the integration operator J, is demonstrated by

$$\mathrm{J}\big(\mathrm{u}(\mathrm{x})\big) = \int_0^{\mathrm{x}} \mathrm{u}(\mathrm{r})\,\mathrm{dr}$$

The main question that fractional derivative aniseed from that, is "Is there any fractional (half) derivative?" or "what is the relation between the derivative and integral?

Can we combine both of them in one operator? " These kind of questions were answered in fractional calculus. Many mathematician have developed this field and we can mention the studies of Euler (1730), Laplace (1812), Fourier (1822), Abel (1823), Liouville and Riemann (1867) (1868), and recently Riesz (1949), Feller (1952). The q-Riemann-Liouville fractional integral operator was introduced by Al-Salam (proc. Am. math. Soc. 17, 616-621, 1966) from that tine several q-analogues of these operator were studied.

The fractional calculus has played a very important role in different fields such as physics, chemistry, mechanics, biology, control theory, etc.

A lot of phenomena at natural life were modeled very well by fractional differential equations which motivates the mathematician to work in this field.

3.1 Classical Fractional Calculus

A main question to ask is whether there exists a linear operator that can extend the meaning of (Annaby & Mansour, 2012):

$$^{d^n y}/_{dx^n}$$

Let f(x) be an integrable function on $\mathbb{R}^+$, then

$$(Ju)_{(x)} = \int_0^x u(r)\,dr \tag{3.1}$$

$$(J^2u)_{(x)} = \int_0^x (Ju)_{(r)}\,dr = \int_0^x \left(\int_0^x f(s)ds\right) dr \tag{3.2}$$

Interchange the boundary of integral leads to

$$\begin{aligned}(J^2u)_{(x)} &= \int_0^x (Ju)_{(r)}\,dr = \int_0^x \int_z^x f(s)ds\,dr \\ &= \int_0^x f(s)(x-s)\,ds \end{aligned} \tag{3.3}$$

As we have seen, one development begins with a generalization of repeated integral. Thus if f is integrable on (d, ∞), then the n-fold iterated integral is

$$\begin{aligned}{}_{\mathrm{d}}\mathrm{D}_{\mathrm{t}}^{-\mathrm{n}}\mathrm{u(x)} &= \int_d^t ds_1 \int_d^{s_1} ds_2 \dots\dots \int_d^{s_{n-1}} u(s_n)\,ds_n \\ &= \frac{1}{(n-1)} \int_d^t (t-s)^{n-1}\,u(s)ds \end{aligned} \tag{3.4}$$

Using (3.4) and the definition of gamma function leads to the definition of Caputo.

3.2 The Modern Approach

The reader who followed the sometimes troubled thoughts of birth from partial calculus, in order to rest should call (Miller K. S., 1993).

$$ {}_{c}D_{x}^{-v}u(x) = \frac{1}{\Gamma(v)}\int_{c}^{x}(x-t)^{v-1}u(t)dt \tag{3.5} $$

The Riemann version and

$$ {}_{-\infty}D_{x}^{-v}u(x) = \frac{1}{\Gamma(v)}\int_{-\infty}^{x}(x-t)^{v-1}u(t)dt \tag{3.6} $$

The Liouville. The case where c=0 in (3.5), namely,

$$ {}_{0}D_{x}^{-v}u(x) = \frac{1}{\Gamma(v)}\int_{0}^{x}(x-t)^{v-1}u(t)dt \tag{3.7} $$

Will be called the Riemann-Liouville fractional integral.

3.3 The Differential Equation Approach and Green Function

In this section we start by discussing about the linear differential equations in the order n. for this reason, let us start by introducing the familiar variation of parameter. Consider the second order linear differential equation as (Kenneth S. Miller, 1952)

$$ Ly = q_0(x)y" + q_1(x)y' + q_2(x) \tag{3.8} $$

Here, we assume that $q_i(x)$ are continuous in some interval say $[a,b]$. By defining this operator we can have the following ordinary differential equation

$$ Ly = u(x), \;\; y(a) = y'(a) = 0 \tag{3.9} $$

First, by using characteristic polynomial, we determine the fundamental set of the solutions for homogenous equation. Then we assume that the constant are the function of x and by putting the corresponding y , y' and y'' we reach to the system of the equation for constant functions. At the end, by taking the integral we can find the solution of the equation. Here, let us consider $A = \{\varphi_1(x), \varphi_2(x)\}$ is the fundamental set of the solution for homogenous equation. The form of the solution is $y = v_1(x)\varphi_1(x){+}v_2(x)\varphi_2(x)$. Now we take the derivative from this expression and let $v_1'(x)\varphi_1(x){+}v_2'(x)\varphi_2(x) = 0$. Then we reach to the following system of the equation

$$\begin{cases} v_1{}'(x)\varphi_1(x) + v_2{}'(x)\varphi_2(x) = 0 \\ v_1{}'(x)\varphi_1{}'(x) + v_2{}'(x)\varphi_2{}'(x) = \dfrac{f(x)}{q_0(x)} \end{cases}, \tag{3.10}$$

We use the Kramer Rule to solve this system and find $v_1{}'(x)$ and $v_2{}'(x)$. Then we have

$$y = -\varphi_1(x)\int_0^x \frac{f(t)}{q_0(t)}\frac{\varphi_2(t)}{W(t)}dt + \varphi_2(x)\int_0^x \frac{f(t)}{q_0(t)}\frac{\varphi_1(t)}{W(t)}dt + A\varphi_1(x) + B\varphi_2(x)$$

Here $W(t)$ denotes Wronskian of $\varphi_1(x)$ and $\varphi_2(x)$ and define as

$$W(t) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1{}'(t) & \varphi_2{}'(t) \end{vmatrix}$$

The idea of the green function is created by this simple example. In differential equation, we are looking for the function like this to describe the solution as an integral of some function. Now let us define the Green function as follow

$$H(x,t) = \frac{-1}{q_0(t)W(t)}\begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1(t) & \varphi_2(t) \end{vmatrix}$$

Then we can write the solution as

$$y(x) = \int_0^x H(x,t)f(t)\,dt + A\varphi_1(x) + B\varphi_2(x)$$

Since we are considering a one-point boundary value problem, this is, our boundary conditions $y(a) = y'(a) = 0$ depend on only one point of the interval, and then we call $H(x,t)$ one-sided Green's function. In the general case, we have the following theorem:

Theorem 3.4: Let $Ly = q_0(x)y^{(n)} + q_1(x)y^{(n-1)} + \cdots + q_n(x)y$ be a linear differential equation with the $q_i(x) \in c^0$ in some closed finite interval $[a,b]$. Let $q_0(x) > 0$ in $[a,b]$. Let $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}$ be n linear independent solutions of $Ly = 0$ with wronskian $W(x)$, (Kenneth S. Miller, 1952)

$$W(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \dots & \varphi_n(x) \\ D\varphi_1(x) & D\varphi_2(x) & \dots & D\varphi_n(x) \\ D^2\varphi_1(x) & D^2\varphi_2(x) & \dots & D^2\varphi_n(x) \\ \dots & \dots & \dots & \dots \\ D^{n-1}\varphi_1(x) & D^{n-1}\varphi_2(x) & \dots & D^{n-1}\varphi_n(x) \end{vmatrix}$$

define

$$H(t,\xi)=\frac{(-1)^{n-1}}{W(\xi)}\begin{vmatrix}\varphi_1(x) & \varphi_2(x) & \dots & \varphi_n(x)\\ \varphi_1(\xi) & \varphi_2(\xi) & \dots & \varphi_n(\xi)\\ D\varphi_1(\xi) & D\varphi_2(\xi) & \dots & D\varphi_2(\xi)\\ D^2\varphi_1(\xi) & D^2\varphi_2(\xi) & \dots & D^2\varphi_2(\xi)\\ \dots & \dots & \dots & \dots\\ D^{n-2}\varphi_1(\xi) & D^{n-2}\varphi_2(\xi) & \dots & D^{n-2}\varphi_2(\xi)\end{vmatrix},$$

If $u(x)$ is any continuous function in $[a,b]$, then the function on $y(x)$,

$$y(x)=\int_a^x H(x,\xi)u(\xi)d\xi$$

Satisfies the non-homogeneous equation $Ly=u(x)$ and the boundary condition

$$y^{(\beta)}(a)=0,\beta=0,1,2,\dots,n-1.$$

Example 3.5: Let $L=D^2-2D+1$ and the equation is determined as (Miller K. S., 1993):

$$\begin{cases} Ly(x)=x\\ y(0)=y'(0)=0\end{cases}$$

Then we have the following characteristic equation:

$$m^2-2m+1=0 \quad\rightarrow\quad (m-1)^2=0 \quad\rightarrow\quad m=1$$

Then fundamental set of solution can be written as $\{e^x,xe^x\}$. we can write all solution as follows:

$$y_c=c_1e^x+c_2xe^x$$

With undetermined coefficient method, it is easily seen that $y_c=c_1e^x+c_2xe^x+x+2$. Now putting the initial value, gives $c_1=-2$ and $c_2=1$.

The I.V.P has a solution in the form of $y_c=-2e^x+xe^x+x+2$. In the discussion that we had it

$$H(t,\xi)=\frac{-1}{W(\xi)}\begin{vmatrix}e^x & xe^x\\ e^\xi & e^\xi\end{vmatrix}=\frac{\xi e^{x+\xi}-xe^{x+\xi}}{-W(\xi)}=\frac{(x-\xi)e^{x+\xi}}{W(\xi)} \qquad (3.11)$$

Where wronskian can be evaluated by

$$W(\xi) = \begin{vmatrix} e^{\xi} & \xi e^{\xi} \\ e^{\xi} & e^{\xi} + \xi e^{\xi} \end{vmatrix} = e^{2\xi} + \xi e^{2\xi} - \xi e^{2\xi} = e^{2\xi}$$

So

$$H(t,\xi) = \frac{(x-\xi)e^{x+\xi}}{e^{2\xi}}$$

And the solution can be written as;

$$y(x) = \int_0^x H(x,\xi)u(\xi)d\xi = xe^x - 2e^x + x + 2$$

In the case that $L = D^n$ we can write the system as following

$$\begin{cases} y^{(n)} = u(x) \\ y(c) = y'(c) = y''(c) = \cdots = y^{(n-1)}(c) = 0 \end{cases}$$

It means that we suppose that L is simply the nth-order derivative operator,

$$L \equiv D^n. \tag{3.12}$$

Then (3.9) may be written as

$$D^n y(x) = u(x), \;\; D^k y(c) = 0, \quad 0 \leqq k \leqq n-1, \tag{3.13}$$

And

$$\{1, x, x^2, \ldots, x^{n-1}\}$$

Is a fundamental set of solutions of $D^n y(x) = 0$.Thus in this special case the one-sided Green's function $H(t,\xi)$ is

$$H(t,\xi) = \frac{(-1)^{n-1}}{W(\xi)} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-2} \\ 0 & 1 & 2\xi & \cdots & (n-1)\xi^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (n-1)\xi! \end{vmatrix}, \tag{3.14}$$

And the Wronskian is

$$W(\xi) = \begin{vmatrix} 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 0 & 1 & 2\xi & \cdots & (n-1)\xi^{n-2} \\ 0 & 1 & 2\xi & \cdots & (n-1)(n-2)\xi^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (n-1)! \end{vmatrix},$$

We easily see that

$$W(\xi) = \prod_{k=0}^{n-1} k! = (n-1)!!, \tag{3.15}$$

Independent of ξ. Now turning to (3.14), we observe that H(x,ξ) may be written as polynomial of degree n-1 in x, whose leading coefficient is

$$\frac{(-1)^{n-1}}{(n-1)!!}[(-1)^{n+1}(n-2)!!] = \frac{1}{(n-1)!}$$

But by direct calculation,

$$\left.\frac{\partial^k}{\partial x^k} H(x,\xi)\right|_{x=\xi} = 0$$

For $k = 0,1, \dots, n-2$. hence ξ is a zero of multiplicity $n-2$, and therefore

$$H(x,\xi) = \frac{1}{(n-1)!}(x-\xi)^{n-1}. \tag{3.16}$$

Thus from (3.10) and (3.13) we arrive

$$D^n y(x) = u(x) \quad y(x) = \frac{1}{(n-1)!}\int_c^x (x-\xi)^{n-1} u(\xi) d\xi. \tag{3.17}$$

Since u is the nth derivative of y, we may interpret (3.17) as the nth integral of u and write it as

$$y(x) = D^{-n}u(x) = \frac{1}{\Gamma(n)}\int_c^x (x-\xi)^{n-1} u(\xi) d\xi. \tag{3.18}$$

CHAPTER 4

q-FRACTIONAL DIFFERENCE EQUATION

4.1 q-Fractional Calculus

Definition 4.1: The Riemann-Liouville type on the fractional q-integral of is defined as (Ferreira, 2011):

$$\left(I_q^0 u\right)(x) = u(t) \text{ and } \left(I_q^\delta u\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q\tau)^{(\delta-1)}\, u(\tau) d_q\tau,$$

$$\delta > 0,\ t \in [0,1] \tag{4.1}$$

Definition 4.2: The Riemann-Liouville type of the fractional q-derivative of order $\delta \geq 0$ is defined as

$$\left(D_q^0 u\right)(t) = u(t) \tag{4.2}$$

$$\left(D_q^\delta u\right)(t) = \left(D_q^m I_q^{m-\delta} u\right)(t), \quad \delta > 0 \tag{4.3}$$

Where m is the smallest integer and $m \geq \ \delta$. (Ferreira, 2011)

Lemma 4.3: There are some similar properties for these operators. Let u be a mapping defined on interval $[0,1]$ such that the Jackson integral which is defined is convergent and $\delta, \eta \geq 0$ then we can list them as follow (Ferreira, 2011):

$$(a)\ \left(I_q^\eta I_q^\delta u\right)(x) = \left(I_q^{\eta+\delta} u\right)(x) \tag{4.4}$$

$$(b)\ \left(D_q^\delta I_q^\delta u\right)(x) = u(x) \tag{4.5}$$

Proof: (a)The operator I_q^δ has the semi-group property, means

$$I_q^\delta I_q^\eta\left(u(x)\right) = I_q^{\delta+\eta}\left(u(x)\right)$$

Let us start by expanding the left side

$$I_q^{\delta} I_q^{\eta}\big(u(x)\big) = \frac{1}{\Gamma_q(\delta)} \int_0^x (x-qt)^{(\delta-1)}\, I_q^{\eta}\big(u(t)\big) d_q t$$
$$= \frac{1}{\Gamma_q(\delta)\Gamma_q(\eta)} \int_0^x (x-qt)^{(\delta-1)} \int_0^t (t-qs)^{(\eta-1)}\, u(s) d_q s d_q t$$

Now apply (lemma 1.45) to change the order of double q-integral

$$I_q^{\delta} I_q^{\eta}\big(f(x)\big) = \frac{1}{\Gamma_q(\delta)\Gamma_q(\eta)} \int_0^x \left(\int_{qs}^x (x-qt)^{(\delta-1)} (t-qs)^{(\eta-1)}\, d_q t \right) u(s)\, d_q s$$

In the aid of (1.54 Lemma) where λ is $\eta - 1$ and a is qs, we have

$$I_q^{\delta} I_q^{\eta}\big(u(x)\big) = \frac{1}{\Gamma_q(\delta)\Gamma_q(\eta)} \int_0^x \left(\frac{\Gamma_q(\delta)\Gamma_q(\eta)}{\Gamma_q(\delta+\eta)} (x-sq)^{(\eta+\delta-1)} \right) u(s)\, d_q s$$
$$= \frac{1}{\Gamma_q(\delta+\eta)} \int_0^x (x-sq)^{(\eta+\delta-1)} u(s)\, d_q s = I_q^{\delta+\eta}\big(u(x)\big).$$

(b) Let us start by expanding the left side

$$\big(D_q^{\delta} I_q^{\delta} u\big)(x) = \big(D_q^{m} I_q^{m-\delta} I_q^{\delta} u\big)(x) = \big(D_q^{m} I_q^{m-\delta+\delta} u\big)(x) = \big(D_q^{m} I_q^{m} u\big)(x) = u(x)$$

Theorem 4.4: Assume that $m \in \mathbb{N}$ be any natural number, then the following identity holds (Ferreira, 2011)

$$\big(I_q^{\delta} D_q^{m} u\big)(x) = \big(D_q^{m} I_q^{\delta} u\big)(x) - \sum_{k=0}^{m-1} \frac{x^{\delta+k-m}}{\Gamma_q(\delta+k-m+1)} \big(D_q^{k} u\big)(0)$$

Proof: we prove the theorem by induction. For the case $m = 1$, we have

$$\big(I_q^{\delta} D_q u\big)(x) = \frac{1}{\Gamma_q(\delta)} \int_0^x (x-tq)^{(\delta-1)} \big(D_q u\big)(t)\, d_q t \tag{4.6}$$

The simple calculation shows what is that the q-derivative of the following function respect to t, we apply the product rule to find

$$\big(D_q\big)\Big((x-t)^{(\delta-1)} u(t)\Big) = (x-tq)^{(\delta-1)} D_q\big(u(t)\big) - [\delta-1]_q (x-tq)^{(\delta-2)} u(t)$$

$$(x-tq)^{(\delta-1)} D_q\big(u(t)\big) = \big(D_q\big)\big((x-t)^{(\delta-1)} u(t)\big) + [\delta-1]_q (x-tq)^{(\delta-2)} u(t)$$

Put this on (4.6) to have

$$\left(I_q^\delta D_q u\right)(x)$$

$$= \frac{1}{\Gamma_q(\delta)}(x-t)^{(\delta-1)}u(t)\Big|_{t=0}^{t=x}$$

$$+\frac{1}{\Gamma_q(\delta-1)}\int_0^x (x-tq)^{(\delta-2)}u(t)\,d_q t$$

$$= \left(0-\frac{x^{\delta-1}}{\Gamma_q(\delta)}u(0)\right)+\left(I_q^{\delta-1}u\right)(x)$$

$$= \left(D_q I_q^\delta u\right)(x) \ -\frac{x^{\delta-1}}{\Gamma_q(\delta)}u(0)$$

Now assume that the induction assumption is true, means the equation holds for $m \in \mathbb{N}$. For $m+1$ we have:

$$\left(I_q^\delta D_q^{m+1}u\right)(x) = \left(I_q^\delta D_q^m D_q u\right)(x)$$

$$= \left(D_q^m I_q^\delta D_q u\right)(x) - \sum_{k=0}^{m-1}\frac{x^{\delta+k-m}}{\Gamma_q(\delta+k-m+1)}\left(D_q^{k+1}u\right)(0)$$

$$= D_q^m\left(\left(D_q I_q^\delta u\right)(x) - \frac{x^{\delta-1}}{\Gamma_q(\delta)}u(0)\right)$$

$$-\sum_{k=0}^{m-1}\frac{x^{\delta+k-m}}{\Gamma_q(\delta+k-m+1)}\left(D_q^{k+1}u\right)(0)$$

$$= \left(D_q^{m+1} D_q I_q^\delta u\right)(x)\left(-\frac{x^{\delta-1-m}}{\Gamma_q(\delta-m)}u(0)\right)$$

$$-\sum_{k=1}^{m}\frac{x^{\delta+k-(m+1)}}{\Gamma_q(\delta+k-(m+1)+1)}\left(D_q^{k}u\right)(0)$$

Which is the proof for case $m+1$, thus the theorem is proved.

CHAPTER 5

MAIN RESULTS FOR q-FRACTIONAL DIFFERENTIAL EQUATION

5.1 Discussion for the Solution of Fractional q-Differential Equation

In this chapter we discuss about the solution of the following q-fractional differential equation and in the aid of Banach fix point theorem, we show the uniqueness and existence of the solution. First of all we start by solving q-fractional differential equation by using theorem 3.4 and direct consequence of definition of q-fractional integral and derivative operators.

Lemma 5.1: Let $0 < \zeta < 1$ and β is a constant such that $1 - \beta\zeta^{\alpha-2} \neq 0$. If $f(t)$ is a continuous function on $[0,1]$, then solution of the following boundary value problem (Zhai & Ren, 2018):

$$\begin{cases} \left(D_q^\alpha u\right)(t) = -f(t) \quad t \in (0,1), \quad \alpha \in (2,3) & (5.1) \\ u(0) = \left(D_q u\right)(0) = 0, \quad \left(D_q u\right)(1) = \beta\left(D_q u\right)(\zeta) & (5.2) \end{cases}$$

Can be written as

$$u(t) = \int_0^1 G(t, qs) f(s) d_q s + \frac{\beta t^{\alpha-1}}{[\alpha-1]_q (1 - \beta\zeta^{\alpha-2})} \int_0^1 H(\zeta, qs) f(s) d_q s$$

Where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-2)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)} t^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \quad (5.3)$$

Also $H(t,s)$ is a q-derivative of $G(t,s)$ respect to t, means

$$H(t,s) = D_q\left(G(t,s)\right)$$
$$= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-2)} t^{\alpha-2} - (t-s)^{(\alpha-2)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)} t^{\alpha-2}, & 0 \le t \le s \le 1, \end{cases}$$

Proof: Put $m = 3$ and $I_q^{3-\alpha}u(t)$ instead of $u(t)$in theorem 4.4 to have

$$I_q^{\alpha} D_q^3 \big(I_q^{3-\alpha} u(t)\big) = D_q^3 I_q^{\alpha} I_q^{3-\alpha} u(t) - A_1 t^{\alpha-1} - A_2 t^{\alpha-2} - A_3 t^{\alpha-3}$$

Now use the semi group property of I_q^{α} in lemma 4.3 to have

$$I_q^{\alpha} D_q^3 \big(I_q^{3-\alpha} u(t)\big) = u(t) - A_1 t^{\alpha-1} - A_2 t^{\alpha-2} - A_3 t^{\alpha-3}$$

In addition, definition of q-fractional derivative 4.2 gives

$$I_q^{\alpha} D_q^3 \big(I_q^{3-\alpha} u(t)\big) = I_q^{\alpha} D_q^{\alpha}(u(t))$$

For solving (5.1), apply I_q^{α} operator in both sides of the equation to have

$$\begin{aligned} I_q^{\alpha} D_q^{\alpha}(u(t)) = I_q^{\alpha} D_q^3 \big(I_q^{3-\alpha} u(t)\big) &= u(t) - A_1 t^{\alpha-1} - A_2 t^{\alpha-2} - A_3 t^{\alpha-3} \\ &= -I_q^{\alpha} f(t) \end{aligned}$$

Therefore we can write $u(t)$ in the form of

$$\begin{aligned} u(t) &= A_1 t^{\alpha-1} + A_2 t^{\alpha-2} + A_3 t^{\alpha-3} - I_q^{\alpha} f(t) \\ &= A_1 t^{\alpha-1} + A_2 t^{\alpha-2} + A_3 t^{\alpha-3} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s \end{aligned}$$

We should find the constants A_i, by using the initial values (5.2). First, let us put $t = 0$ at the above equation. Since $2 < \alpha < 3$, $\alpha - 1$ and $\alpha - 2$ are positive and only $\alpha - 3$ is negative and this implies that $A_3 = 0$. Now we must take q-derivative from $u(t)$ to apply (5.2) and find the constants A_1 and A_2. Here we use fundamental theorem of q-calculus (Kac & Cheung, 2002).

$$\begin{aligned} &D_q \big(u(t)\big) \\ &\quad = A_1 [\alpha-1]_q t^{\alpha-2} + A_2 [\alpha-2]_q t^{\alpha-3} \\ &\quad - \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-2)} f(s) d_q s \end{aligned}$$

Now, initial value of (5.2) implies that $A_2 = 0$, because $\alpha - 3 < 0$ and $D_q\left(u(0)\right) = 0$. In the next step put $t = 1$ then

$$\begin{aligned}
\beta\big(D_q u\big)(\zeta) = \big(D_q u\big)(1) \\
&= A_1[\alpha-1]_q - \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} f(s)d_q s \\
&\Rightarrow A_1 = \frac{\beta\big(D_q u\big)(\zeta)}{[\alpha-1]_q} + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} f(s)d_q s \\
&= A_1\beta\zeta^{\alpha-2} - \frac{\beta}{\Gamma_q(\alpha)}\int_0^\zeta (\zeta-qs)^{(\alpha-2)} f(s)d_q s \\
&+ \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} f(s)d_q s
\end{aligned}$$

Put $A_1\beta\zeta^{\alpha-2}$ at left and factorize A_1 then

$$\begin{aligned}
A_1 = \frac{1}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}\Bigg(&\int_0^1 (1-qs)^{(\alpha-2)} f(s)d_q s \\
&- \int_0^\zeta \beta(\zeta-qs)^{(\alpha-2)} f(s)d_q s\Bigg)
\end{aligned}$$

Thus we find the constants and if we put them in the solution we have

$$\begin{aligned}
u(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}\Bigg(&\int_0^1 (1-qs)^{(\alpha-2)} f(s)d_q s \\
&- \int_0^\zeta \beta(\zeta-qs)^{(\alpha-2)} f(s)d_q s\Bigg) \\
&- \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)d_q s
\end{aligned}$$

If we write $\frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}$ as two fractions $\frac{t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{\beta\zeta^{\alpha-2}t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}$ then lemma is proved.

In the next step, we will investigate the properties of corresponding Green function. The next lemma describes some of these properties.

Lemma 5.2: Given function $G(t,s)$ at (4.3) have the following properties (Zhai & Ren, 2018):

a) $G(t,qs)$ is continuous and nonnegative function for t and s between 0 and 1.

b) Function $G(t,qs)$ is strictly increasing respect to t, means $t_1 < t_2$ implies $G(t_1,qs) < G(t_1,qs)$ where s is fixed.

c) $\frac{1}{\Gamma_q(\alpha)}$ is the upper bound for $G(t,qs)$. By another word, the inequality $G(t,qs) \leq \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1} \leq \frac{1}{\Gamma_q(\alpha)}$ holds.

Proof: a) since $G(t,qs)$ is a combination of preliminary functions, $G(t,qs)$ should be continuous. For positivity of $G(t,qs)$, we study this function in two cases. First, let $0 \leq t \leq s \leq 1$, then

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1} = \frac{1}{\Gamma_q(\alpha)}\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}}t^{\alpha-1}$$

Since $s \leq 1$, the expression $1-sq^m$ is positive for any $m > 0$. In addition, $\frac{1}{\Gamma_q(\alpha)} > 0$ and $t^{\alpha-1} \geq 0$ because $2 < \alpha < 3$ and $0 \leq t$. Therefore in this case $G(t,qs) \geq 0$. Now let $0 \leq s \leq t \leq 1$, then

$$\begin{aligned}
G(t,qs) &= \frac{1}{\Gamma_q(\alpha)}\left((1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}\right) \\
&= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left(\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}} - \prod_{k=0}^{\infty}\frac{1-\frac{s}{t}q^{k+1}}{1-\frac{s}{t}q^{k+\alpha}}\right) \\
&\geq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left(\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}} - \prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha}}\right) \\
&= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}\frac{sq^{\alpha-1}}{1-sq^{\alpha-1}} \geq 0
\end{aligned}$$

Then we conclude that for all $0 \leq s,t \leq 1$, $G(t,qs) \geq 0$.

b) For showing that $G(t,qs)$ is an increasing function, we take q-derivative from $G(t,qs)$ respect to t. First, assume that $0 \leq t \leq s \leq 1$, then

$$
\begin{aligned}
{}_tD_q\left(G(t,qs)\right) &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1} \\
&= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}}\right)t^{\alpha-1} \geq 0
\end{aligned}
$$

Again the conditions of α, t and s imply the last inequality. For $0 \leq s \leq t \leq 1$, we have:

$$
\begin{aligned}
{}_tD_q\left(G(t,qs)\right) &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left((1-sq)^{(\alpha-2)}t^{\alpha-2}-(t-sq)^{(\alpha-2)}\right) \\
&= \frac{t^{\alpha-2}[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}}-\prod_{k=0}^{\infty}\frac{1-\frac{s}{t}q^{k+1}}{1-\frac{s}{t}q^{k+\alpha-1}}\right) \\
&\geq \frac{t^{\alpha-2}[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}}-\prod_{k=0}^{\infty}\frac{1-sq^{k+1}}{1-sq^{k+\alpha-1}}\right) \\
&= 0
\end{aligned}
$$

It is completed the (b) part of lemma.

c) This part is straightforward. For $0 \leq t \leq s \leq 1$, the function is defined by $\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1}$ and for $0 \leq s \leq t \leq 1$ some positive expression is subtracted from this part, so in both cases $G(t,qs)$ should be less than $\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1}$. In addition, $(1-qs)^{(\alpha-2)}t^{\alpha-1} \leq 1$ then $G(t,qs) \leq \frac{1}{\Gamma_q(\alpha)}$. It is easy to see that the similar discussion for $H(t,qs)$ shows that the q- derivative of $G(t,qs)$ which is $H(t,qs)$ is a nonnegative function which is increasing respect to t and $H(t,qs) \leq \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}$.

Remark 5.3: Now we can consider the existence and uniqueness of general form of (4.1) when initial values (4.2) are given. For instance, let us introduce the following function and constants which is temporary used in the main theorem (Zhai & Ren, 2018).

$$r(t) = \frac{b}{\Gamma_q(\alpha)} \left(\frac{(1-\beta\zeta^{\alpha-1})t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\big([\alpha-1]_q\big)} - \frac{t^{\alpha}}{\big([\alpha]_q\big)} \right)$$

Here $b > 0$ and t is a real number between 0 and 1, means $0 \le t \le 1$. $r(t)$ is continuous function on compact set $[0,1]$, so it has maximum and let $\tilde{r}$ be the maximum of this function over this interval. In addition, let $h(t) = Ht^{\alpha-1}$ where H is a constant that is bigger than $\frac{b}{(1-q)^2(1-\beta\zeta^{\alpha-2})([\alpha-1]_q)\Gamma_q(\alpha)}$. In the aid of these definitions, we construct three conditions as follows:

$\boldsymbol{C_1}$) Assume that $f\big(t, u(t)\big)$ be two variables continuous function from $[0,1] \times [-\tilde{r}, \infty)$ to real line. In addition, assume that $f\big(t, u(t)\big)$ be increasing function respect to the second variable, means $f\big(t, u(t)\big) \le f\big(t, v(t)\big)$ where $-\tilde{r} \le u(t) \le v(t) \le \infty$.

$\boldsymbol{C_2}$) Let $0 < \lambda < 1$ be a constant and $0 < y < \tilde{r}$, then there exist $\varphi(\lambda) > \lambda$ such that

$$f(t, \lambda x + (\lambda - 1)y) \ge \varphi(\lambda) f(t, x).$$

$\boldsymbol{C_3}$) Last condition is positivity of $f(t, 0)$, means $f(t, 0) > 0$, specifically $f(t, 0) \ne 0$ for all possible $0 \le t \le 1$.

Theorem 5.4: Assume that all conditions $\boldsymbol{C_1} - \boldsymbol{C_3}$ holds true. Then following q-fractional differential equation problem with given initial values has a unique solution. In addition, following sequence shows successive approximation approach for the solution. (Zhai & Ren, 2018)

$$\begin{cases} \big(D_q^{\alpha} u\big)(t) + f(t, u(t)) = b \qquad t \in (0,1), \qquad \alpha \in (2,3) & (5.4) \\ u(0) = \big(D_q u\big)(0) = 0, \qquad \big(D_q u\big)(1) = \beta\big(D_q u\big)(\zeta) & (5.5) \end{cases}$$

$$v_n(t) = \int_0^1 G(t, qs) f\big(s, v_{n-1}(s)\big) d_q s + \frac{\beta t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\big([\alpha-1]_q\big)} \int_0^1 H(\zeta - qs) f\big(s, v_{n-1}(s)\big) d_q s - r(t)$$

Proof: Assume that $0 \leq t \leq 1$ then according to the definition of $r(t)$ at remark (5.3), we have:

$$\begin{aligned}
r(t) &= \frac{b}{\Gamma_q(\alpha)}\left(\frac{(1-\beta\zeta^{\alpha-1})t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\big([\alpha-1]_q\big)} - \frac{t^{\alpha}}{\big([\alpha]_q\big)}\right) \\
&\geq \frac{bt^{\alpha-1}}{[\alpha-1]_q\Gamma_q(\alpha-1)}\left(\frac{(1-\beta\zeta^{\alpha-1})}{(1-\beta\zeta^{\alpha-2})\big([\alpha-1]_q\big)} - \frac{1}{\big([\alpha]_q\big)}\right) \\
&= \frac{(1-q)^2bt^{\alpha-1}}{\Gamma_q(\alpha-1)}\left(\frac{(1-\beta\zeta^{\alpha-1})}{(1-\beta\zeta^{\alpha-2})(1-q^{\alpha-1})^2} - \frac{1}{(1-q^{\alpha-1})(1-q^{\alpha})}\right) \\
&= \frac{(1-q)^2bt^{\alpha-1}}{\Gamma_q(\alpha-1)}\left(\frac{(1-q^{\alpha})(1-\beta\zeta^{\alpha-1}) - (1-q^{\alpha-1})(1-\beta\zeta^{\alpha-2})}{(1-q^{\alpha})(1-\beta\zeta^{\alpha-2})(1-q^{\alpha-1})^2}\right) \\
&\geq \frac{(1-q^{\alpha-1})(1-q)^2bt^{\alpha-1}}{\Gamma_q(\alpha-1)}\left(\frac{(1-\beta\zeta^{\alpha-1}) - (1-\beta\zeta^{\alpha-2})}{(1-q^{\alpha})(1-\beta\zeta^{\alpha-2})(1-q^{\alpha-1})^2}\right) \\
&\geq \frac{(1-q)^2bt^{\alpha-1}}{\Gamma_q(\alpha-1)}\left(\frac{\beta(\zeta^{\alpha-2}-\zeta^{\alpha-1})}{(1-q^{\alpha})(1-\beta\zeta^{\alpha-2})(1-q^{\alpha-1})}\right) \geq 0. \qquad (5.6)
\end{aligned}$$

The last expression in (5.6) is positive because all terms are positive. Also we have the upper bound for $r(t)$ as we mentioned it at (5.3) with $h(t) = Ht^{\alpha-1}$. So $r(t)$ is bounded as $0 \leq r(t) \leq h(t)$. Here, we consider the Banach space as $(C[0,1], \|.\|)$ which $\|.\|$ is the supreme norm. The cone P is defined as $\{x(t) \in C[0,1] | x(t) \geq 0\}$ which is the standard normal cone. According to (5.3), $r(t)$ is continuous function on $[0,1]$ so (5.6) shows that $r(t) \in P$. In addition, let $P_{h,r} = \{x(t) \in C[0,1] | x(t) + r(t) \in P_h\}$.

Now let us write the solution of the (5.4) with initial values(5.5). In the aid of Lemma (5.1) the solution of (5.4) can be written as

$$\begin{aligned}
u(t) = \int_0^1 G(t,qs)f(s,u(s))d_qs \\
&+ \frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f(s,u(s))d_qs \\
&- b\int_0^1 G(t,qs)d_qs \\
&- \frac{b\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)d_qs \qquad (5.7)
\end{aligned}$$

Now consider the last two integrals of (5.7) and substitute the value of $G(t, qs)$ and $H(\zeta, qs)$ in that expression. We can rewrite the last two integrals as

$$-\frac{bt^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}d_qs + \frac{b}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}d_qs$$
$$-\frac{b\beta t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}\left(\zeta^{\alpha-2}\int_0^1 (1-qs)^{(\alpha-2)}d_qs\right.$$
$$\left.-\int_0^\zeta (\zeta-qs)^{(\alpha-2)}d_qs\right)$$

Now apply Lemma (1.54) to have (for the first integral substitute λ, x, α by $0,1,\alpha-1$ respectively, for the second integral substitute λ, x, α by $0, t, \alpha$, for third integral substitute λ, x, α by $0,1,\alpha-1$ and for the last one substitute λ, x, α by $0, \zeta, \alpha-1$.)

$$-\frac{bt^{\alpha-1}}{\Gamma_q(\alpha)}\frac{\Gamma_q(\alpha-1)\Gamma_q(1)}{\Gamma_q(\alpha-1+1)} + \frac{b}{\Gamma_q(\alpha)}\frac{\Gamma_q(\alpha)\Gamma_q(1)}{\Gamma_q(\alpha+1)}t^\alpha$$
$$-\frac{b\beta t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}\left(\zeta^{\alpha-2}\frac{\Gamma_q(\alpha-1)\Gamma_q(1)}{\Gamma_q(\alpha-1+1)}\right.$$
$$\left.-\frac{\Gamma_q(\alpha-1)\Gamma_q(1)}{\Gamma_q(\alpha)}\zeta^{\alpha-1}\right)$$

We can simplify this expression by using the q-Gamma definition to

$$-\frac{bt^{\alpha-1}\Gamma_q(\alpha-1)}{\left(\Gamma_q(\alpha)\right)^2} + \frac{bt^\alpha}{\Gamma_q(\alpha+1)}$$
$$-\frac{b\beta t^{\alpha-1}}{\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}\left(\frac{\Gamma_q(\alpha-1)\zeta^{\alpha-2}}{\Gamma_q(\alpha)} - \frac{\zeta^{\alpha-1}}{[\alpha-1]_q}\right)$$

If we factorize $-\frac{b(1-q)^2}{\Gamma_q(\alpha-1)}$ from the first two fractions and compute the last bracket, then we have:

$$-\frac{b(1-q)^2}{\Gamma_q(\alpha-1)}\left(\frac{t^{\alpha-1}\Gamma_q(\alpha-1)\Gamma_q(\alpha-1)}{\Gamma_q(\alpha)\Gamma_q(\alpha)(1-q)^2}-\frac{t^{\alpha}\Gamma_q(\alpha-1)}{\Gamma_q(\alpha+1)(1-q)^2}\right)$$

$$-\frac{b\beta t^{\alpha-1}(\zeta^{\alpha-2}-\zeta^{\alpha-1})}{[\alpha-1]_q\Gamma_q(\alpha)(1-\beta\zeta^{\alpha-2})}$$

$$=-\frac{b(1-q)^2}{\Gamma_q(\alpha-1)}\left(\frac{t^{\alpha-1}}{(1-q^{\alpha-1})^2}-\frac{t^{\alpha}}{(1-q^{\alpha-1})(1-q^{\alpha})}\right)$$

$$-\frac{b\beta t^{\alpha-1}(\zeta^{\alpha-2}-\zeta^{\alpha-1})}{\left([\alpha-1]_q\right)^2\Gamma_q(\alpha-1)(1-\beta\zeta^{\alpha-2})}$$

$$=-\frac{b(1-q)^2}{\Gamma_q(\alpha-1)}\left(\frac{t^{\alpha-1}(1-q^{\alpha})-t^{\alpha}(1-q^{\alpha-1})}{(1-q^{\alpha-1})^2(1-q^{\alpha})}\right.$$

$$\left.+\frac{\beta t^{\alpha-1}(\zeta^{\alpha-2}-\zeta^{\alpha-1})}{(1-q^{\alpha-1})^2(1-\beta\zeta^{\alpha-2})}\right)$$

The last expression is as the same as $r(t)$ which we define it before. So the solution of (5.4) should be in the form of

$$u(t)=\int_0^1 G(t,qs)f(s,u(s))d_qs$$
$$+\frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f(s,u(s))d_qs-r(t)$$

Now let us define our operator respect to this solution. For any $v(x)\in P_{h,r}=\{x(t)\in C[0,1]|x(t)+r(t)\in P_h\}$, we define $T: P_{h,r}\to C[0,1]$ as

$$T\big(v(t)\big)$$
$$=\int_0^1 G(t,qs)f\big(s,v(t)\big)d_qs$$
$$+\frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f\big(s,v(t)\big)d_qs-r(t)$$

Therefore, being solution of (5.4) is equivalent to be a fixed point of T. In remain of the proof for theorem we will show that the created operator satisfy the conditions of fixed point theorem in chapter 2. First, let us consider the definition of $\varphi-(h,r)-$concave operator (2.13). Assume that $0<\gamma<1$, in the aid of $\boldsymbol{C_2}$) we have:

$$T\big(\gamma v(t)+(1-\gamma)r(t)\big)$$
$$=\int_0^1 G(t,qs)f\big(s,\gamma v(t)+(1-\gamma)r(t)\big)d_qs$$
$$+\frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f\big(s,\gamma v(t)+(1-\gamma)r(t)\big)d_qs-r(t)$$
$$\geq \varphi(\gamma)\int_0^1 G(t,qs)f\big(s,v(t)\big)d_qs$$
$$+\frac{\varphi(\gamma)\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f\big(s,v(t)\big)d_qs-r(t)$$
$$=\varphi(\gamma)\left(\int_0^1 G(t,qs)f\big(s,v(t)\big)d_qs\right.$$
$$\left.+\frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f\big(s,v(t)\big)d_qs\right)-\varphi(\gamma)r(t)$$
$$+\varphi(\gamma)r(t)-r(t)=\varphi(\gamma)\big(Tv(t)\big)+(\varphi(\gamma)-1)r(t)$$

Therefore, the created operator is $\varphi-(h,r)-$concave operator. In the next step, we should show that T is increasing operator. Condition $\boldsymbol{C_1}$) guarantees that $f(t,u)$ is increasing and this implies increasing of T. But one thing should be clear, in condition $\boldsymbol{C_1}$) the domain of function is restricted to $[0,1]\times[-\tilde{r},\infty)$, so we must show that if $u(t)\in P_{h,r}$ then $u(t)\in[-\tilde{r},\infty)$. Now let $u(t)\in P_{h,r}$ then $u(t)+r(t)\in P_h$ so there exists $\rho>0$ such that $u(t)+r(t)\geq\rho h(t)$, then

$$u(t)\geq\rho h(t)-r(t)\geq -r(t)\geq-\tilde{r} \tag{5.8}$$

In (5.8) we can eliminate $\rho h(t)$ in the second inequality because of $\rho h(t)\in P_h$. Now it is the time to show that $Th\in P_{h,r}$. This is equivalent to show $Th+r\in P_h$. For this reason, let us compute $Th+r$ when according to remark 5.3 we can substitute $h(t)=Ht^{\alpha-1}$:

$$Th(t)+r(t)=T(Ht^{\alpha-1})+r(t)$$
$$=\int_0^1 G(t,qs)f(s,Hs^{\alpha-1})d_qs$$
$$+\frac{\beta t^{\alpha-1}}{[\alpha-1]_q(1-\beta\zeta^{\alpha-2})}\int_0^1 H(\zeta,qs)f(s,Hs^{\alpha-1})d_qs$$

Now use lemma 5.2. part (c), which shows the upper bound for green function. In addition, follow the definition of $H(\zeta, qs)$ at lemma (5.1) to have:

$$Th(t) + r(t) \le \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} t^{\alpha-1} f(s, Hs^{\alpha-1}) d_q s + \frac{\beta t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} f(s, Hs^{\alpha-1}) d_q s$$

Since $0 \le s \le 1$, $Hs^{\alpha-1} \le H$ and condition $\boldsymbol{C_1}$) shows that $f(s, Hs^{\alpha-1}) \le f(s, H)$, so we have:

$$Th(t) + r(t) \le \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} t^{\alpha-1} f(s, H) d_q s + \frac{\beta t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s$$

Since $0 \le t \le 1$ we can follow the inequality as

$$Th(t) + r(t) \le \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\beta t^{\alpha-1}}{(1-\beta\zeta^{\alpha-2})\Gamma_q(\alpha)}\right)\int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s$$

Since $0 \le t \le 1$, $h(t) = Ht^{\alpha-1} \le H$ then $\frac{h(t)}{H} \le 1$, by multiplying this term to both sides of the inequality, we have:

$$Th(t) + r(t) \le \left(\frac{1}{H\Gamma_q(\alpha)} + \frac{\beta t^{\alpha-1}}{H(1-\beta\zeta^{\alpha-2})\Gamma_q(\alpha)}\right)\left(\int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s\right) h(t)$$

Now if we put

$$\xi_1 = \left(\frac{1}{H\Gamma_q(\alpha)} + \frac{\beta t^{\alpha-1}}{H(1-\beta\zeta^{\alpha-2})\Gamma_q(\alpha)}\right)\left(\int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s\right)$$

then $Th(t) + r(t) \leq \ \xi_1 h(t)$, for another side inequality we have:

$$Th(t) + r(t) = T(Ht^{\alpha-1}) + r(t)$$
$$= \int_0^1 G(t, qs) f(s, Hs^{\alpha-1}) d_q s$$
$$+ \frac{\beta t^{\alpha-1}}{[\alpha-1]_q (1 - \beta\zeta^{\alpha-2})} \int_0^1 H(\zeta, qs) f(s, Hs^{\alpha-1}) d_q s$$

Now $Hs^{\alpha-1} \geq 0$ and according to condition $\boldsymbol{C_1}$), $f(t,u)$ is increasing respect to the second variable. So $f(s, Hs^{\alpha-1}) \geq f(s, 0)$. According to lemma (5.1) we define $G(t, qs)$ and $H(\zeta, qs)$, now if we put their values in this expression and use the fact that $\beta > 0$ and $0 < \zeta \leq 1$ then we have:

$$Th(t) + r(t) \geq \frac{1}{\Gamma_q(\alpha)} \int_0^1 \left((1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right) f(s, 0) d_q s \, t^{\alpha-1}$$

In the aid of $h(t) = Ht^{\alpha-1}$, we can define $\xi_2 = \frac{1}{\mathrm{H}\Gamma_q(\alpha)} \int_0^1 \left((1-qs)^{(\alpha-2)} - (1 - qs)^{(\alpha-1)}\right) f(s, 0) d_q s$ such that $Th(t) + r(t) \geq \ \xi_2 h(t)$. In addition, ξ_1 and ξ_2 are positive. We can see that easily because of condition $\boldsymbol{C_3}$) for the function $f(t,u)$ and definition of q-integral. It follows that $Th + r \in P_h$. Now apply theorem 2.15 to see that T has unique fixed point. In addition, the instant consequence of theorem is the recurrence formula for the approximation sequence.

CHAPTER 6

CONCLUSION

6.1 Discussion for Future Works

Now a day, a lot of mathematicians are interested to work in a field of q-fractional differences. Recently a book in this field is published (Annaby & Mansour, 2012).Most of new approaches in this field are focusing to introduce a q-difference integral with different initial values. In fact, in this thesis we introduce a q-fractional difference equation with three boundaries initial values and in the aid of application of fixed point theorem, we discussed about the uniqueness and existence for this difference equation.

Different forms of q-fractional difference equation were studied. New application of q-fractional differences motivates authors to define several forms of these equations. With modifying the initial values a bit, the solution and condition will change totally. Actually, wide study of fractional differential equations is available and it is a good resource to define the q-analogue of them and make the new version of these equations.

In this thesis, we apply another special fixed point theorem. This study based on the generalization of concavity for operator on Banach space. If the operator satisfies the conditions of this definition, we can talk about the uniqueness and existence of fixed point of the operator. We apply this theorem to reduce the conditions and make the uniqueness for fixed point. Most of the papers in this field avoid discussing about the uniqueness of fixed point or solution because of the difficulties of working with q-analogue of fractional differential equation. This thesis can be developed in this field and another q-fractional difference equation can be introduced and in the aid of using this method of concave operator, we can discuss about the uniqueness of solution as well.

Actually, there are two main branch of development in this thesis. First, we can improve the theorem in chapter 2 to cover more operators and work with the conditions for fixed point theorem of concave operator and in the next step we can introduce the general form of q-fractional differential equation with more flexible initial values. The q-fractional difference equation which was discussed in chapter 5 can be developed to a system of q-difference equation and we can solve the q-Cauchy difference equation with more flexible initial values.

Now a day, several method for finding fixed point of operator in Banach space are introduced and according to the application and restriction are used to verify existence and uniqueness of given q-fractional difference equation. We believe that the theorem of fixed point in chapter 2 can make better results with less conditions and can apply easily to guarantees uniqueness and existence of q-fractional difference equations.

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