ON THE ANALYSIS OF ST. PETERSBURG PARADOX AND ITS SOLUTION IN TERMS OF UNIFORM TREATMENT

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
MARWAN JAAFAR FAEQ

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my parents…

ABSTRACT

The St. Petersburg paradox is a gambling game with infinite expected payoff that was first presented in 1713 by Nicholas Bernoulli. Despite the infinite payoff, a reasonable person will hardly pay more than $25 to play the game. This thesis presents a number of ideas that was presented over a period of 300 years to resolve this paradox and also economic and financial aspect of the St. Petersburg paradox is presented. A detailed analysis on the St. Petersburg Paradox and its solutions in terms of uniform treatment using D'Alembert's ratio test is also presented and finally using computer program a simulation of the St. Petersburg game is carried out.

***Keywords*:** St. Petersburg paradox; infinite expected payoff; d'Alembert's ratio test; simulation; maple

ÖZET

Petersburg paradoksu, ilk kez 1713 yılında Nicholas Bernoulli tarafından sunulan sonsuz beklenen kazancı olan bir kumar oyunudur Sınırsız kazanca rağmen, makul bir kişi oyunu oynamak için 25 dolardan fazla para ödeyemez. Bu tez, bu paradoksu çözmek için 300 yıllık bir süre içinde sunulan ve St. Petersburg paradoksunun ekonomik ve finansal yönünü ortaya koyan bir takım fikirler sunar. Petersburg Paradox ve d'Alembert'in oran testi kullanılarak tek tip muamele açısından çözümleri hakkında ayrıntılı bir analiz de sunuldu ve bilgisayar programı kullanılarak St. Petersburg oyununun simülasyonu yapıldı.

Anahtar Kelimeler: St. Petersburg paradoksu; sonsuz beklenen ödeme; d'Alembert'in Oran testi; Simülasyon; Akçaağaç

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Nicholas Bernoulli a Swiss mathematician in 1713 was the first to present the St. Petersburg paradox in a letter to P. R. de Montmort a prominent French mathematician. in 1738 the first academic article about this paradox was published in *Commentaries of the Imperial Academy of Science of Saint Petersburg* by Daniel Bernoulli a cousin of N. Bernoulli (Sergio, D. A., & Raul, G, 2016). In 1768 D'Alembert coined the name of the paradox. D. Bernoulli proposed a game in which a coin is continuously flipped only to be stopped when it comes up tail, where the total of flips *n*, determine the price which equal to $\$2^n$. The number of possible consequences are infinite. For the first time, if the coin comes up tail the price is $\$2^1 = \$2$ and the game is terminated. If head appears on the first flip, it is flipped again, if tail appear on the second flip the game is terminated and the price is $\$2^2 = \$4$. If head appears again on the second flip, it is flipped again and so on (Aumann, 1977).

In his paper (Daniel, 1954), Daniel Bernoulli described the St. Petersburg paradox as follows:

"Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he pays is doubled. Suppose we seek to determine the value of Paul's expectation."

Looking at the game a reasonable person will not be risking to spend much on this game because if the desired payoff is raised the corresponding probabilities decreases very fast. Below is a table that lists the figures for the consequence when $i = 1 \ldots 9$ in the case of a fair coin.

Table 1.1: Payoff values and probabilities

i	P(i)	Price
1	1/2	$2
2	1/4	$4
3	1/8	$8
4	1/16	$16
5	1/32	$32
6	1/64	$64
7	1/128	$128
8	1/256	$256
9	1/512	$512

On the other hand, by the classical probability theory, the "fair price" of playing a game is the mathematical expectation of payoff of the game, which in this game is

$$E = \sum_{i=1}^{\infty} 2^{i-1} \left(\frac{1}{2}\right)^{i} = \infty. \tag{1.1}$$

To reconcile this discrepancy D. Bernoulli made the observation that although the standard calculations shows that the value of expectations is infinitely great it has to be admitted that any fairly reasonable man will hardly be willing to pay even $25 to enter such a game.

The work of D. Bernoulli plays an important role in modern marginal utility theory in economics (Emil, 1953). The focus of this research work is on the mathematical perspective of the St. Petersburg paradox and some proposed methods of resolving it.

The first chapter is an introduction about the origin of St. Petersburg paradox, in the second chapter some outstanding proposals of resolving the paradox are considered, also economic and financial aspect of the St. Petersburg paradox is presented. Chapter 3 is a detailed analysis on the St. Petersburg Paradox and its solutions in terms of uniform treatment using d'Alembert's ratio test and in the fourth chapter a simulation of the St. Petersburg paradox was performed using Maple. The fifth section is the conclusion.

The payoff of a game will be denoted by the random variable X which yields the probability density function

$$P\left(X = 2^{i}\right) = \frac{1}{2^{i+1}}, \quad i = 0,1,2,3, \dots . \tag{1.2}$$

where

$$S_n = \sum_{i=1}^{n} X_i \text{ and } \bar{X}_n = \frac{S_n}{n},$$

represents the total and average payoff of n independent games respectively and X_i, $i = 1,2,3, \dots$ are independent copies of X.

CHAPTER 2

LITERATURE REVIEW

To resolve the St. Petersburg paradox, several proposals have been put forward. In this chapter we consider some of the outstanding ones among these proposals and also economic and financial aspect of the St. Petersburg paradox.

2.1 Expected Utility Approach in Resolving the Paradox

The concept of expected utility was proposed by Daniel Bernoulli in his paper which was published in 1738 (Daniel, 1954). In the paper he argues that to an individual, marginal value of money diminishes as his wealthy increases, a concept which is widely used by economists now known as utility. In his work he states that:

"*the determination of the value of an item must not be based on the price, but rather on the utility it yields …. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount*."

By letting $p(i)$ to denote the probability of the outcome of obtaining head in the i^{th} toss and $f(i)$ to denote the payoff when it happens then the mathematical expectation can be written in the form

$$E = \sum_{n=1}^{\infty} p(i)f(i) \tag{2.1}$$

By modifying either $p(i)$ or $f(i)$ equation (2.2) can be made to converge. Nicholas Bernoulli modified $p(i)$ to obtain a convergent sum while Gabriel Cramer modified $f(i)$ to obtain a convergent sum (Gerard, 1987).

Daniel Bernoulli's proposed solution which considers utility as a function depending on wealth was based on decreasing marginal utility of money based on log utility curve. Using the calculus terminology, his approach can be considered as the rate of change of utility (y)

with respect to wealth (x) is inversely proportional to the initial wealth (Cowen, T. and High, J., 1988).

Mathematically, this can be written as

$$\frac{dy}{dx} = \frac{k}{x}, \qquad k > 0 \tag{2.2}$$

integrating both side of equation (2.2) yields

$$y = kln(x) - kln(c)$$

c is considered to be the initial wealth whose utility $y(c)$ equal to zero.

Payoff in wealth is given by $x - c$. Letting

$$y(i) = kln\left(c + 2^i\right) - kln(c) \tag{2.3}$$

Substituting equation (2.3) into equation (2.2) and simplifying gives

$$k\sum_{i=0}^{\infty}\frac{1}{2^i}\left(ln\left(c + 2^i\right)\right) - kln(c)$$

By letting s to be the suggested stake of the game we have

$$kln(c + s) - kln(c) = k\sum_{i=0}^{\infty}\frac{1}{2^i}\left(ln\left(c + 2^i\right)\right) - kln(c)$$

solving for s gives

$$s = \prod_{i=0}^{\infty}\left(c + 2^i\right)\frac{1}{2^{1+i}} - c \tag{2.4}$$

From equation (2.4) it is clear that based on this approach by D. Bernoulli, a players stake should depend on his initial wealth c.

Estimated price for some given initial wealth are presented in the table below:

Table 2.1: Estimated price for some given initial wealth

Initial wealth *c*	**0**	**10**	10^2	10^3	10^4	10^5	10^6
Suggested stake *s*	2	3	4.4	6	7.6	9.3	10.9

A similar approach was proposed by Cramer in a letter to Nicholas Bernoulli in 1728, where he used a utility function U = $\sqrt{2}$ called the moral value leading to the finite moral expectation

$$\sum_{i=1}^{\infty}\sqrt{2^{i-1}}\left(\frac{1}{2}\right)^{i}=\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{\sqrt{2^{i-1}}}=\frac{1}{2-\sqrt{2}}. \qquad (2.5)$$

and the square utility mean $\left(\frac{1}{2-\sqrt{2}}\right)^{2} \approx 2.914$ to show that the diminishing marginal benefit of gain can resolve the problem. Unlike Daniel Bernoulli, he considered only the gain by lottery instead of the total wealth of the player.

The main criticism in their approach lies in their regards to the correct utility function to be used. Both the logarithmic function and the square root function and many other concave functions would work. Thus, which of the solution will be preferred over the other based on the functions used. Also, by changing 2^n to e^{2n}, the decreasing utility approach used by Daniel Bernoulli will yield an infinite expected utility. That is,

$$\triangle \boldsymbol{E}=\sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^{n}\left(ln(W+e^{2n}-c)\right)\right)-ln=\infty. \qquad (2.6)$$

Thus, the critics argue that it does not matter the type of utility function being used, the result can always be modified to give an infinite expected value. (Peters,O., 2011)

2.2 Probability Weighing Method in Resolving the Paradox

An alternative method for resolving this paradox was proposed by Nicholas Bernoulli. He conjectured that unlikely events will be neglected by people and since only likely events yield high payoff that leads to infinite expected value in St. Petersburg game, the paradox will be resolved by neglecting events of very small probability. Similar concept was stated by his uncle Jacob Bernoulli a concept called moral certitude which appeared in his book Ars. Conjectandi.

Among the draw backs in this solution is that the choice of threshold is subjective, therefore can be very arbitrary. Also, small probability with high payoff have big influence on the outcome of a game.

An underweighting of small probabilities was also suggested by Menger (Menger, 1934). His proposal by todays standard is an S-shaped probability weighting function, although, he assumes a cut-off point beyond which small probabilities are set to be zero.

2.3 Finite St. Petersburg Game

The issue of realism of the game is to be put to question since a player having infinite expected gain implies that the others potential loss will be infinite and since a player cannot have an infinite payout, the game is unrealistic in the real world. Therefore, for the expected value to converge, the potential payout of the player has to depict a real life situation (weiss, 1987).

D. Bernoulli made this observation in a letter he wrote to his cousin in 1731, where he wrote:

"I have no more to say to you, if you do not believe that it is necessary to know the sum that the other is in position to play" (plous, 1993)

Letting W represent the total wealth of the one offering the game, the maximum amount a player can earn will be W. Letting $n^* = 1 + \lfloor log_2(W) \rfloor$, the player will get a payoff of W if he get $n \geq n^*$ even though he is to get 2^n. With this consideration in place the actual payout that can be obtained from the game is now min $(2^n, W)$. This leads to the following finite expected value (Eves, 1990)

$$E = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \min(2^n, W) \ = \sum_{n=1}^{n^*-1} \frac{1}{2^{n+1}} 2^n + \sum_{n=n^*}^{\infty} \frac{1}{2^{n+1}} W$$

Then

$$= \sum_{n=1}^{n^*-1} \frac{1}{2} + W \sum_{n=n^*}^{\infty} \frac{1}{2^{n+1}} = \frac{n^*}{2} + \frac{W}{2^{n^*}} . \qquad (2.4)$$

The expected value E of the game with various players and their bankroll W is shown in the table below (a player will be paid what the bank has if he wins more than the bank roll) (Wikipedi compane, 2018).

Table 2.2: Expected value E of the game with various potential players and their bank roll

Players	**Bankroll**	**Game expected value**	**Consecutive flips to win max**	**Attempts for 50% chance to win max**	**Play time(1 game per minutes)**
Friendly Game	$100	$7.56	6	44	44 minutes
Millionaire	$1 milion	$20.91	19	363.408	256 days
Billionaire	$1 billion	$30.84	29	372,130,559	708 years
Bill Gates (2015)	$79.2 billion	$37.15	36	47,632.711,549	90,625 Years
U.S GDP (2007)	$13.8 trillion	$44.57	43	6,096,987,078,286	11,600,05 2 years
World GDP (2007)	$54.3 trillion	$46.54	45	24,387,948,313,146	46,400,206 Years
Googolnaire	$\$10^{100}$	$333.14	332	1.340E+191	8.48E+180 × life of the universe

2.4 G.L.L Buffon

Buffon made a child play this game 2048 times. He did consider the total number of tosses when it comes up head for the first time instead of the payoff of the St. Petersburg paradox. Buffon published his resolution in a paper of 1771. Below is a table presenting the theoretical and experimental result of Buffon

Table 2.3: Theoretical and experimental result of Buffon

Tosses (k)	**Frequency**	**Payoff (2^k)**
1	1,061	2
2	494	4
3	232	8
4	137	16
5	56	32
6	29	64
7	25	128
8	8	256
9	6	512

Buffon made the conclusion that in practice, the St. Petersburg game becomes fair with an entrance fee of approximately $10.

2.5 Application of Generalized Weak Law of Large Number to St. Petersburg paradox

William Feller applied a generalized weak law of large number to St. Petersburg (William, 1994). In other to have a good understanding of his approach, a good knowledge of probability theory and statistics is necessary. The normal weak law of large number goes as follows:

Suppose $X_1, X_2, X_3, \dots$ are independent and identically distributed random variables with finite common expectation

$$EX_1 = \xi < \infty,$$

Then $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} P(|\bar{X}_n - \xi| < \varepsilon) = 1 \qquad (2.5)$$

Where

$$\bar{X}_n \coloneqq \frac{1}{n}\sum_{i=1}^{n} X_i$$

To be able to apply this to St. Petersburg paradox, Feller generalized the weak law of large number for independent and identically distributed random variables with finite expectation to that with infinite expectation and he used it to obtain a weak law of large number for St. Petersburg paradox (Anders, 1985).

Denoting the payoffs of St. Petersburg paradox as $X_1, X_2, X_3, \dots$, then

$$\lim_{n\to\infty} P(|\bar{X}_n - 1| < \varepsilon) = 1, \qquad (2.6)$$

in the sense of convergence in probability, Feller suggest the fair stake for playing n games should be of order $nlog_2 n$. Using this approaches, the expected value will be infinite when the game of infinite times is possible, and in finite case, the expected value will be a small value.

2.6 Application of Petersburg Paradox to the Stock Market

An interesting application of the St. Petersburg paradox presented by Durand (Durand, 1957) relates to the valuation of a company whose revenue grows significantly faster than the overall economy known as “growth company”. Growth stock is the word used to refer to the stocks or share of the company. Due to St. Petersburg paradox applicability to financial events that occurred at the end of the 1990s and early 2000s, a review of this application is worthwhile.

Early year 2000, an unprecedented increase was recorded over the previous three years in the price of growth stocks, an increase that led to the question of whether the move made by investors to buy the shares was a wise one or was it foolish not to buy more amount of share before prices increased even further.

Among those who were concerned about the inflationary pressure resulting from increase in stock price was Alan Greenspan Chairman of the Board of Governors of the United States Federal Reserve System. He raised the question:

“But how do we know when irrational exuberance has unduly escalated asset values, which then become subject to unexpected and pro longed contractions as they have in Japan over the past decade?"

On 19 November 1999, the Wall Street Journal reported that 59 mutual funds had amassed increase of more than 100% during the period January 1 to November 17, 1999 (Susan, 2005). An example of such fund is the Nicholas-Applegate Global Technology Fund, a fund that specializes in high-tech stocks, it has increased in price during the same period by astonishing amount of 325%. Or about 15 per day.

Applying Durand (1957) method using a modified St. Petersburg game and making the following assumptions:

1. Let player 1 “Peter” be the growth of the company.
2. Let player 2 “Paul” be the prospective purchasers of peters stock.

3. Let the probability of tossing head be $\frac{i}{i+1}$, $i > 0$ (thus, the probability of a tail is $\frac{1}{i+1}$).
4. Corresponding payoffs are series of increasing payments in which "Peter" pays "Paul" S dollars if the toss results in a tail, $S(1+f)$ if the second toss is a tail, $S(1+f)^2$ if the third toss is a tail, and so on, if the toss results in a head the game stops.

Total payments to Paul is given by the equation

$$\sum_{j=0}^{n-2} S(1+f)^j = \frac{[S(1+f)^{n-1} - 1]}{f} \tag{2.7}$$

Equation (2.7) has probability $\frac{i}{(i+1)^n}$ since head and tail appears with probability $\frac{1}{i+1}$ and $\frac{i}{i+1}$ respectively. As observed by Durand (1957), the expected payoffs of Paul are given by the double summation.

$$\sum_{n=1}^{\infty} \frac{i}{(i+1)^n} \sum_{j=0}^{n-2} S(1+f)^j$$

Evaluating the above integral yields

$$\sum_{n=0}^{\infty} \frac{S(1+f)^{n-1}}{(i+1)^n} = \begin{cases} \frac{S}{1-g}, & \text{if } f < i \\ \infty & \text{if } f \geq i. \end{cases} \tag{2.8}$$

Thus the expected payoff of Paul is $\frac{S}{1-f}$ if $f < i$ and his expected payoff is infinite if $f \geq i$.

In terms of financial securities, we have the following:

i. i represents a compound interest rate (or an effective rate of interest).

ii. $\frac{1}{i+1}$ is the present value of a loan of one dollar to be repaid one year in the future.

iii. f is the growth rate of the company as measured by the compound increase in revenue share.

To estimate a fair value for Peter's stock, all future dividends is discarded in perpetuity and his stock is estimated by the present value of all future dividends. Let peter's profit per share in k years be denoted by E_k , B_k his net asset value per share in the same year and D_k his total paid-out dividends per share in year k. Thus, yearly changes in net asset value are equal to the difference between earnings and dividends paid, hence, we have

$$B_{k+1} - B_k = E_k - D_k, \quad for \text{ all } n \geq 1$$

A common practice for Paul to estimate the value for Peter's stock is to make the assumption that $r = \frac{E_k}{B_k}$ and $p = \frac{D_k}{E_k}$ are independent of k. This assumption implies that $B_{k+1} - B_k$ is a constant multiple of E_k

$$B_{k+1} - B_k = E_k - D_k = (1-p)E_k = (1-p)rB_k.$$

Thus Peter's dividends, net asset value, and earnings are all growing at a constant rate,

$f = (1-p)r$. Here equation (4) is interpreted a perpetual series of dividend payments at rate, starting at S $d$ollars, growing at a constant rate f, and discounted at rate i in perpetuity. If $i > f$,

Then equation (4) converges to $\frac{D_1}{1-f} = \frac{pE_1}{1-f}$ an estimation representing a fair value for one of one Peter's stock. Equation (2.8) will diverge if $f \geq i$, this leads to a St. Petersburg paradox in which the practice of discounting future dividends at a uniform rate in perpetuity results to a paradoxical result.

By applying the valuation formula (2.8) to obtain exorbitant estimated valuations for many high-tech growth stocks, stock purchasers bought avidly, thereby forcing prices to extreme levels. By late 2000, stock prices underwent the "prolonged contractions" predicted by Greenspan, with subsequent unprecedented losses to corporate and individual stock buyers.

Three years later, many formerly avidly sought-after high-tech companies and mutual funds were defunct.

2.7 Facebook and the St. Petersburg Paradox

Facebook now the world's largest social networking site created by Mark Zuckerberg originally designed for college students is a site that simplify the connection and sharing of information with friends and family online with over a billion users.

Facebook is a high-growth stock that behaves a lot like St. Petersburg coin St. Petersburg coin with enormous potential payoff but not infinite. Approximately, 608 million people actively uses Facebook on a monthly basis as at the end of 2010, and is predicted that by 2020 over 22 billion people will be Facebook users which is almost three times the population of the world as at 2019.

Charles Lee the former head of equity research at Barclays Global Investments and a professor of accounting at Stanford University Business School said "it's an incredibly difficult thing to forecast the future cash flow of this kind of company, even for a quantitative investor" also he said "once your projections go out beyond two or three years, you're in a very big murky waters."

Any slight change in high growth rates can result in great change in values at fast-moving companies. Just as the coin flip in a St. Petersburg paradox can terminate at any toss, so also the projector in the fastest growing companies can go downward. Facebook recorded a revenue of $3.7 billion and a profit of $ $1 billion in 2011. If Facebook is to continue such a rapid growth over the next decade it will acquire a 26% in annual growth. The question one is to ask is what will happen to Facebook stock?

If the share of Facebook is to rise from $100 billion which is the initial to $190 billion in the course of a decade, 2021 to be precise, it will have 90% cumulative gain for an average annual return of 6.8%.

In this projection, Facebook first public shareholders stand the chance to be richly rewarded over the years to come just as a player playing the St. Petersburg game could become very wealthy.

Table 2.4: Facebook annual revenue (Facebook Revenue, 2009-2018).

2018	$55,838
2017	$40.653
2016	$27,638
2015	$17,928
2014	$12,466
2013	$7,872
2012	$5,089
2011	$3,711
2010	$1,974
2009	$777
2008	$272
2007	$153

2.8 Cumulative Utility Theory and the St. Petersburg paradox

According to CPT (Cumulative Utility Theory) a person utility of the lottery *L* involved in the St. Petersburg paradox is given by formula (2.9) where u: ú$_+$ → ú$_+$ is a person's utility function for gains and w : [0,1] → [0,1] is an individual's probability weighting function for gains (Palvo, 2004).

$$u(L) = \sum_{n=1}^{\infty} u(2^n)[w(2^{1-n}) - w(2^{-n})] \quad (2.9)$$

According to Tversky and Kahneman (Tversky, A. and Kahneman, D., 1992), most of studies uses a power utility function $u(x) = x^\alpha$ and an S-shaped probability weighting function $w(p) = p^\gamma (p^\gamma + (1-p)^\gamma)^{1/\gamma}$ first proposed by Quiggin (Quiggin, 1982). Since the St. Petersburg paradox L involves very small probabilities, Quiggin's function $w(p)$ may be accurately approximated as $w(p) \approx p^\gamma$ due to the fact that the denominator of Quiggin's function $w(p)$ converges to unity for tiny probabilities p. Then, equation (2.9) simplifies into formula (2.10).

$$u(L) \approx (2^\gamma - 1) \sum_{n=1}^{\infty} 2^{(\alpha-1)n} \quad (2.10)$$

It follows from (2.10) that according to CPT a person obtains a bounded utility from lottery L only when $\alpha < \gamma$ *i.e.* when the sum on the right hand side of (2.10) is convergent. Thus, CPT explains the St. Petersburg paradox only when the power coefficient of an individual's utility function is lower than the power coefficient of a person's probability weighting function. Intuitively, an individual's utility function must not simply be concave but it should be concave relative to a person's probability weighting function to avoid the St. Petersburg paradox.

Table 2.4 presents typical values of power coefficients α and γ that were obtained from the best parametric fitting to the experimental data in well-known recent studies. Some studies (*e.g.* Tversky and Fox, 1995) adopted a probability weighting function.

$w(p) = \delta \cdot p^\gamma / (\delta \cdot p^\gamma + (1-p)^\gamma)$, first adopted by Goldstein and Einhorn (Goldstein, W. and Einhorn, H., 1987). For small probabilities a Goldstein-Einhorn function $w(p)$ can be approximated as $w(p) \approx \delta \cdot p^\gamma$. An individual then still obtains a bounded utility from lottery L only when $\alpha < \gamma$. The best fitting estimates of a power coefficient γ for a Goldstein-Einhorn function $w(p)$ are presented in parentheses in the third column of table 2.5.

In all studies from table 2.5 except for Camerer and Ho (Camerer, C. and Ho, T., 1994) and Wu and Gonzalez (Wu, G. and Gonzalez, R., 1996) the estimated best fitting CPT parameters are $\alpha > \gamma$, which implies a divergent sum on the right hand side of equation (2). Thus, conventional parameterizations of CPT predict that a person is willing to pay up to infinity for the St. Petersburg lottery L . This paradoxical result occurs because a conventional inverse S-shaped probability weighting function overweights small probabilities too much for a mildly concave utility function to offset this effect.

Apparently, the parameterization of CPT that accommodates best the available experimental evidence does not explain the oldest and the most famous paradox in decision theory—the St. Petersburg paradox. To accommodate the St. Petersburg paradox CPT must be estimated together with a restriction $\alpha < \gamma$ on its parameters. However, it is not clear if a restricted version of CPT remains descriptively superior to other decision theories.

Table 2.5: Parameterization of CPT that accommodates best the experimental data in well-known recent studies (Palvo, 2004). .

Experimental study	Power of utility function (alpha)	Power of probability weighting function (gamma)
Kahneman and Tversky (1992)	0.88	0.61
Camerer and Ho (1994)	0.37	0.56
Tversky and Fox (1995)	0.88	0.69
Wu and Gonzalez (1996)	0.52	0.71(0.68)
Abdellaoui (2000)	0.89	0.60
Bleichrodt and Pinto (2000)	0.77	0.76(0.55)
Kilka and Weber (2001)	$0.76 - 1.00$	$0.30 - 0.51$
Abdellaoui et al. (2003)	0.91	0.76

CHAPTER 3

ANALYSIS OF ST. PETERSBURG PARADOX

In this chapter, analysis of a two Paul game as well as a four Paul game is carried out and a detailed analysis on the St. Petersburg Paradox and its solutions in terms of uniform treatment using d'Alembert's ratio test

3.1 Two Paul Paradox

Sandor and Gordon (Sandor, C. and Gordon, S., 2002)presented the Two Paul Paradox in their paper as follows:

Suppose Peter agrees to play exactly one St. Petersburg game with each of two players, Paul_1 and Paul_2. Question: Are Paul_1 and Paul_2 better off (i) accepting their individual winning X_1 and X_2, say, or (ii) agreeing, before they play, to divide their total winning in half so that each receives $\frac{X_1+X_2}{2}$?.

Csorgo and Simons (Csörgő, 2005) proceeded to show that Two Pauls adopting strategy (ii) are better off as follows:

Let X, X_1 and X_2 be identically independently distributed random variables with common distribution given by

$$P\left(X = 2^j\right) = \frac{1}{2^{j+1}}, \qquad j = 0,1,\ldots \tag{3.1}$$

For $x > 0$ we have

$$P(X_1 + X_2 \geq x) \geq P(2X_1 \geq x)$$

Proof: From the right side of equation (3.1) we have

$$P(2X_1 \geq x) = P\left(X_1 \geq \frac{x}{2}\right) = \sum_{j=\left\lceil log_2\left(\frac{x}{2}\right)\right\rceil} \frac{1}{2^{j+1}} = \frac{1}{2^{\left\lceil log_2\left(\frac{x}{2}\right)\right\rceil}}$$

$$= \left(\frac{1}{2}\right)^{j_x}, \qquad (3.2)$$

Where

$\lceil x \rceil$ is the ceiling function of x,

$$j_x \coloneqq \lceil log_2(x) \rceil - 1$$

Let $M = \max\{X_1, X_2\}$, and let the events on the left side of equation (3.1) be union of the exclusive events:

(i) $\{M < 2^{j_x}, X_1 + X_2 \geq x\}$
(ii) $\{M > 2^{j_x}, X_1 + X_2 \geq x\}$
(iii) $\{M = 2^{j_x}, X_1 + X_2 \geq x\}$

For case (i)

$M \leq 2^{j_x - 1}$ if $M < 2^{j_x}$, this implies $X_1 + X_2 \leq 2M \leq 2^{j_x} < x$.

Thus,

$X_1 + X_2 \geq x$ implies $M \geq 2^{j_x}$

Hence the probability of evevt (i) is zero y

For case (ii)

$M \geq 2^{j_x+1}$, $then$ $X_1 + X_2 \geq M + 1 \geq 2^{j_x+1} \geq x$.

Thus,

$$P\big(M > 2^{j_x},\ X_1 + X_2 \geq x\big) = P\big(X \geq 2^{j_x+1}\big) = 1 - P(M < 2^{j_x+1})$$
$$= 1 - P(X_1 \leq 2^{j_x})P(X_1 \leq 2^{j_x})$$
$$= 1 - (1 - \frac{1}{2^{j_x+1}})(1 - \frac{1}{2^{j_x+1}}) = \left(\frac{1}{2}\right)^{j_x} - \left(\frac{1}{2}\right)^{2j_x+2}$$

Case (iii) we have

$$X_1 \geq x - 2^{j_x} \Leftrightarrow log_2(\,X_1) \geq \lceil log_2\big(x - 2^{j_x}\big)\rceil - 1 \coloneqq k_x.\ \Leftrightarrow X_1 \geq 2^{kx}$$

Then

$$\{M = 2^{j_x},\ X_1 + X_2 \geq x\} = \big\{\,X_1 = 2^{j_x}, x - 2^{j_x} \leq X_2 \leq 2^{j_x}\big\} \cup \{\,X_2$$
$$= 2^{j_x}, x - 2^{j_x} \leq X_1 \leq 2^{j_x}\}$$
$$= \big\{\,X_1 = 2^{j_x}, 2^{k_x} \leq X_2 \leq 2^{j_x}\big\} \cup \{\,X_2 = 2^{j_x}, 2^{k_x} \leq X_1$$
$$\leq 2^{j_{x-1}}\}$$

Since X_1 and X_2 are independent, we have

$$P\big(M = 2^{j_x}, X_1 + X_2 \geq x\big)$$
$$= P\big(X_1 = 2^{j_x}\big)P\big(2^{k_x} \leq X_2 \leq 2^{j_x}\big)$$
$$+ P\big(X_2 = 2^{j_x}\big)P\big(2^{k_x} \leq X_1 \leq 2^{j_x-1}\big)$$
$$= \frac{1}{2^{j_x+1}}\left(\frac{1}{2^{k_x}} - \frac{1}{2^{j_x+1}}\right) + \frac{1}{2^{j_x+1}}\left(\frac{1}{2^{k_x}} - \frac{1}{2^{j_x}}\right)$$
$$= \left(\frac{1}{2}\right)^{j_x+kx} - \left(\frac{1}{2}\right)^{2j_x+2} + \left(\frac{1}{2}\right)^{2j_x+1}$$

Combining results from (i), (ii) and (iii) yields

$$P(X_1 + X_2 \geq x) = \left(\frac{1}{2}\right)^{j_x} + \left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right)^{k_x} - \left(\frac{1}{2}\right)^{j_x}\right] \qquad (3.3)$$

Comparing (3.2) and (3.3) and the fact that $k_x \leq j_x$ for all $x > 0$ leads to(3.1) This completes the proof.

3.2 Four Paul Paradox

For a Four Paul problem we need to show that for all $x > 0$

$$P(X_1 + X_2 + X_3 + X_4 \geq x) \geq P(4X_1 \geq x) \tag{3.4}$$

To establish the right side of equation (3.4) we proceed as follows:

$$P(4X_1 \geq x) = P\left(X_1 \geq \frac{x}{4}\right) = \sum\nolimits_{j=\lceil log_2(x/4)\rceil} \frac{1}{2^{j+1}} = \frac{1}{2^{\lceil log_2(x/4)\rceil}}$$
$$= \left(\frac{1}{2}\right)^{j_x-1} \tag{3.5}$$

Where

$$j_x \coloneqq \lceil log_2(x)\rceil - 1$$

To find the right hand side of equation (3.4) we use

$$P(X_1 + X_2 + X_3 + X_4 \geq x)$$
$$= \sum_{y=2}^{x-2} P(X_1 + X_2 = y)P(X_3 + X_4 \geq x - y)$$
$$+ P(X_1 + X_2 \geq x - 1) \tag{3.6}$$

$P(X_3 + X_4 \geq x - y)$ *and* $P(X_1 + X_2 \geq x - 1)$ can be found using (3.3) and we can find

$$P(X_1 + X_2 = y) = P(X_1 + X_2 \geq y) - P(X_1 + X_2 \geq x + 1) \tag{3.7}$$

Equation (3.6) cannot be simplified, so we can make use of Maple to calculate their values for each x. Keguo Huang use Mathematica to show the significance of the difference (Keguo, 2013).

3.3 Detailed Analysis of the St. Petersburg Paradox

Theorem 1: Let $\mathbb{N}$ denote the set of natural numbers and let $x: \mathbb{N} \to X \subset \mathbb{R}_+$ denotes a strictly increasing mapping, that is, $x_j > x_i \; \forall \, j > i, \; i, j \in \mathbb{N}$. Let $u: X \to \mathbb{R}_+$ denote a non-decreasing function such that $u(> x_i) < \infty \; \forall \, i < \infty$, and $p: X \to [0,1]$, a non-increasing function such $\sum_{i=1}^{\infty} p(x_i) = 1$, that is, a probability distribution. Then the following hold (Christian, 2013):

Case 1: $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$, if $\exists i^* < \infty$ such that $p(x_i) = 0 \; \forall \, i \, \geq \, i^*$.

Case 2: $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$, if $\exists i^* < \infty$ such that $sup_{i \geq i^*} \, \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} < 1$.

Case 3: $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$, if $\exists i^* < \infty$ such that $inf_{i \geq i^*} \, \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} \geq 1$.

Case 4: $\sum_{i=1}^{\infty} u(x_i)p(x_i)$, may converge or diverge if $\lim_{i \to \infty} \frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = 1$.

Proof: Case 1 is obvious. For case 2, 3 and 4 follow from d'Alembert's ratio test (Stephenson, 1973)

Corollary 1: [St. Petersburg paradox case] for each probability distribution $p(x_i) > 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, which is strictly decreasing for all $i \, \geq \, i^*, i^* < \infty$, there exist functions $u(x_i)$ which is strictly increasing for $i \, \geq \, i^*, i^* < \infty$, such that $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$.

Proof: By applying case 3 of Theorem 1 and considering the function $u(x_i)$ to be such that

$$\frac{u(x_{i+1})}{u(x_i)} > \frac{p(x_i)}{p(x_{i+1})} > 1 \qquad for\ all \ \ i \geq \, i^*,$$

here the growth rate of utility is more than $\frac{p(x_i)}{p(x_{i+1})}$ the shrinkage rate of probabilities (for instance if $\frac{p(x_{i+1})}{p(x_i)} = 3$ this implies that $p(x_i)$ is triple as high as $p(x_{i+1})$) for infinitely many items. Equality in the above relation leads to case 4 of the Theorem 1.

Remark 1: in the original St. Petersburg paradox we have $x_i = i,\ u(x_i) = 2^{i-1}, p(x_i) = \frac{1}{6}\left(\frac{5}{6}\right)^{i-1}$, this gives $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \sum_{i=1}^{\infty} \frac{1}{6}\left(\frac{10}{6}\right)^{i-1} = \infty$. Since $\frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = \frac{10}{6} > 1\ \forall\ i \geq 2$, the original St. Petersburg Paradox becomes case 3 of Theorem 1 $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$.

In Cramer's version of the "Classical" St. Petersburg Paradox the assumptions $x_i = i\ , u(x_i) = 2^{-i}$, which leads to $\sum_{i=1}^{\infty} u(x_i)p(x_i) = 2^{-i} \times \infty = \infty$.

Observe that $\frac{u(x_{i+1})}{u(x_i)} = 2,\ \frac{p(x_i)}{p(x_{i+1})} = 2,$ and $\frac{u(x_{i+1})p(x_{i+1})}{u(x_i)p(x_i)} = 1\ \forall\ i \geq 2$.

Hence, the "classical" St. Petersburg Paradox becomes case 4 of the Theorem 1 for which $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$. Thus, Corollary 1 holds generally for case 3 and may hold for some instances of case 4 Theorem 1.

Corollary 2: Assume a probability distribution for which $E(x) < \infty$. Then for all concave functions $u(\cdot)$ such that $(x_i) < \infty\ \forall\ x_i\ <\ \infty$, we have $E[u(x)] < \infty$.

Proof: applying Jensen's inequality implies $E[u(x)] < u[E(x)]$ and thus, $E[u(x)] < \infty$.

Corollary 3: (Bernoulli-Crammer case) For any non-decreasing $0 < u(x_i) < \infty\ \forall\ i < \infty$ and a strictly decreasing probability distribution $p(x_i) > 0\ \forall\ i\ < \infty$, such that $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$ and $inf_{i\in\mathbb{N}} \frac{p(x_i)}{p(x_{i+1})} > 1$, there exist an increasing transformation of $u(x_i)$, viz. $\varphi[u(x_i)], such\ that\ \sum_{i=1}^{\infty} \varphi[u(x_i)]p(x_i) < \infty$.

Proof: By the given assumption $inf_{i\in\mathbb{N}} \frac{p(x_i)}{p(x_{i+1})} =\ \beta > 1$ and since $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$ this implies for all $i >\ i^*$

$$\frac{u(x_{i+1})}{u(x_i)} > \frac{p(x_i)}{p(x_{i+1})} >\ \beta, \tag{3.8}$$

else the initial series will converge. Constructing $\varphi(\cdot)$ in such a way that for any $u(\cdot)$ having property (3.8)

$$\frac{\varphi[u(x_{i+1})]}{\varphi[u(x_i)]} < \beta. \qquad (3.9)$$

To establish that observe that for any number $\frac{u(x_{i+1})}{u(x_i)} \geq \beta > 1$ there exist a real number $0 < \omega(x_i, x_{i+1}) < 1$ and thus, $1 < \left(\frac{u(x_{i+1})}{u(x_i)}\right)^{\omega(x_i, x_{i+1})} < \beta$. Let $\omega^* = inf_{i>i^*} \omega(x_i, x_{i+1})$, then it can be chosen that $\varphi[u(x_i)] = [u(x_i)]^{\omega^*}$ for $\varphi(\cdot)$ in equation (3.9), this implies that folmula.

$$\frac{\varphi[u(x_{i+1})]p(x_{i+1})}{\varphi[u(x_i)]p(x_i)} < 1 \qquad \forall\, i > i^*.$$

Thus, by Case 2 of Theorem 1 it follows that $\sum_{i=i^*.}^{\infty} \varphi[u(x_i)]p(x_i) < \infty$.

Remark 2: $\varphi[u(x_i)] = \ln[u(x_i)] = \ln(2^{i-1})$ was proposed by Daniel Bernoulli and $\varphi[u(x_i)] = \sqrt{u(x_i)} = \sqrt{2^{i-1}}$. These functions are for $p(x_i) = 2^{-i}$ thus leading to formula

$$\frac{\varphi[u(x_{i+1})]p(x_{i+1})}{\varphi[u(x_i)]p(x_i)} = \frac{i}{2(i-1)} < 1 \text{ for } i > 2 \text{ for that proposed by Bernoulli, we have ıt}$$

$$\frac{\varphi[u(x_{i+1})]p(x_{i+1})}{\varphi[u(x_i)]p(x_i)} = \frac{\sqrt{2}}{2} < 1 \text{ for } i \in \mathbb{N} \text{ for that proposed by Cramer.}$$

Remark 3: The St. Petersburg paradox is obtained about by winning and the probability distribution. Replacing the probabilities of the Cramer solution by $\tilde{p}(x_i) = \frac{1}{L\sqrt{2^i}}$ where $L = \sum_{i=1}^{\infty} 2^{-\frac{i}{2}} = \frac{1}{\sqrt{2}-1}$ denotes the calibrating constant, obviously

$$\sum_{i=1}^{\infty} \tilde{p}(x_,) = \sum_{i=1}^{\infty} \frac{1}{L\sqrt{2^i}} = \sum_{i=1}^{\infty} 1 / \frac{2^{\frac{i}{2}}}{2^{\frac{1}{2}} - 1} = \left(2^{\frac{1}{2}} - 1\right) \sum_{i=1}^{\infty} 2^{-\frac{i}{2}} = 1, \text{and}$$

$$\sum_{i=1}^{\infty} \varphi[u(x_i)]\, \tilde{p}(x_,) = \sum_{i=1}^{\infty} \frac{\sqrt{2^{i-1}}}{\sqrt{2^i}/\sqrt{2-1}} = \sum_{i=1}^{\infty} \left(1 - \frac{1}{\sqrt{2-1}}\right) = \infty,$$

Which reextablishes the St. Petersburg Paradox.

Corollary 4: Case 2 of Theorem 1 obtains if the shrinkage rate of probability exceeds the rate of utility for infinitely many winnings**.**

Case 3 of Theorem 1 obtains if the growth rate of utility exceeds the shrinkage rate of probability for infinitely many winnings.

Case 2 can be transformed into case 3 by an appropriate increasing complex mapping

Corollary 5: (Buffon Case) Consider non-decreasing functions $u(x_i) < \infty, i < \infty$, and probability distribution $p(x_i) > 0, i < \infty$, such that $\sum_{i=1}^{\infty} u(x_i)p(x_i) = \infty$. Suppose a gambler perceives the aggregates upper tail of the probability distribution $0 < \varepsilon := \sum_{i=i^*+1}^{\infty} p(x_i)$ as close to zero, and therefore negligible. Consequently, the gambler perceives all $p(x_i), i > i^*$, as close to zero and negligible. Hence, the probability distribution becomes finite, and $\sum_{i=1}^{\infty} u(x_i)p(x_i)$ is perceived as a finite expression

$$\sum_{i=1}^{i^*} u(x_i)p(x_i) \approx \sum_{i=1}^{i^*} u(x_i)\tilde{p}(x_i) < \infty,$$

where

$$\tilde{p}(x_i) = \begin{cases} \dfrac{p(x_i)}{1-\varepsilon} & \forall\, i \geq i^*, \\ 0 & \forall\, i > i^*. \end{cases}$$

Proof: follows from case 1 of Theorem 1.

Corollary 6: (Menger Case) For all strictly increasing functions $u(\cdot), u(x_i) < \infty, \forall\, i < \infty$, and all probability distribution $p(x_i) \geq 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$ holds if and only if $\exists\, \boldsymbol{B} < \infty$ such that $u(x_i) \leq \boldsymbol{B}\ \forall\, i \in \mathbb{N}$.

Proof: Obviously $u(x_i) \leq \boldsymbol{B} < \infty\ i \in \mathbb{N}$ implies $\sum_{i=1}^{\infty} u(x_i)p(x_i) \leq \sum_{i=1}^{\infty} \boldsymbol{B}p(x_i) = \boldsymbol{B} < \infty$.

Conversely, by assuming $u(x_i) \to \infty$ as $x_i \to \infty$. The case $p(x_i) = 0\ \forall\, i \geq i^*$, leads to Case 1. Since $\sum_{i=1}^{\infty} p(x_i) = 1$ all feasible probability distributions will be strictly decreasing for infinite $x_i's$.thus, $p(x_i) > 0$ is strictly decreasing for all $i > i^*$, $i^* < \infty$.

This leads to $\frac{p(x_i)}{p(x_{i+1})} > 1$ for all $i > i^*$, $i^* < \infty$. Also $\sum_{i=1}^{\infty} p(x_i) = 1$ implies $sup_{i>i^*} \frac{p(x_i)}{p(x_{i+1})} < \infty$. If $inf_{i>i^*} \frac{u(x_{i+1})}{u(x_i)} > sup_{i>i^*} \frac{p(x_i)}{p(x_{i+1})}$, then it can proceed along this path, else a strictly increasing transformation is chosen $\varphi(\cdot)$ such that $inf_{i>i^*} \frac{\varphi[u(x_{i+1})]}{\varphi[u(x_i)]} > sup_{i>i^*} \frac{p(x_i)}{p(x_{i+1})}$. Such transformation $\varphi(\cdot)$ always exist. One of such obvious case is $\varphi[u(x_i)] = [u(x_i)]^{\gamma}, \gamma > 1$. Since $\frac{u(x_{i+1})}{u(x_i)} > 1,$ $,\gamma > 1$ can be selected such that $inf_{i>i^*} \left[\frac{u(x_{i+1})}{u(x_i)}\right]^{\gamma} > sup_{i>i^*} \frac{p(x_i)}{p(x_{i+1})}$. This leads to

$$\frac{\varphi[u(x_{i+1})]}{\varphi[u(x_i)]} > \frac{p(x_i)}{p(x_{i+1})} \Leftrightarrow \frac{\varphi[u(x_{i+1})]p(x_{i+1})}{\varphi[u(x_i)]p(x_i)} > 1 \; \forall i \geq i^*, \qquad i^* < \infty,$$

Which becomes case 3 of Theorem 1 for $i \geq i^*$. Therefore, boundedness of is a necessary and sufficient condition for $\sum_{i=1}^{\infty} u(x_i)p(x_i) < \infty$.

CHAPTER 4

RESULTS

In this chapter we present some of the results of simulated St. Petersburg game using Maple (Adi, 1999).

The plot of the probability of getting n sequential head using maple with n = 10 is given below (see Appendix 1 for the code)**.**

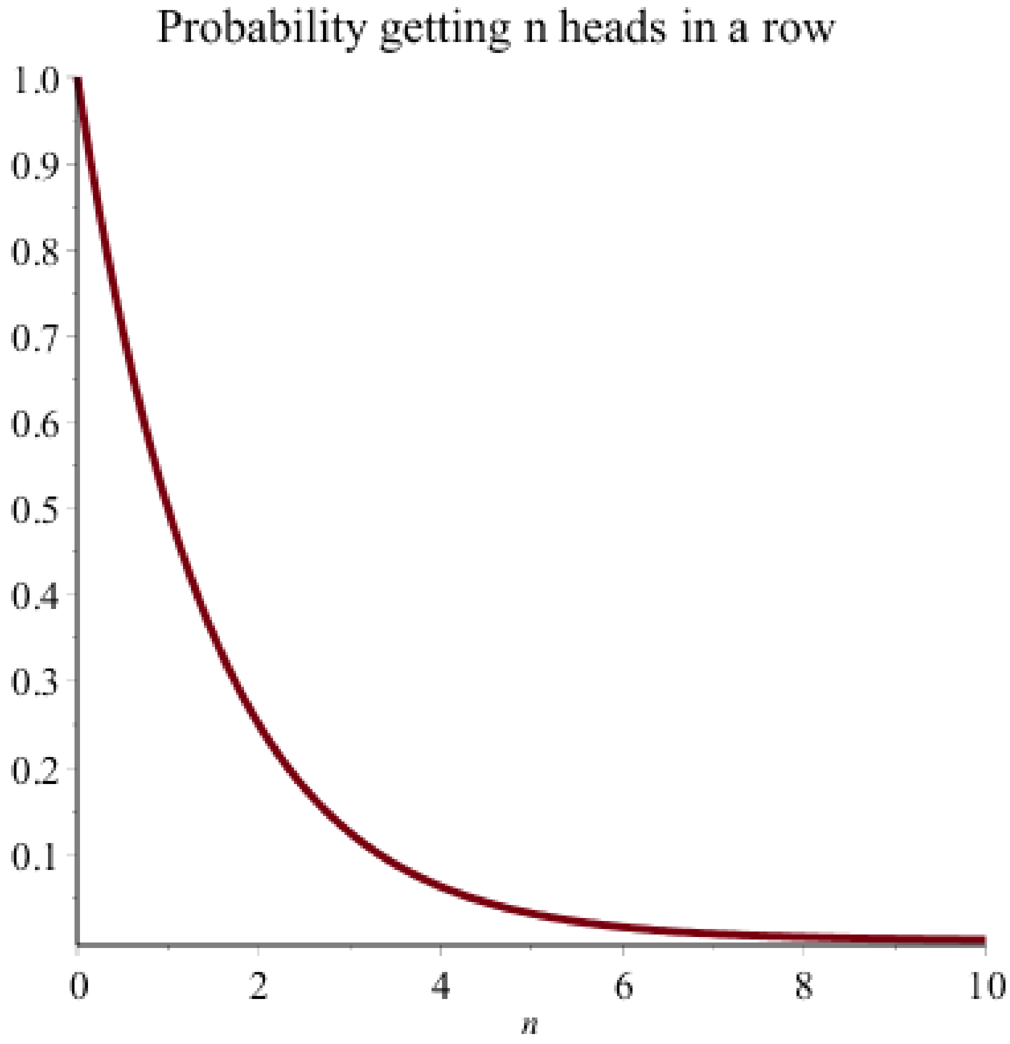

Figure 4.1: Probability of getting *n* head in a row

Setting c = \$10 (cost of playing the game) and n = 150 (number of game played) we obtain the following outcome from the Maple simulation

$$L := [1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1,
1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1,
0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0,
1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0]$$

$$LL := [2, -5, 2, 4, -5, -5, -5, 2, -5, -5, 2, -5, 2, -5, -5, -5, 2, -5, 2, -5, -5, -5, -5, -5, -5,
2, -5, -5, 2, 4, -5, 2, -5, -5, 2, -5, 2, -5, -5, 2, 4, -5, -5, 2, 4, 8, 16, 32, -5, 2, -5, 2, -5,
2, -5, -5, -5, 2, 4, 8, -5, -5, -5, 2, 4, -5, -5, -5, -5, 2, -5, -5, 2, -5, 2, -5, 2, -5, -5, 2,
4, 8, -5, 2, 4, -5, -5, -5, 2, -5, -5, 2, 4, -5, 2, -5, 2, 4, -5, -5, 2, 4, -5, 2, -5, 2, 4, 8, 16,
-5, -5, -5, -5, 2, 4, -5, 2, 4, 8, 16, 32, -5, 2, -5, -5, -5, -5, 2, -5, -5, -5, -5, -5, 2, -5,
-5, 2, 4, -5, -5, -5, -5, -5, 2, 4, -5, -5, -5, -5, 2]$$

Where the entries of L are 0's and 1's, the 0's represents the outcomes where tail are obtained and the 1's represents the outcomes where head are obtained. LL entries represents the payoffs obtained (see Appendix 1 for the code)**.**

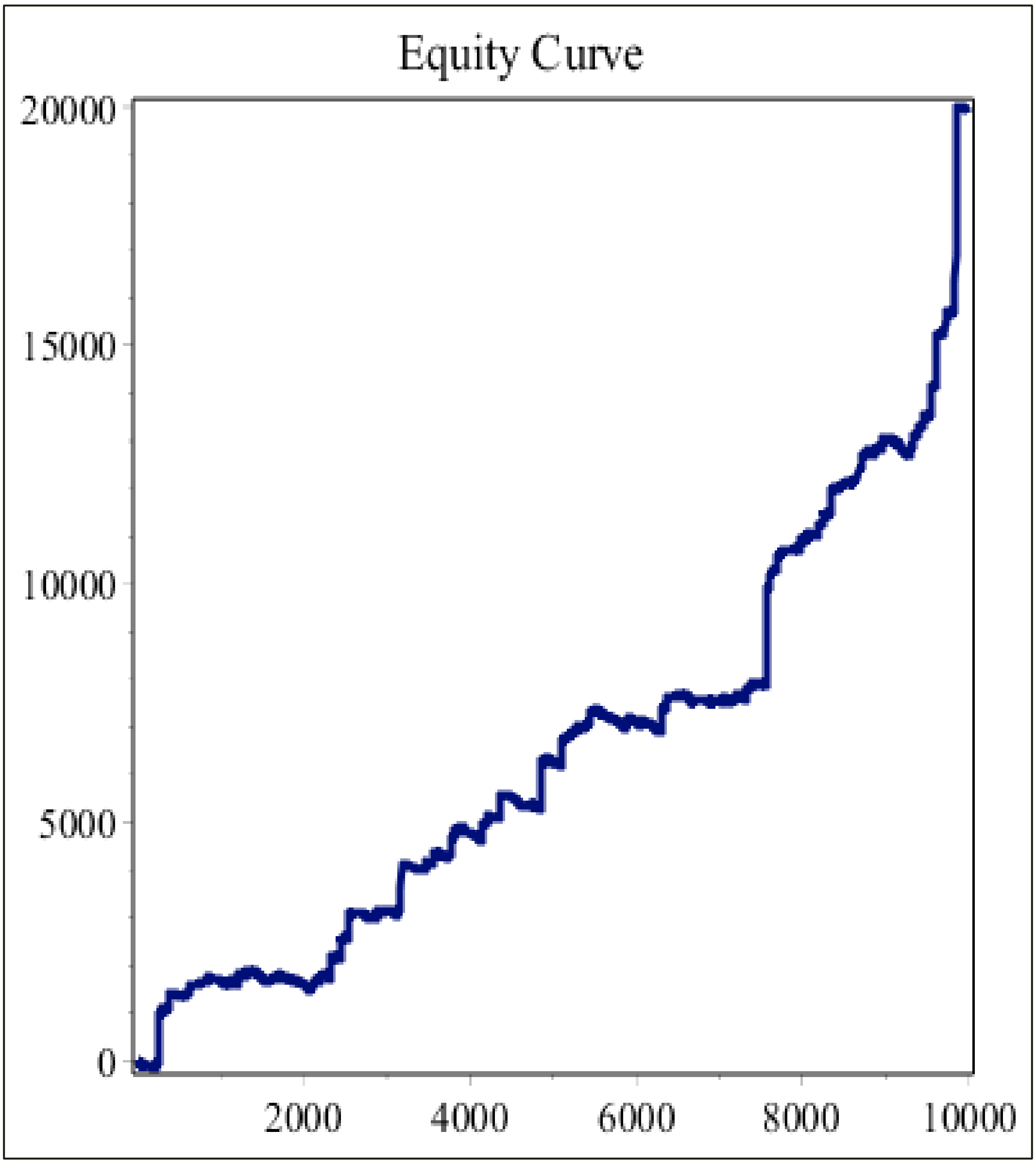

Figure 4.2: Utility Curve

Setting c = \$5 (cost of playing the game) and n = 10000 (number of game played) we obtain the Utility Curve above. The lack of smoothness in the equity curve is brought about by the occasional obtainment of long sequences of head that leads to large profit (see Appendix 1 for the code)**.**

People attitude towards risk can be grouped into three distinct categories in relation to their respective Bernoulli functions.

1. Risk-averse: a risk-averse person is an individual whose utility of the expected value of a game is greater than his expected utility from the game itself.

   A concave utility function is used to capture a risk-averse behavior. Considering a simple game based on a coin toss which a player obtain \$10 if the coin lands heads and \$20 if the coins land tail. The expected value of the game will be $(0.5 \times 10) + (0.5 \times 20) = 1.5$. a risk-averse individual whose Bernoulli utility function functions the form $u(w) = logw, w$ being the outcome will yield the expected utility $(0.5 \times \log(10)) + (0.5 \times \log(20)) = 1.15$

   The utility of the expected value of the gamble is $\log(15) = 1.176$

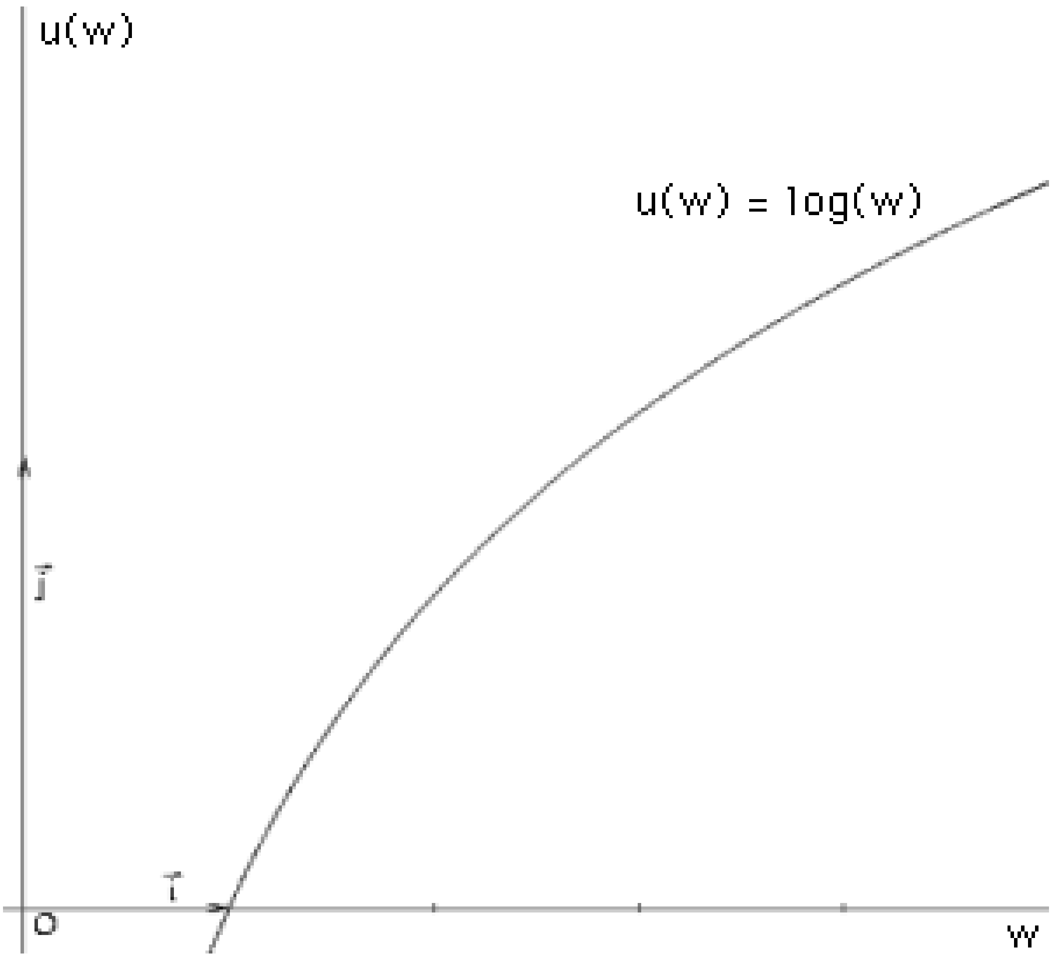

Figure 4.3: Risk-averse curve of the Bernoulli Utility log function

2. Risk-loving: A risk-loving person is an individual whose utility of the expected value of a game is less than his expected utility from the game itself. Based on this definition, a player that is truly risk-loving should be willing to stake all their wealth on a single dice roll.

   We capture this risk-loving behavior using a convex Bernoulli utility function such as the exponential function. A risk-loving individual whose Bernoulli utility function follows the form $u(w) = w^2$ will yield the expected utility over the game of $(0.5 \times 10^2) + (0.5 \times 20^2) = 250$

   The utility of the expected value of the game is 15^2=225

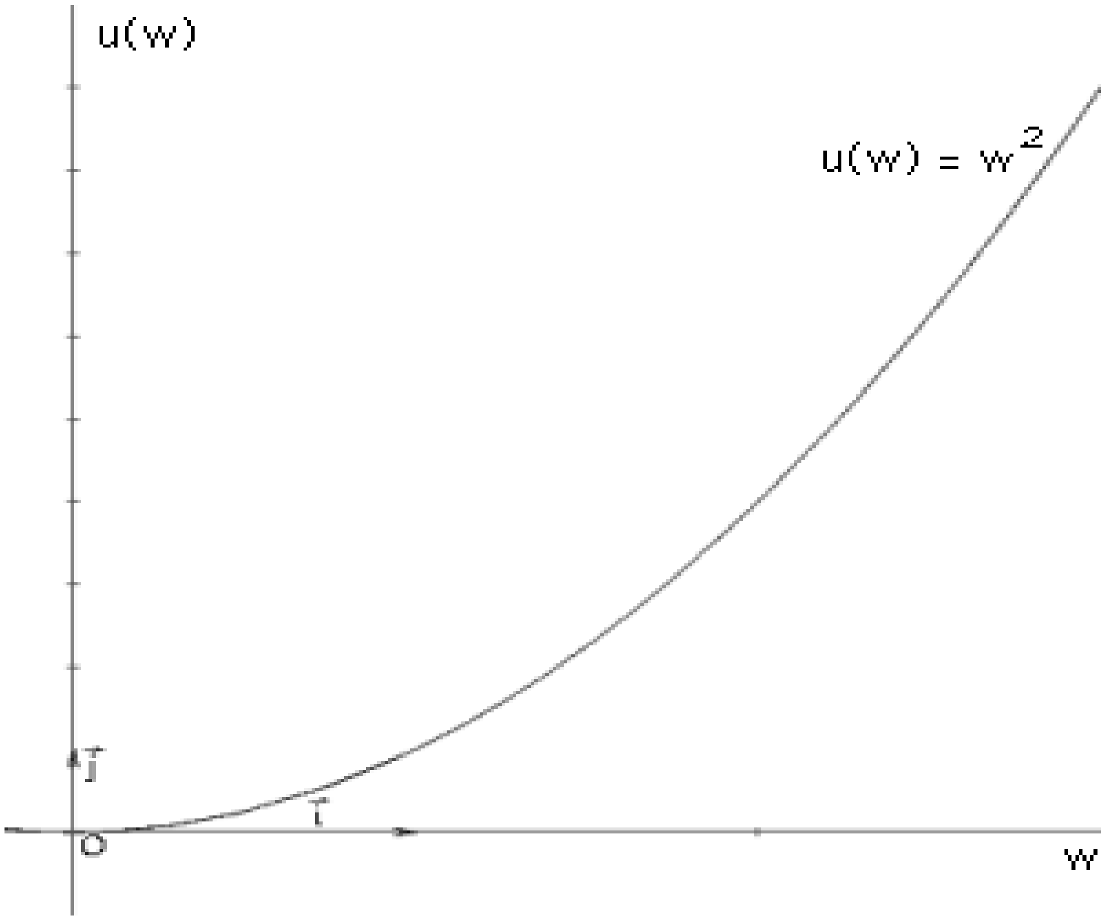

Figure 4.4: Risk-loving curve of the Bernoulli Utility exponential function

3. Risk-neutral: A risk-loving person is an individual whose utility of the expected value of a game is equal his expected utility from the game itself. A lot of financial firms adapt the risk-neutral manner while investing. We capture this risk-loving behavior using a linear Bernoulli utility function. A risk-loving individual whose Bernoulli utility function follows the form $u(w) = 2w$ will yield the expected utility over the game of $(0.5 \times 2 \times 10) + (0.5 \times 2 \times 20) = 30$

   The utility of the expected value of the game is 2×15=30

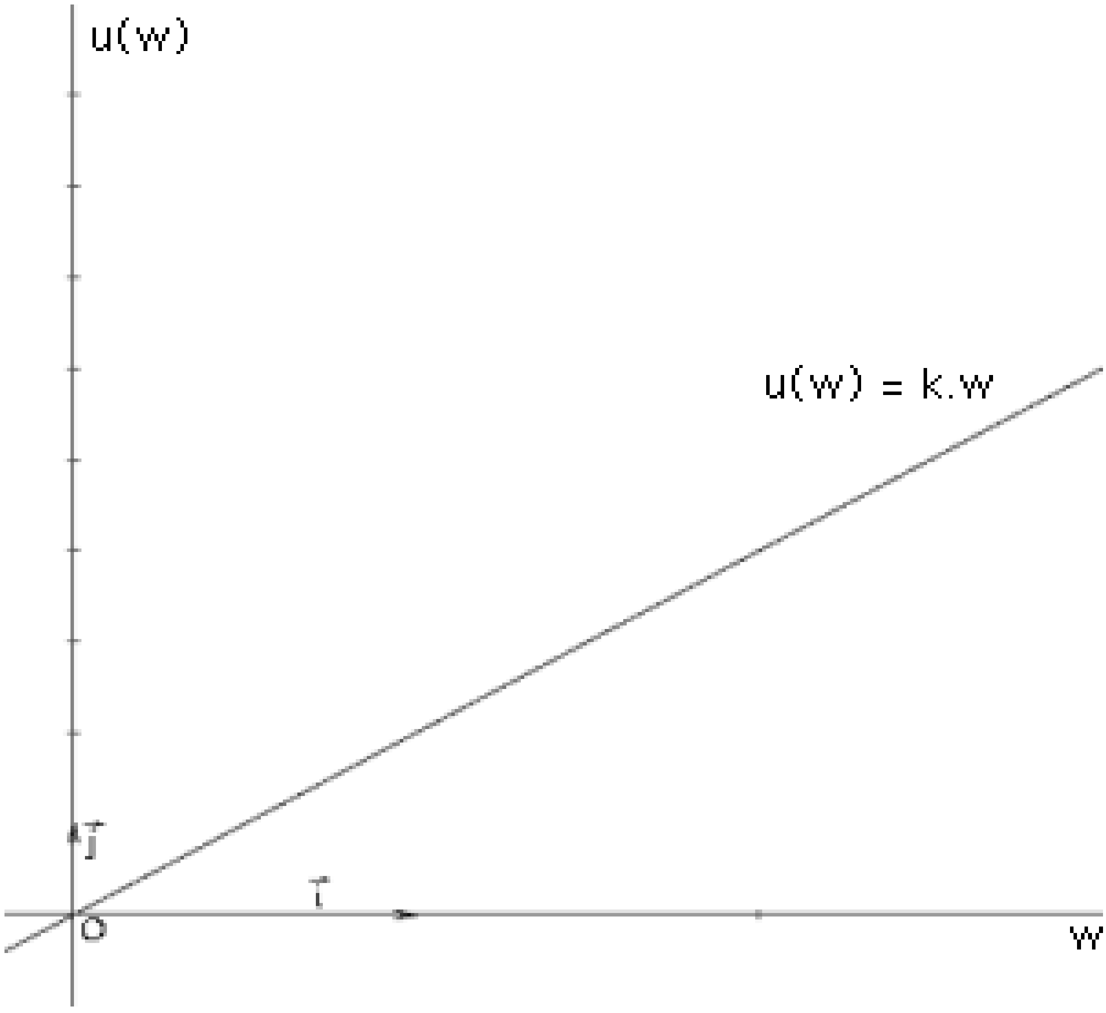

Figure. 4.5: Risk-neutral curve of the Bernoulli Utility linear function

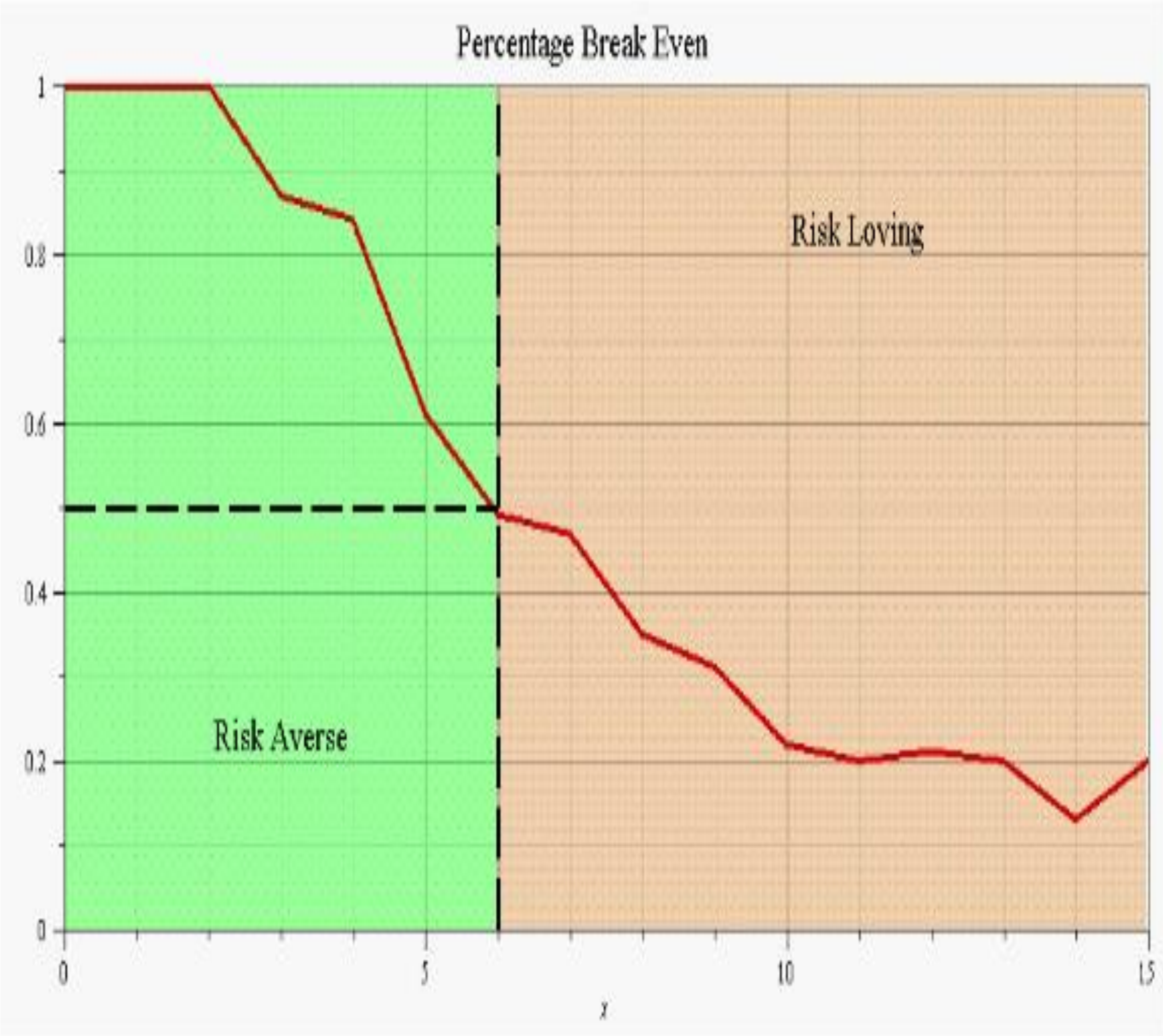

Figure 4.6: Risk Aversion Curve

Setting c = \$0,…,\$5 (cost of playing the game) and n = 100 (number of game played) the percentage number of times a result greater than zero is plotted as can be seen above. It is obvious that a risk aversion player will not be willing to pay more than \$6 to participate in this game (see Appendix 1 for the code)**.**

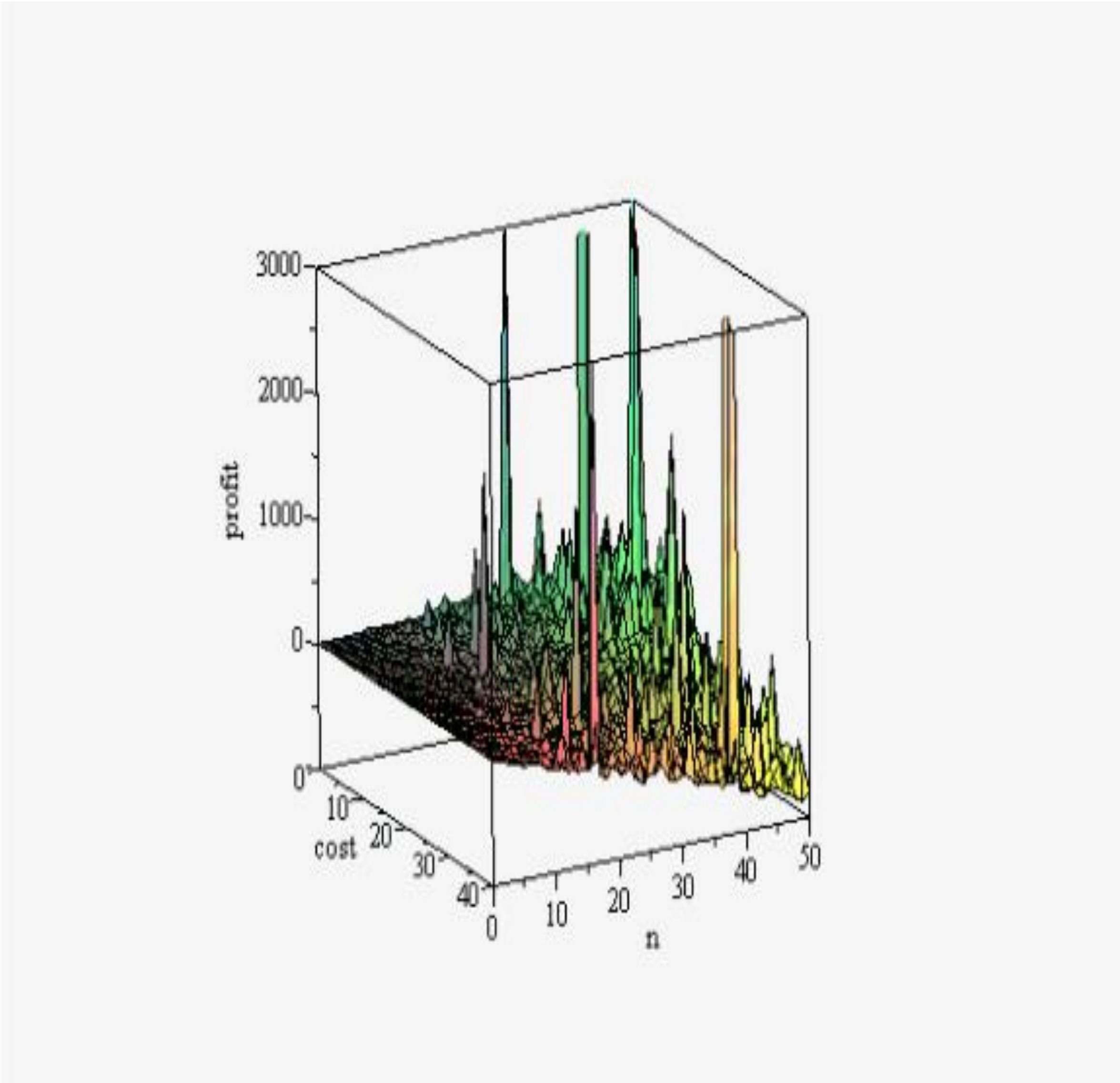

Figure 4.7: Plot of profit as a function of the game cost.

Finally, the plot of profit as a function of the cost of the game and n (number of simulations) shows that the profit is a decreasing function of the cost of the game and the number of times the game is played. Occasionally, long run of head yields a large profit as can be seen from the plot above (see Appendix 1 for the code).

CHAPTER 5
CONCLUSION

Contemporary mathematicians have made the proposal that the stake of a game should be more than the expected value of winning the game. A game of hazard having an infinite expected value was formulated by Nicholas Bernoulli, but no gambler will be willing to bet a stake that is more than a moderate amount to take part in this game of chance. This paradox brought about by this game led to various attempt in obtaining a solution to resolve this paradox. Daniel Bernoulli and Cramer proposed a concave transformation in other to obtain the solution. A method of neglecting some of the probabilities was suggested by Buffon and others. The first person that was successful in showing that a necessary and sufficient condition to obtain a finite expected value for the St. Petersburg paradox is Menger his findings led to the development of expected utility theory by Morgenstern's and von Neumann.

Chapter 3 presents an analysis on a Two-Paul game and a Four-Paul game and also the four possible cases of St. Petersburg paradox have been shown to be uniformly treated by the use of d'Alembert ratio test and finally computer simulation using Maple was carried out to support our result.

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APPENDIX

SIMULATION OF THE ST. PETERSBURG PARADOX USING MAPLE

In this section the Maple codes used to obtain the results in chapter 4 are presented. More detail on the algorithm can be obtain from Maple link (Marcus, 2010).

The code used to obtain the probability of getting *n* sequential head using maple is as follows:

```
> with(plots) :

> plot(0.5^n, n = 0 ..10, thickness = 3, title = "Probability getting n heads in a row", titlefont
      = [times, roman, 14]);
```

The simulation of the outcome of basic fair coin toss is as follows:

```
> restart

> with(Statistics) :

> with(ListTools) :

> randomize( ) :

> with(plots) :

> c := 10 : # cost of the game

> n := 150 :

> L := [seq(rand(0 .. 1)( ) , i = 1 .. n)];

> cc[0] := 0 :

> for i from 1 to n do if L[i] = 0 then cc[i] := 0 : elif L[i] = 1 then cc[i] := cc[i − 1] + 1 :
      end if: end do:
```

```
> LL := [seq(if(cc[i] = 0,-c, 2^cc[i]), i = 1 .. nops(L))];
```

The code used to plot the equality curve in chapter 4 is as follows:

```
> restart

> with(Statistics) :

> with(ListTools) :

> randomize( ) :

> c := 5 : # cost of the game

> n := 10000 :

> L := [seq(rand(0 .. 1)( ) , i = 1 .. n)] :

> cc[0] := 0 :

> for i from 1 to n do if L[i] = 0 then cc[i] := 0 : elif L[i] = 1 then cc[i] := cc[i − 1] + 1 :
      end if: end do:

> LL := [seq(if(cc[i] = 0,-c, 2^cc[i]), i = 1 .. nops(L))] :

> CumulativeSumChart(LL, markers = false, thickness = 3, title = "Equity Curve", titlefont = [times,
      roman, 14]);
```

The code for the risk aversion curve in chapter 4 is as follows:

```
> restart :

> with(plots) :

> with(plottools) :

> with(Statistics) :

> with(ListTools) :

> randomize( ) :

> X := proc(c, nsim) local n; L, cc, i, zz, j;
```

> **for** j **from** 1 **to** $nsim$ **do**

> $n := 100:$

> $L := [seq(rand(0..1)(\,), i = 1..n)];$

> $cc[0] := 0:$

> **for** i **from** 1 **to** n **do**

> **if** $L[i] = 0$ **then** $cc[0] := 0;$

> **elif** $L[i] = 1$ **then** $cc[i] := cc[i-1] + 1:$ **end if**:

> **end do**:

> **if** $add\left('if'\left(cc[i] = 0. - c, 2^{cc[i]}\right), i = 1..nops(L)\right) > 0$ **then** $zz[j] := 1$ **else** $zz[j] := 0$ **end if**:

> **end do**:

> $evalf\left(\left(\frac{add(zz[j], j = 1..nsim)}{nsim}\right), 4\right);$

> **end proc**:

> $data := [seq[c, X(c, 100)],\ c = 0...15]:$

> **for** i **from** 1 **to** 15 **do if** $data[i][2] \leq 0.5$ **then** $rt := data[i][1]$: **break**; **end if** : **end do**:

> $ap1 := plot(data, thickness = 3, color = red):$

> $ap2 := plot(0.5, x = 0..rt, thickness = 3, color = black, linestye = dash):$

> $ap3 := plot([rt, y, y = 0..1],\ thickness = 3, color = black, linestye = dash):$

> $ap4 := polygon([0, 0], [0, 1], [rt, 1], [rt, 0], color = green, transperacy = 0.6):$

> $ap5 := polygon([rt, 0], [rt, 1], [15, 1], [15, 0], color = gold, transperacy = 0.6):$

> $ap6 := textplot([2, 0.2.\text{"Risk Averse"}], align = \{above, right\}, font = [times, roman, 16]):$

> $ap7 := textplot([10, 0.8.\text{"Risk loving"}], align = \{above, right\}, font = [times, roman, 16]):$

```
> display({ap1, ap2 , ap3, ap4, ap5, ap6, ap7}, title = "Percentage break even", titlefont = [times,
      roman, 16], gridlines = true);
```

The code used to plot the profit as a function of the cost of the game and *n* using mapl*e* in chapter 4 is as follows:

```
> restart :

> with(plots) :

> with(plottools) :

> with(Statistics) :

> with(ListTools) :

> randomize( ) :

> XX := proc(cost, n, nsim) local coin; L, cc, i, j, r;

> for j from 1 to nsim do

> L := [seq(rand(0 .. 1) ( ), i = 1 .. n)];

> cc[0] := 0 :

> for i from 1 to n do

> if L[i] = 0 then cc[0] := 0;

> elif L[i] = 1 then cc[i] := cc[i − 1] + 1 : end if:

> end do:

> r[j] := add([seq('if'(cc[i] = 0. − cost, 2^cc[i]), i = 1 .. nops(L) )][i], i = 1 .. n);

> end do:

> ExpectedValue([seq(r[j], j = 1 .. nsim)]);

> end proc:

> data1 := [seq[seq([cost, n, XX(cost, n, 20)], cost = 0 .. 40)], n = 0 ..50)]:
```

> $surfdata(data1,\ axes = boxed, view = [default, default, -1000 ...3000], labels = [\text{"cost"}, \text{"n"}, \text{"profit"}], labeldirections = [horizontal,\ horizontal,\ vertical]);$