TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEMS FOR TELEGRAPH EQUATIONS

**A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES
OF
NEAR EAST UNIVERSITY**

**By
HAITHAM AHMAD AL HAZAIMEH**

**In partial Fulfillment of the Requirements for
the Degree of Master of Science
in
Mathematics**

NICOSIA, 2019

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Haitham Ahmad Al Hazalmeh: TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEMS FOR TELEGRAPH EQUATIONS

Approval of Director of Graduate School of Applied Sciences

Prof.Dr. Nadire ÇAVUŞ

We certify that, this thesis is satisfactory for the of the degree of Master of Science in Mathematics

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct, I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my family...

ABSTRACT

In the present study, a source identification problem for telegraph equations is studied. Using tools of classical approach, we are enabled to obtain the solution of the several source identification problems for telegraph equations. Furthermore, difference schemes for the numerical solution of the source identification problem for telegraph equations are presented. Then, these difference schemes are tested on an example and some numerical results are presented

Keywords: Source identification problems; telegraph equations; Fourier series method; Laplace transform and Fourier transform solutions; difference schemes; modified Gauss elimination method

ÖZET

Bu çalışmada, telegraf denklemleri için kaynak tanımlama problemi incelenmiştir. Klasik yaklaşım araçları, telegraf denklemleri için çeşitli kaynak tanımlama problemlerinin çözümünü elde etmemize imkan tanır. Ayrıca, telegraf denklemleri için kaynak tanımlama probleminin sayısal çözümü için fark şemaları sunulmuştur. Daha sonra bu fark şemaları bir örnek üzerinde test edilip bazı sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Kaynak tanımlama problemleri; telegraf denklemleri; Fourier serisi yöntemi; Laplace dönüşümü ve Fourier dönüşümü çözümleri; fark şemaları; modifiye Gauss eliminasyon yöntemi

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CHAPTER 1
INTRODUCTION

Identification problems take an important place in applied sciences and engineering applications and have been studied by many authors (Prilepko et al.,1987; Kabanikhin and Krivorotko, 2015; Isakov, 1998; Belov, 2002; Kimura and Suzuki, 1993; Gryazin et al., 1999). The theory and applications of source identification problems for partial differential equations have been given in various papers (Orlovskii and Piskarev, 2013; Orlovsky and Piskarev, 2018; Ashyralyyev, 2014; Ashyralyev and Ashyralyyev, 2014; Ashyralyyev and Dedeturk, 2013; Ashyralyyev and Demirdag, 2012; Ashyralyev and Urun, 2014; Kostin, 2013; Eidelman, 1984; Eidelman, 1978; Choulli and Yamamoto, 1999; Ashyralyev et al., 2012; Saitoh et al., 2002; Ivanchov, 1995; Borukhov and Vabishchevich, 2000; Dehghan, 2001; Orazov and Sadybekov, 2012; Ashyralyev, 2011; Ashyralyyev and Akkan, 2015; Ashyralyev and Sazaklioglu, 2017).The well-posedness of the unknown source identification problem for a parabolic equation has been well-investigated when the unknown function p is dependent on space variable (Kostin, 2013; Eidelman, 1984; Eidelman, 1978; Choulli and Yamamoto, 1999; Ashyralyev et al., 2012). Nevertheless when the unknown function p is dependent on t the well-posedness of the source identification problem for a parabolic equation has been investigated in (Saitoh et al., 2002; Ivanchov, 1995; Borukhov and Vabishchevich, 2000; Dehghan, 2001; Orazov and Sadybekov, 2012; Ashyralyev, 2011; Ashyralyyev and Akkan, 2015; Ashyralyev and Sazaklioglu, 2017; Ashyralyev and Erdogan, 2014; Samarskii and Vabishchevich, 2007). Moreover, the well-posedness of the source identification problem for a delay parabolic equation has been investigated in papers (Blasio and Lorenzi, 2007; Ashyralyev and Agirseven, 2014) . There is a great deal of work in analysis of hyperbolic-parabolic equations (Berdyshev, 2005; Kalmenov and Sadybekov, 2017; Sadybekov et al., 2018; Kerbal,Karimov and Rakhmatullaeva,2017). As well as in construction of difference schemes for such equations (Ashyralyev and Ozdemir,2005; Ashyralyev and Ozdemir, 2007; Ashyralyev and Ozdemir,2014; Ashyralyev and Yurtsever, 2001). The theory and applications of source identification problems for hyperbolic-parabolic equations have been well investigated in papers (Ashyralyev and Ashyralyyeva, 2015; Ashyralyyeva and Ashyraliyev, 2018). Direct and inverse boundary value problems for telegraph differential equations have been a major research area in many branches of science and engineering particularly in applied mathematics. The solvability of the inverse problems in various formulations with various overdetermination conditions for telegraph and hyperbolic equations were studied in many works (Anikonov,1976; Ashyralyev and Celik, 2016; Kozhanov and Safiullova, 2017; Ashyralyev and Emharab, 2017; Ashyralyev and Emharab, 2018; Kozhanov and Safiullova, 2010; Kozhanov and Telesheva, 2017). In particular, the well-posedness of the source identification problem for a telegraph equation with unknown parameter p

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au\left(t\right) = p + f\left(t\right), 0 \le t \le T, \\ u\left(0\right) = \varphi, u'\left(0\right) = \psi, u\left(T\right) = \zeta \end{cases}$$

in a Hilbert space H with the self-adjoint positive definite operator A was proved in (Ashyralyev and Celik,2016) . They established stability estimates for the solution of this problem. In applications, three source identification problems for telegraph equations were developed. In (Kozhanov and Safiullova,2017), the authors studied the solvability of the inverse problems on finding a solution $u\left(x,t\right)$ and an unknown coefficient c for a telegraph equation

$$u_{tt} - \Delta u + cu = f\left(x,t\right).$$

Theorems on the existence of the regular solutions were proved.

However, source identification problems for telegraph equations have not been well-investigated so far. In the present study, a source identification problem for telegraph equations is studied. Using tools of classical approach we are enabled to obtain the solution of the several source identification problems for telegraph equations. Furthermore, the first order of accuracy difference scheme for the numerical solution of the source identification problem for telegraph equations is presented. Then, this difference scheme is tested on an example and some numerical results are presented.

CHAPTER 2

METHODS OF SOLUTION OF TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEMS FOR TELEGRAPH EQUATIONS

It is known that identification problems for telegraph differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

2.1 FOURIER SERIES METHOD

We consider Fourier series method for solution of identification problems for telegraph differential equations.

Example 2.1.1. Obtain the Fourier series solution of the identification problems for telegraph differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)\sin x + 2\sin x - e^{-t}, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin x, u_t(0,x) = 0, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \int\limits_0^\pi u(t,x)\,dx = 2, 0 \le t \le 1. \end{cases} \tag{2.1}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u_{xx} - \lambda u(x) + u(x) = 0, 0 < x < \pi, u(0) = u(\pi) = 0$$

generated by the space operator of problem (2.1). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = 1 + k^2, u_k(x) = \sin kx, k = 1, 2, \ldots.$$

Then, we will obtain the Fourier series solution of problem (2.1) by formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx, \tag{2.2}$$

where $A_k(t), k = 1, 2, \ldots$ are unknown functions. Putting (2.2) into main problem and using given initial and boundary conditions, we get

$$\sum_{k=1}^{\infty} A_k''(t)\sin kx + \sum_{k=1}^{\infty} A_k'(t)\sin kx + \sum_{k=1}^{\infty} k^2 A_k(t)\sin kx + \sum_{k=1}^{\infty} A_k(t)\sin kx$$

$$= p(t)\sin x - e^{-t}\sin x + 2\sin x, \tag{2.3}$$

$$u(0,x) = \sum_{k=1}^{\infty} A_k(0)\sin kx = \sin x,$$

$$u_t(0,x) = \sum_{k=1}^{\infty} A_k'(0)\sin kx = 0,$$

$$\int_0^{\pi} u(t,x)\,dx = \int_0^{\pi} \sum_{k=1}^{\infty} A_k(t)\sin kx dx = \sum_{k=1}^{\infty} \left. \frac{-A_k(t)\cos kx}{k} \right|_0^{\pi}$$

$$= \sum_{k=1}^{\infty} A_{2k-1}(t)\frac{2}{2k-1} = 2.$$

Equating coefficients of sinkx, $k = 1, 2...$ to zero,we get

$$\begin{cases} A_k''(t) + A_k'(t) + \left(k^2+1\right)A_k(t) = 0, 0 < t < 1, \\ \\ A_k(0) = A_k'(0) = 0, k \neq 1. \end{cases} \tag{2.4}$$

and

$$\begin{cases} A_1''(t) + A_1'(t) + 2A_1(t) = 2 + p(t) - e^{-t}, 0 < t < 1, \\ \\ A_1(0) = 1, A_1'(0) = 0. \end{cases} \tag{2.5}$$

We will obtain $A_k(t)$. It is clear that for $k \neq 1$, $A_k(t)$ is the solution of the initial value problem (2.4). The auxiliary equation is

$$q^2 + q + \left(k^2+1\right) = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{k^2+\frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{k^2+\frac{3}{4}}.$$

Therefore,

$$A_k(t) = e^{-\frac{t}{2}} \left[c_1 \cos \sqrt{k^2 + \frac{3}{4}} t + c_2 \sin \sqrt{k^2 + \frac{3}{4}} t \right].$$

Applying initial conditions $A_k(0) = A'_k(0) = 0$, we get

$$A_k(0) = c_1 = 0,$$

$$A'_k(0) = c_2 \sqrt{k^2 + \frac{3}{4}} = 0.$$

Then $c_1 = c_2 = 0$ and $A_k(t) = 0$. Now, we obtain $A_1(t)$. Applying $A_k(t) = 0, k \neq 1$ and $\sum_{k=1}^{\infty} A_{2k-1}(t) \frac{2}{2k-1} = 2$, we get $A_1(t) = 1$.
Now, we will obtain $p(t)$. Applying problem (2.5) and $A_1(t) = 1$, we get

$$p(t) = e^{-t}.$$

Therefore,

$$u(t,x) = A_1(t) \sin x = \sin x.$$

So, the exact solution of the problem (2.1) is

$$(u(t,x), p(t)) = \left(\sin x, e^{-t}\right).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(t) q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \ 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, \int\limits_{x \in \overline{\Omega}} ... \int u(t,x) dx_1 ... dx_n = \xi(t), 0 \leq t \leq T \end{cases} \tag{2.6}$$

for the multidimensional telegraph differential equations. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x) \left(t \in (0,T), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \xi(t), \left(t \in [0,T], x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary

$$S, \overline{\Omega} = \Omega \cup S.$$

However Fourier series method described in solving (2.6) can be used only in the case when (2.6) has constant coefficients.

Example 2.1.2. Obtain the Fourier series solution of the identification problem for telegraph differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)(\cos x + 1) + e^{-t} \cos x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \cos x + 1, u_t(0,x) = -(\cos x + 1), 0 \leq x \leq \pi, \\ u_x(t,0) = u_x(t,\pi) = 0, \int\limits_0^\pi u(t,x)\,dx = e^{-t}\pi, 0 \leq t \leq 1 \end{cases} \tag{2.7}$$

Solution. In order to solve problem (2.7), we consider the Sturm-Lioville problem

$$-u_{xx} - \lambda u(x) + u(x) = 0, 0 < x < \pi, u_x(0) = u_x(\pi) = 0$$

generated by the space operator of problem (2.7). It is easy to see that the solution of this Sturm Liouville problem is

$$\lambda_k = 1 + k^2, u_k(x) = \cos kx, \quad k = 0, 1, 2, \ldots.$$

Then, we will obtain the Fourier series solution of problem (2.7) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos kx, \tag{2.8}$$

where $A_k(t)$, $k = 0, 1, 2, \ldots$ are unknown functions. Putting (2.8) into (2.7) and using given initial and boundary conditions, we obtain

$$\sum_{k=0}^{\infty} A_k''(t)\cos kx + \sum_{k=0}^{\infty} A_k'(t)\cos kx + \sum_{k=0}^{\infty} k^2 A_k(t)\cos kx + \sum_{k=0}^{\infty} A_k(t)\cos kx$$

$$= p(t)(\cos x + 1) + e^{-t}\cos x,$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos kx = \cos x + 1,$$

$$u_t(0,x) = \sum_{k=0}^{\infty} A_k'(0)\cos kx = -\cos x - 1,$$

$$A_0(0) = 1, \quad A_0'(0) = -1, \quad A_1(0) = 1, \quad A_1'(0) = -1, \quad A_k(0) = 0, \quad k \neq 0, 1,$$

$$\int_0^{\pi} u(t,x)\,dX = \int_0^{\pi} \sum_{k=0}^{\infty} A_k(t)\cos kx dx = A_0(t)\pi = e^{-t}\pi\ .$$

From that it follows

$$A_0(t) = e^{-t}.$$

Equating coefficients of $\cos kx$, $k = 0, 1, 2, \ldots$ to zero, we get

$$\begin{cases} A_k''(t) + A_k'(t) + \left(k^2 + 1\right) A_k(t) = 0, 0 < t < 1, \\ A_k(0) = 0, A_k'(0) = 0, \ k \neq 0, 1, \end{cases} \tag{2.9}$$

$$\begin{cases} A_1''(t) + A_1'(t) + 2A_1(t) = p(t) + e^{-t}, 0 < t < 1, \\ A_1(0) = 1, A_1'(0) = -1, \end{cases} \tag{2.10}$$

$$\begin{cases} A_0''(t) + A_0'(t) + A_0(t) = p(t), 0 < t < 1, \\ A_0(0) = 1, A_0'(0) = -1. \end{cases} \tag{2.11}$$

First, we obtain $p(t)$. Applying problem (2.11) and $A_0(t) = e^{-t}$, we get

$$p(t) = e^{-t}.$$

Second, we obtain $A_k(t)$, $k \neq 0, 1$. It is clear that for $k \neq 0, 1$, $A_k(t)$ is the solution of the initial value problem (2.9). The auxiliary equation is

$$q^2 + q + \left(k^2 + 1\right) = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{k^2 + \frac{3}{4}}.$$

Therefore,

$$A_k(t) = e^{-\frac{t}{2}} \left[c_1 \cos\sqrt{k^2 + \frac{3}{4}}t + c_2 \sin\sqrt{k^2 + \frac{3}{4}}t \right].$$

Applying initial conditions $A_k(0) = A'_k(0) = 0$ for $k \neq 0, 1$, we get

$$A_k(0) = c_1 = 0,$$

$$A'_k(0) = c_2\sqrt{k^2 + \frac{3}{4}} = 0.$$

Then $c_1 = c_2 = 0$ and $A_k(t) = 0$. Now, we obtain $A_1(t)$ from (2.10). It is the Cauchy problem for the second order linear differential equation. We will seek $A_1(t)$ by formula

$$A_1(t) = A_C(t) + A_P(t),$$

where $A_C(t)$ is general solution of the homogeneous differential equation $A''_1(t) + A'_1(t) + 2A_1(t) = 0$, $A_P(t)$ is particular solution of nonhomogene uos differential equation. The auxiliary equation is

$$q^2 + q + 2 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{\frac{7}{4}}, q_2 = -\frac{1}{2} - i\sqrt{\frac{7}{4}}.$$

Therefore,

$$A_c(t) = e^{-\frac{t}{2}} \left[c_1 \cos\sqrt{\frac{7}{4}}t + c_2 \sin\sqrt{\frac{7}{4}}t \right].$$

Since $-\frac{1}{2} \pm i\sqrt{\frac{7}{4}} \neq -1$, we put

$$A_p(t) = e^{-t}a.$$

Therefore,

$$ae^{-t} - ae^{-t} + 2ae^{-t} = 2e^{-t}.$$

From that it follows $a = 1$ and

$$A_p(t) = e^{-t}.$$

Thus,

$$A_1(t) = e^{-\frac{t}{2}} \left[c_1 \cos\sqrt{\frac{7}{4}}t + c_2 \sin\sqrt{\frac{7}{4}}t \right] + e^{-t}.$$

Applying initial conditions $A_1(0) = 1, A_1'(0) = -1$, we get

$$A_1(0) = c_1 + 1 = 1, \;\; A_1'(0) = \sqrt{\frac{7}{4}}c_2 - 1 = -1.$$

From that it follows $c_1 = c_2 = 0$ and

$$A_1(t) = e^{-t}.$$

Therefore,

$$u(t,x) = A_0(t) + A_1(t)\cos x = e^{-t} + e^{-t}\cos x.$$

So, the exact solution of the problem (2.7) is

$$(u(t,x), p(t)) = \left(e^{-t}(\cos x + 1), e^{-t}\right).$$

Note that using similar procedure one can obtain the solution of the following identification problem.

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)\, q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\; 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega},, \\ \frac{\partial u(t,x)}{\partial \overline{m}} = 0, x \in S, \int\limits_{x\in\overline{\Omega}} ... \int u(t,x)\, dx_1 ... dx_n = \xi(t), 0 \le t \le T \end{cases} \tag{2.12}$$

for the multidimensional telegraph differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x) \left(t \in (0,T), x \in \overline{\Omega}\right), \psi(x), \xi(t), \left(t \in [0,T], x \in \overline{\Omega}\right)$ are given smooth functions. Here $\overline{m}$ is the normal vector to boundary S.

However Fourier series method described in solving (2.12) can be also used only in the case when (2.12) has constant coefficients.

Example 2.1.3. Obtain the Fourier series solution of the identification problem for the telegraph differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) \\ = p(t)(\sin 2x + 1) + 4e^{-t} \sin 2x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin 2x + 1, \quad u_t(0,x) = -(\sin 2x + 1), \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \ 0 \le t \le 1, \\ \int\limits_0^\pi u(t,x)\,dx = e^{-t}\pi, \ 0 \le t \le 1. \end{cases} \tag{2.13}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u_{xx} - \lambda u(x) + u(x) = 0, 0 < x < \pi, \ u(0) = u(x), \ u_x(0) = u_x(\pi)$$

generated by the space operator of problem (2.13). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = 4k^2 + 1, \quad u_k(x) = \cos 2kx, k = 0, 1, 2, ..., \ u_k(x) = \sin 2kx, k = 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem (2.13) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx, \tag{2.14}$$

where $A_k(t), k=0,1,2,...$ and $B_k(t), k=1,2,...$are unknown functions, Putting (2.14) into (2.13) and using given initial and boundary conditions

$$\sum_{k=0}^{\infty} A_k''(t)\cos 2kx + \sum_{k=1}^{\infty} B_k''(t)\sin 2kx + \sum_{k=0}^{\infty} A_k'(t)\cos 2kx + \sum_{k=1}^{\infty} B_k'(t)\sin 2kx$$

$$+\sum_{k=0}^{\infty} 4k^2 A_k(t)\cos 2kx + \sum_{k=1}^{\infty} 4k^2 B_k(t)\sin 2kx + \sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx$$

$$= p(t)(\sin 2x + 1) + 4e^{-t}\sin 2x,$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos 2kx + \sum_{k=1}^{\infty} B_k(0)\sin 2kx = \sin 2x + 1,$$

$$u_t(0,x) = \sum_{k=0}^{\infty} A_k'(0)\cos 2kx + \sum_{k=1}^{\infty} B_k'(0)\sin 2kx = -\sin 2x - 1,$$

$$\int_0^{\pi} u(t,x)\,dx = \int_0^{\pi} \left[\sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx\right] dx$$

$$= A_0(t)\pi + \sum_{k=1}^{\infty} \frac{A_k(t)\sin 2kx}{2k}\Bigg]_0^{\pi} - \sum_{k=1}^{\infty} \frac{B_k(t)\cos 2kx}{2k}\Bigg]_0^{\pi} = A_0(t)\pi = e^{-t}\pi.$$

From that it follows that $A_0(t) = e^{-t}$.Equating the coefficients of $\cos kx, k=0,1,2,...$ and $\sin kx, k=1,2,...$ to zero, we get

$$\begin{cases} A_k''(t) + A_k'(t) + (4k^2+1)A_k(t) = 0, \ 0<t<1, \\ A_k(0) = 0, A_k'(0) = 0, k \neq 0, \end{cases} \tag{2.15}$$

$$\begin{cases} A_0''(t) + A_0'(t) + A_0(t) = p(t), \ 0<t<1, \\ A_0(0) = 1, A_0'(0) = -1, \end{cases} \tag{2.16}$$

$$\begin{cases} B_k''(t) + B_k'(t) + (4k^2+1)B_k(t) = 0, \ 0<t<1, \\ B_k(0) = 0, B_k'(0) = 0, k \neq 0, \end{cases} \tag{2.17}$$

$$\begin{cases} B_1''(t) + B_1'(t) + 5B_1(t) = 5e^{-t}, \ 0 < t < 1, \\ \\ B_1(0) = 1, \ \ B_1'(0) = 1. \end{cases} \tag{2.18}$$

First, we obtain $p(t)$. Applying problem (2.16) and $A_0(t) = e^{-t}$, we get

$$p(t) = e^{-t}.$$

Second, we obtain $A_k(t), k \neq 0$. It is clear that for $k \neq 0, A_k(t)$ is the solution of the initial value problem (2.15) . The auxiliary equation is

$$q^2 + q + 4k^2 + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{4k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{4k^2 + \frac{3}{4}}.$$

Therefore,

$$A_k(t) = e^{-\frac{t}{2}} \left[c_1 \cos\sqrt{4k^2 + \frac{3}{4}}t + c_2 \sin\sqrt{4k^2 + \frac{3}{4}}t \right].$$

Applying initial conditions $A_k(0) = A_k'(0) = 0$, we get

$$A_k(0) = c_1 = 0,$$

$$A_k'(0) = c_2\sqrt{4k^2 + \frac{3}{4}} = 0.$$

Then $c_1 = c_2 = 0$ and $A_k(t) = 0$. Third, we obtain $B_k(t)$. It is clear that for $k \neq 1, B_k(t)$ is the solution of the initial value problem (2.17). The auxiliary equation is

$$q^2 + q + 4k^2 + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{4k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{4k^2 + \frac{3}{4}}.$$

Therefore,

$$B_k(t) = e^{-\frac{t}{2}} \left[c_1 \cos \sqrt{4k^2 + \frac{3}{4}} t + c_2 \sin \sqrt{4k^2 + \frac{3}{4}} t \right].$$

Applying initial conditions $B_k(0) = B'_k(0) = 0$, we get

$$B_k(0) = c_1 = 0,$$

$$B'_k(0) = c_2 \sqrt{4k^2 + \frac{3}{4}} = 0.$$

Then $c_1 = c_2 = 0$ and $B_k(t) = 0$. Now, we obtain $B_1(t)$ from (2.18). It is the Cauchy problem for the second order linear differential equation. We will seek $B_1(t)$ by formula

$$B_1(t) = B_C(t) + B_P(t),$$

where $B_C(t)$ is general solution of the homogeneous differential equation $B''_1(t) + B'_1(t) + 5B_1(t) = 0$ and $B_P(t)$ be particular solution of nonhomogeneuos differential equation. The auxiliary equation is

$$q^2 + q + 5 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{\frac{19}{4}}, q_2 = -\frac{1}{2} - i\sqrt{\frac{19}{4}}.$$

Therefore,

$$B_c(t) = e^{-\frac{t}{2}} \left[c_1 \cos \sqrt{\frac{19}{4}} t + c_2 \sin \sqrt{\frac{19}{4}} t \right].$$

Since $-\frac{1}{2} \pm i\sqrt{\frac{19}{4}} \neq -1$, we put

$$B_p(t) = e^{-t}a.$$

Therefore,

$$ae^{-t} - ae^{-t} + 5ae^{-t} = 5e^{-t}.$$

From that it follows $a = 1$ and

$$B_p(t) = e^{-t}.$$

Thus,

$$B_1(t) = e^{-\frac{t}{2}}\left[c_1 \cos\sqrt{\frac{19}{4}}t + c_2 \sin\sqrt{\frac{19}{4}}t\right] + e^{-t}.$$

Applying initial conditions $B_1(0) = 1, B_1'(0) = -1$, we get

$$B_1(0) = c_1 + 1 = 1,\ B_1'(0) = \sqrt{\frac{19}{4}}c_2 - 1 = -1.$$

From that it follows

$$B_1(t) = e^{-t}.$$

Therefore,

$$u(t,x) = e^{-t}(\sin 2x + 1).$$

So, the exact solution of problem (2.13) is

$$(u(t,x), p(t)) = \left(e^{-t}(\sin 2x + 1), e^{-t}\right).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)\,q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \left.\frac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_1} = \left.\frac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_2}, \\ \int\limits_{x\in\overline{\Omega}} \cdots \int u(t,x)\,dx_1 ... dx_n = \xi(t), 0 \le t \le T \end{cases} \tag{2.19}$$

for the multidimensional telegraph differential equations. Assume that $\alpha_r > \alpha > 0$ and

$f(t,x), q(x)\left(t\in(0,T), x\in\overline{\Omega}\right), \psi(x), \xi(t), \left(t\in[0,T], x\in\overline{\Omega}\right)$ are given smooth functions. Here $S=S_1\cup S_2, S_1\cap S_2=\emptyset$.

However Fourier series method described in solving (2.19) can be used only in the case when (2.19) has constant coefficients.

2.2 LAPLACE TRANSFORM METHOD

We consider Laplace transform solution of identification problems for telegraph differential equations.

Example 2.2.1. Obtain the Laplace transform solution of the following problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}+u(t,x)=p(t)e^{-x}-e^{-t-x}, \\ 0<x<\infty, 0<t<\infty, \\ u(0,x)=e^{-x},\ u_t(0,x)=-e^{-x}, 0\le x<\infty, \\ u(t,0)=e^{-t},\ \ u_x(t,0)=-e^{-t},\ \ 0\le t<\infty, \\ \int\limits_0^\infty u(t,x)\,dx=e^{-t},\ \ 0\le t<\infty \end{cases} \tag{2.20}$$

for the one dimensional telegraph differential equation.

Solution. Here and in future we denote

$$\check{L}\{u(t,x)\}=u(t,s).$$

Using formula

$$\check{L}\{e^{-x}\}=\frac{1}{s+1} \tag{2.21}$$

and taking the Laplace transform of both sides of the differential equation and using conditions

$$u(t,0)=e^{-t},\ u_x(t,0)=-e^{-t},$$

we can write

$$\check{L}\{u_{tt}(t,x)\}+\check{L}\{u_t(t,x)\}-\check{L}\{u_{xx}(t,x)\}+\check{L}\{u(t,x)\}=\left(p(t)-e^{-t}\right)\check{L}\{e^{-x}\},$$

$$\check{L}\left\{u(0,x)\right\} = \check{L}\left\{e^{-x}\right\}, \check{L}\left\{u_t(0,x)\right\} = -\check{L}\left\{e^{-x}\right\}$$

or

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) - s^2 u(t,s) + su(t,0) + u_x(t,0) + u(t,s) \\ = \left(p(t) - e^{-t}\right)\frac{1}{1+s}, t > 0, \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = -\frac{1}{1+s}. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + \left(1 - s^2\right) u(t,s) \\ = (1-s) e^{-t} + \left(p(t) - e^{-t}\right)\frac{1}{1+s}, t > 0, \\ u(0,s) = \frac{1}{1+s} \ , u_t(0,s) = \frac{1}{1+s}. \end{cases} \tag{2.22}$$

Applying the condition

$$\int_0^\infty u(t,x)\,dx = e^{-t}, \ t \geq 0$$

and the definition of the Laplace transform, we get

$$u(t,0) = e^{-t}, \ t \geq 0. \tag{2.23}$$

Now we will obtain the solution of problem (2.22). We denote

$$u(t,s) = v(t,s)\,e^{-\frac{t}{2}}.$$

Then

$$u_t(t,s) = -\frac{1}{2}e^{-\frac{t}{2}}v(t,s) + e^{-\frac{t}{2}}v_t(t,s),$$

$$u_{tt}\ (t,s) = \frac{1}{4}e^{-\frac{t}{2}}v(t,s) - e^{-\frac{t}{2}}v_t(t,s) + e^{-\frac{t}{2}}v_{tt}(t,s).$$

Using in (2.22) we get the following problem

$$\begin{cases} v_{tt}(t,s) + \left(\frac{3}{4} - s^2\right) v(t,s) \\ = \left(1 - s - \frac{1}{1+s}\right) e^{-\frac{t}{2}} + p(t)\, e^{-\frac{t}{2}} \frac{1}{1+s},\ t > 0, \\ v(0,s) = \frac{1}{1+s},\ v_t(0,s) = -\frac{1}{2}\frac{1}{1+s}. \end{cases}$$

Applying the D'Alembert's formula, we get

$$v(t,s) = \frac{1}{1+s} \cos\sqrt{\frac{3}{4} - s^2}\, t + \frac{1}{\sqrt{\frac{3}{4} - s^2}} \sin\sqrt{\frac{3}{4} - s^2}\, t \left(-\frac{1}{2(1+s)}\right) + \frac{1}{\sqrt{\frac{3}{4} - s^2}}$$

$$\times \int_0^t \sin\sqrt{\frac{3}{4} - s^2}\,(t-y) \left\{\left(1 - s - \frac{1}{1+s}\right) e^{-\frac{y}{2}} + p(y)\, e^{\frac{y}{2}} \frac{1}{1+s}\right\} dy. \qquad (2.24)$$

Using condition (2.23), we get $v(t,0) = e^{-\frac{t}{2}}$. Then from (2.24) it follows

$$e^{-\frac{t}{2}} = \cos\sqrt{\frac{3}{4}}\, t + \frac{1}{\sqrt{\frac{3}{4}}} \sin\sqrt{\frac{3}{4}}\, t \left(-\frac{1}{2}\right)$$

$$+\frac{1}{\sqrt{\frac{3}{4}}} \int_0^t \sin\sqrt{\frac{3}{4}}\,(t-y) \left\{p(y)\, e^{\frac{y}{2}}\right\} dy\ . \qquad (2.25)$$

Take the first and second order derivatives, we get

$$-\frac{1}{2} e^{-\frac{t}{2}} = -\frac{\sqrt{3}}{2} \sin\frac{\sqrt{3}}{2} t - \frac{1}{2} \cos\frac{\sqrt{3}}{2} t$$

$$+ \int_0^t \cos\frac{\sqrt{3}}{2}(t-y) \left\{p(y)\, e^{\frac{y}{2}}\right\} dy\ ,$$

$$\frac{1}{4} e^{-\frac{t}{2}} = -\frac{3}{4} \cos\frac{\sqrt{3}}{2} t + \frac{\sqrt{3}}{4} \sin\frac{\sqrt{3}}{2} t$$

$$+p(t)\, e^{\frac{t}{2}} - \frac{\sqrt{3}}{2} \int_0^t \sin\frac{\sqrt{3}}{2}(t-y) \left\{p(y)\, e^{\frac{y}{2}}\right\} dy\ \ . \qquad (2.26)$$

Applying (2.25) and (2.26), we get

$$\frac{1}{4}e^{-\frac{t}{2}} = -\frac{3}{4}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{4}\sin\frac{\sqrt{3}}{2}t + p(t)\,e^{\frac{t}{2}}$$

$$-\frac{\sqrt{3}}{2}\left\{\frac{\sqrt{3}}{2}\left\{e^{-\frac{t}{2}} - \cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right\}\right\}$$

or

$$e^{-\frac{t}{2}} = p(t)\,e^{\frac{t}{2}}.$$

So

$$p(t) = e^{-t}.$$

Now we obtain $v(t,s)$. Applying formula (2.24), we get

$$v(t,s) = \frac{1}{1+s}\cos\sqrt{\frac{3}{4}-s^2}t + \frac{1}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2}t\left(-\frac{1}{2(1+s)}\right)$$

$$+(1-s)\frac{1}{\sqrt{\frac{3}{4}-s^2}}\int_0^t \sin\left(\sqrt{\frac{3}{4}-s^2}\,(t-y)\right)e^{-\frac{y}{2}}dy \quad .$$

We denote that

$$I(t,s) = \int_0^t \frac{1}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}dy.$$

Then

$$v(t,s) = \frac{1}{1+s}\cos\sqrt{\frac{3}{4}-s^2}t$$

$$+\frac{1}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2}t\left(-\frac{1}{2(1+s)}\right) + (1-s)\,I(t,s). \tag{2.27}$$

Now we will compute $I(t,s)$. Actually,

$$I(t,s) = -2\left[\left\{\frac{1}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}\right\}\right]_0^t$$

$$-2\int_0^t \frac{\sqrt{\frac{3}{4}-s^2}}{\sqrt{\frac{3}{4}-s^2}}\cos\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}dy$$

$$=\frac{2}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2t}+4\cos\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}\Big]_0^t$$

$$-4\sqrt{\frac{3}{4}-s^2}\int_0^t \sin\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}dy$$

$$=\frac{2}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2t}+4e^{-\frac{t}{2}}$$

$$-4\cos\sqrt{\frac{3}{4}-s^2t}-4\sqrt{\frac{3}{4}-s^2}\int_0^t \sin\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}dy$$

$$=\frac{2}{\sqrt{\frac{3}{4}-s^2}}\sin\sqrt{\frac{3}{4}-s^2t}+4e^{-\frac{t}{2}}-4\cos\sqrt{\frac{3}{4}-s^2t}$$

$$-4\sqrt{\frac{3}{4}-s^2}\int_0^t \sin\sqrt{\frac{3}{4}-s^2}\,(t-y)\,e^{-\frac{y}{2}}dy.$$

Therefore,

$$I(t,s)=\frac{2}{\sqrt{\frac{3}{4}-s^2}}\sin\left(\sqrt{\frac{3}{4}-s^2t}\right)$$

$$+4e^{-\frac{t}{2}}-4\cos\left(\sqrt{\frac{3}{4}-s^2t}\right)-4\left(\frac{3}{4}-s^2\right)I(t,s).$$

From that it follows

$$(1-s)\,I(t,s)=e^{-\frac{t}{2}}\frac{1}{1+s}$$

$$+\frac{1}{2\sqrt{\frac{3}{4}-s^2}}\sin\left(\sqrt{\frac{3}{4}-s^2t}\right)\frac{1}{1+s}-\cos\left(\sqrt{\frac{3}{4}-s^2t}\right)\frac{1}{1+s}\,. \tag{2.28}$$

Applying (2.27) and (2.28), we get

$$v\,(t,s)=e^{-\frac{t}{2}}\frac{1}{1+s}.$$

From that it follows that

$$u(t,x) = e^{-t}\check{L}^{-1}\left\{\frac{1}{1+s}\right\} = e^{-t-x}.$$

Therefore, the exact solution of problem (2.20) is

$$(u(t,x), p(t)) = \left(e^{-t-x}, e^{-t}\right).$$

Example 2.2.2. Obtain the Laplace trasform solution of the following problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(t)\, e^{-x} - 2e^{-x}, \\ 0 < x < \infty,\ 0 < t < \infty, \\ u(0,x) = e^{-x},\ u_t(0,x) = 0, 0 \le x < \infty, \\ u(t,0) = 1,\ \ u(t,\infty) = 0,\ \ 0 \le t < \infty, \\ \int\limits_0^\infty u(t,x)\, dx = 1,\ 0 \le t < \infty \end{cases} \tag{2.29}$$

for a one dimensional telegraph differential equation.

Solution. Using formula (2.21) and conditions $u(t,0) = 1, u(t,\infty) = 0$, and taking the Laplace transform of both sides of the differential equation and initial conditions, we can write

$$\check{L}\{u_{tt}(t,x)\} + \check{L}\{u_t(t,x)\} - \check{L}\{u_{xx}(t,x)\} = (p(t)-2)\,\check{L}\{e^{-x}\},$$

$$\check{L}\{u(0,x)\} = \check{L}\{e^{-x}\}, \check{L}\{u_t(0,x)\} = 0$$

or

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) - s^2 u(t,s) + s + u_x(t,0) \\ = (p(t)-2)\frac{1}{1+s}, t > 0, \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = 0. \end{cases}$$

We denote that

$$u_x(t,0) = \beta(t). \tag{2.30}$$

Therefore, we get the following problem

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) - s^2 u(t,s) + s + \beta(t) \\ = (p(t) - 2)\frac{1}{1+s}, t > 0, \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = 0. \end{cases}$$

Now, we obtain $\beta(t)$ and $p(t)$. It is clear that $u(t,s)$ is solution of the following source identification problem

$$\begin{cases} u_{tt}(t,s) + u_t(t,x) - s^2 u(t,s) \\ = [p(t) - 2]\frac{1}{s+1} - s - \beta(t), \\ u(0,s) = \frac{1}{s+1}, u_t(0,s) = 0, u(t,0) = 1 \end{cases} \tag{2.31}$$

for $u(t,\infty) = 0$. Taking the Laplace transform with respect to t, we get

$$\begin{aligned} &\omega^2 u(\omega,s) - \omega\frac{1}{s+1} + \omega u(\omega,s) - \frac{1}{s+1} - s^2 u(\omega,s) \\ &= p(\omega)\frac{1}{s+1} - \frac{2}{\omega(s+1)} - \frac{s}{\omega} - \beta(\omega), u(\omega,0) = \frac{1}{\omega}, u(\omega,\infty) = 0. \end{aligned}$$

Then

$$u(\omega,s) = \frac{1}{\omega^2 + \omega - s^2}\left\{\left[\frac{-s^2 - s + \omega^2 + \omega - 2}{\omega(s+1)}\right] + p(\omega)\frac{1}{s+1} - \beta(\omega)\right\}, \tag{2.32}$$

$$u(\omega,0) = \frac{1}{\omega}, u(\omega,\infty) = 0.$$

Since $u(\omega,0) = \frac{1}{\omega}$, we have

$$u(\omega,0) = \frac{1}{\omega^2 + \omega}\left\{\left[\frac{\omega^2 + \omega - 2}{\omega}\right] + p(\omega) - \beta(\omega)\right\}$$

or

$$\frac{1}{\omega} = \frac{1}{\omega} - \frac{2}{\omega(\omega^2+\omega)} + \frac{p(\omega) - \beta(\omega)}{\omega^2+\omega}\ .$$

Hence,

$$p(\omega) = \frac{2}{\omega} + \beta(\omega). \tag{2.33}$$

Then, from (2.32) it follows

$$u(\omega, s) = \frac{1}{\omega^2+\omega-s^2}\left\{\left[\frac{-s^2-s+\omega^2+\omega-2}{\omega(s+1)}\right] + \left(\frac{2}{\omega}+\beta(\omega)\right)\frac{1}{s+1} - \beta(\omega)\right\}.$$

Taking the inverse Laplace transform with respect to x, we get

$$u(\omega, x) = \left[-\beta(\omega) - \frac{1}{\omega}\right]\check{L}^{-1}\left\{\frac{s}{(s+1)(\omega^2+\omega-s^2)}\right\} + \frac{1}{\omega}e^{-x}.$$

Now, we compute $\check{L}^{-1}\left\{\frac{s}{(s+1)(\omega^2+\omega-s^2)}\right\}$. Applying the formula

$$\frac{s}{(s+1)(\omega^2+\omega-s^2)} = \frac{1}{2\sqrt{\omega(\omega+1)}}\left\{\frac{1}{\sqrt{\omega(\omega+1)}-s} + \frac{1}{\sqrt{\omega(\omega+1)}+s}\right\}$$

$$-\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{\left[\frac{1}{\sqrt{\omega(\omega+1)}-s} + \frac{1}{s+1}\right]\frac{1}{1+\sqrt{\omega(\omega+1)}}\right.$$

$$\left.\times \lim_{x\to\infty}\left\{\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{e^{\sqrt{\omega(\omega+1)}x} + e^{-\sqrt{\omega(\omega+1)}x}\right\}\right\}\right\},$$

we get

$$\check{L}^{-1}\left\{\frac{s}{(s+1)(\omega^2+\omega-s^2)}\right\} = \frac{1}{2\sqrt{\omega(\omega+1)}}\left\{e^{\sqrt{\omega(\omega+1)}x} + e^{-\sqrt{\omega(\omega+1)}x}\right\}$$

$$-\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{\left[e^{\sqrt{\omega(\omega+1)}x} + e^{-x}\right]\frac{1}{1+\sqrt{\omega(\omega+1)}} + \right.$$

$$\left.\left[e^{-\sqrt{\omega(\omega+1)}x} - e^{-x}\right]\frac{1}{1-\sqrt{\omega(\omega+1)}}\right\}.$$

Therefore,

$$u(\omega, x) = \frac{1}{\omega}e^{-x} + \left[-\beta(\omega) - \frac{1}{\omega}\right]\left\{\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{e^{\sqrt{\omega(\omega+1)}x} + e^{-\sqrt{\omega(\omega+1)}x}\right\}\right.$$

$$-\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{\left[e^{\sqrt{\omega(\omega+1)}x} + e^{-x}\right]\frac{1}{1+\sqrt{\omega(\omega+1)}}\right.$$

.

$$\left.\left.+\left[e^{-\sqrt{\omega(\omega+1)}x} - e^{-x}\right]\frac{1}{1-\sqrt{\omega(\omega+1)}}\right\}\right\}.$$

Then, using the condition $u(\omega, \infty) = 0$, we get

$$0 = \lim_{x\to\infty} u(\omega, x) = \frac{1}{\omega}e^{-x} + \left[-\beta(\omega) - \frac{1}{\omega}\right]$$

$$\times \lim_{x\to\infty}\left\{\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{e^{\sqrt{\omega(\omega+1)}x} + e^{-\sqrt{\omega(\omega+1)}x}\right\}\right.$$

$$\left.-\frac{1}{2\sqrt{\omega(\omega+1)}}\left\{e^{\sqrt{\omega(\omega+1)}x}\frac{1}{1+\sqrt{\omega(\omega+1)}}\right\}\right\}$$

$$= \left[-\beta(\omega) - \frac{1}{\omega}\right]\frac{1}{2\sqrt{\omega(\omega+1)}}\left[1 - \frac{1}{1+\sqrt{\omega(\omega+1)}}\right]\lim_{x\to\infty} e^{\sqrt{\omega(\omega+1)}x}.$$

Since

$$\frac{1}{2\sqrt{\omega(\omega+1)}}\left[1 - \frac{1}{1+\sqrt{\omega(\omega+1)}}\right] \neq 0,$$

we have that

$$-\beta(\omega) - \frac{1}{\omega} = 0.$$

From that it follows $\beta(\omega) = -\frac{1}{\omega}$. Applying (2.33) and (2.29), we get $p(\omega) = \frac{1}{\omega}$ and

$$u(\omega, s) = \frac{1}{(\omega^2 + \omega - s^2)}\left[\frac{-s^2 - s + \omega^2 + \omega - 2}{\omega(s+1)} + \frac{1}{\omega(s+1)} + \frac{1}{\omega}\right]$$

$$u(\omega, s) = \frac{1}{\omega(s+1)}.$$

Taking the inverse of Laplace transform with respect to x, we get

$$u(\omega, x) = \frac{1}{\omega}\check{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{1}{\omega}e^{-x}.$$

Finally, taking the inverse Laplace transform with respect to t, we get

$$\beta(t) = -1, p(t) = 1$$

and

$$u(t, x) = e^{-x}.$$

Therefore, the exact solution of problem (2.29)

$$(u(t, x), p(t)) = (e^{-x}, 1).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}^+, \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta(t,x), \\ 1 \le r \le n, 0 \le t \le T, x \in S^+, \\ \int\limits_0^{x_1} ... \int\limits_0^{x_n} u(t,x)\, dx_1 ... dx_n = g(t), 0 \le t \le T \end{cases} \tag{2.34}$$

for the multidimensional telegraph differential equation. Assume that $a_r(x) > a > 0$ and $f(t, x), \left(t \in (0, T), x \in \overline{\Omega}^+\right)$, $\varphi(x), \psi(x) \left(x \in \overline{\Omega}^+\right)$, $\alpha(t, x), \beta(t, x)$ $(t \in [0, T], x \in S^+)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (2.34) can be used only in the case when (2.34) has $a_r(x)$ polynomials coefficients.

2.3 FOURIER TRANSFORM METHOD

Third, we consider Fourier transform solution of identification problems for telegraph differential equations.

Example 2.3.1. Obtain the Fourier transform solution of the following problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p\left(t\right) e^{-x^2} + \left(-4x^2+1\right) e^{-t-x^2}, \\ t > 0, x \in \mathbb{R}^1, \\ u\left(0,x\right) = e^{-x^2},\ u_t\left(0,x\right) = -e^{-x^2}, x \in \mathbb{R}^1, \\ \int\limits_{-\infty}^{\infty} u\left(t,x\right) dx = e^{-t}\sqrt{\pi},\quad t \geq 0. \end{cases} \tag{2.35}$$

for a one dimensional telegraph differential equation.

Solution. Here and in future we denote

$$F\left\{u\left(t,x\right)\right\} = u\left(t,s\right).$$

Taking the Fourier transform of both sides of the differential equation (2.35) and using initial conditions, we can obtain

$$\begin{cases} u_{tt}\left(t,s\right) + u_t\left(t,s\right) + s^2 u\left(t,s\right) \\ = p\left(t\right) F\left\{e^{-x^2}\right\} + e^{-t} F\left\{\frac{\partial^2}{\partial x^2}\left(-e^{-x^2}\right)\right\} - e^{-t} F\left\{e^{-x^2}\right\}, t > 0, \\ u\left(0,s\right) = F\left\{e^{-x^2}\right\}, u_t\left(0,s\right) = -F\left\{e^{-x^2}\right\}. \end{cases} \tag{2.36}$$

Applying the formula

$$F\left\{\frac{\partial^2}{\partial x^2}\left(e^{-x^2}\right)\right\} = -s^2 F\left\{e^{-x^2}\right\},$$

we get

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) \\ = \left[p(t) + e^{-t}\left(s^2-1\right)\right] F\left\{e^{-x^2}\right\}, t > 0, \\ u(0,s) = F\left\{e^{-x^2}\right\}, u_t(0,s) = -F\left\{e^{-x^2}\right\}. \end{cases} \tag{2.37}$$

Now, taking the Laplace transform of both sides of the differential equation (2.37) with respect to t, we get

$$\left(\omega^2 + \omega + s^2\right) u(\omega,s) = \left(\omega + p(\omega) + \frac{1}{1+\omega}\left(s^2-1\right)\right) F\left\{e^{-x^2}\right\}. \tag{2.38}$$

Applying condition $\int\limits_{-\infty}^{\infty} u(t,x)\,dx = e^{-t}\sqrt{\pi},\ t \geq 0$ and the definition of Fourier transform, we get

$$u(t,0) = \int\limits_{-\infty}^{\infty} u(t,x)\,dx = e^{-t}\sqrt{\pi},\ t \geq 0.$$

Taking the Laplace transform of both sides of the formula, we get

$$u(\omega,0) = \frac{\sqrt{\pi}}{1+\omega}. \tag{2.39}$$

Therefore, using (2.38), (2.39) and formula

$$F\left\{e^{-x^2}\right\} = \sqrt{\pi}e^{-\frac{s^2}{4}},$$

we get

$$\sqrt{\pi}\omega = \omega\sqrt{\pi}. + \left(p(\omega) - \frac{1}{1+\omega}\right)\sqrt{\pi}.$$

From that it follows

$$p(\omega) = \frac{1}{1+\omega},$$

Now, taking the inverse Laplace transform with respect to t, we get

$$p(t) = e^{-t}.$$

Putting $p(t)$ into the differential equation (2.36), we obtain the following problem

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) = \left(e^{-t} + e^{-t}s^2 - e^{-t}\right) F\left\{e^{-x^2}\right\}, t > 0, \\ u(0,s) = F\left\{e^{-x^2}\right\}, u_t(0,s) = -F\left\{e^{-x^2}\right\} \end{cases}$$

or

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) = e^{-t}s^2 F\left\{e^{-x^2}\right\}, t > 0, \\ u(0,s) = F\left\{e^{-x^2}\right\}, u_t(0,s) = -F\left\{e^{-x^2}\right\}. \end{cases}$$

We will seek the general solution $u(t,s)$ of this equation by the following formula

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

where $u_c(t,s)$ is the solution of homogeneous equation

$$u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) = 0, t > 0$$

and $u_p(t,s)$ is the particular solution of nonhomogeneous equation

$$u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) = e^{-t}s^2 F\left\{e^{-x^2}\right\}, t > 0.$$

The auxiliary equation is

$$q^2 + q + s^2 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + \sqrt{s^2 - \frac{1}{4}}i, q_2 = -\frac{1}{2} - \sqrt{s^2 - \frac{1}{4}}i.$$

Therefore,

$$u_c(t,s) = e^{-\frac{t}{2}}\left[c_1 \cos\sqrt{s^2 - \frac{1}{4}}t + c_2 \sin\sqrt{s^2 - \frac{1}{4}}t\right].$$

Now, we will seek $u_p(t,s)$ by putting the formula

$$u_p(t,s) = A(s)e^{-t}.$$

We have that

$$A(s)e^{-t} - A(s)e^{-t} + s^2 A(s)e^{-t} = e^{-t}s^2 F\left\{e^{-x^2}\right\}.$$

From that it follows

$$A(s) = F\left\{e^{-x^2}\right\}.$$

Therefore, the general solution of this equation is

$$u(t,s) = e^{-\frac{t}{2}}\left[c_1 \cos\sqrt{s^2 - \frac{1}{4}}t + c_2 \sin\sqrt{s^2 - \frac{1}{4}}t\right] + e^{-t}F\left\{e^{-x^2}\right\}.$$

Using initial conditions, we obtain

$$u(0,s) = c_1 + F\left\{e^{-x^2}\right\} = F\left\{e^{-x^2}\right\},$$

$$u_t(0,s) = -\frac{1}{2}c_1 + \sqrt{s^2 - \frac{1}{4}}c_2 - F\left\{e^{-x^2}\right\} = -F\left\{e^{-x^2}\right\}.$$

From that it follows

$$c_1 = 0, \sqrt{s^2 - \frac{1}{4}}c_2 = 0.$$

Therefore $c_1 = c_2 = 0$ and

$$u(t,s) = e^{-t}F\left\{e^{-x^2}\right\}.$$

Taking the inverse Fourier transform with respect to x, we obtain

$$u(t,x) = e^{-t}e^{-x^2}.$$

So, the exact solution of problem (2.35) is

$$(u(t,x), p(t)) = \left(e^{-t-x^2}, e^{-t}\right).$$

Note that using the same manner one obtain the solution of the following boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - \sum\limits_{|r|=2m} \alpha_r \frac{\partial^{|r|}u(t,x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} = p(t)q(x) + f(t,x), \\ 0 < t < T, x, r \in \mathbb{R}^n, |r| = r_1 + \dots + r_n, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \mathbb{R}^n, \\ \int\limits_{\mathbb{R}^n} \dots \int u(t,x)\, dx_1 \cdots dx_n = g(t), 0 \le t \le T \end{cases} \tag{2.40}$$

for a second order in t and $2m - th$ order in space variables multidimensional telegraph differential equation. Assume that $\alpha_r \geq \alpha \geq 0$ and $f(t,x), g(t)$, $(t \in [0,T], x \in \mathbb{R}^n)$, $\varphi(x), \psi(x), (x \in \mathbb{R}^n)$ are given smooth functions.

However Fourier transform method described in solving (2.40) can be used only in the case when (2.40) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients or polynomial coefficients. It is well-known that the most general method for solving partial differential equation with depend on t and in the space variables is finite difference method.

In the next chapter, we consider the time-dependent source identification problem for a one-dimensional telegraph equation. The first and order of accuracy difference schemes for the numerical solution of this source identification problem is presented. Numerical analysis and discussions are presented.

CHAPTER 3

FINITE DIFFERENCE METHOD OF THE SOLUTION OF SOURCE IDENTIFICATION PROBLEMS FOR TELEGRAPH EQUATIONS

In this section, we study the numerical solution of the identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p\left(t\right)\sin x - e^{-t}\sin x, \\ x \in \left(0, \pi\right), t \in \left(0, 1\right), \\ u\left(0, x\right) = \sin x, u_t\left(0, x\right) = -\sin x, x \in \left[0, \pi\right], \\ u\left(t, 0\right) = u\left(t, \pi\right) = 0, t \in \left[0, 1\right], \\ \int_0^\pi u\left(t, x\right) dx = 2e^{-t}, t \in \left[0, 1\right] \end{cases} \tag{3.1}$$

for a telegraph equation. The exact solution pair of this problem is $\left(u\left(t, x\right), p\left(t\right)\right) = \left(e^{-t}\sin x, e^{-t}\right)$.

For the numerical solution of problem (3.1), we present the following first order of accuracy difference scheme for the approximate solution for the problem (3.1)

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1} - u_n^k}{\tau} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} \\ = p_k \sin x_n - e^{-t_{k+1}} \sin x_n, \\ t_k = k\tau, x_n = nh, 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, \\ u_n^0 = \sin x_n, \frac{u_n^1 - u_n^0}{\tau} = -\sin x_n, 0 \leqslant n \leqslant M, Mh = \pi, N\tau = 1, \\ u_0^{k+1} = u_M^{k+1} = 0, \sum_{i=1}^{M-1} u_i^{k+1} h = 2e^{-t_{k+1}}, -1 \leqslant k \leqslant N-1. \end{cases} \tag{3.2}$$

Algorithm for obtaining the solution of identification problem (3.2) $\{u_k\}_{k=0}^{N} = \left\{\{u_n^k\}_{k=0}^{N}\right\}_{n=0}^{M}$ and $\{p_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$u_n^k = w_n^k + \eta_k \sin x_n, 0 \leq k \leq N, 0 \leq n \leq M, \tag{3.3}$$

Applying difference scheme (3.2) and formula (3.3), we will obtain formula

$$\eta_{k+1} = \frac{2e^{-t_{k+1}} - \sum_{i=1}^{M-1} w_i^{k+1} h}{\sum_{i=1}^{M-1} \sin x_i h}, -1 \leq k \leq N-1 \tag{3.4}$$

and the difference scheme

$$\begin{cases} \frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2} + 2\frac{w_n^{k+1}-w_n^k}{\tau} - \frac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2} \\ + \frac{\sum_{i=1}^{M-1} w_i^{k+1} h}{\sum_{i=1}^{M-1} \sin x_i h} \sin x_n \frac{2(\cos h-1)}{h^2} - 2\frac{\sum_{i=1}^{M-1} \frac{w_i^{k+1}-w_i^k}{\tau} h}{\sum_{i=1}^{M-1} \sin x_i h} \sin x_n \\ = -\frac{4\frac{e^{-t_{k+1}}-e^{-t_k}}{\tau}}{\sum_{i=1}^{M-1} \sin x_i h} \sin x_n + \left[\frac{2}{\sum_{i=1}^{M-1} \sin x_i h} \frac{2(\cos h-1)}{h^2} - 1\right] e^{-t_{k+1}} \sin x_n, \\ 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, \\ w_n^0 = \sin x_n, \frac{w_n^1-w_n^0}{\tau} = -\sin x_n, 0 \leqslant n \leqslant M, \\ w_0^{k+1} = w_M^{k+1} = 0, -1 \leqslant k \leqslant N-1. \end{cases} \tag{3.5}$$

In the first stage, we find numerical solution $\left\{\{w_n^k\}_{k=0}^{N}\right\}_{n=0}^{M}$ of corresponding first order of accuracy difference scheme (3.5). For obtaining the solution of difference scheme (3.5), we will write it in the matrix form as

$$\begin{cases} Aw^{k+1} + Bw^k + Cw^{k-1} = f^k, 1 \leqslant k \leqslant N-1, \\ w^0 = \{\sin x_n\}_{n=0}^{M}, w^1 = (1-\tau)\{\sin x_n\}_{n=0}^{M}, \end{cases} \tag{3.6}$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $w^s, s = k, k \pm 1, f^k$ are $(M+1) \times 1$

column matrices and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & c_1 & \cdots & c_1 & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & b+c_2 & \cdots & c_2 & c_2 & c_2 & 0 \\ 0 & c_3 & b+c_3 & a+c_3 & \cdots & c_3 & c_3 & c_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{M-3} & c_{M-3} & c_{M-3} & \cdots & a+c_{M-3} & b+c_{M-3} & c_{M-3} & 0 \\ 0 & c_{M-2} & c_{M-2} & c_{M-2} & \cdots & b+c_{M-2} & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & c_{M-1} & \cdots & c_{M-1} & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

,

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & e+z_1 & z_1 & z_1 & \cdots & z_1 & z_1 & z_1 & 0 \\ 0 & z_2 & e+z_2 & z_2 & \cdots & z_2 & z_2 & z_2 & 0 \\ 0 & z_3 & z_3 & e+z_3 & \cdots & z_3 & z_3 & z_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & z_{M-3} & z_{M-3} & z_{M-3} & \cdots & e+z_{M-3} & z_{M-3} & z_{M-3} & 0 \\ 0 & z_{M-2} & z_{M-2} & z_{M-2} & \cdots & z_{M-2} & e+z_{M-2} & z_{M-2} & 0 \\ 0 & z_{M-1} & z_{M-1} & z_{M-1} & \cdots & z_{M-1} & z_{M-1} & e+z_{M-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

,

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

,

$$f^k=\begin{bmatrix} 0 \\ f(t_k, x_1) \\ . \\ f(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1}$$

,

$$w^s=\begin{bmatrix} w_0^s \\ w_1^s \\ . \\ w_{M-1}^s \\ w_M^s \end{bmatrix}$$

for $s = k, k \pm 1$. Here,

$$a=\frac{1}{\tau^2}+\frac{2}{\tau}+\frac{2}{h^2}, b=-\frac{1}{h^2}, c_n=\frac{h}{d}\sin x_n\left[\frac{2(\cosh -1)}{h^2}-\frac{2}{\tau}\right],$$

$$d=\sum_{i=1}^{M-1}\sin x_i h, e=-\frac{2}{\tau^2}-\frac{2}{\tau}, g=\frac{1}{\tau^2}, z_n=\frac{2h}{d\tau}\sin x_n,$$

$$f\left(t_k,x_n\right)=-\frac{4\frac{e^{-t_{k+1}}-e^{-t_k}}{\tau}}{\sum_{i=1}^{M-1}\sin x_i h}\sin x_n+\left[\frac{2}{\sum_{i=1}^{M-1}\sin x_i h}\frac{2(\cos h-1)}{h^2}-1\right]e^{-t_{k+1}}\sin x_n,$$

$$1\le k\le N-1, 1\le n\le M-1.$$

So, we have the initial value problem for the first order difference equation (3.2) with respect to k with matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain the solution$\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of (3.6) by direct formula

$$w^{k+1}=A^{-1}\left(f^k-Bw^k-Cw^{k-1}\right), k=1,...,N-1.$$

In the second stage, applying formulas

$$p_k=\frac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}, 1\le k\le N-1. \tag{3.7}$$

and (3.4), we can obtain $\{p_k\}_{k=1}^{N-1}$. Finally, in the third stage, we will obtain $\left\{\left\{u_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ by formulas (3.3) and (3.4). The errors are computed by

$$E_u=\max_{0\leqslant k\leqslant N}\left(\sum_{n=1}^{M-1}\left|u\left(t_k,x_n\right)-u_n^k\right|^2 h\right)^{\frac{1}{2}},$$

$$E_p=\max_{1\leqslant k\leqslant N-1}\left|p\left(t_k\right)-p_k\right|,$$

where $u\left(t,x\right), p(t)$ represent the exact solution, u_n^k represent the numerical solutions at $\left(t_k,x_n\right)$ and p_k represent the numerical solutions at t_k. The numerical results are given in the following table.

Table 1. Error analysis

$Error$	$N=M=20$	$N=M=40$	$N=M=80$	$N=M=160$
E_u	0.0179	0.0091	0.0046	0.0023
E_p	0.0936	0.0484	0.0246	0.0124

As it is seen in Table 1, we get some numerical results. If N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $1/2$ for first order difference scheme (3.2).

CHAPTER 4
CONCLUSIONS

This thesis is devoted to time-dependent source identification problem for telegraph equations with unknown parameter $p(t)$. The following results are established:

- The history of direct and inverse boundary value problems for telegraph differential equations are considered.
- Fourier series, Laplace transform and Fourier transform methods are used for solution of six identification problems for telegraph equations.
- The first order of accuracy difference scheme is presented for the approximate solution of the one dimensional time - dependent identification problem for telegraph equation with the Dirichlet condition.
- The Matlab implementation of the numerical solution is added.

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Appendices

APPENDIX 1

MATLAB PROGRAMMING

```
clc; clear all ; close all;

N=160;

M=160;

h=pi/M; tau=1/N;

d=0;

for i=1:M-1;

d=d+h*sin(i*h);

end;

a=(1/(tau^2))+(2/(h^2))+(2/(tau));

b=-1/(h^2);

g=1/(tau^2);

c=(h/d)*((2*(cos(h)-1)/(h^2))-(2/tau));

e=-2/(tau^2)-2/tau;

z=(2*h)/(d*tau);

A=zeros(M+1,M+1);

for i=2:M;

for j=2:M;

A(i,j)=c*sin((i-1)*h);

end;

end;

for i=2:M
```

```
A(i,i)=a+(c*sin((i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(c*sin((i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(c*sin((i-1)*h));
end;
A(1,1)=1;
A(M+1,M+1)=1;
A(2,1)=b;
A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
B(i,j)=z*sin((i-1)*h);
end;
end;
for i=2:M
B(i,i)=e+(z*sin((i-1)*h));
end;
B;
```

```
C=zeros(M+1,M+1);

for n=2:M;

C(n,n)=g;

end;

C;

fii=zeros(M+1,1);

for j=1:M+1;

for k=2:N;

fii(j,k)=((4*(cos(h)-1)/(d*(h^2)))-(4/(d*tau))-1)*exp(-k*tau)*sin((j-
1)*h)+(4/(d*tau))*exp(-(k-1)*tau)*sin((j-1)*h);

end;

end;

fii;

G=inv(A);

W=zeros(M+1,1);

for j=1:M+1;

W(j,1)=sin((j-1)*h);

W(j,2)=(1-tau)*sin((j-1)*h);

for k=3:N+1;

W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));

end;

end;

for k=2:N;
```

```
D=0;
for j=1:M-1;
S(j)=D+W(j,k+1)-2*(W(j,k))+W(j,k-1);
D=S(j);
end;
p(k)=(2*exp(-(k+1)*tau)-4*exp(-k*tau)+2*exp(-(k-1)*tau)-(h*D))/(d*(tau^2));
end;
p(k);
L=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
L(i,j)=0;
end
end;
for i=2:M;
L(i,i)=a;
end;
for i=2:M-1;
L(i,i+1)=b;
end;
for i=3:M;
L(i,i-1)=b;
end;
```

```
L(1,1)=1;
L(M+1,M+1)=1;
L;
R=zeros(M+1,M+1);
for n=2:M;
R(n,n)=e;
end;
R;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii=zeros(M+1,1) ;
for j=1:M+1;
for k=2:N;
x=(j-1)*h;
fii(j,k)=p(k)*sin(x)-exp(-(k)*tau)*sin(x);
end;
end;
fii;
G=inv(L);
u=zeros(M+1,1);
```

```
for j=1:M+1;

x=(j-1)*h;

u(j,1)=sin(x);

u(j,2)=(1-tau)*sin(x);

end;

for k=3:N+1;

u(:,k)=G*(-(R*u(:,k-1))-(C*u(:,k-2))+fii(:,k-1));

end;

%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%

for j=1:M+1;

for k=1:N+1;

t=(k-1)*tau;

x=(j-1)*h;

es(j,k)=(1-t)*sin(x);

eu(j,k)=exp(-t)*sin(x);

end;

end;

for k=2:N;

t=(k-1)*tau;

ep(k)=exp(-t);

end;

% ABSOLUTE DIFFERENCES ;

absdiff=max(max(abs(es-W)))
```

```
absdiff=max(max(abs(ep-p)))

absdiff=max(max(abs(eu-u)))
```