FATHI S. A. EMHARAB

SOURCE IDENTIFICATION PROBLEMS FOR HYPERBOLIC DIFFERENTIAL AND DIFFERENCE EQUATIONS

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A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
FATHI S. A. EMHARAB

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy of Science in Mathematics

Examining Committee in Charge:

Prof. Dr. Evren Hinçal	Committee Chairman, Department of Mathematics, NEU
Prof. Dr. Allaberen Ashyralyev	Supervisor, Department of Mathematics, NEU
Assoc. Prof. Dr. Deniz Ağırseven	Department of Mathematics, Trakya University
Assoc. Prof. Dr. Okan Gerçek	Department of Computer Engineering, Girne American University
Assist. Prof. Dr. Bilgen Kaymakamzade	Department of Mathematics, NEU

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Name, Last name: Fathi S. A. Emharab

Signature:

Date:

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To my parents…

ABSTRACT

In the present study, a source identification problem with local and nonlocal conditions for a one-dimensional hyperbolic equation is investigated. Stability estimates for the solutions of the source identification problems are established. Furthermore, a first and second order of accuracy difference schemes for the numerical solutions of the source identification problems for hyperbolic equations with local and nonlocal conditions are presented. Stability estimates for the solutions of difference schemes are established. Then, these difference schemes are tested on examples and some numerical results are presented.

Keywords: Source identification problem; hyperbolic differential equations; difference schemes; local and nonlocal conditions; stability; accuracy

ÖZET

Bu tezde bir boyutlu bir hiperbolik denklem için yerel ve yerel olmayan koşullu bir kaynak tanımlama problemi araştırılmıştır. Kaynak tanımlama probleminin çözümü için kararlılık kestirimleri oluşturulmuştur. Ayrıca, hiperbolik denklemler için yerel ve yerel olmayan koşullu kaynak tanımlama problemlerinin sayısal çözümleri için birinci ve ikinci dereceden doğruluklu fark şemaları sunulmuştur. Fark şemalarının çözümleri için kararlılık kestirimleri oluşturulmuştur. Daha sonra, bu fark şemaları örnekler üzerinde test edilip bazı sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Kaynak tanımlama problemi; hiperbolik diferansiyel denklemler; fark şemaları; yerel ve yerel olmayan koşullar; kararlılık; doğruluk

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CHAPTER 1
INTRODUCTION

1.1 History

The studies of well-posed and ill-posed boundary value problems for hyperbolic and telegraph partial differential equations are driven not only by a theoretical interest but also by the fact of several phenomena in engineering and various fields of physics and applied sciences. In mathematical modelling, hyperbolic and telegraph partial differential equations are used together with boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe a process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological, or environmental processes often involve nonclassical conditions. Such conditions are usually identified as nonlocal boundary conditions and reflect situations when the data on the boundary of domain cannot be measured directly, or when the data on the inside of the domain. Of great interest is the study of absolutely stable difference schemes of a high order of accuracy for hyperbolic partial differential equations, in which stability was established without any assumptions with respect to the grid steps and such type of stability inequalities for the solutions of the first order of accuracy difference scheme for the differential equations of hyperbolic type were established for the first time (Sobolevskii and Chebotareva, 1977).

The survey paper contains the results on the local and nonlocal well-posed problems for second order differential and difference equations. Results on the stability of differential problems for hyperbolic equations and of difference schemes for approximate solution of the hyperbolic problems were presented (Ashyralyev et al., 2015).

Identification problems take an important place in applied sciences and engineering, and have been studied by many authors (Belov, 2002; Gryazin et al., 1999; Isakov, 1998; Kabanikhin and Krivorotko, 2015; Prilepko, Orlovsky and Vasin, 2000). The theory and applications of source identification problems for partial differential equations have been given in various papers (Anikonov, 1996; Ashyralyev and Ashyralyyev, 2014; Ashyralyyev, 2014; Ashyralyyev and Demirdag, 2012; Kozhanov, 1997; Orlovskii, 2008; Orlovskii and Piskarev, 2013).

In particular, Kozhanov, (1997) applied a new approach for solving elliptic equations which is based on the transition to equations of composite type. The obtained results on solvability of linear inverse problems for elliptic equations are based on the solvability and the properties of solutions of boundary value problems for equations of composite type. The inverse problem of finding the source in an abstract second-order elliptic equation on a finite interval was studied by (Orlovskii, 2008). The additional information given is the value of the solution at an interior point of the interval. Moreover, existence, uniqueness, and Fredholm property theorems for the inverse problem were proved. The authors investigated an inverse problem for an elliptic equation in a Banach space with the Bitsadze-Samarskii conditions. The suggested approach uses the notion of a general approximation scheme, the theory of C_0-semigroups of operators and methods of functional analysis (Orlovskii and Piskarev, 2013).

The well-posedness of the unknown source identification problem for a parabolic equation has been well investigated when the unknown function p is dependent on the space variable (Ashyralyev, 2011; Ashyralyev, Erdogan and Demirdag, 2012; Choulli and Yamamoto, 1999; Èidel'man, 1978; Kostin, 2013). Nevertheless, when the unknown function p is dependent on t, the well-posedness of the source identification problem for a parabolic equation has been investigated by (Ashyralyev and Erdogan, 2014; Borukhov and Vabishchevich, 2000; Dehghan, 2003; Erdogan and Sazaklioglu, 2014; Ivanchov, 1995; Saitoh, Tuan and Yamamoto, 2003; Samarskii and Vabishchevich, 2008). Moreover, the well-posedness of the source identification problem for a delay parabolic equation has also been given by (Ashyralyev and Agirseven, 2014; Blasio and Lorenzi, 2007). The authors studied the inverse problems in determining the coefficients of the equation for the kinetic Boltzmann equation. The Cauchy problem and the boundary value problem for states close to equilibrium have been considered. Theorems of the existence and uniqueness of the inverse problems were proved (Orlovskii and Prilepko, 1987).

The solvability of the inverse problems in various formulations with various overdetermination conditions for telegraph and hyperbolic equations were studied in many works (Anikonov, 1976; Ashyralyev and Çekiç, 2015; Kozhanov and Safiullova, 2010; Kozhanov and Safiullova, 2017; Kozhanov and Telesheva, 2017). In particular, the well-posedness of the source identification problem for a telegraph equation with unknown parameter p

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = p + f(t), 0 \leq t \leq T, \\ v(0) = \varphi, v'(0) = \psi, v(T) = \zeta \end{cases}$$

in a Hilbert space H with the self-adjoint positive-definite operator A was proved by (Ashyralyev and Çekiç, 2015). Here φ, ψ and ζ are given elements of H. They established stability estimates for the solution of this problem. In applications, three source identification problems for telegraph equations are developed. The authors studied the solvability of the inverse problems on finding a solution $v(x,t)$ and an unknown coefficient c for a telegraph equation

$$v_{tt} - \Delta v + cv = f(x,t).$$

Theorems on the existence of the regular solutions are proved. The feature of the problems is a presence of new overdetermination conditions for the considered class of equations (Kozhanov and Safiullova, 2017). The authors studied solvability of the parabolic and hyperbolic inverse problems of finding a solution together with an unknown right-hand side when the general overdetermination condition is given. Some theorems of unique existence of regular solutions were proved (Kozhanov and Safiullova, 2010). The authors considered nonlinear inverse coefficient problems for nonstationary higher-order differential equations of pseudohyperbolic type (Kozhanov and Telesheva, 2017). More precisely, they study the problems of determining both the solution of the corresponding equation, an unknown coefficient at the solution or at the time derivative of the solution in the equation. A distinctive feature of these problems is the fact that the unknown coefficient is a function of time only. Integral overdetermination is used as an additional condition. The existence theorems of regular solutions (those solutions that have all generalized derivatives in the sense of S. L. Sobolev) were proved. The technique of the proof relies on the transition from the original inverse problem to a new direct problem for an auxiliary integral-differential equation, and then on the proof of solvability of the latter and construction of some solution of the original inverse problem from a solution of the auxiliary problem. A theorem on the uniqueness for solutions to an inverse problem for the wave equation has been proved (Anikonov, 1976). Finally, some new representations were given for the solutions and coefficients of the equations of mathematical physics (Anikonov, 1995; Anikonov, 1996; Anikonov and Neshchadim, 2011; Anikonov and Neshchadim, 2012 a, 2012b). Their main direction of study was to search for reciprocal formulas connecting solutions and coefficients, and involving arbitrary functions as well as functions satisfying some differential relations. They gave such formulas for evolution equations of first and second order in time, in particular for parabolic and hyperbolic equations in the linear and nonlinear cases.

In this thesis, we consider the time-dependent source identification problem for a one-dimensional hyperbolic equation with local conditions

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u(t,x)}{\partial x}\right) = p(t)\,q(x) + f(t,x), \\ x \in (0,l),\, t \in (0,T), \\ u(0,x) = \varphi(x),\, u_t(0,x) = \psi(x),\, x \in [0,l], \\ u(t,0) = u(t,l) = 0, \int_0^l u(t,x)\,dx = \zeta(t),\, t \in [0,T] \end{cases} \tag{1.1}$$

and with nonlocal conditions

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u(t,x)}{\partial x}\right) + \delta u(t,x) = p(t)\,q(x) + f(t,x), \\ x \in (0,l),\, t \in (0,T), \\ u(0,x) = \varphi(x),\, u_t(0,x) = \psi(x),\, x \in [0,l], \\ u(t,0) = u(t,l),\, u_x(t,0) = u_x(t,l), \\ \int_0^l u(t,x)\,dx = \zeta(t),\, t \in [0,T], \end{cases} \tag{1.2}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \geq a > 0, \delta \geq 0$, $f(t,x), \zeta(t), \varphi(x)$ and $\psi(x)$ are sufficiently smooth functions, and $q(x)$ is a sufficiently smooth function assuming $\int_0^l q(x)\,dx \neq 0$, and $q(0) = q(l) = 0$ for (1.1), $q(0) = q(l)$, $q'(0) = q'(l)$ for (1.2). At the same time, we note that the inverse problems (1.1) and (1.2) for the hyperbolic equation were not studied before. Basic results of this thesis have been published by the following papers (Ashyralyev and Emharab, 2017; Ashyralyev and Emharab, 2018a, 2018b, 2018c, 2018d). Some results of this work were presented in Mini-symposium "Inverse Ill-posed Problems and its applications" of VI congress of Turkic World Mathematical Society (TWMS 2017), and 2nd International Conference of Mathematical Sciences, 2018.

1.2 Methods of Solution of Source Identification Problem

It is known that local and nonlocal boundary value problems for second order partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples. Let us see how to apply the classic methods, namely, Fourier series, Fourier transform and Laplace transform for obtaining the solution of source identification problem for hyperbolic equations on some examples.

Example 1.2.1 Obtain the Fourier series solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = p(t)\sin x + e^{-t}\sin x, \\ 0 < x < \pi, 0 < t < 1, \\ u(0,x) = \sin x, u_t(0,x) = -\sin x, 0 \le x \le \pi, \\ u(t,\pi) = u(t,0) = 0, \int_0^\pi u(t,x)\,dx = 2e^{-t}, 0 \le t \le 1, \end{cases} \tag{1.3}$$

for a one dimensional hyperbolic equation.

Solution. In order to solve the problem, we consider the Sturm-Liouville problem

$$u_{xx} - \lambda u(x) = 0, 0 < x < \pi,\ u(0) = u(x) = 0,$$

generated by the space operator of problem (1.3). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = -k^2, u_k(x) = \sin kx, k = 1, 2, \ldots.$$

Therefore, we will seek solution $u(t,x)$ using by the Fourier series

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t)\sin kx. \tag{1.4}$$

Here $A_k(t), k = 1, 2\ldots$ are unknown functions. Putting (1.4) into the equation (1.3) and using given initial and boundary conditions, we obtain

$$\sum_{k=1}^{\infty} A_k''(t)\sin kx + \sum_{k=1}^{\infty} k^2 A_k(t)\sin kx = p(t)\sin x + e^{-t}\sin x,$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0) \sin kx = \sin x,$$

$$u_t(0,x) = \sum_{k=0}^{\infty} A_k'(0) \sin kx = -\sin x,$$

$$\int_0^{\pi} u(t,x)\,dx = \int_0^{\pi} \sum_{k=1}^{\infty} A_k(t) \sin kx dx = -\sum_{k=1}^{\infty} \frac{A_k(t)\cos kx}{k}\Bigg|_0^{\pi}$$

$$= \sum_{k=1}^{\infty} A_k(t) \left[\frac{1+(-1)^{k+1}}{k}\right] = 2e^{-t}.$$

Equating the coefficients of $\sin kx, k = 1,2...$ to zero, we get

$$A_k''(t) + k^2 A_k(t) = 0, k \neq 1, \;\; A_1''(t) + A_1(t) = p(t) + e^{-t}, 0 < t < 1,$$

$$A_k(0) = 0, A_k'(0) = 0, k \neq 1, \;\; A_1(0) = 1, A_1'(0) = -1,$$

$$\sum_{k=1}^{\infty} A_k(t) \left[\frac{1+(-1)^{k+1}}{k}\right] = 2e^{-t}, 0 < t < 1. \tag{1.5}$$

First, we will obtain $A_k(t)$ for $k \neq 1$. It is easy to see that $A_k(t)$ is the solution of the following Cauchy problem

$$A_k''(t) + k^2 A_k(t) = 0, \;\; 0 < t < 1, \;\; A_k(0) = 0, A_k'(0) = 0,$$

for the second order differential equations. Its solution is $A_k(t) \equiv 0$, $k \neq 1$. From that and formula (1.5), we get

$$-2A_1(t) = -2e^{-t}.$$

Therefore,

$$A_1(t) = e^{-t}. \tag{1.6}$$

Second,we will obtain $p(t)$. It is clear that $A_1(t)$ is solution of the following Cauchy problem

$$A_1''(t) + A_1(t) = p(t) + 2e^{-t}, \; 0 < t < 1, \; A_1(0) = 1, \; A_1'(0) = -1, \tag{1.7}$$

for the second order differential equations. Applying (1.6) and (1.7), we obtain

$$p(t) = e^{-t}.$$

Then, (1.4) becomes $u(t,x) = A_1(t)\sin x = e^{-t}\sin x$. Therefore, the exact solution of problem (1.3) is $(u(t,x), p(t)) = \left(e^{-t}\sin x, e^{-t}\right)$.

Note that using similar procedure one can obtain the solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in \overline{\Omega}, \\ u(t,x) = 0,\ 0 \le t \le T,\ x \in S, \\ \int\limits_{x\in\overline{\Omega}} \cdots \int u(t,x)\,dx_1...dx_n = \xi(t), 0 \le t \le T, \end{cases} \tag{1.8}$$

for the multidimensional hyperbolic partial differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x), (t \in (0,T), x \in \Omega), \varphi(x), \psi(x), \left(x \in \overline{\Omega}\right), \xi(t), (t \in [0,T])$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with the boundary S and $\overline{\Omega} = \Omega \cup S$.

Example 1.2.2 Obtain the Fourier series solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} + \dfrac{\partial u(t,x)}{\partial x^2} = p(t)(1 + \sin 2x) + 4e^{-t}\sin 2x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = 1 + \sin 2x, u_t(0,x) = -(1 + \sin 2x),\ \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi),\ 0 \le t \le 1, \\ \int_0^\pi u(t,x)\,dx = e^{-t}\pi,\ 0 \le t \le 1, \end{cases} \tag{1.9}$$

for a one dimensional hyperbolic equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$u_{xx} - \lambda u(x) = 0, 0 < x < \pi,\ u(0) = u(x),\ u_x(0) = u_x(\pi)$$

generated by the space operator of problem (1.9). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = 4k^2, u_k(x) = \cos 2kx, k = 0, 1, 2, ..., \; u_k(x) = \sin 2kx, k = 1, 2, ...$$

Therefore, we will seek solution $u(t, x)$ using by the Fourier series

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx, \tag{1.10}$$

where $A_k(t), k = 0, 1, 2, ...$ and $B_k(t), k = 1, 2, ...$ are unknown functions, Putting (1.10) into (1.9) and using given initial and boundary conditions, we get

$$\sum_{k=0}^{\infty} A_k''(t) \cos 2kx + \sum_{k=1}^{\infty} B_k''(t) \sin 2kx + \sum_{k=0}^{\infty} 4k^2 A_k(t) \cos kx$$

$$+\sum_{k=1}^{\infty} 4k^2 B_k(t) \sin 2kx = p(t)(\sin 2x + 1) + 4e^{-t} \sin 2x,$$

$$u(0, x) = \sum_{k=0}^{\infty} A_k(0) \cos 2kx + \sum_{k=1}^{\infty} B_k(0) \sin 2kx = \sin 2x + 1,$$

$$u_t(0, x) = \sum_{k=0}^{\infty} A_k'(0) \cos 2kx + \sum_{k=1}^{\infty} B_k'(0) \sin 2kx = -\sin 2x - 1,$$

$$\int_0^{\pi} u(t, x)\, dx = \int_0^{\pi} \left[\sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx \right] dx$$

$$= A_0(t)\pi + \sum_{k=1}^{\infty} \frac{A_k(t) \sin 2kx}{2k} \Bigg]_0^{\pi} - \sum_{k=1}^{\infty} \frac{B_k(t) \cos 2kx}{2k} \Bigg]_0^{\pi} = A_0(t)\pi = e^{-t}\pi.$$

From that it follows that $A_0(t) = e^{-t}$. Equating the coefficients of $\cos kx, k = 0, 1, 2, ...$ and $\sin kx, k = 1, 2, ...$ to zero, we get

$$\begin{cases} A_k''(t) + 4k^2 A_k(t) = 0, \; 0 < t < 1, \\ A_k(0) = 0, A_k'(0) = 0, k \neq 0, \end{cases} \tag{1.11}$$

$$\begin{cases} A_0''(t) = p(t), \; 0 < t < 1, \\ A_0(0) = 1, A_0'(0) = -1, \end{cases} \tag{1.12}$$

$$\begin{cases} B_k''(t) + 4k^2 B_k(t) = 0,\ 0 < t < 1, \\ B_k(0) = 0, B_k'(0) = 0, \end{cases} \tag{1.13}$$

$$\begin{cases} B_1''(t) + 4B_1(t) = \ p(t) + 4e^{-t},\ 0 < t < 1, \\ \quad B_1(0) = 1, B_1'(0) = -1. \end{cases} \tag{1.14}$$

First, we obtain $p(t)$. Applying problem (1.12) and $A_0(t) = e^{-t}$, we get

$$p(t) = e^{-t}.$$

Second, we obtain $A_k(t), k \neq 0$. It is clear that for $k \neq 0, A_k(t)$ is the solution of the initial value problems (1.11) and (1.12). The auxiliary equation is

$$m^2 + 4k^2 = 0.$$

We have two roots

$$m_1 = 2ik, m_2 = -2ik$$

Therefore,

$$A_k(t) = c_1 \cos 2kt + c_2 \sin 2kt.$$

Applying initial conditions $A_k(0) = A_k'(0) = 0$, we get

$$A_k(0) = c_1 = 0,$$

$$A_k'(0) = 2kc_2 = 0.$$

Then $c_1 = c_2 = 0$ and $A_k(t) = 0, k \neq 0$. Third, we obtain $B_k(t)$. It is clear that for $k \neq 1, B_k(t)$ is the solution of the initial value problem (1.13). The auxiliary equation is

$$m^2 + 4k^2 = 0.$$

We have two roots

$$m_1 = 2ik, m_2 = -2ik.$$

Therefore,

$$B_k(t) = c_1 \cos 2kt + c_2 \sin 2kt.$$

Applying initial conditions $B_k(0) = B_k'(0) = 0$, we get

$$B_k(0) = c_1 = 0,$$

$$B_k'(0) = 2kc_2 = 0.$$

Then $c_1 = c_2 = 0$ and $B_k(t) = 0$. Now, we obtain $B_1(t)$ from (1.14), and $p(t) = e^{-t}$. It is the Cauchy problem for the second order linear differential equation. We will seek $B_1(t)$ by formula

$$B_1(t) = B_C(t) + B_P(t),$$

where $B_C(t)$ is general solution of the homogeneous differential equation $B_1''(t) + 4B_1(t) = 0$ and $B_P(t)$ is particular solution of non-homogeneous differential equation. The auxiliary equation is

$$m^2 + 4 = 0.$$

We have two roots

$$m_1 = 2ik, m_2 = -2ik.$$

Therefore,

$$B_c(t) = c_1 \cos 2kt + c_2 \sin 2kt.$$

Since $\pm i2 \neq -1$, we put

$$B_p(t) = e^{-t}a.$$

Therefore,

$$ae^{-t} + 4ae^{-t} = 5e^{-t}.$$

From that it follows $a = 1$ and

$$B_p(t) = e^{-t}.$$

Thus,

$$B_1(t) = c_1 \cos 2kt + c_2 \sin 2kt + e^{-t}.$$

Applying initial conditions $B_1(0) = 1, B_1'(0) = -1$, we get

$$B_1(0) = c_1 + 1 = 1,\ B_1'(0) = 2kc_2 - 1 = -1.$$

From that it follows

$$B_1(t) = e^{-t}.$$

Therefore,

$$u(t,x) = e^{-t}(\sin 2x + 1).$$

So, the exact solution of problem (1.9) is

$$(u(t,x), p(t)) = \left(e^{-t}(\sin 2x + 1), e^{-t}\right).$$

Note that using similar procedure one can obtain the solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)\, q(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \left.\dfrac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_1} = \left.\dfrac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_2}, \\ \int \cdots \int\limits_{x\in\overline{\Omega}} u(t,x)\, dx_1 ... dx_n = \xi(t), 0 \le t \le T, \end{cases} \tag{1.15}$$

for the multidimensional hyperbolic partial differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x), (t \in (0,T), x \in \Omega), \psi(x), \xi(t), \left(t \in [0,T], x \in \overline{\Omega}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset$, and $\overline{m}$ is the normal vector to S_1 and S_2.

However Fourier series method described in solving (1.8) and (1.15) can be used only in the case when (1.8) and (1.15) have constant coefficients.

Second, we consider the Laplace transform method for solution of the source identification problem for hyperbolic differential equation.

Example 1.2.3 Obtain the Laplace transform solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u\left(t,x\right)}{\partial t^2}-\dfrac{\partial^2 u\left(t,x\right)}{\partial x^2}+2u\left(t,x\right)=p\left(t\right)e^{-x}+e^{-t-x}, \\ 0<x<\infty, 0<t<1, \\ u\left(0,x\right)=e^{-x}, u_t\left(0,x\right)=-e^{-x}, 0\leq x<\infty, \\ u\left(t,0\right)=e^{-t}, u_x\left(t,0\right)=-e^{-t}, 0\leq t\leq 1, \\ \int_0^\infty u\left(t,x\right)dx=e^{-t}, 0\leq t\leq 1, 0\leq x<\infty, \end{cases} \tag{1.16}$$

for a one dimensional hyperbolic equation.

Solution. We will denote

$$\mathcal{L}\left\{u(t,x)\right\}=u(t,s).$$

Using formula

$$\mathcal{L}\left\{e^{-x}\right\}=\frac{1}{s+1} \tag{1.17}$$

and taking the Laplace transform of both sides of the differential equation and conditions $u(t,0)=e^{-t}, u_x(t,0)=-e^{-t}$, we can write

$$\mathcal{L}\left\{u_{tt}\left(t,x\right)\right\}-\mathcal{L}\left\{u_{xx}\left(t,x\right)\right\}+2\mathcal{L}\left\{u\left(t,x\right)\right\}=\left[p\left(t\right)+e^{-t}\right]\mathcal{L}\left\{e^{-x}\right\},$$

$$\mathcal{L}\left\{u\left(0,x\right)\right\}=\mathcal{L}\left\{e^{-x}\right\},\mathcal{L}\left\{u_t\left(0,x\right)\right\}=-\mathcal{L}\left\{e^{-x}\right\}$$

or

$$\begin{cases} u_{tt}\left(t,s\right)-s^2u\left(t,s\right)+su\left(t,0\right)+u_x\left(t,0\right)+2u\left(t,s\right)=p\left(t\right)\frac{1}{s+1}+e^{-t}\frac{1}{s+1}, \\ u(0,s)=\dfrac{1}{s+1}, u_t(0,s)=-\dfrac{1}{s+1}. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} u_{tt}\left(t,s\right)+\left(2-s^2\right)u\left(t,s\right)+se^{-t}-e^{-t}=p\left(t\right)\frac{1}{s+1}+e^{-t}\frac{1}{s+1}, \\ u(0,s)=\dfrac{1}{s+1}, u_t(0,s)=-\dfrac{1}{s+1}. \end{cases}$$

Applying the condition

$$\int_0^\infty u\left(t,x\right)dx=e^{-t}, 0\leq t\leq 1,$$

and the definition of the Laplace transform, we get

$$u\left(t,0\right)=e^{-t}, 0\leq t\leq 1, \tag{1.18}$$

It is clear that $u(t,s)$ is solution of the following source identification problem

$$\begin{cases} u_{tt}(t,s) + \left(2-s^2\right) u(t,s) = p(t)\frac{1}{s+1} + \frac{2-s^2}{s+1}e^{-t}, \\ u(0,s) = \frac{1}{s+1}, u_t(0,s) = \frac{1}{s+1}. \end{cases} \tag{1.19}$$

Applying the D'Alembert's formula, we obtain

$$u(t,s) = \frac{1}{s+1}\cos\sqrt{2-s^2}t - \frac{1}{s+1}\frac{1}{\sqrt{2-s^2}}\sin\sqrt{2-s^2}t \tag{1.20}$$

$$+\frac{1}{\sqrt{2-s^2}}\int_0^t \sin\sqrt{2-s^2}(t-y)\left\{p(y)\frac{1}{s+1} + \frac{2-s^2}{s+1}e^{-y}\right\}dy.$$

Now, we will apply the condition (1.18) with (1.20), we get

$$e^{-t} = \cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t + \frac{1}{\sqrt{2}}\int_0^t \sin\sqrt{2}(t-y)\left\{p(y) + 2e^{-y}\right\}dy. \tag{1.21}$$

Taking the first and second order derivatives, we get

$$-e^{-t} = -\sqrt{2}\sin\sqrt{2}t - \cos\sqrt{2}t + \int_0^t \cos\sqrt{2}(t-y)\left\{p(y) + 2e^{-y}\right\}dy,$$

$$e^{-t} = -2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t$$

$$-\sqrt{2}\int_0^t \sin\sqrt{2}(t-y)\left\{p(y) + 2e^{-y}\right\}dy + p(t) + 2e^{-t}. \tag{1.22}$$

Applying (1.21) and (1.22), we get

$$e^{-t} = -2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t + p(t) + 2e^{-t}$$

$$-\sqrt{2}\left\{\sqrt{2}\left\{e^{-t} - \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right\}\right\}.$$

From that it follows

$$p(t) = e^{-t}. \tag{1.23}$$

Finally, applying (1.20) and (1.23), we get

$$u(t,s) = \frac{1}{s+1}\cos\sqrt{2-s^2}t - \frac{1}{s+1}\frac{1}{\sqrt{2-s^2}}\sin\sqrt{2-s^2}t$$

$$+\frac{1}{\sqrt{2-s^2}}\int_0^t \sin\sqrt{2-s^2}(t-y)\left\{e^{-y}\left(\frac{3-s^2}{s+1}\right)\right\}dy.$$

Applying the formula

$$\int_0^t \sin\sqrt{2-s^2}\,(t-y)\,e^{-y}dy = \frac{\sin\sqrt{2-s^2}t + e^{-t}\sqrt{2-s^2} - \sqrt{2-s^2}\cos\sqrt{2-s^2}t}{3-s^2},$$

we get

$$u(t,s) = \frac{1}{s+1}\cos\sqrt{2-s^2}t - \frac{1}{s+1}\frac{1}{\sqrt{2-s^2}}\sin\sqrt{2-s^2}t$$

$$+\frac{1}{\sqrt{2-s^2}}\frac{3-s^2}{s+1}\left[\frac{\sin\sqrt{2-s^2}t + e^{-t}\sqrt{2-s^2} - \sqrt{2-s^2}\cos\sqrt{2-s^2}t}{3-s^2}\right]$$

$$= \frac{1}{s+1}\cos\sqrt{2-s^2}t - \frac{1}{s+1}\frac{1}{\sqrt{2-s^2}}\sin\sqrt{2-s^2}t$$

$$+\frac{1}{\sqrt{2-s^2}}\frac{1}{s+1}\sin\sqrt{2-s^2}t + \frac{e^{-t}}{s+1} - \frac{1}{s+1}\cos\sqrt{2-s^2}t = e^{-t}\frac{1}{s+1}. \tag{1.24}$$

From that it follows that

$$u(t,x) = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t-x}.$$

Therefore, the exact solution of problem (1.16) is

$$(u(t,x), p(t)) = \left(e^{-t-x}, e^{-t}\right).$$

Example 1.2.4 Obtain the Laplace transform solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} + \dfrac{\partial u(t,x)}{\partial t} = p(t)(1+\sin 2x) + 4e^{-t}\sin 2x, \\ t > 0, 0 < x < \pi, \\ u(0,x) = 1+\sin 2x, \quad u_t(0,x) = -(1+\sin 2x),\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi),\ t \ge 0, \\ \int_0^\pi u(t,x)\,dx = e^{-t}\pi,\ 0 \le t \le 1, \end{cases} \tag{1.25}$$

for a one dimensional hyperbolic equation.

Solution. We will denote

$$\mathcal{L}\{u(t,x)\} = u(s,x).$$

Using formula

$$\mathcal{L}\left\{e^{-t}\right\}=\frac{1}{s+1}$$

and taking the Laplace transform of both sides of the differential equation and conditions $u(0,x)=1+\sin 2x, u_t(0,x)=-(1+\sin 2x)$, we can write

$$\begin{cases} s^2u\left(s,x\right)-u\left(s,0\right)-u_x\left(t,0\right)+u_{xx}\left(s,x\right) \\ =p\left(s\right)\left(1+\sin 2x\right)+\frac{4}{1+s}\sin 2x, \\ u\left(s,0\right)=u\left(s,\pi\right),u_x\left(s,0\right)=u_x\left(s,\pi\right),\ 0\leq t\leq 1. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} s^2u\left(s,x\right)-s\left(1+\sin 2x\right)+\left(1+\sin 2x\right)-u_{xx}\left(s,x\right) \\ =p\left(s\right)\left(1+\sin 2x\right)+\frac{4}{1+s}\sin 2x, \\ u\left(t,0\right)=u\left(t,\pi\right),u_x\left(t,0\right)=u_x\left(t,\pi\right), \\ \int_0^{\pi}u\left(s,x\right)dx=\frac{\pi}{1+s},\ 0\leq t\leq 1. \end{cases} \tag{1.26}$$

In order to solve this problem, we consider the Sturm-Liouville problem

$$u_{xx}-\lambda u\left(x\right)=0,0<x<\pi,\ u\left(0\right)=u\left(x\right),\ u_x\left(0\right)=u_x\left(\pi\right)$$

generated by the space operator of problem (1.26). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k=4k^2,u_k\left(x\right)=\cos 2kx,k=0,1,2,...,\ \ u_k\left(x\right)=\sin 2kx,k=1,2,....$$

Then, we will obtain the Fourier series solution of problem (1.26) by formula

$$u\left(s,x\right)=\sum_{k=0}^{\infty}A_k\left(s\right)\cos 2kx+\sum_{k=1}^{\infty}B_k\left(s\right)\sin 2kx, \tag{1.27}$$

where $A_k\left(t\right),k=0,1,2,...$ and $B_k\left(t\right),k=1,2,...$are unknown functions, Putting (1.27) into (1.26) and using given initial and boundary conditions, we get

$$s^2\sum_{k=0}^{\infty}A_k\left(s\right)\cos 2kx-s\left(1+\sin 2x\right)+\left(1+\sin 2x\right)$$

$$+s^2\sum_{k=1}^{\infty}B_k\left(s\right)\sin 2kx+\sum_{k=0}^{\infty}4k^2A_k\left(s\right)\cos kx$$

$$+\sum_{k=1}^{\infty} 4k^2 B_k(s)\sin 2kx = p(s)(1+\sin 2x) + \frac{4}{1+s}\sin 2x,$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos 2kx + \sum_{k=1}^{\infty} B_k(0)\sin 2kx = 1 + \sin 2x,$$

$$u_t(0,x) = \sum_{k=0}^{\infty} A'_k(0)\cos 2kx + \sum_{k=1}^{\infty} B'_k(0)\sin 2kx = -1 - \sin 2x,$$

$$\int_0^{\pi} u(s,x)\,dx = \int_0^{\pi} \left[\sum_{k=0}^{\infty} A_k(s)\cos 2kx + \sum_{k=1}^{\infty} B_k(s)\sin 2kx\right] dx$$

$$= A_0(s)\pi + \sum_{k=1}^{\infty} \frac{A_k(s)\sin 2kx}{2k}\Bigg]_0^{\pi} - \sum_{k=1}^{\infty} \frac{B_k(s)\cos 2kx}{2k}\Bigg]_0^{\pi} = A_0(s)\pi = \frac{\pi}{1+s}.$$

From that it follows $A_0(s) = \frac{1}{1+s}$. Equating the coefficients of $\cos kx, k = 0, 1, 2, \ldots$ and $\sin kx, k = 1, 2, \ldots$ to zero, we get

$$\begin{cases} s^2 A_k(s) + 4k^2 A_k(s) = 0, \ k \neq 0, \\ A_k(0) = 0, A'_k(0) = 0, \text{and } A_k(s) = 0, \ k \neq 0, \end{cases} \tag{1.28}$$

$$\begin{cases} s^2 A_0(s) - s + 1 = p(s), \\ A_0(0) = 1, A'_0(0) = -1, \end{cases} \tag{1.29}$$

$$\begin{cases} s^2 B_k(s) + 4k^2 B_k(s) = 0, \\ B_k(0) = 0, B'_k(0) = 0, \text{and } B_k(s) = 0, \ k \neq 0, \end{cases} \tag{1.30}$$

$$\begin{cases} s^2 B_1(s) + 4B_1(s) - s + 1 = p(s) + \frac{4}{s+1}, \\ B_1(0) = 1, \ B'_1(0) = 1. \end{cases} \tag{1.31}$$

First, we obtain $p(s)$. Applying problem (1.29) and $A_0(t) = \frac{1}{1+s}$, we get

$$p(s) = \frac{1}{1+s}.$$

Second, we obtain $B_1(t)$ from (1.31) and $p(s) = \frac{1}{1+s}$, where

$$\left(s^2 + 4\right) B_1(s) - s + 1 = \frac{1}{1+s} + \frac{4}{s+1},$$

Thus,

$$B_1(s) = \frac{1}{1+s}.$$

Then, (1.27) becomes

$$u(s,x) = \frac{1}{1+s} + \frac{1}{1+s}\sin 2x,$$

From that it follows

$$u(t,x) = \mathcal{L}^{-1}\left\{\frac{1}{1+s} + \frac{1}{1+s}\sin 2x\right\} = e^{-t}(1+\sin 2x).$$

Therefore, the exact solution of problem (1.25) is

$$\{u(t,x), p(t)\} = \left\{e^{-t}(1+\sin 2x), e^{-t}\right\}.$$

Note that using similar procedure one can obtain the solution of the following source identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} a_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = p(t)q(x) + f(t,x), \\ \quad x = (x_1, \dots, x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \quad x \in \overline{\Omega}^+, \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta(t,x), \\ 1 \le r \le n, 0 \le t \le T, x \in S^+, \\ \int_0^{x_1} \dots \int_0^{x_n} u(t,x)\, dx_1 \dots dx_n = \xi(t), 0 \le t \le T, \end{cases} \tag{1.32}$$

for the multidimensional hyperbolic partial differential equation. Assume that $a_r > a > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}^+\right), \xi(t), (t \in [0,T]), \varphi(x), \psi(x) \left(x \in \overline{\Omega}^+\right)$, $\alpha(t,x), \beta(t,x)\,(t \in [0,T], x \in S^+)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and $\overline{\Omega}^+ = \Omega^+ \cup S^+$.

However Laplace transform method described in solving (1.32) can be used only in the case when (1.32) has constant or polynomial coefficients.

Third, we consider Fourier transform method for solution of the source identification problem for hyperbolic differential equations.

Example 1.2.5 Obtain the Fourier transform solution of the following source identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(t)\, e^{-x^2} - \left(4x^2+2\right) e^{-t-x^2}, \\ -\infty < x < \infty, 0 < t < 1, \\ u(0,x) = e^{-x^2}, u_t(0,x) = -e^{-x^2}, x \in (-\infty,\infty), \\ \int_{-\infty}^{\infty} u(t,x)\, dx = e^{-t}\sqrt{\pi}, t \geq 0, \end{cases} \tag{1.33}$$

for a one dimensional hyperbolic equation.

Solution. Let us denote

$$\mathcal{F}\{u(t,x)\} = u(t,\mu).$$

Taking the Fourier transform of both sides of the differential equation and initial conditions (1.33), we can obtain

$$\begin{cases} u_{tt}(t,\mu) + \mu^2 u(t,\mu) = p(t)\,\mathcal{F}\left\{e^{-x^2}\right\} \\ -e^{-t}\mathcal{F}\left\{\frac{d^2}{dx^2}\left(e^{-x^2}\right)\right\}, 0 < t < 1, \\ u(0,\mu) = \mathcal{F}\left\{e^{-x^2}\right\}, u_t(0,\mu) = -\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases} \tag{1.34}$$

Then, we obtain $u(t,\mu)$ as the solution of the following Cauchy problem

$$\begin{cases} u_{tt}(t,\mu) + \mu^2 u(t,\mu) = \left(p(t) + \mu^2 e^{-t}\right) \mathcal{F}\left\{e^{-x^2}\right\}, 0 < t < 1, \\ u(0,\mu) = \mathcal{F}\left\{e^{-x^2}\right\}, u_t(0,\mu) = -\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases} \tag{1.35}$$

Using the D'Alembert's formula, we obtain

$$u(t,\mu) = \frac{e^{i\mu t} + e^{-i\mu t}}{2}\mathcal{F}\left\{e^{-x^2}\right\} - \frac{e^{i\mu t} - e^{-i\mu t}}{2i\mu}\mathcal{F}\left\{e^{-x^2}\right\}$$

$$+\int_0^t \frac{e^{i\mu(t-s)} - e^{-i\mu(t-s)}}{2i\mu}\left[\left(p(s) + \mu^2 e^{-s}\right)\mathcal{F}\left\{e^{-x^2}\right\}\right] ds$$

or

$$u(t,\mu) = \frac{e^{i\mu t} + e^{-i\mu t}}{2}\mathcal{F}\left\{e^{-x^2}\right\} - \frac{1}{2}\int_{-t}^{t} e^{i\mu y} dy \left(\mathcal{F}\left\{e^{-x^2}\right\}\right)$$

$$+\frac{1}{2}\int_0^t \int_{-(t-s)}^{t-s} e^{i\mu y} dy \left[\left(p(s) + \mu^2 e^{-s}\right)\mathcal{F}\left\{e^{-x^2}\right\}\right] ds.$$

Using the formula $\mathcal{F}\{f(x \pm t)\} = e^{\pm i\mu t}\mathcal{F}\{f(x)\}$, we obtain

$$u(t,\mu) = \frac{1}{2}\left[\mathcal{F}\left\{e^{-(x+t)^2}\right\} + \mathcal{F}\left\{e^{-(x-t)^2}\right\}\right] - \frac{1}{2}\int_{-t}^{t}\mathcal{F}\left\{e^{-(x+y)^2}\right\}dy$$

$$+\frac{1}{2}\int_0^t\int_{-(t-s)}^{t-s}p(s)\mathcal{F}\left\{e^{-(x+y)^2}\right\}dyds + \frac{1}{2}\int_0^t\int_{-(t-s)}^{t-s}e^{-s}\mu^2\mathcal{F}\left\{e^{-(x+y)^2}\right\}dyds.$$

Since $\mu^2\mathcal{F}\left\{e^{-x^2}\right\} = -\mathcal{F}\left\{\frac{d^2}{dx^2}\left(e^{-x^2}\right)\right\}$, we have that

$$u(t,\mu) = \frac{1}{2}\left[\mathcal{F}\left\{e^{-(x+t)^2}\right\} + \mathcal{F}\left\{e^{-(x-t)^2}\right\}\right] - \frac{1}{2}\int_{-t}^{t}\mathcal{F}\left\{e^{-(x+y)^2}\right\}dy$$

$$+\frac{1}{2}\int_0^t p(s)\int_{-(t-s)}^{t-s}\mathcal{F}\left\{e^{-(x+y)^2}\right\}dyds - \frac{1}{2}\int_0^t e^{-s}\int_{-(t-s)}^{t-s}\mathcal{F}\left\{\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right)\right\}dyds.$$

Taking the inverse Fourier transform, we obtain

$$u(t,x) = \mathcal{F}^{-1}\{u(t,\mu)\} = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] - \frac{1}{2}\int_{-t}^{t}e^{-(x+y)^2}dy$$

$$+\frac{1}{2}\int_0^t p(s)\int_{-(t-s)}^{t-s}e^{-(x+y)^2}dyds - \frac{1}{2}\int_0^t e^{-s}\int_{-(t-s)}^{t-s}\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right)dyds.$$

Now, applying condition $\int_{-\infty}^{\infty}u(t,x)\,dx = e^{-t}\sqrt{\pi}$, we obtain

$$e^{-t}\sqrt{\pi} = \frac{1}{2}\left[\int_{-\infty}^{\infty}e^{-(x+t)^2}dx + \int_{-\infty}^{\infty}e^{-(x-t)^2}dx\right] - \frac{1}{2}\int_{-t}^{t}\int_{-\infty}^{\infty}e^{-(x+y)^2}dxdy$$

$$+\frac{1}{2}\int_0^t p(s)\int_{-(t-s)}^{t-s}\int_{-\infty}^{\infty}e^{-(x+y)^2}dxdyds - \frac{1}{2}\int_0^t e^{-s}\int_{-(t-s)}^{t-s}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right)dxdyds.$$

Since $\int_{-\infty}^{\infty}e^{-x^2}dx = \sqrt{\pi}$, we can write

$$e^{-t}\sqrt{\pi} = \sqrt{\pi} - \sqrt{\pi}t + \sqrt{\pi}\int_0^t(t-s)\,p(s)\,ds - \frac{1}{2}\int_0^t e^{-s}\int_{-(t-s)}^{t-s}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right)dxdyds.$$

Since

$$\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right) = 4(x+y)^2e^{-(x+y)^2} - 2e^{-(x+y)^2},$$

we have that

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial y^2}\left(e^{-(x+y)^2}\right)dx = 4\int_{-\infty}^{\infty}(x+y)^2e^{-(x+y)^2}dx - 2\int_{-\infty}^{\infty}e^{-(x+y)^2}dx$$

$$= 4\int_{-\infty}^{\infty} p^2 e^{-p^2} dp - 2\int_{-\infty}^{\infty} e^{-p^2} dp = -2\int_{-\infty}^{\infty} pd\left(e^{-p^2}\right) - 2\sqrt{\pi}$$
$$= 2\int_{-\infty}^{\infty} e^{-p^2} dp - 2\sqrt{\pi} = 2\sqrt{\pi} - 2\sqrt{\pi} = 0.$$

Therefore,

$$e^{-t} = 1 - t + \int_0^t (t-s)\, p(s)\, ds.$$

From that it follows

$$-e^{-t} = -1 + \int_0^t p(s)\, ds.$$

Taking the derivative, we obtain

$$e^{-t} = p(t).$$

Putting $p(t)$ into the given differential equation (1.35), we obtain the following Cauchy problem

$$\begin{cases} u_{tt}(t,\mu) + \mu^2 u(t,\mu) = e^{-t}\left(1+\mu^2\right)\mathcal{F}\left\{e^{-x^2}\right\}, 0 < t < 1, \\ u(0,\mu) = \mathcal{F}\left\{e^{-x^2}\right\}, u_t(0,\mu) = -\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases}$$

We will seek the general solution $u(t,\mu)$ of this equation by the following formula

$$u(t,\mu) = u_c(t,\mu) + u_p(t,\mu),$$

where $u_c(t,\mu)$ is the solution of homogeneous equation

$$u_{tt}(t,\mu) + \mu^2 u(t,\mu) = 0, 0 < t < 1$$

and $u_p(t,\mu)$ is the particular solution of nonhomogeneous equation

$$u_{tt}(t,\mu) + \mu^2 u(t,\mu) = e^{-t}\left(1+\mu^2\right)\mathcal{F}\left\{e^{-x^2}\right\}, 0 < t < 1.$$

Then we have that

$$u_c(t,\mu) = c_1 e^{i\mu t} + c_2 e^{-i\mu t}.$$

Now, we will seek $u_p(t,\mu)$ by putting the formula $u_p(t,\mu) = A(\mu)\, e^{-t}$. We have that

$$A(\mu)\, e^{-t} + \mu^2 A(\mu)\, e^{-t} = e^{-t}\left(1+\mu^2\right)\mathcal{F}\left\{e^{-x^2}\right\}.$$

From that it follows

$$A(\mu) = \mathcal{F}\left\{e^{-x^2}\right\}.$$

Therefore, the general solution of this equation is

$$u(t,\mu) = c_1 e^{i\mu t} + c_2 e^{-i\mu t} + e^{-t}\mathcal{F}\left\{e^{-x^2}\right\}.$$

Using initial conditions, we obtain the system of the equations

$$\begin{cases} u(0,\mu) = c_1 + c_2 + \mathcal{F}\left\{e^{-x^2}\right\} = \mathcal{F}\left\{e^{-x^2}\right\}, \\ u_t(0,\mu) = i\mu(c_1 - c_2) - \mathcal{F}\left\{e^{-x^2}\right\} = -\mathcal{F}\left\{e^{-x^2}\right\} \end{cases}$$

or

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 - c_2 = 0. \end{cases}$$

Solving this system, we get

$$c_1 = c_2 = 0.$$

Therefore,

$$u(t,\mu) = e^{-t}\mathcal{F}\left\{e^{-x^2}\right\}.$$

Taking the inverse Fourier transform, we obtain

$$u(t,x) = e^{-t}e^{-x^2}.$$

So, the exact solution of problem (1.33) is

$$(u(t,x), p(t)) = \left(e^{-(t+x^2)}, e^{-t}\right).$$

Note that using the same manner one obtain the solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{|r|=2m} \alpha_r \dfrac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = p(t)q(x) + f(t,x), \\ 0 < t < T, x, r \in \mathbb{R}^n, |r| = r_1 + \ldots + r_n, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in \mathbb{R}^n, \\ \int\limits_{\mathbb{R}^n} \cdots \int u(t,x)\, dx_1 \ldots dx_n = \xi(t), 0 \le t \le T, \end{cases} \tag{1.36}$$

for a second order in t and $2m-th$ order in space variables multidimensional hyperbolic differential equation. Assume that $\alpha_r \geq \alpha \geq 0$ and $f(t,x), \xi(t), (t \in [0,T], x \in \mathbb{R}^n)$, $\varphi(x), \psi(x), (x \in \mathbb{R}^n)$ are given smooth functions.

However Fourier transform method described in solving (1.36) can be used only in the case when (1.36) has constant coefficients. So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients. It is well-known that the most general method for solving partial differential equation with dependent in t and in the space variables is operator method.

1.3 The Aim of the Thesis

Now, let us briefly describe the contents of the various chapters of the thesis. It consists of four chapters.

First chapter is the introduction.

Second chapter the theorem on stability of problem (1.1) with local conditions is established. The first and second order of accuracy difference schemes for the numerical solution of identification hyperbolic problem (1.1) are presented. The theorems on the stability estimates for the solution of these difference schemes are established. Numerical results are provided.

Third chapter the theorem on stability of problem (1.2) with nonlocal conditions is established. The first and second order of accuracy difference schemes for the numerical solution of identification hyperbolic problem (1.2) are presented. The theorems on the stability estimates for the solution of these difference schemes are proved. Numerical results are provided.

Fourth chapter contains conclusion.

CHAPTER 2
STABILITY OF THE HYPERBOLIC DIFFERENTIAL AND DIFFERENCE EQUATION WITH LOCAL CONDITIONS

2.1 Introduction

In this chapter, we consider the source identification problem for a one-dimensional hyperbolic equation with local conditions

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u(t,x)}{\partial x}\right) = p(t)\,q(x) + f(t,x), \\ x \in (0,l), t \in (0,T), \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in [0,l], \\ u(t,0) = u(t,l) = 0, \int_0^l u(t,x)\,dx = \zeta(t), t \in [0,T], \end{cases} \tag{2.1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \geq a > 0$, $f(t,x), \zeta(t), \varphi(x)$ and $\psi(x)$ are sufficiently smooth functions, and $q(x)$ is a sufficiently smooth function assuming $q(0) = q(l) = 0$ and $\int_0^l q(x)\,dx \neq 0$.

2.2 Stability of the Differential Problem (2.1)

To formulate our results, we introduce the Banach space $C(H) = C([0,T], H)$ of all abstract continuous functions $\phi(t)$ defined on $[0,T]$ with values in H, equipped with the norm

$$\|\phi\|_{C(H)} = \max_{0\leqslant t\leqslant T} \|\phi(t)\|_H .$$

Let $L_2[0,l]$ be the space of all square-integrable functions $\gamma(x)$ defined on $[0,l]$, equipped with the norm

$$\|\gamma\|_{L_2[0,l]} = \left(\int_0^l |\gamma(x)|^2\,dx\right)^{\frac{1}{2}},$$

and let $W_2^1[0,l], W_2^2[0,l]$ be Sobolev spaces with norms

$$\|\gamma\|_{W_2^1[0,l]} = \left(\int_0^l \left[\gamma^2(x) + \gamma_x^2(x)\right] dx\right)^{\frac{1}{2}},$$

$$\|\gamma\|_{W_2^2[0,l]} = \left(\int_0^l \left[\gamma^2(x) + \gamma_{xx}^2(x)\right] dx\right)^{\frac{1}{2}},$$

respectively. We introduce the differential operator A defined by the formula

$$Au(x) = -\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) \tag{2.2}$$

with the domain

$$D(A) = \{u : u, u'' \in L_2[0,l], u(0) = u(l) = 0\}.$$

It is easy that A is the self-adjoint positive-definite operator in $H = L_2[0,l]$. Actually, for all $u, v \in L_2[0,l]$ we have that

$$\begin{aligned}
\langle Au, v\rangle &= \int_0^l A(u)v(x)\,dx = -\int_0^l \frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right)v(x)\,dx \\
&\quad - \left(a(x)\frac{du(x)}{dx}\right)v(x)\Big|_0^l + \int_0^l a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx \\
&\quad -a(l)\frac{du(l)}{dx}v(l) + a(0)\frac{du(0)}{dx}v(0) + \int_0^l a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx \\
&= \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx.
\end{aligned}$$

$$\begin{aligned}
\langle u, Av\rangle &= \int_0^l u(x)\,Av(x)dx = -\int_0^l u(x)\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right)dx \\
&\quad - u(x)\left(a(x)\frac{dv(x)}{dx}\right)\Big|_0^l + \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx \\
&\quad -u(l)a(l)\frac{dv(l)}{dx} + u(0)a(0)\frac{dv(0)}{dx} + \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx \\
&= \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx.
\end{aligned}$$

From that it follows

$$\langle Au, v\rangle = \langle u, Av\rangle$$

and

$$\langle Au, u\rangle = \int_0^l a(x)\frac{du(x)}{dx}\frac{du(x)}{dx}dx \ \geq a\int_0^l \frac{du(x)}{dx}\frac{du(x)}{dx}dx = a\langle u', u'\rangle. \tag{2.3}$$

Moreover, using the condition $u(0) = 0$, we get

$$u(y) = \int_0^y \frac{du(x)}{dx}dx = \int_0^y \frac{du(y-t)}{dt}dt.$$

We will introduce the following function u_* defined by formula

$$\frac{du_*(y-t)}{dt} = \begin{cases} \dfrac{du(y-t)}{dt}, 0 \le t \le y, y \in [0,l], \\ 0, \text{otherwise}. \end{cases}$$

Then

$$u(y) = \int_0^l \frac{du_*(y-t)}{dt} dt.$$

Applying the Minkowsky inequality and the definition of the function $u_*(x)$, we get

$$\begin{aligned} \left(\int_0^l u^2(y)\, dy \right)^{\frac{1}{2}} &\le \int_0^l \left(\int_0^l \left(\frac{du_*(y-t)}{dt} \right)^2 dy \right)^{\frac{1}{2}} dt \\ &\le \int_0^l \left(\int_0^l \left(\frac{du(x)}{dx} \right)^2 dx \right)^{\frac{1}{2}} dt \\ &= l \left(\int_0^l \left(\frac{du(x)}{dx} \right)^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle u,u \rangle = \int_0^l u^2(y)\, dy \le l^2 \int_0^l \left(\frac{du(x)}{dx} \right)^2 dxtext \\ = l^2 \int_0^l \frac{du(x)}{dx} \frac{du(x)}{dx} dx = l^2 \langle u', u' \rangle. \end{aligned} \tag{2.4}$$

Applying inequalities (2.3) and (2.4), we get

$$\langle Au,u \rangle \ge \frac{a}{l^2} \langle u,u \rangle.$$

For the self adjoint positive definite operator A we will introduce $c(t)$ and $s(t)$ operator functions defined by formulas $u(t) = c(t)\varphi$ and $v(t) = s(t)\psi$, where abstract functions $u(t)$ and $v(t)$ are solutions of the following Cauchy problems in a Hilbert space H

$$u''(t) + Au(t) = 0, t > 0, u(0) = \varphi, u'(0) = 0, \tag{2.5}$$

$$v''(t) + Av(t) = 0, t > 0, v(0) = 0, v'(0) = \psi, \tag{2.6}$$

respectively. We have the following formulas

$$c(t) = \frac{e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}}{2}, s(t) = A^{-\frac{1}{2}} \frac{e^{itA^{\frac{1}{2}}} - e^{-itA^{\frac{1}{2}}}}{2i}$$

and the following estimates hold

$$\|c(t)\|_{H\to H} \le 1, \left\|A^{\frac{1}{2}}s(t)\right\|_{H\to H} \le 1. \tag{2.7}$$

It is based on the spectral represents of unit self adjoint positive definite operator A and

$$\|f(A)\|_{H\to H} \le \sup_{\delta\le\lambda<\infty} |f(\lambda)|.$$

Here f is the bounded function on $[\delta, \infty)$.

Moreover, for the differential operator A defined by formula $Au = -u''(x)$ with the domain $D(A) = \{u : u(x), u'(x), u''(x) \in E\}$, we can obtain the following estimates

$$\|c(t)\|_{E\to E} \le 1, \left\|A^{\frac{1}{2}}s(t)\right\|_{E\to E} \le 1.$$

Here, $E = L_p\left(R^1\right), 1 \le p < \infty$, $C^\alpha\left(R^1\right), 0 \le \alpha < 1, R^1 = (-\infty, \infty)$. The proof of these estimates is based on the triangle inequality and the following lemma.

Lemma 2.2.1 The following formulas hold:

$$c(t)\varphi(x) = \frac{\varphi(x+t) + \varphi(x-t)}{2}, \tag{2.8}$$

$$s(t)\varphi(x) = \frac{1}{2}\int_{x-t}^{x+t} \varphi(z)dz, \tag{2.9}$$

$$A^{\frac{1}{2}}s(t)\varphi(x) = \frac{\varphi(x+t) - \varphi(x-t)}{2i}. \tag{2.10}$$

Proof. First, we will proof the formula (2.8). Using the definition of operator function $c(t)$, we can write

$$u(t,x) = c(t)\varphi(x),$$

where $u(t,x)$ is the solution of the following Cauchy problem

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, t > 0, x \in R^1, u(0,x) = \varphi(x), u_t(0,x) = 0 \tag{2.11}$$

for the hyperbolic equation with smooth $\varphi(x)$. Assume that $\varphi(\pm\infty) = 0$.

Taking the Fourier transform, we get the following Cauchy problem

$$u_{tt}(t,s) + s^2u(t,s) = 0, t > 0, u(0,s) = \{\varphi(x)\}, u_t(0,s) = 0$$

for the second order differential equation. Taking the Laplace transform, we get

$$\mu^2 u(\mu, s) - \mu \{\varphi(x)\} + s^2 u(\mu, s) = 0$$

or

$$u(\mu, s) = \frac{\mu}{\mu^2 + s^2} \{\varphi(x)\} .$$

Since

$$\frac{\mu}{\mu^2 + s^2} = \frac{1}{2} \left[\frac{1}{\mu - is} + \frac{1}{\mu + is} \right],$$

we have that

$$u(\mu, s) = \frac{1}{2} \left[\frac{1}{\mu - is} + \frac{1}{\mu + is} \right] \{\varphi(x)\} .$$

Applying the inverse Laplace transform, we get

$$u(t, s) = \frac{1}{2} \left[e^{it} \{\varphi(x)\} + e^{-it} \{\varphi(x)\} \right] .$$

Using the shift rule, we get

$$u(t, s) = \frac{1}{2} \left[\{\varphi(x + t)\} + \{\varphi(x - t)\} \right] .$$

Applying the inverse Fourier transform, we get formula (2.8).

Second, we will proof the formula (2.9). Using the definition of operator function $s(t)$, we can write

$$u(t, x) = s(t)\psi(x),$$

where $u(t, x)$ is the solution of the following Cauchy problem

$$u_{tt}(t, x) - u_{xx}(t, x) = 0, t > 0, x \in R^1, u(0, x) = 0, u_t(0, x) = \psi(x) \tag{2.12}$$

for the hyperbolic equation with smooth $\psi(x)$. Assume that $\psi(\pm\infty) = 0$.

Taking the Fourier transform, we get the following Cauchy problem

$$u_{tt}(t, s) + s^2 u(t, s) = 0, t > 0, u(0, s) = 0, u_t(0, s) = \{\psi(x)\}$$

for the second order differential equation. Taking the Laplace transform, we get

$$\mu^2 u(\mu, s) - \{\psi(x)\} + s^2 u(\mu, s) = 0$$

or

$$u(\mu, s) = \frac{1}{\mu^2 + s^2} \{\psi(x)\}.$$

Since

$$\frac{1}{\mu^2 + s^2} = \frac{1}{2is}\left[\frac{1}{\mu - is} - \frac{1}{\mu + is}\right],$$

we have that

$$u(\mu, s) = \frac{1}{2is}\left[\frac{1}{\mu - is} - \frac{1}{\mu + is}\right]\{\psi(x)\}.$$

Applying the inverse Laplace transform, we get

$$u(t, s) = \frac{1}{2is}\left[e^{ist}\{\psi(x)\} - e^{-ist}\{\psi(x)\}\right] = \frac{1}{2}\int_{-t}^{t} e^{isy}dy\,\{\psi(x)\}.$$

Using the shift rule, we get

$$u(t, s) = \frac{1}{2}\left\{\int_{-t}^{t}\{\psi(x+y)\}\,dy\right\}.$$

Applying the inverse Fourier transform, we get formula

$$u(t, x) = \frac{1}{2}\int_{-t}^{t}\psi(x+y)dy.$$

From that it follows formula (2.9). Lemma 2.2.1 is proved.

Throughout the present thesis, M denotes positive constants, which may differ in time, and thus are not a subject of precision. However, we will use the notation $M(\alpha, \beta, \gamma, ...)$ to stress the fact that the constant depends only on $\alpha, \beta, \gamma, ...$.

We have the following theorem on the stability of problem (2.1):

Theorem 2.2.2 Assume that $\varphi \in W_2^2[0,l], \psi \in W_2^1[0,l]$ and $f(t,x)$ is a continuously differentiable function in t and square-integrable in x, and $\zeta(t)$ is a twice continuously differentiable function. Suppose that $q(x)$ is a sufficiently smooth function assuming $q(0) = q(l) = 0$ and $\int_0^l q(x)\,dx \neq 0$. Then, for the solution of problem (2.1) the following stability estimates hold:

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{C(L_2[0,l])} + \|u\|_{C(W_2^2[0,l])} \leqslant M_1(q)\left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}\right. \tag{2.13}$$

$$+\left\|f(0,.)\right\|_{L_2[0,l]}+\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])}+\left\|\zeta''\right\|_{C[0,T]}\Bigg],$$

$$\|p\|_{C[0,T]}\leqslant M_2(q)\left[\|\varphi\|_{W_2^2[0,l]}+\|\psi\|_{W_2^1[0,l]}+\|\zeta''\|_{C[0,T]}\right. \tag{2.14}$$

$$\left.+\|f(0,.)\|_{L_2[0,l]}+\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])}\right].$$

Proof. We will use the substitution

$$u(t,x)=w(t,x)+\eta(t)\,q(x), \tag{2.15}$$

where $\eta(t)$ is the function defined by formula

$$\eta(t)=\int_0^t (t-s)\,p(s)\,ds, \eta(0)=\eta'(0)=0. \tag{2.16}$$

It is easy to see that $w(t,x)$ is the solution of the mixed problem

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2}-\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w(t,x)}{\partial x}\right)=f(t,x) \\ +\eta(t)\left[\dfrac{d}{dx}\left(a(x)\,q'(x)\right)\right], x\in(0,l), t\in(0,T), \\ w(0,x)=\varphi(x), w_t(0,x)=\psi(x), x\in[0,l], \\ w(t,0)=w(t,l)=0, t\in[0,T]. \end{cases} \tag{2.17}$$

Now, we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition $\int_0^l u(t,x)\,dx=\zeta(t)$ and substitution (2.15), we get

$$\eta(t)=\frac{\zeta(t)-\int_0^l w(t,x)\,dx}{\int_0^l q(x)\,dx}.$$

From that and $p(t)=\eta''(t)$, it follows

$$p(t)=\frac{\zeta''(t)-\int_0^l \frac{\partial^2}{\partial t^2}w(t,x)\,dx}{\int_0^l q(x)\,dx}.$$

Then, using the Cauchy–Schwarz inequality and the triangle inequality, we obtain

$$|p(t)|\leqslant\frac{|\zeta''(t)|+\int_0^l\left|\frac{\partial^2}{\partial t^2}w(t,x)\right|dx}{\left|\int_0^l q(x)\,dx\right|} \tag{2.18}$$

$$\leqslant\frac{1}{\left|\int_0^l q(x)\,dx\right|}\left[|\zeta''(t)|+\sqrt{l}\left(\int_0^l\left(\frac{\partial^2 w(t,x)}{\partial t^2}\right)^2dx\right)^{\frac{1}{2}}\right]$$

$$\leq M\left(q\right)\left[\left|\zeta''\left(t\right)\right|+\left\|\frac{\partial^{2}w\left(t,.\right)}{\partial t^{2}}\right\|_{L_{2}[0,l]}\right]$$

for all $t\in[0,T]$. From that, it follows

$$\|p\|_{C[0,T]}\leqslant M\left(q\right)\left[\|\zeta''\|_{C[0,T]}+\left\|\frac{\partial^{2}w}{\partial t^{2}}\right\|_{C(L_{2}[0,l])}\right]. \tag{2.19}$$

Now, using substitution (2.15), we get

$$\frac{\partial^{2}u\left(t,x\right)}{\partial t^{2}}=\frac{\partial^{2}w\left(t,x\right)}{\partial t^{2}}+p\left(t\right)q\left(x\right).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^{2}u}{\partial t^{2}}\right\|_{C(L_{2}[0,l])}\leqslant\left\|\frac{\partial^{2}w}{\partial t^{2}}\right\|_{C(L_{2}[0,l])}+\|p\|_{C[0,T]}\|q\|_{L_{2}[0,l]}. \tag{2.20}$$

Therefore, the proof of estimates (2.13) and (2.14) is based on equation (2.1), the triangle inequality, estimates (2.19), (2.20) and on the stability estimate

$$\left\|\frac{\partial^{2}w}{\partial t^{2}}\right\|_{C(L_{2}[0,l])}\leqslant M_{3}(q,a)\left[\|\varphi\|_{W_{2}^{2}[0,l]}+\|\psi\|_{W_{2}^{1}[0,l]}\right. \tag{2.21}$$

$$\left.+\|f\left(0,.\right)\|_{L_{2}[0,l]}+\left\|\frac{\partial f}{\partial t}\right\|_{C(L_{2}[0,l])}+\|\zeta''\|_{C[0,T]}\right],$$

for the solution of problem (2.17). This completes the proof of Theorem 2.2.2.

Now, we will prove the estimate (2.21) for the solution of problem (2.17).

Firstly, it is easy to see that problem (2.17) can be written as the abstract Cauchy problem

$$\frac{d^{2}w\left(t\right)}{dt^{2}}+Aw\left(t\right)=f\left(t\right)-\eta\left(t\right)Aq,t\in\left(0,T\right), \tag{2.22}$$

$$w\left(0\right)=\varphi,w'\left(0\right)=\psi,$$

in a Hilbert space H with positive-definite self-adjoint operator A defined by formula (2.2). Here,

$$w\left(t\right)=w(t,x),\ \ f\left(t\right)=f\left(t,x\right)$$

are unknown and known abstract functions defined on $(0,T)$ with values in $H=L_{2}\left[0,l\right]$, respectively, and $\varphi=\varphi(x),\psi=\psi(x),q=q(x)$ are given.

Secondly, applying the approaches of (Ashyralyev and Sobolevskii, 2004) , we will prove a lemma that will be needed in the sequel.

Lemma 2.2.3 Assume that $\varphi \in D(A), \psi, q \in D(A^{\frac{1}{2}}), f(t)$ is a continuously differentiable abstract function in t with values in H, and $\eta(t)$ is a twice continuously differentiable function defined by formula (2.16). Then, for the solution of problem (2.22), the following stability estimate holds

$$\left\| \frac{d^2 w(t)}{dt^2} \right\|_H \leqslant \|A\varphi\|_H + \left\| A^{\frac{1}{2}}\psi \right\|_H + \|f(0)\|_H + T \left\| \frac{df}{dt} \right\|_{C(H)} + M_4(q) \int_0^t |\eta''(y)|\, dy \tag{2.23}$$

for any $t \in [0,T]$.

Proof. We have that (see Ashyralyev and Sobolevskii, 2004)

$$w(t) = c(t)\varphi + s(t)\psi + \int_0^t s(t-y)[f(y) - \eta(y) Aq]\, dy, \tag{2.24}$$

where

$$c(t) = \frac{e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}}{2} \quad \text{and} \quad s(t) = A^{-\frac{1}{2}} \frac{e^{itA^{\frac{1}{2}}} - e^{-itA^{\frac{1}{2}}}}{2i}.$$

Applying equation (2.22) and formula (2.24), we get

$$\frac{d^2 w(t)}{dt^2} = f(t) - \eta(t) Aq - Ac(t)\varphi - As(t)\psi - \int_0^t As(t-y)[f(y) - \eta(y) Aq]\, dy.$$

Integrating by parts, we get

$$\begin{aligned} \frac{d^2 w(t)}{dt^2} &= f(t) - \eta(t) Aq - Ac(t)\varphi - As(t)\psi - f(t) \\ &+ c(t) f(0) + \eta(t) Aq - c(t)\eta(0) Aq \\ &+ \int_0^t c(t-y) f'(y)\, dy - \int_0^t c(t-y)\eta'(y) Aq dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d^2 w(t)}{dt^2} &= -Ac(t)\varphi - As(t)\psi + c(t) f(0) \\ &+ \int_0^t c(t-y) f'(y)\, dy - \int_0^t c(t-y)\eta'(y) Aq dy. \end{aligned}$$

From that it follows

$$\frac{d^2w(t)}{dt^2} = -Ac(t)\varphi - As(t)\psi + c(t)f(0) + \int_0^t c(t-y)f'(y)\,dy$$
$$+\left[s(t-y)\eta'(y)Aq\right]_0^t - \int_0^t s(t-y)\eta''(y)Aq\,dy$$

or

$$\begin{aligned}\frac{d^2w(t)}{dt^2} &= -Ac(t)\varphi - As(t)\psi + c(t)f(0)\\ &\quad+\int_0^t c(t-y)f'(y)\,dy - \int_0^t s(t-y)\eta''(y)Aq\,dy\\ &= \sum_{i=1}^4 G_i(t),\end{aligned}$$

where

$$\begin{aligned}&G_1(t) = c(t)(f(0) - A\varphi),\ \ G_2(t) = -As(t)\psi,\\ &G_3(t) = \int_0^t c(t-y)f'(y)\,dy,\ G_4(t) = -\int_0^t s(t-y)\eta''(y)Aq\,dy\end{aligned}$$

Now, applying the triangle inequality, we obtain

$$\left\|\frac{d^2w(t)}{dt^2}\right\|_H \leqslant \sum_{i=1}^4 \|G_i(t)\|_H$$

for any $t \in [0,T]$. Therefore, we will estimate $\|G_i(t)\|_H, i = 1,2,3,4$, separately. Firstly, using the estimates (2.7), we obtain

$$\begin{aligned}&\|G_1(t)\|_H \leqslant \|c(t)\|_{H\to H}\|A\varphi - f(0)\|_H \leqslant \|A\varphi\|_H + \|f(0)\|_H,\\ &\|G_2(t)\|_H = \left\|A^{\frac12}A^{\frac12}s(t)\psi\right\|_H \leqslant \left\|A^{\frac12}s(t)\right\|_{H\to H}\left\|A^{\frac12}\psi\right\|_H \leqslant \left\|A^{\frac12}\psi\right\|_H\end{aligned}$$

for any $t \in [0,T]$. Secondly, using the triangle inequality and estimates (2.7), we get

$$\begin{aligned}\|G_3(t)\|_H &\leq \int_0^t \|c(t-y)\|_{H\to H}\|f'(y)\|_H\,dy \leqslant \int_0^t \|f'(y)\|_H\,dy\\ &\leq t\max_{0\leq y\leq t}\|f'(y)\|_H \leq T\max_{0\leq y\leq T}\|f'(y)\|_H = T\left\|\frac{df}{dt}\right\|_{C(H)}\end{aligned}$$

for any $t \in [0,T]$. Thirdly, using the triangle inequality and estimates (2.7), we get

$$\begin{aligned}\|G_4(t)\|_H &\le \int_0^t \left\|A^{\frac{1}{2}} s(t-y)\right\|_{H\to H} \left\|A^{\frac{1}{2}} q\right\|_H |\eta''(y)|\, dy \\ &\leqslant \int_0^t |\eta''(y)|\, dy \left\|A^{\frac{1}{2}} q\right\|_H \le M_4(q) \int_0^t |\eta''(y)|\, dy\end{aligned}$$

for any $t \in [0,T]$. Combining these estimates, we obtain estimate (2.23) for the solution of problem (2.22) for any $t \in [0,T]$.

Theorem 2.2.4 Assume that all conditions of Theorem 2.2.1 are satisfied. Then, for the solution of problem (2.17) the stability estimate (2.21) holds.

Proof. Putting $H = L_2[0,l], \varphi = \varphi(x), \psi = \psi(x), q = q(x), f(t) = f(t,x), w(t) = w(t,x)$ and applying estimates (2.18), (2.23), we get

$$\begin{aligned}\left\|\frac{\partial^2 w(t)}{\partial t^2}\right\|_{L_2[0,l]} &\leqslant M_5(a)\left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}\right] \\ &\quad + \|f(0,\cdot)\|_{L_2[0,l]} + T\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])} + M(q)\, M_4(q) \\ &\quad \times \left[\int_0^t |\zeta''(y)|\, dy + \int_0^t \left\|\frac{\partial^2 w(y,\cdot)}{\partial y^2}\right\|_{L_2[0,l]} dy\right]\end{aligned}$$

for any $t \in [0,T]$. By the Gronwall's inequality, we conclude that, for any $t \in [0,T]$, the following estimate for the solution of problem (2.17) holds:

$$\begin{aligned}\left\|\frac{\partial^2 w(t)}{\partial t^2}\right\|_{L_2[0,l]} &\leqslant \Big\{M_5(a)\left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}\right] + \|f(0,\cdot)\|_{L_2[0,l]} \\ &\quad + T\left[\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])} + M(q)\, M_4(q)\, \|\zeta''\|_{C[0,T]}\right]\Big\}\, e^{M(q) M_4(q) t}.\end{aligned}$$

This completes the proof of Theorem 2.2.4.

2.3 Stability of Difference Scheme

To formulate our results on a difference problem, we introduce the Banach space $C_\tau(H) = C([0,T]_\tau, H)$ of all abstract grid functions $\phi^\tau = \{\phi(t_k)\}_{k=0}^N$ defined on

$$[0,T]_\tau = \{t_k = k\tau, 0 \leqslant k \leqslant N, N\tau = T\},$$

with values in H, equipped with the norm

$$\|\phi^{\tau}\|_{C_{\tau}(H)}=\max_{0\leqslant k\leqslant N}\|\phi(t_k)\|_H .$$

Moreover, $L_{2h}=L_2[0,l]_h$ is the Hilbert space of all grid functions $\gamma^h(x)=\{\gamma_n\}_{n=0}^M$ defined on

$$[0,l]_h=\{x_n=nh,0\leqslant n\leqslant M,Mh=l\},$$

equipped with the norm

$$\left\|\gamma^h\right\|_{L_{2h}}=\left\{\sum_{i=0}^{M}|\gamma_i|^2h\right\}^{\frac{1}{2}},$$

and $W_{2h}^1=W_2^1[0,l]_h, W_{2h}^2=W_2^2[0,l]_h$ are the discrete analogues of Sobolev spaces of all grid functions $\gamma^h(x)=\{\gamma_n\}_{n=0}^M$ defined on $[0,l]_h$ with norms

$$\|\gamma^h\|_{W_{2h}^1}=\left\{\sum_{i=0}^{M}|\gamma_i|^2h+\sum_{i=1}^{M}\left|\frac{\gamma_i-\gamma_{i-1}}{h}\right|^2h\right\}^{\frac{1}{2}},$$

$$\|\gamma^h\|_{W_{2h}^2}=\left\{\sum_{i=0}^{M}|\gamma_i|^2h+\sum_{i=1}^{M-1}\left|\frac{\gamma_{i+1}-2\gamma_i+\gamma_{i-1}}{h^2}\right|^2h\right\}^{\frac{1}{2}},$$

respectively. For the differential operator A defined by (2.2), we introduce the difference operator A_h defined by formula

$$A_h\varphi^h(x)=\left\{-\frac{1}{h}\left(a(x_{n+1})\frac{\varphi_{n+1}-\varphi_n}{h}-a(x_n)\frac{\varphi_n-\varphi_{n-1}}{h}\right)\right\}_{n=1}^{M-1},$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi_n\}_{n=0}^M$ defined on $[0,l]_h$, satisfying the conditions $\varphi_M=\varphi_0=0$.

It is easy that A_h is the self-adjoint positive-definite operator in $H=L_{2h}=L_2[0,l]_h$. Actually, we have that

$$\begin{aligned}\left\langle A_hu^h,v^h\right\rangle&=\sum_{x\in[0,l]_h}A_hu^h(x)v^h(x)h\\&=-\sum_{n=1}^{M-1}\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}-u_n}{h}-a(x_n)\frac{u_n-u_{n-1}}{h}\right)v_nh\\&=-\sum_{n=1}^{M-1}a(x_{n+1})\frac{u_{n+1}-u_n}{h}v_n+\sum_{n=1}^{M-1}a(x_n)\frac{u_n-u_{n-1}}{h}v_n\\&=-\sum_{n=2}^{M}a(x_n)\frac{u_n-u_{n-1}}{h}v_{n-1}+\sum_{n=1}^{M-1}a(x_n)\frac{u_n-u_{n-1}}{h}v_n\end{aligned}$$

$$
\begin{aligned}
&= -a\left(x_M\right)\frac{u_M - u_{M-1}}{h}v_{M-1} - \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}v_{n-1} \\
&\quad + \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}v_n \\
&= \frac{1}{h}a\left(x_M\right)u_{M-1}v_{M-1} + \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h \\
&= a\left(x_M\right)\frac{u_M - u_{M-1}}{h}\frac{v_M - v_{M-1}}{h}h + \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h \\
&= \sum_{n=1}^{M} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h,
\end{aligned}
$$

$$
\begin{aligned}
\left\langle u^h, A_h v^h\right\rangle &= \sum_{x\in[0,l]_h} u^h(x)A_h v^h(x)h \\
&= -\sum_{n=1}^{M-1} u_n\frac{1}{h}\left(a\left(x_{n+1}\right)\frac{v_{n+1} - v_n}{h} - a\left(x_n\right)\frac{v_n - v_{n-1}}{h}\right)h \\
&= -\sum_{n=1}^{M-1} u_n a\left(x_{n+1}\right)\frac{v_{n+1} - v_n}{h} + \sum_{n=1}^{M-1} u_n a\left(x_n\right)\frac{v_n - v_{n-1}}{h} \\
&= -\sum_{n=2}^{M} u_{n-1} a\left(x_n\right)\frac{v_n - v_{n-1}}{h} + \sum_{n=1}^{M-1} u_n a\left(x_n\right)\frac{v_n - v_{n-1}}{h} \\
&= -u_{M-1}a\left(x_M\right)\frac{v_M - v_{M-1}}{h} - \sum_{n=1}^{M-1} a\left(x_n\right)\frac{v_n - v_{n-1}}{h}u_{n-1} \\
&\quad + \sum_{n=1}^{M-1} a\left(x_n\right)u_n\frac{v_n - v_{n-1}}{h} \\
&= \frac{1}{h}a\left(x_M\right)u_{M-1}v_{M-1} + \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h \\
&= a\left(x_M\right)\frac{u_M - u_{M-1}}{h}\frac{v_M - v_{M-1}}{h}h + \sum_{n=1}^{M-1} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h \\
&= \sum_{n=1}^{M} a\left(x_n\right)\frac{u_n - u_{n-1}}{h}\frac{v_n - v_{n-1}}{h}h,
\end{aligned}
$$

From that it follows

$$
\left\langle A_h u^h, v^h\right\rangle = \left\langle u^h, A_h v^h\right\rangle
$$

and

$$\left\langle A_h u^h, u^h \right\rangle = \sum_{n=1}^{M} a\left(x_n\right) \frac{u_n - u_{n-1}}{h} \frac{u_n - u_{n-1}}{h} h \tag{2.25}$$

$$\geq a \sum_{n=1}^{M} \frac{u_n - u_{n-1}}{h} \frac{u_n - u_{n-1}}{h} h = a \left\langle D_h u^h, D_h u^h \right\rangle.$$

Here

$$D_h u^h(x) = \left\{ \frac{u_n - u_{n-1}}{h} \right\}_{n=1}^{M}.$$

Moreover, using the condition $u_0 = 0$, we get

$$u_m = \sum_{n=1}^{m} \frac{u_n - u_{n-1}}{h} h = \sum_{i=1}^{m} \frac{u_{m-i+1} - u_{m-i}}{h} h.$$

We will introduce the mesh function $u_*^h(x)$ defined by the following formula

$$\left(\frac{u_{m-i+1} - u_{m-i}}{h} \right)_* = \begin{cases} \dfrac{u_{m-i+1} - u_{m-i}}{h}, 1 \leq i \leq m, 1 \leq m \leq M, \\ 0, \text{otherwise.} \end{cases}$$

Then

$$u_m = \sum_{i=1}^{M} \left(\frac{u_{m-i+1} - u_{m-i}}{h} \right)_* h.$$

Applying the discrete analogue of Minkowsky inequality and the definition of the mesh function $u_*^h(x)$, we get

$$\left(\sum_{m=1}^{M-1} u_m^2 h \right)^{\frac{1}{2}} \leq \sum_{i=1}^{M} \left(\sum_{m=1}^{M-1} \left(\frac{u_{m-i+1} - u_{m-i}}{h} \right)_*^2 h \right)^{\frac{1}{2}} h$$

$$\leq \sum_{i=1}^{M} h \left(\sum_{n=1}^{M} \left(\frac{u_n - u_{n-1}}{h} \right)^2 h \right)^{\frac{1}{2}} = Mh \left(\sum_{n=1}^{M} \left(\frac{u_n - u_{n-1}}{h} \right)^2 h \right)^{\frac{1}{2}}$$

$$= l \left(\sum_{n=1}^{M} \left(\frac{u_n - u_{n-1}}{h} \right)^2 h \right)^{\frac{1}{2}}.$$

Therefore,

$$\left\langle u^h, u^h \right\rangle = \sum_{m=1}^{M-1} u_m^2 h \leq l^2 \sum_{n=1}^{M} \left(\frac{u_n - u_{n-1}}{h} \right)^2 h \tag{2.26}$$

$$= l^2 \sum_{n=1}^{M} \frac{u_n - u_{n-1}}{h} \frac{u_n - u_{n-1}}{h} h = l^2 \left\langle D_h u^h, D_h u^h \right\rangle.$$

Applying inequalities (2.25) and (2.26), we get

$$\left\langle A_h u^h, u^h \right\rangle \geq \frac{a}{l^2} \left\langle u^h, u^h \right\rangle .$$

2.3.1 The first order of accuracy difference scheme

For the numerical solution $\left\{ \left\{ u_n^k \right\}_{k=0}^{N} \right\}_{n=0}^{M}$ of problem (2.1), we consider the first order of accuracy difference scheme

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \left(a\left(x_{n+1}\right) \dfrac{u_{n+1}^{k+1} - u_n^{k+1}}{h^2} - a\left(x_n\right) \dfrac{u_n^{k+1} - u_{n-1}^{k+1}}{h^2} \right) \\ = p_k q\left(x_n\right) + f\left(t_k, x_n\right), \\ t_k = k\tau, x_n = nh, 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, N\tau = T, \\ u_n^0 = \varphi\left(x_n\right), \dfrac{u_n^1 - u_n^0}{\tau} = \psi\left(x_n\right), 0 \leqslant n \leqslant M, Mh = l, \\ u_0^{k+1} = u_M^{k+1} = 0, \sum_{i=1}^{M-1} u_i^{k+1} h = \zeta\left(t_{k+1}\right), -1 \leqslant k \leqslant N-1. \end{cases} \tag{2.27}$$

Here, it is assumed that $q_M = q_0 = 0$, and $\sum_{i=1}^{M-1} q_i \neq 0$. We have the following theorem on the stability of the difference scheme (2.27):

Theorem 2.3.1 For the solution of difference scheme (2.27), the following stability estimates hold:

$$\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ u_{k+1}^h \right\}_{k=1}^{N-1} \right\|_{C_\tau(W_{2h}^2)} \tag{2.28}$$

$$\leqslant M_6(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} + \left\| f_1^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=2}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right],$$

$$\left\| \left\{ p_k \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \leqslant M_7(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| f_1^h \right\|_{L_{2h}} \right. \tag{2.29}$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=2}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right].$$

Here and throughout this subsection $f_k^h(x) = \left\{ f\left(t_k, x_n\right) \right\}_{n=0}^{M}, 1 \leq k \leq N-1.$

Proof. We will use the substitution

$$u_n^k = w_n^k + \eta_k q_n, \tag{2.30}$$

where

$$q_n = q(x_n),$$

and

$$\eta_{k+1} = \sum_{i=1}^{k} (k+1-i) p_i \tau^2, 1 \leqslant k \leqslant N-1, \eta_0 = \eta_1 = 0. \tag{2.31}$$

It is easy to see that $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ is the solution of the difference problem

$$\begin{cases} \dfrac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} - \dfrac{1}{h}\left(a(x_{n+1})\dfrac{w_{n+1}^{k+1} - w_n^{k+1}}{h} - a(x_n)\dfrac{w_n^{k+1} - w_{n-1}^{k+1}}{h}\right) \\ = f(t_k, x_n) + \dfrac{1}{h}\left[a(x_{n+1})\dfrac{q_{n+1} - q_n}{h} - a(x_n)\dfrac{q_n - q_{n-1}}{h}\right]\eta_{k+1}, \\ 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, \\ w_n^0 = \varphi(x_n), \dfrac{w_n^1 - w_n^0}{\tau} = \psi(x_n), 0 \leqslant n \leqslant M, \\ w_0^{k+1} = w_M^{k+1} = 0, -1 \leqslant k \leqslant N-1. \end{cases} \tag{2.32}$$

Now, we will take an estimate for $|p_k|$ .Using the overdetermined condition $\sum_{i=1}^{M-1} u_i^{k+1} h = \zeta(t_{k+1})$ and substitution (2.30), one can obtain

$$\eta_{k+1} = \frac{\zeta_{k+1} - \sum_{i=1}^{M-1} w_i^{k+1} h}{\sum_{i=1}^{M-1} q_i h}. \tag{2.33}$$

Then, using the formulas $p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}$ and (2.33), we get

$$p_k = \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1} - \sum_{i=1}^{M-1}\left(w_i^{k+1} - 2w_i^k + w_i^{k-1}\right)h}{\tau^2 \sum_{i=1}^{M-1} q_i h}.$$

Then, applying the discrete analogue of the Cauchy–Schwartz inequality and the triangle inequality, we obtain

$$\begin{aligned} |p_k| &\leqslant \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|} \\ &\quad \times \left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sum_{i=1}^{M-1}\left|\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right| h\right] \\ &\leqslant \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|}\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sqrt{l}\left\|\left\{\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right\}_{i=0}^M\right\|_{L_{2h}}\right] \end{aligned} \tag{2.34}$$

$$\leqslant M_8(q)\left[\left|\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right|+\left\|\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right]$$

for all $1 \leqslant k \leqslant N-1$. From that, it follows

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \leqslant M_8(q)\left[\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right. \tag{2.35}$$
$$\left.+\left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right].$$

Now, using substitution (2.30), we get

$$\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}=\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}+p_kq(x_n).$$

Applying the triangle inequality, we obtain

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant \left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \tag{2.36}$$
$$+\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\left\|\{q(x_n)\}_{n=0}^{M}\right\|_{L_{2h}}.$$

Therefore, the proof of estimates (2.28) and (2.29) is based on equation (2.27), the triangle inequality, estimates (2.35), (2.36) and on the following stability estimate for the solution of difference problem (2.32):

$$\left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant M_{10}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}\right. \tag{2.37}$$
$$\left.+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right].$$

This completes the proof of Theorem 3.2.1.

Now, we will prove estimate (2.37) for the solution of difference problem (2.32). Firstly, it is easy to see that difference scheme (2.32) can be written as the abstract Cauchy difference problem

$$\begin{cases} \dfrac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}+Aw_{k+1}=f_k-\eta_{k+1}Aq, 1\leqslant k\leqslant N-1, \\ w_0=\varphi, \dfrac{w_1-w_0}{\tau}=\psi \end{cases} \tag{2.38}$$

in a Hilbert space $H = L_{2h}$ with positive-definite self-adjoint operator A. Here,

$$\{w_k\}_{k=0}^{N} = \left\{w_k^h\right\}_{k=0}^{N}, \{f_k\}_{k=1}^{N-1} = \left\{f_k^h\right\}_{k=1}^{N-1}$$

are unknown and known abstract mesh functions defined on $[0,T]_\tau$ with values in $H = L_{2h}$, respectively, and $\varphi = \varphi^h, \psi = \psi^h, q = q^h$ are given elements.

Secondly, applying the approaches of (Ashyralyev and Sobolevskii, 2004) we will prove a lemma that will be needed in the sequel.

Lemma 2.3.2 Assume that $\varphi \in D(A), \psi, q \in D(A^{\frac{1}{2}})$. Then, for the solution of difference problem (2.38), the following stability estimate holds for any $1 \leqslant k \leqslant N-1$:

$$\begin{aligned} \left\| \frac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2} \right\|_H &\leqslant \|A\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H \quad + \|f_1\|_H \\ &+ |\eta_2| \, \|Aq\|_H \quad + T \left\| \left\{ \frac{f_k - f_{k-1}}{\tau} \right\}_{k=2}^{N} \right\|_{C(H)} \\ &+ \sum_{s=1}^{k} \left| \frac{\eta_{s+1} - 2\eta_s + \eta_{s-1}}{\tau^2} \right| \tau \left\|A^{\frac{1}{2}} q\right\|_H \end{aligned} \tag{2.39}$$

Proof. It is clear that there exists a unique solution of this initial value problem, and for the solution of (2.38), the following formula is satisfied (see Ashyralyev and Sobolevskii, 2004)

$$w_0 = \varphi, \; w_1 = \varphi + \tau\psi,$$

$$\begin{aligned} w_k = &\frac{1}{2}[R^{k-1} + \widetilde{R}^{k-1}]\varphi + \tau(R - \widetilde{R})^{-1}\left[R^k - \widetilde{R}^k\right]\psi \\ &+ \sum_{s=1}^{k-1} R\widetilde{R}\left(R - \widetilde{R}\right)^{-1}\left[R^{k-s} - \widetilde{R}^{k-s}\right]\{f_s - \eta_{s+1}Aq\}\,\tau^2, \end{aligned} \tag{2.40}$$

$$2 \leq k \leq N,$$

where $R = \left(I + i\tau A^{\frac{1}{2}}\right)^{-1}$ and $\widetilde{R} = \left(I - i\tau A^{\frac{1}{2}}\right)^{-1}$. Using the spectral property of the self-adjoint positive-definite operator, we get

$$\|R\|_{H\to H} \leq 1, \left\|\tau A^{\frac{1}{2}} R\right\|_{H\to H} \leq 1. \tag{2.41}$$

$$\left\|\widetilde{R}\right\|_{H\to H} \leq 1, \left\|\tau A^{\frac{1}{2}} \widetilde{R}\right\|_{H\to H} \leq 1. \tag{2.42}$$

Now, we will establish estimates for $\left\| \frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2} \right\|_H, 1 \le k \le N-1$. Applying equation (2.38) and formula (2.40), we get

$$\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2} \tag{2.43}$$

$$= f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi - \tau A(R-\widetilde{R})^{-1}\left[R^{k+1} - \widetilde{R}^{k+1}\right]\psi$$

$$-\sum_{s=1}^{k} AR\widetilde{R}\left(R-\widetilde{R}\right)^{-1} \times \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right]\{f_s - \eta_{s+1}Aq\}\tau^2$$

$$. = f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi \; - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k\widetilde{R}^{-1}\right]A^{\frac{1}{2}}\psi$$

$$+\left(\frac{\tau A^{\frac{1}{2}}}{2i}\right)\left(i\tau A^{\frac{1}{2}}\right)^{-1}\left\{\sum_{s=1}^{k} R^{k-s}(I-R) + \sum_{s=1}^{k}\widetilde{R}^{k-s}\left(I-\widetilde{R}\right)\right\}\{f_s - \eta_{s+1}Aq\}$$

$$= f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi \; - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k\widetilde{R}^{-1}\right]A^{\frac{1}{2}}\psi$$

$$-\frac{1}{2}\left\{\sum_{s=1}^{k}\left[R^{k-s} - R^{k+1-s}\right] + \sum_{s=1}^{k}\left[\widetilde{R}^{k-s} - \widetilde{R}^{k+1-s}\right](f_s - \eta_{s+1}Aq)\right\}$$

$$= f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi \; - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k\widetilde{R}^{-1}\right]A^{\frac{1}{2}}\psi \tag{2.44}$$

$$-\frac{1}{2}\left\{\sum_{s=1}^{k}\left[R^{k-s} + \widetilde{R}^{k-s}\right](f_s - \eta_{s+1}Aq) - \sum_{s=1}^{k}\left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right]\right\}(f_s - \eta_{s+1}Aq)$$

$$2 \le k \le N.$$

Applying Abel's formula to (2.44), we can write

$$\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}$$

$$= f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi \; - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k\widetilde{R}^{-1}\right]A^{\frac{1}{2}}\psi$$

$$-\frac{1}{2}\left\{\sum_{s=2}^{k+1}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right] - \sum_{s=1}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right]\right\}(f_{s-1} - \eta_s Aq)$$

$$= f_k - \eta_{k+1}Aq - \frac{1}{2}[R^k + \widetilde{R}^k]A\varphi \; - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k\widetilde{R}^{-1}\right]A^{\frac{1}{2}}\psi$$

$$-\frac{1}{2}\left\{\sum_{s=2}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](f_{s-1} - \eta_s Aq) + 2(f_k - \eta_{k+1}Aq)\right.$$

$$\left.-\sum_{s=2}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](f_s - \eta_{s+1}Aq) - \left[R^k + \widetilde{R}^k\right](f_1 - \eta_2 Aq)\right\}$$

$$
\begin{aligned}
&= -\frac{1}{2}[R^k + \widetilde{R}^k]A\varphi - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - R^k \widetilde{R}^{-1}\right] A^{\frac{1}{2}}\psi \\
&-\frac{1}{2}\sum_{s=2}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](f_{s-1} - f_s) + \frac{1}{2}\left[R^k + \widetilde{R}^k\right] f_1 \\
&+\frac{1}{2}\sum_{s=2}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](\eta_s - \eta_{s+1}) Aq - \frac{1}{2}\left[R^k + \widetilde{R}^k\right]\eta_2 Aq,
\end{aligned}
$$

Now, we have that

$$
\begin{aligned}
&\frac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2} \\
&= -\frac{1}{2}[R^k + \widetilde{R}^k]A\varphi - \frac{1}{2i}\left[\widetilde{R}^k R^{-1} - \widetilde{R}^{-1} R^k\right] A^{\frac{1}{2}}\psi \\
&+\frac{1}{2}\left[R^k + \widetilde{R}^k\right] f_1 - \frac{1}{2}\sum_{s=2}^{k}\left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](f_{s-1} - f_s) \\
&+\frac{1}{2}\sum_{s=1}^{k}\left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](\eta_s - \eta_{s+1}) Aq \\
&= \sum_{i=1}^{4} G_i(k), 2 \le k \le N,
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(k) &= -\frac{1}{2}[R^k + \widetilde{R}^k]A\varphi + \frac{1}{2}[R^k + \widetilde{R}^k]f_1, \\
G_2(k) &= -\frac{1}{2i}\left[\widetilde{R}^k R^{-1} - \widetilde{R}^{-1} R^k\right] A^{\frac{1}{2}}\psi, \\
G_3(k) &= -\frac{1}{2}\sum_{s=2}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](f_{s-1} - f_s), \\
G_4(k) &= \frac{1}{2}\sum_{s=1}^{k}\left[R^{k+1-s} + \widetilde{R}^{k+1-s}\right](\eta_s - \eta_{s+1}) Aq.
\end{aligned}
$$

Now, applying the triangle inequality, we obtain

$$
\begin{aligned}
\|G_1(k)\|_H &\leqslant \frac{1}{2}\left[\left\|R^k\right\|_{H\to H} + \left\|\widetilde{R}^k\right\|_{H\to H}\right]\left[\|A\varphi\|_H + \|f_1\|_H\right] \\
&\le \|A\varphi\|_H + \|f_1\|_H \ , \\
\|G_2(k)\|_H &\leqslant \frac{1}{2}\left[\left\|\widetilde{R}^k R^{-1}\right\|_{H\to H} + \left\|\widetilde{R}^{-1} R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}\psi\right\|_H \\
&\le \left\|A^{\frac{1}{2}}\psi\right\|_H
\end{aligned}
$$

for any $1 \le k \le N-1$. Secondly, using the triangle inequality and estimates (2.41), (2.42), we get

$$\begin{aligned}
\|G_3(k)\|_H &\leqslant \sum_{s=2}^{k} \frac{1}{2}\left[\left\|R^{k+1-s}\right\|_{H\to H} + \left\|\widetilde{R}^{k+1-s}\right\|_{H\to H}\right] \|f_{s-1} - f_s\|_H \\
&\leqslant \sum_{s=2}^{k} \|f_{s-1} - f_s\|_H \le T \left\| \left\{ \frac{f_k - f_{k-1}}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(H)}
\end{aligned}$$

for any $1 \le k \le N-1$. Thirdly, applying Abel's formula to the sum $G_4(k)$, we can write

$$\begin{aligned}
G_4(k) &= \frac{1}{2}\left(i\tau A^{\frac{1}{2}}\right)^{-1} \left\{ \sum_{s=1}^{k} R^{k-s}(I-R) - \sum_{s=1}^{k} \widetilde{R}^{k-s}\left(I - \widetilde{R}\right) \right\} \ (\eta_s - \eta_{s+1}) Aq \\
&= \frac{1}{2}\left(i\tau A^{\frac{1}{2}}\right)^{-1} \left\{ \sum_{s=1}^{k} \left[R^{k-s} - \widetilde{R}^{k-s}\right](\eta_s - \eta_{s+1}) Aq \right. \\
&\quad \left. - \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right] \right\} (\eta_s - \eta_{s+1}) Aq \\
&= \frac{1}{2}\left(i\tau A^{\frac{1}{2}}\right)^{-1} \left\{ \sum_{s=2}^{k+1} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](\eta_{s-1} - \eta_s) Aq \right. \\
&\quad \left. - \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right] \right\} (\eta_s - \eta_{s+1}) Aq \\
&= \frac{1}{2}\left(i\tau A^{\frac{1}{2}}\right)^{-1} \left\{ \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](\eta_{s-1} - \eta_s) Aq \right. \\
&\quad - \left[R^{k} - \widetilde{R}^{k}\right]\Big\} (\eta_0 - \eta_1) Aq \\
&\quad \left. - \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](\eta_s - \eta_{s+1}) Aq \right\} \\
&= \frac{1}{2}\left(i\tau A^{\frac{1}{2}}\right)^{-1} \left\{ \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right](\eta_{s-1} - 2\eta_s - \eta_{s+1}) Aq \right\} \\
&= \left(i\tau A^{\frac{1}{2}}\right)^{-1} \frac{1}{2} \left\{ \sum_{s=1}^{k} \left[R^{k+1-s} - \widetilde{R}^{k+1-s}\right] \left(\frac{\eta_{s+1} - 2\eta_s + \eta_{s-1}}{\tau^2} \right) \tau^2 Aq \right\}
\end{aligned}$$

Finally, using the triangle inequality and estimates (2.41), (2.42), we get

$$\|G_4(k)\|_H \leqslant \sum_{s=1}^{k} \frac{1}{2}\left[\left\|R^{k+1-s}\right\|_{H\to H} + \left\|\widetilde{R}^{k+1-s}\right\|_{H\to H}\right] \left|\frac{\eta_{s+1} - 2\eta_s + \eta_{s-1}}{\tau^2}\right| \tau \left\|A^{\frac{1}{2}} q\right\|_H$$

$$\leqslant \sum_{s=1}^{k} \left|\frac{\eta_{s+1} - 2\eta_s + \eta_{s-1}}{\tau^2}\right| \tau \left\|A^{\frac{1}{2}} q\right\|_H$$

for any $1 \leq k \leq N-1$. Combining these estimates, we obtain estimate (2.39) for the solution of problem (2.38) for any $1 \leq k \leq N-1$.

Theorem 2.3.3 For the solution of difference problem (2.32), the stability estimate (2.38) holds.

Proof. Putting $H = L_{2h}, \varphi = \varphi^h, \psi = \psi^h, q = q^h, A\varphi = A_h\varphi^h, w_k = w_k^h, f_k = f_k^h$ and applying estimates (3.35) and (2.39), we get

$$\left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}$$

$$\leqslant M_{12}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right]$$

$$+ \left\|f_1^h\right\|_{L_{2h}} + T\left\|\left\{\frac{f_k^h - f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})} + M_{13}(q) M_8(q)$$

$$\times \sum_{s=1}^{k}\left\{\left[\left|\frac{\zeta_{s+1} - 2\zeta_s + \zeta_{s-1}}{\tau^2}\right| + \left\|\frac{w_{s+1}^h - 2w_s^h + w_{s-1}^h}{\tau^2}\right\|_{L_{2h}}\right]\right\}\tau$$

for any $1 \leq k \leq N-1$. By the difference analogue of Gronwall's inequality, we conclude that

$$\left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}$$

$$\leqslant \frac{1}{1 - M_{13}(q) M_8(q)\tau}\left\{M_{12}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right]\right.$$

$$+ \left\|f_1^h\right\|_{L_{2h}} + T\left\|\left\{\frac{f_k^h - f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}$$

$$\left. + M_{13}(q) M_8(q) T\left\|\left\{\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right\} e^{(k-1)\tau\frac{M_{13}(q)M_8(q)}{1-M_{13}(q)M_8(q)\tau}}$$

for any $1 \leq k \leq N-1$.This completes the proof of Theorem 2.3.3.

2.3.2 The second order of accuracy difference scheme

For the numerical solution $\left\{\left\{u_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of problem (2.1), we consider the second order of accuracy difference scheme

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{1}{2h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^k-u_n^k}{h}-a\left(x_n\right)\dfrac{u_n^k-u_{n-1}^k}{h}\right) \\ -\dfrac{1}{4h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k+1}-u_n^{k+1}+u_{n+1}^{k-1}-u_n^{k-1}}{h}\right. \\ \left.-a\left(x_n\right)\dfrac{u_n^{k+1}-u_{n-1}^{k+1}+u_n^{k-1}-u_{n-1}^{k-1}}{h}\right)=p_kq_n+f\left(t_k,x_n\right), \\ N\tau=T,1\leqslant n\leqslant M-1,Mh=l,t_k=k\tau,1\leqslant k\leqslant N-1, \\ u_n^0=\varphi\left(x_n\right),0\leqslant n\leqslant M, \\ \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^1-u_{n+1}^0-u_n^1+u_n^0}{h}\right. \\ \left.-a\left(x_n\right)\dfrac{u_n^1-u_n^0-u_{n-1}^1+u_{n-1}^0}{h}\right)=\psi\left(x_n\right)+\dfrac{\tau}{2}f\left(0,x_n\right) \\ +\dfrac{\tau}{2}\left[\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^0-u_n^0}{h}-a\left(x_n\right)\dfrac{u_n^0-u_{n-1}^0}{h}\right)+p_0q_n\right], \\ 1\leqslant n\leqslant M-1, \\ u_0^{k+1}=u_M^{k+1}=0,\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta\left(t_{k+1}\right),-1\leqslant k\leqslant N-1 \end{cases} \tag{2.45}$$

Here, it is assumed that $q_M=q_0=0$, and $\sum_{i=1}^{M-1}q_i\neq 0$. We have the following theorem on the stability of the difference scheme (2.45):

Theorem 2.3.4 For the solution of difference scheme (2.45), the following stability estimates hold:

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{u_{k+1}^h+2u_k^h+u_{k-1}^h}{4}\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)} \tag{2.46}$$

$$\leqslant M_{14}\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|f_0^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right],$$

$$\left\|\left\{p_k\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leqslant M_{15}\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_0^h\right\|_{L_{2h}}\right. \tag{2.47}$$

$$+\left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \Bigg].$$

Here and throughout this subsection $f_k^h(x) = \{f(t_k, x_n)\}_{n=0}^M, 1 \le k \le N-1$.

Proof. We will use the substitution

$$u_n^k = w_n^k + \eta_k q_n, \tag{2.48}$$

where

$$\eta_k = \sum_{i=1}^{k} \left\{ \frac{(k-i)p_i + (k-(i-1))p_{i-1}}{2} \right\} \tau^2, 1 \leqslant k \leqslant N, \eta_0 = 0, \tag{2.49}$$

It is easy to see that $\left\{ \{w_n^k\}_{k=0}^N \right\}_{n=0}^M$ is the solution of the difference problem

$$\begin{cases} \dfrac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} - \dfrac{1}{2h}\left(a(x_{n+1}) \dfrac{w_{n+1}^k - w_n^k}{h} - a(x_n) \dfrac{w_n^k - w_{n-1}^k}{h} \right) \\ -\dfrac{1}{4h}\left(a(x_{n+1}) \dfrac{w_{n+1}^{k+1} - w_n^{k+1}}{h} - a(x_n) \dfrac{w_n^{k+1} - w_{n-1}^{k+1}}{h} \right) \\ -\dfrac{1}{4h}\left(a(x_{n+1}) \dfrac{w_{n+1}^{k-1} - w_n^{k-1}}{h} - a(x_n) \dfrac{w_n^{k-1} - w_{n-1}^{k-1}}{h} \right) = f(t_k, x_n) \\ +\dfrac{1}{h}\left[a(x_{n+1}) \dfrac{q_{n+1} - q_n}{h} - a(x_n) \dfrac{q_n - q_{n-1}}{h} \right] \times \dfrac{1}{4}(\eta_{k+1} + 2\eta_k + \eta_{k-1}), \\ 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, \\ w_n^0 = \varphi(x_n), 0 \leqslant n \leqslant M, \\ \dfrac{w_n^1 - w_n^0}{\tau} - \dfrac{\tau}{h}\left(a(x_{n+1}) \dfrac{w_{n+1}^1 - w_{n+1}^0 - w_n^1 + w_n^0}{h} \right. \\ \left. -a(x_n) \dfrac{w_n^1 - w_n^0 - w_{n-1}^1 + w_{n-1}^0}{h} \right) - \dfrac{\tau}{h}\left(a(x_{n+1}) \dfrac{q_{n+1} - q_n}{h} \right. \\ \left. -a(x_n) \dfrac{q_n - q_{n-1}}{h} \right) \eta_1 = \psi(x_n) + \dfrac{\tau}{2} f(0, x_n) \\ +\dfrac{\tau}{2}\left[\dfrac{1}{h}\left(a(x_{n+1}) \dfrac{w_{n+1}^0 - w_n^0}{h} - a(x_n) \dfrac{w_n^0 - w_{n-1}^0}{h} \right) \right], \\ 1 \leqslant n \leqslant M-1, \\ w_0^{k+1} = w_M^{k+1} = 0, -1 \leqslant k \leqslant N-1. \end{cases} \tag{2.50}$$

Now, we will take an estimate for $|p_k|$ .Using the overdetermined condition $\sum_{i=1}^{M-1} u_i^{k+1} h = \zeta(t_{k+1})$ and substitution (2.48), one can obtain

$$\eta_k = \frac{\zeta_k - \sum_{i=1}^{M-1} w_i^k h}{\sum_{i=1}^{M-1} q_i h}. \tag{2.51}$$

Then, using the formulas $p_k = \frac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}$ and (2.51), we get

$$p_k = \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1} - \sum_{i=1}^{M-1}\left(w_i^{k+1} - 2w_i^k + w_i^{k-1}\right)h}{\tau^2 \sum_{i=1}^{M-1} q_i h}.$$

Then, applying the discrete analogue of the Cauchy–Schwarz inequality and the triangle inequality, we obtain

$$\begin{aligned} |p_k| &\leqslant \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|} \times \left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sum_{i=1}^{M-1}\left|\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right| h\right] \\ &\leqslant \frac{1}{\left|\sum_{i=1}^{M-1} q_i h\right|}\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \sqrt{l}\left\|\left\{\frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2}\right\}_{i=0}^{M}\right\|_{L_{2h}}\right] \\ &\leqslant M_{16}(q)\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right] \end{aligned} \tag{2.52}$$

for all $1 \leqslant k \leqslant N-1$. From that it follows

$$\begin{aligned} \left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \leqslant M_{16}(q) &\left[\left\|\left\{\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right. \\ &\left. + \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right]. \end{aligned} \tag{2.53}$$

Now, using substitution (2.48), we get

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + p_k q(x_n).$$

Applying the triangle inequality, we obtain

$$\begin{aligned} \left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} &\leqslant \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \\ &\quad + \left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\left\|\{q(x_n)\}_{n=0}^{M}\right\|_{L_{2h}}. \end{aligned} \tag{2.54}$$

Therefore, the proof of estimates (2.46) and (2.47) is based on equation (2.45), the triangle inequality, estimates (2.53), (2.54) and on the following stability estimate for the solution of difference problem (2.50):

$$\left\| \left\{ \frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} \leqslant M_{17}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} \right. \tag{2.55}$$

$$\left. + \left\| f_0^h \right\|_{L_{2h}} + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} . \right].$$

This completes the proof of Theorem 2.3.4.

Now, we will prove estimate (2.55) for the solution of difference problem (2.50). Firstly, it is easy to see that difference scheme (2.50) can be written as the abstract Cauchy difference problem

$$\begin{cases} \dfrac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2} + \frac{1}{4} A \left(w_{k+1} + 2w_k + w_{k-1} \right) = \theta_k, \\ \theta_k = f_k - \frac{1}{4} \left(\eta_{k+1} + 2\eta_k + \eta_{k-1} \right) Aq, \\ \theta_k = \theta(t_k), t_k = k\tau, 1 \leqslant k \leqslant N-1, N\tau = 1, \\ \left(I + \tau^2 A \right) \tau^{-1} \left(w_1 - w_0 \right) = \frac{\tau}{2} \left(\theta_0 - Aw_0 \right) + \psi, \\ \theta_0 = f(0) + p_0 q, w_0 = \varphi, \end{cases} \tag{2.56}$$

in a Hilbert space $H = L_{2h}$ with positive-definite self-adjoint operator A. Here,

$$\{w_k\}_{k=0}^N = \left\{ w_k^h \right\}_{k=0}^N, \{f_k\}_{k=1}^{N-1} = \left\{ f_k^h \right\}_{k=1}^{N-1}$$

are unknown and known abstract mesh functions defined on $[0,T]_\tau$ with values in $H = L_{2h}$, respectively, and $\varphi = \varphi^h, \psi = \psi^h, q = q^h$ are given elements.

Secondly, applying the approaches of (Ashyralyev and Sobolevskii, 2004)) we will prove a lemma that will be needed in the sequel.

Lemma 2.3.5 Assume that $\varphi \in D(A), \psi, q \in D(A^{\frac{1}{2}})$. Then, for the solution of difference problem (2.56), the following stability estimate holds for any $1 \leqslant k \leqslant N-1$:

$$\left\| \frac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2} \right\|_H \leqslant M(q) \left[\|A\varphi\|_H + \left\| A^{\frac{1}{2}} \psi \right\|_H + \|f_0\|_H + |\eta_1| \right. \tag{2.57}$$

$$\left. + \left\| \left\{ \frac{f_k - f_{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C(H)} + \sum_{s=1}^k \left| \frac{\eta_{s+1} - 2\eta_s + \eta_{s-1}}{\tau^2} \right| \tau \right]$$

for any $1 \leqslant k \leqslant N-1$.

Proof. It is clear that there exists a unique solution of this initial value problem, and for the solution of (2.56), the following formula is satisfied (see Ashyralyev and Sobolevskii, 2004)

$$w_0 = \varphi, w_1 = \left(I + \tau^2 A\right)^{-1} \left[\left(I + \frac{\tau^2}{2} A\right) \varphi + \tau\psi + \frac{\tau^2}{2}\theta_0\right] \tag{2.58}$$

$$\begin{aligned}
w_k = &\left[R^k + \frac{1}{2i} A^{-\frac{1}{2}} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) \left[R^k - \widetilde{R}^k\right]\right. \\
&\left.\times \left(\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) \frac{\tau}{2} A - iA^{\frac{1}{2}} \left(I + \tau^2 A\right)\right) \left(I + \tau^2 A\right)^{-1}\right] \varphi \\
&+ \frac{i}{2} A^{-\frac{1}{2}} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) \left[R^k - \widetilde{R}^k\right] \left(I + \tau^2 A\right)^{-1} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) \psi \\
&+ \frac{i}{2} A^{-\frac{1}{2}} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) \left[R^k - \widetilde{R}^k\right] \left(I + \tau^2 A\right)^{-1} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) \frac{\tau}{2}\theta_0 \\
&- \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{k-s} - \widetilde{R}^{k-s}\right] \theta_s, \\
&\qquad 2 \le k \le N.
\end{aligned}$$

Applying Abel's formula, we can write

$$\begin{aligned}
w_k = &\left\{R^k - \frac{1}{2} i A^{-\frac{1}{2}} B \left[R^k - \widetilde{R}^k\right] \left(\widetilde{B} \frac{\tau}{2} A - iA^{\frac{1}{2}} \left(I + \tau^2 A\right)\right) \left(I + \tau^2 A\right)^{-1}\right\} \varphi \\
&+ \left\{\frac{i}{2} A^{-\frac{1}{2}} B\widetilde{B} \left[R^k - \widetilde{R}^k\right] \left(I + \tau^2 A\right)^{-1}\right\} \psi \\
&+ \left\{\frac{i}{2} A^{-\frac{1}{2}} B\widetilde{B} \left[R^k - \widetilde{R}^k\right] \left(I + \tau^2 A\right)^{-1}\right\} \frac{\tau}{2}\theta_0 \\
&+ \frac{A^{-1}}{2} \left\{\sum_{s=2}^{k-1} \left[BR^{k-s} - \widetilde{B}\widetilde{R}^{k-s}\right] (\theta_{s-1} - \theta_s)\right\}, \\
&+ \frac{A^{-1}}{2} \left\{\left[B + \widetilde{B}\right] \theta_{k-1} - \left[BR^{k-1} - \widetilde{B}\widetilde{R}^{k-1}\right] \theta_1\right\}, \\
&\qquad 2 \le k \le N,
\end{aligned} \tag{2.59}$$

where

$$R = B\widetilde{B}^{-1} = \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}$$

$$\widetilde{R} = \widetilde{B}B^{-1} = \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}.$$

Using the spectral property of the self-adjoint positive-definite operator, we get

$$\|R\|_{H\to H} \leq 1, \left\|\widetilde{R}\right\|_{H\to H} \leq 1, \left\|\left(I \pm \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right\|_{H\to H} \leq 1, \tag{2.60}$$

$$\left\|\left(I \pm i\tau A^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H} \leq 1, \left\|\tau A^{\frac{1}{2}}\left(I \pm i\tau A^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H} \leq 1. \tag{2.61}$$

Now, we will establish estimates for $\left\|\frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2}\right\|_H, 1 \leq k \leq N-1$. Applying equation (2.56), formula (2.59) and identities

$$I + R = 2\widetilde{B}^{-1},\ I + \widetilde{R} = 2B^{-1}, \tag{2.62}$$

we get

$$\begin{aligned}
& A\frac{w_{k+1} + w_k + w_{k-1}}{4} \qquad (2.63)\\
= & \left\{B^{-1}R^k\widetilde{B}^{-1} - \frac{1}{4}i\tau A^{\frac{1}{2}}\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\right.\\
& \left.\times\left(I + \tau^2 A\right)^{-1} - \frac{1}{2}\left[\widetilde{B}^{-1}R^k - B^{-1}\widetilde{R}^{k-1}\right]\right\}A\varphi\\
& +\frac{1}{2}\left\{\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\left(I + \tau^2 A\right)^{-1}\right\}iA^{\frac{1}{2}}\psi\\
& +\frac{1}{4}\left\{i\tau A^{\frac{1}{2}}\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\left(I + \tau^2 A\right)^{-1}\right\}\theta_0\\
& +\frac{1}{2}\sum_{s=2}^{k-1}\left[\widetilde{B}^{-1}R^{k-s} + B^{-1}\widetilde{R}^{k-s}\right](\theta_{s-1} - \theta_s) + \frac{1}{4}\left[\tau^2 AB^{-1}\widetilde{B}^{-1}\right]\theta_k\\
& +\frac{1}{2}\left[B^{-1} + \widetilde{B}^{-1}\right]\theta_{k-1} - \frac{1}{2}\left[\widetilde{B}^{-1}R^{k-1} + B^{-1}\widetilde{R}^{k-1}\right]\theta_1\\
& = \sum_{i=1}^{4} J_i(k),
\end{aligned}$$

where

$$\begin{aligned}
J_1(k) = & \left\{B^{-1}R^k\widetilde{B}^{-1} - \frac{1}{4}i\tau A^{\frac{1}{2}}\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\left(I + \tau^2 A\right)^{-1}\right.\\
& \left.-\frac{1}{2}\left[\widetilde{B}^{-1}R^k - B^{-1}\widetilde{R}^{k-1}\right]\right\}A\varphi,\\
J_2(k) = & \frac{1}{2}\left\{\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\left(I + \tau^2 A\right)^{-1}\right\}iA^{\frac{1}{2}}\psi,\\
J_3(k) = & \frac{1}{4}\left\{i\tau A^{\frac{1}{2}}\left[BR^{k-1}\widetilde{B}^{-1} - B^{-1}\widetilde{R}^{k-1}\widetilde{B}\right]\left(I + \tau^2 A\right)^{-1}\right\}\theta_0
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\left[\tau^2 AB^{-1}\widetilde{B}^{-1}\right]\theta_k+\frac{1}{2}\left[B^{-1}+\widetilde{B}^{-1}\right]\theta_{k-1}\\
&-\frac{1}{2}\left[\widetilde{B}^{-1}R^{k-1}+B^{-1}\widetilde{R}^{k-1}\right]\theta_1,\\
J_4(k)&=\frac{1}{2}\sum_{s=2}^{k-1}\left[\widetilde{B}^{-1}R^{k-s}+B^{-1}\widetilde{R}^{k-s}\right](\theta_{s-1}-\theta_s).
\end{aligned}
$$

Now, applying the triangle inequality, we obtain

$$
\left\|A\frac{w_{k+1}+w_k+w_{k-1}}{4}\right\|_H \leqslant \sum_{i=1}^{4}\|J_i(k)\|_H \tag{2.64}
$$

for any $1 \le k \le N-1$. Therefore, we will estimate $\|J_i(k)\|_H, i = 1,2,3,4$, separately. Firstly, using estimates (2.60), (2.61), we get

$$
\begin{aligned}
\|J_1(k)\|_H &\leqslant \left\|B^{-1}R^k\widetilde{B}^{-1}\right\|_{H\to H}+\frac{1}{4}\left[\left\|i\tau A^{\frac{1}{2}}BR^{k-1}\widetilde{B}^{-1}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\right.\\
&\left.+\left\|i\tau A^{\frac{1}{2}}B^{-1}\widetilde{R}^{k-1}\widetilde{B}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\right]\\
&+\frac{1}{2}\left[\left\|\widetilde{B}^{-1}R^k\right\|_{H\to H}+\left\|B^{-1}\widetilde{R}^{k-1}\right\|_{H\to H}\right]\|A\varphi\|_H\\
&\le M\|A\varphi\|_H,\\
\|J_2(k)\|_H &\leqslant \frac{1}{2}\left[\left\|BR^{k-1}\widetilde{B}^{-1}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\right.\\
&\left.+\left\|B^{-1}\widetilde{R}^{k-1}\widetilde{B}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\right]\\
&\le M\left\|A^{\frac{1}{2}}\psi\right\|_H
\end{aligned}
$$

for any $1 \le k \le N-1$. Secondly, using the triangle inequality and estimates (2.60), (2.61), we get

$$
\begin{aligned}
\|J_3(k)\|_H &\leqslant \frac{1}{4}\left[\left\|BR^{k-1}\widetilde{B}^{-1}\tau A^{\frac{1}{2}}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\right.\\
&\left.+\left\|B^{-1}\widetilde{R}^{k-1}\widetilde{B}\tau A^{\frac{1}{2}}\left(I+\tau^2A\right)^{-1}\right\|_{H\to H}\right]\|\theta_0\|_H\\
&+\frac{1}{4}\left[\left\|\tau^2AB^{-1}\widetilde{B}^{-1}\right\|_{H\to H}\|\theta_k\|_H\right]\\
&+\frac{1}{2}\left[\left\|B^{-1}\right\|_{H\to H}+\left\|\widetilde{B}^{-1}\right\|_{H\to H}\right]\|\theta_{k-1}\|_H\\
&+\frac{1}{2}\left[\left\|\widetilde{B}^{-1}R^{k-1}\right\|_{H\to H}+\left\|B^{-1}\widetilde{R}^{k-1}\right\|_{H\to H}\right]\|\theta_1\|_H
\end{aligned}
$$

$$\leq \frac{11}{4}\left[\|\theta_0\|_H + \sum_{i=1}^{k}\|\theta_i - \theta_{i-1}\|_H\right],$$

$$\|J_4(k)\|_H \leqslant \sum_{s=2}^{k-1}\frac{1}{2}\left[\left\|\widetilde{B}^{-1}R^{k-s}\right\|_{H\to H} + \left\|B^{-1}\widetilde{R}^{k-s}\right\|_{H\to H}\right] \|\theta_{s-1} - \theta_s\|_H$$

$$\leqslant \sum_{s=2}^{k-1}\|\theta_{s-1} - \theta_s\|_H$$

for any $1 \leq k \leq N-1$. Using the triangle inequality and estimate (2.64), we get

$$A\left\|\frac{w_{k+1} + 2w_k + w_{k-1}}{4}\right\|_H \leq M\|A\varphi\|_H + M\left\|A^{\frac{1}{2}}\psi\right\|_H + \sum_{s=2}^{k-1}\|\theta_{s-1} - \theta_s\|_H \qquad (2.65)$$

$$+\frac{11}{4}\left[\|\theta_0\|_H + \sum_{i=1}^{k}\|\theta_i - \theta_{i-1}\|_H\right]$$

for any $1 \leq k \leq N-1$.Using the triangle inequality and estimate, we get

$$\left\|\frac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2}\right\|_H \leq A\left\|\frac{w_{k+1} + 2w_k + w_{k-1}}{4}\right\|_H + \|\theta_k\|_H$$

$$\leq M\|A\varphi\|_H + M\left\|A^{\frac{1}{2}}\psi\right\|_H + \sum_{s=2}^{k-1}\|\theta_{s-1} - \theta_s\|_H$$

$$+\frac{15}{4}\left[\|\theta_0\|_H + \sum_{i=1}^{k}\|\theta_i - \theta_{i-1}\|_H\right]$$

for any $1 \leq k \leq N-1$. Combining these estimates, we obtain estimate (2.57) for the solution of difference problem (2.56) for any $1 \leq k \leq N-1$.

Theorem 2.3.6 For the solution of difference problem (2.50), the stability estimate (2.56) holds.

Proof. Putting $H = L_{2h}, \varphi = \varphi^h, \psi = \psi^h, q = q^h, A\varphi = A_h\varphi^h, w_k = w_k^h, f_k = f_k^h$ and applying estimates (2.52) and (2.57), we get

$$\left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}} \leqslant M_{20}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right]$$

$$+\left\|f_0^h\right\|_{L_{2h}} + T\left\|\left\{\frac{f_k^h - f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + M_{21}(q)M_{16}(q)$$

$$\times\sum_{s=1}^{k}\left\{\left[\left|\frac{\zeta_{s+1} - 2\zeta_s + \zeta_{s-1}}{\tau^2}\right| + \left\|\frac{w_{s+1}^h - 2w_s^h + w_{s-1}^h}{\tau^2}\right\|_{L_{2h}}\right]\right\}\tau$$

for any $1 \leq k \leq N-1$. By the difference analogue of Gronwall's inequality, we conclude that

$$\left\| \frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2} \right\|_{L_{2h}} \leqslant \frac{1}{1 - M_{21}(q) M_{16}(q) \tau} \left\{ M_{20}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} \right] \right.$$

$$+ \left\| f_0^h \right\|_{L_{2h}} + T \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + M_{21}(q) M_{16}(q) T$$

$$\left. \times \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right\} e^{(k-1)\tau \frac{M_{21}(q) M_{16}(q)}{1 - M_{21}(q) M_{16}(q) \tau}}$$

for any $1 \leq k \leq N-1$.This completes the proof of Theorem 2.3.6.

Applying same approach with this work we can study another the second order of accuracy difference scheme for the numerical solution of problem (2.1).

$$\left\{ \begin{array}{l} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \dfrac{1}{2h} \left(a(x_{n+1}) \dfrac{u_{n+1}^{k+1} - u_n^{k+1} + u_{n+1}^{k-1} - u_n^{k-1}}{h} \right. \\ \left. -a(x_n) \dfrac{u_n^{k+1} - u_{n-1}^{k+1} + u_n^{k-1} - u_{n-1}^{k-1}}{h} \right) \\ = p_k q_n + f(t_k, x_n), q_n = q(x_n), x_n = nh, N\tau = T, \\ 1 \leqslant n \leqslant M-1, Mh = l, t_k = k\tau, 1 \leqslant k \leqslant N-1, \\ u_n^0 = \varphi(x_n), 0 \leqslant n \leqslant M, \\ \dfrac{u_n^1 - u_n^0}{\tau} - \dfrac{\tau}{h} \left(a(x_{n+1}) \dfrac{u_{n+1}^1 - u_{n+1}^0 - u_n^1 + u_n^0}{h} \right. \\ \left. -a(x_n) \dfrac{u_n^1 - u_n^0 - u_{n-1}^1 + u_{n-1}^0}{h} \right) + \psi(x_n) + \dfrac{\tau}{2} f(0, x_n) \\ +\dfrac{\tau}{2} \left[\dfrac{1}{h} \left(a(x_{n+1}) \dfrac{u_{n+1}^0 - u_n^0}{h} - a(x_n) \dfrac{u_n^0 - u_{n-1}^0}{h} \right) + p_0 q_n \right], \\ 1 \leqslant n \leqslant M-1, \\ u_0^{k+1} = u_M^{k+1} = 0, \sum_{i=1}^{M-1} u_i^{k+1} h = \zeta(t_{k+1}), -1 \leqslant k \leqslant N-1 \end{array} \right. \tag{2.66}$$

Here, it is assumed that $q_M = q_0 = 0$, and $\sum_{i=1}^{M-1} q_i \neq 0$. We have the following theorem on the stability of the difference scheme (2.66):

Theorem 2.3.7 For the solution of difference scheme (2.66), the following stability estimates hold:

$$\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{u_{k+1}^h + u_{k-1}^h}{2} \right\}_{k=1}^{N-1} \right\|_{C_\tau\left(W_{2h}^2\right)}$$

$$\leqslant M_{14}(q)\left[\left\|\varphi^{h}\right\|_{W_{2h}^{2}}+\left\|\psi^{h}\right\|_{W_{2h}^{1}}+\left\|f_{0}^{h}\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{\frac{f_{k}^{h}-f_{k-1}^{h}}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_{\tau}(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_{k}+\zeta_{k-1}}{\tau^{2}}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_{\tau}}\right],$$

$$\left\|\{p_{k}\}_{k=1}^{N-1}\right\|_{C[0,T]_{\tau}}\leqslant M_{15}(q)\left[\left\|\varphi^{h}\right\|_{W_{2h}^{2}}+\left\|\psi^{h}\right\|_{W_{2h}^{2}}+\left\|f_{0}^{h}\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{\frac{f_{k}^{h}-f_{k-1}^{h}}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_{\tau}(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_{k}+\zeta_{k-1}}{\tau^{2}}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_{\tau}}\right].$$

Now, we consider one more the second order of accuracy difference scheme

$$\begin{cases}
\dfrac{u_{n}^{k+1}-2u_{n}^{k}+u_{n}^{k-1}}{\tau^{2}}-\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k}-u_{n}^{k}}{h}-a\left(x_{n}\right)\dfrac{u_{n}^{k}-u_{n-1}^{k}}{h}\right)\\
+\dfrac{\tau^{2}}{4}\left\{\dfrac{1}{h}a\left(x_{n+1}\right)\dfrac{1}{h}\left[\dfrac{1}{h}\left(a\left(x_{n+2}\right)\dfrac{u_{n+2}^{k+1}-u_{n+1}^{k+1}}{h}-a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k+1}-u_{n}^{k+1}}{h}\right)\right.\right.\\
\left.-\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k}-u_{n}^{k}}{h}-a\left(x_{n}\right)\dfrac{u_{n}^{k}-u_{n-1}^{k}}{h}\right)\right]\\
-\dfrac{1}{h}a\left(x_{n}\right)\left[\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k}-u_{n}^{k}}{h}-a\left(x_{n-}\right)\dfrac{u_{n}^{k}-u_{n-1}^{k}}{h}\right)\right.\\
\left.\left.-\dfrac{1}{h}\left(a\left(x_{n}\right)\dfrac{u_{n}^{k+1}-u_{n-2}^{k+1}}{h}-a\left(x_{n-1}\right)\dfrac{u_{n-1}^{k+1}-u_{n-2}^{k+1}}{h}\right)\right]\right\}\\
=p_{k}q_{n}+f\left(t_{k},x_{n}\right),q_{n}=q\left(x_{n}\right),x_{n}=nh,N\tau=T,\\
1\leqslant n\leqslant M-1,Mh=l,t_{k}=k\tau,1\leqslant k\leqslant N-1,\\
u_{n}^{0}=\varphi\left(x_{n}\right),0\leqslant n\leqslant M,\\
\dfrac{u_{n}^{1}-u_{n}^{0}}{\tau}-\dfrac{\tau}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{1}-u_{n+1}^{0}-u_{n}^{1}+u_{n}^{0}}{h}\right.\\
\left.-a\left(x_{n}\right)\dfrac{u_{n}^{1}-u_{n}^{0}-u_{n-1}^{1}+u_{n-1}^{0}}{h}\right)+\psi\left(x_{n}\right)+\dfrac{\tau}{2}f\left(0,x_{n}\right)\\
+\dfrac{\tau}{2}\left[\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{0}-u_{n}^{0}}{h}-a\left(x_{n}\right)\dfrac{u_{n}^{0}-u_{n-1}^{0}}{h}\right)+p_{0}q_{n}\right],\\
1\leqslant n\leqslant M-1,\\
u_{0}^{k+1}=u_{M}^{k+1}=0,\sum_{i=1}^{M-1}u_{i}^{k+1}h=\zeta\left(t_{k+1}\right),-1\leqslant k\leqslant N-1
\end{cases}\tag{2.67}$$

Here, it is assumed that $q_M = q_0 = 0$, and $\sum_{i=1}^{M-1} q_i \neq 0$. We have the following theorem on the stability of the difference scheme (2.45):

Theorem 2.3.8 For the solution of difference scheme (2.67), the following stability estimates hold:

$$\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ u_k^h \right\}_{k=1}^{N-1} \right\|_{C_\tau(W_{2h}^2)}$$

$$\leqslant M_{14}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} + \left\| f_0^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right],$$

$$\left\| \{p_k\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \leqslant M_{15}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| f_0^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right].$$

2.4 Numerical Experiments

In this section, we study the numerical solution of the identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = p(t) \sin x + e^{-t} \sin x, \\ x \in (0,\pi), t \in (0,1), \\ u(0,x) = \sin x, u_t(0,x) = -\sin x, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, t \in [0,1], \\ \int_0^\pi u(t,x)\, dx = 2e^{-t}, t \in [0,1] \end{cases} \tag{2.68}$$

for a hyperbolic differential equation. The exact solution pair of this problem is $(u(t,x), p(t)) = \left(e^{-t} \sin x, e^{-t}\right)$.

Firstly, for the numerical solution of problem (2.68), we present the following first order of accuracy difference scheme for the approximate solution for problem (2.68):

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\
=p_k\sin x_n+e^{-t_{k+1}}\sin x_n, \\
t_k=k\tau, x_n=nh, 1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1, \\
u_n^0=\sin x_n, \dfrac{u_n^1-u_n^0}{\tau}=-\sin x_n, 0\leqslant n\leqslant M, Mh=\pi, N\tau=1, \\
u_0^{k+1}=u_M^{k+1}=0, \sum_{i=1}^{M-1}u_i^{k+1}h=2e^{-t_{k+1}}, -1\leqslant k\leqslant N-1.
\end{cases} \tag{2.69}
$$

The algorithm for obtaining the solution of identification problem (2.69) contains three stages. Actually, let us define

$$
u_n^k=w_n^k+\eta_k\sin x_n, 0\leq k\leq N, 0\leq n\leq M, \tag{2.70}
$$

Applying difference scheme (2.69) and formula (2.70), we will obtain

$$
\eta_{k+1}=\frac{2e^{-t_{k+1}}-\sum_{i=1}^{M-1}w_i^{k+1}h}{\sum_{i=1}^{M-1}\sin x_i h}, -1\leq k\leq N-1 \tag{2.71}
$$

and

$$
\begin{cases}
\dfrac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\dfrac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2} \\
+\dfrac{\sum_{i=1}^{M-1}w_i^{k+1}h}{\sum_{i=1}^{M-1}\sin x_i h}\sin x_n\dfrac{2(\cos h-1)}{h^2} \\
=\left[\dfrac{2}{\sum_{i=1}^{M-1}\sin x_i h}\dfrac{2(\cos h-1)}{h^2}+1\right]e^{-t_{k+1}}\sin x_n, \\
1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1, \\
w_n^0=\sin x_n, \dfrac{w_n^1-w_n^0}{\tau}=-\sin x_n, 0\leqslant n\leqslant M, \\
w_0^{k+1}=w_M^{k+1}=0, -1\leqslant k\leqslant N-1.
\end{cases} \tag{2.72}
$$

In the first stage, we find the solution $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of the corresponding first order of accuracy difference scheme (2.72). For obtaining it, we will write difference scheme (2.72) in matrix form as

$$
\begin{cases}
Aw^{k+1}+Bw^k+Cw^{k-1}=\varphi^k, 1\leqslant k\leqslant N-1, \\
w^0=\{\sin x_n\}_{n=0}^M, w^1=(1-\tau)\{\sin x_n\}_{n=0}^M,
\end{cases} \tag{2.73}
$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $w^s, s = k, k\pm 1, \varphi^k$ are $(M+1)\times 1$ column matrices, and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} \quad C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & g & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & g & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & g & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & g & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} \quad w^s = \begin{bmatrix} 0 \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1}, \text{ for } s = k, k\pm 1.$$

Here,

$$a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2},$$
$$d = \sum_{i=1}^{M-1} \sin x_i h, c_n = \sin x_n \frac{2(\cos h - 1)}{dh}, 1 \leqslant n \leqslant M-1,$$

$$\varphi_n^k = \left[\frac{4(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} + 1\right] e^{-t_{k+1}} \sin x_n, 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1.$$

So, we have the initial value problem for the second-order difference equation (2.73) with respect to k with the matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain $\left\{\left\{w_n^k\right\}_{k=0}^{N}\right\}_{n=0}^{M}$ by (2.73).

Now, applying formula (2.31), we can obtain

$$p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}, 1 \leq k \leq N-1. \tag{2.74}$$

In the second stage, we will obtain $\{p_k\}_{k=1}^{N-1}$ by formulas (2.71) and (2.74). Finally, in the third stage, we will obtain $\left\{\left\{u_n^k\right\}_{k=0}^{N}\right\}_{n=0}^{M}$ by formulas (2.70) and (2.71). The errors are computed by

$$E_u = \max_{0 \leqslant k \leqslant N} \left(\sum_{n=1}^{M-1} \left| u\left(t_k, x_n\right) - u_n^k \right|^2 h \right)^{\frac{1}{2}},$$

$$E_p = \max_{1 \leqslant k \leqslant N-1} \left| p\left(t_k\right) - p_k \right|,$$

where $u\left(t, x\right), p(t)$ represent the exact solution, u_n^k represents the numerical solutions at $\left(t_k, x_n\right)$ and p_k represents the numerical solutions at t_k. The numerical results are given in Table 1.

Table 1: Error analysis for difference scheme (2.69)

Error	**N = M = 20**	**N = M = 40**	**N = M = 80**	**N = M = 160**
E_u	0.0347	0.0181	0.0092	0.0047
E_p	0.0462	0.0240	0.0123	0.0062

As can be seen in Table 1, if N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $\frac{1}{2}$ for the first order difference scheme (2.69).

Secondly, for the numerical solution of problem (2.68), we present the following second order of accuracy difference schemes for the approximate solution for problem (2.68):

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{1}{2}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\ -\dfrac{1}{4}\left[\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right] \\ =p_k\sin x_n+e^{-t_k}\sin x_n, \\ t_k=k\tau, 1\leqslant k\leqslant N-1, N\tau=1, x_n=nh, 1\leqslant n\leqslant M-1, Mh=\pi, \\ u_n^0=\sin x_n, \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{h^2}\left(u_{n+1}^1-u_{n+1}^0-2u_n^1+2u_n^0+u_{n-1}^1-u_{n-1}^0\right) \\ =-\sin x_n+\dfrac{\tau}{2}\left(\dfrac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}+p_0\sin x_n+\sin x_n\right), 0\leqslant n\leqslant M, \\ u_0^{k+1}=u_M^{k+1}=0, \sum_{i=1}^{M-1}u_i^{k+1}h=2e^{-t_{k+1}}, -1\leqslant k\leqslant N-1 \end{cases} \tag{2.75}$$

and

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{1}{2}\left[\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right] \\ =p_k\sin x_n+e^{-t_k}\sin x_n, \\ t_k=k\tau, 1\leqslant k\leqslant N-1, N\tau=1, x_n=nh, 1\leqslant n\leqslant M-1, Mh=\pi, \\ u_n^0=\sin x_n, \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{h^2}\left(u_{n+1}^1-u_{n+1}^0-2u_n^1+2u_n^0+u_{n-1}^1-u_{n-1}^0\right) \\ =-\sin x_n+\dfrac{\tau}{2}\left(\dfrac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}+p_0\sin x_n+\sin x_n\right), 0\leqslant n\leqslant M, \\ u_0^{k+1}=u_M^{k+1}=0, \sum_{i=1}^{M-1}u_i^{k+1}h=2e^{-t_{k+1}}, -1\leqslant k\leqslant N-1 \end{cases} \tag{2.76}$$

The algorithm for obtaining the solution of identification problem (2.75) contains three stages. Actually, let us define

$$u_n^k=w_n^k+\eta_k\sin x_n, 0\leq k\leq N, 0\leq n\leq M, \tag{2.77}$$

Applying the second order of accuracy difference scheme(2.75) and formula (2.77), we will obtain

$$\eta_k=\frac{2e^{-t_k}-\sum_{i=1}^{M-1}w_i^k h}{\sum_{i=1}^{M-1}\sin x_i h}, 0\leq k\leq N \tag{2.78}$$

and the difference scheme

$$
\begin{cases}
\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\frac{1}{2}\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}\\
-\frac{1}{4}\left[\frac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2}+\frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{h^2}\right]\\
+\frac{1}{2}\frac{(\cos h-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\sin x_n\\
\times\left(\sum_{i=1}^{M-1}w_i^{k+1}h+2\sum_{i=1}^{M-1}w_i^k h+\sum_{i=1}^{M-1}w_i^{k-1}h\right)\\
=e^{-t_k}\sin x_n+\left[e^{-t_{k+1}}+2e^{-t_k}+e^{-t_{k-1}}\right]\frac{(\cos h-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\sin x_n\\
1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1,\\
w_n^0=\sin x_n, 0\leqslant n\leqslant M,\\
\frac{w_n^1-w_n^0}{\tau}-\frac{\tau}{h^2}\left(w_{n+1}^1-2w_n^1+w_{n-1}^1\right)\\
+\frac{2\tau(\cosh-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\sin x_n\sum_{i=1}^{M-1}w_i^1 h\\
+\frac{\tau}{2h^2}\left(w_{n+1}^0-2w_n^0+w_{n-1}^0\right)=\left(\frac{\tau}{2}-1\right)\sin x_n\\
+\frac{4\tau e^{-t_1}(\cosh-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\sin x_n, 0\leqslant n\leqslant M,\\
w_0^{k+1}=w_M^{k+1}=0, -1\leqslant k\leqslant N-1.
\end{cases}
\tag{2.79}
$$

In the first stage, we find the solution $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of the corresponding second order of accuracy difference scheme (2.79). For obtaining it, we will write difference scheme (2.79), in matrix form as

$$
\begin{cases}
Aw^{k+1}+Bw^k+Cw^{k-1}=\varphi^k, 1\leqslant k\leqslant N-1,\\
w^0, w^1 \text{ are given},
\end{cases}
\tag{2.80}
$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $w^s, s=k, k\pm1, \varphi^k$ are $(M+1)\times 1$ column matrices and

$$
A=\begin{bmatrix}
1 & 0 & 0 & \cdot & 0 & 0 & 0\\
b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & c_1\\
0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & c_2\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0\\
0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b\\
0 & 0 & 0 & \cdot & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ r & q+e_1 & r+e_1 & \cdot & e_1 & e_1 & e_1 \\ 0 & r+e_2 & q+e_2 & \cdot & e_2 & e_2 & e_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & e_{M-2} & e_{M-2} & \cdot & q+e_{M-2} & r+e_{M-2} & 0 \\ 0 & e_{M-1} & e_{M-1} & \cdot & r+e_{M-1} & q+e_{M-1} & r \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, \quad w^s = \begin{bmatrix} 0 \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1,$$

$$w^0 = \begin{bmatrix} \sin x_0 \\ \sin x_1 \\ \cdot \\ \sin x_{M-1} \\ \sin x_M \end{bmatrix},$$

where,

$$a = \frac{1}{\tau^2} + \frac{1}{2h^2}, b = -\frac{1}{4h^2}, r = -\frac{1}{2h^2}, q = \frac{1}{h^2} - \frac{2}{\tau^2},$$

$$c_n = \frac{1}{2} \sin x_n \frac{(\cos h - 1)}{dh}, e_n = \sin x_n \frac{(\cos h - 1)}{dh},$$

$$d = \sum_{i=1}^{M-1} \sin x_i h, 1 \leqslant n \leqslant M - 1,$$

$$\varphi_n^k = e^{-t_k} \sin x_n + \left(e^{-t_{k+1}} + 2e^{-t_k} + e^{-t_{k-1}}\right) \frac{(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n,$$

$$1 \leqslant k \leqslant N - 1, 1 \leqslant n \leqslant M - 1.$$

Finally, we obtain w^1.Applying the formula

$$\frac{w_n^1 - w_n^0}{\tau} - \frac{\tau}{h^2}\left(w_{n+1}^1 - 2w_n^1 + w_{n-1}^1\right) + \frac{2\tau(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n \sum_{i=1}^{M-1} w_i^1 h$$

$$+\frac{\tau}{2h^2}\left(w_{n+1}^0 - 2w_n^0 + w_{n-1}^0\right) = \left(\frac{\tau}{2} - 1\right) \sin x_n + \frac{4\tau e^{-t_1}(\cos h - 1)}{h^2 \sum_{i=1}^{M-1} \sin x_i h} \sin x_n,$$

and conditions $w_0^1 = w_M^1 = 0$, we get

$$Ew^1 + Fw^0 = \gamma. \tag{2.81}$$

Here

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ y & x + z_1 & y + z_1 & \cdot & z_1 & z_1 & 0 \\ 0 & y + z_2 & x + z_2 & \cdot & z_2 & z_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & z_{M-2} & z_{M-2} & \cdot & x + z_{M-2} & y + z_{M-2} & 0 \\ 0 & z_{M-1} & z_{M-1} & \cdot & y + z_{M-1} & x + z_{M-1} & y \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$x=-\frac{2\tau}{h^2}+\frac{1}{\tau}, y=-\frac{\tau}{h^2}, z_n=\frac{2\tau(\cos h-1)}{dh}\sin x_n,$$
$$u=-\frac{\tau}{2h^2}, v=\frac{1}{\tau}+\frac{\tau}{h^2}, v=-\frac{3}{2\tau}-\frac{2\tau}{h^2}+1,$$

$$F=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ u & v & u & \cdot & 0 & 0 & 0 \\ 0 & u & v & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & v & u & 0 \\ 0 & 0 & 0 & \cdot & u & v & u \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\gamma=\begin{bmatrix} 0 \\ \nabla_1 \\ \cdot \\ \nabla_{M-1} \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\nabla_n=\left(\frac{\tau}{2}-1+\frac{4\tau e^{-t_1}(\cos h-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\right)\sin x_n.$$

From that, it follows

$$w^1=E^{-1}\left(\gamma-Fw^0\right). \tag{2.82}$$

So, we have the initial value problem for the second-order difference equation (2.80) with respect to k with the matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain $\left\{\{w_n^k\}_{k=0}^N\right\}_{n=0}^M$ by (2.80)

Now, applying formula (2.49), we can obtain

$$p_k=\frac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}, 1\le k\le N-1, p_0=\frac{2}{\tau^2}\eta_1 \tag{2.83}$$

In the second stage, we will obtain $\{p_k\}_{k=1}^{N-1}$ by formulas (2.78) and (2.83). Finally, in the third stage, we will obtain $\left\{\{u_n^k\}_{k=0}^N\right\}_{n=0}^M$ by formulas (2.77) and (2.78).

In the same manner, we obtain algorithm for obtaining the solution of identification problem (2.76) $\{u_k\}_{k=0}^{N} = \left\{\{u_n^k\}_{k=0}^{N}\right\}_{n=0}^{M}$ and $\{p_k\}_{k=1}^{N-1}$ by three stages. The errors are computed by

$$E_u = \max_{0 \leqslant k \leqslant N} \left(\sum_{n=1}^{M-1} \left| u\left(t_k, x_n\right) - u_n^k \right|^2 h \right)^{\frac{1}{2}},$$

$$E_p = \max_{1 \leqslant k \leqslant N-1} \left| p\left(t_k\right) - p_k \right|,$$

where $u\left(t,x\right), p(t)$ represent the exact solution, u_n^k represents the numerical solutions at (t_k, x_n), and p_k represents the numerical solutions at t_k. The numerical results are given in the following Tables 2 and 3.

Table 2: Error analysis for difference scheme (2.75)

Error	**N = M = 20**	**N = M = 40**	**N = M = 80**	**N = M = 160**
E_u	0.0016	$4.2029e-04$	$1.0648e-04$	$2.6796e-05$
E_p	0.0027	$7.0457e-04$	$1.7835e-04$	$4.4867e-05$

Table 3: Error analysis for difference scheme (2.76)

Error	**N = M = 20**	**N = M = 40**	**N = M = 80**	**N = M = 160**
E_u	0.0016	$4.2011e-04$	$1.0647e-04$	$2.6795e-05$
E_p	0.0033	$8.5650e-04$	$2.1690e-04$	$5.4570e-05$

As can be seen in Tables 2 and 3, if N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $\frac{1}{4}$ for the second order difference schemes (2.75) and (2.76), respectively.

CHAPTER 3

STABILITY OF THE HYPERBOLIC DIFFERENTIAL AND DIFFERENCE EQUATION WITH NONLOCAL CONDITIONS

3.1 Introduction

In this chapter, we consider the source identification problem for a one-dimensional hyperbolic equation with nonlocal conditions

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) + \delta u(t,x) \\ = p(t)q(x) + f(t,x), x \in (0,l), t \in (0,T), \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in [0,l], \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), \\ \int_0^l u(t,x)\,dx = \zeta(t), t \in [0,T], \end{cases} \tag{3.1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \geq a > 0, a(l) = a(0), \delta > 0$, $f(t,x), \zeta(t), \varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions and $q(x)$ is a given sufficiently smooth function assuming $q(0) = q(l)$, $q'(0) = q'(l)$ and $\int_0^l q(x)\,dx \neq 0$.

3.2 Stability of the Differential Problem (3.1)

To formulate our results, we introduce the differential operator A defined by the formula

$$Au = -\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x) \tag{3.2}$$

with the domain

$$D(A) = \{u : u, u'' \in L_2[0,l], u(0) = u(l), u'(0) = u'(l)\}.$$

It is easy that A is the self-adjoint positive-definite operator in $H = L_2[0,l]$. Actually, for all $u, v \in L_2[0,l]$ we have that

$$\begin{aligned} \langle Au, v\rangle &= \int_0^l A(u)v(x)\,dx = \int_0^l \left[-\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x)\right] v(x)\,dx \\ &= -\left(a(x)\frac{du(x)}{dx}\right)v(x)\Big|_0^l + \int_0^l a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx \\ &+\delta\int_0^l u(x)v(x)\,dx \end{aligned}$$

$$= -a(l)\frac{du(l)}{dx}v(l) + a(0)\frac{du(0)}{dx}v(0) + \int_0^l a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$
$$+\delta\int_0^l u(x)v(x)\,dx$$

Since

$$a(l) = a(0), u(0) = u(l), u'(0) = u'(l), \tag{3.3}$$

we have that

$$-a(l)\frac{du(l)}{dx}v(l) + a(0)\frac{du(0)}{dx}v(0) = 0.$$

Therefore,

$$\langle Au, v\rangle = \int_0^l a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \delta\int_0^l u(x)v(x)\,dx.$$

Using (3.3), we get

$$\begin{aligned}\langle u, Av\rangle &= \int_0^l u(x)A(v)dx = \int_0^l u(x)\left[-\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right) + \delta v(x)\right]dx\\ &= -\,u(x)\left(a(x)\frac{dv(x)}{dx}\right)\Big|_0^l + \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx\\ &\quad +\delta\int_0^l v(x)u(x)\,dx\\ &= -u(l)a(l)\frac{dv(l)}{dx} + u(0)a(0)\frac{dv(0)}{dx} + \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx\\ &= \int_0^l a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx + \delta\int_0^l v(x)u(x)\,dx.\end{aligned}$$

From that it follows

$$\langle Au, v\rangle = \langle u, Av\rangle$$

and

$$\begin{aligned}\langle Au, u\rangle &= \int_0^l a(x)\frac{du(x)}{dx}\frac{du(x)}{dx}dx + \delta\int_0^l u(x)u(x)\,dx \qquad (3.4)\\ &\geq \delta\int_0^l u(x)u(x)\,dx\\ &= \delta\langle u, u\rangle.\end{aligned}$$

We have the following theorem on the stability of problem (3.1).

Theorem 3.2.1 Assume that $\varphi \in W_2^2[0,l], \psi \in W_2^1[0,l]$ and $f(t,x)$ is a continuously differentiable function in t and square integrable in x, and $\zeta(t)$ is a twice continuously differentiable function. Suppose that $q(x)$ is a sufficiently smooth function assuming $q(0) = q(l)$, $q'(0) = q'(l)$ and $\int_0^l q(x)\,dx \neq 0$. Then, for the solution of problem (3.1), the following stability estimates hold:

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{C(L_2[0,l])} + \|u\|_{C(W_2^2[0,l])} \leqslant K_1(q)\left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}\right. \tag{3.5}$$

$$\left. + \|f(0,.)\|_{L_2[0,l]} + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])} + \|\zeta''\|_{C[0,T]}\right],$$

$$\|p\|_{C[0,T]} \leqslant K_2(q)\left[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]} + \|\zeta''\|_{C[0,T]}\right. \tag{3.6}$$

$$\left. + \|f(0,.)\|_{L_2[0,l]} + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])}\right].$$

Proof. We will use the following substitution

$$u(t,x) = w(t,x) + \eta(t)\,q(x), \tag{3.7}$$

where $\eta(t)$ is the function defined by formula

$$\eta(t) = \int_0^t (t-s)\,p(s)\,ds, \eta(0) = \eta'(0) = 0. \tag{3.8}$$

It is easy to see that $w(t,x)$ is the solution of problem

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w(t,x)}{\partial x}\right) + \delta w(t,x) \\ = f(t,x) + \eta(t)\left[\dfrac{d}{dx}\left(a(x)\,q'(x)\right) - \delta q(x)\right], \\ x \in (0,l), t \in (0,T), \\ w(0,x) = \varphi(x), w_t(0,x) = \psi(x), x \in [0,l], \\ w(t,0) = w(t,l), w_x(t,0) = w_x(t,l), t \in [0,T]. \end{cases} \tag{3.9}$$

Applying the integral overdetermined condition $\int_0^l u(t,x)\,dx = \zeta(t)$ and substitution (3.7), we get

$$\eta(t) = \frac{\zeta(t) - \int_0^l w(t,x)\,dx}{\int_0^l q(x)\,dx}.$$

From that and $p(t) = \eta''(t)$ it follows

$$p(t) = \frac{\zeta''(t) - \int_0^l \frac{\partial^2}{\partial t^2} w(t,x)\,dx}{\int_0^l q(x)\,dx}.$$

Applying $\int_0^l q(x)\,dx \neq 0$, we get estimate

$$|p(t)| \leq K_3(q)\left[|\zeta''(t)| + \left\|\frac{\partial^2 w(t,.)}{\partial t^2}\right\|_{L_2[0,l]}\right] \tag{3.10}$$

for all $t \in [0,T]$. From that it follows

$$\|p\|_{C[0,T]} \leqslant K_3(q)\left[\|\zeta''\|_{C[0,T]} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])}\right]. \tag{3.11}$$

Now, using substitution (3.7), we get

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 w(t,x)}{\partial t^2} + p(t)\,q(x).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{C(L_2[0,l])} \leqslant \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])} + \|p\|_{C[0,T]}\,\|q\|_{L_2[0,l]}. \tag{3.12}$$

Therefore, the proof of estimates (3.5) and (3.6) is based on equation (3.1), the triangle inequality, estimates (3.11), (3.12) and on the following stability estimate

$$\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{C(L_2[0,l])} \leqslant K_4(q,a)\Big[\|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]} \tag{3.13}$$

$$+ \|f(0,.)\|_{L_2[0,l]} + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])} + \|\zeta''\|_{C[0,T]}\Big],$$

for the solution of problem (3.9). It was proved in Section 2.2 for the identification hyperbolic problem with local boundary condition. The proof of (3.13) is carried out according to the same approach. This completes the proof of Theorem 3.2.1.

3.3 Stability of the Difference Scheme

To formulate our results for the differential operator A defined by (3.2), we introduce the difference operator A_h defined by the formula

$$A_h\varphi^h(x) = \left\{-\frac{1}{h}\left(a(x_{n+1})\frac{\varphi_{n+1} - \varphi_n}{h} - a(x_n)\frac{\varphi_n - \varphi_{n-1}}{h}\right) + \delta\varphi_n\right\}_{n=1}^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_{n=0}^M$ defined on $[0,l]_h$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$.

It is easy that A_h is the self-adjoint positive-definite operator in $H = L_{2h} = L_2[0,l]_h$. Actually,

we have that

$$\left\langle A_h u^h, v^h \right\rangle = \sum_{x\in[0,l]_h} A_h u^h(x) v^h(x) h$$

$$\begin{aligned}
&= \sum_{n=1}^{M-1}\left[-\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}-u_n}{h} - a(x_n)\frac{u_n-u_{n-1}}{h}\right) + \delta u_n\right] v_n h \\
&= -\sum_{n=1}^{M-1} a(x_{n+1})\frac{u_{n+1}-u_n}{h} v_n + \sum_{n=1}^{M-1} a(x_n)\frac{u_n-u_{n-1}}{h} v_n + \delta\sum_{n=1}^{M-1} u_n v_n h \\
&= -\sum_{n=2}^{M} a(x_n)\frac{u_n-u_{n-1}}{h} v_{n-1} + \sum_{n=1}^{M-1} a(x_n)\frac{u_n-u_{n-1}}{h} v_n + \delta\sum_{n=1}^{M-1} u_n v_n h \\
&= -a(x_M)\frac{u_M-u_{M-1}}{h} v_{M-1} + a(x_1)\frac{u_1-u_0}{h} v_0 \\
&\quad -\sum_{n=2}^{M-1} a(x_n)\frac{u_n-u_{n-1}}{h} v_{n-1} + \sum_{n=2}^{M-1} a(x_n)\frac{u_n-u_{n-1}}{h} v_n + \delta\sum_{n=1}^{M-1} u_n v_n h \\
&= -a(x_M)\frac{u_M-u_{M-1}}{h} v_M + a(x_1)\frac{u_1-u_0}{h} v_0 \\
&\quad +\sum_{n=2}^{M} a(x_n)\frac{u_n-u_{n-1}}{h}\frac{v_n-v_{n-1}}{h} h + \delta\sum_{n=1}^{M-1} u_n v_n h.
\end{aligned}$$

Since

$$a(x_M) = a(x_1), \varphi_0 = \varphi_M, \varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}, \tag{3.14}$$

we have that

$$-a(x_M)\frac{u_M-u_{M-1}}{h} v_M + a(x_1)\frac{u_1-u_0}{h} v_0 = 0.$$

$$\left\langle A_h u^h, v^h \right\rangle = \sum_{n=2}^{M} a(x_n)\frac{u_n-u_{n-1}}{h}\frac{v_n-v_{n-1}}{h} h + \delta\sum_{n=1}^{M-1} u_n v_n h.$$

Using (3.14), we get

$$\left\langle u^h, A_h v^h \right\rangle = \sum_{x\in[0,l]_h} u^h(x) A_h v^h(x) h$$

$$
\begin{aligned}
&= \sum_{n=1}^{M-1} u_n \left[-\frac{1}{h} \left(a\left(x_{n+1}\right) \frac{v_{n+1}-v_n}{h} - a\left(x_n\right) \frac{v_n - v_{n-1}}{h} \right) + \delta v_n \right] h \\
&= -\sum_{n=1}^{M-1} u_n a\left(x_{n+1}\right) \frac{v_{n+1}-v_n}{h} + \sum_{n=1}^{M-1} u_n a\left(x_n\right) \frac{v_n - v_{n-1}}{h} + \delta \sum_{n=1}^{M-1} u_n v_n h \\
&= -\sum_{n=2}^{M} u_{n-1} a\left(x_n\right) \frac{v_n - v_{n-1}}{h} + \sum_{n=1}^{M-1} u_n a\left(x_n\right) \frac{v_n - v_{n-1}}{h} + \delta \sum_{n=1}^{M-1} u_n v_n h \\
&= -u_{M-1} a\left(x_M\right) \frac{v_M - v_{M-1}}{h} + u_1 a\left(x_1\right) \frac{v_1 - v_0}{h} - \sum_{n=2}^{M-1} a\left(x_n\right) \frac{v_n - v_{n-1}}{h} u_{n-1} \\
&\quad + \sum_{n=2}^{M-1} a\left(x_n\right) u_n \frac{v_n - v_{n-1}}{h} h + \delta \sum_{n=1}^{M-1} u_n v_n h \\
&= a\left(x_M\right) \frac{u_M - u_{M-1}}{h} \frac{v_M - v_{M-1}}{h} + \sum_{n=2}^{M-1} a\left(x_n\right) \frac{u_n - u_{n-1}}{h} \frac{v_n - v_{n-1}}{h} h \\
&\quad + \delta \sum_{n=1}^{M-1} u_n v_n h \\
&= \sum_{n=2}^{M} a\left(x_n\right) \frac{u_n - u_{n-1}}{h} \frac{v_n - v_{n-1}}{h} h + \delta \sum_{n=1}^{M-1} u_n v_n h.
\end{aligned}
$$

From that it follows

$$
\left\langle A_h u^h, v^h \right\rangle = \left\langle u^h, A_h v^h \right\rangle
$$

and

$$
\begin{aligned}
\left\langle A_h u^h, u^h \right\rangle &= \sum_{n=2}^{M} a\left(x_n\right) \frac{u_n - u_{n-1}}{h} \frac{u_n - u_{n-1}}{h} h + \delta \sum_{n=1}^{M-1} u_n u_n h \\
&\geq \delta \sum_{n=1}^{M-1} u_n u_n h \\
&= \delta \left\langle u^h, u^h \right\rangle .
\end{aligned}
$$

3.3.1 The first order of accuracy difference scheme

For the numerical solution $\left\{ \left\{ u_n^k \right\}_{k=0}^{N} \right\}_{n=0}^{M}$ of problem (3.1), we consider the first order of accuracy difference scheme

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k+1}-u_n^{k+1}}{h^2}-a\left(x_n\right)\dfrac{u_n^{k+1}-u_{n-1}^{k+1}}{h^2}\right) \\ +\delta u_n^{k+1}=p_kq\left(x_n\right)+f\left(t_k,x_n\right),t_k=k\tau,x_n=nh, \\ 1\leqslant k\leqslant N-1,1\leqslant n\leqslant M-1,N\tau=T, \\ u_n^0=\varphi\left(x_n\right),\dfrac{u_n^1-u_n^0}{\tau}=\psi\left(x_n\right),0\leqslant n\leqslant M,Mh=l, \\ u_0^{k+1}=u_M^{k+1},u_1^{k+1}-u_0^{k+1}=u_M^{k+1}-u_{M-1}^{k+1}, \\ \sum_{i=1}^{M-1}u_i^{k+1}h=\zeta\left(t_{k+1}\right),-1\leqslant k\leqslant N-1. \end{cases} \tag{3.15}$$

Here, it is assumed that

$$q_0=q_M,q_1-q_0=q_M-q_{M-1}$$

and $\sum_{i=1}^{M-1}q_i\neq 0$. We have the following theorem on the stability of difference scheme (3.15).

Theorem 3.3.1 For the solution of difference scheme (3.15)

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_{k+1}^h\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)} \tag{3.16}$$

$$\leqslant M_5(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|f_1^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right],$$

$$\left\|\left\{p_k\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leqslant M_6(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}\right. \tag{3.17}$$

$$\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=2}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right].$$

Here and throughout this subsection $f_k^h(x)=\left\{f\left(t_k,x_n\right)\right\}_{n=0}^M, 1\leq k\leq N-1$.

Proof. We will use the following substitution

$$u_n^k=w_n^k+\eta_kq_n, \tag{3.18}$$

where

$$q_n=q\left(x_n\right),$$

and

$$\eta_{k+1}=\sum_{i=1}^{k}(k+1-i)\,p_i\tau^2, 1\leqslant k\leqslant N-1, \eta_0=\eta_1=0. \tag{3.19}$$

It is easy to see that $\left\{\left\{w_n^k\right\}_{k=0}^{N}\right\}_{n=0}^{M}$ is the solution of difference problem

$$\begin{cases} \dfrac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{w_{n+1}^{k+1}-w_n^{k+1}}{h}-a\left(x_n\right)\dfrac{w_n^{k+1}-w_{n-1}^{k+1}}{h}\right) \\ +\delta w_n^{k+1}=f\left(t_k,x_n\right)+\dfrac{1}{h}\left[a\left(x_{n+1}\right)\dfrac{q_{n+1}-q_n}{h}-a\left(x_n\right)\dfrac{q_n-q_{n-1}}{h}-\delta hq_n\right]\eta_{k+1}, \\ 1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1, \\ w_n^0=\varphi\left(x_n\right), \dfrac{w_n^1-w_n^0}{\tau}=\psi\left(x_n\right), 0\leqslant n\leqslant M, \\ w_0^{k+1}=w_M^{k+1}, w_1^{k+1}-w_0^{k+1}=w_M^{k+1}-w_{M-1}^{k+1}, -1\leqslant k\leqslant N-1. \end{cases} \tag{3.20}$$

Applying the overdetermined condition $\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta\left(t_{k+1}\right)$ and substitution (3.18), one can obtain that

$$\eta_{k+1}=\frac{\zeta_{k+1}-\sum_{i=1}^{M-1}w_i^{k+1}h}{\sum_{i=1}^{M-1}q_ih}. \tag{3.21}$$

Then, using formulas $p_k=\dfrac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}$ and (3.21), we get

$$p_k=\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}-\sum_{i=1}^{M-1}\left(w_i^{k+1}-2w_i^k+w_i^{k-1}\right)h}{\tau^2\sum_{i=1}^{M-1}q_ih}.$$

$$|p_k|\leqslant K_7\left(q\right)\left[\left|\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right|+\left\|\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right] \tag{3.22}$$

for all $1\leqslant k\leqslant N-1$. From that it follows

$$\begin{aligned} \left\|\left\{p_k\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leq K_7\left(q\right)&\left[\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right. \\ &\left.+\left\|\left\{\frac{w_{k+1}^h-2w_k^h+w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right]. \end{aligned} \tag{3.23}$$

Now, using substitution (3.18), we get

$$\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}=\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}+p_kq\left(x_n\right).$$

Applying the triangle inequality, we obtain

$$\left\| \left\{ \left\{ \frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\}_{n=0}^{M} \right\|_{C_\tau(L_{2h})} \tag{3.24}$$

$$\leqslant \left\| \left\{ \left\{ \frac{w_i^{k+1} - 2w_i^k + w_i^{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\}_{n=0}^{M} \right\|_{C_\tau(L_{2h})}$$

$$+ \left\| \{p_k\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \left\| \{q(x_n)\}_{n=1}^{M-1} \right\|_{L_{2h}}.$$

Therefore, the proof of estimates (3.16) and (3.17) is based on equation (3.15), the triangle inequality, estimates (3.23), (3.24) and on the following stability estimate

$$\left\| \left\{ \frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} \tag{3.25}$$

$$\leqslant K_8(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} + \left\| f_1^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=2}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right]$$

for the solution of difference problem (3.20). It was proved in Section 2.3 for the identification hyperbolic problem with local boundary condition. The proof of (3.25) is carried out according to the same approach. This completes the proof of Theorem 3.3.1.

3.3.2 The second order of accuracy difference scheme

We have the following approximate formulas

$$\begin{aligned} &\frac{-u(2h) + 4u(h) - 3u(0)}{2h} - u_x(0) = o(h^2), \\ &\frac{3u(l) - 4u(l-h) + u(l-2h)}{2h} - u_x(l) = o(h^2). \end{aligned} \tag{3.26}$$

Applying (3.26) for the numerical solution $\left\{ \{u_n^k\}_{k=0}^N \right\}_{n=0}^M$ of problem (3.1), we consider the second order of accuracy difference scheme

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{1}{2h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^k-u_n^k}{h}-a\left(x_n\right)\dfrac{u_n^k-u_{n-1}^k}{h}\right)\\
-\dfrac{1}{4h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^{k+1}-u_n^{k+1}+u_{n+1}^{k-1}-u_n^{k-1}}{h}\right.\\
\left.-a\left(x_n\right)\dfrac{u_n^{k+1}-u_{n-1}^{k+1}+u_n^{k-1}-u_{n-1}^{k-1}}{h}\right)+\dfrac{1}{2}\delta u_n^k+\delta\dfrac{u_n^{k+1}+u_n^{k-1}}{4}\\
=p_kq_n+f\left(t_k,x_n\right),x_n=nh,t_k=k\tau,\\
N\tau=T,1\leqslant n\leqslant M-1,Mh=l,t_k=k\tau,1\leqslant k\leqslant N-1,\\
u_n^0=\varphi\left(x_n\right),0\leqslant n\leqslant M,\\
\dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^1-u_{n+1}^0-u_n^1+u_n^0}{h}\right.\\
\left.-a\left(x_n\right)\dfrac{u_n^1-u_n^0-u_{n-1}^1+u_{n-1}^0}{h}+\delta h(u_n^0-u_n^1)\right)=\psi\left(x_n\right)+\dfrac{\tau}{2}\left[f\left(0,x_n\right)+p_0q_n\right]\\
-\dfrac{\tau}{2}\left[-\dfrac{1}{h}\left(a\left(x_{n+1}\right)\dfrac{u_{n+1}^0-u_n^0}{h}-a\left(x_n\right)\dfrac{u_n^0-u_{n-1}^0}{h}\right)+\delta u_n^0\right],1\leqslant n\leqslant M-1,\\
u_0^{k+1}=u_M^{k+1},-u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}=3u_M^{k+1}-4u_{M-1}^{k+1}+u_{M-2}^{k+1},\\
\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta\left(t_{k+1}\right),-1\leqslant k\leqslant N-1
\end{cases}
\tag{3.27}
$$

Here, it is assumed that

$$
q_M=q_0,-q_2+4q_1-3q_0=3q_M-4q_{M-1}+q_{M-2} \tag{3.28}
$$

and $\sum_{i=1}^{M-1}q_i\neq 0$. We have the following theorem on the stability of the difference scheme (3.27):

Theorem 3.3.2 For the solution of difference scheme (3.27), the following stability estimates hold:

$$
\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{u_{k+1}^h+2u_k^h+u_{k-1}^h}{4}\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)} \tag{3.29}
$$

$$
\leqslant M_9\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|f_0^h\right\|_{L_{2h}}\right.
$$

$$
\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right],
$$

$$
\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leqslant M_{10}\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_0^h\right\|_{L_{2h}}\right. \tag{3.30}
$$

$$+\left\|\left\{\frac{f_k^h - f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\left\{\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\Bigg].$$

Here and throughout this subsection $f_k^h(x) = \{f(t_k, x_n)\}_{n=0}^{M}, 1 \le k \le N-1$.

Proof. We will use the substitution

$$u_n^k = w_n^k + \eta_k q_n, \tag{3.31}$$

where

$$\eta_k = \sum_{i=1}^{k}\left\{\frac{(k-i)\,p_i + (k-(i-1))\,p_{i-1}}{2}\right\}\tau^2, 1 \leqslant k \leqslant N, \eta_0 = 0, \tag{3.32}$$

It is easy to see that $\left\{\left\{w_n^k\right\}_{k=0}^{N}\right\}_{n=0}^{M}$ is the solution of the difference problem

$$\begin{cases} \dfrac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} - \dfrac{1}{2h}\left(a(x_{n+1})\dfrac{w_{n+1}^k - w_n^k}{h} - a(x_n)\dfrac{w_n^k - w_{n-1}^k}{h}\right) \\ -\dfrac{1}{4h}\left(a(x_{n+1})\dfrac{w_{n+1}^{k+1} - w_n^{k+1} + w_{n+1}^{k-1} - w_n^{k-1}}{h}\right. \\ \left.-a(x_n)\dfrac{w_n^{k+1} - w_{n-1}^{k+1} + w_n^{k-1} - w_{n-1}^{k-1}}{h}\right) + \dfrac{1}{2}\delta w_n^k + \delta\dfrac{w_n^{k+1} + w_n^{k-1}}{4} \\ = \dfrac{1}{h}\left[a(x_{n+1})\dfrac{q_{n+1} - q_n}{h} - a(x_n)\dfrac{q_n - q_{n-1}}{h} - \delta h q_n\right]\dfrac{1}{4}(\eta_{k+1} + 2\eta_k + \eta_{k-1}) \\ +f(t_k, x_n), 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1, \\ w_n^0 = \varphi(x_n), 0 \leqslant n \leqslant M, \\ \dfrac{w_n^1 - w_n^0}{\tau} - \dfrac{\tau}{h}\left(a(x_{n+1})\dfrac{w_{n+1}^1 - w_{n+1}^0 - w_n^1 + w_n^0}{h}\right. \\ \left.-a(x_n)\dfrac{w_n^1 - w_n^0 - w_{n-1}^1 + w_{n-1}^0}{h} + \delta h(w_n^0 - w_n^1)\right) - \dfrac{\tau}{h}\left(a(x_{n+1})\dfrac{q_{n+1} - q_n}{h}\right. \\ \left.-a(x_n)\dfrac{q_n - q_{n-1}}{h} - \delta h q_n\right)\eta_1 = \psi(x_n) + \dfrac{\tau}{2}f(0, x_n) \\ -\dfrac{\tau}{2}\left[-\dfrac{1}{h}\left(a(x_{n+1})\dfrac{w_{n+1}^0 - w_n^0}{h} - a(x_n)\dfrac{w_n^0 - w_{n-1}^0}{h}\right) + \delta w_n^0\right], 1 \leqslant n \leqslant M-1, \\ w_0^{k+1} = w_M^{k+1}, -w_2^{k+1} + 4w_1^{k+1} - 3w_0^{k+1} = 3w_M^{k+1} - 4w_{M-1}^{k+1} + w_{M-2}^{k+1}, \\ -1 \leqslant k \leqslant N-1. \end{cases} \tag{3.33}$$

Now, we will take an estimate for $|p_k|$ .Using the overdetermined condition $\sum_{i=1}^{M-1} u_i^{k+1} h =$

$\zeta\left(t_{k+1}\right)$ and substitution (3.31), one can obtain

$$\eta_k = \frac{\zeta_k - \sum_{i=1}^{M-1} w_i^k h}{\sum_{i=1}^{M-1} q_i h}. \tag{3.34}$$

Then, using the formulas $p_k = \frac{\eta_{k+1}-2\eta_k+\eta_{k-1}}{\tau^2}$ and (3.34), we get

$$p_k = \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1} - \sum_{i=1}^{M-1}\left(w_i^{k+1} - 2w_i^k + w_i^{k-1}\right) h}{\tau^2 \sum_{i=1}^{M-1} q_i h}.$$

Then, applying the discrete analogue of the Cauchy–Schwarz inequality and the triangle inequality, we obtain

$$|p_k| \leqslant M_{11}(q)\left[\left|\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right| + \left\|\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\|_{L_{2h}}\right] \tag{3.35}$$

for all $1 \leqslant k \leqslant N-1$. From that, it follows

$$\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \leqslant K_{11}(q)\left[\left\|\left\{\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right. \tag{3.36}$$

$$\left. + \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}\right].$$

Now, using substitution (3.31), we get

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} = \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + p_k q\left(x_n\right).$$

Applying the triangle inequality, we obtain

$$\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant \left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \tag{3.37}$$

$$+ \left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau} \left\|\{q(x_n)\}_{n=0}^{M}\right\|_{L_{2h}}.$$

Therefore, the proof of estimates (3.29) and (3.30) is based on equation (3.27), the triangle inequality, estimates (3.36), (3.37) and on the following stability estimate

$$\left\|\left\{\frac{w_{k+1}^h - 2w_k^h + w_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})} \leqslant M_{12}(q)\ \left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^1}\right. \tag{3.38}$$

$$+\left\|f_0^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}.\Bigg].$$

for the solution of difference problem (3.33). It was proved in Section 2.3 for the identification hyperbolic problem with local boundary condition. The proof of (3.38) is carried out according to the same approach. This completes the proof of Theorem 3.3.2.

Applying (3.26) and same approach with this work we can study another the second order of accuracy difference scheme

$$\left\{\begin{array}{l}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{1}{2h}\Bigg(a\left(x_{n+1}\right)\frac{u_{n+1}^{k+1}-u_n^{k+1}+u_{n+1}^{k-1}-u_n^{k-1}}{h}\\
-a\left(x_n\right)\frac{u_n^{k+1}-u_{n-1}^{k+1}+u_n^{k-1}-u_{n-1}^{k-1}}{h}\Bigg)+\delta\frac{u_n^{k+1}+u_n^{k-1}}{2}\\
=p_kq_n+f\left(t_k,x_n\right),q_n=q\left(x_n\right),x_n=nh,N\tau=T,\\
1\leqslant n\leqslant M-1,Mh=l,t_k=k\tau,1\leqslant k\leqslant N-1,\\
u_n^0=\varphi\left(x_n\right),0\leqslant n\leqslant M,\\
\frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{h}\Bigg(a\left(x_{n+1}\right)\frac{u_{n+1}^1-u_{n+1}^0-u_n^1+u_n^0}{h}\\
-a\left(x_n\right)\frac{u_n^1-u_n^0-u_{n-1}^1+u_{n-1}^0}{h}+\delta h(u_n^0-u_n^1)\Bigg)=\psi\left(x_n\right)+\frac{\tau}{2}\left[f\left(0,x_n\right)+p_0q_n\right]\\
-\frac{\tau}{2}\left[-\frac{1}{h}\left(a\left(x_{n+1}\right)\frac{u_{n+1}^0-u_n^0}{h}-a\left(x_n\right)\frac{u_n^0-u_{n-1}^0}{h}\right)+\delta u_n^0\right],1\leqslant n\leqslant M-1,\\
u_0^{k+1}=u_M^{k+1},-u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}=3u_M^{k+1}-4u_{M-1}^{k+1}+u_{M-2}^{k+1},\\
\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta\left(t_{k+1}\right),-1\leqslant k\leqslant N-1
\end{array}\right. \tag{3.39}$$

Here, it is assumed that the formula (3.28) holds and $\sum_{i=1}^{M-1}q_i\neq 0$. We have the following theorem on the stability of the difference scheme (3.39):

Theorem 3.3.3 For the solution of difference scheme (3.39), the following stability estimates hold:

$$\left\|\left\{\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{u_{k+1}^h+u_{k-1}^h}{2}\right\}_{k=1}^{N-1}\right\|_{C_\tau\left(W_{2h}^2\right)}$$

$$\leqslant M_{14}\left(q\right)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|f_0^h\right\|_{L_{2h}}\right.$$

$$
+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\Bigg],
$$

$$
\left\|\{p_k\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\leqslant M_{15}(q)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_0^h\right\|_{L_{2h}}\right.
$$

$$
\left.+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\left\|\left\{\frac{\zeta_{k+1}-2\zeta_k+\zeta_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C[0,T]_\tau}\right].
$$

Now, we consider one more the second order of accuracy difference scheme

$$
\begin{cases}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^k-u_n^k}{h}-a(x_n)\frac{u_n^k-u_{n-1}^k}{h}\right)+\delta u_n^k \\
+\frac{\tau^2}{4}\left\{\frac{1}{h}a(x_{n+1})\frac{1}{h}\left[\frac{1}{h}\left(a(x_{n+2})\frac{u_{n+2}^{k+1}-u_{n+1}^{k+1}}{h}-a(x_{n+1})\frac{u_{n+1}^{k+1}-u_n^{k+1}}{h}\right)\right.\right. \\
\left.-\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^k-u_n^k}{h}-a(x_n)\frac{u_n^k-u_{n-1}^k}{h}\right)\right] \\
-\frac{1}{h}a(x_n)\left[\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^k-u_n^k}{h}-a(x_{n-})\frac{u_n^k-u_{n-1}^k}{h}\right)\right. \\
\left.-\frac{1}{h}\left(a(x_n)\frac{u_n^{k+1}-u_{n-2}^{k+1}}{h}-a(x_{n-1})\frac{u_{n-1}^{k+1}-u_{n-2}^{k+1}}{h}\right)\right] \\
\left.-2\delta\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^{k+1}-u_n^{k+1}}{h}-a(x_n)\frac{u_n^{k+1}-u_{n-1}^{k+1}}{h}\right)+\delta^2u_n^{k+1}\right\} \\
=p_kq_n+f(t_k,x_n),q_n=q(x_n),x_n=nh,N\tau=T, \\
1\leqslant n\leqslant M-1,Mh=l,t_k=k\tau,1\leqslant k\leqslant N-1, \\
u_n^0=\varphi(x_n),0\leqslant n\leqslant M, \\
\frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{h}\left(a(x_{n+1})\frac{u_{n+1}^1-u_{n+1}^0-u_n^1+u_n^0}{h}\right. \\
\left.-a(x_n)\frac{u_n^1-u_n^0-u_{n-1}^1+u_{n-1}^0}{h}+\delta h(u_n^0-u_n^1)\right)=\psi(x_n)+\frac{\tau}{2}\left[f(0,x_n)+p_0q_n\right] \\
-\frac{\tau}{2}\left[-\frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^0-u_n^0}{h}-a(x_n)\frac{u_n^0-u_{n-1}^0}{h}\right)+\delta u_n^0\right],1\leqslant n\leqslant M-1, \\
u_0^{k+1}=u_M^{k+1},-u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}=3u_M^{k+1}-4u_{M-1}^{k+1}+u_{M-2}^{k+1}, \\
\sum_{i=1}^{M-1}u_i^{k+1}h=\zeta(t_{k+1}),-1\leqslant k\leqslant N-1
\end{cases}
\tag{3.40}
$$

Here, it is assumed that formula (3.28) holds and $\sum_{i=1}^{M-1} q_i \neq 0$. We have the following theorem on the stability of the difference scheme (3.40):

Theorem 3.3.4 For the solution of difference scheme (3.40), the following stability estimates hold:

$$\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ u_k^h \right\}_{k=1}^{N-1} \right\|_{C_\tau(W_{2h}^2)}$$

$$\leqslant M_{14}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^1} + \left\| f_0^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right],$$

$$\left\| \{p_k\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \leqslant M_{15}(q) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| f_0^h \right\|_{L_{2h}} \right.$$

$$\left. + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\tau^2} \right\}_{k=1}^{N-1} \right\|_{C[0,T]_\tau} \right].$$

3.4 Numerical Experiments

In this section, we study the numerical solution of the identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = p(t)(4 + 4\sin 2x) + 4e^{-2t}\sin 2x, \\ x \in (0,\pi), t \in (0,1), \\ u(0,x) = 1 + \sin 2x, u_t(0,x) = -2(1 + \sin 2x), x \in [0,\pi], \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi)\, t \in [0,1], \\ \int_0^\pi u(t,x)\,dx = \pi e^{-2t}, t \in [0,1] \end{cases} \tag{3.41}$$

for a hyperbolic differential equation with non-local conditions. The exact solution pair of this problem is $(u(t,x), p(t)) = \left(e^{-2t}(1 + \sin 2x), e^{-2t}\right)$.

Firstly, for the numerical solution of problem (3.41), we present the following first order of accuracy difference scheme for the approximate solution for problem (3.41):

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^{k}+u_n^{k-1}}{\tau^2}-\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}\\
=p_k\left(4+4\sin 2x_n\right)+4e^{-2t_{k+1}}\sin 2x_n,\\
t_k=k\tau, x_n=nh, 1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1,\\
u_n^0=1+\sin 2x_n, \dfrac{u_n^1-u_n^0}{\tau}=-2\left(1+\sin 2x_n\right),\\
0\leqslant n\leqslant M, Mh=\pi, N\tau=1,\\
u_0^{k+1}=u_M^{k+1}, u_1^{k+1}-u_0^{k+1}=u_M^{k+1}-u_{M-1}^{k+1},\\
\sum_{i=1}^{M-1}u_i^{k+1}h=\pi e^{-2t_{k+1}}, -1\leqslant k\leqslant N-1.
\end{cases}
\tag{3.42}
$$

Algorithm for obtaining the solution of identification problem (3.42) contains three stages. Actually, let us define

$$
u_n^k=w_n^k+4\eta_k\left(1+\sin 2x_n\right), 0\leq k\leq N, 0\leq n\leq M,
\tag{3.43}
$$

Applying difference scheme (3.42) and formula (3.43), we will obtain formula

$$
\eta_{k+1}=\frac{\pi e^{-2t_{k+1}}-\sum_{i=1}^{M-1}w_i^{k+1}h}{4\sum_{i=1}^{M-1}\left(1+\sin 2x_i\right)h}, -1\leq k\leq N-1
\tag{3.44}
$$

and the difference scheme

$$
\begin{cases}
\dfrac{w_n^{k+1}-2w_n^{k}+w_n^{k-1}}{\tau^2}-\dfrac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2}\\
+\dfrac{\sum_{i=1}^{M-1}w_i^{k+1}h}{\sum_{i=1}^{M-1}\left(1+\sin 2x_i\right)h}\sin 2x_n\dfrac{2(\cos 2h-1)}{h^2}\\
=\left[\dfrac{2\pi(\cos 2h-1)}{h^2\sum_{i=1}^{M-1}\left(1+\sin 2x_i\right)h}+4\right]e^{-2t_{k+1}}\sin 2x_n,\\
1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1,\\
w_n^0=1+\sin 2x_n, \dfrac{w_n^1-w_n^0}{\tau}=-2\left(1+\sin 2x_n\right), 0\leqslant n\leqslant M,\\
w_0^{k+1}=w_M^{k+1}, w_1^{k+1}-w_0^{k+1}=w_M^{k+1}-w_{M-1}^{k+1},\\
-1\leqslant k\leqslant N-1.
\end{cases}
\tag{3.45}
$$

In the first stage, we find numerical solution $\left\{\left\{w_n^k\right\}_{k=0}^{N}\right\}_{n=0}^{M}$ of corresponding first order of accuracy auxiliary difference scheme (3.45). For obtaining the solution of difference scheme (3.45), we will write it in the matrix form as

$$
\begin{cases} Aw^{k+1} + Bw^k + Cw^{k-1} = \varphi^k, 1 \leqslant k \leqslant N-1, \\ w^0 = \{1 + \sin 2x_n\}_{n=0}^M, w^1 = (1-2\tau)\{1 + \sin 2x_n\}_{n=0}^M, \end{cases} \tag{3.46}
$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $w^s, s = k, k \pm 1, f^k$ are $(M+1) \times 1$ column matrices and

$$
A = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & -1 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ -1 & 1 & 0 & \cdot & 0 & 1 & -1 \end{bmatrix}_{(M+1)\times(M+1)},
$$

$$
B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & e & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & e & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & e & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} \quad C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & g & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & g & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & g & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & g & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},
$$

$$
\varphi^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} \quad w^s = \begin{bmatrix} 0 \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1}, \text{ for } s = k, k \pm 1.
$$

Here,

$$
a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, c_n = \frac{2(\cos 2h - 1)}{hd} \sin 2x_n,
$$

$$
d = \sum_{i=1}^{M-1} (1 + \sin 2x_i)\, h, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2},
$$

$$\varphi_n^k = \left[\frac{2\pi(\cos 2h - 1)}{h^3 \sum_{i=1}^{M-1} (1 + \sin 2x_i)} + 4 \right] e^{-2t_{k+1}} \sin 2x_n, 1 \leqslant k \leqslant N-1, 1 \leqslant n \leqslant M-1.$$

So, we have the initial value problem for the second order difference equation (3.46) with respect to k with matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain $\left\{ \{ w_n^k \}_{k=0}^{N} \right\}_{n=0}^{M}$ by (3.46).

Now, applying formula (3.32), we can obtain

$$p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}, 1 \le k \le N-1. \tag{3.47}$$

In the second stage, we will obtain $\{p_k\}_{k=1}^{N-1}$ by formulas (3.44) and (3.47). Finally, in the third stage, we will obtain $\left\{ \{ u_n^k \}_{k=0}^{N} \right\}_{n=0}^{M}$ by formulas (3.43) and (3.44). The errors are computed by

$$E_u = \max_{0 \leqslant k \leqslant N} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}},$$

$$E_p = \max_{1 \leqslant k \leqslant N-1} \left| p(t_k) - p_k \right|,$$

where $u(t,x), p(t)$ represent the exact solution, u_n^k represent the numerical solutions at (t_k, x_n) and p_k represent the numerical solutions at t_k. The numerical results are given in the following table.

Table 4: Error analysis for difference scheme (3.42)

Error	**N = M = 20**	**N = M = 40**	**N = M = 80**	**N = M = 160**
E_u	0.0560	0.0289	0.0147	0.0075
E_p	0.0476	0.0244	0.0123	0.0062

As can be seen in Table 4, if N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $1/2$ for the first order difference scheme (3.42).

Secondly, for the numerical solution of problem (3.41), we present the following second order of accuracy difference scheme for the approximate solution for problem (3.41):

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\dfrac{1}{2}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\ -\dfrac{1}{4}\left[\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right] \\ =p_k\left(4+4\sin 2x_n\right)+4e^{-2t_{k+1}}\sin 2x_n, \\ t_k=k\tau, x_n=nh, 1\leqslant k\leqslant N-1,, 1\leqslant n\leqslant M-1, \\ u_n^0=1+\sin 2x_n, \\ \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{h^2}\left(u_{n+1}^1-u_{n+1}^0-2u_n^1+2u_n^0+u_{n-1}^1-u_{n-1}^0\right) \\ =-2\left(1+\sin 2x_n\right)+\dfrac{\tau}{2}\left\{\dfrac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}\right. \\ \left.+4p_0\left(1+\sin 2x_n\right)+4\sin 2x_n\right\} 0\leqslant n\leqslant M, Mh=\pi, N\tau=1, \\ u_0^{k+1}=u_M^{k+1}, \\ -u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}=3u_M^{k+1}-4u_{M-1}^{k+1}+u_{M-2}^{k+1}, \\ \sum_{i=1}^{M-1}u_i^{k+1}h=\pi e^{-t_{k+1}}, -1\leqslant k\leqslant N-1 \end{cases} \tag{3.48}$$

The algorithm for obtaining the solution of identification problem (3.48) contains three stages. Actually, let us define

$$u_n^k=w_n^k+4\eta_k\left(1+\sin 2x_n\right), 0\leq k\leq N, 0\leq n\leq M, \tag{3.49}$$

Applying the second order of accuracy difference scheme(2.75) and formula (2.77), we will obtain

$$\eta_k=\frac{\pi e^{-2t_k}-\sum_{i=1}^{M-1}w_i^k h}{4\sum_{i=1}^{M-1}\left(1+\sin 2x_i\right)h}, 0\leq k\leq N \tag{3.50}$$

and the difference scheme

$$
\begin{cases}
\dfrac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}-\dfrac{1}{2}\dfrac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}\\
-\dfrac{1}{4}\left[\dfrac{w_{n+1}^{k+1}-2w_n^{k+1}+w_{n-1}^{k+1}}{h^2}+\dfrac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{h^2}\right]\\
+\dfrac{1}{2}\dfrac{(\cos 2h-1)}{h^2\sum_{i=1}^{M-1}(1+\sin x_i)\,h}\sin 2x_n\\
\times\left(\sum_{i=1}^{M-1}w_i^{k+1}h+2\sum_{i=1}^{M-1}w_i^k h+\sum_{i=1}^{M-1}w_i^{k-1}h\right)\\
=4e^{-2t_k}\sin x_n+\left[e^{-2t_{k+1}}+2e^{-2t_k}+e^{-2t_{k-1}}\right]\times\\
\dfrac{1}{2}\dfrac{\pi(\cos 2h-1)}{h^2\sum_{i=1}^{M-1}(1+\sin x_i)\,h}\sin 2x_n, 1\leqslant k\leqslant N-1, 1\leqslant n\leqslant M-1,\\
w_n^0=1+\sin 2x_n, 0\leqslant n\leqslant M,\\
\dfrac{w_n^1-w_n^0}{\tau}-\dfrac{\tau}{h^2}\left(w_{n+1}^1-2w_n^1+w_{n-1}^1\right)\\
+\dfrac{2\tau(\cosh-1)}{h^2\sum_{i=1}^{M-1}\sin x_i h}\sin 2x_n\sum_{i=1}^{M-1}w_i^1 h\\
+\dfrac{\tau}{2h^2}\left(w_{n+1}^0-2w_n^0+w_{n-1}^0\right)=2\tau\sin 2x_n-2\left(1+\sin 2x_n\right)\\
+\dfrac{2\tau\pi e^{-2t_1}(\cos 2h-1)}{h^2\sum_{i=1}^{M-1}(1+\sin x_i)\,h}\sin 2x_n, 0\leqslant n\leqslant M,\\
w_0^{k+1}=w_M^{k+1},\\
-w_2^{k+1}+4w_1^{k+1}-3w_0^{k+1}=3w_M^{k+1}-4w_{M-1}^{k+1}+w_{M-2}^{k+1},\\
-1\leqslant k\leqslant N-1.
\end{cases}
\tag{3.51}
$$

In the first stage, we find the solution $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ of the corresponding second-order-of-accuracy difference scheme (3.51). For obtaining it, we will write difference scheme (3.51), in matrix form as

$$
\begin{cases}
Aw^{k+1}+Bw^k+Cw^{k-1}=\varphi^k =, 1\leqslant k\leqslant N-1,\\
w^0, w^1 \text{ are given,}
\end{cases}
\tag{3.52}
$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $w^s, s=k, k\pm 1, \varphi^k$ are $(M+1)\times 1$ column matrices and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & c_1 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ r & q+e_1 & r+e_1 & \cdot & e_1 & e_1 & e_1 \\ 0 & r+e_2 & q+e_2 & \cdot & e_2 & e_2 & e_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & e_{M-2} & e_{M-2} & \cdot & q+e_{M-2} & r+e_{M-2} & 0 \\ 0 & e_{M-1} & e_{M-1} & \cdot & r+e_{M-1} & q+e_{M-1} & r \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-2} & c_{M-2} & \cdot & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdot & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \cdot \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, \quad w^s = \begin{bmatrix} 0 \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ 0 \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1.$$

Here,

$$
\begin{aligned}
a &= \frac{1}{\tau^2} + \frac{1}{2h^2}, b = -\frac{1}{4h^2},, r = -\frac{1}{2h^2}, q = \frac{1}{h^2} - \frac{2}{\tau^2},\\
c_n &= \frac{1}{2}\sin x_n \frac{(\cos 2h - 1)}{dh}, e_n = \sin x_n \frac{(\cos 2h - 1)}{dh},\\
d &= \sum_{i=1}^{M-1} (1 + \sin x_i)\, h, 1 \leqslant n \leqslant M - 1,
\end{aligned}
$$

$$
\varphi_n^k = 4e^{-2t_k} \sin x_n + \left(e^{-2t_{k+1}} + 2e^{-2t_k} + e^{-2t_{k-1}}\right) \frac{(\cos 2h - 1)}{h^2 \sum_{i=1}^{M-1} (1 + \sin x_i)\, h} \sin 2x_n,
$$

$$
1 \leqslant k \leqslant N - 1, 1 \leqslant n \leqslant M - 1.
$$

Finally, we obtain w^1.Applying the formula

$$
\begin{aligned}
&\frac{w_n^1 - w_n^0}{\tau} - \frac{\tau}{h^2}\left(w_{n+1}^1 - 2w_n^1 + w_{n-1}^1\right) + \frac{2\tau(\cos 2h - 1)}{h^2 \sum_{i=1}^{M-1} (1 + \sin x_i)\, h} \sin 2x_n \sum_{i=1}^{M-1} w_i^1 h\\
&+\frac{\tau}{2h^2}\left(w_{n+1}^0 - 2w_n^0 + w_{n-1}^0\right) = 2\tau \sin 2x_n - 2\,(1 + \sin 2x_n) + \frac{2\tau\pi e^{-2t_1}(\cos 2h - 1)}{h^2 \sum_{i=1}^{M-1} (1 + \sin x_i)\, h} \sin 2x_n,
\end{aligned}
$$

and conditions $w_0^1 = w_M^1, w_1^1 - w_0^1 = w_M^1 - w_{M-1}^1$we get

$$
Ew^1 + Fw^0 = \gamma. \tag{3.53}
$$

Here

$$
E = \begin{bmatrix}
1 & 0 & 0 & \cdot & 0 & 0 & 0\\
y & x + z_1 & y + z_1 & \cdot & z_1 & z_1 & 0\\
0 & y + z_2 & x + z_2 & \cdot & z_2 & z_2 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & z_{M-2} & z_{M-2} & \cdot & x + z_{M-2} & y + z_{M-2} & 0\\
0 & z_{M-1} & z_{M-1} & \cdot & y + z_{M-1} & x + z_{M-1} & y\\
0 & 0 & 0 & \cdot & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$
\begin{aligned}
x = -\frac{2\tau}{h^2} + \frac{1}{\tau}, y = -\frac{\tau}{h^2}, z_n = \frac{2\tau(\cos 2h - 1)}{dh} \sin 2x_n,\\
u = -\frac{\tau}{2h^2}, v = \frac{1}{\tau} + \frac{\tau}{h^2}, v = -\frac{3}{2\tau} - \frac{2\tau}{h^2} + 1,
\end{aligned}
$$

$$F=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ u & v & u & \cdot & 0 & 0 & 0 \\ 0 & u & v & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & v & u & 0 \\ 0 & 0 & 0 & \cdot & u & v & u \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\gamma=\begin{bmatrix} 0 \\ \nabla_1 \\ \cdot \\ \nabla_{M-1} \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\nabla_n = 2\tau\sin 2x_n - 2\left(1+\sin 2x_n\right) + \frac{2\tau\pi e^{-2t_1}(\cos 2h - 1)}{h^2\sum_{i=1}^{M-1}\left(1+\sin x_i\right)h}\sin 2x_n.$$

From that, it follows

$$w^1 = E^{-1}\left(\gamma - Fw^0\right). \tag{3.54}$$

So, we have the initial value problem for the second-order difference equation (3.52) with respect to k with the matrix coefficients A, B and C. Since $w^0$and $w^1$are given, we can obtain $\left\{\left\{w_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ by (3.52). Now, applying formula (3.32), we can obtain

$$p_k = \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\tau^2}, 1 \le k \le N-1, p_0 = \frac{2}{\tau^2}\eta_1 \tag{3.55}$$

In the second stage, we will obtain $\{p_k\}_{k=1}^{N-1}$ by formulas (3.50) and (3.55). Finally, in the third stage, we will obtain $\left\{\left\{u_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ by formulas (3.49) and (3.50).

The errors are computed by

$$E_u = \max_{0\leqslant k\leqslant N}\left(\sum_{n=1}^{M-1}\left|u\left(t_k, x_n\right) - u_n^k\right|^2 h\right)^{\frac{1}{2}},$$

$$E_p = \max_{1\leqslant k\leqslant N-1}\left|p\left(t_k\right) - p_k\right|,$$

where $u(t,x), p(t)$ represent the exact solution, u_n^k represents the numerical solutions at (t_k, x_n), and p_k represents the numerical solutions at t_k. The numerical results are given in the following Table.

Table 5: Error analysis for difference scheme (3.48)

Error	**N = M = 20**	**N = M = 40**	**N = M = 80**	**N = M = 160**
E_u	0.00254	$6.355e-04$	$1.5675e-04$	$3.9723e-05$
E_p	0.0028	$7.0523e-04$	$1.76857e-04$	$4.4758e-05$

As can be seen in Table 5, if N and M are doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately $\frac{1}{4}$ for the second-order difference scheme (3.48).

CHAPTER 4
CONCLUSION

This thesis is devoted to study the source identification problem for hyperbolic differential equations with unknown parameter $p(t)$. The following results are obtained:

• The history of direct and inverse boundary value problems for hyperbolic differential equations are studied.

• Fourier series, Laplace transform and Fourier transform methods are applied for the solution of several identification problems for hyperbolic differential equations.

• The theorem on the stability estimates for the solution of the source identification problem for hyperbolic differential equations with local and nonlocal conditions is proved.

• The first and second order of accuracy difference schemes for the approximate solution of the one dimensional identification problem for hyperbolic differential equation with local and nonlocal conditions are presented.

• The theorem on the stability estimates for these difference schemes for the numerical solution of identification problems for hyperbolic differential equations with local and nonlocal conditions is established.

• The Matlab implementation of these difference schemes is presented.

• The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

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APPENDICES

Appendix 1

Matlab Programming

Matlab Implementation of Difference Schemes (2.69)

```
clc; clear all ; close all;
N=160;
M=160;
h=pi/M; tau=1/N;
a=(1/(tau^2))+(2/(h^2));
e=-2/(tau^2);
b=-1/(h^2);
g=1/(tau^2);
d=0;
for i=1:M-1;
d=d+h*sin(i*h);
end;
z=2*(cos(h)-1)/(d*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=z*sin((i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+(z*sin((i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(z*sin((i-1)*h));
end;
for i=3:M;
```

```
A(i,i-1)=b+(z*sin((i-1)*h));
end;
A(1,1)=1;A(M+1,M+1)=1;A(2,1)=b;A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii=zeros(M+1,1);
for j=1:M+1;
for k=2:N;
fii(j,k)=((4*(cos(h)-1)/(d*(h^2)))+1)*exp(-k*tau)*sin((j-1)*h);
end;
end;
fii;
G=inv(A);
W=zeros(M+1,1);
 for j=1:M+1;
 W(j,1)=sin((j-1)*h);
 W(j,2)=(1-tau)*sin((j-1)*h);
for k=3:N+1;
W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));
end;
end;
for k=2:N;
```

```
D=0;
for j=1:M-1;
S(j)=D+W(j,k+1)-2*(W(j,k))+W(j,k-1);
D=S(j);
end;
p(k)=(2*exp(-(k+1)*tau)-4*exp(-k*tau)+2*exp(-(k-1)*tau)-(h*D))/(d*(tau^2));
end;
p(k);
L=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
L(i,j)=0;
end
end;
for i=2:M;
L(i,i)=a;
end;
for i=2:M-1;
L(i,i+1)=b;
end;
for i=3:M;
L(i,i-1)=b;
end;
L(1,1)=1;
L(M+1,M+1)=1;
L;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
```

```
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii=zeros(M+1,1) ;
for j=1:M+1;
for k=2:N;
x=(j-1)*h;
fii(j,k)=exp(-k*tau)*sin(x)+(p(k)*sin(x));
end;
end;
fii;
G=inv(L);
u=zeros(M+1,1);
for j=1:M+1;
x=(j-1)*h;
u(j,1)=sin(x);
u(j,2)=(1-tau)*sin(x);
end;
for k=3:N+1;
u(:,k)=G*(-(B*u(:,k-1))-(C*u(:,k-2))+fii(:,k-1));
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(j,k)=(1-t)*sin(x);
eu(j,k)=exp(-t)*sin(x);
end;
```

```
end;
for k=2:N;
t=(k-1)*tau;
 ep(k)=exp(-t);
end;
% ABSOLUTE DIFFERENCES ;
absdiff=max(max(abs(es-W)))
absdiff=max(max(abs(ep-p)))
absdiff=max(max(abs(eu-u)))
```

Appendix 2

Matlab Programming

Matlab Implementation of Difference Schemes (2.75)

```
clc; clear all ; close all;
N=20;M=20;
h=pi/M; tau=1/N;
d=0;
for j=1:M-1;
d=d+h*sin((j)*h);
end;
a=1/(tau^2)+1/(2*(h^2));q=1/(h^2)-2/(tau^2);
b=-1/(4*(h^2));r=-1/(2*(h^2));
c=(cos(h)-1)/(2*d*h);e=(cos(h)-1)/(d*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=c*sin((i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+c*sin((i-1)*h);
end;
for i=2:M-1;
A(i,i+1)=b+c*sin((i-1)*h);
end;
for i=3:M;
A(i,i-1)=b+c*sin((i-1)*h);
end;
A(1,1)=1;A(M+1,M+1)=1;A(2,1)=b;A(M,M+1)=b;
```

```
A;
B=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
B(i,j)=e*sin((i-1)*h);
end;
end;
for i=2:M
B(i,i)=q+e*sin((i-1)*h);
end;
for i=2:M-1;
B(i,i+1)=r+e*sin((i-1)*h);
end;
for i=3:M;
B(i,i-1)=r+e*sin((i-1)*h);
end;
B(2,1)=r;B(M,M+1)=r;
B;
C=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
C(i,j)=c*sin((i-1)*h);
end;
end;
for i=2:M
C(i,i)=a+c*sin((i-1)*h);
end;
for i=2:M-1;
C(i,i+1)=b+c*sin((i-1)*h);
end;
for i=3:M;
```

```
C(i,i-1)=b+c*sin((i-1)*h);
end;
C(2,1)=b;C(M,M+1)=b;
C;
fii=zeros(M+1,1);
for j=1:M+1;
for k=2:N;
fii(j,k)=exp(-(k-1)*tau)*sin((j-1)*h)+(exp(-k*tau)+2*exp(-(k-1)*tau)+exp(-(k-
    2)*tau))*sin((j-1)*h)*((cos(h)-1)/(d*(h^2)));
end;
end;
e1=-tau/(h^2);e2=(2*tau)/(h^2)+1/tau;
y=(2*tau*(cos(h)-1))/(d*h);
E=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
E(i,j)=y*sin((i-1)*h);
end;
end;
for i=2:M
E(i,i)=e2+y*sin((i-1)*h);
end;
for i=2:M-1;
E(i,i+1)=e1+y*sin((i-1)*h);
end;
for i=3:M;
E(i,i-1)=e1+y*sin((i-1)*h);
end;
E(1,1)=1;E(M+1,M+1)=1;E(2,1)=e1;E(M,M+1)=e1;
E;
f1=tau/(2*(h^2));f2=-tau/(h^2)-1/tau;
```

```
F=zeros(M+1,M+1);
for i=2:M
F(i,i)=f2;
end;
for i=2:M-1;
F(i,i+1)=f1;
end;
for i=3:M;
F(i,i-1)=f1;
end;
F(2,1)=f1;F(M,M+1)=f1;
F;
phy=zeros(M+1,1);
for j=1:M+1;
phy(j)=(tau/2-1)*sin((j-1)*h)+((2*y)/h)*(exp(-tau)*sin((j-1)*h));
end;
Z=inv(E);G=inv(A);
W=zeros(M+1,1);
for j=1:M+1;
W(j,1)=sin((j-1)*h);
end;
W(:,2)=Z*(phy-F*W(:,1));
for k=3:N+1;
W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));
end;

for k=2:N;
D=0;
for j=1:M-1;
S(j)=D+W(j,k+1)-2*(W(j,k))+W(j,k-1);
D=S(j);
```

```
end;
p(k)=(2*exp(-(k)*tau)-(4*exp(-(k-1)*tau))+(2*exp(-(k-2)*tau))-(h*D))/(d*(tau^2));
end;
p(k);
R=zeros(M+1,M+1);
for i=2:M
R(i,i)=a;
end;
for i=2:M-1;
R(i,i+1)=b;
end;
for i=3:M;
R(i,i-1)=b;
end;
R(1,1)=1;R(M+1,M+1)=1;R(2,1)=b;R(M,M+1)=b;
R;
L=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
L(i,j)=0;
end;
end;
for i=2:M
L(i,i)=q;
end;
for i=2:M;
L(i,i+1)=r;
end;
for i=3:M;
L(i,i-1)=r;
end;
```

```
L(2,1)=r;L(M,M+1);
L;
Q=zeros(M+1,M+1);
for i=2:M
Q(i,i)=a;
end;
for i=2:M-1;
Q(i,i+1)=b;
end;
for i=3:M;
Q(i,i-1)=b;
end;
Q(2,1)=b;Q(M,M+1)=b;
Q;
fii=zeros(M+1,1) ;
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii(j,k)=exp(-(k-1)*tau)*sin(x)+(p(k)*sin(x));
end;
end;
G=inv(R);
u=zeros(M+1,1);
for k=3:N+1;
for j=1:M+1;
x=(j-1)*h;
u(j,1)=sin(x);
u(j,2)=(1-tau+((tau^2)/(2)))*sin((j-1)*h);
u(:,k)=G*(-(L*u(:,k-1))-(Q*u(:,k-2))+fii(:,k-1));
end;
end;
```

```
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau; x=(j-1)*h;
es(j,k)=(1-t)*sin(x);
eu(j,k)=exp(-t)*sin(x);
end;
end;
for k=2:N;
t=(k-1)*tau;
ep(k)=exp(-t);
end
% ABSOLUTE DIFFERENCES ;
absdiff=max(max(abs(es-W)))
absdiff=max(max(abs(ep-p)))
absdiff=max(max(abs(eu-u)))
```

Appendix 3

Matlab Programming

Matlab Implementation of Difference Schemes (2.76)

```
clc; clear all ; close all;
N=160;
M=160;
h=pi/M; tau=1/N;
d=0;
for i=1:M-1;
d=d+h*sin(i*h);
end;
a=(1/(tau^2))+(1/(h^2));b=-1/(2*(h^2));g=-2/(tau^2);c=(cos(h)-1)/(d*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=c*sin((i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+(c*sin((i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(c*sin((i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(c*sin((i-1)*h));
end;
A(1,1)=1;
A(M+1,M+1)=1;
```

```
A(2,1)=b;
A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=g;
end;
B;
C=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
C(i,j)=c*sin((i-1)*h);
end;
end;
for i=2:M
C(i,i)=a+(c*sin((i-1)*h));
end;
for i=2:M-1;
C(i,i+1)=b+(c*sin((i-1)*h));
end;
for i=3:M;
C(i,i-1)=b+(c*sin((i-1)*h));
end;
C(2,1)=b;
C(M,M+1)=b;
C;
fii=zeros(M+1,1);
for j=1:M+1;
for k=2:N;
fii(j,k)=(exp(-(k-1)*tau)+((2*c)/h)*exp(-k*tau)+((2*c)/h)*exp(-(k-2)*tau))*sin((j-1)*h);
```

```
end;
end;
e1=-tau/(h^2);
e2=(2*tau)/(h^2)+1/tau;
y=(2*tau*(cos(h)-1))/(d*h);
E=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
E(i,j)=y*sin((i-1)*h);
end;
end;
for i=2:M
E(i,i)=e2+y*sin((i-1)*h);
end;
for i=2:M-1;
E(i,i+1)=e1+y*sin((i-1)*h);
end;
for i=3:M;
E(i,i-1)=e1+y*sin((i-1)*h);
end;
E(1,1)=1;
E(M+1,M+1)=1;
E(2,1)=e1;
E(M,M+1)=e1;
E;
f1=tau/(2*(h^2));
f2=-tau/(h^2)-1/tau;
F=zeros(M+1,M+1);
for i=2:M
F(i,i)=f2;
end;
```

```
for i=2:M-1;
F(i,i+1)=f1;
end;
for i=3:M;
F(i,i-1)=f1;
end;
F(2,1)=f1;
F(M,M+1)=f1;
F;
phy=zeros(M+1,1);
for j=1:M+1;
phy(j)=(tau/2-1)*sin((j-1)*h)+((2*y)/h)*(exp(-tau)*sin((j-1)*h));
end;
Z=inv(E);
G=inv(A);
W=zeros(M+1,1);
for j=1:M+1;
 W(j,1)=sin((j-1)*h);
end;
W(:,2)=Z*(phy-F*W(:,1));
for k=3:N+1;
W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));
end;
for k=2:N;
D=0;
for j=1:M-1;
S(j)=D+W(j,k+1)-2*(W(j,k))+W(j,k-1);
D=S(j);
end;
p(k)=(2*exp(-(k)*tau)-(4*exp(-(k-1)*tau))+(2*exp(-(k-2)*tau))-(h*D))/(d*(tau^2));
end;
```

```
p(k);
R=zeros(M+1,M+1);
for i=2:M
R(i,i)=a;
end;
for i=2:M-1;
R(i,i+1)=b;
end;
for i=3:M;
R(i,i-1)=b;
end;
R(1,1)=1;
R(M+1,M+1)=1;
R(2,1)=b;
R(M,M+1)=b;
R;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=g;
end;
B;
Q=zeros(M+1,M+1);
for i=2:M
Q(i,i)=a;
end;
for i=2:M-1;
Q(i,i+1)=b;
end;
for i=3:M;
Q(i,i-1)=b;
end;
```

```
Q(2,1)=b;
Q(M,M+1)=b;
Q ;
fii=zeros(M+1,1) ;
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii(j,k)=exp(-(k-1)*tau)*sin(x)+(p(k)*sin(x));
end;
end;
G=inv(R);
u=zeros(M+1,1);
for k=3:N+1;
for j=1:M+1;
x=(j-1)*h;
u(j,1)=sin(x);
u(j,2)=(1-tau+((tau^2)/(2)))*sin((j-1)*h);
u(:,k)=G*(-(B*u(:,k-1))-(Q*u(:,k-2))+fii(:,k-1));
end;
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(j,k)=(1-t)*sin(x);
% ep(k)=exp(-t);
eu(j,k)=exp(-t)*sin(x);
end;
end;
for k=2:N;
```

```
t=(k-1)*tau;
ep(k)=exp(-t);
end
% ABSOLUTE DIFFERENCES ;
absdiff=max(max(abs(es-W)))
absdiff=max(max(abs(ep-p)))
absdiff=max(max(abs(eu-u)))
```

Appendix 4

Matlab Programming

Matlab Implementation of Difference Schemes (3.42)

```
N=160;M=160;
h=pi/M; tau=1/N;
a=(1/(tau^2))+(2/(h^2));e=-2/(tau^2);b=-1/(h^2);g=1/(tau^2);
r=0;
for i=1:M-1;
r=r+h*(1+sin(2*(i)*h));
end;
r;
z=2*(cos(2*h)-1)/(r*(h));
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=z*sin(2*(i-1)*h);
end;
end;
for i=2:M;
A(i,i)=a+(z*sin(2*(i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(z*sin(2*(i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(z*sin(2*(i-1)*h));
end;
A(1,1)=1; A(M+1,M+1)=-1; A(1,M+1)=-1; A(M+1,1)=-1; A(M+1,M)=1; A(M+1,2)=1;
A(2,1)=b; A(M,M+1)=b;A;
```

```
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii=zeros(M+1,1) ;
for j=2:M;
for k=2:N;
t=(k)*tau; x=(j-1)*h;
fii(j,k)=((2*(pi)*(cos(2*h)-1)/(r*(h^2)))+4)*exp(-2*(k)*tau)*sin(2*(j-1)*h);
end;
end;
fii;
G=inv(A);
W=zeros(M+1,1);
forj=1:M+1;
x=(j-1)*h;
W(j,1)=1+sin(2*(j-1)*h);
W(j,2)=(1-2*tau)*(1+sin(2*(j-1)*h));
for k=3:N+1;
W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));
end;
end;
D=0;
for k=2:N;
for j=1:M-1;
```

```
S(j)=D+(W(j,k+1)-2*(W(j,k))+W(j,k-1));
D=S(j);
end;
p(k)=(pi)*(exp(-2*(k-1)*tau))*(1/r)-((h*D)/(4*r)*(tau^2));
end;
p(k);
L=zeros(M+1,M+1);
for i=2:M;
L(i,i)=a;
end;
for i=2:M;
L(i,i+1)=b;
end;
for i=2:M;
L(i,i-1)=b;
end;
L(1,1)=1; L(M+1,M+1)=-1; L(1,M+1)=-1;L(M+1,1)=-1; L(M+1,M)=1; L(M+1,2)=1;
L;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii=zeros(M+1,M+1) ;
for j=2:M;
for k=2:N;
```

```
x=(j-1)*h; t=(k)*tau;
fii(j,k)=4*exp(-2*t)*sin(2*(j-1)*h)+p(k)*(4+4*sin(2*(j-1)*h));
end;
end;
fii;
G=inv(L);
u=zeros(M+1,1);
for j=1:M+1;
x=(j-1)*h;
u(j,1)=(1+sin(2*(j-1)*h));
u(j,2)=(1-2*tau)*(1+sin(2*(j-1)*h));
for k=3:N+1;
u(:,k)=G*(-(B*u(:,k-1))-(C*u(:,k-2))+fii(:,k-1));
end;
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau; x=(j-1)*h;
es(j,k)=(-2*t+1)*(1+sin(2*(j-1)*h));
eu(j,k)=exp(-2*t)*(1+sin(2*(j-1)*h));
 end;
end;
for k=2:N;
t=(k-1)*tau;
ep(k)=exp(-2*t);
end;
% ABSOLUTE DIFFERENCES ;
absdiff=max(max(abs(es-W)))
absdiff=max(max(abs(ep-p)))
absdiff=max(max(abs(eu-u)))
```

Appendix 5

Matlab Programming

Matlab Implementation of Difference Schemes (3.48)

```
clc; clear all ; close all;
N=80;M=80;
h=pi/M; tau=1/N;
d=0;
for i=1:M;
d=d+h*(1+sin(2*(i-1)*h));
end;
a=(1/(tau^2))+(1/(2*(h^2)));
b=-1/(4*(h^2));
q=(1/(h^2))-(2/(tau^2));
r=-1/(2*(h^2));
e=(cos(2*h)-1)/(d*h);
c=(cos(2*h)-1)/(2*d*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=c*sin(2*(i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+c*sin(2*(i-1)*h);
end;
for i=2:M-1;
A(i,i+1)=b+c*sin(2*(i-1)*h);
end;
for i=3:M;
```

```
A(i,i-1)=b+c*sin(2*(i-1)*h);
end;
A(1,1)=1; A(1,M+1)=-1; A(2,1)=b; A(M,M+1)=b;
A(M+1,1)=-3; A(M+1,2)=4; A(M+1,3)=-1;
A(M+1,M-1)=-1;A(M+1,M)=4;A(M+1,M+1)=-3;
B=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
B(i,j)=e*sin(2*(i-1)*h);
end;
end;
for i=2:M
B(i,i)=q+(e*sin(2*(i-1)*h));
end;
for i=2:M-1;
B(i,i+1)=r+(e*sin(2*(i-1)*h));
end;
for i=3:M;
B(i,i-1)=r+(e*sin(2*(i-1)*h));
end;
B(2,1)=r;B(M,M+1)=r;
B;
C=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
C(i,j)=c*sin(2*(i-1)*h);
end;
end;
for i=2:M
C(i,i)=a+(c*sin(2*(i-1)*h));
end;
```

```
for i=2:M-1;
C(i,i+1)=b+(c*sin(2*(i-1)*h));
end;
for i=3:M;
C(i,i-1)=b+(c*sin(2*(i-1)*h));
end;
C(2,1)=b;C(M,M+1)=b;
fii=zeros(M+1,1);
for j=2:M;
for k=1:N+1;
fii(j,k)=(4*exp(-2*(k-1)*tau))*sin(2*(j-1)*h)
+((c*(pi))/h)*(exp(-2*(k)*tau)+2*exp(-2*(k-1)*tau)+exp(-2*(k-2)*tau))*sin(2*(j-1)*h);
end;
end;
ee1=-tau/(h^2);
ee2=(2*tau)/(h^2)+1/tau;
y=((2*tau)*(cos(2*h)-1))/(d*h);
E=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
E(i,j)=y*sin(2*(i-1)*h);
end;
end;
for i=2:M
E(i,i)=ee2+y*sin(2*(i-1)*h);
end;
for i=2:M-1;
E(i,i+1)=ee1+y*sin(2*(i-1)*h);
end;
for i=3:M;
E(i,i-1)=ee1+y*sin(2*(i-1)*h);
```

```
end;
E(1,1)=1; E(1,M+1)=-1; E(2,1)=ee1; E(M,M+1)=ee2;
E(M+1,1)=-3; E(M+1,2)=4;E(M+1,3)=-1;
E(M+1,M-1)=-1;E(M+1,M)=4;E(M+1,M+1)=-3;
f1=tau/(2*(h^2));f2=-tau/(h^2)-1/tau;
F=zeros(M+1,M+1);
for i=2:M
F(i,i)=f2;
end;
for i=2:M-1;
F(i,i+1)=f1;
end;
for i=3:M;
F(i,i-1)=f1;
end;
F(2,1)=f1;
F(M,M+1)=f1
phy=zeros(M+1,1);
for j=1:M+1;
phy(j)=-2*(1+sin(2*(j-1)*h))+(2*tau)*sin(2*(j-1)*h)+((y*pi)/h)*(exp(-2*tau))*sin(2*(j-
        1)*h);
end;
Z=inv(E);
G=inv(A);
W=zeros(M+1,1);
for j=1:M+1;
W(j,1)=1+sin(2*(j-1)*h);
end;
W(:,2)=Z*(phy-F*W(:,1));
for k=3:N+1;
W(:,k)=G*(-(B*W(:,k-1))-(C*W(:,k-2))+fii(:,k-1));
```

```
end;
for k=2:N;
 D=0;
for j=1:M-1;
S(j)=D+W(j,k+1)-2*(W(j,k))+W(j,k-1);
D=S(j);
end;
p(k)=((pi)*(exp(-(k)*tau)-2*exp(-(k-1)*tau)+exp(-(k-2)*tau))-(h*D))/(4*d*(tau^2));
end;
p(k);
R=zeros(M+1,M+1);
for i=2:M
R(i,i)=a;
end;
for i=2:M-1;
R(i,i+1)=b;
end;
for i=3:M;
R(i,i-1)=b;
end;
R(1,1)=1;
R(M+1,M+1)=1;
R(2,1)=b;
R(M,M+1)=b;
R;
L=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
L(i,j)=0;
end;
end;
```

```
for i=2:M
L(i,i)=q;
end;
for i=2:M;
L(i,i+1)=r;
end;
for i=3:M;
L(i,i-1)=r;
end;
L(2,1)=r;
L(M,M+1);
L;
Q=zeros(M+1,M+1);
for i=2:M
Q(i,i)=a;
end;
for i=2:M-1;
Q(i,i+1)=b;
end;
for i=3:M;
Q(i,i-1)=b;
end;
Q(2,1)=b;
Q(M,M+1)=b;
Q ;
fii=zeros(M+1,M+1) ;
for j=2:M;
for k=2:N;
x=(j-1)*h;
t=(k)*tau;
fii(j,k)=4*exp(-2*t)*sin(2*(j-1)*h)+4*p(k)*(1+sin(2*(j-1)*h));
```

```
end;
end;
fii;
G=inv(R);
u=zeros(M+1,1);
for k=3:N+1;
for j=1:M+1;
x=(j-1)*h;
u(j,1)=1+sin(2*(j-1)*h);
u(j,2)=(1-2*tau+2*(tau^2))*(1+sin(2*(j-1)*h));
u(:,k)=G*(-(L*u(:,k-1))-(Q*u(:,k-2))+fii(:,k-1));
end;
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE' \%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(j,k)=(-2*t+1)*(1+sin(2*(j-1)*h));
eu(j,k)=exp(-2*t)*(1+sin(2*(j-1)*h));
end;
end;
 for k=2:N;
 t=(k-1)*tau;
 ep(k)=exp(-2*t);
 end;
% ABSOLUTE DIFFERENCES ;
absdiff=max(max(abs(es-W)))
absdiff=max(max(abs(ep-p)))
absdiff=max(max(abs(eu-u)))
```