STRONGLY POSITIVE OPERATORS WITH NONLOCAL CONDITIONS AND THEIR APPLICATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
AYMAN OMAR ALI HAMAD

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy of Science in Mathematics

Examining Committee in Charge:

Prof. Dr. Evren Hinçal	Committee Chairman, Department of Mathematics, NEU
Prof. Dr. Allaberen Ashyralyev	Supervisor, Department of Mathematics, NEU
Assoc. Prof. Dr. Okan Gerçek	Department of Computer Engineering, Girne American University
Assoc. Prof. Dr. Murat Tezer	Department of Primary Mathematics Teaching, University NEU
Assist. Prof. Dr. Bilgen Kaymakamzade	Department of Mathematics, NEU

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Name, Last name: Ayman Omar Ali Hamad

Signature:

Date:

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To my parents…

ABSTRACT

The present thesis deals with strongly positive operators with nonlocal conditions and their applications. The structure of fractional powers of positive operators in fractional spaces are given. The well-posedness of the abstract nonlocal boundary value problem for differential equation of the elliptic type

$$-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T), v(0) = v(T) + \varphi, \int_0^T v(s)ds = \psi$$

in an arbitrary Banach space E with the positive operator A is established. The coercive stability estimates in Hölder norms for the solution of three type elliptic problems are obtained. The second order of approximation two-step difference scheme for the numerical solution of a nonlocal boundary value problem is presented. The well-posedness of difference problems in Banach spaces is established. The stability, almost coercive stability and coercive stability estimates for the solutions of difference schemes for the numerical solution of elliptic problems are obtained. Illustrative numerical results for two and three dimensional case are provided.

Keywords: Fractional powers; interpolation spaces; fractional derivatives; positive operators; elliptic operators; well-posedness; coercive stability; difference scheme

ÖZET

Bu tez, yerel olmayan koşullar ve bunların uygulamaları ile birlikte güçlü pozitif operatörler ile ilgilidir. Kesirli uzaylarda pozitif operatörlerin kesirli mertebelerinin yapısı verilmiştir. Eliptik tipin diferansiyel denklemi için yerel olmayan sınır değer probleminin iyi kuruluşu

$$-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T), v(0) = v(T) + \varphi, \int_0^T v(s)ds = \psi$$

keyfi bir Banach uzayında E pozitif operatör A ile kurulur. Üç tip eliptik problemin çözümü için Hölder normlarında zorunlu olarak istikrarlı tahminler elde edilmiştir. Yerel olmayan bir sınır değer probleminin sayısal çözümü için yakınsak iki aşamalı fark şemasının ikinci mertebesi sunulmuştur. Banach uzaylarındaki farklılık sorunlarının iyi oluşu kurulmuştur. Kararlılık, neredeyse zorlayıcı kararlılık tahminleriyle, eliptik problemlerin sayısal çözümü için fark şemalarının çözümlerinin tahminleri elde edilmiştir. İki ve üç boyutlu durumlar için açıklayıcı sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Kesirli mertebeler; interpolasyon uzayları; kesirli türevler; pozitif operatörler; eliptik operatörler; iyi konumlanmışlık; zorlayıcı kararlılık; fark şemaları

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CHAPTER 1
INTRODUCTION

The method of operators as a tool for the investigation of the solution to partial differential equations in Hilbert and Banach spaces, has been systematically developed by many authors. It is well-known that various local and nonlocal problems for partial differential equations can be reduced to local and problems for ordinary differential equations in Hilbert or Banach spaces with unbounded positive operator. The role played by positivity property of differential and difference operators in Hilbert and Banach spaces in the study of various properties of boundary value problems for partial differential equations, of stability of difference schemes for partial differential equations, and of summation Fourier series is well-known (Ashyralyev & Sobolevskii, 1994; Ashyralyev& Sobolevskii, 2004; Krasnosel'skii, et al., 1966; Sobolevskii, 2005).

Important progress has been made in the study of positive operators from the view-point of the stability analysis of difference schemes for partial differential equations. It is well known that the most useful methods for stability analysis of difference schemes are difference analogue of maximum principle and energy method. The application of theory of positive difference operators allows us to investigate the stability and coercive stability properties of difference schemes in various norms for partial differential equations especially when one can not use a maximum principle and energy method. Moreover, the structure of fractional spaces generated by positive differential and difference operators and its applications to partial differential equations has been investigated by many researchers. Finally, a survey of results in fractional spaces generated by positive operators and their applications to partial differential equations was given in paper of Ashyralyev, 2015. Nevertheless, structure of fractional powers generated by differential and difference operators and its applications to partial differential equations has not been investigated a sufficiently.

The present work is devoted to the study of applications of second order differential operator with nonlocal conditions. Investigation of the structure of fractional spaces generated by positive operator with nonlocal conditions in a Banach space. It consists five chapters. The first chapter is introduction. In the second chapter we consider the definitions of positive operator in a Banach space, of fractional power of positive operator, of fractional spaces genareted by positive opaters and essential statements and estimates which will be useful in the sequel. The simply two differential positive operators in Banach and Hilbert spaces are considered. The structure of fractional spaces generated by positive operator in a Banach space is investigated. In applications, we give the structure of fractional powers of elliptic operators in Banach norms. In the third chapter five nonlocal boundary value problems are solved analytically by Fourier series, Fourier transform and Laplace transform methods. We consider the nonlocal boundary value problem for elliptic equations in a Banach space. The well-posedness of the differential problem in various Banach spaces is established. In applications, the new coercive stability estimates in Hölder norms for the solutions of the mixed type nonlocal boundary value problems for elliptic equations are obtained. In the fourth chapter we present second order of accuracy two-step difference scheme for the approximate solution of the nonlocal boundary value problem for elliptic equations in a Banach space. The well-posedness of the difference problem in various Banach spaces is established. In applications, the new stability, almost coercive stability and coercive stability estimates in Hölder norms for the solutions of the difference schemes for the approximate solution of the nonlocal boundary value problem for elliptic equations are obtained. Numerical analysis is given. The fifth chapter is conclusions. Basic results of this thesis have been published by the following papers (Ashyralyev and Hamad, 2017; Ashyralyev and Hamad, 2018a, 2018b, 2018c; Ashyralyev and Hamad, 2019). Some results of this work were presented in seminar “Analysis and Applied Mathematics Seminar Series” of Department of Mathematics, Near East University and in VI congress of Turkic World Mathematical Society (TWMS 2017), and Fourth International Conference on Analysis and Applied Mathematics (ICAAM 2018), and in 2nd International Conference of Mathematical Sciences, Maltepe University, Istanbul in International summer mathematical school in memoriam V.A. Plotnikov, Odessa National University, Odessa, Ukraine.

CHAPTER 2

STRUCTURE OF FRACTIONAL SPACES AND THEIR APPLICATIONS

This chapter consists three sections, In the first section we consider the definition of positive operators, the fractional power of positive operator, statements and estimates concerning the semigroup $\exp\{-tA\}(t \geq 0)$ from (Ashyralyev and Sobolevskii, 2012; Krasnosel'skii et al., 1966) which will be useful in the sequel. The some examples are given for explanation their. In the second, the main Theorem on the structure of fractional spaces D(A^{β},$E_{\alpha,q}$(E,A)) is proved. Applications of this theorem are included in the third section.

2.1 INTRODUCTION

Definition. The operator A is said to be strongly positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ, $0 < 2\phi < \pi$, symmetric with respect to the real axis and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \leq \rho < \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \leq \rho < \infty\}$ and outside of it, the resolvent $(\lambda - A)^{-1}$ is subject to the bound

$$\left\|(\lambda - A)^{-1}\right\|_{E\to E} \leq \frac{M(\phi)}{1 + |\lambda|}. \tag{2.1}$$

The infimum of all such angles ϕ is called the spectral angle of the strongly positive operator A and is denoted by $\phi(A) = \phi(A, E)$. Since the spectrum $\sigma(A)$ is a closed set, it lies inside the sector formed by the rays $S_1(\phi(A))$ and $S_2(\phi(A))$ and some neighborhood of the apex of this sector does not intersect $\sigma(A)$. We shall consider contours $\Gamma = \Gamma(\phi, r)$ composed by the rays $S_1(\phi)$, $S_2(\phi)$ and an arc of circle of radius r centered at the origin; ϕ and r will be chosen so that $\phi(A) < |\phi| < \pi/2$ and the arc of circle of radius r lies in the resolvent set $\rho(A)$ of the operator A.

Let $f(z)$ be an analytic function on the set bounded by such a contour Γ and suppose that f satisfies estimate

$$|f(z)| \leq M|z|^{-\varepsilon}$$

for some $\varepsilon > 0$. Then the operator Cauchy-Riesz integral

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(z - A)^{-1}\,dz \tag{2.2}$$

converges in the operator norm and defines a bounded linear operator $f(A)$ which is a function of the strongly positive operator A. If $f(z)$ is continuous in a neighbourhood of the origin, then in (2.2) we shall consider that $r = 0$, i.e., $\Gamma = S_1(\phi) \cup S_2(\phi)$.

As in the case of a bounded operator A one shows that $f(A)$ does not depend on the choice of the contour Γ in the domain of analyticalness of the function $f(z)$ and that the correspondence between the function $f(z)$ and the operator $f(A)$ is linear and multiplicative.

The function $f(z) = z^{-\alpha}$ defines a bounded operator $A^{-\alpha}$ whenever $\alpha > 0$. Here the contour Γ is chosen with $r > 0$. By the multiplicative property, $A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta} = A^{-\beta}A^{-\alpha}$ is satisfied for any powers of the strongly positive operator A and not only for negative integer ones. From this identity it follows (when $\alpha + \beta$ is an integer) that the equation $A^{-\alpha}x = 0$ has the unique solution $x = 0$. Hence, the positive powers $A^{\alpha} = (A^{-\alpha})^{-1}$ of the strongly positive operator are defined. The operators A^{α} $(\alpha > 0)$ are unbounded if A is unbounded; they have dense domains $D(A^{\alpha})$ and one has the continuous embeddings $D(A^{\alpha}) \subset D\left(A^{\beta}\right)$ if $\beta < \alpha$.

Now let us consider the function $f(z) = e^{-tz}$. For any $t > 0$ this function tends to zero faster any power $z^{-\alpha}$ as $|z| \to \infty$ and its values lie inside any sector bounded by a contour Γ. Therefore, formula (2.2) can be used to define the function $\exp\{-tA\}$ of the strongly positive operator A. By multiplicative, the semigroup property holds:

$$\exp\{-(t_1 + t_2)A\} = \exp\{-t_1A\}\exp\{-t_2A\}, \ t_1, t_2 > 0.$$

Consider the function $\Psi(z) = z^{\alpha}e^{-tz}$ for some $\alpha > 0$ and $t > 0$. Since, obviously, $\Psi(z) \to 0$ faster than any negative power of z as $|z| \to \infty$, $\Psi(z)$ defines the operator function

$$\Psi(A) = \frac{1}{2\pi i}\int_{\Gamma} z^{\alpha}e^{-tz}(z - A)^{-1}\,dz. \tag{2.3}$$

Let us show that the operator $\exp\{-tA\}$ maps E into $D(A^{\alpha})$ and $A^{\alpha}\exp\{-tA\} = \Psi(A)$. Let x be an arbitrary element of E. By the multiplicativity property, (2.3) implies that

$$A^{-\alpha}\Psi(A)\,x = \frac{1}{2\pi i}\int_{\Gamma} e^{-tz}(z - A)^{-1}\,xdz = \exp\{-tA\}\,x$$

which proves our assertion. Thus, we have the formula

$$A^{\alpha}\exp\{-tA\} = \frac{1}{2\pi i}\int_{\Gamma} z^{\alpha}e^{-tz}(z - A)^{-1}\,dz. \tag{2.4}$$

In the above argument we must assume that the contour Γ contains an arc of radius r, since we applied the operator $A^{-\alpha}$, which corresponds to the function $z^{-\alpha}$. The final formula (2.4) is valid for any (small) $r > 0$. Since the integrand in (2.4) is continuous at the point $z = 0$, letting $z \to 0$ we obtain the formula

$$A^{\alpha} \exp\{-tA\} = \frac{1}{2\pi i}\left[\int_{\infty}^{0} \rho^{\alpha} e^{i\alpha\phi} e^{-t\rho e^{i\phi}} \left(\rho e^{i\phi} - A\right)^{-1} d\rho\right.$$

$$\left. + \int_{0}^{\infty} \rho^{\alpha} e^{-i\alpha\phi} e^{-t\rho e^{-i\phi}} \left(\rho e^{-i\phi} - A\right)^{-1} d\rho\right]$$

for some $0 < \phi < \pi/2$. From this and the estimate (2.1) it follows that

$$\left\|A^{\alpha} \exp\{-tA\}\right\|_{E\to E} \le \frac{M(\phi)}{\pi} \int_{0}^{\infty} \rho^{\alpha-1} e^{-t\rho\cos\phi} d\rho = \frac{M(\phi)\,\Gamma(\alpha)}{\pi(\cos\phi)^{\alpha}} t^{-\alpha}.$$

In particular, we have the estimate

$$\left\|\exp\{-tA\}\right\|_{E\to E} \le \frac{M(\phi)}{\pi}. \tag{2.5}$$

Let us show that the estimate (2.5) can be sharpened by a factor that decays exponentially when $t \to +\infty$.

Let A be a strongly positive operator. We claim that for sufficiently small $\delta > 0$ the operator $A - \delta$ is also strongly positive and $\phi(A - \delta) = \phi(A)$. Indeed, let $\lambda \in \Gamma(\phi)$. Consider the equation $\lambda x - (A - \delta)x = y$ for an arbitrary $y. \in E$. The substitution $\lambda x - Ax = z$ yields the equation $z + \delta(\lambda - A)^{-1} z = y$. Since

$$\left\|\delta(\lambda - A)^{-1}\right\|_{E\to E} \le \delta M(\phi)$$

if $\lambda \in \Gamma(\phi)$, we see that for $\delta \le [2M(\phi)]^{-1}$the equation for z has a unique solution and $\|z\| \le 2\|y\|$. Consequently, the equation for x has a unique solution and

$$\|x\| \le M(\phi)\,[|\lambda| + 1]^{-1}\,\|z\| \le 2M(\phi)\,[|\lambda| + 1]^{-1}\,\|y\|\,.$$

This means that the operator $\lambda - (A + \delta)$ has a bounded inverse for
$0 < \delta \le [2M(\phi)]^{-1}$ and

$$\left\|[\lambda - (A - \delta)]^{-1}\right\|_{E\to E} \le 2M(\phi)\,[|\lambda| + 1]^{-1}\,.$$

Thus, we have shown that $A - \delta$ is a strongly positive operator. Hence, by (2.5), we have the estimate

$$\left\|\exp\{-(A-\delta)\,t\}\right\|_{E\to E} \le \frac{2M(\phi)}{\pi}.$$

This obviously yields

$$\left\|\exp\{-At\}\right\|_{E\to E} \le \frac{2M(\phi)}{\pi}e^{-\delta t}, \tag{2.6}$$

where we can put $\delta = [2M(\phi)]^{-1}$.

Let $t > 1$. Then, using the semigroup property, we can write

$$\exp\{-tA\} = \exp\{-A\}\exp\{-(t-1)\,A\}.$$

Next, applying the estimates (2.5) with $t = 1$ and (2.6), we obtain

$$\left\|A^{\alpha}\exp\{-tA\}\right\|_{E\to E} \le \frac{M(\phi)}{\pi(\cos\phi)^{\alpha}}\frac{2M(\phi)}{\pi}e^{-\delta(t-1)}.$$

Hence, the following estimate holds for $t > 1$:

$$\left\|A^{\alpha}\exp\{-tA\}\right\|_{E\to E} \le M_1(\phi)\,e^{-\delta t}.$$

If $0 < t \le 1$, then estimate (2.5) trevails. Combining these two estimates, we conclude that

$$\left\|A^{\alpha}\exp\{-tA\}\right\|_{E\to E} \le \tilde{M}(\phi)\,e^{-\delta t}t^{-\alpha} \tag{2.7}$$

for some $\tilde{M}(\phi) > 0$ and $\delta > 0$.

Further, formula (2.2) allows us to establish that the operator- valued function $\exp\{-tA\}$ is differentiable in the operator norm for $t > 0$ and

$$\frac{d}{dt}\exp\{-tA\} = -A\exp\{-tA\}. \tag{2.8}$$

In particular, this implies that $\exp\{-tA\}$ is continuous in the operator norm. Using the semigroup property we deduce that the derivative of $\exp\{-tA\}$ is also continuous in the operator norm for $t > 0$. Finally, formula (2.8) shows that the operator-valued function $\exp\{-tA\}$ has derivative of arbitrary order in the operator norm for $t > 0$.

Now, let $x \in D(A)$. Then the ($E-$valued) function $\exp\{-tA\}\, x$ has a derivative for $t > 0$ and, by (2.8),

$$\frac{d}{dt}\exp\{-tA\}\, x = -\exp\{-tA\}\, Ax.$$

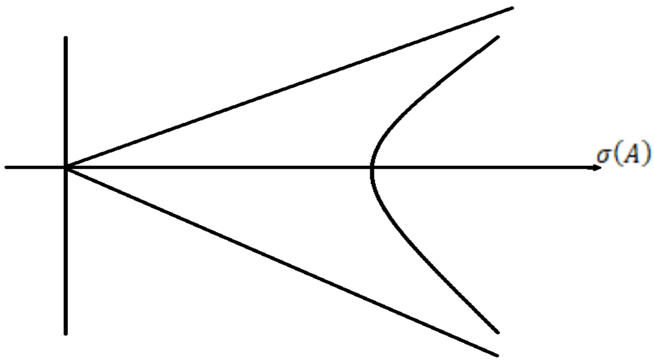
Next, for x as above we can write

$$(z-A)^{-1}\, x = z^{-1}x + z^{-1}(z-A)^{-1}\, Ax.$$

Using formula (2.2), we obtain

$$\exp\{-tA\}\, x = \frac{1}{2\pi i}\int_{\Gamma} e^{-tz}\left[z^{-1}x + z^{-1}(z-A)^{-1}\, Ax\right] dz.$$

Here the contour Γ has the form



Using the Cauchy theorem, we get

$$\exp\{-tA\}\, x = \frac{1}{2\pi i}\int_{\Gamma} e^{-tz}z^{-1}(z-A)^{-1}\, Axdz + x.$$

The estimate (2.1) shows that in the last equality one can pass to the limit under the integral sign when $t \to +0$. Hence, the limit

$$\lim_{t\to+0}\exp\{-tA\}\, x = x + \frac{1}{2\pi i}\int_{\Gamma} z^{-1}(z-A)^{-1}\, Axdz.$$

exists (in the norm of E). By Cauchy's theorem, the integral

$$\vartheta = \frac{1}{2\pi i}\int_{\Gamma} z^{-1}(z-A)^{-1}\, Axdz = \frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} z^{-1}(z-A)^{-1}\, Axdz.$$

for some $\sigma > 0$. Hence, by (2.1),

$$\|\vartheta\|_E \le \frac{M}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{\sigma^2+t^2}\,\|Ax\|_E\,.$$

Since ϑ does not depend on σ, it follows that $\vartheta \equiv 0$. Hence, we proved that

$$\lim_{t\to+0} \exp\{-tA\} x = x \tag{2.9}$$

for any $x \in D(A)$. Since the norm $\|\exp\{-tA\}\|_{E\to E}$ is uniformly bounded for $t > 0$, the limit relation (2.9) holds for any $x \in E$. Thus, if we extend the operator- valued function $U(t) = \exp\{-tA\}$, $t > 0$, at $t = 0$ by $U(0) = I$, we obtain a strongly continuous semigroup. From the estimate (2.7) (with $\alpha = 0$) it follows that this semigroup is analytic. Finally, let us show that its generator is $U'(0) = -A$. From (2.6) and the estimate (2.7) we derive the identity

$$U(t)x - x = -\int_0^t U(s) Axds$$

for $x \in D(A)$. Since $U(t)$ is strongly continuous to the left at the point $t = 0$, this implies that $x \in D(U'(0))$ and $U'(0)x = -Ax$. Hence, $U'(0)$ is an extension of the operator $-A$. By the estimate(2.6), the operator $U'(0) + \lambda$ and $-A + \lambda$ have bounded inverses for any $\lambda < 0$. Therefore, $U'(0) = -A$.

We have shown that the operator-valued function $\exp\{-tA\}$ is an analytic semigroup with generator $-A$ and with an exponentially decaying norm. Operators $-A$ that generate such semigroups were called strongly positive operators.

With the help of a strongly positive operator A we introduce the Banach space $E_{\alpha,q}(E, A)$, $0 < \alpha < 1$, consisting of all $v \in E$ for which the following norms are finite:

$$\|v\|_{E_{\alpha,q}} = \left(\int_0^\infty \left\|\lambda^{1-\alpha} A \exp\{-\lambda A\}v\right\|_E^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}, 1 \le q < \infty,$$

$$\|v\|_{E_\alpha} = \|v\|_{E_{\alpha,\infty}} = \sup_{\lambda>0} \left\|\lambda^{1-\alpha} A \exp\{-\lambda A\}v\right\|_E.$$

For all $v \in E$ with a strongly positive operator A and $-1 < \alpha < 0$, the following norms are finite

$$\|v\|_{E_{\alpha,q}} = \left(\int_0^\infty \left\|\lambda^{-\alpha} \exp\{-\lambda A\}v\right\|_E^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}, 1 \le q < \infty,$$

$$\|v\|_{E_{\alpha,\infty}} = \|v\|_{E_\alpha} = \sup_{\lambda>0} \left\|\lambda^{-\alpha} \exp\{-\lambda A\}v\right\|_E,$$

we define the fractional space $E_{\alpha,q}(E,A), -1 < \alpha < 0$. The replenishment of space E in this norm forms a Banach space $E_{\alpha,q}(E,A), -1 < \alpha < 0, 1 \leq q \leq \infty$.
Clearly, the positive operator commutes A and its resolvent $(A-\lambda)^{-1}$. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(E,A)$, $E_{\alpha,p} = E_{\alpha,p}(E,A), 1 \leq p < \infty, (-1 < \alpha < 1)$, we get

$$\|(A-\lambda)^{-1}\|_{E_\alpha \to E_\alpha}, \|(A-\lambda)^{-1}\|_{E_{\alpha,p} \to E_{\alpha,p}} \leq \|(A-\lambda)^{-1}\|_{E \to E}.$$

Thus, from the positivity of operator A in the Banach space E it follows the positivity of this operator in fractional spaces $E_\alpha = E_\alpha(E,A)$, $E_{\alpha,p} = E_{\alpha,p}(E,A), 1 \leq p < \infty, (-1 < \alpha < 1)$.

Let us consider the selfadjoint positive definite operator A in a Hilbert space H with dense domain $\overline{D(A)} = H$. That means there exists $\delta > 0$ such that $A = A^* \geq \delta I$. Then, applying the spectral representation of the selfadjoint positive definite operator, we can get

$$\left\|(A-\lambda)^{-1}\right\|_{H \to E} \leq \sup_{\delta \leq \mu < \infty} \frac{1}{|\mu - \lambda|}. \tag{2.10}$$

It is easy to see that from (2.10) it follows that the selfadjoint positive definite operator A in a Hilbert space H is the strongly positive operator with the spectral angle $\varphi(A,H) = 0$. Therefore, the positivity of operators in a Banach space is the generalization of the notion of selfadjoint positive definite operators in a Hilbert space.
Now, let us consider two examples of positive operators in Banach spaces.
1. Let $C(R^1)$ be the Banach space of continuous scalar functions $f(x)$ on $R^1 = (-\infty,\infty)$ satisfying condition $f(x) \to 0$ as $|x| \to \infty$, with the norm $\|f\|_{C(R^1)} = \sup_{x \in R^1} |f(x)|$. Let A be the operator acting in $C(R^1)$ according to the rule $Av(x) = -v''(x) + v(x)$, so that we also have $v''(x) \in C(R^1)$. It is easy that A is the self-adjoint positive-definite operator in $L_2(R^1)$. Here $L_2(R^1)$ is the Hilbert space of square-interability scalar functions $f(x)$ on R^1 with the norm

$$\|f\|^2_{L_2(R^1)} = \int\limits_{x \in R^1} |f(x)|^2 \, dx.$$

Actually, for all $u, v \in L_2(R^1)$ we have that

$$\langle Au, v\rangle = \int_{-\infty}^{\infty} Au(x)v(x)\,dx = -\int_{-\infty}^{\infty} \frac{d}{dx}\left(\frac{du(x)}{dx}\right) v(x)\,dx + \int_{-\infty}^{\infty} u(x)v(x)\,dx$$

$$-\left(\frac{du(x)}{dx}\right)v(x)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{du(x)}{dx}\frac{dv(x)}{dx}dx+\int_{-\infty}^{\infty}u(x)v(x)\,dx$$
$$=\int_{-\infty}^{\infty}\frac{du(x)}{dx}\frac{dv(x)}{dx}dx+\int_{-\infty}^{\infty}u(x)v(x)\,dx,$$
$$\langle u,Av\rangle=\int_{-\infty}^{\infty}u(x)\,Av(x)dx=-\int_{-\infty}^{\infty}u(x)\frac{d}{dx}\left(\frac{dv(x)}{dx}\right)dx+\int_{-\infty}^{\infty}u(x)v(x)\,dx$$
$$-u(x)\left(\frac{dv(x)}{dx}\right)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{dv(x)}{dx}\frac{du(x)}{dx}dx+\int_{-\infty}^{\infty}u(x)v(x)\,dx$$
$$=\int_{-\infty}^{\infty}a(x)\frac{dv(x)}{dx}\frac{du(x)}{dx}dx+\int_{-\infty}^{\infty}u(x)v(x)\,dx.$$

From that it follows $\langle Au,v\rangle=\langle u,Av\rangle$ and

$$\langle Au,u\rangle=\int_{-\infty}^{\infty}\frac{du(x)}{dx}\frac{du(x)}{dx}dx+\int_{-\infty}^{\infty}u(x)u(x)\,dx\geq\int_{-\infty}^{\infty}u(x)u(x)\,dx=\langle u,u\rangle. \quad (2.11)$$

For the self adjoint positive definite operator A we will introduce the operator-valued function $\exp\{-tA\}$ defined by formula $u(t)=\exp\{-tA\}\varphi$, where abstract function $u(t)$ is the solution of the following Cauchy problem in a Hilbert space $H=L_2(R^1)$

$$u'(t)+Au(t)=0,t>0,u(0)=\varphi. \quad (2.12)$$

and the following estimates hold

$$\left\|\exp\{-tA\}\right\|_{H\to H}\leq e^{-t},\left\|tA\exp\{-tA\}\right\|_{H\to H}\leq e. \quad (2.13)$$

It is based on the spectral represents of unit self adjoint positive definite operator A and

$$\|f(A)\|_{H\to H}\leq\sup_{\delta\leq\lambda<\infty}|f(\lambda)|.$$

Here f is the bounded function on $[\delta,\infty)$. Therefore, the operator A in a Hilbert space $H=L_2(R^1)$ is the strongly positive operator with the spectral angle $\varphi(A,H)=0$.

Moreover, this differential operator A is the strongly positive operator in Banach spaces $E=L_p\left(R^1\right),1\leq p<\infty,C^{\alpha}\left(R^1\right),0\leq\alpha<1$.

It is based on the triangle inequality and formula

$$\exp\{-tA\}\varphi(x)=\frac{e^{-t}}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}\varphi(y)dy, \quad (2.14)$$

First, we will proof the formula (2.14). Using the definition of operator function $\exp\{-tA\}$, we can write

$$u(t,x) = \exp\{-tA\}\varphi(x),$$

where $u(t,x)$ is the solution of the following Cauchy problem

$$u_t(t,x) - u_{xx}(t,x) + u(t,x) = 0, t > 0, u(0,x) = \varphi(x), x \in R^1 \tag{2.15}$$

for the parabolic equation with smooth $\varphi(x)$. Assume that $\varphi(\pm\infty) = 0$. Taking the Fourier transform, we get the following Cauchy problem

$$u_t(t,s) + s^2 u(t,s) + u(t,s) = 0, t > 0, u(0,s) = F\{\varphi(x)\}$$

for the first order ordinary differential equation. Taking the Laplace transform, we get

$$\mu u(\mu,s) - F\{\varphi(x)\} + s^2 u(\mu,s) + u(\mu,s) = 0$$

or

$$u(\mu,s) = \frac{1}{\mu + s^2 + 1} F\{\varphi(x)\}.$$

Applying the inverse Laplace transform, we get

$$u(t,s) = e^{-(s^2+1)t} F\{\varphi(x)\} = e^{-t} e^{-s^2 t} F\{\varphi(x)\} = e^{-t} \frac{1}{2\sqrt{\pi t}} F\left\{ e^{-\frac{(x)^2}{4t}} \right\} F\{\varphi(x)\}.$$

Applying the inverse Fourier transform, we get formula (2.14). Applying formula (2.14), we can get the following estimates

$$\left\| e^{-tA} \right\|_{C(R^1) \to C(R^1)} \le e^{-t}, t \ge 0, \tag{2.16}$$

$$\left\| A e^{-tA} \right\|_{C(R^1) \to C(R^1)} \le \frac{2e^{-1}}{\sqrt{\pi} t}, t > 0. \tag{2.17}$$

2. Now, let $C(R^{1+})$ be the Banach space of continuous scalar functions $f(x)$ on $R^{1+}=[0,\infty)$ satisfying condition $f(x)\to 0$ as $x\to\infty$, with the norm

$$\|f\|_{C(R^{1+})}=\sup_{x\in R^{1+}}|f(x)|\,.$$

Let A be the operator acting in $C(R^{1+})$ according to the rule $Av(x)=-v'(x)+v(x)$, so that we also have $v'(x)\in C(R^{1+})$. It is easy that A is not self-adjoint, but positive-definite operator in $L_2(R^{1+})$.

Here, $L_2(R^{1+})$ is the Hilbert space of square-interability scalar functions $f(x)$ on R^{1+} with the norm

$$\|f\|^2_{L_2(R^{1+})}=\int_{x\in R^{1+}}|f(x)|^2\,dx.$$

Actually, for all $u,v\in L_2(R^{1+})$ we have that

$$\langle Au,v\rangle=\int_0^\infty A(u)v\,(x)\,dx=-\int_0^\infty\frac{du\,(x)}{dx}v\,(x)\,dx+\int_0^\infty u(x)v\,(x)\,dx$$

$$-\,u(x)v(x)|_0^\infty+\int_0^\infty u\frac{dv\,(x)}{dx}dx+\int_0^\infty u(x)v\,(x)\,dx$$

$$=\int_0^\infty u\frac{dv\,(x)}{dx}dx+\int_0^\infty u(x)v\,(x)\,dx+u(0)v(0),$$

$$\langle u,Av\rangle=\int_0^\infty u\,(x)\,A(v)dx=-\int_0^\infty u\,(x)\,\frac{dv\,(x)}{dx}dx+\int_0^\infty u(x)v\,(x)\,dx$$

$$-\,u(x)v(x)|_0^\infty+\int_0^\infty\frac{du\,(x)}{dx}v(x)dx+\int_0^\infty u(x)v\,(x)\,dx$$

$$=\int_0^\infty\frac{du\,(x)}{dx}v(x)dx+\int_0^\infty u(x)v\,(x)\,dx+u(0)v(0).$$

From that it follows $\langle Au,v\rangle\neq\langle u,Av\rangle$. Moreover,

$$\langle Au,u\rangle=-\int_0^\infty\frac{du\,(x)}{dx}udx+\int_0^\infty u(x)u\,(x)\,dx$$

$$=-\left.\frac{u^2(x)}{2}\right|_0^\infty+\int_0^\infty u(x)u\,(x)\,dx=\int_0^\infty u(x)u\,(x)\,dx+\frac{u^2(0)}{2}\geq\langle u,u\rangle\,.$$

That means A is not self-adjoint, but positive-definite operator in $L_2(R^{1+})$.

Moreover, this differential operator A is the positive operator in Banach spaces $E=L_p\left(R^{1+}\right),1\leq p<\infty,C^\alpha\left(R^{1+}\right),0\leq\alpha<1.$

It is based on the triangle inequality and formula

$$\exp\{-tA\}\varphi(x) = e^{-t}\varphi(x+t). \tag{2.18}$$

First, we will proof the formula (2.18). Using the definition of operator function $\exp\{-tA\}$, we can write

$$u(t,x) = \exp\{-tA\}\varphi(x),$$

where $u(t,x)$ is the solution of the following Cauchy problem

$$u_t(t,x) - u_x(t,x) + u(t,x) = 0, t > 0, u(0,x) = \varphi(x), x \in R^{1+} \tag{2.19}$$

for the transport equation with smooth $\varphi(x)$. Assume that $\varphi(\infty) = 0$.
The associated system of equations are

$$\frac{dt}{1} = \frac{dx}{-1} = \frac{du}{-u}.$$

Applying $\frac{dt}{1} = \frac{dx}{-1}$, we get

$$t + x = c_1.$$

Similarly, applying $\frac{dt}{1} = \frac{du}{-u}$, we get

$$-t = \ln u - \ln c_2$$

or

$$u = c_2 e^{-t}.$$

Therefore,

$$e^t u = c_2.$$

Then, using Lagrange's method, we get

$$c_2 = f(c_1).$$

The general solution of the given equation is

$$u(t,x) = e^{-t} f(t+x).$$

Using the initial condition, we get

$$u(0,x) = f(x) = \varphi(x).$$

Then $f(t+x) = \varphi(t+x)$ and

$$u(t,x) = e^{-t}\varphi(t+x).$$

The formula (2.18) is proved. Applying formula (2.18), we can get the following estimate

$$\left\| e^{-tA} \right\|_{C(R^{1+}) \to C(R^{1+})} \leq e^{-t}, t \geq 0. \tag{2.20}$$

The theory of fractional powers of operators can be constructed for a wider class of positive operatorse (even for a more extensive class-weakly positive operators (Krasnosel'skii and Sobolevskii, 1959). For such operators the estimate (2.1) is required to hold for some ϕ and not only from the interval $[0, \pi/2]$, but from the larger interval $[0, \pi)$. Their domains of definition $D(A^\alpha, E)$ are closely connected with the spaces $E_\alpha(E, A)$. In fact, for arbitrary small $\varepsilon > 0$ the following continuous embeddings hold.

Theorem 2.1.1. *(see for example, Ashyralyev & Sobolevskii, 1994)* .

$$D(A^\alpha, E) \subset E_\alpha(E, A) \subset D(A^{\alpha-\varepsilon}, E),$$

$$D(A^{\alpha+\varepsilon}, E) \subset E_{\alpha,q}(E, A) \subset D(A^{\alpha-\varepsilon}, E), 1 \leq q < \infty$$

for all $0 < \alpha < 1$.

The main aim of this chapter is study structure of fractional powers of positive operators.

2.2 STRUCTURE OF FRACTIONAL SPACES $D(A^\beta, E_{\alpha,Q}(E, A))$

In (Sobolevskii, 1966) embedding theorems were obtained for the domains of definition of fractional powers of elliptic operators. These theorems and embeddings allow one to obtain almost the same (up to ε) embedding theorems for the spaces $E_\alpha(L_p, A)$. In (Smirnitskii and Sobolevskii, 1974) precisely the same embedding theorems for the spaces $E_\alpha(L_p, A)$ as for $D(A^\alpha, L_p)$.

Let us prove the main theorem in this chapter which deal with structure of fractional spaces $D(A^\beta, E_{\alpha,q}(E, A))$ generated by a strongly positive operator A in a Banach space.

Theorem 2.2.1. *$D(A^\beta, E_{\alpha,q}(E, A)) = E_{\alpha+\beta,q}(E, A)$ for all $1 \le q \le \infty$ and*
$0 < |\alpha| < 1, |\beta| < 1, 0 < |\alpha+\beta| < 1.$

Proof. It is clear for $\beta = 0$. Therefore, we will put $\beta \neq 0$. To prove this statement we examine separately the six cases

$$\begin{aligned}
&0 < \alpha,\ \beta < 1,\ 0 < \alpha+\beta < 1;\\
&-1 < \alpha,\ \beta < 0,\ -1 < \alpha+\beta < 0;\\
&0 < \alpha < 1,\ -1 < \beta < 0,\ 0 < \alpha+\beta < 1;\\
&0 < \alpha < 1,\ -1 < \beta < 0,\ -1 < \alpha+\beta < 0;\\
&-1 < \alpha < 0,\ 0 < \beta < 1, 0 < \alpha+\beta < 1;\\
&-1 < \alpha < 0,\ 0 < \beta < 1, -1 < \alpha+\beta < 0.
\end{aligned}$$

let $u \in D(A^\beta, E_{\alpha,q}(E, A))$. Then we will prove that $u \in E_{\alpha+\beta,q}(E, A)$ and the following statement holds

$$D(A^\beta, E_{\alpha,q}(E, A)) \subset E_{\alpha+\beta,q}(E, A). \tag{2.21}$$

In the first case $0 < \alpha, \beta < 1$. Applying formula (see Ashyralyev and Sobolevskii, 1994)

$$A^{-\beta} = \frac{1}{G(\beta)} \int_0^\infty \lambda^{\beta-1} \exp\{-\lambda A\}\, d\lambda \tag{2.22}$$

and the definition of fractional spaces $E_{\alpha,\infty}(E, A)$ and $D(A^\beta, E)$, we get

$$\mu^{1-\alpha-\beta} \left\|A \exp\{-\mu A\}\, u\right\|_E = \mu^{1-\alpha-\beta} \left\|A^{-\beta} A \exp\{-\mu A\}\, A^\beta u\right\|_E$$

$$\le \frac{\mu^{1-\alpha-\beta}}{G(\beta)} \int_0^\infty \lambda^{\beta-1} \left\|A \exp\{-(\lambda+\mu) A\}\, A^\beta u\right\|_E d\lambda$$

$$\leq M\frac{\mu^{1-\alpha-\beta}}{G(\beta)}\int_0^\infty \frac{\lambda^{\beta-1}}{(\lambda+\mu)^{1-\alpha}}d\lambda \left\|u\right\|_{E_{\alpha,\infty}(D(A^\beta,E),A)}.$$

Since

$$\int_0^\infty \frac{\mu^{1-\alpha-\beta}\lambda^{\beta-1}}{(\lambda+\mu)^{1-\alpha}}d\lambda = \int_0^\infty \frac{\rho^{\beta-1}}{(\rho+1)^{1-\alpha}}d\rho$$

$$\leq \int_0^1 \rho^{\beta-1}d\rho + \int_1^\infty \rho^{\alpha+\beta-2}d\rho = \frac{1-\alpha}{\beta(1-\alpha-\beta)}$$

it follows that

$$\mu^{1-\alpha-\beta}\left\|A\exp\{-\mu A\}u\right\|_E \leq M_1(\alpha,\beta)\left\|u\right\|_{E_{\alpha,\infty}(D(A^\beta,E),A)}$$

for any $\mu > 0$. Here and in future $M_1(\alpha,\beta) = M\dfrac{1-\alpha}{G(\beta+1)(1-\alpha-\beta)}$. Therefore,

$$\left\|u\right\|_{E_{\alpha+\beta,\infty}(E,A)} \leq M_1(\alpha,\beta)\left\|u\right\|_{E_{\alpha,\infty}(D(A^\beta,E),A)}$$

which completes the proof of statement (2.21) for $q=\infty$. In the case $1 \leq q < \infty$, applying formula (2.22) and the triangle inequality, we get

$$\left\|\mu^{1-\alpha-\beta}A\exp\{-\mu A\}u\right\|_E = \left\|\mu^{1-\alpha-\beta}A^{-\beta}A\exp\{-\mu A\}A^\beta u\right\|_E$$

$$\leq \frac{\mu^{1-\alpha-\beta}}{G(\beta)}\int_0^\infty \lambda^{\beta-1}\left\|A\exp\{-(\lambda+\mu)A\}A^\beta u\right\|_E d\lambda$$
$$\leq \frac{\mu^{1-\alpha}}{G(\beta)}\int_0^\infty \rho^{\beta-1}\left\|A\exp\{-\mu(\rho+1)A\}A^\beta u\right\|_E d\rho$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$ and using this estimate and Minkowski's inequality, we get

$$\left\|u\right\|_{E_{\alpha+\beta,q}(E,A)} \leq \left(\int_0^\infty\left(\frac{\mu^{1-\alpha}}{G(\beta)}\int_0^\infty \rho^{\beta-1}\left\|A\exp\{-\mu(\rho+1)A\}A^\beta u\right\|_E d\rho\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}} \tag{2.23}$$

$$\leq \frac{1}{G(\beta)}\int_0^\infty \rho^{\beta-1}\left(\int_0^\infty\left(\mu^{1-\alpha}\left\|A\exp\{-\mu(\rho+1)A\}A^\beta u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}d\rho \tag{2.24}$$

$$\leq \frac{1}{G(\beta)}\int_0^\infty \frac{\rho^{\beta-1}d\rho}{(\rho+1)^{1-\alpha}}\left(\int_0^\infty\left(z^{1-\alpha}\left\|A\exp\{-zA\}A^\beta u\right\|_E\right)^q\frac{dz}{z}\right)^{\frac{1}{q}}.$$

Using estimate (2.7), we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} \leq M(\alpha,\beta)\,\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. In the second case $-1 < \alpha,\beta < 0$. Applying estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^{\beta},E)$, we get

$$\begin{aligned}
&\left\|\mu^{-\alpha-\beta}\exp\{-\mu A\}\,u\right\|_{E}\\
&\leq \left\|A^{-\beta}\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\mu^{-\alpha-\beta}\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\\
&\leq M2^{-\beta}\left\|\mu^{-\alpha}\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E} \leq M2^{-\beta-\alpha}\,\|u\|_{E_{\alpha,\infty}(D(A^{\beta},E),A)}
\end{aligned}$$

for any $\mu > 0$. Therefore,

$$\|u\|_{E_{\alpha+\beta,\infty}(E,A)} \leq M2^{-\beta-\alpha}\,\|u\|_{E_{\alpha,\infty}(D(A^{\beta},E),A)}$$

which completes the proof of statement (2.21) for $q = \infty$. In the case $1 \leq q < \infty$, applying formula (2.22), the triangle inequality, and estimate (2.7), we get

$$\begin{aligned}
&\left\|\mu^{-\alpha-\beta}\exp\{-\mu A\}\,u\right\|_{E}\\
&\leq \left\|A^{-\beta}\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\mu^{-\alpha-\beta}\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\\
&\leq M2^{-\beta}\left\|\mu^{-\alpha}\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}
\end{aligned}$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^{\beta},E)$, we get

$$\begin{aligned}
\|u\|_{E_{\alpha+\beta,q}(E,A)} &\leq M2^{-\beta}\left(\int_0^{\infty}\left(\mu^{-\alpha}\left\|\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\right)^{q}\frac{d\mu}{\mu}\right)^{\frac{1}{q}}\\
&= M2^{-\beta-\alpha}\,\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}
\end{aligned}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. In the third case $0 < \alpha < 1$, $-1 < \beta < 0$, $0 < \alpha+\beta < 1$. Using estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^{\beta},E)$, we get

$$\left\|\mu^{1-\alpha-\beta}A\exp\{-\mu A\}\,u\right\|_{E} = \left\|\mu^{1-\alpha-\beta}A^{1-\beta}\exp\{-\mu A\}\,A^{\beta}u\right\|_{E}$$

$$\leq \left\| A^{-\beta} \exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E} \left\|\mu^{1-\alpha-\beta} A \exp\left\{-\frac{\mu}{2}A\right\} A^{\beta}u\right\|_{E}$$
$$\leq M2^{-\beta} \left\|\mu^{1-\alpha} A \exp\left\{-\frac{\mu}{2}A\right\} A^{\beta}u\right\|_{E} \leq M2^{1-\alpha-\beta} \|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

for any $\mu > 0$. From that it follows

$$\|u\|_{E_{\alpha+\beta,\infty}(E,A)} \leq M2^{1-\alpha-\beta} \|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

which completes the proof of statement (2.21) for $q = \infty$. In the case $1 \leq q < \infty$, using formula (2.22) and estimate (2.7), and the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^{\beta},E)$, we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} = \left(\int_0^{\infty} \left(\mu^{1-\alpha-\beta} \left\|A \exp\{-\mu A\} u\right\|_{E}\right)^{q} \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\infty} \left(\mu^{1-\alpha-\beta} \left\|A^{-\beta} \exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E} \left\|A \exp\left\{-\frac{\mu}{2}A\right\} A^{\beta}u\right\|_{E}\right)^{q} \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq M2^{-\beta} \left(\int_0^{\infty} \left(\mu^{1-\alpha} \left\|A \exp\left\{-\frac{\mu}{2}A\right\} A^{\beta}u\right\|_{E}\right)^{q} \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$
$$= M2^{1-\alpha-\beta} \|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. In the fourth case $0 < \alpha < 1$, $-1 < \beta < 0$, $-1 < \alpha + \beta < 0$. Using formula (2.22), estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^{\beta},E)$, we get

$$\mu^{-\alpha-\beta} \left\|\exp\{-\mu A\} u\right\|_{E} = \mu^{-\alpha-\beta} \left\|A^{-(\beta+1)} A \exp\{-\mu A\} A^{\beta}u\right\|_{E}$$
$$\leq \frac{\mu^{-\alpha-\beta}}{G(\beta+1)} \int_0^{\infty} \lambda^{\beta} \left\|A \exp\{-(\lambda+\mu) A\} A^{\beta}u\right\|_{E} d\lambda$$
$$\leq \frac{\mu^{-\alpha-\beta}}{G(\beta+1)} \int_0^{\infty} \frac{\lambda^{\beta}}{(\lambda+\mu)^{1-\alpha}} d\lambda \|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

for any $\mu > 0$. Since

$$\int_0^{\infty} \frac{\mu^{-\alpha-\beta}\lambda^{\beta}}{(\lambda+\mu)^{1-\alpha}} d\lambda = \int_0^{\infty} \frac{\rho^{\beta}}{(\rho+1)^{1-\alpha}} d\rho$$

$$\leq \int_0^1 \rho^{\beta} d\rho + \int_1^{\infty} \rho^{\beta+\alpha-1} d\rho = \frac{\alpha-1}{(\beta+1)(\alpha+\beta)}, \tag{2.25}$$

it follows that

$$\|u\|_{E_{\alpha+\beta,\infty}(E,A)} \leq \frac{1}{G(\beta+1)} \frac{\alpha-1}{(\beta+1)(\alpha+\beta)} \|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

which completes the proof of statement (2.21) for $q = \infty$. In the case $1 \leq q < \infty$, using formula (2.22) and estimate (2.7), we get

$$\mu^{-\alpha-\beta} \left\|\exp\{-\mu A\} u\right\|_E \leq \frac{\mu^{-\alpha-\beta}}{G(\beta+1)} \int_0^{\infty} \lambda^{\beta} \left\|A \exp\{-(\lambda+\mu) A\} A^{\beta} u\right\|_E d\lambda$$

$$\leq \frac{\mu^{1-\alpha}}{G(\beta+1)} \int_0^{\infty} \rho^{\beta} \left\|A \exp\{-\mu(\rho+1) A\} A^{\beta} u\right\|_E d\rho$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^{\beta},E)$, and Minkowski's inequality, we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} = \left(\int_0^{\infty} \left(\mu^{-\alpha-\beta} \left\|\exp\{-\mu A\} u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{G(\beta+1)} \int_0^{\infty} \rho^{\beta} \left(\int_0^{\infty} \left(\mu^{1-\alpha} \left\|A \exp\{-\mu(\rho+1) A\} A^{\beta} u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}} d\rho$$

$$\leq \frac{1}{G(\beta+1)} \int_0^{\infty} \frac{\rho^{\beta} d\rho}{(\rho+1)^{1-\alpha}} \left(\int_0^{\infty} \left(z^{1-\alpha} \left\|A \exp\{-zA\} A^{\beta} u\right\|_E\right)^q \frac{dz}{z}\right)^{\frac{1}{q}}.$$

Using estimate (2.25), we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} \leq \frac{1}{G(\beta+1)} \frac{\alpha-1}{(\beta+1)(\alpha+\beta)} \|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. In the fifth case $-1 < \alpha < 0$, $0 < \beta < 1$, $0 < \alpha+\beta < 1$. Using estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^{\beta},E)$, we get

$$\left\|\mu^{1-\alpha-\beta} A \exp\{-\mu A\} u\right\|_E$$
$$\leq \mu^{1-\alpha-\beta} \left\|A^{1-\beta} \exp\left\{-\frac{\mu}{2} A\right\}\right\|_{E\to E} \left\|\exp\left\{-\frac{\mu}{2} A\right\} A^{\beta} u\right\|_E$$

$$\leq M2^{1-\beta}\sup_{\mu>0}\left\|\mu^{-\alpha}\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\leq M2^{1-\beta-\alpha}\|u\|_{E_{\alpha}(D(A^{\beta},E),A)}$$

for any $\mu > 0$. From that it follows

$$\|u\|_{E_{\alpha+\beta,\infty}(E,A)}\leq M2^{1-\alpha-\beta}\|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

which completes the proof of statement (2.21) for $q = \infty$. In the case $1 \leq q < \infty$, using estimate (2.7) and the definition of fractional spaces $E_{\alpha,q}(E, A)$ and $D(A^{\beta}, E)$, we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)}=\left(\int_{0}^{\infty}\left(\mu^{1-\alpha-\beta}\left\|A^{1-\beta}\exp\{-\mu A\}A^{\beta}u\right\|_{E}\right)^{q}\frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq\left(\int_{0}^{\infty}\left(\mu^{1-\alpha-\beta}\left\|A^{1-\beta}\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\right)^{q}\frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq M2^{1-\beta}\left(\int_{0}^{\infty}\left(\mu^{-\alpha}\left\|\exp\left\{-\frac{\mu}{2}A\right\}A^{\beta}u\right\|_{E}\right)^{q}\frac{d\mu}{\mu}\right)^{\frac{1}{q}}=M2^{1-\alpha-\beta}\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. In the sixth case $-1 < \alpha < 0$, $0 < \beta < 1$, $-1 < \alpha + \beta < 0$. Using formula (2.22) and the definition of fractional spaces $E_{\alpha,\infty}(E, A)$ and $D(A^{\beta}, E)$, we get

$$\mu^{-\alpha-\beta}\left\|\exp\{-\mu A\}u\right\|_{E}=\mu^{-\alpha-\beta}\left\|A^{-\beta}\exp\{-\mu A\}A^{\beta}u\right\|_{E}$$

$$\leq\frac{\mu^{-\alpha-\beta}}{G(\beta)}\int_{0}^{\infty}\lambda^{\beta-1}\left\|\exp\{-(\lambda+\mu)A\}A^{\beta}u\right\|_{E}d\lambda$$

$$\leq\frac{\mu^{-\alpha-\beta}}{G(\beta)}\int_{0}^{\infty}\frac{\lambda^{\beta-1}}{(\lambda+\mu)^{-\alpha}}d\lambda\,\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))}.$$

Since

$$\int_{0}^{\infty}\frac{\mu^{-\alpha-\beta}\lambda^{\beta-1}}{(\lambda+\mu)^{-\alpha}}d\lambda=\int_{0}^{\infty}\frac{\rho^{\beta-1}}{(\rho+1)^{-\alpha}}d\rho$$

$$\leq\int_{0}^{1}\rho^{\beta-1}d\rho+\int_{1}^{\infty}\rho^{\beta+\alpha-1}d\rho=\frac{\alpha}{(\beta+1)(\alpha+\beta)},\qquad(2.26)$$

it follows that

$$\|u\|_{E_{\alpha+\beta,\infty}(E,A)}\leq\frac{1}{G(\beta)}\frac{\alpha}{(\beta+1)(\alpha+\beta)}\|u\|_{D(A^{\beta},E_{\alpha,\infty}(E,A))}$$

which completes the proof of statement (2.21) for $q = \infty$. In the case $1 \leq q < \infty$, using formula (2.22) and estimate (2.7), we get

$$\mu^{-\alpha-\beta} \left\|\exp\{-\mu A\} u\right\|_E \leq \mu^{-\alpha-\beta} \left\|A^{-\beta} \exp\{-\mu A\} A^{\beta} u\right\|_E$$

$$\leq \frac{\mu^{-\alpha-\beta}}{G(\beta)} \int_0^\infty \lambda^{\beta-1} \left\|\exp\{-(\lambda+\mu) A\} A^{\beta} u\right\|_E d\lambda$$

$$\leq \frac{\mu^{-\alpha}}{G(\beta)} \int_0^\infty \rho^{\beta-1} \left\|\exp\{-\mu(\rho+1) A\} A^{\beta} u\right\|_E d\rho$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E, A)$ and $D(A^{\beta}, E)$, and Minkowski's inequality, we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} = \left(\int_0^\infty \left(\mu^{-\alpha-\beta} \left\|\exp\{-\mu A\} u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{G(\beta)} \int_0^\infty \rho^{\beta-1} \left(\int_0^\infty \left(\mu^{-\alpha} \left\|\exp\{-\mu(\rho+1) A\} A^{\beta} u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}} d\rho$$

$$\leq \frac{1}{G(\beta)} \int_0^\infty \frac{\rho^{\beta-1} d\rho}{(\rho+1)^{-\alpha}} \left(\int_0^\infty \left(z^{-\alpha} \left\|\exp\{-zA\} A^{\beta} u\right\|_E\right)^q \frac{dz}{z}\right)^{\frac{1}{q}}.$$

Using estimate (2.26), we get

$$\|u\|_{E_{\alpha+\beta,q}(E,A)} \leq \frac{1}{G(\beta+1)} \frac{\alpha}{(\beta+1)(\alpha+\beta)} \|u\|_{D(A^{\beta}, E_{\alpha,q}(E,A))}$$

which completes the proof of statement (2.21) for $1 \leq q < \infty$. Now, we will prove the opposite inequality. Actually, Let $u \in E_{\alpha+\beta,q}(E, A)$. Then we will prove that $u \in D(A^{\beta}, E_{\alpha,q}(E, A))$ and the following statement

$$E_{\alpha+\beta,q}(E, A) \subset D(A^{\beta}, E_{\alpha,q}(E, A)). \tag{2.27}$$

In the first case $0 < \alpha, \beta < 1,\ 0 < \alpha + \beta < 1$. Applying estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E, A)$ and $D(A^{\beta}, E)$, we get

$$\left\|\mu^{1-\alpha} A \exp\{-\mu A\} A^{\beta} u\right\|_E$$

$$\leq \left\|A^{\beta} \exp\left\{-\frac{\mu}{2} A\right\}\right\|_{E\to E} \mu^{1-\alpha} \left\|A \exp\left\{-\frac{\mu}{2} A\right\} u\right\|_E$$

$$\leq M\mu^{1-\alpha-\beta}\left\|A\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E \leq M2^{1-\alpha-\beta}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$. From that it follows

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} \leq M2^{1-\alpha-\beta}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q = \infty$. In the case $1 \leq q < \infty$, applying estimate (2.7), and the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$, we get

$$\|u\|_{D(A^\beta,E_{\alpha,q}(E,A))} = \left(\int_0^\infty \left(\mu^{1-\alpha}\left\|A\exp\{-\mu A\}A^\beta u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \left(\mu^{1-\alpha}\left\|A^\beta\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|A\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq M\left(\int_0^\infty \left(\mu^{1-\alpha-\beta}\left\|A\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$= M2^{1-\alpha-\beta}\|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1 \leq q < \infty$. In the second case $-1 < \alpha,\beta < 0$. Applying formula (2.22) and estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^\beta,E)$, we get

$$\mu^{-\alpha}\left\|\exp\{-\mu A\}A^\beta u\right\|_E \leq \frac{\mu^{-\alpha}}{G(-\beta)}\int_0^\infty \lambda^{-\beta-1}\left\|\exp\{-(\lambda+\mu)A\}u\right\|_E d\lambda$$

$$\leq \frac{\mu^{-\alpha}}{G(-\beta)}\int_0^\infty \frac{\lambda^{-\beta-1}}{(\lambda+\mu)^{-(\alpha+\beta)}}d\lambda\,\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$.Since

$$\int_0^\infty \frac{\mu^{-\alpha}\lambda^{-\beta-1}}{(\lambda+\mu)^{-(\alpha+\beta)}}d\lambda = \int_0^\infty \frac{z^{-\beta-1}}{(z+1)^{-(\alpha+\beta)}}dz$$

$$\leq \int_0^1 z^{-\beta-1}dz + \int_1^\infty z^{\alpha-1}dz = \frac{-(\alpha+\beta)}{\beta\alpha}, \tag{2.28}$$

it follows that

$$\mu^{-\alpha}\left\|\exp\{-\mu A\}A^\beta u\right\|_E \leq M(\alpha,\beta)\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$. Here and in future $M(\alpha,\beta) = \dfrac{1}{G(-\beta)}\dfrac{-(\alpha+\beta)}{\beta\alpha}$. Therefore,

$$\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))} \leq M(\alpha,\beta)\,\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q = \infty$. In the case $1 \leq q < \infty$, applying formula (2.22) and estimate (2.7), we get

$$\left\|\mu^{-\alpha}\exp\{-\mu A\}A^{\beta}u\right\| \leq \frac{\mu^{-\alpha}}{G(-\beta)}\int_0^{\infty}\lambda^{-\beta-1}\left\|\exp\{-(\mu+\lambda)A\}u\right\|_E d\lambda.$$

$$\leq \frac{\mu^{-\alpha-\beta}}{G(-\beta)}\int_0^{\infty}\rho^{-\beta-1}\left\|\exp\{-\mu(1+\rho)A\}u\right\|_E d\rho$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^{\beta},E)$, we get

$$\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))} = \left(\int_0^{\infty}\left\|\mu^{-\alpha}\exp\{-\mu A\}A^{\beta}u\right\|_E^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{G(-\beta)}\int_0^{\infty}\rho^{-\beta-1}\left(\int_0^{\infty}\left(\mu^{-\alpha-\beta}\left\|\exp\{-\mu(1+\rho)A\}u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}d\rho$$

$$= \frac{1}{G(-\beta)}\left(\int_0^{\infty}\frac{\rho^{-\beta-1}}{(1+\rho)^{-\alpha-\beta}}d\rho\right)\left(\int_0^{\infty}\left(z^{-\alpha-\beta}\left\|\exp\{-zA\}u\right\|_E\right)^q\frac{dz}{z}\right)^{\frac{1}{q}}.$$

Using estimate (2.28), we get

$$\|u\|_{D(A^{\beta},E_{\alpha,q}(E,A))} \leq M(\alpha,\beta)\,\|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1 \leq q < \infty$. In the third case $0 < \alpha < 1$, $-1 < \beta < 0.0 < \alpha+\beta < 1$. Using formula (2.22) ,the triangle inequality and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^{\beta},E)$, we get

$$\mu^{1-\alpha}\left\|A\exp\{-\mu A\}A^{\beta}u\right\|_E$$

$$\leq \frac{\mu^{1-\alpha}}{G(-\beta)}\int_0^{\infty}\lambda^{-\beta-1}\left\|A\exp\{-(\lambda+\mu)A\}u\right\|_E d\lambda$$

$$\leq \frac{\mu^{1-\alpha}}{G(-\beta)}\int_0^{\infty}\frac{\lambda^{-\beta-1}}{(\lambda+\mu)^{1-\alpha-\beta}}d\lambda\,\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$. Since

$$\int_0^\infty \frac{\mu^{1-\alpha}\lambda^{-\beta-1}}{(\lambda+\mu)^{1-\alpha-\beta}}d\lambda = \int_0^\infty \frac{\rho^{-\beta-1}}{(\rho+1)^{1-\alpha-\beta}}d\rho$$

$$\leq \int_0^1 \rho^{-\beta-1}d\rho + \int_1^\infty \rho^{\alpha-2}d\rho = \frac{\alpha+\beta-1}{(1-\alpha)\beta}, \tag{2.29}$$

it follows that

$$\|u\|_{D(A^\beta,E_{\alpha,q}(E,A))} \leq \frac{1-\alpha-\beta}{G(-\beta+1)(1-\alpha)}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q=\infty$. In the case $1 \leq q < \infty$, applying formula (2.22) and the triangle inequality, we get

$$\mu^{1-\alpha}\left\|A\exp\{-\mu A\}A^\beta u\right\|_E \leq \frac{\mu^{1-\alpha}}{G(-\beta)}\int_0^\infty \lambda^{-\beta-1}\left\|A\exp\{-(\lambda+\mu)A\}u\right\|_E d\lambda$$

$$\leq \frac{\mu^{1-\alpha-\beta}}{G(-\beta)}\int_0^\infty \rho^{-\beta-1}\left\|A\exp\{-\mu(\rho+1)A\}u\right\|_E d\rho$$

for any $\mu > 0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$ and using estimate (2.29) and Minkowski's inequality, we get

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))}$$

$$\leq \frac{1}{G(-\beta)}\int_0^\infty \rho^{-\beta-1}\left(\int_0^\infty\left(\mu^{1-\alpha-\beta}\left\|A\exp\{-\mu(\rho+1)A\}u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}d\rho$$

$$= \frac{1}{G(-\beta)}\int_0^\infty \frac{\rho^{-\beta-1}}{(\rho+1)^{1-\alpha-\beta}}d\rho\left(\int_0^\infty\left(z^{1-\alpha-\beta}\left\|A\exp\{-zA\}u\right\|_E\right)^q\frac{dz}{z}\right)^{\frac{1}{q}}$$

$$\leq \frac{1-\alpha-\beta}{G(-\beta+1)(1-\alpha)}\|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1 \leq q < \infty$. In the fourth case $0 < \alpha < 1$, $-1 < \beta < 0$, $-1 < \alpha+\beta < 0$. Using estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^\beta,E)$, we get

$$\left\|\mu^{1-\alpha}A\exp\{-\mu A\}A^\beta u\right\|_E \leq \mu^{1-\alpha}\left\|A^{1+\beta}\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E$$

$$\leq M2^{1+\beta}\sup_{\mu>0}\left\|\mu^{-\alpha-\beta}\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E = M2^{1-\alpha}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$. From that it follows

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} \leq M2^{1-\alpha}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q = \infty$. In the case $1 \leq q < \infty$, using formula (2.22) and estimate (2.7), we get

$$\left\|\mu^{1-\alpha}A\exp\{-\mu A\}A^\beta u\right\|_E \leq \mu^{1-\alpha}\left\|A^{1+\beta}\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E$$

$$\leq M2^{1+\beta}\left\|\mu^{-\alpha-\beta}\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E$$

for any $\mu > 0$. Using the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$, we get

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} \leq M2^{1+\beta}\left(\int_0^\infty\left(\mu^{-\alpha-\beta}\left\|\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$\leq M2^{1-\alpha}\|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1 \leq q < \infty$. In the fifth case $-1 < \alpha < 0$, $0 < \beta < 1$, $0 < \alpha+\beta < 1$. Using formula (2.22) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^\beta,E)$, we get

$$\mu^{-\alpha}\left\|\exp\{-\mu A\}A^\beta u\right\|_E = \mu^{-\alpha}\left\|A^{-(1-\beta)}A\exp\{-\mu A\}u\right\|_E$$

$$\leq \frac{\mu^{-\alpha}}{G(1-\beta)}\int_0^\infty\lambda^{-\beta}\left\|A\exp\{-(\lambda+\mu)A\}u\right\|_E d\lambda$$

$$\leq \frac{\mu^{-\alpha}}{G(1-\beta)}\int_0^\infty\frac{\lambda^{-\beta}}{(\lambda+\mu)^{1-\alpha-\beta}}d\lambda\,\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

for any $\mu > 0$. Since

$$\int_0^\infty\frac{\mu^{-\alpha}\lambda^{-\beta}}{(\lambda+\mu)^{1-\alpha-\beta}}d\lambda = \int_0^\infty\frac{\rho^{-\beta}}{(\rho+1)^{1-\alpha-\beta}}d\rho$$

$$\leq \int_0^1\rho^{-\beta}d\rho + \int_1^\infty\rho^{\alpha-1}d\rho \leq \frac{\alpha+\beta-1}{(1-\beta)\alpha}, \tag{2.30}$$

it follows that

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} \le \frac{\alpha+\beta-1}{G(2-\beta)\alpha}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q=\infty$. In the case $1\le q<\infty$, using formula (2.22) , we get

$$\mu^{-\alpha}\left\|\exp\{-\mu A\}A^\beta u\right\|_E \le \frac{\mu^{-\alpha}}{G(1-\beta)}\int_0^\infty \lambda^{-\beta}\left\|A\exp\{-(\lambda+\mu)A\}u\right\|_E d\lambda$$
$$=\frac{\mu^{1-\alpha-\beta}}{G(1-\beta)}\int_0^\infty \rho^{-\beta}\left\|A\exp\{-\mu(\rho+1)A\}u\right\|_E d\rho$$

for any $\mu>0$. Therefore, applying the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$, using estimate (2.29) and Minkowski's inequality, we get

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))}$$
$$\le \frac{1}{G(1-\beta)}\int_0^\infty \rho^{-\beta}\left(\int_0^\infty\left(\mu^{1-\alpha-\beta}\left\|A\exp\{-\mu(\rho+1)A\}u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}} d\rho$$
$$=\frac{1}{G(1-\beta)}\int_0^\infty \frac{\rho^{-\beta}}{(\rho+1)^{1-\alpha-\beta}}d\rho\left(\int_0^\infty\left(z^{1-\alpha-\beta}\left\|A\exp\{-zA\}u\right\|_E\right)^q\frac{dz}{z}\right)^{\frac{1}{q}}$$
$$=\frac{\alpha+\beta-1}{G(2-\beta)\alpha}\|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1\le q<\infty$. In the sixth case $-1<\alpha<0$, $0<\beta<1$, $-1<\alpha+\beta<0$. Using estimate (2.7) and the definition of fractional spaces $E_{\alpha,\infty}(E,A)$ and $D(A^\beta,E)$, we get

$$\left\|\mu^{-\alpha}\exp\{-\mu A\}A^\beta u\right\|_E \le \mu^{-\alpha}\left\|A^\beta\exp\left\{-\frac{\mu}{2}A\right\}\right\|_{E\to E}\left\|\exp\left\{-\frac{\mu}{2}A\right\}u\right\|_E$$
$$\le M2^\beta\sup_{\mu>0}\left\|\mu^{-\alpha-\beta}\exp\left\{-\frac{\mu}{2}A\right\}A^\beta u\right\|_E \le M2^{-\alpha}\|u\|_{E_{\alpha+\beta}(E,A)}$$

for any $\mu>0$. From that it follows

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} \le M2^{-\alpha}\|u\|_{E_{\alpha+\beta,\infty}(E,A)}$$

which completes the proof of statement (2.27) for $q=\infty$. In the case $1\le q<\infty$, using estimate (2.7) and the definition of fractional spaces $E_{\alpha,q}(E,A)$ and $D(A^\beta,E)$, we get

$$\|u\|_{D(A^\beta,E_{\alpha,\infty}(E,A))} = \left(\int_0^\infty\left(\mu^{-\alpha}\left\|\exp\{-\mu A\}A^\beta u\right\|_E\right)^q\frac{d\mu}{\mu}\right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty \left(\mu^{-\alpha} \left\| A^\beta \exp\{-\mu A\} u \right\|_E \right)^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \left(\mu^{-\alpha} \left\| A^\beta \exp\left\{-\frac{\mu}{2} A\right\} \right\|_{E \to E} \left\| \exp\left\{-\frac{\mu}{2} A\right\} u \right\|_E \right)^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}}$$

$$\leq M 2^\beta \left(\int_0^\infty \left(\mu^{-\alpha-\beta} \left\| \exp\left\{-\frac{\mu}{2} A\right\} u \right\|_E \right)^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \leq M 2^{-\alpha} \|u\|_{E_{\alpha+\beta,q}(E,A)}$$

which completes the proof of statement (2.27) for $1 \leq q < \infty$. Therefore, Theorem 2.2.1 is proved. □

With the help of a positive operator A, we introduce the Banach space $E'_{\alpha,q}(E,A), 0 < \alpha < 1$, consisting of all $v \in E$ for which the following norms are finite:

$$\|v\|_{E'_{\alpha,q}} = \left(\int_0^\infty \left\| \lambda^\alpha A (\lambda + A)^{-1} v \right\|_E^q \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}}, 1 \leq q < \infty,$$

$$\|v\|_{E'_{\alpha,\infty}} = \sup_{\lambda > 0} \left\| \lambda^\alpha A (\lambda + A)^{-1} v \right\|_E .$$

Applying the positive operator A, for all $v \in E$ and $-1 < \alpha < 0$, the following norms are finite

$$\|v\|_{E'_{\alpha,q}} = \left(\int_0^\infty \left\| \lambda^{\alpha-1} (\lambda + A)^{-1} v \right\|_E^q \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}}, 1 \leq q < \infty,$$

$$\|v\|_{E'_{\alpha,\infty}} = \sup_{\lambda > 0} \left\| \lambda^{\alpha-1} (\lambda + A)^{-1} v \right\|_E ,$$

we define the fractional space $E'_{\alpha,q}(E,A), -1 < \alpha < 0$. The replenishment of space E in this norm forms a Banach space $E'_{\alpha,q}(E,A), -1 < \alpha < 0, 1 \leq q \leq \infty$.

The fractional power and structure of fractional spaces generated by the wider class of differential and difference positive operators and their related applications have been investigated by many researchers (see, for example, Simirnitskii, 1983; Bekir, Aksoy and Guner, 2014; Agmon, 1962; Ashyralyev, 2009; Ashyralyev, 2015 and the references given therein).

Theorem 2.2.2. *(see, for example, Ashyralyev and Sobolevskii, 1994).*

$E'_{\alpha,q}(E,A) = E_{\alpha,q}(E,A)$ *for all* $0 < |\alpha| < 1, 1 \leq q \leq \infty$.

Applying Theorems 2.2.1 and 2.2.2 we get the following result.

Theorem 2.2.3. $D(A^{\beta}, E'_{\alpha,q}(E,A)) = E'_{\alpha+\beta,q}(E,A)$ *for all* $1 \leq q \leq \infty$ *and* $0 < |\alpha| < 1, |\beta| < 1, 0 < |\alpha+\beta| < 1$.

Note that positive fractional powers of positive operators in a Banach space and the structure of positive fractional powers of positive operators in fractional spaces $E'_{\alpha,q}(E,A), 1 \leq q \leq \infty$ for $\alpha > 0$ were investigated by Sobolevskii P.E. in papers (Sobolevskii, 1966; Sobolevskii, 1967; Sobolevskii, 1974).

2.3 APPLICATION

Now, we consider the applications of Theorems 2.2.1 and 2.2.3. First, we consider the differential operator A^x defined by the formula

$$A^x u = -u_{xx}(x) + \delta u(x), \ \delta > 0 \tag{2.31}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u'(x), u''(x) \in C[0,2\pi], u(x) = u(x+2\pi), \ \int_0^{2\pi} u(x)dx = 0 \right\}.$$

The resolvent of the operator A^x, i.e.

$$A^x u + \lambda u = \varphi \tag{2.32}$$

or

$$\begin{cases} -u''(x) + (\delta+\lambda)\, u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ \qquad u(0) = u(2\pi), \ \int_0^{2\pi} u(x)dx = 0 \end{cases} \tag{2.33}$$

was investigated in paper (Ashyralyev and Tetikoğlu, 2017). We introduce the Banach spaces $C^{\beta}[0,2\pi]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^{\beta}[0,2\pi]} = \| \varphi \|_{C[0,2\pi]} + \sup_{0 \leq x < x+\tau \leq 2\pi} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^{\beta}},$$

where $C[0,2\pi]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,2\pi]$ with the usual norm

$$\| \varphi \|_{C[0,2\pi]} = \max_{0\leq x\leq 2\pi} |\varphi(x)|.$$

The positivity of the operator A^x in the Banach space $C[0,2\pi]$ was established in paper (Sobolevskii, 2005). In paper (Ashyralyev and Tetikoğlu, 2017), it was proved that for any $\alpha \in (0,\frac{1}{2})$ the norms in space $E_\alpha(C[0,2\pi],A)$ and $C^{2\alpha}[0,2\pi]$ are equivalent. The positivity of A^x in the Hölder spaces of $C^{2\alpha}[0,2\pi]$, $\alpha \in (0,\frac{1}{2})$ was proved. Theorem on the structure of $E_\alpha(C[0,2\pi],A)$ of paper (Ashyralyev and Tetikoğlu, 2017) and Theorem 2.2.1 implies the following result.

Theorem 2.3.1. *$D(A^\beta, E_\alpha(C[0,2\pi],A)) = C^{2(\alpha+\beta)}[0,2\pi]$ for all $0<\alpha<\frac{1}{2}$, $|\beta|<\frac{1}{2}$, $0<\alpha+\beta<\frac{1}{2}$.*

Now, we introduce the Banach space $W_p^\mu[0,2\pi]$ $(0<\mu<1)$ of all integrable functions $\varphi(x)$ defined on $[0,2\pi]$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{W_p^\mu[0,2\pi]} = \left[\int_0^{2\pi}\int_0^{2\pi}\frac{|\varphi(x+y)-\varphi(x)|^p}{|y|^{1+\mu p}}dydx + \|\varphi\|^p_{L_p[0,2\pi]}\right]^{\frac{1}{p}}, 1\leq p<\infty.$$

Here, $L_p[0,2\pi]$, $1\leq p<\infty$ is the space of the all integrable functions $\varphi(x)$ defined on $[0,2\pi]$ with the norm

$$\|\varphi\|_{L_p[0,2\pi]} = \left(\int_0^{2\pi}|\varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

Theorem on the structure of $E_{\alpha,q}(L_q[0,2\pi],A)$ of paper (Ashyralyev and Tetikoğlu, 2017) and Theorem 2.2.1 imply the following result.

Theorem 2.3.2. *$W_q^{2(\alpha+\beta+\varepsilon)}[0,2\pi] \subset D(A^\beta, W_p^\alpha[0,2\pi]) \subset W_q^{2(\alpha+\beta-\varepsilon)}[0,2\pi]$, $1\leq q<\infty$ for all $0<\alpha<\frac{1}{2}$, $|\beta|<\frac{1}{2}$, $0<\alpha+\beta\pm\varepsilon<\frac{1}{2}$.*

Second, we consider the differential operator with constant coefficients of the form

$$B = \sum_{|r|=2m} b_r \frac{\partial^{|r|}}{\partial_{x_1^{r_1}}\ldots\partial_{x_n^{r_n}}}$$

acting on functions defined on the entire space $\mathbb{R}^n$. Here $r \in \mathbb{R}^n$ is a vector with nonnegative integer components, $|r| = r_1 + ... + r_n$. If $\varphi(y)\,(y = (y_1, ..., y_n) \in \mathbb{R}^n)$ is an infinitely differentiable function that decays at infinity together with all its derivatives, then by means of the Fourier transformation one establishes the equality

$$F\left(B_\varphi\right)(\xi) = B(\xi)\,F(\varphi)(\xi).$$

Here the Fourier transform operator is defined by the rule

$$F(\varphi)(\xi) = (2\pi)^{-n/2} \int\limits_{R^n} \exp\{-i(y,\xi)\}\,\varphi(y)\,dy,$$

$$(y,\xi) = y_1\xi_1 + ... + y_n\xi_n.$$

The function $B(\xi)$ is called the symbol of the operator B and is given by

$$B(\xi) = \sum_{|r|=2m} b_r\,(i\xi_1)^{r_1} ... (i\xi_n)^{r_n}.$$

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)\,(i\xi_1)^{r_1} ... (i\xi_n)^{r_n}, \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{2.34}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$.

Then, for sufficiently large positive δ, an elliptic operator $A = B^x + \delta I$ is a strongly positive operator in Banach spaces $C(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n), 1 \le p < \infty$. Here $C(\mathbb{R}^n)$ is the space of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ with the usual norm

$$\|\varphi\|_{C(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |\varphi(x)|,$$

$L_p(\mathbb{R}^n)$ is the space of the all integrable functions $\varphi(x)$ defined on $\mathbb{R}^n$ with the norm

$$\|\varphi\|_{L_p(\mathbb{R}^n)} = \left(\int_{x\in\mathbb{R}^n} |\varphi(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We will introduce the Banach space $C^\mu(\mathbb{R}^n)\,(0 < \mu < 1)$ of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{C^\mu(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |\varphi(x)| + \sup_{\substack{x,y\in\mathbb{R}^n \\ y\neq 0}} \frac{|\varphi(x+y) - \varphi(x)|}{|y|^\mu},$$

the Banach space $W_p^\mu(\mathbb{R}^n)\,(0 < \mu < 1)$ of all integrable functions $\varphi(x)$ defined on $\mathbb{R}^n$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{W_p^\mu(\mathbb{R}^n)} = \left[\int_{x\in\mathbb{R}^n} \int_{\substack{y\in\mathbb{R}^n \\ y\neq 0}} \frac{|\varphi(x+y) - \varphi(x)|^p}{|y|^{n+\mu p}} dy dx + \|\varphi\|_{L_p(\mathbb{R}^n)}^p \right]^{\frac{1}{p}}, 1 \leq p < \infty.$$

Theorem 2.3.3. *(Ashyralyev & Sobolevskii, 1994; Triebel, 1978).*

$$E_\alpha(C(\mathbb{R}^n), A) = C^{2m\alpha}(\mathbb{R}^n),$$

$$E_{\alpha,p}(L_p(\mathbb{R}^n), A) = W_p^{2m\alpha}(\mathbb{R}^n), 1 \leq p < \infty \textit{ for all } 0 < 2m\alpha < 1.$$

Theorem 2.3.3 on the structure of $E_\alpha(C(\mathbb{R}^n), A)$ and $E_{\alpha,p}(L_p(\mathbb{R}^n), A), 1 \leq p < \infty$ of papers (Ashyralyev and Sobolevskii, 1994; Triebel, 1978) and Theorem 2.2.1 imply the following results.

Theorem 2.3.4. $D(A^\beta, C^\alpha(\mathbb{R}^n)) = C^{2m(\alpha+\beta)}(\mathbb{R}^n)$ *for all* $0 < \alpha < \frac{1}{2m}, |\beta| < \frac{1}{2m}, 0 < \alpha + \beta < \frac{1}{2m}$.

Theorem 2.3.5. $W_q^{2m(\alpha+\beta+\varepsilon)}(\mathbb{R}^n) \subset D(A^\beta, W_p^\alpha(\mathbb{R}^n)) \subset W_q^{2m(\alpha+\beta-\varepsilon)}(\mathbb{R}^n), 1 \leq q < \infty$ *for all* $0 < \alpha < \frac{1}{2m}, |\beta| < \frac{1}{2m}, 0 < \alpha + \beta \pm \varepsilon < \frac{1}{2m}$.

CHAPTER 3

WELL-POSEDNESS OF ELLIPTIC DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

3.1 INTRODUCTION

In mathematical modeling, elliptic equations are used together with local boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers, see for example, (Ashyralyev, 2008; Ashyralyev, 2003; Ashyralyev et al., 2004; Ashyralyev & Tetikoglu, 2012; Ashyralyev & Ozturk, 2013; Ashyralyyev, 2017; Ashyralyyev & Akkan, 2015; Kadirkulov & Kirane, 2015; Kirane & Torebek, 2016; Sapagovas et al., 2017; Sapagovas et al., 2016; Shakhmurov & Musaev, 2017; Čiupaila et al., 2013; Wang & Zheng, 2009) and the references given therein.

It is known that the mixed problem for elliptic equations can be solved analytically by Fourier series, Fourier transform and Laplace transform methods. Now, let us illustrate these different analytical methods by examples.

Example 3.1.1. Obtain the Fourier series solution of the nonlocal boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}+u=3\sin t\sin x,\ 0<t<2\pi,\ 0<x<\pi, \\ u(0,x)=u(2\pi,x),\ \int_0^{2\pi} u(s,x)\,ds=0, 0\le x\le \pi, \\ u(t,0)=u(t,\pi)=0,\ 0\le t\le 2\pi \end{cases} \tag{3.1}$$

for the elliptic equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) + u(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi) = 0.$$

generated by the space operator of problem (3.1). So, the nontrivial solutions of this Sturm-Liouville problem are given by formulas

$$u_k(x) = \sin kx,\ \lambda_k = k^2 + 1 \ \text{ where } k = 1, 2, 3, \cdots.$$

Then, we will obtain the Fourier series solution of problem(3.1) by formula

$$u(t, x) = \sum_{k=1}^{\infty} A_k(t) \sin kx,$$

where $A_k(t), k = 1, 2, 3, ...$ are unknown functions. Putting $u(t, x)$ into the equation (3.1) and nonlocal boundary conditions

$$u(0, x) = u(2\pi, x),\ \int_0^{2\pi} u(s, x)\, ds = 0,\ 0 \le x \le \pi, \tag{3.2}$$

we obtain

$$-\sum_{k=1}^{\infty} A_k''(t) \sin kx + k^2 \sum_{k=1}^{\infty} A_k(t) \sin kx + \sum_{k=1}^{\infty} A_k(t) \sin kx$$

$$= 3 \sin t \sin x,\ 0 < t < 2\pi,\ 0 < x < \pi,$$

and

$$\sum_{k=1}^{\infty} A_k(0) \sin kx = \sum_{k=1}^{\infty} A_k(2\pi) \sin kx, 0 \le x \le \pi,$$

$$\int_0^{2\pi} \sum_{k=1}^{\infty} A_k(s) \sin kx ds = \sum_{k=1}^{\infty} \sin kx \int_0^{2\pi} A_k(s)\, ds = 0, 0 \le x \le \pi.$$

Equating coefficients $\sin kx, k = 1, ...$ to zero, we get

$$-A_1''(t) + 2A_1(t) = 3 \sin t, 0 < t < 2\pi,$$

$$-A_k''(t) + \left(k^2 + 1\right) A_k(t) = 0, k \neq 1, 0 < t < 2\pi,$$

$$A_k(0) = A_k(2\pi), \quad \int_0^{2\pi} A_k(s)\,ds = 0, k = 1, 2, 3, ...$$

We will obtain $A_k(t), k = 1, 2, ...$ Firstly, for $k = 1$,we have the following problem

$$\begin{cases} -A_1''(t) + 2A_1(t) = 3\sin t, 0 < t < 2\pi, \\ \\ A_1(0) = A_1(2\pi), \quad \int_0^{2\pi} A_1(s)\,ds = 0. \end{cases}$$

It is easy to obtain that $A_1(t) = \sin t$. Secondly, for $k \neq 1$, we have the following problem

$$\begin{cases} -A_k''(t) + \left(k^2 + 1\right) A_k(t) = 0, 0 < t < 2\pi, \\ \\ A_k(0) = A_k(2\pi), \quad \int_0^{2\pi} A_k(s)\,ds = 0. \end{cases}$$

It is easy to obtain that $A_k(t) = 0$. Thus, the solution of (3.1) is $u(t, x) = \sin t \sin x$.

Note that using similar procedure one can obtain the solution of the following mixed problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t, x), \\ \\ 0 < t < 2\pi, \ x = (x_1, ..., x_n) \in \Omega, \\ u(0, x) = u(2\pi, x) + \varphi(x), \ \int_0^{2\pi} u(s, x)\,ds = \psi(x), \\ \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \\ \\ u(t, x) = 0, x \in S, 0 \leq t \leq 2\pi \end{cases} \tag{3.3}$$

for the multidimensional elliptic differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t, x)\left(t \in (0, 2\pi), x \in \overline{\Omega}\right), \varphi(x), \ \psi(x)\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary$S, \overline{\Omega} = \Omega \cup S$.

However Fourier series method described in solving (3.3) can be used only in the case when (3.3) has constant coefficients.

Example 3.1.2. Obtain the Fourier series solution of the nonlocal boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + u = 3\cos t \cos x,\ 0 < t < 2\pi,\ 0 < x < \pi, \\ u(0,x) = u(2\pi, x),\ \int_0^{2\pi} u(s,x)\,ds = 0, 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0,\ 0 \le t \le 2\pi \end{cases} \tag{3.4}$$

for the elliptic equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) + u(x) - \lambda u(x) = 0, 0 < x < \pi, u'(0) = u'(\pi) = 0.$$

generated by the space operator of problem (3.4). So, the nontrivial solutions of this Sturm-Liouville problem are given by formulas

$$u_k(x) = \cos kx, \lambda_k = k^2 + 1, k = 0, 1, 2, 3, \cdots.$$

Therefore, we will obtain the Fourier series solution of problem (3.4) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos kx,$$

where $A_k(t), k = 0, 1, 2, 3, ...$ are unknown functions. Putting $u(t,x)$ into the equation (3.4) and nonlocal boundary conditions (3.2), we obtain

$$-\sum_{k=0}^{\infty} A_k''(t) \cos kx + k^2 \sum_{k=0}^{\infty} A_k(t) \cos kx + \sum_{k=0}^{\infty} A_k(t) \cos kx$$

$$= 3\cos t \cos x, 0 < t < 2\pi, 0 < x < \pi,$$

$$\sum_{k=0}^{\infty} A_k(0) \cos kx = \sum_{k=0}^{\infty} A_k(2\pi) \cos kx, 0 \le x \le \pi,$$

$$\int_0^{2\pi} \sum_{k=0}^{\infty} A_k(s) \cos kx ds = \sum_{k=0}^{\infty} \cos kx \int_0^{2\pi} A_k(s)\,ds = 0, 0 \le x \le \pi.$$

Equating coefficients $\cos kx, k = 0, 1, ...$ to zero , we get

$$-A_1''(t) + 2A_1(t) = 3\cos t, 0 < t < 2\pi,$$

$$-A_k''(t) + \left(k^2+1\right) A_k(t) = 0, k \neq 1, 0 < t < 2\pi,$$

$$A_k(0) = A_k(2\pi), \ \int_0^{2\pi} A_k(s)\, ds = 0, k = 0, 1, 2, 3, ...$$

We will obtain $A_k(t), k = 0, 1, ...$ Firstly, for $k = 1$,we have the following problem

$$\begin{cases} -A_1''(t) + 2A_1(t) = 3\cos t, 0 < t < 2\pi, \\ \\ A_1(0) = A_1(2\pi), \ \int_0^{2\pi} A_1(s)\, ds = 0. \end{cases}$$

It is easy to obtain that $A_1(t) = \cos t$. Secondly, for $k \neq 1$,we have the following problem

$$\begin{cases} -A_k''(t) + \left(k^2+1\right) A_k(t) = 0, 0 < t < 2\pi, \\ \\ A_k(0) = A_k(2\pi), \ \int_0^{2\pi} A_k(s)\, ds = 0. \end{cases}$$

It is easy to obtain that $A_k(t) = 0$. Thus the solution of (3.4) is $u(t, x) = \cos t \cos x$.

Note that using similar procedure one can obtain the solution of the following mixed problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ \\ 0 < t < 2\pi, \ x = (x_1, ..., x_n) \in \Omega, \\ u(0,x) = u(2\pi, x) + \varphi(x), \ \int_0^{2\pi} u(s,x)\, ds = \psi(x), \\ \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \\ \\ \dfrac{\partial u(t,x)}{\partial \overline{m}} = 0, x \in S, 0 \le t \le 2\pi \end{cases} \tag{3.5}$$

for the multidimensional elliptic differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t, x)\left(t \in (0, 2\pi), x \in \overline{\Omega}\right), \varphi(x), \psi(x) \ \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to S.

However Fourier series method described in solving (3.5) can be used only in the case when (3.5) has constant coefficients.

Example 3.1.3. Obtain the Fourier series solution of the nonlocal boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + u = 6\cos t \cos 2x,\ 0 < t < 2\pi,\ 0 < x < \pi, \\ \\ u(0,x) = u(2\pi,x)\,,\ \int_0^{2\pi} u(s,x)\, ds = 0, 0 \le x \le \pi, \\ \\ u(t,0) = u(t,\pi)\,, u_x(t,0) = u_x(t,\pi)\ \ 0 \le t \le 2\pi \end{cases} \tag{3.6}$$

for the elliptic equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) + u(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi)\,, u'(0) = u'(\pi)\,.$$

generated by the space operator of problem (3.6). So, the nontrivial solutions of this Sturm-Liouville problem are given by formulas

$$\lambda_k = 4k^2 + 1,\ u_k(x) = \cos kx,\ k = 0, 1, 2, 3, \cdots, u_k(x) = \sin kx, k = 1, 2, 3, \cdots.$$

Therefore, we will obtain the Fourier series solution of problem(3.6) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx,$$

where $A_k(t)\,, k = 0, 1, 2, 3, ...$ and $B_k(t)\,, k = 1, 2, 3, ...$ are unknown functions. Putting $u(t,x)$ into the equation (3.6) and nonlocal boundary conditions (3.2), we obtain

$$- \sum_{k=0}^{\infty} A_k''(t) \cos 2kx - \sum_{k=1}^{\infty} B_k''(t) \sin 2kx + 4k^2 \sum_{k=0}^{\infty} A_k(t) \cos 2kx + 4k^2 \sum_{k=1}^{\infty} B_k(t) \sin 2kx$$

$$+ \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx = 6\cos t \cos 2x, 0 < t < 2\pi, 0 < x < \pi,$$

$$\sum_{k=0}^{\infty} A_k(0) \cos 2kx + \sum_{k=1}^{\infty} B_k(0) \sin 2kx = \sum_{k=0}^{\infty} A_k(2\pi) \cos 2kx$$

$$+ \sum_{k=1}^{\infty} B_k(2\pi) \sin 2kx, 0 \le x \le \pi,$$

$$\int_0^{2\pi} \left(\sum_{k=0}^{\infty} A_k(s) \cos 2kx + \sum_{k=1}^{\infty} B_k(s) \sin 2kx \right) ds \tag{3.7}$$
$$= \sum_{k=0}^{\infty} \cos 2kx \int_0^{2\pi} A_k(s)\, ds + \sum_{k=1}^{\infty} \sin 2kx \int_0^{2\pi} B_k(s)\, ds = 0, 0 \le x \le \pi.$$

Equating coefficients $\sin kx, k = 1, ...,$and $\cos kx, k = 0, 1, ...$ to zero, we get

$$-A_1''(t) + 5A_1(t) = 6\cos t, 0 < t < 2\pi,$$
$$-A_k''(t) + \left(4k^2 + 1\right) A_k(t) = 0, k \ne 1, 0 < t < 2\pi,$$
$$A_k(0) = A_k(2\pi), \quad \int_0^{2\pi} A_k(s)\, ds = 0, k = 0, 1, 2, 3, ...,$$
$$-B_k''(t) + \left(4k^2 + 1\right) B_k(t) = 0, 0 < t < 2\pi,$$
$$B_k(0) = B_k(2\pi), \quad \int_0^{2\pi} B_k(s)\, ds = 0, k = 1, 2, 3, ...$$

that is

$$A_k(0) = A_k(2\pi), \quad \int_0^{2\pi} A_k(s)\, ds = 0, k = 0, 1, 2, 3, ...,$$
$$B_k(0) = B_k(2\pi), \quad \int_0^{2\pi} B_k(s)\, ds = 0, k = 1, 2, 3, ...$$

We will obtain $A_k(t), k = 0, 1, ...$ Firstly, for $k = 1$,we have the following problem

$$\begin{cases} -A_1''(t) + 5A_1(t) = 6\cos t, 0 < t < 2\pi, \\ \\ A_1(0) = A_1(2\pi), \quad \int_0^{2\pi} A_1(s)\, ds = 0. \end{cases}$$

It is easy to obtain that $A_1(t) = \cos t$. Secondly, for $k \ne 1$,we have the following problem

$$\begin{cases} -A_k''(t) + \left(4k^2 + 1\right) A_k(t) = 0, 0 < t < 2\pi, \\ \\ A_k(0) = A_k(2\pi), \quad \int_0^{2\pi} A_k(s)\, ds = 0. \end{cases}$$

It is easy to obtain that $A_k(t) = 0$.

Now will obtain $B_k(t), k = 1, ...$ We have the following problem

$$\begin{cases} -B_k''(t) + \left(4k^2 + 1\right) B_k(t) = 0, 0 < t < 2\pi, \\ \\ B_k(0) = B_k(2\pi), \quad \int_0^{2\pi} B_k(s)\, ds = 0. \end{cases}$$

It is easy to obtain that $B_k(t) = 0$. Thus the solution of problem (3.6) is $u(t, x) = \cos t \cos 2x$.

Note that using similar procedure one can obtain the solution of the following mixed problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ 0 < t < 2\pi,\ x = (x_1, ..., x_n) \in \Omega, \\ u(0,x) = u(2\pi, x) + \varphi(x),\ \int_0^{2\pi} u(s,x)\, ds = \psi(x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2},\ \dfrac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_1} = \dfrac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_2},\ 0 \le t \le 2\pi \end{cases} \tag{3.8}$$

for the multidimensional elliptic differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (0, 2\pi), x \in \overline{\Omega}\right), \varphi(x), \psi(x)$ $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, \varnothing = S_1 \cap S_2$.

However Fourier series method described in solving (3.8) can be used only in the case when (3.8) has constant coefficients.

Now, we consider Laplace transform solution of nonlocal problems for elliptic differential equations.

Example 3.1.4. Obtain the Laplace transform solution of the initial-boundary-value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + u = \sin t\, e^{-x},\ 0 < t < 2\pi,\ 0 < x < \infty, \\ u(0,x) = u(2\pi, x),\ \int_0^{2\pi} u(s,x)\, ds = 0, 0 \le x < \infty, \\ u(t,0) = \sin t,\ u_x(t,0) = -\sin t,\ 0 \le t \le 2\pi \end{cases} \tag{3.9}$$

for the elliptic equation.

Solution. We will denote $\mathcal{L}\{u(t,x)\} = u(t,s)$. Using formula

$$\mathcal{L}\{e^{-x}\} = \frac{1}{s+1}$$

and taking the Laplace transform of both sides of partial differential equations and $u(0,x) = u(2\pi,x)$, $\int_0^{2\pi} u(s,x)\,ds = 0$ and using conditions $u(t,0) = \sin t$, $u_x(t,0) = -\sin t$, $0 \le t \le 2\pi$, we obtain

$$-u_{tt}(t,s) - \left(s^2 - 1\right) u(t,s) = \frac{2-s^2}{1+s}\sin t,\ 0 < t < 2\pi,$$

$$u(0,s) = u(2\pi,s),\ \int_0^{2\pi} u(y,s)\,dy = 0.$$

It is easy to obtain that

$$u(t,s) = \frac{1}{1+s}\sin t.$$

Finally, taking the inverse Laplace transform of this equation, we obtain

$$u(t,x) = e^{-x}\sin t.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\left\{\begin{array}{l} -\dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ \\ x = (x_1,...,x_n) \in \Omega^+,\ 0 < t < 2\pi, \\ \\ u(0,x) = u(2\pi,x) + \varphi(x),\ \int_0^{2\pi} u(s,x)\,ds = \psi(x), \\ \\ x = (x_1,...,x_n) \in \overline{\Omega}^+, \\ \\ u(t,x) = \alpha(t,x),\quad u_{x_r}(t,x) = \beta(t,x), \\ \\ 1 \le r \le n, 0 \le t \le 2\pi, x \in S^+ \end{array}\right. \tag{3.10}$$

for the multidimensional elliptic differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\,(t \in (0,2\pi), x \in \Omega^+), \varphi(x), \psi(x)\ \left(x \in \overline{\Omega}^+\right),\ \alpha(t,x), \beta(t,x)\ \ (t \in [0,2\pi], x \in S^+)$ are given smooth functions. Here Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n\,(0 < x_k < \infty, 1 \leq k \leq n)$ with the boundary S^+ and $\overline{\Omega}^+ = \Omega^+ \cup S^+$.
However Laplace transform method described in solving (3.10) can be used only in the case when (3.10) has $a_r(x)$ polynomials coefficients.

Finally, we consider the Fourier transform solution of the nonlocal boundary value problem for elliptic differential equations.

Example 3.1.5. Obtain the Fourier transform solution of the nonlocal boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + u = \left(-4x^2 + 4\right) e^{-x^2} \sin t\,,\ 0 < t < 2\pi,\ \ -\infty < x < \infty, \\ \\ u(0,x) = u(2\pi,x)\,,\ \int_0^{2\pi} u(s,x)\,ds = 0, -\infty \leq x < \infty \end{cases} \tag{3.11}$$

for the elliptic differential equation.

Solution. We denote $\mathcal{F}\{u(t,x)\} = u(t,\mu)$. Then, applying the formula

$$\left(e^{-x^2}\right)'' = \left(4x^2 - 2\right) e^{-x^2},$$

and taking the Fourier transform from both sides of (3.11) and boundary conditions

$$u(0,x) = u(2\pi,x)\,,\ \int_0^{2\pi} u(s,x)\,ds = 0,$$

we get

$$-u_{tt}(t,\mu) + \left(\mu^2 + 1\right) u(t,\mu) = \left(\mu^2 + 2\right) \mathcal{F}\left\{e^{-x^2}\right\} \sin t,\ 0 < t < 2\pi,$$

$$u(0,\mu) = u(2\pi,\mu)\,,\ \int_0^{2\pi} u(s,\mu)\,ds = 0. \tag{3.12}$$

The general solution of this equation is given by formula

$$u(t,\mu) = c_1 e^{\sqrt{\mu^2+1}t} + c_1 e^{-\sqrt{\mu^2+1}t} + \mathcal{F}\left\{e^{-x^2}\right\}\sin t.$$

Then, using the nonlocal boundary conditions (3.12), we get

$$u(t,\mu) = \mathcal{F}\left\{e^{-x^2}\right\}\sin t.$$

Finally, taking the inverse Fourier transform, we get

$$u(t,x) = e^{-x^2}\sin t.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ \\ x = (x_1, ..., x_n) \in \mathbb{R}^n, \ 0 < t < 2\pi, \\ \\ u(0,x) = u(2\pi,x) + \varphi(x), \ \int_0^{2\pi} u(s,x)\,ds = \psi(x), \\ \\ x = (x_1, ..., x_n) \in \mathbb{R}^n \end{cases} \tag{3.13}$$

for the multidimensional elliptic differential equation. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\,(t \in (0,2\pi), x \in \mathbb{R}^n), \varphi(x), \psi(x)\ (x \in \mathbb{R}^n)$ are given smooth functions.

However Fourier transform method described in solving (3.13) can be used only in the case when (3.13) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant or polynomial coefficients. It is well-known that the most general method for solving partial differential equation with dependent in t and in the space variables is operator method.

In Chapter 3, we will study well-posedness of the nonlocal boundary value problem for elliptic equations. Note that the well-posedness of the local boundary value problem for the elliptic equation

$$-v''(t) + Av(t) = f(t)\ (0 \leq t \leq T),\ v(0) = v_0,\ v(T) = v_T \tag{3.14}$$

in an arbitrary Banach space E with the positive operator A and its related applications have been investigated by many researchers, see for example, (Ashyralyev & Sobolevskii, 2004; Lunardi, 1995; Skubachevskii, 1997) and the references given therein.

Here, the abstract nonlocal boundary value problem for differential equation of elliptic type

$$-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T), v(0) = v(T) + \varphi, \int_0^T v(s)ds = \psi \tag{3.15}$$

in an arbitrary Banach space E with the positive operator A is considered. A function $v(t)$ is called a solution of the problem (3.15) if the following conditions are satisfied:

i. $v(t)$ is a twice continuously differentiable on the segment $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii. The element $v(t)$ belongs to $D(A)$ for all $t \in [0, T]$, and the function $Av(t)$ is continuous on the segment $[0, T]$.

iii. $v(t)$ satisfies the equation and boundary conditions (3.15).

A solution of problem (3.15) defined in this manner will from now on be referred to as a solution of problem (3.15) in the space $C(E) = C([0, T], E)$. Here $C(E)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[0, T]$ with values in E equipped with the norm

$$||\varphi||_{C(E)} = \max_{t\in[0,T]} ||\varphi(t)||_E.$$

The well-posedness of the problem (3.15) in various Banach spaces is established. In applications, the new coercive stability estimates in Hölder norms for the solutions of the mixed type nonlocal boundary value problems for elliptic equations are obtained.

3.2 AUXILIARY RESULTS FOR PROBLEM (3.14)

In this section, we give some auxiliary statements from (Ashyralyev & Sobolevskii, 2004) which will be useful in the sequel. The operator $B = A^{\frac{1}{2}}$ has better spectral properties than the positive operator A. Indeed, the operator $-B$ is a generator of an analytic semigroup $\exp\{-tB\}$ $(t \geq 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$, i. e. the following estimates

$$\left\|\exp\,(-tB)\right\|_{E\to E}, \left\|tB\exp(-tB)\right\|_{E\to E} \leq M(B)e^{-\alpha(B)t}(t > 0) \tag{3.16}$$

hold for some $M(B) \in [1, +\infty)$, $a(B) \in (0, +\infty)$. From that it follows that the operator $I - e^{-2TB}$ has the bounded inverse and the following estimate holds:

$$\|(I - e^{-2TB})^{-1}\|_{E\to E} \leq M(B)(1 - e^{-2T\alpha(B)})^{-1}. \tag{3.17}$$

The following formula

$$v(t) = (I - e^{-2TB})^{-1}\{(e^{-tB} - e^{-(2T-t)B})v_0 + (e^{-(T-t)B} - e^{-(T+t)B})v_T$$
$$-(e^{-(T-t)B} - e^{-(T+t)B})(2B)^{-1}\int_0^T (e^{-(T-s)B} - e^{-(T+s)B})f(s)ds\}$$
$$+(2B)^{-1}\int_0^T (e^{-|t-s|B} - e^{-(t+s)B})f(s)ds \tag{3.18}$$

holds for the exact solution of problem (3.14) under sufficiently smooth data v_0, v_T and $f(t)$. We denote by $C^\alpha(E), (0 < \alpha < 1)$, the Banach space obtained by completion of the set of all smooth E-valued functions $\varphi(t)$ on [0, T] in the norm

$$\|\varphi\|_{C^\alpha(E)} = \max_{0\leq t\leq T}\|\varphi(t)\|_E + \sup_{0\leq t<t+\tau\leq T}\frac{\|\varphi(t+\tau) - \varphi(t)\|_E}{\tau^\alpha}.$$

Theorem 3.2.1. *Suppose v_0'', $v_T'' \in E_\alpha$, $f(t) \in C^\alpha(E)(0 < \alpha < 1)$. Then the boundary value problem (3.14) is well-posed in Hölder space $C^\alpha(E)$, if A is the positive operator in Banach space E. For the solution $v(t)$ in $C^\alpha(E)$ of the boundary value problem the coercive inequality*

$$\|v''\|_{C^\alpha(E)} + \|Av\|_{C^\alpha(E)} + \|v''\|_{C(E_\alpha)} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha(E)} + \frac{M}{\alpha}[\|v_0''\|_{E_\alpha} + \|v_T''\|_{E_\alpha}]$$

holds, where M does not depend on α, v_0, v_T and $f(t)$

Here, the Banach space $E_\alpha = E_\alpha(B, E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\| v \|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \| B exp\{-zB\} v \|_E + \| v \|_E$$

is finite. Moreover, the positivity of A is a necessary condition for well-posedness of problem (3.14) in $C(E)$. However, the problem (3.14) is not well posed in $C(E)$ for all positive operators. It turns out that a Banach space E can be restricted to a Banach space E' in such a manner that the restricted problem (3.14) in E' will be well posed in $C(E')$. The role of E' will be played here by the fractional spaces $E_\alpha = E_\alpha(B, E)(0 < \alpha < 1)$.

Theorem 3.2.2. *Let A be the positive operator in a Banach space E and $f(t) \in C(E_\alpha)$* $(0 < \alpha < 1)$*. Then for the solution $v(t)$ in $C(E_\alpha)$ of the boundary value problem (3.14) the coercive inequality*

$$\| v'' \|_{C(E_\alpha)} + \| Av \|_{C(E_\alpha)} \leq M[\| Av_0 \|_{E_\alpha} + \| Av_T \|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \| f \|_{C(E_\alpha)}]$$

holds, where M does not depend on α, v_0, v_T and $f(t)$.

3.3 WELL-POSEDNESS OF PROBLEM (3.15)

We consider the problem (3.15). Using formula (3.18) and nonlocal conditions, $v(0) = v(T) + \varphi$ and $\int\limits_0^T v(s)ds = \psi$, we get

$$v(0) = \frac{1}{2}\left(I - e^{-TB}\right)^{-1}\left(I + e^{-TB}\right)\left(B\psi - B^{-1}\int_0^T f(s)\,ds\right) + \frac{1}{2}\varphi$$

$$+\frac{1}{2}\left(I - e^{-TB}\right)^{-1} B^{-1}\left(\int_0^T e^{-(T-s)B} f(s)\,ds + \int_0^T e^{-sB} f(s)\,ds\right), \tag{3.19}$$

$$v(T) = \frac{1}{2}\left(I - e^{-TB}\right)^{-1}\left(I + e^{-TB}\right)\left(B\psi - B^{-1}\int_0^T f(s)\,ds\right) - \frac{1}{2}\varphi$$

$$+\frac{1}{2}\left(I - e^{-TB}\right)^{-1} B^{-1}\left(\int_0^T e^{-(T-s)B} f(s)\,ds + \int_0^T e^{-sB} f(s)\,ds\right). \tag{3.20}$$

Actually, applying formula (3.18), we get

$$\int_0^T v(y)\,dy = \left(I - e^{-2TB}\right)^{-1} \int_0^T \left(e^{-yB} - e^{-(2T-y)B}\right) dy v(0)$$

$$+ \left(I - e^{-2TB}\right)^{-1} \int_0^T \left(e^{-(T-y)B} - e^{-(T+y)B}\right) dy v(T)$$

$$- \left(I - e^{-2TB}\right)^{-1} (2B)^{-1} \left(\int_0^T \left(e^{-(T-y)B} - e^{-(T+y)B}\right) dy\right)$$

$$\left(\int_0^T \left(e^{-(T-s)B} - e^{-(T+s)B}\right) f(s)\,ds\right)$$

$$+ (2B)^{-1} \int_0^T \int_0^T \left(e^{-|y-s|B} - e^{-(y+s)B}\right) f(s)\,ds dy.$$

By computing and interchanging of the order of integration the following formula yields

$$\int_0^T v(y)\,dy = B^{-1} \left(I - e^{-2TB}\right)^{-1} \left(I - e^{-TB}\right)^2 v(0)$$

$$+ B^{-1} \left(I - e^{-2TB}\right)^{-1} \left(I - e^{-TB}\right)^2 v(T)$$

$$- \frac{1}{2} A^{-1} \left(I - e^{-2TB}\right)^{-1} \left(I - e^{-TB}\right)^2 \int_0^T \left(e^{-(T-s)B} - e^{-(T+s)B}\right) f(s)\,ds$$

$$+ (2B)^{-1} \int_0^T \left(\int_0^s \left(e^{-(s-y)B} - e^{-(y+s)B}\right) dy\right.$$

$$\left. + \int_s^T \left(e^{-(y-s)B} - e^{-(y+s)B}\right) dy\right) f(s)\,ds.$$

It follows that

$$\psi = B^{-1} \left(I - e^{-TB}\right)\left(I + e^{-TB}\right)^{-1} v(0)$$

$$+ B^{-1} \left(I - e^{-TB}\right)\left(I + e^{-TB}\right)^{-1} v(T)$$

$$- \frac{1}{2} A^{-1} \left(I - e^{-TB}\right)\left(I + e^{-TB}\right)^{-1} \int_0^T \left(e^{-(T-s)B} - e^{-(T+s)B}\right) f(s)\,ds$$

$$+ \frac{1}{2} A^{-1} \int_0^T \left(2I - 2e^{-sB} - e^{-(T-s)B} + e^{-(T+s)B}\right) f(s)\,ds.$$

Thus

$$\psi = B^{-1} \left(I - e^{-TB}\right)\left(I + e^{-TB}\right)^{-1} v(0)$$

$$+B^{-1}\left(I-e^{-TB}\right)\left(I+e^{-TB}\right)^{-1}v(T)$$

$$-A^{-1}\left(I+e^{-TB}\right)^{-1}\left(\int_0^T e^{-(T-s)B}f(s)\,ds+\int_0^T e^{-sB}f(s)\,ds\right)$$

$$+A^{-1}\int_0^T f(s)\,ds.$$

Applying the nonlocal condition $v(0)=v(T)+\varphi$, we get

$$\psi=2B^{-1}\left(I-e^{-TB}\right)\left(I+e^{-TB}\right)^{-1}v(0)-B^{-1}\left(I-e^{-TB}\right)\left(I+e^{-TB}\right)^{-1}\varphi$$

$$-A^{-1}\left(I+e^{-TB}\right)^{-1}\left(\int_0^T e^{-(T-s)B}f(s)\,ds+\int_0^T e^{-sB}f(s)\,ds\right)$$

$$+A^{-1}\int_0^T f(s)\,ds.$$

From that they follow formulas (3.19) and (3.20).

It is easy to show that $v(t)$ defined on $[0,T]$ by formulas (3.18), (3.19), and (3.20) is a unique solution in $C(E)$ of problem (3.15) if, for example, $\varphi\in D(A^2), \psi\in D(A^3)$ and $Af(t)\in C(E)$ or $f'(t)\in C(E)$. Sufficient conditions for the well-posedness of the nonlocal boundary value problem (3.15) can be established if one considers this problem in certain spaces of smooth $E-$valued functions defined on $[0,T]$.

Note that for the solution of problem (3.15) the coercivity inequality

$$\|v''\|_{C^\alpha(E)}+\|Av\|_{C^\alpha(E)}\le M_C[\|f\|_{C^\alpha(E)}+\|A\varphi\|_E+\|A\psi\|_E]$$

fails. Nevertheless, we have the following result.

Theorem 3.3.1. *Suppose $A\psi-\int_0^T f(s)\,ds=0$, $A\varphi-f(0)+f(T)\in E_\alpha$, $f(t)\in C^\alpha(E)$ $(0<\alpha<1)$. Then the boundary value problem (3.15) is well-posed in Hölder space $C^\alpha(E)$, if A is the positive operator in Banach space E. For the solution $v(t)$ in $C^\alpha(E)$ of the boundary value problem the coercive inequality*

$$\|v''\|_{C^\alpha(E)}+\|Av\|_{C^\alpha(E)}+\|v''\|_{C(E_\alpha)}\le\frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha(E)}+\frac{M}{\alpha}\|A\varphi-f(0)+f(T)\|_{E_\alpha}$$

holds, where M does not depend on α, φ and $f(t)$.

Proof. By Theorem 3.2.1 we have the following estimate

$$\|v''\|_{C^{\alpha}(E)} + \|Av\|_{C^{\alpha}(E)} + \|v''\|_{C(E_\alpha)}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \| f \|_{C^{\alpha}(E)} + \frac{M}{\alpha}\left[\|Av(0) - f(0)\|_{E_\alpha} + \|Av(T) - f(T)\|_{E_\alpha}\right]$$

for the solution of problem (3.15). Therefore, to prove the theorem it suffices to establish the estimates for $\|Av(0) - f(0)\|_{E_\alpha}$ and $\|Av(T) - f(T)\|_{E_\alpha}$. Applying formula (3.19), we get

$$Av(0) - f(0) = \frac{1}{2}(A\varphi - f(0) + f(T)) + \frac{1}{2}\left(I - e^{-TB}\right)^{-1}$$

$$\times\left(\int_0^T Be^{-(T-s)B}(f(s) - f(T))\,ds + \int_0^T Be^{-sB}(f(s) - f(0))\,ds\right).$$

Then using the triangle inequality, the estimates (3.16), (3.17) and the definition of the spaces $C^{\alpha}(E)$ and E_α, we get

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}(Av(0) - f(0))\right\|_E$$

$$\leq \frac{1}{2}\left\|\lambda^{1-\alpha}Be^{-\lambda B}(A\varphi - f(0) + f(T))\right\|_E + \frac{1}{2}\left\|\left(I - e^{-TB}\right)^{-1}\right\|_{E\to E}$$

$$\times\lambda^{1-\alpha}\left(\int_0^T \left\|B^2e^{-(\lambda+(T-s))B}\right\|_{E\to E}\|f(s) - f(T)\|_E\,ds\right.$$

$$\left.+\int_0^T \left\|B^2e^{-(\lambda+s)B}\right\|_{E\to E}\|f(s) - f(0)\|_E\,ds\right)$$

$$\leq \frac{1}{2}\sup_{\lambda>0}\left\|\lambda^{1-\alpha}Be^{-\lambda B}(A\varphi - f(0) + f(T))\right\|_E$$

$$+\frac{M}{2}\lambda^{1-\alpha}\left(\int_0^T \frac{(T-s)^\alpha}{(\lambda+(T-s))^2}ds\sup_{0\leq s<T}\frac{\|f(s) - f(T)\|_E}{(T-s)^\alpha}\right.$$

$$\left.+\int_0^T \frac{s^\alpha}{(\lambda+s)^2}ds\sup_{0<s\leq T}\frac{\|f(s) - f(0)\|_E}{s^\alpha}\right)$$

$$\leq \frac{1}{2}\|A\varphi - f(0) + f(T)\|_{E_\alpha}$$

$$+\frac{M}{2}\lambda^{1-\alpha}\left(\int_0^T \frac{(T-s)^\alpha}{(\lambda+(T-s))^2}ds + \int_0^T \frac{s^\alpha}{(\lambda+s)^2}ds\right)\|f\|_{C^{\alpha}(E)}$$

$$\leq \frac{1}{2}\|A\varphi - f(0) + f(T)\|_{E_\alpha} + M\int_0^T \frac{\lambda^{1-\alpha}s^\alpha}{(\lambda+s)^2}ds\,\|f\|_{C^{\alpha}(E)}$$

for any $\lambda > 0$. Since

$$\int_0^T \frac{\lambda^{1-\alpha} s^\alpha}{(\lambda + s)^2} ds \leq \int_0^\infty \frac{p^\alpha}{(1+p)^2} dp \leq \frac{2}{(1+\alpha)(1-\alpha)},$$

we have that

$$\left\| \lambda^{1-\alpha} B e^{-\lambda B} (Av(0) - f(0)) \right\|_E$$

$$\leq \frac{1}{2} \|A\varphi - f(0) + f(T)\|_{E_\alpha} + M \frac{2}{(1+\alpha)(1-\alpha)} \|f\|_{C^\alpha(E)}$$

for any $\lambda > 0$. Therefore

$$\|Av(0) - f(0)\|_{E_\alpha} \leq \frac{1}{2} \|A\varphi - f(0) + f(T)\|_{E_\alpha} + \frac{2M}{1-\alpha} \|f\|_{C^\alpha(E)} . \tag{3.21}$$

Applying $v(0) = v(T) + \varphi$, we get

$$Av(T) - f(T) = Av(0) - A\varphi - f(T) .$$

Using the triangle inequality and the estimate (3.21), we get

$$\|Av(T) - f(T)\|_{E_\alpha} \leq \|Av(0) - f(0)\|_{E_\alpha} + \|f(0) - A\varphi - f(T)\|_{E_\alpha}$$

$$\leq \frac{3}{2} \|A\varphi - f(0) + f(T)\|_{E_\alpha} + \frac{2M}{1-\alpha} \|f\|_{C^\alpha(E)} .$$

Therefore, Theorem 3.3.1 is proved. □

Theorem 3.3.2. *Suppose $A\psi - \int_0^T f(s)\, ds = 0$, A is the positive operator in a Banach space E and $f(t) \in C(E_\alpha)$ $(0 < \alpha < 1)$. Then for the solution $v(t)$ in $C(E_\alpha)$ of the boundary value problem (3.15) the coercive inequality*

$$\| v'' \|_{C(E_\alpha)} + \| Av \|_{C(E_\alpha)} \leq M[\| A\varphi \|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \| f \|_{C(E_\alpha)}]$$

holds, where M does not depend on α, φ and $f(t)$.

Proof. By Theorem 3.2.2 we have the following estimate

$$\| v'' \|_{C(E_\alpha)} + \| Av \|_{C(E_\alpha)} \leq M[\| Av(0) \|_{E_\alpha} + \| Av(T) \|_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \| f \|_{C(E_\alpha)}]$$

for the solution of problem (3.15). Therefore, to prove the theorem it suffices to establish the estimates for $\|Av(0)\|_{E_\alpha}$ and $\|Av(T)\|_{E_\alpha}$. Applying formula (3.19), we get

$$Av(0) = \frac{1}{2}A\varphi + \frac{1}{2}\left(I - e^{-TB}\right)^{-1}\left(\int_0^T Be^{-(T-s)B}f(s)\,ds + \int_0^T Be^{-sB}f(s)\,ds\right).$$

Using the triangle inequality, the estimates (3.16), (3.17) and the definition of the spaces E_α, we get

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}Av(0)\right\|_E \leq \frac{1}{2}\left\|\lambda^{1-\alpha}Be^{-\lambda B}A\varphi\right\|_E + \frac{1}{2}\left\|\left(I - e^{-TB}\right)^{-1}\right\|_{E\to E}$$

$$\times\lambda^{1-\alpha}\left(\int_0^T \left\|B^2e^{-(\lambda+(T-s))B}f(s)\right\|_E ds + \int_0^T \left\|B^2e^{-(\lambda+s)B}f(s)\right\|_E ds\right)$$

$$\leq \frac{1}{2}\|A\varphi\|_{E_\alpha} + M\lambda^{1-\alpha}\left(\int_0^T \frac{ds}{(\lambda+T-s)(T-s)^{1-\alpha}}\|f(s)\|_{E_\alpha}\right.$$

$$\left.+\int_0^T \frac{ds}{(\lambda+s)\,s^{1-\alpha}}\|f(s)\|_{E_\alpha}\right)$$

$$\leq \frac{1}{2}\|A\varphi\|_{E_\alpha} + 2M\int_0^T \frac{\lambda^{1-\alpha}ds}{(\lambda+s)\,s^{1-\alpha}}\|f\|_{C(E_\alpha)}$$

for any $\lambda > 0$. Since

$$\int_0^T \frac{\lambda^{1-\alpha}ds}{(\lambda+s)\,s^{1-\alpha}} \leq \int_0^\infty \frac{p^{\alpha-1}}{p+1}dp \leq \frac{1}{\alpha(1-\alpha)},$$

we have that

$$\left\|\lambda^{1-\alpha}Be^{-\lambda B}Av(0)\right\|_E \leq \frac{1}{2}\|A\varphi\|_{E_\alpha} + \frac{2M}{\alpha(1-\alpha)}\|f\|_{C(E_\alpha)}.$$

for any $\lambda > 0$. Therefore

$$\|Av(0)\|_{E_\alpha} \leq \frac{1}{2}\|A\varphi\|_{E_\alpha} + \frac{2M}{\alpha(1-\alpha)}\|f\|_{C(E_\alpha)}. \tag{3.22}$$

Applying $v(0) = v(T) + \varphi$, the triangle inequality and the estimate (3.22), we get

$$\|Av(T)\|_{E_\alpha} \leq \|Av(0)\|_{E_\alpha} + \|A\varphi\|_{E_\alpha} \leq \frac{3}{2}\|A\varphi\|_{E_\alpha} + \frac{2M}{1-\alpha}\|f\|_{C^\alpha(E)}.$$

Therefore, Theorem 3.3.2 is proved. $\square$

3.4 APPLICATIONS

Finally, we consider the applications of Theorems 3.3.1 and 3.3.2 to the elliptic equations. First, we consider the boundary value problems for two dimensional elliptic equations

$$\begin{cases} -\dfrac{\partial^2 u}{\partial y^2} - a(x)\dfrac{\partial^2 u}{\partial x^2} + \delta u = f(y,x),\ 0 < y < T,\ 0 < x < 1, \\ u(0,x) = u(T,x) + \varphi(x), \int\limits_0^T u(s,x)ds = 0,\ 0 \le x \le 1, \\ u(y,0) = u(y,1), \quad u_x(y,0) = u_x(y,1), \quad 0 \le y \le T, \end{cases} \tag{3.23}$$

where $a(x)$, $\varphi(x)$ and $f(y,x)$ are given sufficiently smooth functions and $a(x) > 0,\ \delta > 0$ is a sufficiently large number. We introduce the Banach spaces $C^\beta[0,1]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^\beta[0,1]} = \| \varphi \|_{C[0,1]} + \sup_{0 \le x < x+\tau \le 1} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^\beta},$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,1]$ with the usual norm

$$\| \varphi \|_{C[0,1]} = \max_{0 \le x \le 1} |\varphi(x)|.$$

It is known that the differential expression

$$A^x v = -a(x)v''(x) + \delta v(x)$$

define a positive operator A^x acting in $C^\beta[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the conditions $v(0) = v(1),\ v_x(0) = v_x(1)$. Therefore, we can replace boundary value problems (3.23) by the abstract boundary value problem (3.15). Using the results of Theorems 3.3.1 and 3.3.2, we can obtain that

Theorem 3.4.1. *Assume that* $\int\limits_0^T f(s,x)ds = 0,\ 0 \le x \le 1$. *Then, for the solution of the boundary value problem (3.23) the following coercive inequalities are valid:*

$$\| u \|_{C^{2+\alpha}(C^\mu[0,1])} + \| u \|_{C^\alpha\left(C^{2+\mu}[0,1]\right)}$$

$$\leq M(\alpha)\left[\| f \|_{C^{\alpha}(C^{\mu}[0,1])} + \| -a(\cdot)\varphi''(\cdot) + \delta\varphi(\cdot) - f(0,\cdot) + f(T,\cdot)\|_{C^{2\alpha+\mu}[0,1]}\right],$$

$$\| u \|_{C^{2}\left(C^{2\alpha+\mu}[0,1]\right)} + \| u \|_{C\left(C^{2+2\alpha+\mu}[0,1]\right)}$$

$$\leq M(\alpha)\left[\| f \|_{C\left(C^{2\alpha+\mu}[0,1]\right)} + \|\varphi\|_{C^{2+2\alpha+\mu}[0,1]}\right], \quad 0 < 2\alpha + \mu < 1.$$

Here $M(\alpha)$ is independent of $\varphi(x)$ and $f(y,x)$.

Second, let Ω be the unit open cube in the $n-$dimensional Euclidean space $\mathbf{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$ we consider the mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\dfrac{\partial^2 u(y,x)}{\partial y^2} - \sum\limits_{r=1}^{n} \alpha_r(x) \dfrac{\partial^2 u(y,x)}{\partial x_r^2} + \delta u(y,x) = f(y,x), \\ x = (x_1, \dots, x_n) \in \Omega, 0 < y < T, \\ u(0,x) = u(T,x) + \varphi(x), \int\limits_0^T u(s,x)ds = 0, \ x \in \overline{\Omega}, \\ u(y,x) = 0, \ x \in S, \end{cases} \tag{3.24}$$

where $\alpha_r(x)$ $(x \in \Omega)$ and $f(y,x)$ $(y \in (0,T), \ x \in \Omega)$, $\varphi(x)(x \in \overline{\Omega})$ are given smooth functions and $\alpha_r(x) > 0, \ \delta > 0$ is a sufficiently large number. We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta = (\beta_1, \dots, \beta_n), 0 < x_k < 1, k = 1, \dots, n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta = (\beta_1, \dots, \beta_n), \beta_k \in (0,1), 1 \leq k \leq n$ and with weight $x_k^{\beta_k}(1 - x_k - h_k)^{\beta_k}, 0 \leq x_k < x_k + h_k \leq 1, 1 \leq k \leq n$ which equipped with the norm

$$\| f \|_{C_{01}^{\beta}(\overline{\Omega})} = \| f \|_{C(\overline{\Omega})}$$

$$+ \sup_{0 \leq x_k < x_k + h_k \leq 1, 1 \leq k \leq n} |f(x_1, \dots, x_n) - f(x_1 + h_1, \dots, x_n + h_n)|$$

$$\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k},$$

where $C(\overline{\Omega})$-is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|.$$

It is known that the differential expression

$$A^x v = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(y,x)}{\partial x^2} + \delta v(y,x)$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S. Therefore, we can replace boundary value problems (3.24) by the abstract boundary value problems (3.15). Using the results of Theorems 3.3.1, we can obtain that

Theorem 3.4.2. *Assume that*

$$\int_0^T f(s,x)ds = 0, \ -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 \varphi(x)}{\partial x^2} + \delta\varphi(x) - f(0,x) + f(T,x) = 0, x \in \overline{\Omega}.$$

Then, for the solution of the boundary value problem (3.24) the following coercive inequality is valid:

$$\|u\|_{C^{2+\alpha}\left(C_{01}^{\mu}(\overline{\Omega})\right)} + \sum_{r=1}^{n} \left\|\frac{\partial^2 u}{\partial x_r^2}\right\|_{C^{\alpha}\left(C_{01}^{\mu}(\overline{\Omega})\right)} \le M(\alpha) \|f\|_{C^{\alpha}\left(C_{01}^{\mu}(\overline{\Omega})\right)},$$

$$0 < \alpha < 1, \mu = \{\mu_1, \cdots, \mu_n\}, 0 < \mu_k < 1, 1 \le k \le n,$$

where $M(\alpha)$ is independent of $f(y,x)$.

Third, we consider the boundary value problem on the range

$$\{0 \le y \le T, x \in \mathbf{R}^n\}$$

for $2m-$order multidimensional elliptic equations

$$\begin{cases} -\dfrac{\partial^2 u}{\partial y^2} + \sum\limits_{|r|=2m} a_r(x) \dfrac{\partial^{|\tau|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \delta u(y,x) = f(y,x), \\ 0 < y < T, \ x, r \in \mathbf{R}^n, |r| = r_1 + \cdots + r_n, \\ u(0,x) = u(T,x) + \varphi(x), \int\limits_0^T v(s,x)ds = 0, x \in \mathbf{R}^n, \end{cases} \tag{3.25}$$

where $a_r(x)$ and $f(y,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\alpha_r(x) > 0$, $\delta > 0$ is the sufficiently large number . We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)\,(i\xi_1)^{r_1} \ldots (i\xi_n)^{r_n}\,, \xi = (\xi_1, \cdots, \xi_n) \in R^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{3.26}$$

acting on functions defined on the space $\mathbf{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$. The problem (3.25) has a unique smooth solution. This allows us to reduce the boundary value problem (3.25) to the boundary value problem (3.15) in a Banach space $E = C^\mu(R^n)$ of all continuous bounded functions defined on $\mathbf{R}^n$satisfying a H ölder condition with the indicator $\mu \in (0,1)$ with a strongly positive operator $A^x = B^x + \delta I$ defined by (3.26).

Theorem 3.4.3. *Assume that $\int_0^T f(s,x)ds = 0$, $x \in R^n$. Then, for the solution of the boundary value problem (3.25) the following coercivity inequalities are satisfied*

$$\| u \|_{C^{2+\alpha}(C^\mu(R^n))} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \right\|_{C^\alpha(C^\mu(R^n))} \le M(\alpha) \Big[\|f\|_{C^\alpha(C^\mu(R^n))}$$

$$+ \left\| \sum_{|r|=2m} a_r(\cdot) \frac{\partial^{|\tau|}\varphi(\cdot)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \delta\varphi(\cdot) - f(0,\cdot) + f(T,\cdot) \right\|_{C^{2m\alpha+\mu}(R^n)} \Bigg],$$

$$\| u \|_{C^2\left(C^{2m\alpha+\mu}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\mu}(R^n)\right)}$$

$$\le M(\alpha) \left[\|f\|_{C\left(C^{2m\alpha+\mu}(R^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}\varphi}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\mu}(R^n)\right)} \right], \quad 0 < 2m\alpha + \mu < 1,$$

where $M(\alpha)$ does not depend on $\varphi(x)$ and $f(y,x)$.

The proof of Theorem 3.4.3 is based on the abstract Theorems 3.3.1 and 3.3.2, the positivity of the operator A^x in $C^\mu(R^n)$, the structure of the fractional spaces $E_\alpha((A^x)^{\frac{1}{2}}, C(R^n))$ and the coercivity inequality for an elliptic operator A^x in $C^\mu(R^n)$.

CHAPTER 4

WELL-POSEDNESS OF ELLIPTIC DIFFERENCE EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

4.1 INTRODUCTION

In the present thesis, second order of approximation two-step difference scheme

$$\begin{cases} -\dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=f_k, f_k=f(t_k), t_k=k\tau, 1\leq k\leq N-1, N\tau=T, \\ \\ u_0=u_N+\varphi, \sum\limits_{i=1}^{N} u_i\tau=\psi \end{cases} \tag{4.1}$$

for the approximate solution of problem (3.15) is presented. The well-posedness of the difference scheme (4.1) in Banach spaces is established. In applications, the stability, almost coercive stability and coercive stability estimates in Hölder norms in one variable for the solutions of difference schemes for numerical solution of two type elliptic problems are obtained.

4.2 AUXILIARY RESULTS

In this section, we give some auxiliary statements from (Ashyralyev & Sobolevskii, 2004) which will be useful in the sequel. We consider the second order of accuracy difference scheme

$$-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=f_k,\ f_k=f(t_k),\ t_k=k\tau,\ 1\leq k\leq N-1,\ N\tau=T, \tag{4.2}$$

$$u_0=v_0, u_N=v_T.$$

of approximation solution of the boundary value problem (3.15). This problem is uniquely solvable, and the following formula holds

$$u_k=(I-R^{2N})^{-1}\Big\{(R^k-R^{2N-k})u_0+(R^{N-k}-R^{N+k})u_N \tag{4.3}$$

$$-(R^{N-k}-R^{N+k})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau\Big\}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|k-i|}-R^{k+i})f_i\tau,\ 1\leq k\leq N-1,$$

where

$$B=B(\tau,A)=\frac{\tau A}{2}+\sqrt{\left(\frac{\tau A}{2}\right)^2+A},\ R=(I+\tau B)^{-1}.$$

Note that $B(\tau,A)\neq A^{\frac{1}{2}}$but then $B(\tau,A)\to A^{\frac{1}{2}}$ as $\tau\to 0$ and it has same spectral properties of $A^{\frac{1}{2}}$ under the some assumption for A.

Let us denote by $F_\tau(E)=F([0,T]_\tau,E)$ the space of grid functions $\varphi^\tau=\{\varphi_k\}_{k=1}^{N-1}$ for fixed $\tau=\frac{T}{N}$. Thus, $F_\tau(E)$ is the vector space whose elements are ordered $(N-1)$-tuples of elements of E. The space $F_\tau(E)$ can be equipped with various norms and thus become a normed space. Thus, for instance, the vector space $F_\tau(E)$ generates the normed space $C_\tau(E)=C([0,T]_\tau,E)$ with the norm

$$\parallel\varphi^\tau\|_{C_\tau(E)}=\max_{1\leq k\leq N-1}\parallel\varphi_k\parallel_E.$$

Let us reduce the difference scheme (4.2) to an operator problem in the space $F_\tau(E)$. In addition to the operator D_τ^2, acting from the space $E\times F_\tau(E)\times E$ of vectors $w^\tau=\{w_k\}_{k=0}^N$ into the space $F_\tau(E)$ of vectors $v^\tau=\{v_k\}_{k=1}^{N-1}$ by the rule

$$v^\tau=D_\tau^2u^\tau, v_k=\frac{1}{\tau^2}(w_{k+1}-2w_k+w_{k-1}), k=1,\cdots,N-1,$$

define an operator A_τ from the space $E\times F_\tau(E)\times E$ of vectors $w^\tau=\{w_k\}_{k=0}^N$ into the space $F_\tau(E)$ of vectors $v^\tau=\{v_k\}_{k=1}^{N-1}$ by the rule

$$v^\tau=A_\tau u^\tau,\ v_k=Aw_k,\ k=1,\cdots,N-1.$$

Then the difference scheme (4.2) can obviously be rewritten as the equivalent operator equation

$$-D_\tau^2\Pi(u_0,u_N)u^\tau+A_\tau\Pi(u_0,u_N)u^\tau=f^\tau.$$

Here f^τ is defined by the formula

$$f^\tau=(f_1,\cdots,f_{N-1}).$$

The last operator problem will be considered in the space $F_\tau(E)$. From its unique solvability for any u_0, $u_N \in E$ and $f^\tau \in F_\tau(E)$ it follows that its solution u^τ defines an additive and homogeneous operator $u^\tau(f^\tau, u_0, u_N)$ is continuous.
The boundary value problem (4.2) is said to be stable in $F_\tau(E)$ if we have the inequality

$$\| u^\tau(f^\tau, u_0, u_N) \|_{F_\tau(E)} \leq M[\, \| f^\tau \|_{F_\tau(E)} + \| u_0\|_E + \|u_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N, but also of τ.
The boundary value difference problem (4.2) is said to be well-posed (coercively stable) in $F_\tau(E)$ if we have the coercive inequality

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{F_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{F_\tau(E)}$$

$$\leq M[\| f^\tau \|_{F_\tau(E)} + \| Au_0 \|_E + \|Au_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N, but also of τ.
From the formula (4.3) it follows that the investigation of the stability and well-posedness of difference scheme (4.2) relies in an essential manner on a number of properties of the powers of the operator $R = (I + \tau B)^{-1}$ in the general cases of operator A. We begin by deriving some estimates for powers of the operator $(I + \tau B)^{-1}$ a strongly positive operator A in a Banach space E (Sobolevskii, 2005).

Lemma 4.2.1. *Let A be a strongly positive operator in a Banach space E. Then, $-A$ is a generator of the analitic semigroup exp$\{-tA\}$ $(t \geq 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$, i. e. we have the following estimates*

$$\left\|\exp\{-tA\}\right\|_{E\longrightarrow E} \leq M\, e^{-t\delta}\ (t > 0),$$

$$\left\|tA\exp\{-tA\}\right\|_{E\longrightarrow E} \leq M\, e^{-t\delta}(t > 0)$$

for some $1 \leq M < +\infty$, $0 < \delta < +\infty$. *Here M does not depend on* τ.

Lemma 4.2.2. *Let $-A$ be a generator of the analytic semigroup exp$\{-tA\}$ $(t \geq 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$. Then the following estimates hold for any* $k \geq 1$:

$$\left\|(\lambda I + \tau B)^{-k}\right\|_{E\to E} \leq M[\lambda + \tau a(A)]^{-k},$$

$$\|k\tau B(I+\tau B)^{-k}\|_{E\to E} \le M,$$

where M does not depend on τ.

Let $-A$ be a generator of the analytic semigroup $\exp\{-tA\}$ $(t \ge 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$. Then the following estimates hold for any $k \ge 1$:

$$\left\|(\lambda I+\tau B)^{-k}\right\|_{E\to E} \le M[\lambda+\tau a(A)]^{-k}, \tag{4.4}$$

$$\|k\tau B(I+\tau B)^{-k}\|_{E\to E} \le M, \tag{4.5}$$

where M does not depend on τ.

We have the following results.

Theorem 4.2.3. *Let A be a strongly positive operator in a Banach space E. Then, difference problem (4.2) is stable in $C_\tau(E)$. For the solutions of the difference problem (4.2) satisfy the stability inequalities*

$$\| u^\tau \|_{C_\tau(E)} \le M[\| f^\tau \|_{C_\tau(E)} + \| u_0\|_E + \|u_N\|_E],$$

where M does not depend on f^τ, u_0, u_N and τ.

Theorem 4.2.4. *Let A be a strongly positive operator in a Banach space E and u_0, $u_N \in D(A)$. Then, the solutions of the difference problem (4.2) in $C_\tau(E)$ obey the almost coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$
$$\le M[\min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\} \| f^\tau \|_{C_\tau(E)} + \| Au_0\|_E + \|Au_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N but also of τ.

Theorem 4.2.5. *Let A be a strongly positive operator in a Banach space E and $(I-R)^2\tau^{-2}u_0 - f_1$, $(I-R)^2\tau^{-2}u_N - f_{N-1} \in E'_\alpha$. Then, the solutions of the difference problem (4.2) in $C_\tau^\alpha(E)$ obey the coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_\tau^\alpha(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau^\alpha(E)}$$

$$+ \| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E'_\alpha)}$$
$$\leq \frac{M}{\alpha(1-\alpha)}[\| f^\tau \|_{C^\alpha_\tau(E)} + \| (I-R)^2\tau^{-2}u_0 - f_1 \|_{E'_\alpha}$$
$$+\|(I-R)^2\tau^{-2}u_N - f_{N-1}\|_{E'_\alpha}],$$

where M_1 is independent not only of f^τ, u_0, u_N, α, but also of τ. Here, the Banach space $E'_\alpha = E'_\alpha(B,E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\| v \|_{E'_\alpha} = \sup_{z>0} z^\alpha \| B(zI+B)^{-1}v \|_E + \| v \|_E$$

is finite.

Theorem 4.2.6. *Let A be a strongly positive operator in a Banach space E and $Au_0,\ Au_N \in E'_\alpha$. Then the solutions of the difference problem (4.2) in $C_\tau(E'_\alpha)$ obey the coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E'_\alpha)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E'_\alpha)}$$
$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\| f^\tau \|_{C_\tau(E'_\alpha)} + \|Au_0\|_{E'_\alpha} + \|Au_N\|_{E'_\alpha}\right],$$

where M is independent not only of f^τ, u_0, u_N, α, but also of τ. Here, the Banach space $E'_\alpha = E'_\alpha(B,E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\| v \|_{E'_\alpha} = \sup_{z>0} z^\alpha \| B(zI+B)^{-1}v \|_E + \| v \|_E$$

is finite.

4.3 WELL-POSEDNESS OF DIFFERENCE PROBLEM (4.1)

We consider the difference problem (4.1). Using formula (4.3) and nonlocal conditions

$$u_0 = u_N + \varphi, \sum_{i=1}^{N} u_i\tau = \psi,$$

we get

$$u_0 = (2I+\tau B)^{-1}(I+R^N)\left(I-R^N\right)^{-1}\left\{B\psi - B^{-1}(I+\tau B)\sum_{i=1}^{N-1} f_i\tau\right\} \tag{4.6}$$
$$+ (I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)\varphi$$
$$+ B^{-1}(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1} R^{N-i}f_i\tau + \sum_{i=1}^{N-1} R^i f_i\tau\right\},$$

$$u_N = (2I+\tau B)^{-1}(I+R^N)\left(I-R^N\right)^{-1}\left\{B\psi - B^{-1}(I+\tau B)\sum_{i=1}^{N-1} f_i\tau\right\} \tag{4.7}$$
$$- (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N-1}\right)\varphi$$
$$+ B^{-1}(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1} R^{N-i} f_i\tau + \sum_{i=1}^{N-1} R^i f_i\tau\right\}.$$

Actually, applying formula (4.3), we get

$$\psi = u_N\tau + \sum_{k=1}^{N-1} u_k\tau = (I-R^{2N})^{-1}\left\{\sum_{k=1}^{N-1}(R^k - R^{2N-k})u_0\tau + \sum_{k=1}^{N}(R^{N-k}-R^{N+k})u_N\tau\right.$$
$$\left. - \sum_{k=1}^{N-1}(R^{N-k}-R^{N+k})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2\right\}$$
$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{k=1}^{N-1}\sum_{i=1}^{N-1}(R^{|k-i|}-R^{k+i})f_i\tau^2.$$

By computing and interchange of the order of summation, we obtain

$$\psi = (I-R^{2N})^{-1}\left\{R\left(I+...+R^{N-2}\right) - R^{N+1}\left(R^{N-2}+...+I\right)\right\}u_0\tau$$
$$+ (I-R^{2N})^{-1}\left\{\left(R^{N-1}+...+I\right) - R^{N+1}\left(I+...+R^{N-1}\right)\right\}u_N\tau$$
$$- (I-R^{2N})^{-1}\left\{R\left(R^{N-2}+...+I\right) - R^{N+1}\left(I+...+R^{N-2}\right)\right\}$$
$$\times (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2 + (I+\tau B)(2I+\tau B)^{-1}B^{-1}$$
$$\times \sum_{i=1}^{N-1}\left\{\left(R^{i-1}+...+I\right) + \left(R+...+R^{N-i-1}\right) - \left(R^{1+i}+...+R^{N+i-1}\right)\right\}f_i\tau^2.$$

Therefore,

$$\psi = (I-R^{2N})^{-1}R\left(I-R^N\right)(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau$$
$$+ (I-R^{2N})^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}\left(I-R^N\right)u_N\tau$$
$$- (I-R^{2N})^{-1}R\left(I-R^N\right)(I-R)^{-1}\left(I-R^{N-1}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}$$
$$\times \sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2 + (I+\tau B)(2I+\tau B)^{-1}B^{-1}(I-R)^{-1}$$
$$\times \sum_{i=1}^{N-1}\left\{\left(I-R^i\right) + R\left(I-R^{N-i-1}\right) - R^{1+i}\left(I-R^{N-1}\right)\right\}f_i\tau^2.$$

From that it follows that

$$\psi = (I+R^N)^{-1}R\,(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau + (I+R^N)^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}\,u_N\tau$$
$$-\,(I+R^N)^{-1}\,(I-R)^{-1}\left(I-R^{N-1}\right)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2$$
$$+\,(I+\tau B)(2I+\tau B)^{-1}B^{-1}\,(I-R)^{-1}\sum_{i=1}^{N-1}\left(I-R^i+R-R^{N-i}-R^{i+1}+R^{N+i}\right)f_i\tau^2.$$

Thus

$$\psi = (I+R^N)^{-1}R\,(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau + (I+R^N)^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}\,u_N\tau$$
$$-\,B^{-1}\,(I-R)^{-1}\,(I+R^N)^{-1}\left\{\sum_{i=1}^{N-1}R^{N-i}f_i\tau^2+\sum_{i=1}^{N-1}R^if_i\tau^2\right\}$$
$$+\,B^{-1}(I+\tau B)(2I+\tau B)^{-1}\,(I-R)^{-1}\,(I+R)\sum_{i=1}^{N-1}f_i\tau^2.$$

Since $u_N = u_0-\varphi$, we have that

$$\psi = (I-R)^{-1}\,(I+R^N)^{-1}\,(I+R)\left(I-R^N\right)u_0\tau - (I-R)^{-1}\,(I+R^N)^{-1}\left(I-R^{N+1}\right)\varphi\tau$$
$$-\,B^{-1}\,(I-R)^{-1}\,(I+R^N)^{-1}\left\{\sum_{i=1}^{N-1}R^{N-i}f_i\tau^2+\sum_{i=1}^{N-1}R^if_i\tau^2\right\}+B^{-1}\,(I-R)^{-1}\sum_{i=1}^{N-1}f_i\tau^2.$$

From that they follow formulas (4.6), (4.7).

Theorem 4.3.1. *Let A be a strongly positive operator in a Banach space E and $\psi = A^{-1}\sum_{i=1}^{N-1} f_i\tau$. Then, difference problem (4.1) is stable in $C_\tau(E)$. For the solutions of the difference problem (4.1) satisfy the stability inequality*

$$\| u^\tau \|_{C_\tau(E)} \le M\left[\| f^\tau \|_{C_\tau(E)} + \| \varphi \|_E\right],$$

where M_1 does not depend on f^τ, φ and τ.

Proof. By Theorem 4.2.3, we have the following estimate

$$\| u^\tau \|_{C_\tau(E)} \le M[\| f^\tau \|_{C_\tau(E)} + \| u_0\|_E + \|u_N \|_E]$$

for solution of problem (4.2). Therefore, to prove the theorem it suffices to establish the estimate for $\|u_0\|_E$ and $\|u_N\|_E$. Applying condition $\psi = A^{-1} \sum\limits_{i=1}^{N-1} f_i\tau$, formula (4.6), we get

$$u_0 = (I + \tau B)(2I + \tau B)^{-1} \left(I - R^{N+1}\right)\left(I - R^{N}\right)^{-1} \varphi \tag{4.8}$$
$$+ B^{-1}(I + \tau B)(2I + \tau B)^{-1} \left(I - R^{N}\right)^{-1} \left\{\sum_{i=1}^{N-1} R^{N-i} f_i\tau + \sum_{i=1}^{N-1} R^{i} f_i\tau\right\}.$$

Using formula (4.8) and the triangle inequality, we get

$$\|u_0\|_E \leq \left\|(I + \tau B)(2I + \tau B)^{-1} \left(I - R^{N+1}\right)\left(I - R^{N}\right)^{-1}\right\|_{E\to E} \|\varphi\|_E$$
$$+ \left\|B^{-1}(I + \tau B)(2I + \tau B)^{-1} \left(I - R^{N}\right)^{-1}\right\|_{E\to E}$$
$$\times \sum_{i=1}^{N-1} \left\|R^{N-i}\right\|_{E\to E} \|f_i\|_E\, \tau + \sum_{i=1}^{N-1} \left\|R^{i}\right\|_{E\to E} \|f_i\|_E\, \tau.$$

Using estimate (4.4), (4.5), we get

$$\|u_0\|_E \leq M_1 \|\varphi\|_E + M_2 \max_{1\leq i\leq N-1} \|f_i\|_E \sum_{i=1}^{N-1} \left(\frac{1}{1 + \tau a(A)}\right)^{i} \tau$$
$$\leq M_3 \left(\| f^{\tau} \|_{C_\tau(E)} + \|\varphi\|_E\right).$$

From that and formula $u_0 = u_N + \varphi$ it follows

$$\|u_N\|_E \leq M_3\| f^{\tau} \|_{C_\tau(E)} + (M_3 + 1) \| \varphi \|_E.$$

Therefore, Theorem 4.3.1 is proved. □

Note that for the solution of difference problem (4.1) the coercivity inequality

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$
$$\leq M_C[\|f\|_{C_\tau(E)} + \|A\varphi\|_E + \|A\psi\|_E]$$

fails. Nevertheless, we have the following results.

Theorem 4.3.2. *Let A be a strongly positive operator in a Banach space E and $\psi = A^{-1} \sum\limits_{i=1}^{N-1} f_i\tau$. Then, the solutions of the difference problem (4.1) in $C_\tau(E)$ obey the almost coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$

$$\leq M\left[\min\left\{\ln\frac{1}{\tau},|\ln\|A\|_{E\to E}|\right\}\| f^{\tau}\|_{C_{\tau}(E)}+\| A\varphi\|_{E}\right],$$

where M_4 is independent not only of f^{τ},φ but also of τ.

Proof. By Theorem 4.2.4, we have the following estimate

$$\| Au^{\tau}\|_{C_{\tau}(E)}\leq M[\min\left\{\ln\frac{1}{\tau},|\ln\|A\|_{E\to E}|\right\}\| f^{\tau}\|_{C_{\tau}(E)}+\| Au_0\|_E+\|Au_N\|_E]$$

for solution of problem (4.2). Therefore, to prove the theorem it suffices to establish the estimate for $\|Au_0\|_E$ and $\|A\,u_N\|_E$. Using formula (4.8), and formula $A=B^2R$, we get

$$Au_0=(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^{N}\right)^{-1}A\varphi \tag{4.9}$$
$$+(2I+\tau B)^{-1}\left(I-R^{N}\right)^{-1}\left[\sum_{i=1}^{N-1}BR^{N-i}f_i\tau+\sum_{i=1}^{N-1}BR^{i}f_i\tau\right].$$

Applying the triangle inequality, we get

$$\|Au_0\|_E\leq\|(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^{N}\right)^{-1}\|_{E\to E}\|A\varphi\|_E$$
$$+\|(2I+\tau B)^{-1}\left(I-R^{N}\right)^{-1}\|_{E\to E}\left\{\sum_{i=1}^{N-1}\left\|BR^{N-i}\right\|_{E\to E}\|f_i\|_E\,\tau+\sum_{i=1}^{N-1}\left\|BR^{i}\right\|_{E\to E}\|f_i\|_E\,\tau\right\}.$$

Using estimate (4.4), (4.5), we get

$$\|Au_0\|_E\leq M_5\|A\varphi\|_E+M_6\max_{1\leq i\leq N-1}\|f_i\|_E\ln\frac{1}{\tau},$$

$$\|Au_0\|_E\leq M_5\|A\varphi\|_E+M_6\max_{1\leq i\leq N-1}\|f_i\|_E\left[1+|\ln\|B\|_{E\to E}|\right].$$

Hence

$$\| Au_0\|_E\leq M_7\left[\| A\varphi\|_E+\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{E\to E}|\right\}\| f^{\tau}\|_{C_{\tau}(E)}\right].$$

From that and formula $u_N=u_0-\varphi$ it follows

$$\|Au_N\|_E\leq(M_7+1)\| A\varphi\|_E+M_7\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{E\to E}|\right\}\| f^{\tau}\|_{C_{\tau}(E)}.$$

Therefore, Theorem 4.3.2 is proved. □

Note that for the solution of difference problem (4.1) the coercivity inequality

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C_{\tau}^{\alpha}(E)}+\|\{Au_k\}_1^{N-1}\|_{C_{\tau}^{\alpha}(E)}$$

$$\leq M_C[\|f\|_{C^{\alpha}(E)}+\|A\varphi\|_E+\|A\psi\|_E]$$

fails. Nevertheless, we have the following result.

Theorem 4.3.3. *Let A be a strongly positive operator in a Banach space E and $\psi = A^{-1}\sum\limits_{i=1}^{N-1} f_i\tau$. Then the solutions of the difference problem (4.1) in $C_\tau^\alpha(E)$ obey the coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau^\alpha(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau^\alpha(E)}$$

$$+ \| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E'_\alpha)} \le \frac{M_8}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau^\alpha(E)}$$

$$+ M_8 \left[\|A\varphi + Rf_{N-1} - (I + 3\tau B + (\tau B)^2)Rf_1\|_{E'_\alpha} + \|A\varphi - Rf_1 + (I + 3\tau B + (\tau B)^2)Rf_{N-1}\|_{E'_\alpha} \right],$$

where M_8 is independent not only of f^τ, φ, α, but also of τ.

Proof. By Theorem 4.2.5 we have the following estimate

$$\| \{Au_k\}_1^{N-1} \|_{C_\tau^\alpha(E)}$$

$$\le \frac{M}{\alpha(1-\alpha)} [\| f^\tau \|_{C_\tau^\alpha(E)} + \| (I-R)^2\tau^{-2}u_0 - f_1 \|_{E'_\alpha} + \|(I-R)^2\tau^{-2}u_N - f_{N-1}\|_{E'_\alpha}],$$

for solution of problem (4.2). Therefore, to prove the theorem it suffices to establish the estimates for $\| (I-R)^2\tau^{-2}u_0 - f_1 \|_{E'_\alpha}$ and $\|(I-R)^2\tau^{-2}u_N - f_{N-1}\|_{E'_\alpha}$. Applying formula (4.8) and $(I-R)^2\tau^{-2} = AR$, we get

$$(I-R)^2 \tau^{-2} u_0 - f_1 = (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} \left(I - R^{N+1}\right) A\varphi$$

$$+ (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} R \left\{ \sum_{i=1}^{N-1} BR^{N-i} f_i \tau + \sum_{i=1}^{N-1} BR^i f_i \tau \right\} - f_1$$

asd

$$(I-R)^2 \tau^{-2} u_0 - f_1 = (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} \left(I - R^{N+1}\right) \left[A\varphi + Rf_{N-1} - \left(I + 3\tau B + (\tau B)^2\right) Rf_1\right]$$

$$+ (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} R^2 \left\{ \sum_{i=1}^{N-1} BR^{N-i-1} (f_i - f_{N-1})\, \tau + \sum_{i=1}^{N-1} BR^{i-1} (f_i - f_1)\, \tau \right\}$$

$$+ (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} R^{N-1} (I - R) \left(I - R^2\right) f_1$$

$$- (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} R^N \left(I - R^2\right) f_{N-1} = \sum_{m=1}^{5} H^m,$$

where

$$H^1 = (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)\left\{A\varphi + Rf_{N-1} - (I+3\tau B+(\tau B)^2)Rf_1\right\},$$

$$H^2 = (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\sum_{i=1}^{N-1} BR^{N-i+1}\left(f_i - f_{N-1}\right)\tau,$$

$$H^3 = (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\sum_{i=1}^{N-1} BR^{i+1}\left(f_i - f_1\right)\tau,$$

$$H^4 = (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}R^{N-1}(I-R)\left(I-R^2\right)f_1,$$

$$H^5 = -(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}R^N\left(I-R^2\right)f_{N-1}.$$

Now, let us estimate H^m for any $m = 1,2,3,4,5$ in E'_α, separately. We start with H^1. Using estimate (4.5) and the definition of the spaces E'_α,we get

$$\|H^1\|_{E'_\alpha} \le \|(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)\|_{E\to E}$$

$$\times\|A\varphi + Rf_{N-1} - (I+3\tau B+(\tau B)^2)Rf_1\|_{E'_\alpha} \le M\|A\varphi + Rf_{N-1} - (I+3\tau B+(\tau B)^2)Rf_1\|_{E'_\alpha}.$$

Thus, we have proved that

$$\|H^1\|_{E'_\alpha} \le M\|A\varphi + Rf_{N-1} - (I+3\tau B+(\tau B)^2)Rf_1\|_{E'_\alpha}.$$

Using estimates (4.4) , (4.5) ,

$$\|B^2(\lambda+B)^{-1}R^{N+1-i}(\tau B)\|_{E\to E} \le M\min\left\{\frac{1}{t_{N-i}}, \frac{1}{\lambda t_{N-i}^2}\right\},$$

and the definition of fractional spaces $E'_\alpha(B,E)$ and normed space $C_\tau^\alpha(E)$, we get

$$\| \lambda^\alpha B(\lambda+B)^{-1}H^2 \|_E$$

$$\le \lambda^\alpha\left\|(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\right\|_{E\to E}\left\|\sum_{i=1}^{N-1} B^2(\lambda+B)^{-1}R^{N-i+1}\left(f_i - f_{N-1}\right)\tau\right\|$$

$$\le M_1\lambda^\alpha\sum_{i=1}^{N-1}(t_N-t_i)^\alpha\tau\min\left\{\frac{1}{t_{N-i}}, \frac{1}{\lambda t_{N-i}^2}\right\}\| f^\tau \|_{C_\tau^\alpha(E)}$$

$$\le M_2\lambda^\alpha\left[\sum_{i=1}^{N-1}\frac{\tau}{t_i^{1-\alpha}(1+\lambda t_i)}\right]\| f^\tau \|_{C_\tau^\alpha(E)}$$

for all λ, $\lambda > 0$. The sum enclosed in the right-hand side square brackets is the lower Darboux integral sum for the integral

$$\int_0^{t_N} \frac{ds}{s^{1-\alpha}(1+s\lambda)}.$$

Since

$$\lambda^\alpha \int_0^{t_N} \frac{ds}{s^{1-\alpha}(1+s\lambda)} \leq \int_0^{\infty} \frac{dp}{p^{1-\alpha}(1+p)},$$

it follows that

$$\| \lambda^\alpha B(\lambda + B)^{-1} H^2 \|_E \leq \frac{M_3}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau^\alpha(E)}$$

for all λ, $\lambda > 0$. From that it follows

$$\|H^2\|_{E'_\alpha} \leq \frac{M_3}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau^\alpha(E)}.$$

In a similar manner we can show that

$$\|H^3\|_{E'_\alpha} \leq \frac{M_3}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau^\alpha(E)}.$$

Now, we will estimate $\|H^2\|_{E'_\alpha}$.Using estimates (4.4) , (4.5) ,we get

$$\| \lambda^\alpha B(\lambda + B)^{-1} H^4 \|_E$$

$$\leq \left\| \lambda^\alpha B(\lambda + B)^{-1} (2I + \tau B)^{-1} \left(I - R^N\right)^{-1} R^{N-1} (I - R) \left(I - R^2\right) f_1 \right\|_E$$

$$\leq \lambda^\alpha \left\| (\lambda + B)^{-1} (2I + \tau B)^{-1} (I + \tau B) \left(I - R^N\right)^{-1} (I - R) \left(I - R^2\right) f_1 \right\|_{E \to E} \left\| BR^N f_1 \right\|_E$$

for all λ, $\lambda > 0$. From that it follows

$$\|H^4\|_{E'_\alpha} \leq M_2 \|f_1\|_E .$$

In a similar manner we can show that

$$\|H^5\|_{E'_\alpha} \leq M_3 \| f_{N-1} \|_E .$$

Finally, applying the triangle estimate and estimates for H^m for any $m = 1,2,3,4,5$ in E'_α, we get

$$\| (I-R)^2\tau^{-2}u_0 - f_1 \|_{E'_\alpha}$$

$$\leq \frac{M_4}{\alpha(1-\alpha)} \| f^\tau \|_{C^\alpha_\tau(E)} + M\|A\varphi + Rf_{N-1} - (I+3\tau B+(\tau B)^2)Rf_1\|_{E'_\alpha}.$$

In a similar manner we can show that

$$\left\|(I-R)^2\,\tau^{-2}u_N - f_{N-1}\right\|_{E'_\alpha}$$

$$\leq \frac{M_4}{\alpha(1-\alpha)} \| f^\tau \|_{C^\alpha_\tau(E)} + M\|A\varphi - Rf_1 + (I+3\tau B+(\tau B)^2)Rf_{N-1}\|_{E'_\alpha}.$$

Therefore, Theorem 4.3.3 is proved. □

Theorem 4.3.4. *Let A be a strongly positive operator in a Banach space E and $\psi = A^{-1}\sum\limits_{i=1}^{N-1} f_i\tau$. Then the solutions of the difference problem (4.1) in $C_\tau(E'_\alpha)$ obey the coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_\tau(E'_\alpha)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E'_\alpha)}$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau(E'_\alpha)} + \|A\varphi\|_{E'_\alpha}\right],$$

where M_5 is independent not only of f^τ, φ, α, but also of τ.

Proof. By Theorem 4.2.6 we have the following estimate

$$\| \{Au_k\}_1^{N-1} \|_{C_\tau(E'_\alpha)} \leq M\left[\frac{1}{\alpha(1-\alpha)} \| f^\tau \|_{C_\tau(E'_\alpha)} + \|Au_0\|_{E'_\alpha} + \|Au_N\|_{E'_\alpha}\right],$$

for solution of problem (4.2). Therefore, to prove the theorem it suffices to establish the estimate $\|Au_0\|_{E'_\alpha}$, $\|Au_N\|_{E'_\alpha}$. Applying formula (4.9), the triangle inequality and estimates (4.4), (4.5), we get

$$Au_0 = (I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)A\varphi$$
$$+ (2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1}(I-R)\,R^{N-i-1}f_i + \sum_{i=1}^{N-1}(I-R)\,R^{i-1}f_i\right\}.$$

Applying the triangle inequality, we get

$$\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}Au_0\right\|_E \leq \left\|(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)\right\|_{E\to E}\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}A\varphi\right\|_E$$

$$+\left\|(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\right\|_{E\to E}$$

$$\times\left\{\sum_{i=1}^{N-1}\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}(I-R)R^{N-i-1}f_i\right\|_E+\sum_{i=1}^{N-1}\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}(I-R)R^{i-1}f_i\right\|_E\right\}.$$

To estimate the last two sums we use the following Cauchy-Riesz representation formula for these two operators $\lambda^{\alpha}B(\lambda I + B)^{-1}(I-R(\tau B))R^{N-i-1}(\tau B)f_i$, $\lambda^{\alpha}B(\lambda I+B)^{-1}(I-R(\tau B))R^{i-1}(\tau B)f_i$.

$$B(\lambda I+B)^{-1}(I-R(\tau B))R^{N-i-1}(\tau B)f_i$$

$$=\int_{S_1\cup S_2}(\lambda+s)^{-1}(I-R(\tau s))R^{N-i-1}(\tau s)B(sI-B)^{-1}f_i ds$$

$$=\int_{S_1\cup S_2}(\lambda\tau+z)^{-1}(I-R(z))R^{N-i-1}(z)B\left(\frac{z}{\tau}I-B\right)^{-1}f_i dz$$

where $S_1=\left\{\rho e^{i\psi},0\leq\rho<\infty\right\}$ and $S_2=\left\{\rho e^{-i\psi},0\leq\rho<\infty\right\}$, $0\leq\psi<\frac{\pi}{2l}$. Since $z=\rho e^{\pm i\psi}$, with $|\psi|<\frac{\pi}{2l}$, from the strongly positivity of A it follows that

$$\left|\frac{z}{\tau}\right|^{\alpha}\left\|B\left(\frac{z}{\tau}I-B\right)^{-1}f_i\right\|_E\leq\left|\frac{\rho}{\tau}\right|^{\alpha}\left\|B\left(\frac{\rho}{\tau}I+B\right)^{-1}f_i\right\|_E.$$

From this estimate and the estimates

$$\frac{1}{|\lambda\tau+z|}\leq\frac{M_9}{\lambda\tau+\rho},$$

$$\left|(I-R(z))R^{N-i-1}(z)\right|=\rho\left(1+2\rho\cos\psi+\rho^2\right)^{\frac{-N+i}{2}}$$

it follows that

$$\sum_{i=1}^{N-1}\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}(I-R)R^{N-i-1}f_i\right\|_E$$

$$\leq M_{10}\int_0^{\infty}\sum_{i=1}^{N-1}\frac{\rho^{1-\alpha}}{(1+2\rho\cos\psi+\rho^2)^{\frac{N-i}{2}}}\frac{(\lambda\tau)^{\alpha}}{\lambda\tau+\rho}\left(\frac{\rho}{\tau}\right)^{\alpha}\left\|B\left(\frac{\rho}{\tau}I+B\right)^{-1}f_i\right\|_E d\rho$$

$$\leq M_{10}\int_0^{\infty}\frac{\left(\frac{\lambda\tau}{\rho}\right)^{\alpha}}{\lambda\tau+\rho}d\rho\max_{1\leq i\leq N-1}\|f_i\|_{E'_{\alpha}}=M_{11}\frac{1}{\alpha(1-\alpha)}\|f^{\tau}\|_{C_{\tau}(E'_{\alpha})}.$$

In a similar manner,

$$\sum_{i=1}^{N-1}\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}\left(I-R\right)R^{i-1}f_i\right\|_E\leq M_{12}\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}.$$

Hence

$$\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}Au_0\right\|_E\leq M_{13}\left\{\|A\varphi\|_{E'_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}\right\}.$$

Thus

$$\|\ Au_0\ \|_{E'_\alpha}\leq M_{13}\left\{\|A\varphi\|_{E'_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}\right\}.$$

From that and formula $u_0=u_N+\varphi$ it follows

$$\|\ Au_N\ \|_{E'_\alpha}\leq M_{13}\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}+(M_{13}+1)\,\|A\varphi\|_{E'_\alpha}\,.$$

Therefore, Theorem 4.3.4 is proved. □

4.4 APPLICATIONS

Finally, we consider the applications of Theorems 4.3.1-4.3.4 to the elliptic equations. First, we consider the nonlocal boundary value problem (3.23) for two dimensional elliptic equations.

The discretization of problem (3.23) is carried out in two steps. In the first step, let us define the grid sets

$$[0,1]_h=\{x_n=nh,0\leq n\leq M,Mh=1\}\,.$$

We introduce the Banach spaces $C_h=C\,[0,1]_h$ and $C_h^{\alpha}=C^{\alpha}\,[0,1]_h\,,0\leq\alpha\leq 1$ of the grid functions $\varphi^h(x)=\{\varphi_n\}_{n=0}^{M}$ defined on $[0,1]_h$ equipped with the norms

$$\left\|\varphi^h\right\|_{C_h}=\max_{x\in[0,1]_h}\left|\varphi^h(x)\right|,$$

$$\left\|\varphi^h\right\|_{C_h^{\alpha}}=\left\|\varphi^h\right\|_{C_h}+\sup_{0\leq n<n+r\leq M}\frac{|\varphi_{n+r}-\varphi_n|}{rh},$$

respectively. For the differential operator A defined by (3.23), we assign the difference operator A_h^x defined by the formula

$$A_h^x\varphi^h\left(x\right)=\left\{-a\left(x_n\right)\frac{\varphi_{n+1}-2\varphi_n+\varphi_{n-1}}{h^2}+\delta\varphi_n\right\}_{n=1}^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_{n=0}^{M}$ defined on $[0,1]_h$ satisfying the conditions $\varphi_0 = \varphi_M, \varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$.

With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u^h(y,x)}{dy^2} + A_h^x u^h(y,x) = f^h(y,x), \\ x \in [0,1]_h\,, 0 < y < T, \\ u^h(0,x) = u^h(T,x) + \varphi^h(x), \int\limits_0^T u^h(s,x)ds = 0,\ x \in [0,1]_h\,. \end{cases} \tag{4.10}$$

In the second step we replace problem (4.10) by the difference scheme

$$-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k, x), y_k = k\tau,$$

$$1 \le k \le N-1, N\tau = T, u_0^h(x) = u_N^h(x) + \varphi^h(x),\ \sum_{i=1}^{N} u_i^h(x) = 0,\ x \in [0,1]_h\,. \tag{4.11}$$

Theorem 4.4.1. *Assume that* $\sum\limits_{i=1}^{N} f_i^h(x) = 0,\ x \in [0,1]_h$ *. Let τ and h be sufficiently small numbers. Then, the solutions of difference scheme (4.11) satisfy the following estimates*

$$\max_{0\le k\le N} \left\|u_k^h\right\|_{C_h} \le M_1 \left[\max_{1\le k\le N-1} \left\|f_k^h\right\|_{C_h} + \|\varphi^h\|_{C_h}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau(C_h)} + \|\{A_h^x u_k^h\}_1^{N-1}\|_{C_\tau(C_h)}$$

$$\le M_1 \left[\ln\frac{1}{\tau+h} \|\{f_k^h\}_1^{N-1}\|_{C_\tau(C_h)} + \|\varphi^h\|_{C_h^2}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau^\alpha(C_h)} + \|\{D_h^2 u_k^h\}_1^{N-1}\|_{C_\tau^\alpha(C_h)}$$

$$\le M_2(\alpha)\|\{f_k^h\}_1^{N-1}\|_{C_\tau^\alpha(C_h)} + M_1\|\varphi^h\|_{C_h^{2+2\alpha}}, 0 < \alpha < \frac{1}{2},$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau(C_h^{2\alpha})} + \|\{A_h^x u_k^h\}_1^{N-1}\|_{C_\tau\left(C_h^{2\alpha}\right)}$$

$$\le M_2(\alpha)\|\{f_k^h\}_1^{N-1}\|_{C_\tau\left(C_h^{2\alpha}\right)} + M_1\|\varphi^h\|_{C_h^{2+2\alpha}}, 0 < \alpha < \frac{1}{2}.$$

Here $M_1, M_2(\alpha)$ do not depend on τ, h and $f_k^h, 1 \le k \le N-1$ and φ^h.Here, the difference operator D_h^2 defined by the formula

$$D_h^2\varphi^h(x) = \left\{-\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{h^2} + \delta\varphi_n\right\}_{n=1}^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_{n=0}^{M}$ *defined on* $[0,1]_h$ *satisfying the conditions* $\varphi_0 = \varphi_M, \varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$.

Proof. It is known that A_h^x is a stronly positive operator in C_h . Therefore, we can replace difference scheme (4.11) by difference problem (4.2). Then, the proof of Theorem 4.4.1 is based on the abstract Theorems 4.3.1- 4.3.4, the positivity of the operator A_h^x in C_h, the structure of the fractional spaces $E_\alpha^{'}((A_h^x)^{\frac{1}{2}}, C_h)$ and the following estimate see (Ashyralyev & Sobolevskii, 2004; Ashyralyev, 2015)

$$\min\left\{\ln\frac{1}{\tau}, \left|\ln\left\|A_h^x\right\|_{C_h\to C_h}\right|\right\} \le M_2 \ln\frac{1}{\tau+h}.$$

□

Second, we consider the boundary value problem (3.24) on the range $\{0 \le y \le T,\ x \in \mathbb{R}^n\}$ for $2m-$order multidimensional elliptic equations.

The discretization of problem (3.24) is carried out in two steps. Let us define the grid space $R_h^n\,(0 < h \le h_0)$ as the set of all points of the Euclidean space R^n whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm1, \pm2, \cdots,\ k = 1, \cdots, n.$$

We introduce the Banach spaces $C_h = C\left(R_h^n\right)$ and $C_h^\alpha = C^\alpha\left(R_h^n\right), 0 \le \alpha \le 1$ of the grid functions $\varphi^h(x)$ defined on R_h^n equipped with the norms

$$\left\|\varphi^h\right\|_{C_h} = \max_{x\in R_h^n}\left|\varphi^h(x)\right|,$$

$$\left\|\varphi^h\right\|_{C_h^\alpha} = \left\|\varphi^h\right\|_{C_h} + \sup_{x,y\in R_h^n, x\neq y}\frac{\left|\varphi^h(x)-\varphi^h(y)\right|}{|x-y|^\alpha},$$

respectively. To the differential operator A let us give the difference operator A_h^x by the formula

$$A_h^x u_x^h = \sum_{2m\le|r|\le S} b_r^x D_h^r u_x^h + \delta u_x^h.$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator (Ashyralyev & Sobolevskii, 2004)

$$\sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}...\partial x_n^{r_n}} + \delta.$$

We shall assume that for $|\xi_k h| \leq \pi$ the symbol $A(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M_1 |\xi|^{2m}, |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2l}.$$

With the help of A_h^x we arrive at the boundary value problem

$$-\frac{d^2 v^h(y,x)}{dy^2} + A_h^x v^h(y,x) = f^h(y,x), 0 < y < T, \tag{4.12}$$

$$v^h(0,x) = v^h(T,x) + \varphi^h(x), \int\limits_0^T v^h(s,x)ds = 0, x \in \mathbb{R}_h^n,$$

for an infinite system of ordinary differential equations. In the second step we replace problem (4.12) by the difference scheme

$$-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + \ A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k, x), y_k = k\tau,$$

$$1 \leq k \leq N-1, N\tau = T, u_0^h(x) = u_N^h(x) + \varphi^h(x), \ \sum_{i=1}^N u_i^h(x) = 0, \ x \in R_h^n. \tag{4.13}$$

Theorem 4.4.2. *Assume that* $\sum\limits_{i=1}^N f_i^h(x) = 0, f_1^h(x) = f_{N-1}^h(x) = 0, \ x \in R_h^n$. *Let* τ *and* h *be sufficiently small numbers. Then, the solutions of difference scheme (4.13) satisfy the following estimates*

$$\max_{0\leq k\leq N} \left\|u_k^h\right\|_{C_h} \leq M_1 \left[\max_{1\leq k\leq N-1} \left\|f_k^h\right\|_{C_h} + \|\varphi^h\|_{C_h}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau(C_h)} + \|\{A_h^x u_k^h\}_1^{N-1}\|_{C_\tau(C_h)}$$

$$\leq M_1 \left[\ln \frac{1}{\tau + h} \|\{f_k^h\}_1^{N-1}\|_{C_\tau(C_h)} + \ln \frac{1}{h} \|\varphi^h\|_{C_h^{2m}}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau^\alpha\left(C_h^\beta\right)} + \|\{u_k^h\}_1^{N-1}\|_{C_\tau^\alpha\left(C_h^{2m+\beta}\right)}$$

$$\leq M_2(\alpha) \left[\|\{f_k^h\}_1^{N-1}\|_{C_\tau^\alpha(C_h)} + \|\varphi^h\|_{C_h^{2m+2m\alpha}}\right], 0 < \alpha < \frac{1}{2m}, 0 < \beta < 1.$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau\left(C_h^{2m\alpha}\right)} + \|\{A_h^x u_k^h\}_1^{N-1}\|_{C_\tau\left(C_h^{2m\alpha}\right)}$$

$$\leq M_2(\alpha) \left[\|\{f_k^h\}_1^{N-1}\|_{C_\tau\left(C_h^{2m\alpha}\right)} + \|\varphi^h\|_{C_h^{2m+2m\alpha}}\right], 0 < \alpha < \frac{1}{2m}.$$

Here $M_1, M_2(\alpha)$ *do not depend on* τ, h *and* $f_k^h, 1 \leq k \leq N-1$ *and* φ^h.

Proof. It is known that A_h^x is a stronly positive operator in C_h . Therefore, we can replace difference scheme (4.13) by difference problem (4.2). Then, the proof of Theorem 4.4.2 is based on the abstract Theorems 4.3.1- 4.3.4, the positivity of the operator A_h^x in C_h, and on the almost coercivity inequality for an elliptic operator A_h^x in $C_h$and the following estimate

$$\min\left\{\ln\frac{1}{\tau}, \left|\ln\left\|A_h^x\right\|_{C_h\to C_h}\right|\right\} \le M_2 \ln\frac{1}{\tau+h},$$

and the structure of the fractional spaces $E_\alpha^{'}((A_h^x)^{\frac{1}{2}}, C_h)$ and on the coercivity inequality for an elliptic operator A_h^x in C_h^β, see (Ashyralyev & Sobolevskii, 2004; Triebel, 1978). □

Third, let Ω be the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $\quad (0 < x_k < 1, 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T]\times\Omega$ we consider the nonlocal boundary value problem (3.25) for the multidimensional elliptic equation.

The discretization of problem (3.25) is also carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \left\{x = x_j = (h_1 j_1, ..., h_m j_m),\ j = (j_1, ..., j_m),\right.$$

$$\left. 0 \le j_r \le M_r,\ h_r M_r = 1,\ r = 1, ..., m, \right\},$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega,\ S_h = \overline{\Omega}_h \cap S.$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$, $C_h^\beta = C_{01}^\beta(\overline{\Omega}_h)$ $\quad (\beta = (\beta_1, \ldots, \beta_n), 0 < x_k < 1, k = 1, \ldots, n)$ and $C_h = C(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 j_1, ..., h_m j_m)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2},$$

and

$$\| \varphi^h \|_{C_{01}^\beta(\overline{\Omega}_h)} = \| \varphi^h \|_{C(\overline{\Omega}_h)}$$

$$+ \sup_{0\le x_k < x_k + h_k \le 1, 1\le k\le n} |\varphi^h(x_1, \ldots, x_n) - \varphi^h(x_1 + h_1, \ldots, x_n + h_n)| \prod_{k=1}^n h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k},$$

$$\left\|\varphi^h\right\|_{C_h} = \sup_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|,$$

respectively. To the differential operator A generated by problem (3.25), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{m}\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ ($\forall\ x \in S_h$). It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ and $C(\overline{\Omega}_h)$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u^h(y,x)}{dy^2} + A_h^x u^h(t,x) = f^h(t,x), \\ x \in \Omega_h, , 0 < y < T, \\ u^h(0,x) = u^h(T,x), \int\limits_0^T u^h(s,x)ds = 0,\ x \in \overline{\Omega}_h, \\ u^h(y,x) = 0,\ x \in S_h. \end{cases} \tag{4.14}$$

In the second step, we replace problem (4.14) by second order of accuracy difference scheme (4.1)

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k,x), \\ y_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ u_0^h(x) = u_N^h(x), \sum\limits_{i=1}^{N} u_i^h(x)\tau = 0, x \in \overline{\Omega}_h. \end{cases} \tag{4.15}$$

Theorem 4.4.3. *Assume that* $\sum\limits_{i=1}^{N} f_i^h(x) = 0, f_1^h(x) = f_{N-1}^h(x) = 0,\ x \in \overline{\Omega}_h$.*Let* τ *and* $|h|$ *be sufficiently small numbers. Then, the solutions of difference scheme (4.15) satisfy the following estimates*

$$\|\{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1}\|_{C_\tau^\alpha(L_{2h})} + \|\{u_k^h\}_1^{N-1}\|_{C_\tau^\alpha(W_{2h})}$$

$$\le M_2(\alpha)\|\{f_k^h\}_1^{N-1}\|_{C_\tau^\alpha(L_{2h})},$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau^\alpha\left(C_h^\beta\right)} + \|\{u_k^h\}_1^{N-1}\|_{C_\tau^\alpha\left(C_h^{2+\beta}\right)}$$
$$\leq M_3(\alpha,\beta)\|\{f_k^h\}_1^{N-1}\|_{C_\tau^\alpha\left(C_h^\beta\right)},$$
$$0 < \alpha < 1, \beta = \{\beta_1, \cdots, \beta_n\}, 0 \leq \beta_k < 1, 1 \leq k \leq n.$$

Here $M_2(\alpha), M_3(\alpha,\beta)$ do not depend on τ, h and $f_k^h, 1 \leq k \leq N-1$ and φ^h.

Proof. It is known that A_h^x is a stronly positive operator in C_h and L_{2h}. Therefore, we can replace difference scheme (4.13) by difference problem (4.2). Then, the proof of Theorem 4.4.3 is based on the abstract Theorem 4.4.1, the positivity of the operator A_h^x in C_h and L_h and on the coercivity inequality for an elliptic operator A_h^x in L_{2h} and C_h^β (Ashyralyev & Sobolevskii, 2004) □

4.5 THE ILLUSTRATIVE NUMERICAL RESULT

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. Now, we give results for two and three dimensional NBVP. These numerical results are carried out by using MATLAB program.

4.5.1 Two Dimensional Case

For the approximate solutions of nonlocal boundary problem for two dimensional elliptic equation, the second order of accuracy difference schemes will be used, a procedure of modified Gauss elimination method to solve the problem will be applied, and finally, the error analysis of second order of accuracy difference schemes will be given in present section.

First, we consider the nonlocal boundary problem for two dimensional elliptic equation with Dirichlet boundary condition

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 3\cos t \sin x, 0 < t < 2\pi, 0 < x < 2\pi, \\ u(0,x) = u(2\pi, x), \int_0^{2\pi} u(s,x)\, ds = 0, 0 \leq x \leq 2\pi, \\ u(t,0) = u(t,2\pi) = 0,\ 0 \leq t \leq 2\pi. \end{cases} \tag{4.16}$$

The exact solution of this problem is

$$u(t,x) = \cos t \sin x.$$

For the approximate solution of the nonlocal boundary problem (4.16), we consider the set $[0,2\pi]_\tau \times [0,2\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,2\pi]_\tau \times [0,2\pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \le k \le N, N\tau = 2\pi, \quad x_n = nh, 0 \le n \le M, Mh = 2\pi\}.$$

For the numerical solution, we consider the second order of approximation difference scheme

$$\begin{cases} -\dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k = 3\cos t_k \sin x_n, \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ u_n^0 = u_n^N,\ \sum_{i=0}^{N-1} u_n^i = 0,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N. \end{cases} \tag{4.17}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, 1 \le n \le M-1, \\ u_0 = u_M = 0. \end{cases} \tag{4.18}$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, B = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ c & b & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & b & c & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & c & b & c & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & b & c \\ 0 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where $a = \frac{-1}{h^2},\ b = \frac{2}{\tau^2} + \frac{2}{h^2} + 1,\ c = \frac{-1}{\tau^2}$,

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ . \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} = \begin{bmatrix} 0 \\ 3\cos t_1 \sin x_n \\ . \\ 3\cos t_{N-1} \sin x_n \\ 0 \end{bmatrix}_{(N+1)\times 1},$$

$D = I_{N+1}$ is the identity matrix and

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ . \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1},\ s = n-1,\ n,\ n+1.$$

Therefore, for the solution of the matrix equation (4.18), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, 0$$

where $u_M = 0, \alpha_j \quad (j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices, $\beta_j \ (j = 1, ..., M-1)$ are $(N+1)\times 1$ column matrices, α_1, β_1 are zero matrices and

$$\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A, \beta_{n+1} = \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1, ..., M-1.$$

Now, we give the error analysis between exact solutions $u(t_k, x_n)$ and the approximate solutions u_n^k for the different values of N and M. The errors are computed by the formula

$$E_M^N = \max_{0\le k\le N, 0\le n\le M} \left| u(t_k, x_n) - u_n^k \right|$$

The results are given in the following table that is constructed for $N = M = 20, 40$ and 80.

Table 1: Error analysis for difference scheme (4.17)

	$\mathbf{N = M = 20, 20}$	$\mathbf{N = M = 40, 40}$	$\mathbf{N = M = 80, 80}$
Error	0.0055	0.0014	0.00034274

As it is seen in Table 1 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately$1/4$ for second order difference scheme.

Second, we consider the nonlocal boundary problem for the elliptic equation with Neumann boundary condition

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 3\cos t \cos x, \quad 0 < t < 2\pi,\ 0 < x < 2\pi, \\ \\ u(0,x) = u(2\pi, x), \quad \int_0^{2\pi} u(s,x)\,ds = 0,\ 0 \le x \le 2\pi, \\ \\ u_x(t,0) = u_x(t,2\pi) = 0, \qquad 0 \le x \le 2\pi. \end{cases} \tag{4.19}$$

The exact solution of this problem is

$$u(t,x) = \cos t \cos x.$$

For the approximate solution of the nonlocal boundary problem (4.16), we consider the set $[0,2\pi]_\tau \times [0,2\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,2\pi]_\tau \times [0,2\pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \le k \le N, N\tau = 2\pi, \quad x_n = nh, 0 \le n \le M, Mh = 2\pi\}.$$

For the numerical solution, we consider the difference scheme of the second order of accuracy in t and first order of accuracy in x.

$$\begin{cases} -\dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k = 3\cos t_k \cos x_n, \\ \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ \\ u_n^0 = u_n^N,\ \sum_{i=0}^{N-1} u_n^i = 0,\ 0 \le n \le M, \\ \\ u_1^k - u_0^k = u_M^k - u_{M-1}^k = 0,\ 0 \le k \le N. \end{cases} \tag{4.20}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n,\ 1 \le n \le M-1, \\ \\ u_0 = u_1, u_{M-1} = u_M. \end{cases} \tag{4.21}$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, B = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ c & b & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & b & c & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & c & b & c & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & b & c \\ 0 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

where $a = -\frac{1}{h^2},\ b = \frac{2}{\tau^2} + \frac{2}{h^2} + 1,\ c = -\frac{1}{\tau^2}$,

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ . \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} = \begin{bmatrix} 0 \\ 3\cos t_1 \cos x_n \\ . \\ 3\cos t_{N-1} \cos x_n \\ 0 \end{bmatrix}_{(N+1)\times 1},$$

and $D = I_{N+1}$ is the identity matrix,

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ . \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1,\ n,\ n+1.$$

Therefore, for the solution of the matrix equation (4.18), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, 0, \tag{4.22}$$

where $u_M = (I - \alpha_M)^{-1} \beta_M$, $\alpha_j \quad (j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices, β_j $(j = 1, ..., M-1)$ are $(N+1)\times 1$ column matrices, α_1 is the identity and β_1 are zero matrices and

$$\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A,$$

$$\beta_{n+1} = \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1, ..., M-1.$$

Now, we give the error analysis between exact solutions $u(t_k, x_n)$ and the approximate solutions u_n^k for the different values of N and M. The errors are computed by the formula

$$E_M^N = \max_{0\le k\le N, 0\le n\le M} \left|u(t_k, x_n) - u_n^k\right| \tag{4.23}$$

The numerical results for the difference scheme (4.20) are given in the following tables.

Table 2: Error analysis for difference scheme (4.19)

	$\mathbf{N = M = 20, 20}$	$\mathbf{N = M = 40, 40}$	$\mathbf{N = M = 80, 80}$
Error	0.1329	0.0607	0.0290

Table 3: Error analysis for difference scheme (4.19)

	$\mathbf{N, M = 20, 400}$	$\mathbf{N, M = 40, 1600}$	$\mathbf{N, M = 80, 6400}$
Error	0.0029	$7.1859e-04$	$1.7955e-04$

As it is seen in Table 2, if N and M are doubled, the value of errors decrease by a factor of approximately $1/2$. Moreover, as it is seen in Table 3, if N is doubled and $M \ge N^2$, the value of errors decrease by a factor of approximately $1/4$ difference scheme as the second order of accuracy.

4.5.2 Three Dimensional Case

For the approximate solutions of nonlocal boundary problem for three dimensional elliptic equation, the second order of accuracy difference schemes will be used, a procedure of modified Gauss elimination method to solve the problem will be applied, and the error analysis of second order of accuracy difference schemes will be given in present section.

For numerical analysis, we consider the nonlocal boundary problem for three dimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 4\cos t \sin x \cos y, 0 < t < 2\pi, 0 < x, y < 2\pi, \\ u(0, x, y) = u(2\pi, x, y), \int_0^{2\pi} u(s, x, y)\, ds = 0, 0 \le x \le 2\pi, 0 \le y \le 2\pi, \\ u(t, 0, y) = u(t, 2\pi, y) = 0, \ 0 \le t \le 2\pi, \ 0 \le y \le 2\pi, \\ u(t, x, 0) = u(t, x, 2\pi), \int_0^{2\pi} u(t, x, s)\, ds = 0, \le t \le 2\pi, 0 \le x \le 2\pi. \end{cases}$$

The exact solution of this problem is

$$u(t, x) = \cos t \sin x \cos y.$$

For the approximate solution of the nonlocal boundary problem (4.16), we consider the set $[0, 2\pi]_\tau \times [0, 2\pi]_h \times [0, 2\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0, 2\pi]_\tau \times [0, 2\pi]_h \times [0, 2\pi]_h = \{(t_k, x_n, y_m) : t_k = k\tau, 0 \le k \le N, N\tau = 2\pi,$$

$$x_n = nh, 0 \le n \le M, Mh = 2\pi, y_m = mh, 0 \le m \le M, Mh = 2\pi\}.$$

For the numerical solution, we consider the second order of approximation difference scheme.

$$\begin{cases} -\frac{u_{n,m}^{k+1} - 2u_{n,m}^{k} + u_{n,m}^{k-1}}{\tau^2} - \frac{u_{n+1,m}^{k} - 2u_{n,m}^{k} + u_{n-1,m}^{k}}{h^2} - \frac{u_{n,m+1}^{k} - 2u_{n,m}^{k} + u_{n,m-1}^{k}}{h^2} + u_{n,m}^{k} \\ = 4\cos t_k \sin x_n \cos y_m, 1 \le k \le N-1, 1 \le n, m \le M-1, \\ u_{n,m}^0 = u_{n,m}^N, \ \sum_{i=0}^{N-1} u_{n,m}^i = 0, \ 0 \le n, m \le M, \\ u_{0,m}^k = u_{M,m}^k = 0, \ 0 \le k \le N, 0 \le m \le M, \\ u_{n,0}^k = u_{n,M}^k, \ \sum_{i=0}^{M-1} u_{n,i}^k = 0, 0 \le k \le N, \ 0 \le n \le M. \end{cases} \tag{4.24}$$

It is the system of algebraic equations and it can be written in the matrix form

$$\begin{cases} AU_{n+1} + BU_n + CU_{n-1} = D\Phi_n, 1 \le n \le M-1, \\ \\ U_0 = U_M = 0. \end{cases} \tag{4.25}$$

Here,

$$A = C = \begin{bmatrix} O & O & O & O & . & O & O & O & O \\ O & A_1 & O & O & . & O & O & O & O \\ O & O & A_1 & O & . & O & O & O & O \\ O & O & O & A_1 & . & O & O & O & O \\ . & . & . & . & . & . & . & . & . \\ O & O & O & O & . & A_1 & O & O & O \\ O & O & O & O & . & O & A_1 & O & O \\ O & O & O & O & . & O & O & A_1 & O \\ O & O & O & O & . & O & O & O & O \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} I & O & O & O & . & O & O & O & -I \\ C_1 & B_1 & C_1 & O & . & O & O & O & O \\ O & C_1 & B_1 & C_1 & . & O & O & O & O \\ O & O & C_1 & B_1 & . & O & O & O & O \\ . & . & . & . & . & . & . & . & . \\ O & O & O & O & . & B_1 & C_1 & O & O \\ O & O & O & O & . & C_1 & B_1 & C_1 & O \\ O & O & O & O & . & O & C_1 & B_1 & C_1 \\ O & I & I & I & . & I & I & I & I \end{bmatrix}_{(M+1)\times(M+1)},$$

where $O = O_{(N+1)\times(N+1)}, I = I_{(N+1)\times(N+1)}$,

$$A_1 = C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ c & b & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & b & c & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & c & b & c & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & b & c \\ 0 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where $a = \frac{-1}{h^2},\ b = \frac{2}{\tau^2} + \frac{2}{h^2} + 1,\ c = \frac{-1}{\tau^2}$,

$$\Phi_{n,r} = \begin{bmatrix} \varphi^0_{n,r} \\ \varphi^1_{n,r} \\ . \\ \varphi^{N-1}_{n,r} \\ \varphi^N_{n,r} \end{bmatrix}_{(N+1)\times 1} = \begin{bmatrix} 0 \\ 4\cos t_1 \sin x_n \cos y_r \\ . \\ 4\cos t_{N-1} \sin x_n \cos y_r \\ 0 \end{bmatrix}_{(N+1)\times 1} \qquad 1 \le r \le M-1,$$

$$\Phi_n = \begin{bmatrix} \Phi_{n,0} \\ \Phi_{n,1} \\ \cdot \\ \Phi_{n,M-1} \\ \Phi_{n,M} \end{bmatrix}_{(M+1)\times 1} \qquad 1 \le n \le M-1, \Phi_{n,0} = \Phi_{n,M} = O_{(N+1)\times 1},$$

$$U_{s,r} = \begin{bmatrix} u^0_{s,r} \\ u^1_{s,r} \\ \cdot \\ u^{N-1}_{s,r} \\ u^N_{s,r} \end{bmatrix}_{(N+1)\times 1} \qquad 0 \le r \le M,\ s = n-1,\ n,\ n+1,$$

$D = I_{(N+1)(M+1)}$ is the identity matrix and

$$U_s = \begin{bmatrix} U_{s,0} \\ U_{s,1} \\ \cdot \\ U_{s,M-1} \\ U_{s,M} \end{bmatrix}_{(M+1)\times 1} \qquad ,\ s = n-1,\ n,\ n+1.$$

Therefore, for the solution of the matrix equation (4.25), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, 0,$$

where $U_M = 0$, $\alpha_j \ \ (j = 1, ..., M-1)$ are $(N+1)(M+1) \times (N+1)(M+1)$ square matrices, $\beta_j \ \ (j = 1, ..., M-1)$ are $(N+1)(M+1) \times 1$ column matrices, α_1, β_1 are zero matrices and

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1} A,$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), n = 1, ..., M-1.$$

Now, we give the error analysis between exact solutions $u(t_k, x_n, y_m)$ and the approximate solutions $u^k_{n,m}$ for the different values of N and M. The errors are computed by the formula

$$E^N_M = \max_{0 \le k \le N, 0 \le n,m \le M} \left| u(t_k, x_n, , y_m) - u^k_{n,m} \right|$$

The results are given in the following table that is constructed for $N = M = 20, 40$ and 80.

Table 4: Error analysis for difference scheme of the problem (4.24)

	N = M = 20, 20	**N = M = 40, 40**	**N = M = 80, 80**
Error	0.0062	0.0015	0.00038560

As it is seen in Table 4 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately$1/4$ for second order difference scheme.

Second, we consider the nonlocal boundary problem for three dimensional elliptic equation with Neumann boundary condition

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 4\cos t \cos x \cos y, 0 < t < 2\pi, 0 < x, y < 2\pi, \\ u(0, x, y) = u(2\pi, x, y), \int_0^{2\pi} u(s, x, y)\, ds = 0, 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi, \\ u_x(t, 0, y) = u_x(t, 2\pi, y) = 0,\ 0 \leq t \leq 2\pi,\ 0 \leq y \leq 2\pi, \\ u(t, x, 0) = u(t, x, 2\pi), \int_0^{2\pi} u(t, x, s)\, ds = 0, \leq t \leq 2\pi, 0 \leq x \leq 2\pi. \end{cases} \tag{4.26}$$

The exact solution of this problem is

$$u(t, x) = \cos t \cos x \cos y.$$

For the approximate solution of the nonlocal boundary problem (4.26), we consider the set $[0, 2\pi]_\tau \times [0, 2\pi]_h \times [0, 2\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0, 2\pi]_\tau \times [0, 2\pi]_h \times [0, 2\pi]_h = \{(t_k, x_n, y_m) : t_k = k\tau, 0 \leq k \leq N, N\tau = 2\pi,$$

$$x_n = nh, 0 \leq n \leq M, Mh = 2\pi, y_m = mh, 0 \leq m \leq M, Mh = 2\pi\}.$$

For the numerical solution, we consider the second order of approximation difference scheme.

$$
\begin{cases}
-\dfrac{u_{n,m}^{k+1}-2u_{n,m}^{k}+u_{n,m}^{k-1}}{\tau^2}-\dfrac{u_{n+1,m}^{k}-2u_{n,m}^{k}+u_{n-1,m}^{k}}{h^2}-\dfrac{u_{n,m+1}^{k}-2u_{n,m}^{k}+u_{n,m-1}^{k}}{h^2}+u_{n,m}^{k} \\
\qquad = 4\cos t_k \cos x_n \cos y_m, 1 \le k \le N-1, 1 \le n,m \le M-1, \\
u_{n,m}^{0}=u_{n,m}^{N}, \ \sum_{i=0}^{N-1} u_{n,m}^{i}=0, \ 0 \le n,m \le M, \\
u_{1,m}^{k}-u_{0,m}^{k}=u_{M,m}^{k}-u_{M-1,m}^{k}=0, \ 0 \le k \le N, 0 \le m \le M, \\
u_{n,0}^{k}=u_{n,M}^{k}, \ \sum_{i=0}^{M-1} u_{n,i}^{k}=0, 0 \le k \le N, \ 0 \le n \le M.
\end{cases}
\tag{4.27}
$$

It is the system of algebraic equations and it can be written in the matrix form

$$
\begin{cases}
AU_{n+1}+BU_n+CU_{n-1}=D\Phi_n, 1 \le n \le M-1, \\
u_0=u_1, u_{M-1}=u_M. \\
U_0=U_1, U_{M-1}=U_M.
\end{cases}
\tag{4.28}
$$

Here,

$$
A=C=\begin{bmatrix}
O & O & O & O & . & O & O & O & O \\
O & A_1 & O & O & . & O & O & O & O \\
O & O & A_1 & O & . & O & O & O & O \\
O & O & O & A_1 & . & O & O & O & O \\
. & . & . & . & . & . & . & . & . \\
O & O & O & O & . & A_1 & O & O & O \\
O & O & O & O & . & O & A_1 & O & O \\
O & O & O & O & . & O & O & A_1 & O \\
O & O & O & O & . & O & O & O & O
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$B = \begin{bmatrix} I & O & O & O & . & O & O & O & -I \\ C_1 & B_1 & C_1 & O & . & O & O & O & O \\ O & C_1 & B_1 & C_1 & . & O & O & O & O \\ O & O & C_1 & B_1 & . & O & O & O & O \\ . & . & . & . & . & . & . & . & . \\ O & O & O & O & . & B_1 & C_1 & O & O \\ O & O & O & O & . & C_1 & B_1 & C_1 & O \\ O & O & O & O & . & O & C_1 & B_1 & C_1 \\ O & I & I & I & . & I & I & I & I \end{bmatrix}_{(M+1)\times(M+1)},$$

where $O = O_{(N+1)\times(N+1)}, I = I_{(N+1)\times(N+1)}$,

$$A_1 = C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ c & b & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & b & c & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & c & b & c & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & b & c \\ 0 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where $a = \frac{-1}{h^2},\ b = \frac{2}{\tau^2} + \frac{2}{h^2} + 1,\ c = \frac{-1}{\tau^2}$,

$$\Phi_{n,r} = \begin{bmatrix} \varphi_{n,r}^{0} \\ \varphi_{n,r}^{1} \\ \cdot \\ \varphi_{n,r}^{N-1} \\ \varphi_{n,r}^{N} \end{bmatrix}_{(N+1)\times 1} = \begin{bmatrix} 0 \\ 4\cos t_1 \cos x_n \cos y_r \\ \cdot \\ 4\cos t_{N-1} \cos x_n \cos y_r \\ 0 \end{bmatrix}_{(N+1)\times 1} \quad 1 \le r \le M-1,$$

$$\Phi_{n} = \begin{bmatrix} \Phi_{n,0} \\ \Phi_{n,1} \\ \cdot \\ \Phi_{n,M-1} \\ \Phi_{n,M} \end{bmatrix}_{(M+1)\times 1} \quad 1 \le n \le M-1, \Phi_{n,0} = \Phi_{n,M} = O_{(N+1)\times 1},$$

$$U_{s,r} = \begin{bmatrix} u_{s,r}^{0} \\ u_{s,r}^{1} \\ \cdot \\ u_{s,r}^{N-1} \\ u_{s,r}^{N} \end{bmatrix}_{(N+1)\times 1} \quad 0 \le r \le M,\ s = n-1,\ n,\ n+1,$$

$D = I_{(N+1)(M+1)}$ is the identity matrix and

$$U_{s} = \begin{bmatrix} U_{s,0} \\ U_{s,1} \\ \cdot \\ U_{s,M-1} \\ U_{s,M} \end{bmatrix}_{(M+1)\times 1} \quad ,\ s = n-1,\ n,\ n+1.$$

Therefore, for the solution of the matrix equation (4.28), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, 0,$$

where $U_M = 0$, $\alpha_j \quad (j = 1, ..., M-1)$ are $(N+1)(M+1) \times (N+1)(M+1)$ square matrices, $\beta_j \ (j = 1, ..., M-1)$ are $(N+1)(M+1) \times 1$ column matrices, α_1, β_1 are zero matrices and

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1} A,$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1} (D\varphi_n - C\beta_n), n = 1, ..., M - 1.$$

Now, we give the error analysis between exact solutions $u(t_k, x_n, y_m)$ and the approximate solutions $u^k_{n,m}$ for the different values of N and M. The errors are computed by the formula

$$E^N_M = \max_{0 \leq k \leq N, 0 \leq n,m \leq M} \left| u(t_k, x_n, , y_m) - u^k_{n,m} \right|$$

The results are given in the following table that is constructed for $N = M = 20, 40$ and 80.

Table 5: Error analysis for difference scheme of the problem (4.27)

	N = M = 20, 20	**N = M = 40, 40**
Error	0.1125	0.0504

As it is seen in Table 5 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately1/2 for first order difference scheme.

CHAPTER 5
CONCLUSIONS

The present thesis deals with strongly positive operators with nonlocal conditions and their applications. The following results are obtained:

- Fourier series, Laplace transform and Fourier transform methods are applied for the solution of several problems for elliptic differential equations with nonlocal boundary conditions.

- The theorem on the structure of fractional with powers of strongly positive operators in fractional spaces is established.

- Structure of fractional powers of elliptic operators is studied.

- The well-posedness of the abstract nonlocal boundary value problem for the elliptic equation in an arbitrary Banach space with positive operator in various Banach spaces is established.

- The theorems on coercive stability estimates for the solutions of three type elliptic differential nonlocal problems are proved.

- The second order of approximation two-step difference scheme is presented. The well-posedness of this difference scheme in various Banach spaces is established.

- The theorems on stability, almost coercive stability and coercive stability estimates for the solutions of difference schemes for the three type elliptic differential nonlocal problems are proved.

- Illustrative numerical results for two and three dimensional case are provided. The Matlab implementation of these difference schemes is presented.

- The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

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APPENDICES

APPENDIX A

MATLAB PROGRAMMING

In this part, Matlab programs are presented for the first and second orders of accuracy difference schemes.

1. Matlab Implementation of the Second Order of Accuracy Difference Scheme of Problem (4.16)

```
clear all; clc; close all; delete '*.asv';
N=40; M= N;
h=(2*pi)/M;
tau=(2*pi)/N;
c=-1/(tau^2);
a=-1/(h^2);
b=(2/(h^2))+(2/(tau^2))+1;
a1=-((h^2)/(2*tau^2));
b1=((h^2)/(tau^2))+((h^2)/2)+1;
for k=2:N;
a(k)=(-1/(h^2)); A(k,k)=a(k); A(N+1,N+1)=0;
end; A;
for k=2:N;
E(k,k)=b1; E(k,k-1)=a1; E(k,k+1)=a1;
E(N+1,k)=1; E(N+1,N+1)=0; E(1,1)=1;
E(1,N+1)=-1; E(N+1,1)=1;
B(k,k)=b; B(k,k-1)=c; B(k,k+1)=c;
B(N+1,k)=1; B(N+1,N+1)=0; B(1,1)=1;
B(1,N+1)=-1; B(N+1,1)=1;
end; B;
for k=2:N;
```

```
F(k,k)=1;
F(N+1,k)=0; F(N+1,N+1)=0; F(1,1)=0;
F(1,N+1)=-0; F(N+1,1)=0;
end
C=A; C;
for i=1:N+1;
D(i,i)=1;
end; D; D;
for j=1:M-1;
for k=2:N;
t=(k-1)*tau; x=(j)*h;
phy(k,j:j)=3*cos(t)*cos(x);
end;
end;
for j=1:M-1;
phy(1,j:j)=0; phy(N+1,j:j)=0;
end; phy;
for k=2:N;
t=(k-1)*tau;
phy0(k,1:1)=3*cos(t);
phy0(1,1:1)=0;
phy0(N+1,1:1)=0;
end;
I=eye(N+1,N+1);
alpha1=inv(E)*F;
betha1=inv(E)*((h^2)/2)*phy0(:,1:1);
for j=1:M-1;
alphaj+1=inv(B+C*alphaj)*(-A);
bethaj+1=inv(B+C*alphaj)*(I*phy(:,j:j)-C*bethaj);
```

```
end;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=cos(t)*cos(x);
end;
end;
UM=inv(((alphaM-1-4*I))*alphaM+3*I)*(-bethaM-1-((alphaM-1-4*I)*bethaM));
for Z=M-1:-1:1;
UZ=alphaZ+1*UZ+1+bethaZ+1;
end;
for Z=1:M;
p(:,Z+1)=UZ;
end;
p(:, 1)=UM;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=cos(t)*cos(x);
end;
end;
abs(es-p);
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)))
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror];
```

```
p; es;
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es; p]; table(1:2:end,:)=es; table(2:2:end,:)=p;
```

APPENDIX B

MATLAB PROGRAMMING

2. Matlab Implementation of the First Order of Accuracy Difference Scheme of Problem (4.19)

```
clear all; clc; close all; delete '*.asv';
N=80; M= 80;
h=(2*pi)/M;
tau=(2*pi)/N;
c=-1/(tau^2);
a=-1/(h^2);
b=(2/(h^2))+(2/(tau^2))+1;
for k=2:N;
a(k)=(-1/(h^2)); A(k,k)=a(k); A(N+1,N+1)=0;
end; A;
for k=2:N;
B(k,k)=b; B(k,k-1)=c; B(k,k+1)=c;
B(N+1,k)=1; B(N+1,N+1)=0; B(1,1)=1;
B(1,N+1)=-1; B(N+1,1)=1;
end; B;
C=A; C;
for i=1:N+1;
D(i,i)=1;
end; D; D;
for j=1:M-1;
for k=2:N;
t=(k-1)*tau; x=(j)*h;
phy(k,j:j)=3*cos(t)*sin(x);
```

```
end;
end;
for j=1:M-1;
phy(1,j:j)=0; phy(N+1,j:j)=0;
end; phy;
I=eye(N+1,N+1);
alpha1=zeros(N+1,N+1);
betha1=zeros(N+1,1);
for j=1:M-1;
alphaj+1=inv(B+C*alphaj)*(-A);
bethaj+1=inv(B+C*alphaj)*(I*phy(:,j:j)-C*bethaj);
end;
UM=zeros(N+1,1);
for Z=M-1:-1:1;
UZ=alphaZ+1*UZ+1+bethaZ+1;
end;
for Z=1:M;
p(:,Z+1)=UZ;
end;
p(:,1)=zeros(N+1,1);
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=cos(t)*sin(x);
end;
end;
es;
p;
```

```
abs(es-p);
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)))
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror];
p; es;
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es; p]; table(1:2:end,:)=es; table(2:2:end,:)=p;
```

APPENDIX C

MATLAB PROGRAMMING

In this chapter, Matlab programs are presented for the first and second orders of accuracy difference schemes.

3. Matlab Implementation of the Second Order of Accuracy Difference Scheme of Problem (4.19)

```
clear all; clc; close all; delete '*.asv';
N=40; M= N^2;
h=(2*pi)/M;
tau=(2*pi)/N;
c=-1/(tau^2);
a=-1/(h^2);
b=(2/(h^2))+(2/(tau^2))+1;
for k=2:N;
a(k)=(-1/(h^2)); A(k,k)=a(k); A(N+1,N+1)=0;
end; A;
for k=2:N;
B(k,k)=b; B(k,k-1)=c; B(k,k+1)=c;
B(N+1,k)=1; B(N+1,N+1)=0; B(1,1)=1;
B(1,N+1)=-1; B(N+1,1)=1;
end; B;
C=A; C;
for i=1:N+1;
D(i,i)=1;
end; D; D;
for j=1:M-1;
for k=2:N;
```

```
t=(k-1)*tau; x=(j)*h;
phy(k,j:j)=3*cos(t)*cos(x);
end;
end;
for j=1:M-1;
phy(1,j:j)=0; phy(N+1,j:j)=0;
end; phy;
I=eye(N+1,N+1);
alpha1=eye(N+1,N+1);
betha1=zeros(N+1,1);
for j=1:M-1;
alphaj+1=inv(B+C*alphaj)*(-A);
bethaj+1=inv(B+C*alphaj)*(I*phy(:,j:j)-C*bethaj);
end;
UM=inv(I-alphaM)*bethaM;
for Z=M-1:-1:1;
UZ=alphaZ+1*UZ+1+bethaZ+1;
end;
for Z=1:M;
p(:,Z+1)=UZ;
end;
p(:, 1)=U1;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=cos(t)*cos(x);
end;
end;
```

```
abs(es-p);
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)))
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror];
p; es;
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es; p]; table(1:2:end,:)=es; table(2:2:end,:)=p;
```

APPENDIX D

MATLAB PROGRAMMING

4. Matlab Implementation of the Second Order of Accuracy Difference Schemeof Problem (4.24)

```
clear all; clc; close all;delete '*.asv'
M=40; N=M;
h=(2*pi)/M;
tau=(2*pi)/N;
a=-1/(h^2);
b=(2/(h^2))+(2/(h^2))+(2/(tau^2))+1;
c=-1/(tau^2);
A=zeros((N+1)*(N+1),(N+1)*(N+1));
B=eye((N+1)*(N+1),(N+1)*(N+1));
for k=2:N;
A1(k,k)=a; A1(N+1,N+1)=0;
B1(k,k)=b; B1(k,k-1)=c; B1(k,k+1)=c;
B1(N+1,k)=1; B1(1,1)=1;
B1(1,N+1)=-1; B1(N+1,N+1)=1;
end;B1;
for j=N+2:N+1:N*(N+1);
A(j:j+N,j:j+N)=A1;
B(1:N+1,1:N+1)=eye(N+1,N+1);B(1:N+1,N*(N+1)+1:(N+1)*(N+1))=-eye(N+1,N+1);
B(j:j+N,j:j+N)=B1;
B(j:j+N,j-N-1:j-1)=A1;
B(j:j+N,j+N+1:j+2*N+1)=A1;
B(N*(N+1)+1:(N+1)*(N+1),j:j+N)=eye(N+1,N+1);
end
```

```
C=A;C;
NK=(N+1)*(M+1);
phy=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h;
for j=2:M;
i1=(N+1)*(j-1);
y=(j-1)*h;
for k=2:N;
t=(k-1)*tau;
phy(i1+k,n)=4*cos(t)*sin(x)*cos(y);
end;
end;
end;
R=eye(NK,NK);
alphaf1=zeros(NK,NK);
bethaf1=zeros(NK,1);
for j=2:M
alphafj=-inv(B+C*alphafj-1)*A;
bethafj=inv(B+C*alphafj-1)*(R*(phy(:,j))-C*bethafj-1);
end;
U=zeros(NK,M+1);
for j=M:-1:1;
U(:,j)=alphafj*U(:,j+1)+bethafj;
end
p=zeros(N+1,M+1,M+1);
for n=1:M+1;
x=(n-1)*h;
for j=1:M+1;
```

```
i1=(N+1)*(j-1);
y=(j-1)*h;
for k=1:N+1;
t=(k-1)*tau;
p(k,n,j)=U(i1+k,n);
es(k,n,j)=cos(t)*sin(x)*cos(y);
end;
end;
end;
maxerror=max(max(max(abs(es-p))))
```

APPENDIX E

MATLAB PROGRAMMING

5. Matlab Implementation of the Second Order of Accuracy Difference Schemeof Problem (4.27)

```
clear all; clc; close all;delete '*.asv'
M=20; N=M;
h=(2*pi)/M;
tau=(2*pi)/N;
a=-1/(h^2);
b=(2/(h^2))+(2/(h^2))+(2/(tau^2))+1;
c=-1/(tau^2);
A=zeros((N+1)*(N+1),(N+1)*(N+1));
B=eye((N+1)*(N+1),(N+1)*(N+1));
for k=2:N;
A1(k,k)=a; A1(N+1,N+1)=0;
B1(k,k)=b; B1(k,k-1)=c; B1(k,k+1)=c;
B1(N+1,k)=1; B1(1,1)=1;
B1(1,N+1)=-1; B1(N+1,N+1)=1;
end;B1;
for j=N+2:N+1:N*(N+1);
A(j:j+N,j:j+N)=A1;
B(1:N+1,1:N+1)=eye(N+1,N+1);B(1:N+1,N*(N+1)+1:(N+1)*(N+1))=-eye(N+1,N+1);
B(j:j+N,j:j+N)=B1;
B(j:j+N,j-N-1:j-1)=A1;
B(j:j+N,j+N+1:j+2*N+1)=A1;
B(N*(N+1)+1:(N+1)*(N+1),j:j+N)=eye(N+1,N+1);
end
```

```
C=A;C;
NK=(N+1)*(M+1);
phy=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h;
for j=2:M;
i1=(N+1)*(j-1);
y=(j-1)*h;
for k=2:N;
t=(k-1)*tau;
phy(i1+k,n)=4*cos(t)*cos(x)*cos(y);
end;
end;
end;
R=eye(NK,NK);
alphaf1=eye(NK,NK);
bethaf1=zeros(NK,1);
for j=2:M
alphafj=-inv(B+C*alphafj-1)*A;
bethafj=inv(B+C*alphafj-1)*(R*(phy(:,j))-C*bethafj-1);
end;
U(:,M+1)=inv(R-alphafM)*bethafM;
for j=M:-1:1;
U(:,j)=alphafj*U(:,j+1)+bethafj;
end
p=zeros(N+1,M+1,M+1);
for n=1:M+1;
x=(n-1)*h;
for j=1:M+1;
```

```
i1=(N+1)*(j-1);
y=(j-1)*h;
for k=1:N+1;
t=(k-1)*tau;
p(k,n,j)=U(i1+k,n);
es(k,n,j)=cos(t)*cos(x)*cos(y);
end;
end;
end;
maxerror=max(max(max(abs(es-p))))
```

Ödevler
Öğrenciler
Not Defteri
Kütüphaneler
Takvim
Tartışma
Tercihler

Bu sayfa hakkında

Bu sizin ödev kutunuzdur. Bir yazılı ödevi görüntülemek için yazılı ödevin başlığını seçin. Bir Benzerlik Raporunu görüntülemek için yazılı ödevin benzerlik sütunundaki Benzerlik Raporu ikonunu seçin. Tıklanabilir durumda olmayan bir ikon Benzerlik Raporunun henüz oluşturulmadığını gösterir.

Ayman

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

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Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
Ayman Hamad	ABSTRACT	%0 %0	0%	0%	0%	--	--	ödev indir	1070120114	29-Oca-2019
Ayman Hamad	CONCLUSION	%0 %0	0%	0%	0%	--	--	ödev indir	1070108484	29-Oca-2019
Ayman Hamad	CHAPTER 2	%5 %5	3%	3%	0%	--	--	ödev indir	1070103237	29-Oca-2019
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Ayman Hamad	FULL THESIS	%9 %9	8%	3%	0%	--	--	ödev indir	1070119895	29-Oca-2019
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Ayman Hamad	CHAPTER 4	%11 %11	3%	10%	1%	--	--	ödev indir	1070103987	29-Oca-2019

CURRICULUM VITAE

Personal information:

First name: Ayman Omar Ali **surname**: Hamad

Nationality: Libyan

Data and Place of birth: 29-02-1976

Marital Status: Married

E-mailaddress: ayman2952000@gmail.com

Education:

Degree	Institute	Year of Graduation
M.Sc.	University of Tripoli, Department of Mathematics	2008
B.Sc.	Omar Al-Mukhtar, Department of Mathematics	1999

WORK EXPERIENCE:

Years	Place	Enrollment
2008/2009	University of Tripoli	Assistant Lecture
2009/2010	University of Tripoli	Assistant Lecture
2010/2011	University of Tripoli	Assistant Lecture
2011/2012	University of Benghazi	Assistant Lecture
2012/2013	Omar Al-Mukhtar University	Assistant Lecture
2013/2014	Omar Al-Mukhtar University	Lecture

LANGUAGES:

- Arabic, speak and written.
- English, speak and written.

INTERNATIONAL PUBLICATIONS:

Ashyralyev, A., & Hamad, A. (2017, September). Fractional powers of strongly positive operators and their applications. In AIP Conference Proceedings (Vol. 1880, No. 1, p. 050001). AIP Publishing.

Ashyralyev, A., & Hamad, A. (2018, August). Numerical solution of nonlocal elliptic problems. In AIP Conference Proceedings (Vol. 1997, No. 1, p. 020081). AIP Publishing.

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COURSES GIVE:

- Calculus
- Linear algebra
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