THE SPACE DEPENDENT SOURCE IDENTIFICATION PARABOLIC-ELLIPTIC PROBLEM

**A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES
OF
NEAR EAST UNIVERSITY**

**By
ÇAĞIN ARIKAN**

**In partial Fulfillment of the Requirements for
the Degree of Master of Science
in
Mathematics**

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

Prof.Dr. Nadire ÇAVUŞ

We certify that, this thesis is satisfactory for the of the degree of Master of Science in Mathematics

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct, I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my family...

ABSTRACT

In the present study, a source identification problem for parabolic-elliptic equations is studied. Using tools of classical approximate, we are active to obtain the solution of any source identification problems for parabolic-elliptic equations. Moreover, numerical solutions of identification parabolic-elliptic equations are investigated. The first and second order of accuracy difference schemes are presented for the solution of the identification problem for a one-dimensional parabolic-elliptic equations and the numerical procedure for application of these schemes is discussed.

Keywords: Source identification problems; parabolic-elliptic differential equations; Fourier series method; Laplace transform and Fourier transform solutions; difference schemes; numerical experience

ÖZET

Bu çalışmada, parabolik-eliptik denklemleri için kaynak tanımlama problemi incelenmiştir. Klasik yaklaşım araçları, parabolik-eliptik denklemleri için çeşitli kaynak tanımlama problemlerinin çözümünü elde etmemize imkan tanır. Ayrıca , parabolik-eliptik problemlerin tanımlanmasında sayısal çözümler incelenmiştir. Bir boyutlu parabolik-eliptik denklemlerde tanımlama probleminin çözümü için birinci ve ikinci dereceden doğruluk farkı şemaları sunulmuş ve bu şemaların uygulanması için sayısal prosedür ele alınmıştır.

.***Anahtar Kelimeler:*** Kaynak tanımlama problemleri; parabolik-eliptik denklemleri; Fourier serisi yöntemi; Laplace dönüşümü ve Fourier dönüşümü çözümleri; fark şemaları; sayısal deneyim

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CHAPTER 1
INTRODUCTION

Partial differential equations are ubiquitous in the applied sciences where they ensure a mathematical description of phenomena in the physical, natural and social sciences. Scientists have tried to study the real life problems applying mathematical models which consist of partial differential equations. Lately, there have been many papers on inverse problems with applications (A. K. Urinov and S. T. Nishonova, 2017 ;M. Kohlmann, 2015; R. Denk, T. Seger, 2014; I. C. Kim, 2013; A. Ashyralyev, 2011; A. Ashyralyev and O.Gercek,2010; M. Stiemer,2010; A. Ashyralyev and P.E. Sobolevskii,2004).

One of the main classes of inverse problems is the source identification problem. Since many phenomena of the sciences and engineering are modeled by inverse problems, the apply of more applicable and true algorithms that use some given investigation at accessible parts of the domain of the problem to determine the unknown function has been becoming very important. The study of well-posedness of the problem plays a vital role in obtaining a numerical solution of the problem, as well (A. K. Urinov, S. T. Nishonova, 2017). As it is known, if a problem is ill-posed, it is quite possible to have some difficulties with using the numerical methods for the numerical solution of this problem.

The unknown source term of a parabolic equation may based on time, space or time and space as the unknown variables. Several methods have been recommended or studied for the reconstruction of the space-dependent source term. The traditional approach in solving problems of source identification approximately consists in degradation of the inverse problem to the Volterra integral equation of the first kind appyling the Green function (M. Kohlmann, 2015). Another approximation is to use numerical schemes such as finite difference method and finite elements method. Borukhov and Vabishchevich (I. C. Kim, 2013) presented a numerical algorithm depending the passing to the problem of the loaded parabolic equation and used a difference scheme to solve the non-classical problem. A numerical algorithm on the basis of the Landweber iteration is applied to take care of with the problem (R. Denk, T. Seger, 2014). Recovering the space-dependent source and the theoretical and numerical methods interrelated to it are investigated by several writers. In these works, finite difference method (A. Ashyralyev, 2011), the radial basis functions method (M. Stiemer, 2010), the boundary elements method, the combination of the boundary elements method and variational method are applied. For a general survey about inverse and ill-posed problems, we cite (A. Ashyralyev and P.E. Sobolevskii,2004).

In the present paper, we deal with an inverse problem for mixed partial differential equations. The problem of identifying the pair $(u(t), p)$ is stated as follows:

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = p + f(t), 0 < t < 1, \\ u(0+) = u(0-), u'(0+) = u'(0-), \\ \frac{du(t)}{dt} - Au(t) = -p + g(t), -1 < t < 0, \\ u(-1) = \varphi, u(1) = \psi \end{cases} \tag{1.1}$$

for the parabolic-elliptic differential equation in a Hilbert space H with self adjoint positive definite operator A.

Using tools of classical approach we are enabled to obtain the solution of the several source identification problems for parabolic-elliptic equations. Furthermore, the first and second order of accuracy difference schemes for the numerical solution of the one-dimensional parabolic-elliptic source identification problem are presented. Then, these difference schemes are tested on an example and some numerical results are presented.

CHAPTER 2

METHODS OF SOLUTION OF SPACE DEPENDENT SOURCE IDENTIFICATION PROBLEMS FOR PARABOLIC-ELLIPTIC EQUATIONS

It is known that identification problems for parabolic-elliptic differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

2.1 FOURIER SERIES METHOD

We consider Fourier series method for solution of identification problems for parabolic-elliptic differential equations.

Example 2.1.1. Obtain the Fourier series solution of the boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + (2t-1)\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x) + (-2t+2)\sin x, \\ -1 < t < 0, 0 < x < \pi, \\ \\ u(-1,x) = -\sin x, u(1,x) = \sin x, 0 \le x \le \pi, \\ \\ u(t,0) = u(t,\pi) = 0, -1 \le t \le 1 \end{cases} \tag{2.1}$$

for the parabolic-elliptic differential equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem for the differential operator A defined by formula

$$Au(x) = -\frac{d^2u(x)}{dx^2} + u(x)$$

with the domain

$$D(A) = \left\{ u : u(x), \frac{d^2u(x)}{dx^2} \in L_2[0,\pi], u(0) = u(\pi) = 0 \right\}.$$

Actually, we will obtain all $(u(x), \lambda)$ such that

$$-u''(x) + u(x) = -\lambda u(x), 0 < x < \pi, u(0) = u(\pi) = 0, u(x) \neq 0.$$

It is easy to see that the solution of this Sturm-Liouville problem is

$$(u_k(x), \lambda_k) = \left(\sin kx, -k^2 - 1\right), k = 1, 2, ...$$

Therefore, we will seek solution $u(t, x)$ of problem (2.1) using by the Fourier series

$$u(t, x) = \sum_{k=1}^{\infty} A_k(t) \sin kx. \tag{2.2}$$

and

$$p(x) = \sum_{k=1}^{\infty} p_k \sin kx. \tag{2.3}$$

Here $A_k(t), p_k, k = 1, 2, ...$ are unknown functions and parameters. Applying formula (2.2), given boundary conditions, we get

$$u(-1, x) = \sum_{k=1}^{\infty} A_k(-1) \sin kx = -\sin x,$$

$$u(1, x) = \sum_{k=1}^{\infty} A_k(1) \sin kx = \sin x.$$

From that it follows

$$A_1(-1) = -1, A_k(-1) = 0, k \neq 1 \tag{2.4}$$

and

$$A_1(1) = 1, A_k(1) = 0, k \neq 1, \tag{2.5}$$

respectively. Applying formulas (2.2), (2.3) and given equations, we obtain

$$-\sum_{k=1}^{\infty} A_k''(t) \sin kx + \sum_{k=1}^{\infty} A_k(t) k^2 \sin kx + \sum_{k=1}^{\infty} A_k(t) \sin kx \tag{2.6}$$
$$= \sum_{k=1}^{\infty} p_k \sin kx + (2t - 1) \sin x, 0 < t < 1,$$

$$\sum_{k=1}^{\infty} A_k'(t)\sin kx - \sum_{k=1}^{\infty} A_k(t)k^2 \sin kx - \sum_{k=1}^{\infty} A_k(t)\sin kx \tag{2.7}$$
$$= -\sum_{k=1}^{\infty} p_k \sin kx + (-2t+2)\sin x, -1 < t < 0.$$

Equating coefficients of $\sin kx$, $k = 1, 2, ...,$ to zero, we get

$$-A_1''(t) + 2A_1(t) = p_1 + (2t-1), k = 1, \tag{2.8}$$
$$-A_k''(t) + (k^2+1)A_k(t) = p_k, k \neq 1, 0 < t < 1,$$
$$A_1'(t) - 2A_1(t) = -p_1 + (-2t+2), k = 1, \tag{2.9}$$
$$A_k'(t) - (k^2+1)A_k(t) = -p_k, k \neq 1, -1 < t < 0.$$

First, we obtain $A_1(t)$. Applying (2.4), (2.5), (2.8) and (2.9), we get the following boundary value problem

$$\begin{cases} -A_1''(t) + 2A_1(t) = p_1 + (2t-1), 0 < t < 1, \\ A_1'(t) - 2A_1(t) = -p_1 - 2t + 2, -1 < t < 0, \\ A_1(-1) = -1, A_1(1) = 1 \end{cases}$$

for a mixed ordinary differential equations. We will seek the general solution of this mixed equation by the formula

$$A_1(t) = A_c(t) + A_p(t),$$

where $A_c(t)$ of homogeneous equation

$$-A_1''(t) + 2A_1(t) = 0, 0 < t < 1,$$
$$A_1'(t) - 2A_1(t) = 0, -1 < t < 0$$

and $A_p(t)$ is the particular solution of non-homogeneous mixed equation. It is clear that

$$A_c(t) = \begin{cases} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}, 0 < t < 1, \\ c_3 e^{2t}, -1 < t < 0. \end{cases} \tag{2.10}$$

For obtaining $A_p(t)$, we put

$$A_p(t) = at + b.$$

Putting it into the first equation, we get

$$2(at + b) = p_1 + 2t - 1.$$

Then $a = 1, b = \frac{p_1-1}{2}$ and $A_p(t) = t + \frac{p_1-1}{2}$.
Let $0 < t < 1$. Using (2.10), we get the general solution of this equation by the following formula

$$A_1(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + t + \frac{p_1 - 1}{2}. \tag{2.11}$$

Then,

$$A_1'(t) = \sqrt{2}c_1 e^{\sqrt{2}t} - \sqrt{2}c_2 e^{-\sqrt{2}t} + 1. \tag{2.12}$$

Assume that $-1 < t < 0$. Using (2.10), we get the general solution of this equation by the following formula

$$A_1(t) = c_3 e^{2t} + t + \frac{p_1 - 1}{2}. \tag{2.13}$$

Then,

$$A_1'(t) = 2c_3 e^{2t} + 1. \tag{2.14}$$

Using (2.11),(2.13), (2.12),(2.14), boundary conditions $A_1(-1) = -1, A_1(1) = 1$, and continuity conditions at $t = 0$, we get the following system of equations

$$\begin{cases} c_3 e^{-2} - 1 + \frac{p_1-1}{2} = -1, \\ c_1 e^{\sqrt{2}} + c_2 e^{-\sqrt{2}} + 1 + \frac{p_1-1}{2} = 1, \\ c_3 + \frac{p_1-1}{2} = c_1 + c_2 + \frac{p_1-1}{2}, \\ 2c_3 + 1 = \sqrt{2}c_1 - \sqrt{2}c_2 + 1. \end{cases}$$

From that it follows

$$\begin{cases} c_3e^{-2} + \frac{p_1-1}{2} = 0, \\ c_1e^{\sqrt{2}} + c_2e^{-\sqrt{2}} + \frac{p_1-1}{2} = 0, \\ -c_3 + c_1 + c_2 = 0, \\ -2c_3 + \sqrt{2}c_1 - \sqrt{2}c_2 = 0 \end{cases}$$

or

$$\begin{cases} c_3e^{-2} + \frac{p_1-1}{2} = 0, \\ c_1e^{\sqrt{2}} + c_2e^{-\sqrt{2}} - c_3e^{-2} = 0, \\ -c_3 + c_1 + c_2 = 0, \\ -2c_3 + \sqrt{2}c_1 - \sqrt{2}c_2 = 0. \end{cases}$$

Since

$$\begin{vmatrix} e^{\sqrt{2}} & e^{-\sqrt{2}} & -e^{-2} \\ 1 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & -2 \end{vmatrix}$$

$$= -2e^{\sqrt{2}} - \sqrt{2}e^{-\sqrt{2}} + \sqrt{2}e^{-2} + \sqrt{2}e^{-2} + 2e^{-\sqrt{2}} - \sqrt{2}e^{\sqrt{2}} \neq 0,$$

we have that

$$c_1 = c_2 = c_3 = 0\,.$$

From that and equation $c_3e^{-2} + \frac{p_1-1}{2} = 0$ it follows that $p_1 = 1$and $A_1(t) = t$.

Second, we obtain $A_k(t), k \neq 1$. Applying (2.4), (2.5), (2.8) and (2.9), we get the following boundary value problem

$$\begin{cases} -A_k''(t) + (k^2+1)A_k(t) = p_k, 0 < t < 1, \\ \\ A_k'(t) - (k^2+1)A_k(t) = -p_k, -1 < t < 0, \\ \\ A_k(-1) = 0, A_k(1) = 0 \end{cases}$$

for a mixed ordinary differential equations. We will seek the general solution of this mixed equation by the formula

$$A_1(t) = A_c(t) + A_p(t),$$

where $A_c(t)$ of homogeneous equation

$$-A_k''(t) + (k^2+1)A_k(t) = 0, 0 < t < 1,$$

$$A_k'(t) - (k^2+1)A_k(t) = 0, -1 < t < 0$$

and $A_p(t)$ is the particular solution of non-homogeneous mixed equation. It is clear that

$$A_c(t) = \begin{cases} c_1 e^{\sqrt{k^2+1}t} + c_2 e^{-\sqrt{k^2+1}t}, 0 < t < 1, \\ \\ c_3 e^{(k^2+1)t}, -1 < t < 0. \end{cases} \tag{2.15}$$

For obtaining $A_p(t)$, we put

$$A_p(t) = b.$$

Let $0 < t < 1$.Putting it into the first equation, we get

$$(k^2+1)b = p_k.$$

Then $b = \frac{p_k}{k^2+1}$. Using (2.15), we get the general solution of this equation by the following formula

$$A_k(t) = c_1 e^{\sqrt{k^2+1}t} + c_2 e^{-\sqrt{k^2+1}t} + \frac{p_k}{k^2+1}. \tag{2.16}$$

Then,

$$A_k'(t) = c_1\sqrt{k^2+1}e^{\sqrt{k^2+1}t} - c_2\sqrt{k^2+1}e^{-\sqrt{k^2+1}t}. \tag{2.17}$$

Let $-1 < t < 0$. Putting into $A_p(t) = b$ the second equation, we get

$$-(k^2+1)b = -p_k.$$

Then $b = \frac{p_k}{k^2+1}$. Using (2.15), we get the general solution of this equation by the following formula

$$A_1(t) = c_3 e^{(k^2+1)t} + \frac{p_k}{k^2+1}. \tag{2.18}$$

Then,

$$A_1'(t) = c_3(k^2+1)e^{(k^2+1)t}. \tag{2.19}$$

Using (2.16),(2.17), (2.18),(2.19), boundary conditions $A_k(-1) = 0, A_k(1) = 0$, and continuity conditions at $t = 0$, we get the following system of equations

$$\begin{cases} c_3 e^{-(k^2+1)} + \frac{p_k}{k^2+1} = 0, \\ c_1 e^{\sqrt{k^2+1}} + c_2 e^{-\sqrt{k^2+1}} + \frac{p_k}{k^2+1} = 0, \\ c_3 + \frac{p_k}{k^2+1} = c_1 + c_2 + \frac{p_k}{k^2+1}, \\ (k^2+1)c_3 = c_1\sqrt{k^2+1} - c_2\sqrt{k^2+1}. \end{cases}$$

From that it follows

$$\begin{cases} c_3 e^{-(k^2+1)} + \frac{p_k}{k^2+1} = 0, \\ c_1 e^{\sqrt{k^2+1}} + c_2 e^{-\sqrt{k^2+1}} - c_3 e^{-(k^2+1)} = 0, \\ -c_3 + c_1 + c_2 = 0, \\ -(k^2+1)c_3 + c_1\sqrt{k^2+1} - c_2\sqrt{k^2+1} = 0. \end{cases}$$

Since

$$\begin{vmatrix} e^{\sqrt{k^2+1}} & e^{-\sqrt{k^2+1}} & -e^{-(k^2+1)} \\ 1 & 1 & -1 \\ \sqrt{k^2+1} & -\sqrt{k^2+1} & -(k^2+1) \end{vmatrix}$$

$$= -(k^2+1)e^{\sqrt{k^2+1}} - \sqrt{k^2+1}e^{-\sqrt{k^2+1}} + \sqrt{k^2+1}e^{-(k^2+1)}$$

$$+ \sqrt{k^2+1}e^{-(k^2+1)} + (k^2+1)e^{-\sqrt{k^2+1}} - \sqrt{k^2+1}e^{\sqrt{k^2+1}} \neq 0,$$

we have that

$$c_1 = c_2 = c_3 = 0 .$$

From that and equation $c_3 e^{-(k^2+1)} + \frac{p_k}{k^2+1} = 0$ it follows that $p_k = 0$ and $A_k(t) = 0$.Then, $A_k(t) = 0, p_k = 0, k \neq 1$ and the exact solution of the problem (2.1) is

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx = A_1(t) \sin x = t \sin x, \tag{2.20}$$

$$p(x) = \sum_{k=1}^{\infty} p_k \sin kx = p_1 \sin x = \sin x.$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ \\ \frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = -p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ -T < t < 0, \\ \\ u(-T,x) = \psi(x), u(T,x) = \psi(x), x \in \overline{\Omega}, \\ \\ u(t,x) = 0, -T \le t \le T, \quad x \in S \end{cases} \tag{2.21}$$

for the multidimensional parabolic-elliptic partial differential equation. Here $\alpha_r > \alpha > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x), \psi(x) \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary

$$S, \overline{\Omega} = \Omega \cup S.$$

However, Fourier series method described in solving (2.21) can be used only in the case when (2.21) has constant coefficients.

Example 2.1.2. Obtain the Fourier series solution of the identification problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + (2t-1)\cos x, \\ 0 < t < 1, 0 < x < \pi, \\ \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x) - 2t\cos x, \\ -1 < t < 0, 0 < x < \pi, \\ \\ u(-1,x) = -\cos x, u(1,x) = \cos x, 0 \leq x \leq \pi, \\ \\ u_x(t,0) = u_x(t,\pi) = 0, -1 \leq t \leq 1 \end{cases} \tag{2.22}$$

for parabolic-elliptic equations.

Solution. In order to solve the problem, first we consider the Sturm-Liouville problem for the operator A defined by formula

$$Au(x) = -\frac{d^2 u(x)}{dx^2} + u(x)$$

with the domain

$$D(A) = \left\{ u : u(x), \frac{d^2 u(x)}{dx^2} \in L_2[0,\pi], u'(0) = u'(\pi) = 0 \right\}.$$

Actually, we will obtain all $(u(x), \lambda)$ such that

$$-u''(x) + u(x) = -\lambda u(x), 0 < x < \pi, u'(0) = u'(\pi), u(x) \neq 0.$$

It is easy to see that the solution of this Sturm-Liouville problem is

$$(u_k(x), \lambda_k) = \left(\cos kx, -k^2 - 1\right), k = 0, 1, ...$$

Therefore, we will seek solution $u(t, x)$ of problem (2.22) using by the Fourier series

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos kx. \tag{2.23}$$

and

$$p(x) = \sum_{k=0}^{\infty} p_k \cos kx. \tag{2.24}$$

Here $A_k(t), p_k, k = 0, 1, 2, ...$ are unknown functions and parameters. Applying formula (2.23), given boundary conditions, we get

$$u(-1, x) = \sum_{k=0}^{\infty} A_k(-1) \cos kx = -\cos x,$$

$$u(1, x) = \sum_{k=0}^{\infty} A_k(1) \cos kx = \cos x.$$

From that it follows

$$A_1(-1) = -1, A_k(-1) = 0, k \neq 1 \tag{2.25}$$

and

$$A_1(1) = 1, A_k(1) = 0, k \neq 1, \tag{2.26}$$

respectively. Applying formulas (2.23), (2.24) and given equations, we obtain

$$-\sum_{k=0}^{\infty} A_k''(t) \cos kx + \sum_{k=0}^{\infty} A_k(t) k^2 \cos kx + \sum_{k=0}^{\infty} A_k(t) \cos kx \tag{2.27}$$

$$= \sum_{k=0}^{\infty} p_k \cos kx + (2t - 1) \cos x, 0 < t < 1,$$

$$\sum_{k=0}^{\infty} A_k'(t) \cos kx - \sum_{k=0}^{\infty} A_k(t) k^2 \cos kx - \sum_{k=0}^{\infty} A_k(t) \cos kx \tag{2.28}$$

$$= -\sum_{k=0}^{\infty} p_k \cos kx + -2t \cos x, -1 < t < 0.$$

Equating coefficients of $\cos kx$, $k = 0, 1, 2, ...$, to zero, we get

$$-A_1''(t) + 2A_1(t) = p_1 + (2t - 1), k = 1, -A_k''(t) + (k^2+1)A_k(t) = p_k, k \neq 1, 0 < t < 1, \quad (2.29)$$

$$A_1'(t) - 2A_1(t) = -p_1 - 2t, k = 1, A_k'(t) - (k^2+1)A_k(t) = -p_k, k \neq 1, -1 < t < 0. \quad (2.30)$$

First, we obtain $A_1(t)$. Applying (2.25), (2.26), (2.29) and (2.30), we get the following boundary value problem

$$\begin{cases} -A_1''(t) + 2A_1(t) = p_1 + (2t - 1), 0 < t < 1, \\ A_1'(t) - 2A_1(t) = -p_1 - 2t, -1 < t < 0, \\ A_1(-1) = -1, A_1(1) = 1 \end{cases}$$

for a mixed ordinary differential equations. We will seek the general solution of this mixed equation by the formula

$$A_1(t) = A_c(t) + A_p(t),$$

where $A_c(t)$ of homogeneous equation

$$-A_1''(t) + 2A_1(t) = 0, 0 < t < 1,$$

$$A_1'(t) - 2A_1(t) = 0, -1 < t < 0$$

and $A_p(t)$ is the particular solution of non-homogeneous mixed equation. It is clear that

$$A_c(t) = \begin{cases} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}, 0 < t < 1, \\ c_3 e^{2t}, -1 < t < 0. \end{cases} \quad (2.31)$$

For obtaining $A_p(t)$, we put

$$A_p(t) = at + b.$$

Assume that $0 < t < 1$.Putting it into the first equation, we get

$$2(at + b) = p_1 + 2t - 1.$$

Then $a = 1, b = \frac{p_1-1}{2}$. Using (2.31), we get the general solution of this equation by the following formula

$$A_1(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + t + \frac{p_1 - 1}{2}. \tag{2.32}$$

Then,

$$A_1'(t) = \sqrt{2} c_1 e^{\sqrt{2}t} - \sqrt{2} c_2 e^{-\sqrt{2}t} + 1. \tag{2.33}$$

Assume that $-1 < t < 0$.Putting into $A_p(t) = at + b$ the second equation, we get

$$a - 2(at + b) = -p_1 - 2t.$$

Then $a = 1, b = \frac{p_1-1}{2}$. Using (2.31), we get the general solution of this equation by the following formula

$$A_1(t) = t + \frac{1}{2}(1 - e^{2+2t})(1 - p_1). \tag{2.34}$$

Then,

$$A_1'(t) = 1 - e^{2+2t}(1 - p_1) \tag{2.35}$$

Using (2.32),(2.33), (2.34),(2.35), boundary conditions $A_1(-1) = -1$, and continuity conditions at $t = 0$, we get the following system of equations

$$\begin{cases} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + t + \frac{p_1-1}{2}, \\ c_1 e^{\sqrt{2}} + c_2 e^{-\sqrt{2}} + 1 + \frac{p_1-1}{2} = 1, \\ c_1 + c_2 + \frac{p_1-1}{2} = \frac{1}{2}(1 - e^2)(1 - p_1), \\ \sqrt{2} c_1 - \sqrt{2} c_2 + 1 = 1 - e^2(1 - p_1). \end{cases}$$

From that it follows

$$\begin{cases} c_1 e^{\sqrt{2}} + c_2 e^{-\sqrt{2}} - \frac{p_1 - 1}{2} = 0, \\ \\ c_1 + c_2 + (p_1 - 1)(\frac{1}{2} + \frac{1}{2} - \frac{e^2}{2}) = 0, \\ \\ \sqrt{2}c_1 - \sqrt{2}c_2 + (p_1 - 1)(-e^2) = 0. \end{cases}$$

Since

$$\begin{vmatrix} e^{\sqrt{2}} & e^{-\sqrt{2}} & \frac{1}{2} \\ 1 & 1 & 1 - \frac{e^2}{2} \\ \sqrt{2} & -\sqrt{2} & -e^2 \end{vmatrix}$$

$$= -\sqrt{2} + e^{2-\sqrt{2}} - e^{2+\sqrt{2}} + \sqrt{2}(1 - \frac{e^2}{2})(e^{\sqrt{2}} + \frac{1}{e^{\sqrt{2}}} \neq 0,$$

we have that

$$c_1 = c_2 = p_1 - 1 = 0\,.$$

It follows that $p_1 = 1$ and $A_1(t) = t$.

Second, we obtain $A_k(t), k \neq 1$. Applying (2.25), (2.26), (2.29) and (2.30), we get the following boundary value problem

$$\begin{cases} -A_k''(t) + (k^2 + 1)A_k(t) = p_k, 0 < t < 1, \\ \\ A_k'(t) - (k^2 + 1)A_k(t) = -p_k, -1 < t < 0, \\ \\ A_k(-1) = 0, A_k(1) = 0 \end{cases}$$

for a mixed ordinary differential equations. We will seek the general solution of this mixed equation by the formula

$$A_1(t) = A_c(t) + A_p(t)\,,$$

where $A_c(t)$ of homogeneous equation

$$-A_k''(t) + (k^2 + 1)A_k(t) = 0, 0 < t < 1, \tag{2.36}$$

$$\tag{2.37}$$

$$A_k'(t) - (k^2 + 1)A_k(t) = 0, -1 < t < 0.$$

and $A_p(t)$ is the particular solution of non-homogeneous mixed equation. It is clear that

$$A_c(t) = \begin{cases} c_1 e^{\sqrt{k^2+1}t} + c_2 e^{-\sqrt{k^2+1}t}, 0 < t < 1, \\ \\ c_3 e^{(k^2+1)t}, -1 < t < 0. \end{cases} \tag{2.38}$$

For obtaining $A_p(t)$, we put

$$A_p(t) = b.$$

Assume that $0 < t < 1$. Putting it into the first equation, we get

$$(k^2 + 1)b = p_k.$$

Then $b = \frac{p_k}{k^2+1}$. Using (2.38), we get the general solution of this equation by the following formula

$$A_k(t) = c_1 e^{\sqrt{k^2+1}t} + c_2 e^{-\sqrt{k^2+1}t} + \frac{p_k}{k^2 + 1}. \tag{2.39}$$

Then,

$$A_k'(t) = c_1 \sqrt{k^2 + 1} e^{\sqrt{k^2+1}t} - c_2 \sqrt{k^2 + 1} e^{-\sqrt{k^2+1}t}. \tag{2.40}$$

Assume that $-1 < t < 0$. Putting into $A_p(t) = b$ the second equation, we get

$$-(k^2 + 1)b = -p_k.$$

Then $b = \frac{p_k}{k^2+1}$. Using (2.38), we get the general solution of this equation by the following formula

$$A_k(t) = (-1 + e^{(k^2+1)(t+1)})\frac{p_k}{k^2 + 1}. \tag{2.41}$$

Then,

$$A_k'(t) = (k^2 + 1)(e^{(k^2+1)(t+1)})p_k. \tag{2.42}$$

Using (2.39),(2.41), (2.40),(2.42), boundary conditions $A_k(-1) = 0, A_k(1) = 0$, and continuity conditions at $t = 0$, we get the following system of equations

$$\begin{cases} c_1 e^{\sqrt{k^2+1}} + c_2 e^{-\sqrt{k^2+1}} + \frac{p_k}{k^2+1} = 0, \\ \\ \frac{1}{k^2+1}(-1 + e^{k^2+1})p_k = c_1 + c_2 + \frac{p_k}{k^2+1}, \\ \\ e^{k^2+1} p_k = c_1 \sqrt{k^2+1} - c_2 \sqrt{k^2+1}. \end{cases}$$

From that it follows

$$\begin{cases} c_1 e^{\sqrt{k^2+1}} + c_2 e^{-\sqrt{k^2+1}} + \frac{p_k}{k^2+1} = 0, \\ \\ c_1 + c_2 + p_k \left(\frac{2-e^{k^2+1}}{k^2+1} \right) = 0, \\ \\ c_1 \sqrt{k^2+1} - c_2 \sqrt{k^2+1} - e^{k^2+1} p_k = 0. \end{cases}$$

Since

$$\begin{vmatrix} e^{\sqrt{k^2+1}} & e^{-\sqrt{k^2+1}} & \frac{1}{k^2+1} \\ 1 & 1 & \frac{2-e^{k^2+1}}{k^2+1} \\ \sqrt{k^2+1} & -\sqrt{k^2+1} & -e^{k^2+1} \end{vmatrix}$$

$$= 2\sqrt{k^2+1} \left(\frac{2 - e^{k^2+1}}{k^2+1} \right) - \frac{2\sqrt{k^2+1}}{k^2+1} \neq 0,$$

we have that

$$c_1 = c_2 = 0 \,.$$

From that and equation $\frac{p_k}{k^2+1} = 0$ it follows that $p_k = 0$ and $A_k(t) = 0$. Then, $A_k(t) = 0, p_k = 0, k \neq 1$ and the exact solution of the problem (2.22) is

$$u(t, x) = \sum_{k=1}^{\infty} A_k(t) \cos kx = A_1(t) \cos x = t \cos x, \tag{2.43}$$

$$p(x) = \sum_{k=1}^{\infty} p_k \cos kx = p_1 \cos x = \cos x.$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ \frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = -p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ -T < t < 0, \\ u(-T,x) = \psi(x), u(T,x) = \psi(x), x \in \overline{\Omega}, \\ \frac{\theta u(t,x)}{\partial \overline{m}} = 0, -T \le t \le T, \quad x \in S \end{cases} \tag{2.44}$$

for the multidimensional mixed partial differential equation. Here $\alpha_r > \alpha > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x), \psi(x) \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here $\overline{m}$ is the normal vector to boundary S.

However Fourier series method described in solving (2.44) can be also used only in the case when (2.44) has constant coefficients.

Example 2.1.3. Obtain the Fourier series solution of the identification problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + (4e^{-t} - 1)\sin 2x, \\ 0 < t < 1, 0 < x < \pi, \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x) + (-6e^{-t} + 1)\sin 2x, \\ -1 < t < 0, 0 < x < \pi, \\ u(-1,x) = e \sin 2x, u(1,x) = e^{-1} \sin 2x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), -1 \le t \le 1 \end{cases} \tag{2.45}$$

for the parabolic-elliptic differential equation.

Solution. In order to solve the problem, first we consider the Sturm-Liouville problem for the operator A defined by formula

$$Au(x) = -\frac{d^2u(x)}{dx^2} + u(x)$$

with the domain

$$D(A) = \left\{u : u(x), \frac{d^2u(x)}{dx^2} \in L_2[0,\pi], u(0) = u(\pi), u_x(0) = u_x(\pi)\right\}.$$

Actually, we will obtain all $(u(x), \lambda)$ such that

$$-u''(x) + u(x) = -\lambda u(x), 0 < x < \pi, u(0) = u(\pi), u_x(x) = u_x(\pi).$$

It is easy to see that the solution of this Sturm-Liouville problem is

$$(u_k(x), \lambda_k) = \left(\sin 2kx, -4k^2 - 1\right), k = 1, 2, ...,$$

$$(u_k(x), \lambda_k) = \left(\cos 2kx, -4k^2 - 1\right), k = 0, 1, ...$$

Therefore, we will seek solution $u(t, x)$ of problem (2.45) using by the Fourier series

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx. \tag{2.46}$$

and

$$p(x) = \sum_{k=0}^{\infty} p_k \cos 2kx + \sum_{k=1}^{\infty} q_k \sin 2kx. \tag{2.47}$$

Here $A_k(t), p_k, k = 0, 1, 2, ...$ and $B_k(t), q_k, k = 1, 2, ...$ are unknown functions and parameters. Applying formula (2.46), given boundary conditions, we get

$$u(-1, x) = \sum_{k=0}^{\infty} A_k(-1) \cos 2kx + \sum_{k=1}^{\infty} B_k(-1) \sin 2kx = e \sin 2x,$$

$$u(1, x) = \sum_{k=0}^{\infty} A_k(1) \cos 2kx + \sum_{k=1}^{\infty} B_k(1) \sin 2kx = e^{-1} \sin 2x.$$

From that it follows

$$A_1(-1) = 0, A_k(-1) = 0, B_1(-1) = e, B_k(-1) = 0, k \neq 1 \tag{2.48}$$

and

$$A_1(1) = 0, A_k(1) = 0, B_1(1) = e^{-1}, B_k(1) = 0, k \neq 1, \tag{2.49}$$

respectively. Applying formulas (2.46), (2.47) and given equations, we obtain

$$- \sum_{k=0}^{\infty} A_k''(t) \cos 2kx + \sum_{k=0}^{\infty} A_k(t) 4k^2 \cos 2kx + \sum_{k=0}^{\infty} A_k(t) \cos 2kx$$

$$- \sum_{k=1}^{\infty} B_k''(t) \sin 2kx + \sum_{k=1}^{\infty} B_k(t) 4k^2 \sin 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx$$

$$= \sum_{k=0}^{\infty} p_k \cos 2kx + \sum_{k=1}^{\infty} q_k \sin 2kx + (4e^{-t} - 1) \sin 2x, 0 < t < 1,$$

$$\sum_{k=0}^{\infty} A_k'(t) \cos 2kx - \sum_{k=0}^{\infty} A_k(t) 4k^2 \cos 2kx - \sum_{k=0}^{\infty} A_k(t) \cos 2kx$$

$$+ \sum_{k=1}^{\infty} B_k'(t) \sin 2kx - \sum_{k=1}^{\infty} B_k(t) 4k^2 \sin 2kx - \sum_{k=1}^{\infty} B_k(t) \sin 2kx$$

$$= - \sum_{k=0}^{\infty} p_k \cos 2kx - \sum_{k=1}^{\infty} q_k \sin 2kx + (-6e^{-t} + 1) \sin 2x, -1 < t < 0.$$

Equating coefficients of $\sin 2kx$, $k = 1, 2, ..., \cos 2kx, k = 0, 1, ...,$ to zero, we get

$$\begin{cases} A_k'(t) - 4k^2 A_k(t) - A_k(t) = -p_k, k = 0, 1, ..., \\ B_k'(t) - 4k^2 B_k(t) - B_k(t) = -q_k, k \neq 1, -1 < t < 0, \\ B_1'(t) - 5B_1(t) = -q_1 + (-6e^{-t} + 1), \\ A_k(-1) = 0, A_k(1) = 0, k = 0, 1, ..., \\ B_1(-1) = e, B_1(1) = e^{-1}, \\ B_k(-1) = 0, B_k(1) = 0, k \neq 1. \end{cases} \tag{2.50}$$

$$\begin{cases} -A_k''(t) + 4k^2 A_k(t) + A_k(t) = -p_k, k = 0, 1, ..., \\ \\ -B_k''(t) + 4k^2 B_k(t) + B_k(t) = -q_k, k \neq 1, 0 < t < 1, \\ \\ B_1''(t) - 5B_1(t) = -q_1 + (4e^{-t} - 1), \\ \\ A_k(-1) = 0, A_k(1) = 0, \forall k, B_1(-1) = e, B_1(1) = e^{-1}, \\ \\ B_k(-1) = 0, B_k(1) = 0, k \neq 1. \end{cases} \tag{2.51}$$

First, we obtain $A_k(t), k \neq 1$. Applying (2.46), (2.47), (2.50) and (2.51), we get the following boundary value problem

$$\begin{cases} -A_k''(t) + (4k^2 + 1)A_k(t) = p_k, 0 < t < 1, \\ \\ A_k'(t) - (4k^2 + 1)A_k(t) = -p_k, -1 < t < 0, \\ \\ A_k(-1) = 0, A_k(1) = 0. \end{cases}$$

for a mixed ordinary differential equations. Assume that $-1 < t < 0$.

$$A_k'(t) - (4k^2 + 1)A_k(t) = -p_k.$$

It is clear that

$$A_k(t) = c_1 e^{(4k^2+1)t} + \frac{p_k}{4k^2 + 1},$$

and $0 < t < 1$.

$$-A_k''(t) + (4k^2 + 1)A_k(t) = p_k$$

also it is clear that

$$A_k(t) = c_2 e^{\sqrt{(4k^2+1)}t} + c_3 e^{-\sqrt{(4k^2+1)}t} + \frac{p_k}{4k^2+1}.$$

and we use boundary conditions $A_k(-1) = 0, A_k(1) = 0$. we get

$$\begin{cases} A_k(-1) = c_1 e^{-(4k^2+1)} + \frac{p_k}{4k^2+1} = 0, \\ \\ A_k(1) = c_2 e^{\sqrt{(4k^2+1)}} + c_3 e^{-\sqrt{(4k^2+1)}} + \frac{p_k}{4k^2+1} = 0. \end{cases} \tag{2.52}$$

Using (2.52) and continuity conditions at $t = 1$, we get the following system of equations

$$\begin{cases} c_1 = c_2 + c_3, \\ \\ c_1 e^{-(4k^2+1)} = c_2 e^{-(4k^2+1)} + c_3 e^{-(4k^2+1)}, \\ \\ (4k^2+1)c_1 = \sqrt{(4k^2+1)}c_2 - \sqrt{(4k^2+1)}c_3. \end{cases}$$

or

$$\begin{cases} c_1 - c_2 - c_3 = 0, \\ \\ c_1 e^{-(4k^2+1)} - c_2 e^{(4k^2+1)} - c_3 e^{-(4k^2+1)} = 0, \\ \\ (4k^2+1)c_1 - \sqrt{(4k^2+1)}c_2 + \sqrt{(4k^2+1)}c_3 = 0. \end{cases}$$

since

$$\begin{vmatrix} 1 & -1 & -1 \\ e^{-(4k^2+1)} & -e^{(4k^2+1)} & -e^{-(4k^2+1)} \\ 4k^2+1 & -\sqrt{(4k^2+1)} & \sqrt{(4k^2+1)} \end{vmatrix}$$

$$= \sqrt{(4k^2+1)}\left(e^{-(4k^2+1)} - e^{(4k^2+1)}\right) \neq 0.$$

we have that

$$c_1 = c_2 = c_3 = 0.$$

From that and equation $A_k(t) = \frac{p_k}{4k^2+1}, \forall k$. it follows that $p_k = 0$and $A_k(t) = 0$.
Second, we obtain $B_k(t)$. It is cleat that for $k \neq 1, B_k(t)$ is the solution of the initial value problem (2.46), (2.47), (2.50) and (2.51), we get the following boundary value problem

$$\begin{cases} -B_k''(t) + (4k^2+1)B_k(t) = q_k, 0 < t < 1, \\ B_k'(t) - (4k^2+1)B_k(t) = -q_k, -1 < t < 0, \\ B_k(-1) = 0, B_k(1) = 0. \end{cases}$$

for a mixed ordinary differential equations. Assume that $-1 < t < 0$.

$$B_k'(t) - (4k^2+1)B_k(t) = -q_k$$

It is clear that

$$B_k(t) = c_1 e^{(4k^2+1)t} + \frac{q_k}{4k^2+1},$$

and $0 < t < 1$.

$$-B_k''(t) + (4k^2+1)B_k(t) = q_k$$

also it is clear that

$$B_k(t) = c_2 e^{\sqrt{(4k^2+1)}t} + c_3 e^{-\sqrt{(4k^2+1)}t} + \frac{q_k}{4k^2+1}.$$

and we use boundaey conditions $B_k(-1) = 0, B_k(1) = 0$. We get

$$\begin{cases} B_k(-1) = c_1 e^{-(4k^2+1)} + \frac{q_k}{4k^2+1} = 0, \\ B_k(1) = c_2 e^{\sqrt{(4k^2+1)}} + c_3 e^{-\sqrt{(4k^2+1)}} + \frac{q_k}{4k^2+1} = 0. \end{cases} \tag{2.53}$$

Using (2.53) and continuity conditions at $t = 1$, we get the following system of equations

$$\begin{cases} c_1 = c_2 + c_3, \\ c_1 e^{-(4k^2+1)} = c_2 e^{-(4k^2+1)} + c_3 e^{-(4k^2+1)}, \\ (4k^2+1)c_1 = \sqrt{(4k^2+1)}c_2 - \sqrt{(4k^2+1)}c_3. \end{cases}$$

or

$$\begin{cases} c_1 - c_2 - c_3 = 0, \\ c_1 e^{-(4k^2+1)} - c_2 e^{(4k^2+1)} - c_3 e^{-(4k^2+1)} = 0, \\ (4k^2+1)c_1 - \sqrt{(4k^2+1)}c_2 + \sqrt{(4k^2+1)}c_3 = 0. \end{cases}$$

since

$$\begin{vmatrix} 1 & -1 & -1 \\ e^{-(4k^2+1)} & -e^{(4k^2+1)} & -e^{-(4k^2+1)} \\ 4k^2+1 & -\sqrt{(4k^2+1)} & \sqrt{(4k^2+1)} \end{vmatrix}$$

$$= \sqrt{(4k^2+1)}\left(e^{-(4k^2+1)} - e^{(4k^2+1)}\right) \neq 0.$$

we have that

$$c_1 = c_2 = c_3 = 0.$$

From that and equation $B_k(t) = \frac{q_k}{4k^2+1}, k \neq 1$. it follows that $q_k = 0$and $B_k(t) = 0$.

Third, we obtain $B_k(t), k = 1$. Applying (2.46), (2.47), (2.50) and (2.51), we get the following boundary value problem

$$\begin{cases} -B_1''(t) + 5B_1(t) = q_1 + (4e^{-t} - 1), 0 < t < 1, \\ B_1'(t) - 5B_1(t) = -q_1, +(6e^{-t} + 1), -1 < t < 0, \\ B_1(-1) = e, B_1(1) = e^{-1} \end{cases}$$

for a mixed ordinary differential equations. We will seek the general solution of this mixed equation by the formula

$$B_1 p(t) = a + be^{-t}.$$

Assume that $-1 < t < 0$. It is clear that

$$-be^{-t} - 5(a + be^{-t}) = -q_1 + 1 - 6e^{-t},$$

$$b = 1, a = \frac{q-1}{5}.$$

we get

$$B_1(t) = e^{5t}c_1 + \frac{q-1}{5} + e^{-t}, -1 < t < 0, \tag{2.54}$$

$$B_1(-1) = e^{-5}c_1 + \frac{q-1}{5} = 0 \tag{2.55}$$

Now we use again differential equation for $0 < t < 1$

$$-be^{-t} - 5(a + be^{-t}) = -q_1 + 1 - 6e^{-t},$$

$$b = 1, a = \frac{q-1}{5}.$$

we get

$$B_1(t) = c_2 e^{\sqrt{5}t} + c_3 e^{-\sqrt{5}t} + \frac{q-1}{5} + e^{-t}, 0 < t < 1, \tag{2.56}$$

$$B_1(1) = c_2 e^{\sqrt{5}} + c_3 e^{-\sqrt{5}} + \frac{q-1}{5} = 0. \tag{2.57}$$

Using (2.54),(2.55), (2.56),(2.57), we get the following system of equations

$$\begin{cases} c_1 = c_2 + c_3, \\ 5c_1 = \sqrt{5}c_2 - \sqrt{5}c_3, \\ e^{-5}c_1 = c_2 e^{\sqrt{5}} + c_3 e^{-\sqrt{5}}. \end{cases}$$

From that it follows

$$\begin{cases} c_1 - c_2 - c_3 = 0, \\ 5c_1 - \sqrt{5}c_2 + \sqrt{5}c_3 = 0, \\ e^{-5}c_1 - c_2e^{\sqrt{5}} - c_3e^{-\sqrt{5}} = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & -1 & -1 \\ 5 & -\sqrt{5} & \sqrt{5} \\ e^{-5} & -e^{\sqrt{5}} & -e^{-\sqrt{5}} \end{vmatrix}$$

$$(\sqrt{5} - 5)e^{-\sqrt{5}} + (\sqrt{5} + 5)e^{\sqrt{5}} - 2\sqrt{5}e^{-5} \neq 0,$$

we have that

$$c_1 = c_2 = c_3 = 0, q_1 = 1.$$

From that it follows $q_1 = 1$and $A_k(t) = 0$.Then, $A_k(t) = e^{-t}, k = 1$ and the exact solution of the problem (2.45) is

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx = A_1(t) \sin 2x = e^{-t} \sin 2x, \tag{2.58}$$

$$p(x) = \sum_{k=0}^{\infty} p_k \cos 2kx + \sum_{k=1}^{\infty} q_k \sin 2kx = q_1 \sin 2x = \sin 2x.$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$
\begin{cases}
-\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\
\\
\frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = -p(x) + g(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega},\ -T < t < 0, \\
\\
u(-T,x) = \psi(x), u(T,x) = \psi(x), x \in \overline{\Omega}, \\
\\
u(t,x) \mid_{S_1} = u(t,x) \mid_{S_2}, \frac{\partial u(t,x)}{\partial \overline{m}} \mid_{S_1} = \frac{\partial u(t,x)}{\partial \overline{m}} \mid_{S_2} -T \le t \le T
\end{cases}
\tag{2.59}
$$

for the multidimensional mixed partial differential equation. Here $\alpha_r > \alpha > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x), \psi(x), x \in \overline{\Omega}$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with the boundary

$$S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset.$$

However, Fourier series method described in solving (2.59) can be used only in the case when (2.59) has constant coefficients.

2.2 LAPLACE TRANSFORM METHOD

We consider Laplace transform solution of identification problems for parabolic-elliptic differential equations.

Example 2.2.1. Obtain the Laplace transform solution of the following problem source identification problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) - e^{-x}, \\ 0 < x < \infty, 0 < t < 1, \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x), \\ 0 < x < \infty, -1 < t < 0, \\ u(-1,x) = -e^{-x}, u(1,x) = e^{-x}, 0 \le x < \infty, \\ u(t,0) = t, u_x(t,0) = -t, -1 \le t \le 1 \end{cases} \tag{2.60}$$

for a one dimensional parabolic-elliptic equation.

Solution. Here and in future, we will denote

$$\mathcal{L}\{u(t,x)\} = u(t,s).$$

Using formula

$$\mathcal{L}\{e^{-x}\} = \frac{1}{s+1} \tag{2.61}$$

and taking the Laplace transform of both sides of the differential equation and using conditions

$$u(t,0) = t,\ u_x(t,0) = -t,$$

we can write

$$\begin{cases} -u_{tt}(t,s) - \left[s^2 u(t,s) - su(t,0) - u_x(t,0)\right] + u(t,s) \\ = p(s) - \frac{1}{s+1}, 0 < t < 1, \\ u_t(t,s) + \left[s^2 u(t,s) - su(t,0) - u_x(t,0)\right] - u(t,s) \\ = -p(s), -1 < t < 0, \\ u(-1,s) = -\frac{1}{1+s}, u(1,s) = \frac{1}{1+s}. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} -u_{tt}(t,s) + (-s^2+1)u(t,s) \\ = p(s) - st + t - \frac{1}{s+1}, 0 < t < 1, \\ u_t(t,s) + (s^2-1)u(t,s) \\ = -p(s) + st - t, -1 < t < 0, \\ u(-1,s) = -\frac{1}{1+s} \ , u(1,s) = \frac{1}{1+s}. \end{cases} \tag{2.62}$$

Now we will obtain the solution of problem (2.62). Let $-1 \leq t \leq 0$.Then, we have the following initial value problem

$$\begin{cases} u_t(t,s) + (s^2-1)u(t,s) = -p(s) + (s-1)t, -1 < t < 0, \\ u(-1,s) = -\frac{1}{1+s}. \end{cases} \tag{2.63}$$

Intergrating it, we get

$$u(t,s) = -e^{-(s^2-1)(t+1)}u(-1,s) + \int_{-1}^{t} e^{-(s^2-1)(t-y)} \{p(s) + (s-1)y\} \, dy$$

$$= -e^{-(s^2-1)(t+1)}\left[-\frac{p(s)}{s^2-1}+\frac{1}{(s+1)(s^2-1)}\right]+\frac{p(s)}{s^2-1}+\frac{t}{s+1}-\frac{1}{(s+1)(s^2-1)}.$$

From that it follows

$$u(0,s) = e^{-(s^2-1)}\left[-\frac{p(s)}{s^2-1}+\frac{1}{(s+1)(s^2-1)}\right]+\frac{p(s)}{s^2-1}-\frac{1}{(s+1)(s^2-1)}, \tag{2.64}$$

$$u_t(0,s) = e^{-(s^2-1)}\left[p(s)-\frac{1}{s+1}\right]+\frac{1}{s+1}. \tag{2.65}$$

Now, let $0 \le t \le 1$. Applying (2.63) , (2.64) and (2.65) , we get the following initial value problem

$$\begin{cases} u_{tt}(t,s)+(s^2-1)u(t,s) = -p(s)+st-t+\frac{1}{s+1}, 0<t<1, \\ u(0,s) = e^{-(s^2-1)}\left[-\frac{p(s)}{s^2-1}+\frac{1}{(s+1)(s^2-1)}\right]+\frac{p(s)}{s^2-1}-\frac{1}{(s+1)(s^2-1)}, \\ u_t(0,s) = e^{-(s^2-1)}\left[p(s)-\frac{1}{s+1}\right]+\frac{1}{s+1}. \end{cases} \tag{2.66}$$

Applying the D'Alembert's formula , we get

$$u(t,s) = \cos\sqrt{s^2-1}te^{-(s^2-1)}\left[\frac{-p(s)}{s^2-1}+\frac{1}{(s+1)(s^2-1)}\right] \tag{2.67}$$

$$+\frac{1}{\sqrt{s^2-1}}\sin\sqrt{s^2-1}te^{-(s^2-1)}\left[p(s)-\frac{1}{s+1}\right]$$

$$+\frac{1}{s^2-1}\left[\left(\frac{1}{s+1}-p(s)\right)\right]+\frac{t}{s+1}.$$

Putting $t=1$ and using $u(1,s)=\frac{1}{s+1}$, we get

$$\frac{\cos\sqrt{s^2-1}e^{-(s^2-1)}}{s^2-1}\left[\frac{1}{1+s}-p(s)\right]+\frac{1}{\sqrt{s^2-1}}\sin\sqrt{s^2-1}\left[p(s)-\frac{1}{s^2+1}\right]$$

$$+\frac{1}{s^2-1}\left[\frac{1}{s+1}-p(s)\right]+\frac{1}{s+1}=\frac{1}{s+1}.$$

Then

$$\left[\frac{1}{1+s}-p(s)\right]\left\{\frac{\cos\sqrt{s^2-1}}{s^2-1}e^{-(s^2-1)}-\frac{1}{\sqrt{s^2-1}}\sin\sqrt{s^2-1}+\frac{1}{s^2-1}\right\}=0.$$

Since,

$$\cos\sqrt{s^2-1}e^{-(s^2-1)}-\sqrt{s^2-1}\sin\sqrt{s^2-1}+1\neq 0,$$

we have that

$$\frac{1}{1+s}-p(s)=0.$$

Therefore, $p(s)=\frac{1}{1+s}$ and

$$p(x)=\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=e^{-x}. \tag{2.68}$$

Using $p(s)=\frac{1}{1+s}$, we get

$$u(t,s)=\begin{cases}\frac{t}{1+s}, 0\leq t\leq 1,\\ \\ \frac{t}{1+s}, -1\leq t\leq 0.\end{cases}=\frac{t}{1+s}. \tag{2.69}$$

Taking the inverse Laplace transform with respect to x , we get

$$u(t,x)=t\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=te^{-x}, -1\leq t\leq 1.$$

Thus, the exact solution of problem (2.60) is

$$(u(t,x),p(x))=\left(te^{-x},e^{-x}\right).$$

Example 2.2.2. Obtain the Laplace transform solution of the following problem source identification problem

$$\begin{cases}-\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}=p\left(x\right)-e^{-x}-2e^{-(t+x)},\\ \\ 0<x<\infty, 0<t<1,\\ \\ \frac{\partial u(t,x)}{\partial t}+\frac{\partial^2 u(t,x)}{\partial x^2}=-p\left(x\right)+e^{-x},\\ \\ 0<x<\infty, -1<t<0,\\ \\ u\left(-1,x\right)=e^{1-x}, u\left(1,x\right)=e^{-1-x}, 0\leq x<\infty,\\ \\ u\left(t,0\right)=e^{-t}, u_x\left(t,\infty\right)=-e^{-t}, -1\leq t\leq 1\end{cases} \tag{2.70}$$

for a one dimensional parabolic-elliptic equation.

Solution. Taking the Laplace transform of both sides of the differential equation and using conditions $u(t,0)=e^{-t}, u_x(t,\infty)=-e^{-t}$, we can write

$$\begin{cases} -u_{tt}(t,s)-\left[s^2u(t,s)-su(t,0)-u_x(t,0)\right] \\ =p(s)-\frac{1}{s+1}-\frac{2e^{-t}}{1+s}, 0<t<1, \\ u_t(t,s)+\left[s^2u(t,s)-su(t,0)-u_x(t,0)\right] \\ =-p(s)+\frac{1}{1+s}, -1<t<0, \\ u(-1,s)=\frac{e}{1+s}, u(1,s)=\frac{e^{-1}}{1+s}. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} -u_{tt}(t,s)+-s^2u(t,s) \\ =p(s)-se^{-t}+e^{-t}-\frac{1}{s+1}-\frac{2e^{-t}}{1+s}, 0<t<1, \\ u_t(t,s)+s^2u(t,s) \\ =-p(s)+se^{-t}+e^{-t}+\frac{1}{1+s}, -1<t<0, \\ u(-1,s)=\frac{e}{1+s} \ , u(1,s)=\frac{e^{-1}}{1+s}. \end{cases} \tag{2.71}$$

Now, we will obtain the solution of problem (2.71). Let $-1\leq t\leq 0$.Then, we have the following initial value problem

$$\begin{cases} u_t(t,s)+s^2u(t,s)=-p(s)+e^{-t}(s-1)+\frac{1}{1+s}, -1<t<0, \\ u(-1,s)=\frac{e}{1+s}. \end{cases} \tag{2.72}$$

Solving it, we get

$$u(t,s) = e^{-s^2(t+1)}u(-1,s) + \int_{-1}^{t} e^{-s^2(t-y)}\left(p(s) + e^{-y}(s-1) + \frac{1}{1+s}\right)dy \tag{2.73}$$

$$= \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left\{1 - e^{-s^2(t+1)}\right\} + \frac{1}{s+1}e^{-t}.$$

From that it follows

$$u(0,s) = \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left\{1 - e^{-s^2}\right\} + \frac{1}{s+1}, \tag{2.74}$$

$$u_t(0,s) = \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)s^2e^{-s^2} - \frac{1}{s+1}. \tag{2.75}$$

Now, let $0 \leq t \leq 1$. Applying (2.72) , (2.74) and (2.75) , we get the following initial value problem

$$\begin{cases} u_{tt}(t,s) + s^2u(t,s) = -p(s) + (s-1)e^{-t} + \frac{1+2e^{-t}}{1+s}, 0 < t < 1, \\ u(0,s) = \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left(1 - e^{-s^2}\right) + \frac{1}{s+1}, \\ u_t(0,s) = \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)s^2e^{-s^2} - \frac{1}{s+1}. \end{cases} \tag{2.76}$$

Applying the D'Alembert's formula , we get

$$u(t,s) = \cos st\left\{\frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left(1 - e^{-s^2}\right) + \frac{1}{s+1}\right\} \tag{2.77}$$

$$+ \frac{1}{s}\sin st\left\{\frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)s^2e^{-s^2} - \frac{1}{s+1}\right\}$$

$$+ \frac{1}{s}\int_{0}^{t} \sin s(t-y)\left(\frac{1+2e^{-y}}{s+1} - p(s) + (s-1)e^{-y}\right)dy.$$

Integrating by parts, we get

$$I(t) = \int_0^t \sin s(t-y)e^{-y}dy = \frac{\sin st + se^{-t} - s\cos st}{1+s^2}.$$

Then,

$$u(t,s) = \cos st \left\{ \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left(1 - e^{-s^2}\right) + \frac{1}{s+1} \right\} \tag{2.78}$$

$$+\frac{1}{s}\sin st \left\{ \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right) s^2 e^{-s^2} - \frac{1}{s+1} \right\}$$

$$-\left(1 - \frac{1}{s^2}\cos st\right)\frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right) + \frac{1+s^2}{s(1+s)}\left(\frac{\sin st + se^{-t} - s\cos st}{1+s^2}\right).$$

Putting $t = 1$ and using $u(1,s) = \frac{e^{-1}}{s+1}$, we get

$$\cos s \left\{ \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right)\left(1 - e^{-s^2}\right) + \frac{1}{s+1} \right\}$$

$$+\frac{1}{s}\sin s \left\{ \frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right) s^2 e^{-s^2} - \frac{1}{s+1} \right\}$$

$$-\left(1 - \frac{1}{s^2}\cos s\right)\frac{1}{s^2}\left(\frac{1}{1+s} - p(s)\right) + \frac{1+s^2}{s(1+s)}\left(\frac{\sin s + se^{-1} - s\cos s}{1+s^2}\right) = \frac{e^{-1}}{s+1}.$$

From that it follows

$$\left[\frac{1}{1+s} - p(s)\right]\left\{ \frac{\cos s}{s^2}(1 - e^{-s^2}) + \frac{1}{s}\sin s e^{-s^2} - \frac{1}{s^2} + \frac{\cos s}{s^2} \right\} = 0.$$

Since,

$$\cos s(2 - e^{-s^2}) + s\sin s e^{-s^2} - 1 \neq 0,$$

we have that

$$\frac{1}{1+s} - p(s) = 0.$$

Therefore, $p(s) = \frac{1}{1+s}$ and

$$p(x) = \mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\} = e^{-x}. \tag{2.79}$$

Using $p(s)=\frac{1}{1+s}$, we get

$$u(t,s)=\begin{cases}\frac{t}{1+s}, 0\le t\le 1,\\ \\ \frac{t}{1+s}, -1\le t\le 0.\end{cases}=\frac{t}{1+s}. \tag{2.80}$$

Taking the inverce Laplace transform with respect to x , we get

$$u(t,x)=e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=e^{-t}e^{-x}, -1\le t\le 1.$$

Thus, the exact solution of problem (2.70) is

$$(u(t,x),p(x))=\left(e^{-(t+x)},e^{-x}\right).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases}-\frac{\partial^2 u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{n}a_r\frac{\partial^2 u(t,x)}{\partial x_r^2}=p(x)+f(t,x),\\ x=(x_1,...,x_n)\in\overline{\Omega}^+,\ 0<t<T,\\ \\ \frac{\partial u(t,x)}{\partial t}+\sum\limits_{r=1}^{n}a_r\frac{\partial^2 u(t,x)}{\partial x_r^2}=-p(x)+g(t,x),\\ x=(x_1,...,x_n)\in\overline{\Omega}^+,\ -T<t<0,\\ \\ u(-T,x)=\psi(x), u(T,x)=\psi(x), x\in\overline{\Omega}^+,\\ \\ u(t,x)=\alpha(t,x), u_{x_r}(t,x)=\beta(t,x), 1\le r\le n,\\ \\ -T\le t\le T,\quad x\in S^+\end{cases} \tag{2.81}$$

for the multidimensional parabolic-elliptic partial differential equation. Here $a_r>a>0$ and $f(t,x),\left(t\in(0,T), x\in\overline{\Omega}^+\right), g(t,x),\left(t\in(-T,0), x\in\overline{\Omega}^+\right)$, $\varphi(x),\psi(x)\left(x\in\overline{\Omega}^+\right), \alpha(t,x),\beta(t,x)(-T\le t\le T,\quad x\in S^+)$, are given smooth functions. Here and in future Ω^+ is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<\infty, 1\le k\le n)$ with the boundary

$$S^+, \overline{\Omega}^+=\Omega^+\cup S^+.$$

However, Laplace transform method described in solving (2.81) can be used only in the case when (2.81) has constant and polynomial coefficients.

2.3 FOURIER TRANSFORM METHOD

We consider Fourier transform solution of identification problems for parabolic-elliptic differential equations.

Example 2.3.1. Obtain the Fourier transform solution of the following problem source identification problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial u^2(t,x)}{\partial x^2} + u(t,x) = p(x) - e^{-x^2} + \left(-4x^2+2\right) e^{-t-x^2}, \\ 0 < t < 1, x \in \mathbb{R}^1, \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x) - e^{-x^2} + \left(4x^2-2\right) e^{-t-x^2}, \\ -1 < t < 0, x \in \mathbb{R}^1, \\ u(-1,x) = e^{1-x^2},\ u(1,x) = e^{-1-x^2}, x \in \mathbb{R}^1 \end{cases} \tag{2.82}$$

for a one dimensional parabolic-elliptic differential equation.

Solution. Here and in future we denote

$$\mathcal{F}\{u(t,x)\} = u(t,s).$$

Taking the Fourier transform of both sides of the differential equation (2.82) and using initial conditions, we can obtain

$$\begin{cases} -u_{tt}(t,s) - \mathcal{F}\left\{\frac{\partial^2 u(t,s)}{\partial x^2}\right\} + \mathcal{F}\left\{u(t,s)\right\} = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} \\ +e^{-t}\mathcal{F}\left\{(-4x^2+2)e^{-x^2}\right\}, 0 < t < 1, \\ u_t(t,s) + \mathcal{F}\left\{\frac{\partial^2 u(t,s)}{\partial x^2}\right\} - \mathcal{F}\left\{u(t,s)\right\} = -p(s) - \mathcal{F}\left\{e^{-x^2}\right\}, \\ +e^{-t}\mathcal{F}\left\{(4x^2-2)e^{-x^2}\right\}, -1 < t < 0, \\ u(-1,s) = e\mathcal{F}\left\{e^{-x^2}\right\}, u(1,s) = e^{-1}\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases} \tag{2.83}$$

We denote that

$$\mathcal{F}\left\{e^{-x^2}\right\} = q(s).$$

Then

$$\begin{cases} -u_{tt}(t,s) + s^2 u(t,s) + u(t,s) = p(s) - q(s) - \mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\left(e^{-x^2}\right)\right\}, \\ 0 < t < 1, \\ u_t(t,s) - s^2 u(t,s) + u(t,s) = p(s) - q(s) + e^{-t}\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\left(e^{-x^2}\right)\right\}, \\ -1 < t < 0, \\ u(-1,s) = eq(s), u(1,s) = e^{-1}q(s). \end{cases} \tag{2.84}$$

Let $-1 \le t \le 0$. Then we have the identification problem

$$\begin{cases} u_t(t,s) - (s^2-1)u(t,s) \\ = p(s) - q(s) - e^{-t}s^2 q(s), -1 < t < 0, \\ u(-1,s) = eq(s) \end{cases} \tag{2.85}$$

for ordinary differential equations. Solving the problem, we get

$$u(t,s)=e^{(s^2-1)(t+1)}eq(s)+\int\limits_{-1}^{t}e^{(s^2-1)(t-y)}\left(p(s)-q(s)-e^{-y}s^2q(s)\right)dy. \tag{2.86}$$

$$=e^{(s^2-1)(t+1)}eq(s)-\frac{p(s)-q(s)}{s^2-1}\left(1-e^{(s^2-1)(t+1)}\right)+q(s)e^{s^2-1)t}\left[e^{-s^2t}-e^{s^2}\right].$$

Then,

$$u(0,s)=q(s)-\frac{p(s)-q(s)}{s^2-1}\left(1-e^{(s^2-1)}\right), \tag{2.87}$$

$$u_t(0,s)=-q(s)-\frac{p(s)-q(s)}{s^2-1}\left(-(s^2-1)e^{(s^2-1)}\right). \tag{2.88}$$

Now, let $0\le t\le 1$. Applying (2.85) , (2.87) and (2.88) , we get the following initial value problem

$$\begin{cases} u_{tt}(t,s)-(s^2-1)u(t,s)=-\left\{p(s)-q(s)+e^{-t}s^2q(s)\right\} \\ u(0,s)=q(s)-\frac{p(s)-q(s)}{s^2-1}\left(1-e^{(s^2-1)}\right), \\ u_t(0,s)=-q(s)-\frac{p(s)-q(s)}{s^2-1}\left(-(s^2-1)e^{(s^2-1)}\right). \end{cases} \tag{2.89}$$

Applying the D'Alembert's formula , we get

$$u(t,s)=\cosh\sqrt{s^2-1}tu(0,s)+\frac{1}{\sqrt{s^2-1}}\sinh\sqrt{s^2-1}tu_t(0,s) \tag{2.90}$$

$$-\frac{1}{\sqrt{s^2-1}}\int\limits_{0}^{t}\sinh\sqrt{s^2-1}(t-y)\left\{p(s)-q(s)+e^{-y}s^2q(s)\right\}dy.$$

Integrating by parts, we get

$$u(t,s) = \cosh\sqrt{s^2-1}t\left\{q(s) - -\frac{p(s)-q(s)}{s^2-1}\left(1-e^{(s^2-1)}\right)\right\}$$

$$+\frac{\sinh\sqrt{s^2-1}t}{\sqrt{s^2-1}}\left\{-q(s) - \frac{p(s)-q(s)}{s^2-1}\left(-(s^2-1)e^{(s^2-1)}\right)\right\} \tag{2.91}$$

$$+\frac{p(s)-q(s)}{s^2-1}\left(-1+\cosh\sqrt{s^2-1}t\right)$$

$$-\frac{q(s)}{\sqrt{s^2-1}}\left\{\sinh\sqrt{s^2-1}t + \sqrt{s^2-1}\left(e^{-t}-\cosh\sqrt{s^2-1}t\right)\right\}.$$

Putting $t = 1$ and using $u(1,s) = e^{-1}q(s)$, we get

$$\cosh\sqrt{s^2-1}\left\{q(s) - -\frac{p(s)-q(s)}{s^2-1}\left(1-e^{(s^2-1)}\right)\right\}$$

$$+\frac{\sinh\sqrt{s^2-1}}{\sqrt{s^2-1}}\left\{-q(s) - \frac{p(s)-q(s)}{s^2-1}\left(-(s^2-1)e^{(s^2-1)}\right)\right\}$$

$$+\frac{p(s)-q(s)}{s^2-1}\left(-1+\cosh\sqrt{s^2-1}\right) \tag{2.92}$$

$$-\frac{q(s)}{\sqrt{s^2-1}}\left\{\sinh\sqrt{s^2-1} + \sqrt{s^2-1}\left(e^{-1}-\cosh\sqrt{s^2-1}\right)\right\}$$

$$= e^{-1}q(s).$$

Then,

$$\frac{p(s)-q(s)}{s^2-1}\left[-\cosh\sqrt{s^2-1}\left(1-e^{(s^2-1)}\right) - \frac{\sinh\sqrt{s^2-1}}{\sqrt{s^2-1}}\left(-(s^2-1)e^{(s^2-1)}\right)\right. \tag{2.93}$$

$$\left.-1+\cosh\sqrt{s^2-1}\right] = 0.$$

Since

$$-\cosh\sqrt{s^2-1}\left(1-e^{(s^2-1)}\right)-\frac{\sinh\sqrt{s^2-1}}{\sqrt{s^2-1}}\left(-(s^2-1)e^{(s^2-1)}\right)+\left(-1+\cosh\sqrt{s^2-1}\right)\neq 0,$$

we have that

$$\frac{p(s)-q(s)}{s^2-1}=0.$$

Therefore $p(s)=q(s)$ and

$$p(x)=q(x)=\mathcal{F}^{-1}\mathcal{F}\left\{e^{-x^2}\right\}=e^{-x^2}. \tag{2.94}$$

Using $p(s)=\mathcal{F}\left\{e^{-x^2}\right\}$, we get

$$u(t,s)=\begin{cases} q(s)e^{-t}, 0<t<1, \\ \\ q(s)e^{-t}, -1<t<0. \end{cases} = q(s)e^{-t}. \tag{2.95}$$

Taking the inverse Fourier transform with respect to x , we get

$$u(t,x)=\mathcal{F}^{-1}\left\{\mathcal{F}\left\{e^{-(t+x^2)}\right\}\right\}=e^{-(t+x^2)}, -1\leq t\leq 1.$$

Thus , the exact solution of problem (2.82) is

$$(u(t,x),p(x))=\left(e^{-(t+x^2)},e^{-x^2}\right).$$

Note that using similar procedure one can obtain the solution of the following boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}=p(x)+f(t,x), \\ x=(x_1,...,x_n)\in\mathbb{R}^n,\ 0<t<T, \\ \\ \frac{\partial u(t,x)}{\partial t}+\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}=-p(x)+g(t,x), \\ x=(x_1,...,x_n)\in\mathbb{R}^n,\ -T<t<0, \\ \\ u(-T,x)=\psi(x), u(T,x)=\psi(x), x\in\mathbb{R}^n \end{cases} \tag{2.96}$$

for the multidimensional parabolic-elliptic partial differential equation. Here $\alpha_r > \alpha > 0$ and $f(t,x), (t \in (0,T), x \in \mathbb{R}^n), g(t,x), (t \in (-T,0), x \in \mathbb{R}^n), \varphi(x), \psi(x) (x \in \mathbb{R}^n)$ are given smooth functions.
However, Fourier series method described in solving (2.96) can be used only in the case when (2.96) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients or polynomial coefficients. It is well-known that the most general method for solving partial differential equation with depend on t and in the space variables is finite difference method. In the next chapter, we consider the source identification problem for a one dimensional parabolic-elliptic equation. The first and second order of accuracy difference schemes for the numerical solution of this source identification problem for a one dimensional parabolic-elliptic equation is presented. Numerical analysis and discussions are presented.

CHAPTER 3

FINITE DIFFERENCE METHOD OF THE SOLUTION OF SOURCE IDENTIFICATION PROBLEMS FOR PARABOLIC-ELLIPTIC EQUATIONS

In this section, we study the numerical solution of the identification problem for parabolic-elliptic equations

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + f(t,x), \\ 0 < t < 1, 0 < x < \pi, \\ \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = -p(x) + g(t,x), \\ -1 < t < 0, 0 < x < \pi, \\ u(0^+,x) = u(0^-,x), u_t(0^+,x) = u_t(0^-,x), 0 \le x \le \pi, \\ u(-1,x) = \varphi(x), u(1,x) = \psi(x), 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, -1 \le t \le 1, \end{cases} \tag{3.1}$$

where $p(x)$ is an unknown source term. Problem (3.1) has a unique smooth solution $\{u(t,x), p(x)\}$ for the smooth functions $\varphi(x), \psi(x), f(t,x)$ and $g(t,x)$. We construct the first and second order of accuracy difference schemes for the approximate solutions of the identification problem (3.1). We discuss the numerical procedure for implementation of these schemes on the computer. We provide with numerical illustration for simple test problem.

3.1 THE NUMERICAL ALGORITHM

The solution of problem (3.1) can be obtained

$$u(t,x) = w(t,x) + q(x), 0 \leq x \leq \pi, -1 \leq t \leq 1 \tag{3.2}$$

where $q(x)$ is the solution of identification problem

$$-q^{''}(x) = p(x), 0 < x < \pi, q(0) = q(\pi) = 0 \tag{3.3}$$

and $w(t,x)$ is the solution of the nonlocal boundary value problem

$$\begin{cases} -w_{tt} - w_{xx} = f(t,x), 0 < t < 1, 0 < x < \pi, \\ w_t + w_{xx} = g(t,x), -1 < t < 0, 0 < x < \pi, \\ w(0^+,x) = w(0^-,x), w_t(0^+,x) = w_t(0^-,x), 0 \leq x \leq \pi, \\ w(-1,x) - w(1,x) = \varphi(x) - \psi(x), 0 \leq x \leq \pi, \\ u(t,0) = u(t,1) = 0, -1 \leq t \leq 1. \end{cases} \tag{3.4}$$

Note that from (3.1)-(3.3) it follows that

$$p(x) = w_{xx}(1,x) - \psi^{''}(x), 0 \leq x \leq \pi. \tag{3.5}$$

Taking into account all of the above, the following numerical algorithm can be used for the approximate solutions of the identification problem (3.1):

1. Find the approximate solution of the nonlocal boundary value problem (3.4).
2. Approximate the source function $p(x)$ by the formula (3.5).
3. Find the approximate solutions of identification problem (3.3).
4. Find the approximate solution of identification problem (3.1) by the formula (3.2).

For the numerical solution of problem (3.1), we consider grid spaces

$$[-1,1]\,\tau = \{t : t_k = k\tau, -N \leq k \leq N, N\tau = 1, \}$$

$[0,\pi]_h = \{x : x_n = nh, 0 \le n \le M, Mh = \pi.\}$

For the numerical solution of problem (3.1), we present stable two-step difference schemes

$$\begin{cases} -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} = p_n + f(t_k, x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \\ \frac{u_n^k-u_n^{k-1}}{\tau} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} = -p_n + g(t_k, x_n), \\ -N+1 \le k \le 0, 1 \le n \le M-1, \\ \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1}, 0 \le n \le M, \\ u_n^{-N} = e \sin x_n, u_n^N = e^{-1} \sin x_n, 0 \le n \le M, \\ u_0^k = u_M^k = 0, -N \le k \le N \end{cases} \tag{3.6}$$

of the first order of accuracy in t and the second order of accuracy in x and

$$\begin{cases} -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} = p_n + f(t_k, x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \\ \frac{u_n^k-u_n^{k-1}}{\tau} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2} + \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} = -p_n + g(t_k - \frac{\tau}{2}, x_n), \\ -N+1 \le k \le -1, 1 \le n \le M-1, \\ \\ -3u_n^0 + 4u_n^1 - u_n^2 = 3u_n^0 - 4u_n^{-1} + u_n^{-2}, 0 \le n \le M, \\ u_n^{-N} = e \sin x_n, u_n^N = e^{-1} \sin x_n, 0 \le n \le M, \\ u_0^k = u_M^k = 0, -N \le k \le N \end{cases} \tag{3.7}$$

of the second order of accuracy in t and in x. Therefore, in the first step for the approximate solution of nonlocal boundary value problem (3.4) we have the following stable two-step difference schemes

$$\begin{cases} -\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2} - \frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2} = f(t_k, x_n), \\ 1 \leq k \leq N-1, 1 \leq n \leq M-1, \\ \\ \frac{w_n^k-w_n^{k-1}}{\tau} + \frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{h^2} = g(t_k, x_n), \\ -N+1 \leq k \leq 0, 1 \leq n \leq M-1, \\ \\ w_n^1 - w_n^0 = w_n^0 - w_n^{-1}, 0 \leq n \leq M, \\ w_n^{-N} - w_n^N = \left(e - e^{-1}\right) \sin x_n, 0 \leq n \leq M, \\ w_0^k = w_M^k = 0, -N \leq k \leq N \end{cases} \tag{3.8}$$

of the first order of accuracy in t and the second order of accuracy in x and

$$\begin{cases} -\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2} - \frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2} = f(t_k, x_n), \\ 1 \leq k \leq N-1, 1 \leq n \leq M-1, \\ \\ \frac{w_n^k-w_n^{k-1}}{\tau} + \frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{2h^2} + \frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{2h^2} = g(t_k - \frac{\tau}{2}, x_n), \\ -N+1 \leq k \leq -1, 1 \leq n \leq M-1, \\ \\ -3w_n^0 + 4w_n^1 - w_n^2 = 3w_n^0 - 4w_n^{-1} + w_n^{-2}, 0 \leq n \leq M, \\ w_n^{-N} - w_n^N = \left(e - e^{-1}\right) \sin x_n, 0 \leq n \leq M, \\ w_0^k = w_M^k = 0, -N \leq k \leq N \end{cases} \tag{3.9}$$

of the second order of accuracy in t and in x. Difference schemes (3.8) and (3.9) can be written in the matrix form

$$\begin{cases} Aw_{n+1} + Bw_n + Cw_{n-1} = \varphi_n, 1 \leq n \leq M-1, \\ \\ w_0 = w_M = 0, \end{cases} \tag{3.10}$$

where $\widetilde{0}$ is a zero vector and

$$w_n = \begin{bmatrix} w_n^{-N} \\ w_n^{-N+1} \\ w_n^{-N+2} \\ . \\ . \\ . \\ w_n^{0} \\ w_n^{1} \\ w_n^{2} \\ w_n^{3} \\ . \\ . \\ . \\ w_n^{N-1} \\ w_n^{N} \end{bmatrix}_{(2N+1)\times 1} ,$$

$$A = C = \begin{bmatrix}
0 & 0 & 0 & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & & 0 \\
a & 0 & 0 & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & & 0 \\
0 & a & 0 & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & & 0 \\
0 & 0 & a & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & & 0 \\
\cdot & \cdot & \cdot & \cdot & & & & \cdot & & & & & & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & & & \cdot & & & & & & & \cdot \\
\cdot & \cdot & \cdot & & & \cdot & & \cdot & & & & & & & \cdot \\
0 & 0 & 0 & \cdot & \cdot & \cdot & a & 0 & & & \cdot & \cdot & \cdot & & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & & \cdot & b & & \cdot & \cdot & \cdot & & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & & \cdot & & b & \cdot & \cdot & \cdot & & 0 \\
\cdot & \cdot & \cdot & & & & & \cdot & & & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & & & & & 0 & & & & \cdot & & & \cdot \\
\cdot & \cdot & \cdot & & & & & 0 & & & & & \cdot & & \cdot \\
0 & 0 & 0 & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & b & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & & 0 & & & \cdot & \cdot & \cdot & & 0
\end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$, \ \varphi_n = \begin{bmatrix} \psi(x_n) - \varphi(x_n) \\ \tau g(t_{-N+1}, x_n) \\ \tau g(t_{-N+2}, x_n) \\ . \\ . \\ . \\ \tau g(t_1, x_n) \\ \tau g(t_0, x_n) \\ \tau^2 f(t_1, x_n) \\ \tau^2 f(t_2, x_n) \\ . \\ . \\ . \\ \tau^2 f(t_{N-2}, x_n) \\ 0 \end{bmatrix}_{(2N+1)\times 1} ,$$

$$
B = \begin{bmatrix}
1 & 0 & 0 & . & . & . & & 0 & & & . & . & . & & -1 \\
c & d & 0 & . & . & . & & 0 & & & . & . & . & & 0 \\
0 & c & d & . & . & . & & 0 & & & . & . & . & & 0 \\
 & & . & . & & & & & & & & & & & \\
. & . & . & . & . & & & . & & & & & & & . \\
. & . & . & & . & . & & . & & & & & & & . \\
. & . & . & & & . & . & . & & & & & & & . \\
0 & 0 & 0 & . & . & . & c & d & & & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & e & f & e & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & 0 & e & f & e & . & . & & 0 \\
. & . & . & & & & & . & & . & . & . & & & . \\
. & . & . & & & & & . & & & . & . & . & & . \\
. & . & . & & & & & . & & & & . & . & . & . \\
0 & 0 & 0 & . & . & . & & & & & . & . & e & f & e \\
0 & 0 & 0 & . & . & . & -1 & 2 & -1 & & . & . & . & & 0
\end{bmatrix}_{(2N+1)\times(2N+1)}
$$

$a = \frac{1}{h^2}, b = -\frac{1}{h^2}, c = \frac{1}{\tau}, d = -\frac{1}{\tau} - \frac{2}{h^2}, e = -\frac{1}{\tau^2}$ and $f = \frac{2}{\tau^2} + \frac{2}{h^2}$ for difference scheme (3.8) and

$$
A = C = \begin{bmatrix}
0 & 0 & 0 & . & . & . & & 0 & & & . & . & . & & 0 \\
a & a & 0 & . & . & . & & 0 & & & . & . & . & & 0 \\
0 & a & a & . & . & . & & 0 & & & . & . & . & & 0 \\
 & & . & . & . & . & & 0 & & & . & . & . & & 0 \\
. & . & . & . & . & & & . & & & & & & & . \\
. & . & . & & . & . & & . & & & & & & & . \\
. & . & . & & & . & . & . & & & & & & & . \\
0 & 0 & 0 & . & . & . & a & a & & & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & . & b & & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & . & & b & . & . & . & & 0 \\
. & . & . & & & & & . & & & . & & & & . \\
. & . & . & & & & & 0 & & & & . & & & . \\
. & . & . & & & & & 0 & & & & & . & & . \\
0 & 0 & 0 & . & . & . & & 0 & & & . & . & . & b & 0 \\
0 & 0 & 0 & . & . & . & & 0 & & & . & . & . & & 0
\end{bmatrix}_{(2N+1)\times(2N+1)}
$$

$$
B=\begin{bmatrix}
1 & 0 & 0 & . & . & . & & 0 & & & . & . & . & & -1 \\
c & d & 0 & . & . & . & & 0 & & & . & . & . & & 0 \\
0 & c & d & . & . & . & & 0 & & & . & . & . & & 0 \\
 & & . & . & & & & & & & & & & & \\
. & . & . & . & . & & & . & & & & & & & . \\
. & . & . & & . & . & & . & & & & & & & . \\
. & . & . & & & . & . & . & & & & & & & . \\
0 & 0 & 0 & . & . & . & c & d & & & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & e & f & e & . & . & . & & 0 \\
0 & 0 & 0 & . & . & . & & 0 & e & f & e & . & . & & 0 \\
. & . & . & & & & & . & & . & . & . & & & . \\
. & . & . & & & & & . & & & . & . & . & & . \\
. & . & . & & & & & . & & & & . & . & . & . \\
0 & 0 & 0 & . & . & . & & & & & . & . & e & f & e \\
0 & 0 & 0 & . & . & 1 & -4 & 6 & -4 & 1 & . & . & . & & 0
\end{bmatrix}_{(2N+1)\times(2N+1)},
$$

$a=\frac{1}{2h^2}, b=-\frac{1}{2h^2}, c=\frac{1}{\tau}-\frac{1}{h^2}, d=-\frac{1}{\tau}-\frac{1}{h^2}, e=-\frac{1}{\tau^2}, f=\frac{2}{\tau^2}+\frac{2}{h^2},$

$$
\varphi_n = \begin{bmatrix} \psi(x_n) - \varphi(x_n) \\ \tau g(t_{-N+1+\frac{\tau}{2}}, x_n) \\ \tau g(t_{-N+2+\frac{\tau}{2}}, x_n) \\ . \\ . \\ . \\ \tau g(t_{1+\frac{\tau}{2}}, x_n) \\ \tau g(t_{0+\frac{\tau}{2}}, x_n) \\ \tau^2 f(t_1, x_n) \\ \tau^2 f(t_2, x_n) \\ . \\ . \\ . \\ \tau^2 f(t_{N-2}, x_n) \\ 0 \end{bmatrix}_{(2N+1)\times 1}
$$

for difference scheme (3.9). For the solution of the matrix equation (3.10), we use the modified Gauss elimination method. We seek a solution of the matrix equation (3.10) by the following form:

$$
\begin{cases} w_n = \alpha_{n+1} w_{n+1} + \beta_{n+1}, n = M-1, ..., 2, 1 \\ \\ w_M = \widetilde{0} \end{cases} \tag{3.11}
$$

where $\alpha_n (1 \leq n \leq M)$ are $(2N+1)x(2N+1)$ square matrices and $\beta_n (1 \leq n \leq M)$ are $(2N+1)x1$ column vectors, calculated as,

$$
\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \\ \\ \beta_{n+1} = (B + C\alpha_n)^{-1} \left[D\varphi_n - C\beta_n \right] \end{cases} \tag{3.12}
$$

for $n = 1, 2, ...M-1$. Here α_1 is a zero matrix and β_1 is a zero matrix.

In the second step, we obtain $\{p_n\}_1^{M-1}$ by formula

$$
p_n = \frac{w_{n+1}^N - 2w_n^N + w_{n-1}^N}{h^2} - \psi''(x_n), n = 1, 2, ...M-1. \tag{3.13}
$$

In the third step, we obtain $\{q_n\}_0^M$ by formula

$$\begin{cases} p_n = -\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}, 1 \le n \le M-1, \\ \\ \qquad q_0 = q_m = 0 \end{cases} \tag{3.14}$$

or

$$\begin{bmatrix} -2s & s & & & & & \\ s & -2s & s & & & & \\ & s & -2s & s & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & s & -2s & s \\ & & & & & s & -2s \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_{M-2} \\ q_{M-1} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ . \\ p_{M-2} \\ p_{M-1} \end{bmatrix},$$

where $s = -\frac{1}{h^2}$. In the fourth step, we obtain solution of difference schemes (3.8) and (3.9), we will use formula

$$u_n^k = w_n^k + q_n, n = 0, 1, ..., M, k = -N, ..., N. \tag{3.15}$$

For the numerical test, we consider the example when $f(t, x) = -\sin x, g(t, x) = -2e^{-t}\sin x + \sin x, \varphi(x) = e\sin x$ and $\psi(x) = e^{-1}\sin x$.The exact solution of this problem is

$$u(t, x) = e^{-t}\sin x, -1 \le t \le 1, 0 \le x \le \pi, p(x) = \sin x, 0 \le x \le \pi. \tag{3.16}$$

The numerical solutions are computed using the first and second order of accuracy schemes for different values of M and N. We compute the error between the exact solution and numerical solution by formulas

$$\|E_u\|_\infty = \max_{-N\le k\le N, 0\le n\le M} \left|u(t_k, x_n) - u_n^k\right|, \tag{3.17}$$

$$\left\|E_p\right\|_\infty = \max_{1\le n\le M-1} |p(x_n) - p_n|, \tag{3.18}$$

where $u(t_k, x_n)$ is the exact value of $u(t, x)$ at (t_k, x_n) and $p(x_n)$ is the exact value of source $p(x)$ at $x = x_n$; u_n^k and p_n represent the corresponding numerical solutions by these difference schemes. Applying the numerical algorithm on the above, we get the following numerical results:

TABLE 1. **Difference Scheme** (3.8)

Errors	**N = M = 40, 40**	**N = M = 80, 80**	**N = M = 160, 160**
$\|\|E_w\|\|$	0, 0527	0, 0265	0, 0133
$\|\|E_p\|\|$	0, 0521	0, 0264	0, 0132
$\|\|E_u\|\|$	0, 0277	0, 0140	0, 0070

TABLE 2. **Difference Scheme** (3.9)

Errors	**N = M = 40, 40**	**N = M = 80, 80**	**N = M = 160, 160**
$\|\|E_w\|\|$	$1,5271x10^{-4}$	$3,8366x10^{-5}$	$9,6146x10^{-6}$
$\|\|E_p\|\|$	$6,6656x10^{-4}$	$1,6687x10^{-4}$	$4,1742x10^{-5}$
$\|\|E_u\|\|$	$1,7230x10^{-4}$	$4,2859x10^{-5}$	$1,0687x10^{-5}$

Tables 1 and 2 are constructed for the obtained errors in maximum norm of solution of problem (3.1) with difference schemes (3.8) and (3.9), respectively. As it is seen in Table 1 and 2, if N and M are doubled, the value of errors decreases by a factor of approximately $1/2$ for the difference scheme (3.8) and $1/4$ for the difference scheme (3.9), respectively. Thus,

the results show that the second order of accuracy difference scheme (3.9) is more accurate comparing with the first order of accuracy difference scheme (3.8).

CHAPTER 4
CONCLUSIONS

This thesis is seperated to the source identification problem for parabolic-elliptic equations with unknown parameter $p(x)$. The following conslusions are established:

- The literature of direct and invert boundary value problems for parabolic-elliptic equations was studied.
- Fourier series, Laplace transform and Fourier transform methods are used for the solving of six identification problems for parabolic-elliptic equations.
- The first and the second order of accuracy difference schemes are presented for the approach solution of the one dimensional identification problem for the parabolic-elliptic equation with the Dirichlet condition.
- The Matlab application of the numerical solution is added.

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Appendices

APPENDIX 1

MATLAB PROGRAMMING

```
function AA(N,M)
if nargin < 1; end;
    close;close;
  % first order, first equation
  tau=1/N;
  h =pi/M;
    A = zeros(2*N+1,2*N+1);
    for i=2:N+1;
        A(i,i-1)=(1/(h^2));
    end;
    for i=N+2:2*N;
        A(i,i)=-(1/h^2);
    end;
    C=A;
    A;
  B=zeros(2*N+1,2*N+1);
    for i=2:N+1;
        B(i,i)=(1/tau);
        B(i,i-1)=-(1/tau)-(2/(h^2));
    end;
    for i=N+2:2*N;
        B(i,i-1)=-(1/(tau^2));
        B(i,i)=(2/(tau^2))+(2/(h^2));
        B(i,i+1)=-(1/(tau^2));
    end;
  B(1,1)=1;
  B(1,2*N+1)=-1;
  B(2*N+1,N)=-1;
  B(2*N+1,N+1)=2;
  B(2*N+1,N+2)=-1;
  B;
     D=eye(2*N+1,2*N+1);
    for j=1:M+1;
         fii(1,j)=(exp(1)-exp(-1))*sin((j-1)*h);
    for k=2:N+1;
         fii(k,j)=-2*exp(-(k-1-N)*tau)*sin((j-1)*h)+sin((j-1)*h);
    end;
    for k=N+2:2*N;
         fii(k,j)=-sin((j-1)*h);
    end;
         fii(2*N+1,j)=0;
    end;
    alpha{1}=zeros(2*N+1,2*N+1);
    betha{1}=zeros(2*N+1,1);
    for j=2:M;
        Q=inv(B+C*alpha{j-1});
        alpha{j}=-Q*A;
        betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
    end;
    U=zeros(2*N+1,M+1);
    for j=M:-1:1;
        U(:,j)=alpha{j}*U(:,j+1)+betha{j};
```

```
end;
'EXACT SOLUTION OF THIS PROBLEM';
 for j=1:M+1;
    for k=1:2*N+1;
     esw(k,j)=(exp(-(k-1-N)*tau)-1)*sin((j-1)*h);
    end;
 end;
figure;
m(1,1)=min(min(U))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(esw) ; rotate3d ;axis tight;
figure;
surf(m);
hold;
surf(U); rotate3d; axis tight;
title('FIRST ORDER');
maxes=max(max(esw));
maxerror=max(max(abs(esw-U)))
relativeerror=maxerror/maxes;
cevap1 = [maxerror,relativeerror];
```

APPENDIX 2

MATLAB PROGRAMMING

```
function CC(N,M)
if nargin < 1; end;
    close;close;
  % first order, first equation
  tau=1/N;
  h =pi/M;
    A = zeros(2*N+1,2*N+1);
    for i=2:N+1;
        A(i,i)=1/(2*(h^2));
        A(i,i-1)=1/(2*(h^2));
    end;
    for i=N+2:2*N;
        A(i,i)=-1/(h^2);
    end;
    C=A;
    A;
  B=zeros(2*N+1,2*N+1);
    for i=2:N+1;
        B(i,i)=(1/tau)-(1/h^2);
        B(i,i-1)=-(1/tau)-(1/(h^2));
    end;
    for i=N+2:2*N;
        B(i,i-1)=-(1/(tau^2));
        B(i,i)=(2/(tau^2))+(2/(h^2));
        B(i,i+1)=-(1/(tau^2));
    end;
    B(1,1)=1;
    B(1,2*N+1)=-1;
    B(2*N+1,N-1)=1;
    B(2*N+1,N)=-4;
    B(2*N+1,N+1)=6;
    B(2*N+1,N+2)=-4;
    B(2*N+1,N+3)=1;
  B;
     D=eye(2*N+1,2*N+1);
    for j=1:M+1;
         fii(1,j)=(exp(1)-exp(-1))*sin((j-1)*h);
    for k=2:N+1;
         fii(k,j)=-2*(exp((-(k-1-N)*tau)+(tau/2)))*sin((j-1)*h)+sin((j-
1)*h);
    end;
    for k=N+2:2*N;
         fii(k,j)=-sin((j-1)*h);
    end;
         fii(2*N+1,j)=0;
    end;
    alpha{1}=zeros(2*N+1,2*N+1);
    betha{1}=zeros(2*N+1,1);
    for j=2:M;
        Q=inv(B+C*alpha{j-1});
        alpha{j}=-Q*A;
        betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
```

```
end;
w=zeros(2*N+1,M+1);
for j=M:-1:1;
    w(:,j)=alpha{j}*w(:,j+1)+betha{j};
end;
'EXACT SOLUTION OF THIS PROBLEM';
 for j=1:M+1;
    for k=1:2*N+1;
     esw(k,j)=(exp(-(k-1-N)*tau)-1)*sin((j-1)*h);
    end;
 end;
 for j=1:M+1;
     for k=1:2*N+1;
         esU(k,j)=(exp(-(k-1-N)*tau))*sin((j-1)*h);
     end;
 end;
 for j=1:M+1;
          ep(j)=sin((j-1)*h);
     end;
       for j=1:M+1;
      q(j)=-w(2*N+1,j)+((exp(-1))*sin((j-1)*h));
  end;
  for j=2:M;
      p(j)=-(((q(j+1))-(2*q(j))+(q(j-1)))/(h^2));
  end;
    p(1)=0;
   p(M+1)=0;
  for k=1:2*N+1;
  for j=1:M+1;
      U(k,j)=w(k,j)+q(j);
  end;
  end;
figure;
m(1,1)=min(min(w))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(esw) ; rotate3d ;axis tight;
figure;
surf(m);
hold;
surf(w); rotate3d; axis tight;
title('FIRST ORDER');
maxes=max(max(esw));
maxerror=max(max(abs(esw-w)))
 maxerror=max(max(abs(esU-U)))
 maxerror=max(max(abs(ep-p)))
relativeerror=maxerror/maxes;
cevap1 = [maxerror,relativeerror];
```