PARABOLIC TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

AMER MOHAMMED SAEED AHMED

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

AMER MOHAMMED SAEED AHMED
PARABOLIC TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS
NEU
2019

PARABOLIC TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
AMER MIHAMMED SAEED AHMED

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

Amer Mohammed Saeed AHMED: PARABOLIC TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics Department

Examining Committee in Charge:

Prof. Dr. Evren Hinçal	Committee Chairman, Department of Mathematics, NEU
Prof. Dr. Allaberen Ashyralyev	Supervisor, Department of Mathematics, NEU
Assoc. Prof. Dr. Okan Gerçek	Department of Computer Engineering, Girne American University

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Amer Mohammed Saeed AHMED

Signature:

Date:

ACKNOWLEDGMENTS

First and foremost, Glory to my parent, for protecting me, granting me strength and courage to complete my study and in every step of my life. I would like to express my deepest appreciation and thanks to my Supervisor Prof. Dr. Allaberen Ashyralyev. I would like to thank him not only for abetting me on my Thesis but also for encouraging me to look further in the field my career development. His advice on the Thesis as well as on the career I chose has been splendid. In addition, I am very lucky to have a very supportive family and group of friends who have endured my varying emotion during the process of completing this piece of work and I would like to thank them sincerely for their support and help during this period.

To my family...

ABSTRACT

In this thesis, a parabolic type involutory partial differential equation is investigated. Applying Fourier series, Laplace and Fourier transform methods, we obtain the solution of several parabolic type involutory differential problems. Furthermore, the first and second order of accuracy difference schemes for the numerical solution of the initial boundary value problem for one dimensional parabolic type involutory partial differential equation are presented. Numerical results are given.

Keywords: Parabolic involutory differential equations; Fourier series method; Laplace transform solution; Fourier transform solution; Difference scheme

ÖZET

Bu tezde parabolik tipi involüsyon kısmi diferansiyel denklemi incelenmiştir. Fourier serileri, Laplace ve Fourier dönüşüm yöntemlerini uygulayarak, birkaç parabolik tipi involüsyon kısmi diferansiyel problemlerin çözümü elde edilmiştir. Ayrıca, bir boyutlu parabolik tipi involüsyon kısmi diferansiyel başlangıç sınır değer problemin sayısal çözümü için birinci ve ikinci dereceden doğruluklu farkı şemaları sunulmuştur. Sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Parabolik invitatör diferansiyel denklemler; Fourier serisi yöntemi; Laplace dönüşümü çözümü; Fourier dönüşümü çözümü; Fark şeması

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ABBREVIATIONS

DDE: Delay Differential Equation

FDE: Functional Differential Equation

IDE: Involutory Differential Equation

CHAPTER 1
INTRODUCTION

Time delay is a universal phenomenon existing in almost every practical engineering systems (Bhalekar and Patade 2016; Kuralay, 2017; Vlasov and Rautian 2016; Sriram and Gopinathan 2004; Srividhya and Gopinathan 2006). The value of unknown function on one point is not enough for finding of solutions of delay equations. In an experiment measuring the population growth of a species of water fleas, Nesbit (1997), used a DDE model in his study. In simplified form his population equation was

$$N'(t) = aN(t-d) + bN(t).$$

He came into difficulty with this model because he did not have a reasonable history function to carry out the solution of this equation. To overcome this roadblock he proposed to solve a "time reversal" problem in which he sought the solution to functional differential equations. He used a "time reversal" equation to get the juvenile population prior to the beginning time $t = 0$. The time reversal problem is a special case of a type of equation called an involutory differential equation. These are defined as equations of the form

$$y'(t) = f(t; y(t); y(u(t)), y(t_0) = y_0. \tag{1.1}$$

Here $u(t)$ is involutory, that is $u(u(t)) = t$, and t_0 is a fixed point of u. For the "time reversal" problem, we have the simplest involutory differential equation, one in which the deviating argument is $u(t) = -t$. This function is involutory since

$$u(u(t)) = u(-t) = -(-t) = t.$$

We consider the simplest involutory differential equation, one in which the deviating argument is $u(t) = d - t$. This function is involutory since $u(u(t)) = u(d - t)$, which is $d - (d - t) = t$. Note $d - t$ is not the "delay" function as $t - d$.
The existence and uniqueness of a bounded solution was established for a nonlinear delay one dimensional parabolic and hyperbolic differential equations with constant coefficients on $[0, \infty) \times (-\infty, \infty)$ in, S. M. Shah, H. Poorkarimi, J. Wiener, (1986). Note that the approach of

these papers is not applicable for studying a wider class of multidimensional delay nonlinear differential equations and with local and nonlocal boundary conditions.

The discussions of time delay issues are significant due to the presence of delay normally makes systems less effective and less stable. Especially, for hyperbolic systems, only a small time delay may cause the energy of the controlled systems increasing exponentially. The stabilization problem of one dimensional parabolic equation subject to boundary control is concerned in the paper Gordeziani and Avalishvili, 2005. The control input is suffered from time delay. A partial state predictor is designed for the system and undelayed system is deduced. Based on the undelayed system, a feedback control strategy is designed to stabilize the original system. The exact observability of the dual one of the undelayed system is checked. Then it is shown that the system can be stabilized exponentially under the feedback control.

Ashyralyev and Sobolevskii (2001) consider the initial-value problem for linear delay partial differential equations of the parabolic type and give a sufficient condition for the stability of the solution of this initial-value problem. They obtain the stability estimates in Holder norms for the solutions of the problem. Applications, theorems on stability of several types of initial and boundary value problems for linear delay multidimensional parabolic equations are established.

Time delay linear and nonlinear parabolic equations with local and nonlocal boundary conditions have been investigated by many researchers (D. Agirseven, 2012; H. Poorkarimi, J. Wiener, 1999; A. Ashyralyev, A. M. Sarsenbi, 2017; X. Lu, Combined, 1998; H. Bhrawy, M.A. Abdelkawy, 2015; H. Egger, H. W. Eng and M. V. Klibanov, 2004; V. L. Kamynin, 2003; Orazov and M. A. Sadybekov, 2012; D. Guidetti, 2012; M. Ashyralyyeva and M. Ashyraliyev, 2016). Ashyralyev and Agirseven (2014) investigated several types of initial and boundary value problems for linear delay parabolic equations. They give theorems on stability and convergence of difference schemes for the numerical solution of initial and boundary value problems for linear parabolic equations with time delay. As noted above for the solution of delay differential equations we need given values of unknown function from history. Twana Abbas (2019) in his master thesis investigated a Schrödinger type ivolutory partial differential equation. He obtained the solutions of several Schrödinger type ivolutory ordinary and partial differential problems. The first order of accuracy difference scheme for the numerical solution of the initial boundary value problem for involutory one dimensional a Schrödinger type partial differential equations was presented. Moreover, this difference scheme was tested on an example and some numerical results were presented.

In the present study, an involutory parabolic partial differential equation is investigated. Applying tools of the classical integral transform approach we obtain the solution of the six parabolic type involutory differential problems. Furthermore, the first and second order of accuracy difference schemes for the numerical solution of the initial boundary value problem for involutory parabolic type partial differential equations are presented. Then, these difference schemes are tested on an example and some numerical results are presented.

The thesis is organized as follows. Chapter 1 is introduction. In chapter 2, a involutory ordinary differential equations and involutory parabolic type partial differential equations are investigated. Using tools the classical methods we obtain the solution of the several parabolic type involutory differential problems. In chapter 3, numerical analysis and discussions are presented. Finally, chapter 4 is conclusion.

CHAPTER 2

METHODS OF SOLUTION OF PARABOLIC TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

2.1 Involutory ordinary differential equations

In this section we consider the parabolic type involutory ordinary differential equations

$$y_0(t) = f(t; y(t); y(u(t)), y(t_0) = y_0.$$

Here $u(t)$ is involutory, that is $u(u(t)) = t$, and t_0 is a fixed point of u.

Example 2.1. Consider the initial value problem for the first order ordinary differential equation

$$y'(t) = 5y(\pi - t) + 4y(t) \quad \text{on} \quad I = (-\infty, \infty), \quad y(\frac{\pi}{2}) = 1.$$

Solution. We will obtain the initial value problem for the second order differential equation which is equivalent to the given problem. Substituting $\pi - t$ for t into this equation, we get

$$y^{'}(\pi - t) = 5y'(t) + 4y(\pi - t).$$

Differentiating the given equation, we get

$$y^{''}(t) = -5y^{'}(\pi - t) + 4y'(t).$$

Using these equations, we can eliminate the terms of $y(\pi - t)$ and $y^{'}(\pi - t)$. Really, using formula

$$y(\pi - t) = \frac{1}{5}\{y'(t) - 4y(t)\},$$

we get

$$y'(\pi - t) = 5y'(t) + \frac{4}{5}y'(t) - \frac{16}{5}y(t) = \frac{9}{5}y(t) + \frac{4}{5}y'(t).$$

Therefore

$$y^{''}(t) = -5\left\{\frac{9}{5}y(t) + \frac{4}{5}y'(t)\right\} + 4y'(t)$$

or

$$y''(t) = -9y(t).$$

Using initial condition $y(\frac{\pi}{2}) = 1$ and given equation, we get

$$y'(\frac{\pi}{2}) = 5y(\frac{\pi}{2}) + 4y(\frac{\pi}{2}) = 9$$

or

$$y'(\frac{\pi}{2}) = 9.$$

Therefore, we have the following initial value problem for the second order differential equation

$$y''(t) + 9y(t) = 0,\ t \in I, y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = 9.$$

The auxiliary equation is

$$m^2 + 9 = 0.$$

There are two roots $m_1 = 3i$ and $m_2 = -3i$. Therefore, the general solution is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t.$$

Differentiating this equation, we get

$$y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t.$$

Using initial conditions $y(\frac{\pi}{2}) = 1$and $y'(\frac{\pi}{2}) = 9$, we get

$$y(\frac{\pi}{2}) = c_1 \cos \frac{3\pi}{2} + c_2 \sin \frac{3\pi}{2} = -c_2 = 1,$$

$$y'(\frac{\pi}{2}) = -3c_1 \sin \frac{3\pi}{2} + 3c_2 \cos \frac{3\pi}{2} = -3c_1 = 9.$$

From that it follows $c_1 = -3, c_2 = -1$. Therefore, the exact solution of this problem is

$$y(t) = -3 \cos 3t - \sin 3t.$$

Example 2.2. Consider the initial value problem

$$y'(t) = by(\pi - t) + ay(t) + f(t) \quad \text{on } I = (-\infty, \infty), y(\frac{\pi}{2}) = 1. \tag{2.1}$$

Solution. In the same manner, we will obtain equivalent to (2.1) initial value problem for the second order differential equation. Differentiating equation (2.1), we get

$$y''(t) = -by'(\pi - t) + ay'(t) + f'(t).$$

Substituting $\pi - t$ for t into equation (2.1), we get

$$y'(\pi - t) = by(t) + ay(\pi - t) + f(\pi - t).$$

Using these equations, we can eliminate the $y(\pi - t)$ and $y'(\pi - t)$ terms. Actually, using formula

$$y(\pi - t) = \frac{1}{b}\{y'(t) - ay(t) - f(t)\},$$

we get

$$y'(\pi - t) = by(t) + \frac{a}{b}y'(t) - \frac{a^2}{b}y(t) - \frac{a}{b}f(t) + f(\pi - t)$$

$$= \frac{b^2 - a^2}{b}y(t) + \frac{a}{b}y'(t) - \frac{a}{b}f(t) + f(\pi - t).$$

Therefore

$$y''(t) = (a^2 - b^2)y(t) + af(t) - bf(\pi - t) + f'(t)$$

or

$$y''(t) - (a^2 - b^2)y(t) = af(t) - bf(\pi - t) + f'(t).$$

Putting initial condition $y(\frac{\pi}{2}) = 1$ into equation (2.1), we get

$$y'(\frac{\pi}{2}) = a + b + f(\frac{\pi}{2}).$$

We denote

$$F(t) = af(t) - bf(\pi - t) + f'(t).$$

Then, we have the following initial value problem for the second order ordinary differential equation

$$y^{''}(t) + (b^2 - a^2)y(t) = F(t), t \in I, y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = a + b + f(\frac{\pi}{2}). \tag{2.2}$$

Now, we obtain the solution of problem (2.2). There are three cases:$b^2 - a^2 > 0$, $b^2 - a^2 = 0$, $b^2 - a^2 < 0$.

In the first case $b^2 - a^2 = m^2 > 0$. Substituting m^2 for $b^2 - a^2$ into equation (2.2), we get

$$y^{''}(t) + m^2 y(t) = F(t).$$

We will obtain Laplace transform solution of problem (2.2). Here and in future

$$u(s) = L\{u(t)\}.$$

Applying the Laplace transform, we get

$$s^2 y(s) - sy(0) - y'(0) + m^2 y(s) = F(s)$$

or

$$(s^2 + m^2)y(s) = sy(0) + y'(0) + F(s).$$

Then,

$$y(s) = \frac{s}{s^2 + m^2}y(0) + \frac{1}{s^2 + m^2}y'(0) + \frac{1}{s^2 + m^2}F(s).$$

Applying formulas

$$L\{\cos mt\} = \frac{s}{s^2 + m^2},$$

$$L\{\sin mt\} = \frac{m}{s^2 + m^2},$$

$$L\{(f * g)(t)\} = L\left\{\int_0^t f(p)g(t-p)dp\right\} = L\{f(t)\}L\{g(t)\}, \tag{2.3}$$

we get,

$$y(s) = L\{\cos mt\}y(0) + \frac{1}{m}L\{\sin mt\} + \frac{1}{m}L\left\{\int_0^t \sin(m(t-p))\,F(p)dp\right\}.$$

Taking the inverse Laplace transform, we get

$$y(t) = \cos(mt)\,y(0) + \frac{1}{m}\sin(mt)\,y'(0) + \frac{1}{m}\int_0^t \sin(m(t-p))\,F(p)dp.$$

Now, we obtain $y(0)$ and $y'(0)$. Taking the derivative, we get

$$y'(t) = -m\sin(mt)\,y(0) + \cos(mt)\ \ y'(0) + \int_0^t \cos(m(t-p))\,F(p)dp.$$

Putting $F(p) = af(p) - bf(\pi - p) + f^{'}(p)$, we get

$$y(t) = \cos(mt)\,y(0) + \frac{1}{m}\sin(mt)\,y'(0)$$

$$+\frac{1}{m}\int_0^t \sin m(t-p)\,[af(p) - bf(\pi - p) + f'(p)]\,dp, \tag{2.4}$$

$$y'(t) = -m\sin(mt)\ \ y(0) + \cos(mt)\ \ y'(0)$$

$$+\int_0^t \cos(m(t-p))\,[af(p) - bf(\pi - p) + f'(p)]\,dp. \tag{2.5}$$

Substituting $\frac{\pi}{2}$ for t into equations (2.4) and (2.5) we get

$$y(\frac{\pi}{2}) = \cos m\frac{\pi}{2}\ y(0) + \frac{1}{m}\sin m\frac{\pi}{2}\ y'(0)$$

$$+\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin m(\frac{\pi}{2} - p)\,[af(p) - bf(\pi - p) + f'(p)]\,dp,$$

$$y'(\frac{\pi}{2}) = -m\sin m\frac{\pi}{2}\ y(0) + \cos m\frac{\pi}{2}\ y'(0)$$

$$+\int_0^{\frac{\pi}{2}} \cos m(\frac{\pi}{2}-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

Applying initial conditions $y(\frac{\pi}{2})=1, y'(\frac{\pi}{2})=a+b+f(\frac{\pi}{2})$, we obtain

$$\begin{cases} \cos\left(\frac{m\pi}{2}\right)\, y(0)+\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\, y'(0)=1-\alpha_1, \\ \\ -m\sin\left(\frac{m\pi}{2}\right)\, y(0)+\cos\left(\frac{m\pi}{2}\right)\, y'(0)=a+b+f(\frac{\pi}{2})-\alpha_2. \end{cases}$$

Here

$$\alpha_1=\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin\Big(m(\frac{\pi}{2}-p)\Big)[af(p)-bf(\pi-p)+f'(p)]\,dp,$$

$$\alpha_2=\int_0^{\frac{\pi}{2}} \cos\Big(m(\frac{\pi}{2}-p)\Big)[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

Since

$$\Delta=\begin{vmatrix} \cos\left(\frac{m\pi}{2}\right) & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ -m\sin\left(\frac{m\pi}{2}\right) & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix}=\cos^2 m\frac{\pi}{2}+\sin^2 m\frac{\pi}{2}=1\neq 0,$$

we have that

$$y(0)=\frac{\Delta_0}{\Delta}=\begin{vmatrix} 1-\alpha_1 & \frac{1}{m}\sin\left(\frac{m\pi}{2}\right) \\ a+b+f(\frac{\pi}{2})-\alpha_2 & \cos\left(\frac{m\pi}{2}\right) \end{vmatrix}$$

$$=\cos\left(\frac{m\pi}{2}\right)[1-\alpha_1]-\frac{1}{m}\sin\left(\frac{m\pi}{2}\right)\Big[a+b+f(\frac{\pi}{2})-\alpha_2\Big],$$

$$y'(0)=\frac{\Delta_1}{\Delta}=\begin{vmatrix} \cos\left(\frac{m\pi}{2}\right) & 1-\alpha_1 \\ -m\sin\left(\frac{m\pi}{2}\right) & a+b+f(\frac{\pi}{2})-\alpha_2 \end{vmatrix}$$

$$=\cos\left(\frac{m\pi}{2}\right)\Big[a+b+f(\frac{\pi}{2})-\alpha_2\Big]+m\sin\left(\frac{m\pi}{2}\right)[1-\alpha_1]\,.$$

Putting $y(0)$ and $y'(0)$ into equation (2.4), we get

$$y(t)=\cos(mt)\Bigg\{\cos\left(\frac{m\pi}{2}\right)\Bigg\{1-\frac{1}{m}\int_0^{\frac{\pi}{2}} \sin\Big(m(\frac{\pi}{2}-p)\Big)$$

$$\times [af(p) - bf(\pi - p) + f'(p)]\, dp\}\} - \frac{1}{m} \sin\left(\frac{m\pi}{2}\right)$$

$$\times \left\{ a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp \right\}\Bigg\}$$

$$+\frac{1}{m} \sin(mt) \left\{ \cos\left(\frac{m\pi}{2}\right) \left\{ a + b + f(\frac{\pi}{2}) \right.\right.$$

$$\left. - \int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp \right\}$$

$$\left. + m \sin\left(\frac{m\pi}{2}\right) \left\{ 1 - \frac{1}{m} \int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp \right\}\right\}$$

$$+\frac{1}{m} \int_0^{t} \sin(m(t - p)) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$= \cos(mt) \cos\left(\frac{m\pi}{2}\right) + \sin(mt) \sin\left(\frac{m\pi}{2}\right)$$

$$+\frac{1}{m} \left[-\cos(mt) \sin\left(\frac{m\pi}{2}\right) + \sin(mt) \cos\left(\frac{m\pi}{2}\right) \right] \left[a + b + f(\frac{\pi}{2}) \right]$$

$$-\frac{1}{m} \cos(mt) \cos\left(\frac{m\pi}{2}\right) \int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$+\frac{1}{m} \cos(mt) \sin\left(\frac{m\pi}{2}\right) \int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$-\frac{1}{m} \sin(mt) \cos\left(\frac{m\pi}{2}\right) \int_0^{\frac{\pi}{2}} \cos\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$-\frac{1}{m} \sin(mt) \sin\left(\frac{m\pi}{2}\right) - \frac{1}{m} \int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2} - p)\right) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$+\frac{1}{m} \int_0^{t} \sin(m(t - p)) [af(p) - bf(\pi - p) + f'(p)]\, dp$$

$$-\cos(mt)\,\frac{1}{m}\int_0^{\frac{\pi}{2}}\sin\left(m(\frac{\pi}{2}-p)\right)[-af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$-\frac{1}{m}\sin(mt)\int_0^{\frac{\pi}{2}}\cos\left(m(\frac{\pi}{2}-p)\right)[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_0^{t}\sin m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$=\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[\{a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\cos m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\sin m(\frac{\pi}{2}-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\sin m(t-\frac{\pi}{2})\int_0^{\frac{\pi}{2}}\cos m(\frac{\pi}{2}-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_0^{t}\sin m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$=\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_0^{\frac{\pi}{2}}\sin m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_0^{t}\sin m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

Therefore, the exact solution of this problem is

$$y(t)=\cos m(t-\frac{\pi}{2})+\frac{1}{m}\sin m(\frac{\pi}{2}-t)\left[a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_t^{\frac{\pi}{2}}\sin m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

In the second case $b^2-a^2=0$. Then,

$$y''(t) = F(t).$$

Applying the Laplace transform, we get

$$s^2 y(s) - sy(0) - y'(0) = F(s).$$

Then

$$y(s) = y(0)L\{1\} + y'(0)L\{t\} + L\{t\}\, L\{F(t)\}$$

Taking the inverse Laplace transform, we get

$$y(t) = y(0) + ty'(0) + \int_0^t (t-p)F(p)dp. \qquad (2.6)$$

From that it follows

$$y'(t) = y'(0) + \int_0^t F(p)\, dp.$$

Applying initial conditions $y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = a + b + f(\frac{\pi}{2}), F(p) = af(p) - bf(\pi - p) + f'(p)$, we obtain

$$1 = y(\frac{\pi}{2}) = y(0) + \frac{\pi}{2}\, y'(0) + \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - p\right)\left[af(p) - bf(\pi - p) + f'(p)\right] dp,$$

$$a + b + f(\frac{\pi}{2}) = y'(\frac{\pi}{2}) = y'(0) + \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f'(p)\right] dp.$$

Therefore,

$$y'(0) = a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f'(p)\right] dp,$$

$$y(0) = 1 - \frac{\pi}{2}\left\{a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f'(p)\right] dp\right\}$$

$$- \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - p\right)\left[af(p) - bf(\pi - p) + f'(p)\right] dp.$$

Putting $y(0)$ and $y'(0)$ into equation (2.6), we get

$$y(t) = 1 - \frac{\pi}{2}\left\{a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp\right\}$$

$$- \int_0^{\frac{\pi}{2}} (\frac{\pi}{2} - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp$$

$$+t\left\{a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp\right\}$$

$$+ \int_0^{t} (t - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp = 1 + \left(t - \frac{\pi}{2}\right)$$

$$\times\left\{a + b + f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp\right\}$$

$$- \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - p\right)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp$$

$$+ \int_0^{t} (t - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp$$

$$= 1 + \left(t - \frac{\pi}{2}\right)\left[a + b + f(\frac{\pi}{2})\right]$$

$$- \int_0^{\frac{\pi}{2}} (t - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp$$

$$+ \int_0^{t} (t - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp$$

$$= 1 + \left(t - \frac{\pi}{2}\right)\left[a + b + f(\frac{\pi}{2})\right]$$

$$- \int_t^{\frac{\pi}{2}} (t - p)\left[af(p) - bf(\pi - p) + f^{'}(p)\right] dp.$$

In the third case $b^2 - a^2 = m^2 < 0$. Substituting $-m^2$ for $b^2 - a^2$ into equation (2.2), we get

$$y''(t) - m^2 y(t) = F(t).$$

Applying Laplace transform, we get

$$s^2 y(s) - sy(0) - y'(0) - m^2 y(s) = F(s)$$

or

$$y(s) = \frac{s}{s^2 - m^2} y(0) + \frac{1}{s^2 - m^2} y'(0) + \frac{1}{s^2 - m^2} F(s).$$

Applying (2.3) and formulas

$$L\{\cosh mt\} = \frac{s}{s^2 - m^2},$$

$$L\{\sinh mt\} = \frac{m}{s^2 - m^2},$$

we get

$$y(s) = L\{\cosh mt\} y(0) + \frac{1}{m} L\{\sinh mt\} + \frac{1}{m} L\left\{\int_0^t \sinh(m(t-p)) F(p) dp\right\}.$$

Taking the inverse Laplace transform, we get

$$y(t) = \cosh(mt) y(0) + \frac{1}{m} \sinh(mt) y'(0) + \frac{1}{m} \int_0^t \sinh(m(t-p)) F(p) dp.$$

Now, we obtain $y(0)$ and $y'(0)$. Taking the derivative, we get

$$y'(t) = m \sinh(mt) y(0) + \cosh(mt)\ y'(0) + \int_0^t \cosh(m(t-p)) F(p) dp.$$

Putting $F(p) = af(p) - bf(\pi - p) + f'(p)$, we get

$$y(t) = \cosh(mt) y(0) + \frac{1}{m} \sinh(mt) y'(0) \tag{2.7}$$

$$+ \frac{1}{m} \int_0^t \sinh(m(t-p)) \left[af(p) - bf(\pi - p) + f'(p)\right] dp,$$

$$y'(t) = m \sinh(mt)\ y(0) + \cosh(mt)\ y'(0)$$

$$+\int_0^t \cosh\left(m(t-p)\right)\left[af(p)-bf(\pi-p)+f'(p)\right]dp.$$

Substituting $\frac{\pi}{2}$ for t into equations (*2.7*) and (**??**), we get

$$y(\frac{\pi}{2}) = \cosh m\frac{\pi}{2}\, y(0) + \frac{1}{m}\sinh m\frac{\pi}{2}\, y'(0)$$

$$+\frac{1}{m}\int_0^{\frac{\pi}{2}} \sinh m(\frac{\pi}{2}-p)\left[af(p)-bf(\pi-p)+f'(p)\right]dp,$$

$$y'(\frac{\pi}{2}) = m\sinh m\frac{\pi}{2}\, y(0) + \cosh m\frac{\pi}{2}\, y'(0)$$

$$+\int_0^{\frac{\pi}{2}} \cosh m(\frac{\pi}{2}-p)\left[af(p)-bf(\pi-p)+f'(p)\right]dp.$$

Applying initial conditions $y(\frac{\pi}{2}) = 1, y'(\frac{\pi}{2}) = a+b+f(\frac{\pi}{2})$, we obtain

$$\begin{cases} \cosh\left(\frac{m\pi}{2}\right)\, y(0) + \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)\, y'(0) = 1-\alpha_1, \\ m\sinh\left(\frac{m\pi}{2}\right)\, y(0) + \cosh\left(\frac{m\pi}{2}\right)\, y'(0) = \left\{a+b+f(\frac{\pi}{2})\right\} - \alpha_2. \end{cases}$$

Here

$$\alpha_1 = \frac{1}{m}\int_0^{\frac{\pi}{2}} \sinh\left(m(\frac{\pi}{2}-p)\right)\left[af(p)-bf(\pi-p)+f'(p)\right]dp,$$

$$\alpha_2 = \int_0^{\frac{\pi}{2}} \cosh\left(m(\frac{\pi}{2}-p)\right)\left[af(p)-bf(\pi-p)+f'(p)\right]dp.$$

Since

$$\Delta = \begin{vmatrix} \cosh\left(\frac{m\pi}{2}\right) & \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right) \\ m\sinh\left(\frac{m\pi}{2}\right) & \cosh\left(\frac{m\pi}{2}\right) \end{vmatrix} = \cosh^2 m\frac{\pi}{2} - \sinh^2 m\frac{\pi}{2} = 1 \neq 0,$$

we have that

$$y(0) = \frac{\Delta_0}{\Delta} = \begin{vmatrix} 1-\alpha_1 & \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right) \\ \{a+b+f(\frac{\pi}{2})\} - \alpha_2 & \cosh\left(\frac{m\pi}{2}\right) \end{vmatrix}$$

$$= \cosh\left(\frac{m\pi}{2}\right)[1-\alpha_1] - \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)\left[\left\{a+b+f(\frac{\pi}{2})\right\} - \alpha_2\right],$$

$$y'(0) = \frac{\Delta_1}{\Delta} = \begin{vmatrix} \cosh\left(\frac{m\pi}{2}\right) & 1-\alpha_1 \\ m\sinh\left(\frac{m\pi}{2}\right) & \left\{a+b+f(\frac{\pi}{2})\right\} - \alpha_2 \end{vmatrix}$$

$$= \cosh\left(\frac{m\pi}{2}\right)\left[\left\{a+b+f(\frac{\pi}{2})\right\} - \alpha_2\right] - m\sinh\left(\frac{m\pi}{2}\right)[1-\alpha_1].$$

Putting $y(0)$ and $y'(0)$ into equation (2.7), we get

$$y(t) = \cosh(mt)\left\{\cosh\left(\frac{m\pi}{2}\right)\left\{1 - \frac{1}{m}\int_0^{\frac{\pi}{2}} \sinh\left(m(\frac{\pi}{2}-p\right)\right.\right.$$

$$\times [af(p) - bf(\pi-p) + f'(p)]\,dp\}\} - \frac{1}{m}\sinh\left(\frac{m\pi}{2}\right)$$

$$\left.\times\left\{a+b+f(\frac{\pi}{2}) - \int_0^{\frac{\pi}{2}} \cosh\left(m(\frac{\pi}{2}-p)\right)[af(p) - bf(\pi-p) + f'(p)]\,dp\right\}\right\}$$

$$+\frac{1}{m}\sinh(mt)\left\{\cosh\left(\frac{m\pi}{2}\right)\left\{a+b+f(\frac{\pi}{2})\right.\right.$$

$$\left.-\int_0^{\frac{\pi}{2}} \cosh\left(m(\frac{\pi}{2}-p)\right)[af(p) - bf(\pi-p) + f'(p)]\,dp\right\}$$

$$\left.-m\sinh\left(\frac{m\pi}{2}\right)\left\{1 - \frac{1}{m}\int_0^{\frac{\pi}{2}} \sin\left(m(\frac{\pi}{2}-p)\right)[af(p) - bf(\pi-p) + f'(p)]\,dp\right\}\right\}$$

$$+\frac{1}{m}\int_0^{t} \sin(m(t-p))[af(p) - bf(\pi-p) + f'(p)]\,dp$$

$$= \cosh(mt)\cosh\left(\frac{m\pi}{2}\right) - \sinh(mt)\sinh\left(\frac{m\pi}{2}\right)$$

$$-\frac{1}{m}\left[\cosh(mt)\sinh\left(\frac{m\pi}{2}\right) + \frac{1}{m}\sinh(mt)\cosh\left(\frac{m\pi}{2}\right)\right]\left\{a+b+f(\frac{\pi}{2})\right\}$$

$$-\frac{1}{m}\cosh(mt)\cosh\left(\frac{m\pi}{2}\right)\int_0^{\frac{\pi}{2}} \sinh\left(m(\frac{\pi}{2}-p)\right)[af(p) - bf(\pi-p) + f'(p)]\,dp$$

$$-\frac{1}{m}\cosh(mt)\sin(\frac{m\pi}{2})\int_0^{\frac{\pi}{2}} \cosh\left(m(\frac{\pi}{2}-p)\right)[af(p) - bf(\pi-p) + f'(p)]\,dp$$

$$-\frac{1}{m}\sinh(mt)\cosh\left(\frac{m\pi}{2}\right)\int_{0}^{\frac{\pi}{2}}\cosh\left(m(\frac{\pi}{2}-p)\right)[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$-\frac{1}{m}\sinh(mt)\sinh\left(\frac{m\pi}{2}\right)\int_{0}^{\frac{\pi}{2}}\sin\left(m(\frac{\pi}{2}-p)\right)[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_{0}^{t}\sin\left(m(t-p)\right)[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\cosh m(t-\frac{\pi}{2})\int_{0}^{\frac{\pi}{2}}\sinh m(\frac{\pi}{2}-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$-\frac{1}{m}\sinh m(t-\frac{\pi}{2})\int_{0}^{\frac{\pi}{2}}\cosh m(\frac{\pi}{2}-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_{0}^{t}\sinh m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_{0}^{\frac{\pi}{2}}\sinh m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp$$

$$+\frac{1}{m}\int_{0}^{t}\sinh m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

Therefore, the exact solution of this problem is

$$y(t)=\cosh m(t-\frac{\pi}{2})-\frac{1}{m}\sinh m(\frac{\pi}{2}-t)\left[a+b+f(\frac{\pi}{2})\right]$$

$$-\frac{1}{m}\int_{t}^{\frac{\pi}{2}}\sinh m(t-p)\,[af(p)-bf(\pi-p)+f'(p)]\,dp.$$

2.2 Parabolic Type Involutory Partial Differential Equations

It is known that initial value problems for parabolic type involutory partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

First, we consider Fourier series method for solution of problems for parabolic type involutory partial differential equations.

Example 3.1. Consider the initial-boundary-value problem for parabolic type involutory partial differential equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) \\ = (-1+a)e^{-t}\sin(x) + be^{t}\sin(x), \\ x \in (0,\pi), -\infty < t < \infty, \\ u(0,x) = \sin(x), \ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, t \in (-\infty,\infty). \end{cases} \tag{2.8}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, \ \ 0 < x < \pi, \ u(0) = u(\pi) = 0$$

generated by the space operator of problem (2.8). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = k^2, \ u_k(x) = \sin kx, \ k = 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem by formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx,$$

where $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$\sum_{k=1}^{\infty} A_k'(t)\sin kx + a\sum_{k=1}^{\infty} k^2 A_k(t)\sin kx + b\sum_{k=1}^{\infty} k^2 A_k(-t)\sin kx$$

$$= (-1+a)\,e^{-t}\sin(x) + be^{t}\sin(x)\,, x\in(0,\pi)\,, -\infty<t<\infty,$$

$$\sum_{k=1}^{\infty} A_k(0)\sin kx = \sin(x), x\in[0,\pi].$$

Equating coefficients $\sin kx, k = 1, 2, ...$ to zero, we get the initial value problems

$$\begin{cases} A_1'(t) + aA_1(t) + bA_1(-t) = (-1+a)\,e^{-t} + be^{t}, -\infty<t<\infty, \\ \\ A_1(0) = 1 \end{cases} \tag{2.9}$$

$$\begin{cases} A_k'(t) + ak^2A_1(t) + bk^2A_k(-t) = 0, -\infty<t<\infty, k\neq 1, \\ \\ A_k(0) = 0 \end{cases} \tag{2.10}$$

for involutory ordinary differential equations. We will obtain $A_1(t)$. The equivalent to (2.9) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.9), we get

$$A_1''(t) + aA_1'(t) - bA_1'(-t) = (1-a)e^{-t} + be^{t}. \tag{2.11}$$

Putting $-t$ instead of t in (2.9), we get

$$A_1'(-t) + aA_1(-t) + bA_1(t) = (-1+a)\,e^{t} - be^{-t}. \tag{2.12}$$

Multiplying equation (2.12) by b, we get

$$bA_1'(-t) + abA_1(-t) + b^2A_1(t) = b\,(-1+a)\,e^{t} + b^2e^{-t}.$$

Adding last equation with (2.11), we get

$$A_1''(t) + aA_1'(t) + abA_1(-t) + b^2A_1(t) = (1-a+b^2)e^{-t} + abe^{t}. \tag{2.13}$$

Multiplying equation (2.9) by $(-a)$, we get

$$-aA_1'(t) - a^2A_1(t) - abA_1(-t) = \left(a-a^2\right)e^{-t} - abe^{t}.$$

Then adding these equations, we get

$$A_1''(t) + \left(b^2 - a^2\right) A_1(t) = \left(b^2 - a^2 + 1\right) e^{-t}. \tag{2.14}$$

Using equation (2.9) and $A_1(0) = 1$, we get

$$A_1'(0) + aA_1(0) + bA_1(0) = -1 + a + b,$$

$$A_1'(0) = -1.$$

So, we have the following problem

$$A_1''(t) + \left(b^2 - a^2\right) A_1(t) = \left(1 + b^2 - a^2\right) e^{-t}, \quad -\infty < t < \infty, \quad A_1(0) = 1, A_1'(0) = -1.$$

There are three cases : $b^2 - a^2 > 0, \quad b^2 - a^2 = 0, \quad b^2 - a^2 < 0.$
In the first case $b^2 - a^2 = m^2 > 0$. Substituting m^2 for$b^2 - a^2$ into equation (2.14), we get

$$A_1''(t) + m^2 A_1(t) = \left(m^2 + 1\right) e^{-t}. \tag{2.15}$$

We will obtain Laplace transform solution of problem (2.15). We have that

$$s^2 A_1(s) - sA_1(0) - A_1'(0) + m^2 A_1(s) = \left(m^2 + 1\right) e^{-s}$$

or

$$(s^2 + m^2)A(s) = sA_1(0) + A_1'(0) + \left(m^2 + 1\right) e^{-s}.$$

Then,

$$A(s) = \frac{s}{s^2 + m^2} A_1(0) + \frac{1}{s^2 + m^2} A_1'(0) + \frac{1}{s^2 + m^2}\left(m^2 + 1\right) e^{-s}.$$

Applying formulas

$$L\{\cos mt\} = \frac{s}{s^2 + m^2}, \tag{2.16}$$

$$\frac{1}{m} L\{\sin mt\} = \frac{1}{m}\frac{m}{s^2 + m^2}, \tag{2.17}$$

$$L\{(f * g)(t)\} = L\left\{ \int_0^t f(p)g(t - p)dp \right\} = L\{f(t)\}L\{g(t)\}, \tag{2.18}$$

we get

$$A_1(s) = L\{\cos mt\}A_1(0) + \frac{1}{m}L\{\sin mt\} + \frac{m^2+1}{m}L\left\{\int_0^t \sin m(t-p)e^{-s}\right\}.$$

Taking the inverse Laplace transorm, we get

$$A_1(t) = \cos(mt)\,A_1(0) + \frac{1}{m}\sin(mt)\,A_1'(0) + \frac{\left(m^2+1\right)}{m}\int_0^t \sin m(t-p)e^{-p}dp. \qquad (2.19)$$

Applying initial conditions $A_1(0) = 1, A_1'(0) = -1$, we obtain

$$A_1(t) = \cos(mt) - \frac{1}{m}\sin(mt) + \frac{\left(m^2+1\right)}{m}\int_0^t \sin(m(t-p))\,e^{-p}dp.$$

We denote

$$I = \int \sin(m(t-p)e^{-p}dp.$$

We have that

$$I = -\sin m(t-p)e^{-p} - m\int \cos m(t-p)e^{-p}dp$$

$$= -\sin m(t-p)e^{-p} + m\cos m(t-p)e^{-p} - m^2\int \sin m(t-p)e^{-p}dp.$$

Therefore,

$$I\left(m^2+1\right) = -\sin m(t-p)e^{-p} + m\cos m(t-p)e^{-p}$$

or

$$I = \frac{1}{m^2+1}\left\{-\sin m(t-p)e^{-p} + m\cos m(t-p)e^{-p}\right\}. \qquad (2.20)$$

From that it follows

$$\int_0^t \sin m(t-p)e^{-p}dp = \frac{1}{m^2+1}\left\{me^{-t} + \sin mt - m\cos mt\right\}.$$

Then,

$$A_1(t) = \cos(mt) - \frac{1}{m}\sin(mt) + \frac{\left(m^2+1\right)}{m}\left\{\frac{1}{m^2+1}\left\{me^{-t} + \sin mt - m\cos mt\right\}\right\}$$

$$\cos(mt) - \frac{1}{m}\sin(mt) + e^{-t} + \frac{1}{m}\sin mt - \cos mt = e^{-t}.$$

Therefore

$$A_1(t) = e^{-t}.$$

It is easy to see that $A_1(t) = e^{-t}$ for $b^2 - a^2 = 0$, $b^2 - a^2 < 0$.

Now we will obtain $A_k(t)$ for $k \neq 1$. The equivalent to (2.10) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.10), we get

$$A_k''(t) + ak^2 A_k'(t) - bk^2 A_k'(-t) = 0. \tag{2.21}$$

Putting $-t$ instead of t in (2.10), we get

$$A_k'(-t) + ak^2 A_k(-t) + bk^2 A_k(t) = 0. \tag{2.22}$$

Multiplying equation (2.22) by bk^2, we get

$$bk^2 A_k'(-t) + abk^4 A_k(-t) + b^2k^4 A_k(t) = 0.$$

Adding last equation with (2.21), we get

$$A_k''(t) + ak^2 A_k'(t) + abk^4 A_k(-t) + b^2k^4 A_k(t) = 0.$$

Multiplying equation (2.10) by $(-ak^2)$, we get

$$-ak^2 A_k'(t) - a^2k^4 A_k(t) - abk^4 A_k(-t) = 0.$$

Then from these equations, we get

$$A_k''(t) + \left(b^2 - a^2\right)k^4 A_k(t) = 0.$$

Using equation (2.10) and $A_k(0) = 0$, we get

$$A_k'(0) + ak^2 A_k(0) + bk^2 A_k(-0) = 0$$

or

$$A_k'(0) = 0.$$

We have the following problem

$$A_k''(t) + \left(b^2 - a^2\right) k^4 A_k(t) = 0, \quad A_k(0) = 0, A_k'(0) = 0.$$

From that it follows $A_k(t) = 0, k \neq 1$. In the same manner $A_k(t) = 0, k \neq 1$ for $b^2 - a^2 = 0$ and $b^2 - a^2 < 0$.

Therefore, the exact solution of problem (2.8) is

$$u(t,x) = A_1(t) \sin x = e^{-t} \sin x.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \overline{\Omega}, d \geq 0, \\ u(t,x) = 0, x \in S, t \in (-\infty, \infty) \end{cases} \tag{2.23}$$

for the multidimensional involutory parabolic type equation. Assume that $a_r > a_0 > 0$ and $f(t,x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right), \varphi(x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary

$$S, \overline{\Omega} = \Omega \cup S.$$

However Fourier series method described in solving (2.23) can be used only in the case when (2.23) has constant coefficients.

Example 3.2. Consider the initial-boundary-value problem for parabolic type involutory partial differential equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = (-1+a)e^{-t}\cos(x) + be^{t}\cos(x), \\ x \in (0,\pi), -\infty < t < \infty, \\ u(0,x) = \cos(x), \ x \in [0,\pi], \\ u_x(t,0) = u_x(t,\pi) = 0, t \in (-\infty,\infty). \end{cases} \tag{2.24}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, \ \ 0 < x < \pi, \ u_x(0) = u_x(\pi) = 0$$

generated by the space operator of problem. It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = k^2, \ u_k(x) = \cos kx, \ k = 0, 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos kx,$$

Here $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$\sum_{k=0}^{\infty} A_k'(t)\cos kx + a\sum_{k=0}^{\infty} k^2 A_k(t)\cos kx$$

$$+b\sum_{k=o}^{\infty} k^2 A_k(-t)\cos kx = (-1+a)e^{-t}\cos(x) + be^{t}\cos(x),$$

$$\sum_{k=o}^{\infty} A_k(0)\cos kx = \cos(x), x \in [0,\pi].$$

Equating coefficients $\cos kx, k = 0, 1, 2, ...$ to zero, we get

$$\begin{cases} A_1'(t) + aA_1(t) + bA_1(-t) = (-1+a)e^{-t} + be^{t}, \\ A_1(0) = 1, \end{cases} \tag{2.25}$$

$$\begin{cases} A_k'(t) + ak^2 A_1(t) + bk^2 A_k(-t) = 0, k \neq 1, \\ A_k(0) = 0 \end{cases} \tag{2.26}$$

for involutory ordinary differential equations. We will obtain $A_1(t)$. The equivalent to (2.25) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.25), we get

$$A_1''(t) + aA_1'(t) - bA_1'(-t) = (1-a)e^{-t} + be^{t}. \tag{2.27}$$

Putting $-t$ instead of t in (2.25), we get

$$A_1'(-t) + aA_1(-t) + bA_1(t) = (-1+a)\, e^{t} - be^{-t}. \tag{2.28}$$

Multiplying equation (2.28) by b, we get

$$bA_1'(-t) + abA_1(-t) + b^2 A_1(t) = b\,(-1+a)\, e^{t} + b^2 e^{-t}.$$

Adding last equation with (2.27), we get

$$A_1''(t) + aA_1'(t) + abA_1(-t) + b^2 A_1(t) = (1 - a + b^2)e^{-t} + abe^{t}.$$

Multiplying equation (2.25) by $(-a)$, we get

$$-aA_1'(t) - a^2 A_1(t) - abA_1(-t) = \left(a - a^2\right) e^{-t} - abe^{t}.$$

Then adding these equations, we get

$$A_1''(t) + \left(b^2 - a^2\right) A_1(t) = \left(b^2 - a^2 + 1\right) e^{-t} \tag{2.29}$$

Using equation (2.25), $A_1\,(0) = 1$, we get

$$A_1'\,(0) + a + b = (-1+a) + b$$

or

$$A_1'\,(0) = -1.$$

So, we have the following problem

$$A_1''(t) + \left(b^2 - a^2\right) A_1(t) = \left(1 + b^2 - a^2\right) e^{-t}, \quad A_1\,(0) = 1, A_1'\,(0) = -1.$$

There are three cases : $b^2 - a^2 > 0, \quad b^2 - a^2 = 0, \quad b^2 - a^2 < 0.$

In the first case $b^2 - a^2 = m^2 > 0$. Substituting m^2 for $b^2 - a^2$ into equation (2.29), we get

$$A_1''(t) + m^2 A_1(t) = \left(m^2 + 1\right) e^{-t}. \tag{2.30}$$

We will obtain Laplace transform solution of problem (2.30). We have that

$$s^2 A_1(s) - sA_1(0) - A_1'(0) + m^2 A_1(s) = \left(m^2 + 1\right) e^{-s}$$

or

$$(s^2 + m^2)A(s) = sA_1(0) + A_1'(0) + \left(m^2 + 1\right) e^{-s}.$$

Then,

$$A(s) = \frac{s}{s^2 + m^2} A_1(0) + \frac{1}{s^2 + m^2} A_1'(0) + \frac{1}{s^2 + m^2}\left(m^2 + 1\right) e^{-s}.$$

Applying formulas (2.16), (2.17) and (2.18), we get

$$A_1(s) = L\{\cos mt\} A_1(0) + \frac{1}{m} L\{\sin mt\} + \frac{m^2 + 1}{m} L\left\{\int_0^t \sin m(t - p) e^{-p} dp\right\}.$$

Taking the invers Laplace transorm, we get

$$A_1(t) = \cos(mt) A_1(0) + \frac{1}{m}\sin(mt) A_1'(0) + \frac{\left(m^2 + 1\right)}{m}\int_0^t \sin m(t - p) e^{-p} dp. \tag{2.31}$$

Applying initial conditions $A_1(0) = 1, A_1'(0) = -1$, we obtain $A_1(t) = \cos(mt) - \frac{1}{m}\sin(mt) + \frac{(m^2+1)}{m}\int_0^t \sin(m(t - p)) e^{-p} dp.$

Applying (2.20), we get

$$A_1(t) = \cos(mt) - \frac{1}{m}\sin(mt) + \frac{\left(m^2 + 1\right)}{m}\left\{\frac{1}{m^2 + 1}\{me^{-t} + \sin mt - m\cos mt\}\right\},$$

$$\cos(mt) - \frac{1}{m}\sin(mt) + e^{-t} + \frac{1}{m}\sin mt - \cos mt = e^{-t}.$$

Therefore

$$A_1(t) = e^{-t}.$$

It is easy to see that $A_1(t) = e^{-t}$ for $b^2 - a^2 = 0$, $b^2 - a^2 < 0$.

Now we will obtain $A_k(t)$ for $k \neq 1$. The equivalent to (2.26) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.26), we get

$$A_k''(t) + ak^2 A_k'(t) + bk^2 A_k'(-t) = 0. \tag{2.32}$$

Putting $-t$ instead of t in (2.26), we get

$$A_k'(-t) + ak^2 A_k(-t) + bk^2 A_k(t) = 0. \tag{2.33}$$

Multiplying equation equation (2.33) by bk^2, we get

$$bk^2 A_k'(-t) + abk^4 A_k(-t) + b^2 k^4 A_k(t) = 0.$$

Adding last equation with (2.32), we get

$$A_k''(t) + ak^2 A_k'(t) + abk^4 A_k(-t) + b^2 k^4 A_k(t) = 0.$$

Multiplying equation (2.26) by $(-ak^2)$, we get

$$-ak^2 A_k'(t) - a^2 k^4 A_k(t) - abk^4 A_k(-t) = 0.$$

Then from these equations, we get

$$A_k''(t) + \left(b^2 - a^2\right) k^4 A_k(t) = 0.$$

Substituting t=0 in equation (2.26), we get

$$A_k'(0) + ak^2 A_k(0) + bk^2 A_k(0) = 0$$

or

$$A_k'(0) = 0.$$

We have the following problem

$$A_k''(t) + \left(b^2 - a^2\right) k^4 A_k(t) = 0, \quad A_k(0) = 0, A_k'(0) = 0.$$

From that it follows $A_k(t) = 0, k \neq 1$. In the same manner $A_k(t) = 0, k \neq 1$ for $b^2 - a^2 = 0$ and $b^2 - a^2 < 0$.

Therefore, the exact solution of problem(2.24) is

$$u(t, x) = A_1(t)\cos x = e^{-t}\cos x.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t, x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \overline{\Omega}, d \geq 0, \\ \frac{\partial u(t,x)}{\partial \overline{p}} = 0, x \in S, t \in (-\infty, \infty) \end{cases} \tag{2.34}$$

for the multidimensional involutory parabolic type equation. Assume that $a_r > a_0 > 0$ and $f(t, x)\left(t \in (-\infty, \infty), x \in \overline{\Omega}\right), \varphi(x)\left(t \in (-\infty, \infty), x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{p}$ is the normal vector to S. However Fourier series method described in solving (2.34) can be used only in the case when (2.34) has constant coefficients.

Example 3.3. Consider the initial-boundary-value problem for parabolic type involutory partial differential equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t, x) - bu_{xx}(-t, x) = e^{-t}(-1 + 4a)\cos 2x + 4e^t \cos 2x, \\ x \in (0, \pi), -\infty < t < \infty, \\ u(0, x) = \cos 2x, x \in [0, \pi] \\ u(t, 0) = u(t, \pi), u_x(t, 0) = u_x(t, 0) = u_x(t, \pi), t \in (-\infty, \infty). \end{cases} \tag{2.35}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, \ \ 0 < x < \pi, \ u(0) = u(\pi), \ u_x(0) = u_x(\pi)$$

generated by the space operator of problem (2.35). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = 4k^2, \ u_k(x) = \cos 2kx, \ k = 0, 1, 2, ..., \ u_k(x) = \sin 2kx, \ k = 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem (2.35) by formula

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx, \tag{2.36}$$

where $A_k(t), k = 0, 1, 2, ...,$ and $B_k(t), k = 1, 2, ...$ are unknown functions.
Putting formula (2.36) into the main problem and using given initial condition, we obtain

$$\sum_{k=0}^{\infty} A_k'(t) \cos 2kx + \sum_{k=1}^{\infty} B_k'(t) \sin 2kx + a \sum_{k=0}^{\infty} 4k^2 A_k(t) \cos 2kx$$

$$+a \sum_{k=1}^{\infty} 4k^2 B_k(t) \sin 2kx + b \sum_{k=0}^{\infty} 4k^2 A_k(-t) \cos 2kx + b \sum_{k=1}^{\infty} 4k^2 B_k(-t) \sin 2kx$$

$$= e^{-t}(-1 + 4a) \cos 2x + 4e^t \cos 2x, x \in (0, \pi), t \epsilon I, \ x \epsilon (0, \pi),$$

$$\sum_{k=0}^{\infty} A_k(0) \cos 2kx + \sum_{k=1}^{\infty} B_k(0) \sin 2kx = \cos 2x, x \in [0, \pi].$$

Equating the coefficients of $\cos kx, k = 0, 1, 2, ...,$ and $\sin kx, k = 1, 2, ...$ to zero, we get

$$\begin{cases} B_k'(t) + 4ak^2 B_k(t) + 4bk^2 B_k(-t) = 0, \ t \epsilon (-\infty, \infty), \\ \\ B_k(0) = 0, \ \ k = 1, 2, ..., \end{cases} \tag{2.37}$$

$$\begin{cases} A_1'(t) + 4aA_1(t) + 4bA_1(-t) = e^{-t}(-1 + 4a) + 4e^t, \ t \epsilon (-\infty, \infty), \\ \\ A_1(0) = 1, \end{cases} \tag{2.38}$$

$$\begin{cases} A_k'(t) - 4ak^2 A_k(t) - 4bk^2 A_k(-t) = 0, k \neq 1,\ t \,\epsilon(-\infty, \infty), \\ \\ A_k(0) = 0,\ \ k = 1, 2, ... \end{cases} \tag{2.39}$$

for involutory ordinary differential equations. We will obtain $A_1(t)$. The equivalent to (2.38) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.38), we get

$$A_1''(t) + 4aA_1'(t) - 4bA_1'(-t) = (1 - 4a)e^{-t} + 4be^{t}. \tag{2.40}$$

Putting $-t$ instead of t (2.38), we get

$$A_1'(-t) + 4aA_1(-t) + 4bA_1(t) = (-1 + 4a)\, e^{t} - 4be^{-t}. \tag{2.41}$$

Multiplying equation (2.41) by $4b$, we get

$$4bA_1'(-t) + 16abA_1(-t) + 16b^2 A_1(t) = 4b\,(-1 + 4a)\, e^{t} + 16b^2 e^{-t}.$$

Adding last equation with (2.40), we get

$$A_1''(t) + 4aA_1'(t) + 16abA_1(-t) + 16b^2 A_1(t) = (1 - 4a + 16b^2)e^{-t} + 16abe^{t}.$$

Multiplying equation (2.38) by $(-4a)$, we get

$$-4aA_1'(t) - 16a^2 A_1(t) - 16abA_1(-t) = \left(4a - 16a^2\right) e^{-t} - 16abe^{t}.$$

Then adding these equations, we get

$$A_1''(t) + \left(16b^2 - 16a^2\right) A_1(t) = \left(1 + 16b^2 - 16a^2\right) e^{-t}. \tag{2.42}$$

Using equation (2.38) and $A_1(0) = 1$, we get

$$A_1'(0) + aA_1(0) + bA_1(0) = -1 + a + b,$$

$$A_1'(0) = -1.$$

So, we have the following problem

$$A_1''(t)+\left(16b^2-16a^2\right)A_1(t)=\left(1+16b^2-16a^2\right)e^{-t},\ -\infty<t<\infty,\quad A_1(0)=1, A_1'(0)=-1.$$

There are three cases : $b^2-a^2>0,\quad b^2-a^2=0,\quad b^2-a^2<0.$
In the first case $b^2-a^2=m^2>0$. Substituting $16m^2$ for $16b^2-16a^2$ into equation (2.42), we get

$$A_1''(t)+16m^2A_1(t)=\left(1+16m^2\right)e^{-t}. \tag{2.43}$$

We will obtain Laplace transform solution of problem (2.43). We have that

$$s^2A_1(s)-sA_1(0)-A_1'(0)+16m^2A_1(s)=\left(1+16m^2\right)e^{-s}$$

or

$$(s^2+(4m)^2)A(s)=sA_1(0)+A_1'(0)+\left((4m)^2+1\right)e^{-s}.$$

Then,

$$A(s)=\frac{s}{s^2+(4m)^2}A_1(0)+\frac{1}{s^2+(4m)^2}A_1'(0)+\frac{1}{s^2+(4m)^2}\left((4m)^2+1\right)e^{-s}.$$

Applying formulas (2.16), (2.17) and (2.18), we get

$$A_1(s)=L\{\cos 4m)\}A_1(0)+\frac{1}{4m}L\{\sin 4mt\}+\frac{(4m)^2+1}{4m}L\left\{\int_0^t \sin 4m(t-p))e^{-s}\right\}.$$

Taking the inverse Laplace transorm, we get

$$A_1(t)=\cos(4mt)\ A_1(0)+\frac{1}{4m}\sin(4mt)\ A_1'(0)+\frac{(4m)^2+1}{4m}\int_0^t \sin(4m(t-p))\,e^{-p}dp. \tag{2.44}$$

Applying initial conditions $A_1(0)=1, A_1'(0)=-1$, we obtain

$$A_1(t)=\cos(4mt)-\frac{1}{4m}\sin(4mt)\ +\frac{(4m)^2+1}{4m}\int_0^t \sin(4m(t-p))\,e^{-p}dp.$$

We denote

$$I=\int \sin(4m(t-p)e^{-p}dp.$$

We have that

$$I = -\sin 4m(t-p)e^{-p} - 4m\int \cos 4m(t-p)e^{-p}dp$$

$$= -\sin 4m(t-p)e^{-p} + 4m\cos 4m(t-p)e^{-p} - 16m^2\int \sin 4m(t-p)e^{-p}dp.$$

Therefore,

$$I\left(16m^2+1\right) = -\sin 4m(t-p)e^{-p} + 4m\cos 4m(t-p)e^{-p}$$

or

$$I = \frac{1}{16m^2+1}\left\{-\sin 4m(t-p)e^{-p} + 4m\cos 4m(t-p)e^{-p}\right\}. \tag{2.45}$$

Therefore,

$$\int_0^t \sin 4m(t-p)e^{-p}dp = \frac{1}{16m^2+1}\left\{4me^{-t} + \sin 4mt - 4m\cos 4mt\right\}.$$

$$A_1(t) = \cos(4mt) - \frac{1}{4m}\sin(4mt) + \frac{\left(16m^2+1\right)}{4m}\left\{\frac{1}{16m^2+1}\left\{4me^{-t} + \sin 4mt - 4m\cos 4mt\right\}\right\}$$

$$= \cos(4mt) - \frac{1}{4m}\sin(4mt) + e^{-t} + \frac{1}{4m}\sin 4mt - \cos 4mt = e^{-t}.$$

It is easy to see that $A_1(t) = e^{-t}$ for $16b^2 - 16a^2 = 0, 16b^2 - 16a^2 < 0$.

Now we will obtain $A_k(t)$ for $k \neq 1$. The equivalent to (2.39) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.39), we get

$$A_k''(t) + 4ak^2A_k'(t) + 4bk^2A_k'(-t) = 0. \tag{2.46a}$$

Putting $-t$ instead of t (2.39), we get

$$A_k'(-t) + 4ak^2A_k(-t) + 4bk^2A_k(t) = 0. \tag{2.47}$$

Multiplying equation (2.40) by $4bk^2$, we get

$$4bk^2A_k'(-t) + 16abk^4A_k(-t) + 16b^2k^4A_k(t) = 0.$$

Adding this equation with (2.46a), we get

$$A_k''(t) + 4ak^2A_k'(t) + 16abk^4A_k(-t) + 16b^2k^4A_k(t) = 0.$$

Multiplying equation (2.39) by $(-4ak^2)$, we get

$$-4ak^2A_k'(t) - 16a^2k^4A_k(t) - 16abk^4A_k(-t) = 0.$$

Then from these equations, we get

$$A_k''(t) + \left(16b^2 - 16a^2\right)k^4A_k(t) = 0.$$

Substituting t=0 in equation (2.39), we get

$$A_k'(0) + 4ak^2A_k(0) + 4bk^2A_k(-0) = 0$$

or

$$A_k'(0) = 0.$$

We have the following problem

$$A_k''(t) + \left(16b^2 - 16a^2\right)k^4A_k(t) = 0, \quad A_k(0) = 0, A_k'(0) = 0.$$

From that it follows $A_k(t) = 0, k \neq 1$. In the same manner $A_k(t) = 0, k \neq 1$ for $16b^2 - 16a^2 = 0$ and $16b^2 - 16a^2 < 0$.

Therefore, the exact solution of problem (2.35) is

$$u(t,x) = A_1(t)\cos 2x = e^{-t}\cos 2x.$$

Note that using similar procedure one can obtain the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r\frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} a_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \overline{\Omega}, d \geq 0, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2},\ \frac{\partial u(t,x)}{\partial \overline{p}}\Big|_{S_1} = \frac{\partial u(t,x)}{\partial \overline{p}}\Big|_{S_2}, t \in (-\infty, \infty) \end{cases} \tag{2.48}$$

for the multidimensional involutory parabolic type equation. Assume that $a_r > a_0 > 0$ and $f(t,x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right), \varphi(x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, \varnothing = S_1 \cap S_2$. However Fourier series method described in solving (2.48) can be used only in the case when (2.48) has constant coefficients.

Second, we consider Laplace transform solution of problems for parabolic type involutory partial differential equations.

Example 3.4. Consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = (-1-a)e^{-t}e^{-x} - be^{t}e^{-x}, \\ x \in (0,\infty), t \in (-\infty,\infty), \\ u(0,x) = e^{-x}, \ x \in [0,\infty), \\ u(t,0) = e^{-t}, u_x(t,0) = -e^{-t}, t \in (-\infty,\infty) \end{cases} \tag{2.49}$$

for one dimensional involutory parabolic equation.

Solution. We will obtain Laplace transform solution of problem (2.49). We denote that

$$u(t,s) = L\{u(t,x)\}.$$

Taking the Laplace transform with respect to x, we get

$$u_t(t,s) - a\left\{s^2u(t,s) - se^{-t} - (-e^{-t})\right\} - b\left\{s^2u(-t,s) - se^{-t} - (-e^{-t})\right\}$$

$$= -(1+a)e^{-t}\frac{1}{1+s} - be^{t}\frac{1}{1+s}, u(0,s) = \frac{1}{1+s}.$$

From that it follows the initial value problem

$$u_t(t,s) - as^2u(t,s) - bs^2u(-t,s) = a(s)e^{-t} - b(s)e^{t}, u(0,s) = \frac{1}{1+s} \tag{2.50}$$

for involutory ordinary differential equations. Here

$$a(s) = -\frac{as^2+1}{1+s}, b(s) = \frac{bs^2}{1+s}. \tag{2.51}$$

We will obtain $u(t,s)$. The equivalent to (2.50) initial value problem for the second order differential equation can be obtain. Taking the derivative of (2.50), we get

$$u_{tt}(t,s) - as^2 u_t(t,s) + bs^2 u_t(-t,s) = -a(s)e^{-t} - b(s)e^{t}. \tag{2.52}$$

Putting $-t$ instead of t equation (2.50), we get

$$u_t(-t,s) - as^2 u(-t,s) - bs^2 u(t,s) = a(s)e^{t} - b(s)e^{-t}. \tag{2.53}$$

Multiplying equation (2.53) by $\left(bs^2\right)$, we get

$$-bs^2 u_t(-t,s) + abs^4 u(-t,s) + b^2 s^4 u(t,s) = bs^2 a(s)e^{t} - bs^2 b(s)e^{-t}.$$

Adding this equation (2.52), we get

$$u_{tt}(t,s) - as^2 u_t(t,s) + abs^4 u(-t,s) + b^2 s^4 u(t,s)$$

$$= \left(bs^2 b(s) - a(s)\right)e^{-t} - \left(b(s) + bs^2 a(s)\right)e^{t}.$$

Multiplying equation (2.50) by $\left(as^2\right)$, we get

$$as^2 iu_t(t,s) - a^2 s^4 u(t,s) - abs^4 u(-t,s) = as^2 a(s)e^{-t} - as^2 b(s)e^{t}.$$

Adding two last equations, we get

$$u_{tt}(t,s) + b^2 s^4 u(t,s) - a^2 s^4 u(t,s)$$

$$= \left(as^2 a(s) - a(s) + bs^2 b(s)\right)e^{-t} - \left(as^2 b(s) - b(s) - bs^2 a(s)\right)e^{t}.$$

Using notations (2.51), we get

$$u_{tt}(t,s) + \left(b^2 s^4 - a^2 s^4\right)u(t,s) = \left(-as^2\frac{as^2+1}{1+s} + \frac{as^2+1}{1+s} + bs^2\frac{bs^2}{1+s}\right)e^{-t}$$

$$-\left(as^2\frac{bs^2}{1+s} - \frac{bs^2}{1+s} + -bs^2\frac{as^2+1}{1+s}\right)e^{t}$$

or

$$u_{tt}(t,s) + \left(b^2 s^4 - a^2 s^4\right)u(t,s) = \frac{\left(a^2 - b^2\right)s^4 - 1}{1+s}e^{-t}.$$

Using $u(0,s)=\frac{1}{1+s}$ and equation (2.50), we get

$$u_t(0,s)=-\frac{1}{1+s}.$$

Then, we have the following initial value problem for the second order ordinary differential equation

$$\begin{cases} u_{tt}(t,s)+\left(b^2-a^2\right)s^4u(t,s)=\frac{\left(a^2-b^2\right)s^4+1}{1+s}e^{-t}, t\in I, \\ \\ u(0,s)=\frac{1}{1+s}, u_t(0,s)=-\frac{1}{1+s}. \end{cases} \tag{2.54}$$

Now, we obtain the solution of problem (2.54). There are three cases: $b^2-a^2=0$, $b^2-a^2>0$, $b^2-a^2<0$.

Substituting m^2 for $\left(b^2-a^2\right)s^4$ into equation (2.54), we get

$$u_{tt}(t,s)+m^2u(t,s)=\frac{m^2+1}{1+s}e^{-t}, t\in I, u(0,s)=\frac{1}{1+s}, u_t(0,s)=-\frac{1}{1+s}.$$

We have that

$$u(t,s)=u_c(t,s)+u_p(t,s),$$

where $u_c(t,s)$ is the general solution of homogenous equation

$$u_{tt}(t,s)+m^2u(t,s)=0.$$

and $u_p(t,s)$ is the particular solution of nonhomogenous equation. The auxillary equation is

$$p^2+m^2=0.$$

In the first case, we have that $p_{1,2}=0,0$ and

$$u_c(t,s)=c_1+c_2t.$$

In the second case $p_{1,2}=\pm im$. Then

$$u_c(t,s)=c_1\cos mt+c_2\sin mt.$$

In the third case $p_{1,2} = \pm m$. Then

$$u_c(t,s) = c_1 e^{mt} + c_2 e^{-mt}.$$

Now, we will obtain the particular solution $u_p(t,s)$ by formula

$$u_p(t,s) = w(s)\, e^{-t}.$$

Putting it into nonhomogenous equation, we get

$$w(s)\, e^{-t} + m^2 w(s)\, e^{-t} = \frac{m^2+1}{1+s} e^{-t}$$

or

$$\left\{1 + m^2\right\} w(s) = \frac{m^2+1}{1+s}.$$

Then

$$w(s) = \frac{1}{1+s}$$

and

$$u_p(t,s) = \frac{1}{1+s} e^{-t}.$$

In the first case, we have

$$u(t,s) = c_1 + c_2 t + \frac{1}{1+s} e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + \frac{1}{1+s} = \frac{1}{1+s},$$

$$u_t(0,s) = c_2 - \frac{1}{1+s} = -\frac{1}{1+s}.$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = w(s) e^{-t}.$$

In the second case, we have

$$u(t,s) = c_1 \cos mt + c_2 \sin mt + \frac{1}{1+s} e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + \frac{1}{1+s} = \frac{1}{1+s},$$

$$u_t(0,s) = c_2 - \frac{1}{1+s} = -\frac{1}{1+s}.$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = w(s)e^{-t}.$$

In the third case, we have

$$u(t,s) = c_1 e^{mt} + c_2 e^{-mt} + \frac{1}{1+s} e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + c_2 + \frac{1}{1+s} = \frac{1}{1+s},$$

$$u_t(0,s) = m(c_1 - c_2) - \frac{1}{1+s} = -\frac{1}{1+s}.$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = w(s)e^{-t}.$$

Therefore,

$$u(t,s) = w(s)e^{-t} = e^{-t} L\{e^{-x}\}$$

and

$$u(t,x) = L^{-1}\{e^{-t} L\{e^{-x}\}\} = e^{-t}e^{-x}.$$

Therefore,

$$u(t,x) = e^{-t}e^{-x}.$$

is the exact solution of problem (2.49).

Example 3.5. Consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = -(a+b)e^{-x}, \\ x \in (0,\infty), -\infty < t < \infty, \\ u(0,x) = e^{-x}, \ x \in [0,\infty), \\ u(t,0) = 1, u(t,\infty) = 0, t \in (-\infty,\infty) \end{cases} \tag{2.55}$$

for one dimensional involutory parabolic equation.

Solution. We will obtain Laplace transform solution of problem (2.55). Taking the Laplace transform, we get

$$\begin{cases} u_t(t,s) - a\left[s^2u(t,s) - s - \beta(t)\right] - b\left[s^2u(-t,s) - s - \beta(-t)\right] \\ = -\frac{a+b}{1+s}, -\infty < t < \infty, \beta(t) = u_x(t,0), \\ u(0,s) = \frac{1}{1+s} \end{cases}$$

or

$$\begin{cases} u_t(t,s) - as^2u(t,s) - bs^2u(-t,s) \\ = -(a+b)s - a\beta(t) - b\beta(-t) - \frac{a+b}{1+s}, -\infty < t < \infty, \\ u(0,s) = \frac{1}{1+s}. \end{cases} \tag{2.56}$$

From (2.56) it follows that

$$u_t(0,s) = \frac{(a+b)s^2}{1+s} - (a+b)s - (a+b)\beta(0) - \frac{a+b}{1+s}$$

or

$$u_t(0,s) = -(a+b)\left[1+\beta(0)\right].$$

Taking the derivative of (2.56), we get

$$u_{tt}(t,s) - as^2 u_t(t,s) + bs^2 u_t(-t,s) = -a\beta'(t) + b\beta'(-t). \tag{2.57}$$

Putting $-t$ instead of t equation (2.56), we get

$$u_t(-t,s) - as^2 u(-t,s) - bs^2 u(t,s) = -(a+b)s - a\beta(-t) - b\beta(t) - \frac{a+b}{1+s}. \tag{2.58}$$

Multiplying equation (2.58) by $\left(-bs^2\right)$, we get

$$-bs^2 u_t(-t,s) + abs^4 u(-t,s) + b^2 s^4 u(t,s)$$

$$= (a+b)bs^3 + abs^2\beta(-t) + b^2 s^2 \beta(t) + \left(\frac{a+b}{1+s}\right)bs^2.$$

Adding last equation with (2.59), we get

$$u_{tt}(t,s) - as^2 u_t(t,s) + abs^4 u(-t,s) + b^2 s^4 u(t,s)$$

$$= -a\beta'(t) + b\beta'(-t) + (a+b)bs^3 + abs^2\beta(-t) + b^2 s^2\beta(t) + \left(\frac{abs^2 + bbs^2}{1+s}\right).$$

Multiplying equation (2.56) by $\left(as^2\right)$, we get

$$as^2 u_t(t,s) - a^2 s^4 u(t,s) - abs^4 u(-t,s)$$

$$= -(a+b)as^3 - a^2 s^2 \beta(t) - as^2 b\beta(-t) - \frac{a^2 s^2 + bas^2}{1+s}.$$

Adding these equations , we get

$$u_{tt}(t,s) + \left(b^2 - a^2\right)u(t,s) = -a\beta'(t) + b\beta'(-t) + (a+b)bs^3 + b^2 s^2\beta(t)$$

$$-(a+b)as^3 - a^2 s^2\beta(t) + \frac{abs^2 + b^2 s^2 - a^2 s^2 - abs^2}{1+s}$$

or

$$u_{tt}(t,s) + \left(b^2 - a^2\right)u(t,s) = -a\beta'(t) + b\beta'(-t) + \left(b^2 - a^2\right)s^3 + \frac{\left(b^2 - a^2\right)}{1+s}s^2 + \left(b^2 - a^2\right)s^2\beta(t).$$

There are three cases: $b^2 - a^2 = 0$, $b^2 - a^2 > 0$, $b^2 - a^2 < 0$. In the first case $b^2 - a^2 = 0$. Then

$$\begin{cases} u_{tt}(t,s) = -a\beta'(t) + b\beta'(-t). \\ \\ u(0,s) = \frac{1}{1+s}, u_t(0,s) = -(a+b)[1 + \beta(0)]. \end{cases}$$

Applying formula

$$p(t) = p_0 + t\, p_0' + \int_0^t (t-s)\, p''(s) ds,$$

we get

$$u(t,s) = \frac{1}{1+s} + t\, \{-(a+b)\,[1+\beta(0)]\} + \int_0^t (t-p)\{-a\beta'(p) + b\beta'(-p)\}\, dp$$

$$= \frac{1}{1+s} + t\, \{-(a+b)\,[1+\beta(0)]\} + [(t-p)\, a\beta(t) + b\beta(-t)]_0^t - \int_0^t (a\beta(p) + b\beta(-p))\, dp$$

$$= \frac{1}{1+s} + t\, \{-(a+b)\,[1+\beta(0)]\} + t\, (a+b)\beta(0) + \int_0^t a\beta(p) + b\beta(-p)\, dp$$

$$= \frac{1}{1+s} - t\,(a+b) - \int_0^t (a\beta(p) + b\beta(-p))\, dp.$$

Therefore,

$$u(t,s) - \frac{1}{1+s} = -t\,(a+b) - \int_0^t (a\beta(p) + b\beta(-p))\, dp.$$

Putting

$$A(t) = -t\,(a+b) - \int_0^t (a\beta(p) + b\beta(-p))\, dp,$$

we get

$$u(t,s) - \frac{1}{1+s} = A(t).$$

Taking invers the Laplase transform, we get

$$u(t,x) - e^{-x} = L^{-1}\{A(t)\}. \tag{2.59}$$

Applying $x \to \infty$, we get

$$0 = L^{-1}\{A(t)\}.$$

Then,

$$LL^{-1}\{A(t)\} = 0$$

or

$$A(t) = 0.$$

Putting $A(t)$ into (2.59), we get

$$u(t,x) - e^{-x} = 0.$$

$$u(t,x) = e^{-x}.$$

In the same manner we can obtain

$$u(t,x) = e^{-x}$$

for $b^2 - a^2 > 0$ and $b^2 - a^2 < 0$. Therefore,

$$u(t,x) = e^{-x}$$

is the exact solution of problem (2.55).

Note that using similar procedure one can obtain the solution of the following problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \overline{\Omega}^+, \\ u(t,x) = \ a(t,x), \quad u_{x_r}(t,x) = \beta_r(t,x), 1 \le r \le n, t \in I, x \in S^+ \end{cases} \tag{2.60}$$

for the multidimensional parabolic type involutory partial differential equations. Assume that $a_r > a_0 > 0$ and $f(t,x)\left(t \in I, x \in \overline{\Omega}^+\right), \varphi(x)\left(x \in \overline{\Omega}^+\right), \alpha(t,x), \beta_r(t,x)\,(t \in I, x \in S^+)$

are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n\,(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and $\overline{\Omega}^+ = \Omega^+ \cup S^+$. However Laplace transform method described in solving (2.60) can be used only in the case when (2.60) has constant or polynomial coefficients.

Third, we consider Fourier transform solution of the problem for parabolic type involutory partial differential equations.

Example 3.6. Consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) \\ = \left(-1 - a(4x^2-2)\right)e^{-t}e^{-x^2} - b(4x^2-2)e^{t}e^{-x^2}, \\ x \in (-\infty,\infty), -\infty < t < \infty, \\ u(0,x) = e^{-x^2},\ x \in (-\infty,\infty) \end{cases} \tag{2.61}$$

for one dimensional involutory parabolic equation.

Solution. We will obtain Fourier transform solution of problem (2.61). Taking the Fourier transform, we get

$$\begin{cases} u_t(t,s) + as^2u(t,s) + bs^2u(-t,s) \\ = -e^{-t}q(s) + as^2e^{-t}q(s) + bs^2e^{t}q(s), \\ u(0,s) = q(s). \end{cases} \tag{2.62}$$

Here

$$u(t,s) = F\{u(t,x)\}, q(s) = F\left\{e^{-x^2}\right\}.$$

From (2.62) it follows that

$$u_t(0,s) = -as^2q(s) - bs^2q(s) - q(s) + as^2q(s) + bs^2q(s) = -q(s)$$

or

$$u_t(0,s) = -q(s).$$

Taking the derivative of (2.62), we get

$$u_{tt}(t,s) + as^2 u_t(t,s) - bs^2 u_t(-t,s) = e^{-t}q(s) + as^2 e^{-t} + be^t q(s). \tag{2.63}$$

Putting $-t$ instead of t equation (2.62), we get

$$u_t(-t,s) + s^2\left(au(-t,s) + bu(t,s)\right) = -e^t q(s) + s^2\left(ae^t + be^{-t}\right)q(s). \tag{2.64}$$

Multiplying equation (2.64) by $\left(bs^2\right)$, we get

$$s^2 bu_t(-t,s) + s^4 b\left(au(-t,s) + bu(t,s)\right) = -s^2 be^t q(s) + s^4 b\left(ae^t + be^{-t}\right)q(s).$$

Adding last equation with (2.63), we get

$$u_{tt}(t,s) - s^2 au_t(t,s) - s^4 bau(-t,s) - s^4 b^2 u(t,s)$$

$$= e^{-t}q(s) - s^2 ae^{-t}q(s) + bas^4 e^t q(s) + s^4 b^2 e^{it} q(s).$$

Multiplying equation (2.62) by $\left(-as^2\right)$, we get

$$-as^2 u_t(t,s) - a^2 s^4 u(t,s) - abs^4 u(-t,s) = as^2 e^{-t}q(s) - a^2 s^4 e^{-t}q(s) - abs^4 e^t q(s).$$

Adding two last equations, we get the following problem

$$\begin{cases} u_{tt}(t,s) + \left(b^2 - a^2\right)s^4 u(t,s) = e^{-t}q(s)\left\{1 + \left(b^2 - a^2\right)s^4\right\}, \\ \\ \qquad u(0,s) = q(s), \quad u_t(0,s) = -q(s). \end{cases} \tag{2.65}$$

Now we will obtain initial value problem for the second order ordinary differential equation of the problem (2.65). There are three cases: $\left(b^2 - a^2\right)s^4 = 0$, $\left(b^2 - a^2\right)s^4 > 0$, $\left(b^2 - a^2\right)s^4 < 0$.

Substituting m^2 for $\left(b^2 - a^2\right)s^4$ into equation (2.65), we get

$$u_{tt}(t,s) + m^2 u(t,s) = e^{-t}q(s)\left\{1 + m^2\right\}.$$

We have that

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

where $u_c(t,s)$ is the general solution of homogenous equation

$$u_{tt}(t,s) + m^2 u(t,s) = 0$$

and $u_p(t,s)$ is the particular solution of nonhomogenous equation. The auxillary equation is

$$p^2 + m^2 = 0.$$

In the first case $m^2 = 0$. Then $p_{1,2} = 0, 0$.

$$u_c(t,s) = c_1 + c_2 t.$$

In the second case $p_{1,2} = \pm i m^2$. Then

$$u_c(t,s) = c_1 \cos m^2 t + c_2 \sin m^2 t.$$

In the third case $p_{1,2} = \pm m^2$. Then

$$u_c(t,s) = c_1 e^{m^2 t} + c_2 e^{-m^2 t}.$$

Now, we will obtain the particular solution $u_p(t,s)$ by formula

$$u_p(t,s) = A(s)\, e^{-t}.$$

Putting it into nonhomogenous equation, we get

$$A(s)\, e^{-t} + m^2 A(s)\, e^{-t} = e^{-t} q(s) \left\{1 + m^2\right\}$$

or

$$\left\{1 + m^2\right\} A(s) = q(s) \left\{-1 + m^2\right\}.$$

Therefore

$$A(s) = q(s)$$

and

$$u_p(t,s) = q(s)e^{-t}.$$

In the first case, we have

$$u(t,s) = c_1 + c_2 t + q(s)e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + q(s) = q(s),$$

$$u_t(0,s) = c_2 - q(s) = -q(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{-t}.$$

In the second case, we have

$$u(t,s) = c_1 \cos mt + c_2 \sin mt + q(s)e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + q(s) = q(s),$$

$$u_t(0,s) = c_2 m - q(s) = -q(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{-t}.$$

In the third case, we have

$$u(t,s) = c_1 e^{mt} + c_2 e^{-mt} + q(s)e^{-t}.$$

Applying initial conditions, we get

$$u(0,s) = c_1 + c_2 + q(s) = q(s),$$

$$u_t(0,s) = m(c_1 - c_2) - q(s) = -q(s).$$

From that it follows $c_1 = c_2 = 0$ and

$$u(t,s) = q(s)e^{-t}.$$

Therefore,

$$u(t,s) = q(s)e^{-t} = e^{-t}F\left\{e^{-x^2}\right\}$$

and

$$u(t,x) = F^{-1}\left\{e^{-t}F\left\{e^{-x^2}\right\}\right\} = e^{-t}e^{-x^2}.$$

Therefore,

$$u(t,x) = e^{-t}e^{-x^2}$$

is the exact solution of problem (2.61).

Note that using similar procedure one can obtain the solution of the following problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \mathbb{R}^n, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \varphi(x), x \in \mathbb{R}^n \end{cases} \tag{2.66}$$

for the multidimensional parabolic type involutory partial differential equations. Assume that $a_r > a_0 > 0$ and $f(t,x)$ $(t \in I, x \in \mathbb{R}^n)$, $\varphi(x)$ $(x \in \mathbb{R}^n)$ are given smooth functions. However Fourier transform method described in solving (2.66) can be used only in the case when (2.66) has constant coefficients.

CHAPTER 3
NUMERICAL ALGORITHM FOR THE SOLUTION OF THE INVOLUTORY PARTIAL DIFFERENTIAL EQUATION

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of the local and nonlocal problems for the parabolic type involutory partial differential equations play an important role in applied mathematics. In this section, we present the algorithm for the numerical solution of the initial boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - u_{xx}(t,x) - u_{xx}(-t,x) = \cos t \sin x, \\ \\ 0 < x < \pi, \quad -\pi < t < \pi, \\ \\ u(0,x) = 0, \ 0 \le x \le \pi, \\ \\ u(t,0) = u(t,\pi) = 0, -\pi \le t \le \pi \end{cases} \tag{3.1}$$

for the one dimensional parabolic type involutory partial differential equation. The exact solution of problem (3.1) is $u(t,x) = \sin t \sin x, 0 \le x \le \pi, -\pi \le t \le \pi$.

For the approximate solutions of the problem (3.1), we will apply Gauss elimination method to solve the problem. Using the set of grid points,

$$[-\pi,\pi]_\tau \times [0,\pi]_h = \{(t_k,x_n) : t_k = k\tau, -N \le k \le N, N\tau = \pi, x_n = nh, \ 0 \le n \le M, Mh = \pi\},$$

we get the first order of accuracy in t difference scheme

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} - \frac{u_{n+1}^{-k}-2u_n^{-k}+u_{n-1}^{-k}}{h^2} = \cos t_k \sin x_n,\ t_k = k\tau, x_n = nh, N\tau = \pi, Mh = \pi, \\ -N+1 \le k \le N,\ \ 1 \le n \le M-1, \\ u_n^0 = 0,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ \ -N \le k \le N \end{cases} \tag{3.2}$$

and second order of accuracy in t difference scheme

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{1}{2}\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} - \frac{1}{2}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\frac{1}{2}\frac{u_{n+1}^{-k}-2u_n^{-k}+u_{n-1}^{-k}}{h^2} - \frac{1}{2}\frac{u_{n+1}^{-k+1}-2u_n^{-k+1}+u_{n-1}^{-k+1}}{h^2} \\ = \cos(t_k - \frac{\tau}{2})\sin x_k,\ t_k = k\tau,\ x_n = nh,\ N\tau = \pi, \\ -N+1 \le k \le N,\ 1 \le n \le M-1, \\ u_n^0 = 0, 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ \ -N \le k \le N. \end{cases} \tag{3.3}$$

They are systems of algebraic equations and they can be written in the matrix form

$$Au_{n-1} + Bu_n + Cu_{n+1} = D\varphi_n,\ 1 \le n \le M-1, u_0 = \overrightarrow{0},\ \ u_M = \overrightarrow{0}, \tag{3.4}$$

where A, B, C are $(2N+1)\times(2N+1)$ matrices and $D = I_{2N+1}$ is the identity matrix, φ_n and u_s are $(2N+1)\times 1$ column vectors

$$D\varphi_n = \begin{bmatrix} 0 \\ \cos t_{-N+1}\sin x_n \\ \cdot \\ \cos t_{N-1}\sin x_n \\ \cos t_N \sin x_n \end{bmatrix}_{(2N+1)\times 1},\ \ u_s = \begin{bmatrix} u_s^{-N} \\ u_s^{-N+1} \\ \cdot \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(2N+1)\times 1}$$

and

$$
A = C = \begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\
0 & a & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & a & 0 \\
0 & 0 & a & \cdot & 0 & 0 & 0 & \cdot & a & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & a & 0 & a & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & 0 & 2a & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & a & 0 & a & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & a & \cdot & 0 & 0 & 0 & \cdot & a & 0 & 0 \\
0 & a & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & a & 0 \\
a & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & a
\end{bmatrix}_{(2N+1)\times(2N+1)},
$$

$$
B = \begin{bmatrix}
0 & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\
b & d & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & c & 0 \\
0 & b & d & \cdot & 0 & 0 & 0 & \cdot & c & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & d & 0 & c & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & b & d+c & 0 & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & c & b & d & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & c & \cdot & 0 & 0 & 0 & \cdot & d & 0 & 0 \\
0 & c & 0 & \cdot & 0 & 0 & 0 & \cdot & b & d & 0 \\
c & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & b & d
\end{bmatrix}_{(2N+1)\times(2N+1)},
$$

$a = -\frac{1}{h^2}$, $b = -\frac{1}{\tau}$, $c = \frac{2}{h^2}$ and $d = \frac{2}{h^2} + \frac{1}{\tau}$ for the difference scheme (3.2) and

$$A = C = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ a & a & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & a & a \\ 0 & a & a & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & a & a & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a & a & 0 & a & a & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a & 2a & a & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & a & 2a & a & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & a & a & 0 & a & a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & a & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & a & 0 & 0 \\ 0 & a & a & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & a & a & 0 \\ a & a & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & a & a \end{bmatrix}_{(2N+1)(2N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ c & d & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & b & b \\ 0 & c & d & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & b & b & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & c & d & 0 & b & b & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & c & b+d & b & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & b & b+c & d & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & b & b & 0 & c & d & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & b & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & d & 0 & 0 \\ 0 & b & b & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & c & d & 0 \\ b & b & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & c & d \end{bmatrix}_{(2N+1)(2N+1)},$$

$a = -\frac{1}{2h^2}$, $b = \frac{1}{h^2}$, $c = \frac{1}{h^2} - \frac{1}{\tau}$ and $d = \frac{1}{h^2} + \frac{1}{\tau}$ for the difference scheme (3.3).

Therefore, for the solution of the matrix equation (3.4), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 1, \tag{3.5}$$

where $u_M = \overrightarrow{0}$, α_j $(j = 1, ..., M-1)$ are $(2N+1)\times(2N+1)$ square matrices, β_j $(j = 1, ..., M-1)$ are $(2N+1)\times 1$ column matrices, α_1, β_1 are zero matrices and

$$\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n + C\beta_n), \ n = 1, ..., M-1. \end{cases}$$

NUMERICAL ANALYSIS

The numerical solutions are recorded for different values of N and M, and u_n^k represents the numerical solution of this difference scheme at $u(t_k, x_n)$. Table 1 is constructed for $N = M = 40, 80, 160$ respectively and the errors are computed by

$$E_M^N = \max_{-N\leq k\leq N,\ 1\leq n\leq M-1} |u(t_k, x_n) - u_n^k|.$$

If N and M are doubled, the values of the errors are decreases by a factor of approximately $1/2$ for the first order difference scheme (3.2) and $1/4$ for the second order of accuracy difference scheme (3.3). The errors presented in this table indicates the accuracy of difference schemes. We conclude that, the accuracy increases with the second order approximation.

Table 1: Error analysis E_M^N

	N = M = 40, 40	**N = M = 80, 80**	**N = M = 160, 160**
(3.2)	0.3015	0.1565	0.0798
(3.3)	2.5707×10^{-4}	6.4258×10^{-5}	1.6×10^{-5}

CHAPTER 4
CONCLUSION

This thesis is devoted to initial boundary value problem for parabolic type involutory differential equations: The following results are obtained: The history of involutory differential equations is studied. Fourier series, Laplace transform and Fourier transform methods are applied for the solution of six parabbolic type involutory partial differential equations. The first and second order of accuracy difference schemes are presented for the approximate solution of the one dimensional parabolic type involutory partial differential equation with Dirichlet condution. Numerical results are given. The Matlab implementation of the numerical solution is presented.

REFERENCES

A.Ardito, P. Ricciardi. (1980). Nonlinear Analysis: Theory, *Methods & Applications* 4 (2), 411-414.

O. Arino, M.L. Hbid, E. Ait Dads . (2006). Delay Differential Equations and Applications, (Springer, Berlin).

G. Di Blasio. (2003). Delay differential equations with unbounded operators on delay terms, *Nonlinear Analysis-Theory and Applications* 52(2), 1-18

A.L. Skubachevskii. (1994). On the problem of attainment of equilibrium for control-system with delay, *Doklady Akademii Nauk* 335(2), 157-160.

D. Agirseven. (2012). Approximate Solutions of delay parabolic equations with the Dirichlet condition, *Abstract and Applied Analysis*, 10(1155), 682752.

A. Ashyralyev, D. Agirseven. (2014). On source identifcation problem for a delay parabolic equation, *Nonlinear Analysis: Modelling and Control* 19(3), 335-349.

H. Poorkarimi, J. Wiener. (1999). Bounded solutions of nonlinear parabolic equations with time delay, 15-th Annual Conference of Applied Mathematics, Univ. of Central Oklahoma, *Electronic Journal of Differential Equations Conference* 02, 87-91.

A.H. Bhrawy, M.A. Abdelkawy, F. Mallawi. (2015). An accurate Chebyshev pseudospectral scheme formulti-dimensional parabolic problems with time delays, *Boundary Value Problems*, 2015(103).

V. Borukhov and P. Vabishchevich (2000). Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation.*Comput. Phys. Commun*.126, 32-36.

R. Nesbit, Delay Differential Equations for Structured Populations,Structured Population Models in Marine, Terrestrial and Freshwater Systems, Tuljapurkar & Caswell, ITP, 1997.

A. Ashyralyev, A. M. Sarsenbi. (2015). Well-posedness of an elliptic equation with involution, *Electronic Journal of Differential Equations* 2015 (284), 1-8.

.A. Ashyralyev, A. M. Sarsenbi. (2017). Well-posedness of a parabolic equation with the involution, *Numerical Functional Analysis and Optimization* 38(10), 1295-1304.

H. Poorkarimi, J. Wiener. (1999). Bounded solutions of nonlinear parabolic equations with time delay, textrm15th Annual Conference of Applied Mathematics, Univ. of Central Oklahoma, *Electronic Journal of Differential Equations Conference* 02 (1999), 87-91.

X. Lu, Combined. (1998). iterative methods for numerical solutions of parabolic problems with time delays, *Appl. Math. Comput.* 89 (1998) 213-224.

H. Egger, H. W. Engl, and M. V. Klibanov (2004). Global uniqueness and Hölder stability for recovering a nonlinear source term in a parabolic equation. *Inverse Probl.* 21:271.

Orazov and M. A. Sadybekov. (2012). On a class of problems of determining the temperature and density of heat sources given initial and nal temperature. *Siberian Math*. J. 53: 146-151.

D. Guidetti. (2012). Determining the source term in an abstract parabolic problem from a time integral of the solution. *Mediterr. J. Math*. 9(6), 11-633.

M. Ashyralyyeva and M. Ashyraliyev. (2016). On a second order of accuracy stable difference scheme for the solution of a source identi cation problem for hyperbolic-parabolic equations. *AIP Conf. Proc*. 1759:020023.

Bhalekar, S., & Patade, J. (2016). Analytical solutions of nonlinear equations with proportional delays. *Appl Comput Math*, 15, 331-445.

A. Ashyralyev, A. S. Erdogan, and O. Demirdag. (2012). On the determination of the right-hand side in a parabolic equation. Appl. *Numer. Math.* 62(11), 1672-1683.

C. Ashyralyyev, A. Dural, and Y. Sozen (2012). Finite difference method for the reverse parabolic problem with Neumann condition. *AIP Conf. Proc.* 1470, 102-105.

O. Sh.Mukhtarov and H. Demir. (1999). Coersiveness of the discontinuous initial-boundary value problem for parabolic equations. *Israel J. Math*. 114, 239-252.

D. Guidetti. (2012). Determining the source term in an abstract parabolic problem from a time integral of the solution. *Mediterr. J. Math.* 9(611)-633.

V.V. Vlasov, N.A. Rautian. (2016). Spectral Analysis of Functional Differential Equations, *Monograph, M.:MAKS Press* 488p. (Russian).

M. Sapagovas and K. Jakubelien˙e. (2012). Alternating direction method for two dimensional parabolic equation with nonlocal integral condition. Nonlinear Anal. *Model. Control*, 17(1):91-98.

S. Svajunas. (2012). Stability of the weighted splitting nite-difference scheme for a two-dimensional parabolic equation with two nonlocal integral conditions. *Comput.Math. Appl*, 64(11):3485-3499.

A. Ashyralyev, P. E. Sobolevskii. (2001). On the stability of the linear delay di erential and difference equations,Abstract and Applied Analysis 6(5), 267-297.

Twana Abbas. (2019). Schrödinger type Ivolutoury Partial Differential Equation, Master Thesis, Near East Universty, Nicosia, 62p.

K. Sriram., & M.S. Gopinathan. (2004). A two variable delay model for the circadian rhythm of Neurospora crassa, 231 (2004) 23-38.

J. Srividhyaa., & M.S. Gopinathan. (2006). A simple time delay model for eukaryotic cell cycle,241(2006), 617-627.

S. M. Shah, H. Poorkarimi, J. Wiener. (1986). Bounded solutions of retarded nonlinear hyperbolice quations, Bull. *Allahabad Math.* Soc. 1 (1986), 1-14.

A. Ashyralyev. (2011). On the problem of determining the parameter of a parabolic equation.*Ukr. Math. J*. 62(9), 1397-1408.

M. Sapagovas and K. Jakubelien˙e. (2012). Alternating direction method for two dimensional parabolic equation with nonlocal integral condition. *Nonlinear Anal. Model. Control*, 17(1):91-98.

APPENDICES

APPENDIX 1

MATLAB PROGRAMMING

In this part, Matlab programs are presented for the first and second orders of accuracy difference schemes.

1. Matlab Implementation of the First Order of Accuracy Difference Scheme of Problem (3.1)

```
clear all; clc; close all; delete '*.asv';
N=40;
M=N;
tau=pi/N;
h=pi/M;
a=-1/(h^2);
b=-1/tau;
c=2/h^2;
d=(1/tau)+(2/h^2);
A=zeros(2*N+1,2*N+1);
for k=2:N;
A(N+1,N+1)=2*a;
A(k,k)=a;
A(k,2*N+2-k)=a;
end;
for k=N+2: 2*N+1;
A(k,k)=a;
A(k,2*N+2-k)=a;
end;
A;
C=A;
B=zeros(2*N+1,2*N+1);
B(1,N+1)=1;
for k=2:N;
```

```
B(k,k-1)=b;
B(N+1,N+1)=c+d;
B(N+1,N)=b;
B(k,k)=d;
B(k,2*N+2-k)=c;
end;
for k=N+2:2*N+1;
B(k,k)=d;
B(k,2*N+2-k)=c;
B(k,k-1)=b;
end;
B;
D=eye(2*N+1,2*N+1);
for j=1:M+1;
for k=2:2*N+1;
fii(k,j)=cos((k-1-N)*tau)*sin((j-1)*h);

end;
fii(1,j)=0;
end;
alpha{1}=zeros(2*N+1,2*N+1);
betha{1}=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(2*N+1,M+1);
for j=M:-1:1;
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
```

```
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
es(k,j)=sin((k-1-N)*tau)*sin((j-1)*h);
end;
end;
%.ERROR ANALYSIS.;
maxes=max(max(abs(es)));
maxerror=max(max(abs(es-U)))
relativeerror=maxerror/maxes;

cevap1=[maxerror,relativeerror] ;
%figure;
%m(1,1)=min(min(abs(U)))-0.01;
%m(2,2)=nan;
%surf(m);
%hold;
%surf(es);rotate3d;axis tight;
%title('Exact Solution');
%figure;
%surf(m);
%hold;
%surf(U);rotate3d;axis tight;
%title('FIRST ORDER');
```

APPENDIX 2

MATLAB PROGRAMMING

2. Matlab Implementation of the second Order of Accuracy Difference Scheme of Problem (3.1)

```
clear all; clc; close all; delete '*.asv';
N=160;
M=N;
tau=pi/N;
h=pi/M;
a=-1/2*h^2;
b=1/h^2;
c=(1/h^2)-(1/tau);
d=(1/h^2)+(1/tau);
A=zeros(2*N+1,2*N+1);
for k=2:N;
A(N+1,N+1)=2*a;
A(N+2,N+1)=2*a;
A(N+2,N)=a;
A(N+1,N)=a;
A(N+2,N+2)=a;
A(N+1,N+2)=a;
A(k,k-1)=a;
A(k,k)=a;
A(k,2*N+3-k)=a;
A(k,2*N+3-k-1)=a;
end;
for k=N+3: 2*N+1;
A(k,k)=a;
A(k,k-1)=a;
A(k,2*N+3-k)=a;
```

```
A(k,2*N+2-k)=a;
end;
A;
C=A;
B=zeros(2*N+1,2*N+1);
B(1,N+1)=1;
for k=2:N;
B(N+1,N+1)=b+d;
B(N+2,N+1)=b+c;
B(N+2,N)=b;
B(N+1,N)=c;
B(N+2,N+2)=d;
B(N+1,N+2)=b;
B(k,k-1)=c;
B(k,k)=d;
B(k,2*N+3-k)=b;
B(k,2*N+3-k-1)=b;
end;
for k=N+3:2*N+1;
B(k,k)=d;
B(k,k-1)=c;
B(k,2*N+3-k)=b;
B(k,2*N+2-k)=b;
end;
B;
D=eye(2*N+1,2*N+1);
for j=1:M+1;
for k=2:2*N+1;
fii(k,j)=cos((k-1-N)*tau-tau/2)*sin((j-1)*h);

end;
```

```
fii(1,j)=0;
end;
alpha{1}=zeros(2*N+1,2*N+1);
betha{1}=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alpha{j-1});
alpha{j}=-Q*A;
betha{j}=Q*(D*(fii(:,j))-C*betha{j-1});
end;
U=zeros(2*N+1,M+1);
for j=M:-1:1;
U(:,j)=alpha{j}*U(:,j+1)+betha{j};
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
es(k,j)=sin((k-1-N)*tau)*sin((j-1)*h);
end;
end;
%.ERROR ANALYSIS.;
maxes=max(max(abs(es)));
maxerror=max(max(abs(es-U)))
relativeerror=maxerror/maxes;

cevap1=[maxerror,relativeerror] ;
%figure;
%m(1,1)=min(min(abs(U)))-0.01;
%m(2,2)=nan;
%surf(m);
%hold;
%surf(es);rotate3d;axis tight;
```

```
%title('Exact Solution');
%figure;
%surf(m);
%hold;
%surf(U);rotate3d;axis tight;
%title('SECOND ORDER');
```