AN APPROXIMATE SOLUTION OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR GENERAL SECOND ORDER LINEAR ELLIPTIC EQUATION

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

**By
MFON AUGUSTINE ESSIEN**

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2020

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Master of Science in Mathematics

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I declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

This study present an approximate solution of a nonlocal boundary value problem for second order linear elliptic equation on a rectangular domain. Dirichlet boundary condition were applied to obtain the solution of the problems given on the three sides of the rectangle, while on the fourth side the unknown function f was set and obtained which define the trace of the solution parallel at the midline of the rectangle.

We assume that, the boundary functions on three sides of the rectangle belong to the Hölder classes $C^{2,\lambda}$, $0 < \lambda < 1$. On the fourth side, the desired function f gives rise for a simple prove of the existence and uniqueness of the solution. The proposed method help in constructing the 5-point finite difference approximate solution of the general second order linear elliptic equation.

Keywords: Elliptic equation; nonlocal boundary value problem in a rectangular domain; finite difference method; Dirichlet problem; numerical solution

ÖZET

Bu çalışma, dikdörtgen bir alanda ikinci mertebeden doğrusal eliptik denklem için yerel olmayan sınır değer probleminin yaklaşık bir çözümünü sunmaktadır. Dikdörtgenin üç kenarında verilen problemlerin çözümünü elde etmek için Dirichlet sınır koşulu uygulanmış, dördüncü kenarda ise dikdörtgenin orta çizgisinde paralel çözeltinin izini tanımlayan bilinmeyen fonksiyon f ayarlanmış ve elde edilmiştir.

Dikdörtgenin üç kenarında sınır fonksiyonlarının Hölder $C^{2,\lambda}$, $0 < \lambda < 1$, sınıflarına ait olduğunu varsayıyoruz. Dördüncü kenarda, arzu edilen f fonksiyonu, çözümün varlığının ve tekliğinin basit bir kanıtını doğurmaktadır. Önerilen yöntem, genel ikinci mertebeden lineer eliptik denklemin 5 noktalı sonlu fark yaklaşık çözümünün oluşturulmasına yardımcı olur.

Anahtar Kelimeler: Eliptik denklem; dikdörtgensel bir alanda yerel olmayan sınır değer problemi; sonlu farklar yöntemi; Dirichlet problemi; sayısal çözüm

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CHAPTER 1
INTRODUCTION

Modeling real life situation in diverse disciplines like Economics, Applied Science, Engineering etc. most often lead to partial differential equation of different orders. Most of these partial differential equation cannot be solved analytically that's give rise to a numerical approximations. Obviously many numerical approach has been developed to replace analytical approach using partial differential equation.

Bitsadze and Samarskii (1969) introduced nonlocal boundary value problem for simplest generalizations of linear elliptic problems. Still on the research of reducing a nonlocal to local value problem, many authors were investigated the generalization of nonlocal boundary value problem and their approximate solution. Due to the simplicity of nonlocal condition difficulties arises in solving the exact and numerical solution of this problem. Furthermore, (Volkov; 2013) continues this problem on approximate grid solution of nonlocal boundary value problem for Laplace's equation on a rectangle. On the rectangle, Dirichlet boundary conditions were given on the three sides. At the fourth side, a function f was set to be unknown function using the condition that the equal to the trace of the solution on the parallel midline of the rectangle. The existence and uniqueness of this function were stated and the proposed method was used to generalize 5-point finite difference for the approximate solution of nonlocal boundary value problem. Volkov and Dosiyev (2016) proposed on the numerical solution of a multilevel nonlocal problem. 5-point approximate solution of the multilevel nonlocal boundary value problem for Laplace's equation using a Dirichlet problems was stated. Uniform estimate of the error of the approximate solution find from multilevel nonlocal problem is of order $O\left(h^2\right)$, where h is the mesh step.

In (Volkov; 2013) contraction mapping principle for solvability analysis of a nonlocal boundary value problem is investigated. For simplicity it was assumed that the boundary values given on the three sides of the rectangle have a second derivatives satisfying a Hölder condition. In particular, approximate solution was proved to converge uniformly on the grid to the solution of the differential problem at an $O\left(h^2\right)$ rate, where h is the mesh side.

Volkov et al (2013) focuses on the solution of a nonlocal probem. They were concluded that the solution of this problem defined as a solution of the first boundary value problem on the rectangle, by finding a function given as the boundary value on those side of the rectangle where the nonlocal condition was given. The proposed work was justified through the numerical experiments which support the analysis made. Gordeziani et al (2005) also worked on finite-difference methods for solution of nonlocal boundary value problems.

In chapter 2, on a solution of a nonlocal boundary value problem for general second order linear elliptic equation is considered. It was assumed that the boundary values given on the three sides of the rectangle were given and have a second derivatives satisfying a Hölder condition. On the fourth side of the rectangle the unknown function f is obtained by solving the Dirichlet boundary value problem on the rectangle. A special method was applied to find a continuous function. The solution of the nonlocal problem is defined as a sum of two Dirichlet problems. The solution of a considered problem give rise to a simple proof of the necessary and sufficient condition to prove ours claimed. See Theorem 1 and 2, simple proves for existence and uniqueness condition of a continuous function is stated and well proved. The desired function is obtained through the limit of infinite sequences of a continuous function.

In chapter 3, an approximate solution of the nonlocal boundary value problem for general second order linear elliptic equation on a rectangle using a finite difference method is considered. We assumed that the approximate continuous functions on th rectangle satisfy Hölder condition. We define L_h to be a new linear differential operator of averaging over four neighboring grid nodes. By the use of $n-$th iteration of the convergent fixed-point iterations the desired function was obtained.

CHAPTER 2
ON THE SOLUTION OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR GENERAL ELLIPTIC EQUATION

Our aim is to find the function from this domain which are continuous boundary value given on the three sides of the rectangle. The fourth side, the boundary function coincide with those on the middle of the rectangle parallel to this side. Our expected function seen not to be harmonic on open rectangle then continuous on the closed rectangle.

2.1 NONLOCAL BOUNDARY VALUE PROBLEM

Consider a linear space C^0 to be a space of continuous function with a close interval $x \in [0, 1]$, vanish at n-th of this interval. For any arbitrary function $f \in C^0$, the given function equipped with the norm is defined as,

$$\| f \|_{c^0} = \max_{0 \leq x \leq 1} | f(x) | . \tag{2.1}$$

Let R be an open rectangle with two variables x and y, then R can be defined as,

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\} . \tag{2.2}$$

On R, let γ^m denote the sides of the rectangle, where $m = 1, ..., 4$ from the right side numbered in a clockwise direction.

Again on R we take γ^j as a given continuous functions on the three sides where $j = 1, 2, 3$. It follow that

$$\begin{aligned} &\varphi^2 = \varphi^2 (x) , 0 \leq x \leq 1 \\ &\varphi^k = \varphi^k (y) , k = 1, 3., \ 0 \leq y \leq 2 \end{aligned} \tag{2.3}$$

$$\varphi^1 (2) = \varphi^2 (0)$$

$$\varphi^3 (2) = \varphi^2 (1)$$

generally

$$\varphi^k(0) = \varphi^k(1) = 0, k = 1, 3. \tag{2.4}$$

Now consider the boundary value problem

$$LU = g \text{ on } R \tag{2.5}$$

$$U = \varphi^1 \text{ on } \gamma^1 \tag{2.6}$$

$$U = \varphi^2 \text{ on } \gamma^2 \tag{2.7}$$

$$U = \varphi^3 \text{ on } \gamma^3 \tag{2.8}$$

$$U(x, 0) = U(x, 1) \text{ on } \gamma^4, \tag{2.9}$$

where

$$LU = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + a(x, y)\frac{\partial U}{\partial x} + b(x, y)\frac{\partial U}{\partial y} + c(x, y)U$$

with $C(x, y) \leq 0$.

For any $f \in C^0$, problem (2.5) – (2.9) has a unique classical solution U in the open rectangle R and continuous on the closed rectangle $\overline{R}$. Our interest is to obtain the desired function $f \in C^0$, which

$$U(x, 0) = U(x, 1), 0 \leq x \leq 1, \tag{2.10}$$

where U is the problem of (2.5) − (2.9).

Clearly, function U can be written as a sum of two functions V and W

$$U(x, y) = V(x, y) + W(x, y). \tag{2.11}$$

Consider a function $f \in C^0$ and $W(x,y)$ to be the solution of the Dirichlet problem

$$LW = 0 \text{ on } R \tag{2.12}$$

$$W = 0 \text{ on } \gamma^m,\ m = 1,2,3 \tag{2.13}$$

$$W = f \text{ on } \gamma^4, \tag{2.14}$$

where

$$LW = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + a(x,y)\frac{\partial W}{\partial x} + b(x,y)\frac{\partial W}{\partial y} + c(x,y)W.$$

Let B be the linear operator from C^0 to C^0, we define

$$Bf(x) = W(x,1) \in C^0, \tag{2.15}$$

where Bf denote the trace of the solution to Dirichlet problem (2.12) on the interval $\gamma^0 = \{(x,y) : 0 \le x \le 1, y = 1\} \subset \overline{R}$. We see that in problem (2.12) the boundary value problem are zero on the three sides γ^m, $m = 1,2,3$. The following inequality is true on Γ

$$|W(x,y)| \le \frac{1}{2} \|f\|_{c^0} (2-y),\ (x,y) \in \Gamma.$$

Consider the function

$$\overline{W} = \frac{1}{2} \|f\|_{c^0} (2-y),\ (x,y) \in R. \tag{2.16}$$

We have

$$|W(x,y)| \le \overline{W}(x,y) \text{ on } \Gamma.$$

It follows from the maximum principle see (Ber's book 1971).

Let us prove that

$$|W(x,y)| \le \overline{W}(x,y) \text{ on } R \tag{2.17}$$

hold, where

$$LW = 0. \tag{2.18}$$

Since (2.16) is

$$\overline{W} = \frac{1}{2} \parallel f \parallel_{c^0} (2 - y). \tag{2.19}$$

clearly

$$\overline{W} \geq 0. \tag{2.20}$$

We consider the following Lemma:

Lemma 2.1.1. *If* $\overline{W}$ *is non constant* $LW \geq 0$ *then* $W(x, y)$ *can't take its positive maximum on* R*.*

Lemma 2.1.2. *If* $\overline{W}$ *is non constant* $LW \leq 0$ *then* $W(x, y)$ *can't take its negative minimum on* R*.*

Lemma 2.1.3. *Let* $\mid \overline{W}(x, y) \mid= \frac{1}{2} \parallel f \parallel_{c^0} (2 - y)$ *and let* $W(x, y)$ *be the solution to the problem* $(2.12) - (2.14)$*. Assume that* $b(x, y) \geq 0$ *on* R *in* (2.12)*. Then the inequality* $\mid W(x, y) \mid \leq \overline{W}(x, y)$ *on* R *hold,*

Proof. We define

$$\overline{W}(x, y) = \frac{1}{2} \parallel f \parallel_{c^0} (2 - y).$$

Then it is clear that

$$\mid W(x, y) \mid \leq \overline{W}(x, y) \text{ on } \gamma^m, m = 1, 2, ..., 4. \tag{2.21}$$

Let us consider the function

$$h^{\mp}(x, y) = \overline{W}(x, y) \pm \mid W(x, y) \mid.$$

It implies that

$$h^{\mp}(x, y) \geq 0, \text{on } \gamma^m, m = 1, 2, ..., 4,$$

where

$$Lh^{\mp}(x,y) = \frac{\partial^2 h^{\mp}}{\partial x^2} + \frac{\partial^2 h^{\mp}}{\partial y^2} + a(x,y)\frac{\partial h^{\mp}}{\partial x} + b(x,y)\frac{\partial h^{\mp}}{\partial y} + c(x,y)h^{\mp}. \tag{2.22}$$

We take the partial derivative of $L\overline{W}(x,y)$ and $LW(x,y)$ with respect to x and y, we have

$$L\overline{W} \pm LW = -b(x,y)\left(-\frac{1}{2}\parallel f \parallel\right) + c(x,y)\frac{1}{2}\parallel f \parallel (2-y) \pm 0,$$

where $LW = 0$ in (2.18).

We assume that $b(x,y) \geq 0$, then

$$Lh^{\mp}(x,y) \leq 0 \text{ on } R \tag{2.23}$$

$$h^{\mp}(x,y) \geq 0 \text{ on } \gamma^m, m = 1,2,...,4.$$

By using the maximum principle see (Ber's et al 1971), directly

$$h^{\mp}(x,y) \geq 0 \text{ on } \overline{R},$$

$$|\ W(x,y)\ | \leq \overline{W}(x,y) \text{ on } \overline{R}.$$

Since

$$\overline{W} = \frac{1}{2}\parallel f \parallel_{c^0}(2-y) - W(x,y) \geq 0 \text{ on } \Gamma,$$

it follows that

$$\overline{W} = \frac{1}{2}\parallel f \parallel_{c^0}(2-y) - W(x,y) \geq 0 \text{ on } \overline{R}$$

and

$$-W(x,y) \leq \frac{1}{2}\parallel f \parallel_{c^0}(2-y)$$

$$|\ W(x,y)\ | \leq \frac{1}{2}\parallel f \parallel_{c^0}(2-y).$$

Therefore

$$|\ W\ | \leq \overline{W}. \tag{2.24}$$

Since

$$|\ W(x,y)\ |\leq \frac{1}{2}\ \|\ f\ \|_{C^0}\ (2-y)\ ,\ (x,y)\ \text{on}\ R,$$

by (2.15) we have

$$\|\ Bf\ \|\leq \frac{1}{2}\ \|\ f\ \|_{C^0}, f\in C^0 \tag{2.25}$$

which shows

$$|\ B\ |\leq \frac{1}{2}. \tag{2.26}$$

Then, the norm of operation B is at most $\frac{1}{2}$. □

Consider the Dirichlet problem

$$LV = g \text{ on } R, \tag{2.27}$$

$$V = \varphi^m \text{ on } \gamma^m\ , m = 1, 2, 3,$$

$$V = 0 \text{ on } \gamma^4, \tag{2.28}$$

where φ^m, $m = 1, 2, 3$ denote boundary value functions and

$$LV = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + a(x,y)\frac{\partial V}{\partial x} + b(x,y)\frac{\partial V}{\partial y} + c(x,y)V.$$

We set

$$\sigma^0 = \sigma^0(x) = V(x,1) \in C^0. \tag{2.29}$$

From (2.10) it show that

$$f(x) = U(x,0), 0 \leq x \leq 1.$$

Let

$$\varphi(x) = f(x) = U(x,0). \tag{2.30}$$

By virtue of equation (2.11) and put $y = 1$ we have

$$U(x,1) = \sigma^0(x) + B\varphi(x)\ , 0 \leq x \leq 1, \tag{2.31}$$

where U is the solution of (2.5), σ^0 is the function given in (2.29) and φ use as a boundary values in equation (2.5).

2.2 EXISTENCE AND UNIQUENESS OF A SOLUTION

2.2.1 EXISTENCE OF THE SOLUTION

Theorem 2.2.1. *For any arbitrary function* $\varphi \in C^0$*, with equality* $U(x,0) = \varphi(x) = U(x,1)$*,* $0 \le x \le 1$*, holds if and only if* φ *satisfy*

$$\varphi(x) = \sigma^0(x) + B\varphi(x) \ , 0 \le x \le 1 \tag{2.32}$$

Where U *is the solution of problem* (2.5).

Proof. The proof is trivi'al □

Theorem 2.2.2. *There is unique function* $\varphi \in C^0$ *for which equation* (2.32) *holds.*

Proof. Consider a linear space C^0 consisting an infinite sequence of functions $\left\{\psi^k\right\}_{k=0}^{\infty}$ where $k = 0, 1, 2, ...$ for $k = 0$, $\psi^0 = 0$, for $k = n$, we have

$$\psi^n = B\left(\sigma^0 + \psi^{n-1}\right), n = 1, 2, \dots . \tag{2.33}$$

For $k = n + 1$, we have

$$\psi^{n+1} = B\left(\sigma^0 + \psi^n\right), n = 1, 2, \dots \ . \tag{2.34}$$

We subtract (2.33) from (2.34) , it yield

$$\psi^{n+1} - \psi^n = B\left(\sigma^0 + \psi^n\right) - B\left(\sigma^0 + \psi^{n-1}\right).$$

By virtue of (2.33) and (2.34) ,

$$\| \psi^{n+1} - \psi^n \|_{C^0} = \| B\left(\psi^n - \psi^{n-1}\right) \|_{C^0} \ , n = 1, 2, \dots .$$

From (2.26)

$$\| \psi^{n+1} - \psi^n \|_{C^0} \le \frac{1}{2} \| \left(\psi^n - \psi^{n-1}\right) \|_{C^0} \ , n = 1, 2, \dots . \tag{2.35}$$

It follows that the sequence of a function is a Cauchy sequence and is convergence. Hence (2.33) is fundamental and it therefore has a limit.

$$\lim_{n\to\infty} \psi^n = \psi \in C^0. \tag{2.36}$$

Again we consider a linear operator B with ψ^n from C^0 to C^0 as,

$$\left\{B\psi^k\right\}_{k=0}^{\infty},$$

where $k = 0, 1, 2, ...$, for $k = 0$, $B\psi^0 = 0$, for $k = n$,

$$B\psi^n = B\left(B\left(\sigma^0 + \psi^{n-1}\right)\right), n = 1, 2, ... \ .$$

For $k = n + 1$,

$$B\psi^{n+1} = B\left(B\left(\sigma^0 + \psi^n\right)\right), n = 1, 2, ... \ .$$

We subtract $B\psi - B\psi^n$ yield

$$\| B\psi - B\psi^n \|_{C^0} = \| B(\psi - \psi^n) \|_{C^0}, n = 1, 2, ... \ .$$

And (2.26)

$$\| B(\psi - \psi^n) \|_{C^0} \leq \frac{1}{2} \| (\psi - \psi^n) \|_{C^0}, n = 1, 2, ... \ .$$

The following limit exist

$$\lim_{n\to\infty} B\psi^n = B\psi \in C^0. \tag{2.37}$$

Putting (2.33) , (2.36) and (2.37) together we have

$$\psi^n = B\left(\sigma^0 + \psi^{n-1}\right), n = 1, 2, ... \ .$$

By virtue of (2.36) it become

$$\psi = B\left(\sigma^0 + \psi\right). \tag{2.38}$$

Analogy (2.38) become

$$\varphi = \sigma^0 + \psi \tag{2.39}$$

$$\varphi = \sigma^0 + B\varphi.$$

It show that equation (2.39) satisfies equality (2.32) which is the desired function. □

2.2.2 THE UNIQUENESS OF THE SOLUTION

Proof. Assume that there are two functions $\varphi^k \in C^0$, $k = 1, 2$ holds in (2.32) that is

$$\varphi^1(x) = \sigma^0(x) + B\varphi^1(x),$$
$$\varphi^2(x) = \sigma^0(x) + B\varphi 2(x). \tag{2.40}$$

We are to show that

$$\varphi^1(x) = \varphi^2(x).$$

We subtract $\varphi^1(x) - \varphi^2(x)$ and by virtue of (2.39) we have

$$\| \varphi^1(x) - \varphi^2(x) \|_{C^0} = \| B\varphi^1(x) - B\varphi^2(x) \|_{C^0} = \| B\left(\varphi^1(x) - \varphi^2(x)\right) \|_{C^0} \tag{2.41}$$
$$\leq \frac{1}{2} \| \varphi^1(x) - \varphi^2(x) \|_{C^0}.$$

Clearly, we have

$$\| \varphi^1(x) - \varphi^2(x) \|_{C^0} \leq 0 \tag{2.42}$$
$$\| \varphi^1(x) - \varphi^2(x) \|_{C^0} = 0.$$

Hence it become

$$\varphi^1(x) = \varphi^2(x).$$

Equation (2.40) holds. □

CHAPTER 3
APPROXIMATE SOLUTION OF NONLOCAL BOUNDARY VALUE PROBLEM BY THE FINITE DIFFERENCE METHOD

Everywhere below we can consider a function f on the set E to belong in a class of $C^{k,\lambda}(E)$ if f has k-th derivative of E satisfy the Hölder condition with an exponent $0 < \lambda < 1$. We assume that the function φ^m in (2.6) and (2.8) are in the class $C^{2,\lambda}(\gamma^m)$, $m = 1, 2, 3$.

3.1 FINITE DIFFERENCE METHOD

Lemma 3.1.1. *The function* $\sigma^0(x)$ *defined by* (2.29) *belongs to* $C^{2,\lambda}$,$0 < \lambda < 1$ *on the interval* $0 \leq x \leq 1$.

Lemma 3.1.2. *The function* $\psi = \psi(x)$ *found as limit* (2.36) *of an infinite sequence of continuous functions is in the class* $C^{2,\lambda}(\gamma^m)$, $0 < \lambda < 1$, $m = 1, 2, 3$

Proof. We consider equation (2.38) defined by $\psi = B\left(\sigma^0 + \psi\right)$, the function ψ is the trace on the interval $\gamma^0 = \{(x, y) : 0 \leq x \leq 1, y = 1\} \subset \overline{R}$, of the solution to problem (2.12), where $f = \sigma^0 + \psi$ is in C^0. We see that the boundary values on the sides γ^1 and γ^3 in (2.12) are zero. Hence this problem (2.12) can be extended through γ^1 and γ^3 to the domain in which γ^0 is strictly in its interior (R). Then equation (2.36) is in the class $C^{2,\lambda}$,$0 < \lambda < 1$, on the interval $0 \leq x \leq 1$. □

Now, we construct a square mesh D_h, obtain with the lines x,$y = 0, h, 2h, ...$ let $h = \frac{1}{N}$ denote a mesh side, where $N > 2$, is positive integer. D_h denote the set of nodes of the square mesh or grid. R_h denote the set of grid on γ^m.

$$R_h = R \cap D_h$$

$$\overline{R}_h = \overline{R} \cap D_h$$

$$\gamma_h^m = \gamma^m \cap D_h, m = 1, ..., 4.$$

Let C_h^0 denote the linear of grid function on the interval $[0,1]_h$ where x vanish at $x = 0$ and $x = 1$. For any arbitrary given function $f_h \in C_h^0$. We define a space together with a norm as,

$$\| f_h \|_{C_h^0} = \max_{x \in [0,1]_h} | f_h(x) | . \tag{3.1}$$

It is obvious that the space is complete equipped with this norm. Consider $C = C_0, C_1 ...$ to be a constants which are independent of h.

Consider L_h to be a new linear difference operator of averaging over four neighboring grid nodes.

$$\begin{aligned} L_h U_h(x,y) = h^{-2}[&U_h(x+h,y) + U_h(x-h,y) + U_h(x,y+h) \\ &+U_h(x,y-h) - 4U_h(x,y)] \\ &+a(x,y)\left[\frac{U_h(x+h,y) - U_h(x-h,y)}{2h}\right] \\ &+b(x,y)\left[\frac{U_h(x,y+h) - U_h(x,y-h)}{2h}\right] \\ &+c(x,y)U_h(x,y). \end{aligned} \tag{3.2}$$

Let V_h be a solution of the finite difference problem. It follow that the boundary value problem (2.27) is approximated by the system of grid equation.

$$\begin{aligned} L_h V_h &= g_h \text{ on } R_h \\ V_h &= \varphi^m \text{ on } \gamma_h^m,\ m = 1,2,3 \\ V_h &= 0 \text{ on } \gamma_h^4. \end{aligned} \tag{3.3}$$

We define

$$\widetilde{\sigma}_h^0 = \widetilde{\sigma}_h^0(x) = V_h(x,1) \in C_h^0,\ x \in [0,1]_h, \tag{3.4}$$

where V_h is a solution of the grid equation (3.3).

Lemma 3.1.3. *It is true that*

$$\| \widetilde{\sigma}_h^0 - \sigma_h^0 \|_{C_h^0} \le C_1 h^2$$

where $\widetilde{\sigma}_h^0$ *is defined in* (3.4) *and* σ_h^0 *is a trace of function* (2.29) *on* $[0,1]_h$, C_1 *is a constant independent of* h.

Let B_h be a linear operator from C_h^0 to C_h^0.
Consider $f_h \in C_h^0$ from equation (2.12), we defined a new system of grid equations as,

$$
\begin{aligned}
&W_h = B_h W_h \text{ on } R_h\\
&W_h = 0 \text{ on } \gamma_h^m,\ m = 1, 2, 3\\
&W_h = f_h \text{ on } \gamma_h^4.
\end{aligned}
\tag{3.5}
$$

We set

$$B_h f_h(x) = W_h(x, 1) \in C_h^0, \tag{3.6}$$

where W_h is he solution to problem (3.5), by virtue to inequality (2.25), similarly we have

$$\| B_h f_h(x) \|_{C_h^0} \leq \frac{1}{2} \| f_h(x) \|_{C_h^0}, f_h \in C_h^0. \tag{3.7}$$

Let $\left\{\widetilde{\psi}_h^k\right\}_{k=0}^{\infty} \in C_h^0$ be an infinite sequence and for $k = 0, 1, 2, ...$

$$
\begin{aligned}
&k = 0, \widetilde{\psi}_h^0 = 0\\
&k = n, \widetilde{\psi}_h^n = B_h\left(\widetilde{\sigma}_h^0 + \widetilde{\psi}_h^{n+1}\right), n = 1, 2...,
\end{aligned}
\tag{3.8}
$$

where

$$\widetilde{\sigma}_h^0 = \widetilde{\sigma}_h^0(x) = V_h(x, 1) \in C_h^0. \tag{3.9}$$

We compute the elements on the grid $[0, 1]_h$ of (3.8) with the corresponding elements of sequence (2.33) of continuous functions. When $k = n$ then ψ_h^n is a trace of the function ψ^n on a close interval $0 < x_h < 1$, also σ_h^0 and $\left(B\sigma^0\right)_h$ be the trace of the function σ^0 and $B\sigma^0$ on $0 < x_h < 1$.
We take the norm difference of each correspond element in (3.8) and (2.33).
For

$$k = 0, \| \widetilde{\psi}_h^0 - \psi_h^0 \|_{C_h^0} = 0, \tag{3.10}$$

$$k = 1, \widetilde{\psi}_h^1 - \psi_h^1 = B_h \widetilde{\sigma}_h^0 - \left(B\sigma^0\right)_h.$$

We add to both side

$$-B_h\sigma_h^0 + B_h\sigma_h^0.$$

We have

$$\widetilde{\psi}_h^1 - \psi_h^1 = B_h\left(\widetilde{\sigma}_h^0 - \sigma_h^0\right) + B_h\sigma_h^0 - \left(B\sigma^0\right)_h.$$

By taking the norm and it become

$$\| \widetilde{\psi}_h^1 - \psi_h^1 \|_{C_h^0} \leq \| B_h\left(\widetilde{\sigma}_h^0 - \sigma_h^0\right) \|_{C_h^0} + \| B_h\sigma_h^0 - \left(B\sigma^0\right)_h \|_{C_h^0}. \tag{3.11}$$

From (3.11) we see that

$$\| \widetilde{\psi}_h^1 - \psi_h^1 \|_{C_h^0} \leq \frac{1}{2} \| B_h\left(\widetilde{\sigma}_h^0 - \sigma_h^0\right) \|_{C_h^0}.$$

By virtue of lemma (3.1.3) , we have the inequality

$$\| B_h\left(\widetilde{\sigma}_h^0 - \sigma_h^0\right) \|_{C_h^0} \leq \frac{c_1h^2}{2}. \tag{3.12}$$

Also by virtue of Lemma 3.1.1 the function defined in (2.29) $\sigma^0(x)$ belongs to a class $C^{2,\lambda}, 0 < \lambda < 1$ on the interval $0 \leq x \leq 1$. Hence $\sigma^0(x) \in C^{2,\lambda}, 0 < \lambda < 1$.
From Theorem 1.1 (see in (Volkov, 1979)), we have

$$\max_{\overline{R}^h} | V_h - V | \leq Ch^2.$$

Analogy we have

$$\| B_h\sigma_h^0 - \left(B\sigma^0\right)_h \|_{C_h^0} \leq C_2h^2. \tag{3.13}$$

Putting (3.12) and (3.13) into equation (3.11) yield

$$\| \widetilde{\psi}_h^1 - \psi_h^1 \|_{C_h^0} \leq \left(\frac{c_1h^2}{2} + C_2h^2\right) \leq C_3h^2, C_3 = \frac{c_1}{2} + C_2, \tag{3.14}$$

where C_3 is a constant independent of h.
We consider when $k = n \geq 2$

$$\widetilde{\psi}_h^n - \psi_h^n = B_h\left(\widetilde{\sigma}_h^0 + \widetilde{\psi}_h^{n-1}\right) - \left(B\left(\sigma^0 + \psi^{n-1}\right)\right)_h. \tag{3.15}$$

We add $-B_h\left(\sigma^0+\psi_h^{n-1}\right)+B_h\left(\sigma^0+\psi_h^{n-1}\right)$ in (3.15) then it become

$$\widetilde{\psi}_h^n-\psi_h^n=B_h\left(\widetilde{\sigma}_h^0+\widetilde{\psi}_h^{n-1}\right)-B_h\left(\sigma^0+\psi_h^{n-1}\right)+B_h\left(\sigma^0+\psi_h^{n-1}\right)-\left(B\left(\sigma^0+\psi^{n-1}\right)\right)_h. \quad (3.16)$$

From (3.16) the right handset become

$$B_h\widetilde{\sigma}_h^0-\left(B\sigma^0\right)_h+B_h\left(\widetilde{\psi}_h^{n-1}-\psi_h^{n-1}\right)+B_h\psi_h^{n-1}-\left(B\psi^{n-1}\right)_h.$$

By virtue of Lemma (3.1.3) equation (3.15) become

$$\begin{aligned}&\| \widetilde{\psi}_h^n-\psi_h^n \|_{C_h^0}\leq \| B_h\widetilde{\sigma}_h^0-\left(B\sigma^0\right)_h \|_{C_h^0}+\| B_h\left(\widetilde{\psi}_h^{n-1}-\psi_h^{n-1}\right) \|_{C_h^0}\\ &+\| B_h\psi_h^{n-1}-\left(B\psi^{n-1}\right)_h \|_{C_h^0}, n\geq 2.\end{aligned} \quad (3.17)$$

But

$$\| B_h\widetilde{\sigma}_h^0-\left(B\sigma^0\right)_h \|_{C_h^0}\leq C_3h^2. \quad (3.18)$$

We further estimate the second and the third terms. Then we take the sup norm of ψ^n defined by (2.33) as follows:

$$\sup_{0\leq n<\infty} \| \psi^n \|_{C^0}\leq \| \sigma^0 \|_{C^0}.$$

Again

$$\begin{aligned}\psi&=B\left(\sigma^0+\psi\right)\\ \psi^n&=B\left(\sigma^0+\psi^n\right).\end{aligned} \quad (3.19)$$

Taking the sup norm we have

$$\begin{aligned}\sup_{0\leq n<\infty} \| \sigma^0+\psi^n \|_{C^0}&=\sup_{0\leq n<\infty} \| \sigma^0 \|_{C^0}+\sup_{0\leq n<\infty} \| \psi^n \|_{C^0}\\ &=\| \sigma^0 \|_{C^0}+\| \sigma^0 \|_{C^0}\leq 2 \| \sigma^0 \|_{C^0},\end{aligned} \quad (3.20)$$

where $\sigma^0=\sigma^0(x)=V(x,1)\in C^0$.

Let B be from C^0 to C^0 then for $k=n-1$,

$$\psi^{n-1}=B\left(\sigma^0+\psi^{n-2}\right), n\geq 2$$

be the trace on the interval

$$\gamma^0 = \{(x, y) : 0 \leq x \leq 1, y = 1\} \subset \overline{R}$$

for $n \geq 2$, we consider a new system of Dirichlet boundary problem.

$$\frac{\partial^2 V^n}{\partial x^2} + \frac{\partial^2 V^n}{\partial y^2} + a(x, y)\frac{\partial V^n}{\partial x} + b(x, y)\frac{\partial V^n}{\partial y} + c(x, y)V^n = g(x, y) \text{ on } R. \tag{3.21}$$

$$V^n = 0 \text{ on } \gamma^m, m = 1, 2, 3$$

$$V^n = \sigma^0 + \psi^{n-2} \text{ on } \gamma^4,$$

where V^n=$V^n(x, y)$ is the solution of problem (3.21) which is extended to an odd function from R to R_1 through the sides γ^1 and γ^3, where

$$R_1 = \{(x, y) : -1 < x < 2, 0 < y < 2\}. \tag{3.22}$$

Analogy by estimate of (3.20) and (3.21) become

$$\sup_{(x,y)\in R_1} | V^n |= \sup_{(x,y)\in R} | V^n |=\| \sigma^0 + \psi^{n-2} \|_{C^0} \leq 2 \| \sigma^0 \|_{C^0}, n \geq 2... \ . \tag{3.23}$$

By virtue of Lemma (3.1.1) consider the open rectangle of R_1, the distance between γ^0 to R_1 is positive . We state the following estimate, by taking the derivative of $\psi^{n-1}(x), 0 \leq x \leq 1$

$$\frac{d\psi^{n-1}}{dx} + \frac{d^2\psi^{n-1}}{dx^2} + ... + \frac{d^m\psi^{n-1}}{dx^m}.$$

It follow by maximum principle

$$\max_{0\leq x\leq 1} | \frac{d^m\psi^{n-1}}{dx^m} |\leq C_m \| \sigma^0 \|_{C^0}, n \geq 2, m \geq 4, \tag{3.24}$$

where

$$\sigma^0 = \sigma^0(x) = V(x, 1) \in C^{\cdot 0}.$$

For $n \geq 2$.

We take a new Dirichlet boundary value problem,

$$\frac{\partial^2 Z^n}{\partial x^2} + \frac{\partial^2 Z^n}{\partial y^2} + a(x, y)\frac{\partial Z^n}{\partial x} + b(x, y)\frac{\partial Z^n}{\partial y} + c(x, y)Z^n = 0 \text{ on } R \tag{3.25}$$

$Z^n = 0$ on γ^m , $m = 1, 2, 3$

$Z^n = \psi^{n-1}$ on γ^4.

Consider the approximate grid of (3.25)

$$Z_h^n = B_h Z_h^n \text{ on } R_h,$$

$$Z_h^n = 0 \text{ on } \gamma^m , m = 1, 2, 3$$

$$Z_h^n = \psi^{n-1} \text{ on } \gamma^4. \quad (3.26)$$

Clearly, solution (3.24) satisfies Hölder condition with an interval $0 < \lambda < 1, 0 \leq x \leq 1$. From (3.21) , γ^1 and γ^3 are zero which take the derivative $\frac{d^{2q}\psi^{n-1}}{dx^{2q}}$ where $q = 0, 1, 2$ with $0 \leq x \leq 1$. From (3.21) and (3.24) the estimate hold by using the maximum principle

$$\max_{(x,y)\in\overline{R}} \left| \frac{\partial^4 Z^n (x, y)}{dx^4} \right| = \max_{(x,y)\in\overline{R}} \left| \frac{\partial^4 Z^n (x, y)}{dy^4} \right|$$
$$= \max_{0\leq x\leq 1} \left| \frac{d^4\psi^{n-1}}{dx^4} \right| \leq C_4 \parallel \sigma^0 \parallel_{C^0}, n \geq 2, \quad (3.27)$$

where

$$\sigma^0 = \sigma^0 (x) = V (x, 1) \in C^0$$

and C_4 is a constant independent of h.

Now we take the estimate $\parallel B_h\psi_h^{n-1} - \left(B\psi^{n-1}\right)_h \parallel_{C_h^0}$.

Lemma 3.1.4. *Consider Q_h and $\overline{Q}_h$ to be the solution of the system of grid equations*

$$Q_h = B_h Q_h + \xi_h \text{ on } R_h, \quad Q_h = 0 \text{ on } \Gamma_h$$
$$\overline{Q}_h = B_h \overline{Q}_h + \overline{\xi}_h \text{ on } \overline{R}_h, \quad \overline{Q}_h = 0 \text{ on } \Gamma_h, \quad (3.28)$$

where $\Gamma_h = \overline{R}_h \backslash R_h$, ξ_h and $\overline{\xi}_h$ are the function given on $R_h \in | \xi_h | \leq \overline{\xi}_h$ on R_h and $| Q_h | \leq \overline{Q}_h$ on $\overline{R}_h$. The proof (3.28) *follow directly from finite difference methods for elliptic equation by Samarski and Andreev,*(1976)

Now we defined

$$W_h (x, y) = \left\langle \begin{matrix} 0, & (x,y)\in\Gamma_h \\ C_4 \frac{h^2}{12} \|\sigma^0\|_{C^0} (2--y), & (x,y)\in R_h, \end{matrix} \right. , \quad (3.29)$$

where

$$\sigma^0 = \sigma^0(x) = V(x,1) \in C^0$$

and C_4 is a constant independent of h. Clearly (3.29) is a function which satisfies the system of the grid functions

$$\begin{aligned} &W_h = B_h W_h + \beta_h \text{ on } R_h \\ &W_h = 0 \text{ on } \Gamma_h. \end{aligned} \tag{3.30}$$

We take the

$$\min_{(x,y)\in\overline{R}} \beta_h \geq C_4 \frac{h^2}{24} \parallel \sigma^0 \parallel_{C^0} . \tag{3.31}$$

From (3.2) we have

$$\begin{aligned} LU_{i,j} = &\left(\frac{1}{h^2} + \frac{a_{i,j}}{2h}\right) U_{i+1,j} + \left(\frac{1}{h^2} - \frac{a_{i,j}}{2h}\right) U_{i-1,j} + \left(\frac{1}{h^2} + \frac{b_{i,j}}{2h}\right) U_{i,j+1} \\ &+ \left(\frac{1}{h^2} - \frac{b_{i,j}}{2h}\right) U_{i,j-1} + \left(c_{i,j} + \frac{-4}{h^2}\right) U_{i,j}. \end{aligned} \tag{3.32}$$

Let

$$BU_{i,j}$$

denote the right hand side of (3.32), we have

$$LU_{i,j} = BU_{i,j}\ ,\ LU_{i,j} = f_{i,j.}$$

It implies that

$$\begin{aligned} &U_{i,j} = BU_{i,j} + f_{i,j}, \quad U_{i,j} = 0 \\ &V_{i,j} = BV_{i,j} + f_{i,j}, \quad V_{i,j} = 0. \end{aligned} \tag{3.33}$$

Lemma 3.1.5. *If* $| f_{i,j} | \leq \overline{f}_{i,j}$ *then* $| U_{i,j} | \leq U_{i,j}$

Proof. The inequality $U_{i,j} \geq 0$ on $\varphi + \gamma$ but $U_{i,j} = V_{i,j} + W_{i,j}$ and $V_{i,j} = U_{i,j} - W_{i,j}$ since $LU_{i,j} = f_{i,j} = \overline{f}_{i,j} + f_{i,j} \geq 0$ then $LV_{i,j} = f_{i,j} = \overline{f}_{i,j} - f_{i,j} \geq 0$.
Consider a boundary condition, directly we have $\overline{U}_{i,j} + U_{i,j} \geq 0$, $\overline{U}_{i,j} - U_{i,j} \geq 0$, $\overline{V}_{i,j} + V_{i,j} \geq 0$, $\overline{V}_{i,j} - V_{i,j} \geq 0$ Clearly $U_{i,j} \geq 0$, implies that the $| U_{i,j} | \leq V_{i,j}$. ☐

We assume that

$$\left(\frac{1}{h^2} \pm \frac{a_{i,j}}{2h}\right) \geq 0, \ \left(\frac{1}{h^2} \pm \frac{b_{i,j}}{2h}\right) \geq 0, \ \left(c_{i,j} \pm \frac{-4}{h^2}\right) \geq 0. \tag{3.34}$$

We take the

$$M = \max\left\{\max_{\overline{R}} \mid a_{i,j} \mid, \max_{\overline{R}} \mid b_{i,j} \mid\right\}.$$

Consider a Taylor expansion of (3.32) up to fourth derivative on $(x \pm h, y)$ and $(x, y \pm h)$.

$$Z^n(x+h,y) = Z^n(x,y) + \frac{h\partial Z^n(x,y)}{\partial x} + \frac{h^2\partial^2 Z^n(x,y)}{2!\partial x^2} \tag{3.35}$$

$$+\frac{h^3\partial^3 Z^n(x,y)}{3!\partial x^3} + \frac{h^4\partial^4 Z^n(\xi,y)}{4!\partial x^4} \tag{3.36}$$

$$Z^n(x,y+h) = Z^n(x,y) + \frac{h\partial Z^n(x,y)}{\partial y} + \frac{h^2\partial^2 Z^n(x,y)}{2!\partial y^2} \tag{3.37}$$

$$+\frac{h^3\partial^3 Z^n(x,y)}{3!\partial y^3} + \frac{h^4\partial^4 Z^n(x,\xi)}{4!\partial y^4}$$

$$AZ^n = 2Z^n(x,y) + \frac{h^4}{4!}\left(\frac{\partial^4 Z^n(\xi,y)}{\partial x^4} + \frac{\partial^4 Z^n(x,\xi)}{\partial y^4}\right)$$

$$Z_h^n(x,y) = AZ^n - \frac{h^4}{24}\left(\frac{\partial^4 Z^n(\xi,y)}{\partial x^4} + \frac{\partial^4 Z^n(x,\xi)}{\partial y^4}\right)$$

$$Z_h^n(x,y) = AZ_h^n - \frac{h^4}{24}\left(\frac{\partial^4 Z_h^n(\xi,y)}{\partial x^4} \pm \frac{\partial^4 Z_h^n(x,\xi)}{\partial y^4}\right). \tag{3.38}$$

Let

$$\varepsilon_h^n = Z_h^n - Z^n \text{ on } \overline{R}_h, n \geq 2 \tag{3.39}$$

where Z_h^n and Z^n are the solution of grid in equation (3.26) and (3.25) we have from (3.38)

$$\varepsilon_h^n = A\varepsilon_h^n - \frac{h^4}{24}\left(\frac{\partial^4 Z_h^n(\xi,y)}{\partial x^4} + \frac{\partial^4 Z_h^n(x,\xi)}{\partial y^4}\right).$$

Let

$$\eta = \frac{h^4}{24}\left(\frac{\partial^4 Z_h^n(\xi,y)}{\partial x^4} \pm \frac{\partial^4 Z_h^n(x,\xi)}{\partial y^4}\right).$$

So that

$$\max_{(x,y)R_h} \mid \eta_h^n \mid \leq \frac{h^4}{24}\left(\max_{(x,y)\overline{R}} \mid \frac{\partial^4 Z_h^n(\xi,y)}{\partial x^4} \mid + \max_{(x,y)\overline{R}} \mid \frac{\partial^4 Z_h^n(x,\xi)}{\partial y^4} \mid\right) \tag{3.40}$$

$$\leq \frac{C_4 h^4}{24} \parallel \sigma^0 \parallel_{C^0} \leq \min_{(x,y)R_h} \beta_h, n \geq 2.$$

From Lemma (3.1.5) we have

$$\max_{(x,y)\bar{R}_h} \mid Z_h^n(x,y) - Z^n(x,y) \mid \leq \max_{(x,y)\bar{R}_h} W_h(x,y) \leq C_4^* h^2 \parallel \sigma^0 \parallel_{C^0}, n \geq 2, \tag{3.41}$$

where

$$C_4^* = \frac{h^4}{12}.$$

In equation (3.29) and equation (3.25) , we set

$$Z^n = \psi^{n-1} \text{ on } \gamma^4.$$

Consider the definition of operators B and B_h yield

$$B_h \psi_h^{n-1} = Z_h^n(x,y), \ x \in [0,1]_h$$

$$B_h \psi^{n-1} = Z^n(x,y), \ x \in [0,1]. \tag{3.42}$$

Putting (3.41) and (3.42) into consideration we have

$$\parallel B_h \psi_h^{n-1} - \left(B\psi^{n-1}\right)_h \parallel_{C_h^0} = \max_{x \in [o,1]_h} \mid B_h \psi_h^{n-1} - \left(B\psi^{n-1}\right)_h \mid_{C_h^0}$$

$$= \max_{x \in [o,1]_h} \mid Z_h^n(x,1) - Z^n(x,1) \mid \leq \max_{(x,y)\bar{R}_h} \mid Z_h^n(x,y) - Z^n(x,y) \mid$$

$$\leq C_4^* h^2 \parallel \sigma^0 \parallel_{C^0}, n \geq 2. \tag{3.43}$$

Since (3.43) is bounded above by

$$\max_{(x,y)\bar{R}_h} \mid Z_h^n(x,y) - Z^n(x,y) \mid \leq B_h \parallel \left(\widetilde{\psi}_h^{n-1} - \psi_h^{n-1}\right) \parallel_{C_h^0} \tag{3.44}$$

$$\leq \frac{1}{2} \parallel \left(\widetilde{\psi}_h^{n-1} - \psi_h^{n-1}\right) \parallel_{C_h^0} \leq \frac{1}{2} C_4^* h^2 \parallel \sigma^0 \parallel_{C^0}.$$

Then equation (3.17) become

$$\parallel \widetilde{\psi}_h^n - \psi_h^n \parallel_{C_h^0} \leq C_3 h^2 + \frac{1}{2} C_4^* h^2 \parallel \sigma^0 \parallel_{C^0} \leq C_0 h^2,$$

$$\parallel \widetilde{\psi}_h^n - \psi_h^n \parallel_{C^0} \leq C_0 h^2, n \geq 2 \tag{3.45}$$

where

$$C_0 = C_3 + \frac{1}{2}C_4^* \parallel \sigma^0 \parallel_{C^0},$$

and $\widetilde{\psi}_h^n$ is the nth element of sequence (3.8) and ψ_h^n is the trace of nth element of sequence (2.33). From (2.33) we take the estimate

$$\psi^0 = 0,\ \psi^k = B\left(\sigma^0 + \psi^{k-1}\right).$$

For $k = 1$ we have

$$\psi^1 = B\sigma^0.$$

By virtue of (3.45), we have

$$\parallel \psi^1 \parallel_{C^0} \leq \frac{1}{2} \parallel \sigma^0 \parallel_{C^0} . \tag{3.46}$$

For $k \geq 2$ we have the following

$$\psi^n = B\left(\sigma^0 + \psi^{n-1}\right),\ \ \psi^{n-1} = B\left(\sigma^0 + \psi^{n-2}\right)$$

$$\psi^n - \psi^{n-1} = B\left(\sigma^0 + \psi^{n-1}\right) - B\left(\sigma^0 + \psi^{n-2}\right)$$

$$\psi^n - \psi^{n-1} = B\left(\psi^{n-1} - \psi^{n-2}\right). \tag{3.47}$$

Since the sequence is fundamental,satisfying cauchy sequence we take the limit of (3.47) and its become

$$\parallel \psi^n - \psi^{n-1} \parallel_{C^0} \leq \frac{n}{2} \parallel \sigma^0 \parallel_{C^0} . \tag{3.48}$$

Consider $k = n + m$, we have

$$\psi^{n+m} = B\left(\sigma^0 + \psi^{(n+m)-1}\right).$$

By taking a norm, yield

$$\parallel \psi^{n+m} - \psi^{n-} \parallel_{C^0} = \parallel B\left(\sigma^0 + \psi^{(n+m)-1}\right) - B\left(\sigma^0 + \psi^{n-1}\right) \parallel_{C^0} .$$

By virtue of (3.48) we have

$$\| \psi^{n+m} - \psi^{n-} \|_{C^0} \leq 2^{-n} \left(1 - 2^{-m}\right) \| \sigma^0 \|_{C^0}, \tag{3.49}$$

where n and m are positive integer and $\sigma^0 = V(x,1)$.

Now consider the triangle inequality

$$\| \psi^n - \psi \|_{C^0} \leq \| \psi^{n+m} - \psi^n \|_{C^0} + \| \psi^{n+m} - \psi \|_{C^0} \tag{3.50}$$

We take the limit in (3.50) as $m \to \infty$ we have

$$\| \psi^n - \psi \|_{C^0} \leq 2^{-n} \| \sigma^0 \|_{C^0}, n \geq 1. \tag{3.51}$$

Putting (3.45) and (3.51) corresponding inequality

$$\| f_h \|_{C_h^0} \leq \| f \|_{C^0} .$$

We find that

$$\| \widetilde{\psi}_h^n - \psi_h \|_{C_h^0} \leq \| \widetilde{\psi}_h^n - \psi_h^n \|_{C_h^0} + \| \psi_h^n - \psi_h \|_{C_h^0} \leq \| \widetilde{\psi}_h^n - \psi_h^n \|_{C_h^0}$$

$$+ \| \psi^n - \psi \|_{C_h^0} \leq 2^{-n} \| \sigma^0 \|_{C^0} + C_0 h^2, n \geq 2. \tag{3.52}$$

Consider equation (2.38) and (2.32), for $n \geq 2$.

Let

$$\widetilde{\varphi}_h^n = \widetilde{\sigma}_h^n + \widetilde{\psi}_h^n \tag{3.53}$$

where $\widetilde{\sigma}_h^n = \widetilde{\sigma}_h^n(x) = \widetilde{V}_h^n(x)$on $[0,1]_h$. take the difference of (3.53) and (2.32) we have

$$\widetilde{\varphi}_h^{n-} - \psi_h = \left(\widetilde{\sigma}_h^{n-} + \widetilde{\psi}_h^{n-}\right) - \left(\sigma_h^0 - \psi_h\right) = \left(\widetilde{\sigma}_h^{n-} - \sigma_h^0\right) + \left(\widetilde{\psi}_h^{n-} - \psi_h\right)$$

where φ_h is the trace of the desired function φ on the grid $[0,1]_h$. We have

$$\| \widetilde{\psi}_h^n - \psi_h \|_{C_h^0} \leq \| \widetilde{\sigma}_h^n - \sigma_h^0 \|_{C_h^0} + \| \widetilde{\psi}_h^n - \psi_h \|_{C_h^0} \tag{3.54}$$

$$\leq C_1 h^2 + 2^{-n} \| \sigma^0 \|_{C^0} + C_0 h^2 \tag{3.55}$$

$$\| \widetilde{\psi}_h^n - \psi_h \|_{C_h^0} \leq (C_1 + C_0) h^2 + 2^{-n} \| \sigma^0 \|_{C^0} .$$

Let

$$C' = C_1 + C_0.$$

Hence we have

$$\| \widetilde{\psi}_h^n - \psi_h \|_{C_h^0} \leq 2^{-n} \| \sigma^0 \|_{C^0} + C'h^2. \tag{3.56}$$

Let $U(x, y)$ be the solution from (2.5)-(2.9) with $\varphi = \sigma^0 + \psi$ which satisfying $U(x, y)$, Let $U_h(x, y)$ be the solution of system of grid equations

$$U_h = B_h U_h \text{ on } R_h$$

$$U_h = \varphi^m \text{ on } \gamma_h^m,\ m = 1, 2, 3 \tag{3.57}$$

$$U_h = \varphi \text{ on } \gamma_h^4. \tag{3.58}$$

Clearly the desired function φ^m, $m = 1, 2, 3$, are in the class $C^{2,\lambda}, 0 < \lambda < 1$ on R. From Lemma(3.1.1) and Lemma (3.1.2) seen that σ^0 and ψ are also in the class $C^{2,\lambda}, 0 < \lambda < 1$. Directly in theorem (1.1) 'the method of composite regular nets for the Laplace's equation on polygons'. We have the inequality

$$\max_{(x,y)\overline{R}_h} | U_h(x, y) - U(x, y) | \leq C''h^2, \tag{3.59}$$

where U is the solution of problem (3.5)-(3.14), U_h is the solution of system (3.57) and C'' is a constant independent of h.

From (3.57) take $n \geq 2$, then the actual system of the grid equations become

$$\widetilde{U}_h^n = B_h \widetilde{U}_h^n \text{ on } R_h$$

$$\widetilde{U}_h^n = \varphi^m \text{ on } \gamma^m,\ m = 1, 2, 3 \tag{3.60}$$

$$\widetilde{U}_h^n = \widetilde{\varphi}_h^n \text{ on } \gamma_h^4. \tag{3.61}$$

By virtue of inequality (3.56) yield

$$\max_{(x,y)\overline{R}_h} | \widetilde{U}_h^n(x, y) - U_h(x, y) | \leq 2^{-n} \| \sigma^0 \|_{C^0} + C'h^2, n \geq 2 \tag{3.62}$$

where U_h and $\widetilde{U}_h^n$ are the respective solutions of system (3.57) ,(3.61) and C' is a constant independent of h.

Finally by estimation of (3.59) and (3.62) we have the final estimate

$$\max_{(x,y)\bar{R}_h} \mid \widetilde{U}_h^n(x,y) - U(x,y) \mid \leq 2^{-n} \parallel \sigma^0 \parallel_{C^0} + Ch^2, n \geq 2 \tag{3.63}$$

where $U(x,y)$ and $\widetilde{U}_h^n$ are the respective solutions of systems (2.5)-(2.9) ,(3.61) , $\varphi = \sigma^0 + \psi$ is a desired function and $C = C' + C''$ is a constant independent of n or h

$$n = \left[\frac{2\ln h^{-1}}{\ln 2}\right] + 1$$

where the right hand side of (3.63) is $O\left(h^2\right)$.

CHAPTER 4
ON THE NUMERICAL SOLUTION OF A MULTILEVEL NONLOCAL PROBLEM FOR ELLIPTIC EQUATION

4.1 NONLOCAL BOUNDARY VALUE PROBLEM

We take R to be an open rectangle define as,

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}. \tag{4.1}$$

From R, let $\gamma^d, d = 1, 2, 3, 4$ denote it sides including its end point, numerated from the right hand side by starting with the side which lies on the y-axis. Let $\gamma = U_{d=1}^{4} \gamma^d$ denote the boundary of R . take $\overline{R} = R \cup \gamma$. Consider $(\eta_1, \eta_2, ..., \eta_m)$ and $(\alpha_1, \alpha_2, ..., \alpha_m)$ to be a given numbers which satisfy some fixed number say $\delta > 0$, then the inequalities hold

$$0 < \delta \leq \eta_1 < \eta_2 < ... < \eta_m < 2 \tag{4.2}$$

$$\left(1 - \frac{\eta_1}{2}\right) \sum_{k=1}^{m} |\ \alpha_k \ |< 1. \tag{4.3}$$

(4.2) and (4.3) is denoted by

$$R_\delta = \{(x, y) : 0 < x < 1, \delta < y < 2\} \tag{4.4}$$

$$Y_\mu = \left\{\left(x, \eta_\mu\right) : 0 \leq x \leq 1\right\}, \mu = 1, 2, ..., m. \tag{4.5}$$

Let C^0called a linear space of functions on x variable which are continuous on the closed interval $0 \leq x \leq 1$, and vanish at $x = 0$ and $x = 1$.

For any arbitrary function $f \in C^0$ with a norm , C^0 is said to be a complete space which define as,

$$\|\ f \ \|_{C^0} = \max_{0 \leq x \leq 1} |\ f(x)\ | .$$

We consider a multilevel nonlocal boundary value problem on R,

$$LU = g \text{ on } R \tag{4.6}$$

$$U = 0 \text{ on } \gamma^1 \cup \gamma^3$$
$$U = \tau(x) \text{ on } \gamma^2 \tag{4.7}$$

$$\sum_{k=1}^{m} \alpha_k U(x, \eta_k) = U(x, 0), \; 0 \leq x \leq 1 \text{ on } \gamma^4, \tag{4.8}$$

Where

$$LU = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + a(x,y)\frac{\partial U}{\partial x} + b(x,y)\frac{\partial U}{\partial y} + c(x,y)U \text{ and } \tau(x) \in C^0$$

is a given function.

The existence and uniqueness of problem (4.6) and (4.8) are given in chapter 2 .

The problem (4.6) and (4.8) can be written as a sum of two functions

$$U(x,y) = U(x,y) + W(x,y). \tag{4.9}$$

$V(x,y)$ is the multilevel problem define as,

$$LV = g \text{ on } R$$
$$V = 0 \text{ on } \gamma \backslash \gamma^3$$
$$V = \tau(x) \text{ on } \gamma^2, \tag{4.10}$$

where

$$LV = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + a(x,y)\frac{\partial V}{\partial x} + b(x,y)\frac{\partial V}{\partial y} + c(x,y)V.$$

$W(x,y)$ is a solution of problem

$$LW = 0 \text{ on } R$$
$$W = 0 \text{ on } \gamma \backslash \gamma^4$$
$$W = f \text{ on } \gamma^4, \tag{4.11}$$

where

$$LW = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + a(x,y)\frac{\partial W}{\partial x} + b(x,y)\frac{\partial W}{\partial y} + c(x,y)W.$$

For any arbitrary function $f \in C^0$, let B_i be an operator as

$$B_i f(x) = W(x, \eta_i) \in C^0,\ i = 1, 2, ..., m. \tag{4.12}$$

The norm $| B_i |$ hold by the inequality

$$| B_i | < \left(1 - \frac{\eta_i}{2}\right),\ i = 1, 2, ..., m. \tag{4.13}$$

The arbitrary function $f \in C^0$ in (4.11) is given by

$$f = \varphi + \sum_{k=1}^{m} \alpha_k \psi_k \tag{4.14}$$

where

$$\varphi = \sum_{k=1}^{m} \alpha_k V(x, \eta_k). \tag{4.15}$$

Consider an infinite sequences $\left\{\psi_i^n\right\}_{n=1}^{\infty}$, where the system of nonlocal equations with $\psi_i \in C^0, i = 1, 2, ..., m$ unknown function.

We define

$$\psi_i = B_i\left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k\right), i = 1, 2, ..., m \tag{4.16}$$

Where φ is given in (4.15). In (4.16) we use the method of fixed point iteration to find the solution of the system

Let

$$\psi_i^0 = 0$$

$$\psi_i^n = B_i\left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k^{n-1}\right), i = 1, 2, ..., m.\ n = 1, 2, ... \tag{4.17}$$

and

$$\psi_i^{n+1} = B_i\left(\varphi + \sum_{k=1}^{m}\alpha_k\psi_k^n\right), i = 1, 2, ..., m. \; n = 1, 2, ... \; . \tag{4.18}$$

We subtract (4.17) and (5.1)

$$\psi_i^{n+1} - \psi_i^n = B_i\sum_{k=1}^{m}\alpha_k\left(\psi_k^n - \psi_k^{n-1}\right), i = 1, 2, ..., m, n \geq 1. \tag{4.19}$$

Since

$$\max_{0\leq i\leq m} \| \psi_i^1 - \psi_i^0 \| \leq \| \varphi \| \, .$$

It follow from (4.19)

$$\max_{0\leq i\leq m} \| \psi_i^{n+1} - \psi_i^n \| \leq q\max_{0\leq i\leq m} \| \psi_i^{n+1} - \psi_i^n \| \leq q^n \| \varphi \|, n \geq 1$$

where

$$q = | B_1 | \sum_{k=1}^{m} | \alpha_k | < 1.$$

By simplicity, the sequence of function (4.16) are fundamental. It has the limit

$$\lim_{n\to\infty}\psi_i^n = \psi_i \in C^0, i = 1, 2, ..., m. \tag{4.20}$$

Since

$$\| B_i\psi_k^n - B_i\psi_k \| = \| B_i\left(\psi_k^n - \psi_k\right) \| \leq \| \psi_k^n - \psi_k \|,$$

then, it also has the limit

$$\lim_{n\to\infty} B_i\psi_k^n = B_i\psi_k \in C^0, i, k = 1, 2, ..., m.$$

It follow that the limit as $n \to \infty$ then (4.17) become

$$\psi_i = B_i\left(\varphi + \sum_{k=1}^{m}\alpha_k\psi_k\right), i = 1, 2, ..., m$$

where $\psi_i \in C^0$ is defined.

We seek for the existence and uniqueness of the solution.

Theorem 4.1.1. *The nonlocal boundary value problem* (4.6) *and* (4.8) *can have only one classical solution*

Proof. We consider problem (4.6) and (4.8) to be a two classical solution. We take $\widetilde{U}$ to be their difference, clearly $\widetilde{U}$ is also a classical solution, now since

$$\widetilde{U}(x,\eta_i) = \widetilde{\psi}_i(x), 0 \leq x \leq 1, i = 1,2,...,m, \tag{4.21}$$

where $\widetilde{\psi}_i \in C^0$. Then we have

$$\widetilde{U}(x,0) = \sum_{i=1}^{m} \alpha_i \widetilde{\psi}_i, 0 \leq x \leq 1. \tag{4.22}$$

But

$$\widetilde{\psi}_i = B_i \sum_{k=1}^{m} \alpha_k \widetilde{\psi}_k, i = 1,2,...,m.$$

Clearly

$$\max_{0\leq i\leq m} \| \widetilde{\psi}_i \| \leq | B_1 | \sum_{k=1}^{m} \alpha_k \widetilde{\psi}_k \leq | B_1 | \sum_{k=1}^{m} | \alpha_k | \max_{0\leq k\leq m} \| \widetilde{\psi}_k \| \tag{4.23}$$

$$= q \max_{0\leq k\leq m} \| \widetilde{\psi}_k \|, q < 1.$$

It hold if and only if

$$\max_{0\leq i\leq m} \| \widetilde{\psi}_i \| = 0, i = 1,2,...,m.$$

It implies that

$$\widetilde{\psi}_i = 0, i = 1,2,...,m.$$

Hence we have

$$\widetilde{U}(x,0) = 0, 0 \leq x \leq 1.$$

Therefore $\widetilde{U}$ is continuous on $\overline{R}$. □

Theorem 4.1.2. *The nonlocal boundary value problem* (4.6) *and* (4.8) *has a unique classical solution*

Proof. We consider the existence of the solution define, $\psi_i = \psi_1, \psi_2, ..., \psi_n \in C^0$ but

$$\varphi = \sum_{k=1}^{m} \alpha_k \psi_k.$$

From (4.10) and f define as,

$$f = \varphi + \sum_{k=1}^{m} \alpha_k \psi_k \in C^0.$$

By the the definition of operator B_i we have

$$W(x, \eta_i) = B_i \left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k \right), i, k = 1, 2, ..., m.$$

But

$$\psi_i = B_i \left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k \right), i, k = 1, 2, ..., m.$$

Clearly we have

$$W(x, \eta_i) = \psi_i.$$

Analogy

$$V(x, \eta_i) = \varphi_i.$$

Since $U(x, y)$ can be express as a sum of tow finite function

$$U(x, y) = V(x, y) + W(x, y). \tag{4.24}$$

It follow that

$$\sum_{i=1}^{m} \alpha_i U(x, \eta_i) = \sum_{i=1}^{m} \alpha_i V(x, \eta_i) + \sum_{i=1}^{m} \alpha_i W(x, \eta_i) \tag{4.25}$$

$$= \sum_{i=1}^{m} \alpha_i \varphi_i + \sum_{i=1}^{m} \alpha_i \psi_i \tag{4.26}$$

$$= f.$$

We have

$$f = W(x, 0) = U(x, 0), 0 \leq x \leq 1.$$

Since $U(x, 0)$ is non-zero then we conclude that (4.24) is a classical solution of the (4.6) and (4.8). □

CHAPTER 5

APPROXIMATE SOLUTION OF THE MULTILEVEL NONLOCAL PROBLEM BY THE FINITE DIFFERENCE METHOD

Since $\phi \in C^{k,\lambda}(D)$, if ϕ take $k-$derivative on D and satisfy Hölder condition on $0 < x < 1$. Then, from (4.6) and (4.8) a given function $\tau(x)$ on the side γ^2 of rectangle R, belongs to the class $C^{2,\lambda}, 0 < x < 1$.

Lemma 5.0.1. *The function* $\varphi = \sum_{k=1}^{m} \alpha_k V(x, \eta_k) \in C^{2,\lambda}, 0 < x < 1$, *on the interval* $0 \leq x \leq 1$

Proof. Since $\tau(x) \in C^{2,\lambda}, 0 < x < 1$, on γ^2 and the problem $V(x, y)$ is continuous on $\overline{R}$, directly it follows from theorem 8.1 differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle'. That is $V(x, y) \in C^{2,\lambda}(\overline{R}_\delta), 0 < x < 1$, but R_δ is defined in (4.4). Clearly from (4.15), each of the functions $V(x, \eta_k) \in C^{2,\lambda}, 0 < x < 1, k = 1, 2, ..., m$ on the interval $0 \leq x \leq 1$. □

Lemma 5.0.2. *The functions* $\psi_i = B_i\left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k\right)$, *where* $i = 1, 2, ..., m$ *are the limits of the sequences* (4.17) *are in the class* $C^{2,\lambda}, 0 < x < 1$ *on the interval* $[0, 1]$.

Proof. Since the function ψ_i is the trace of the solution $W(x, y)$ of problem (4.11) on P_i and $f = \left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k\right) \in C^0$, from (4.6) $W = 0$ on $\gamma^1 \cup \gamma^3$, $P_i \subset R$, $i = 1, 2, ..., m$ also from Lemma (3.1.2) $\psi_i = B_i\left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k\right) = B_i f(x)$. We have $\psi_i = B_i f(x)$, but (4.12) we have $B_i f(x) = W(x, \eta_i) \in C^0$, $i = 1, 2, ..., m$ hence $\psi_i = W(x, \eta_i) \in C^{2,\lambda}, 0 < x < 1$ on the interval $0 \leq x \leq 1$ $\forall$ $i = 1, 2, ..., m$. □

We set D_h to be a square mesh, find with $x, y = 0, h, 2h, ...$ Let $h = \frac{1}{N}$ denote step size, $N > 2$ is an integer. Then h gets less than half of the minimum length of the interval $[0, \eta_1], [\eta_1, \eta_2], ..., [\eta_m, 2]$ which denoted by P_j. We have

$$P_j h \leq \eta_j < \left(P_j + 1\right) h. \tag{5.1}$$

Let $R_h = D_h \cap R$, γ_h^d be the set of grids on γ^d, $d = 1, 2, 3, 4$ but $\gamma_h = U_{d=1}^4 \gamma_h^d$ and $\overline{R}_h = R_h \cup \gamma_h$ the interval $1 \leq \mu \leq m$, where μ is an integer. Consider Y_μ^h to be the intersection points of the line $y = \eta_\mu$ with the grid line $x = ih, i = 0, 1, ..., N$. We take an approximate closed interval as

$$[0,1]_h = \left\{ x = x_i,\ x_i = ih, i = 0, 1, ..., N, h = \frac{1}{N} \right\}$$

to be the set of nodes on the interval $[0, 1]$, where h is called the step size.
Let C_h^0 denote a set of grid functions f_h on $0 \leq x_h \leq 1$ but $f_h(0) = f_h(1) = 0$. Then the space equipped with the norm can be define as ,

$$\| f_h \|_{c_h^0} = \max_{0 \leq x_h \leq 1} | f_h(x) | .$$

Let C_k, $k = 0, 1, 2, ...$ be a constant which are independent of h.
Let V_h be the problem of the finite difference problem

$$\begin{aligned} L_h V_h &= g_h \text{ on } R_h \\ V_h &= \tau_h(x) \text{ on } \gamma_h^2 \\ V_h &= 0 \text{ on } \gamma_h \backslash \gamma_h^2, \end{aligned} \tag{5.2}$$

Where

$$L_h V_h = \frac{\partial^2 V_h}{\partial x^2} + \frac{\partial^2 V_h}{\partial y^2} + a(x,y)\frac{\partial V_h}{\partial x} + b(x,y)\frac{\partial V_h}{\partial y} + c(x,y) V_h \text{ and } \tau_h(x)$$

is the trace of $\tau(x)$ on γ_h^2.
We consider the approximate solution of $U_h(x, y)$ as,

$$\begin{aligned} L_h U_h(x,y) = h^{-2[} & U_h(x+h,y) + U_h(x-h,y) \\ & + U_h(x,y+h) + U_h(x,y-h) - 4U_h(x,y)] \\ & + a(x,y)\left[\frac{U_h(x+h,y) - U_h(x-h,y)}{2h}\right] \\ & + b(x,y)\left[\frac{U_h(x,y+h) - U_h(x,y-h)}{2h}\right] \\ & + c(x,y) U_h(x,y). \end{aligned} \tag{5.3}$$

Directly from chapter 2 problem (5.2) has a unique solution. From theorem [1.1] on the method of composite meshes for Laplace's equation on polygon' the estimate follow directly

$$\max_{(x,y)\in \overline{R}_h} | V_h - v_h | \leq C_1 h^2, \tag{5.4}$$

where V_h is the problem (5.2) and v_h is the trace of problem (4.10) on $\overline{R}_h$.
We set

$$\widetilde{\varphi}_j(x) = \left(V\left(x, p_j h\right) \frac{\left(p_j + 1\right) h - \eta_j}{h} + V\left(x, \left(p_j + 1\right) h\right) \frac{\eta_j - p_j h}{h} \right)$$

$$\in C^0 \text{ on } x \in [0,1], j = 1, 2, ..., m. \tag{5.5}$$

We compare (4.10) and (5.5) its follow that the estimate (6.1) on differential properties of solutions of the Laplace's and Poisson equations on a parallelepiped and efficient error estimates of the method of nets we have,

$$\| \widetilde{\varphi}_j(x) - V\left(x, \eta_j\right)_h \|_{C_h^0} \leq C_2 h^2, j = 1, 2, ..., m, \tag{5.6}$$

where $\left(\widetilde{\varphi}_j - \varphi_j\right)_h$ is the trace of $\left(\widetilde{\varphi}_j - \varphi_j\right)$ on $[0,1]_h$.
We consider the approximate solution of $\widetilde{\varphi}_j, h(x)$ as,

$$\widetilde{\varphi}_j, h(x) = \left(V_h\left(x, p_j h\right) \frac{\left(p_j + 1\right) h - \eta_j}{h} + V_h\left(x, \left(p_j + 1\right) h\right) \frac{\eta_j - p_j h}{h} \right)$$

$$\in C_h^0 \text{ on } x \in [0,1]_h, j = 1, 2, ..., m \tag{5.7}$$

where V_h is defined in problem (5.2).
Now we compare (5.4) and (5.6) its become

$$\| \widetilde{\varphi}_j, h(x) - V\left(x, \eta_j\right)_h \|_{C_h^0} \leq C_3 h^2, j = 1, 2, ..., m. \tag{5.8}$$

We construct another finite problem W_h as,

$$W_h = AW_h \text{ on } R_h$$

$$W_h = 0 \text{ on } \gamma_h^m, m = 1, 2, 3$$

$$W_h = f_h \text{ on } \gamma_h^4, \tag{5.9}$$

where $f_h \in C_h^0$, is an arbitrary function, since $f_h = f_h(x) \in C_h^0$ and $B_j^h : C_h^0 \to C_h^0$.
Then

$$B_j^h f_h = \left(W_h\left(x, p_j h\right) \frac{\left(p_j + 1\right) h - \eta_j}{h} + W_h\left(x, \left(p_j + 1\right) h\right) \frac{\eta_j - p_j h}{h} \right)$$

$$\in C_h^0 \text{ on } x \in [0,1]_h, j = 1, 2, ..., m, \tag{5.10}$$

where f_h, B_j and W_h are called a grid function, a linear operator and a solution to a given problem see (5.9) .

Its follows from chapter (2) and by virtue of lemma [1 − 3] we estimate the inequality by analogy yield,

$$| W_h\left(x, p_j h\right) |\leq \frac{1}{2} \| f_h \|_{C_h^0} \left(2 - p_j h\right)$$
$$| W_h\left(x, \left(p_j + 1\right) h\right) |\leq \frac{1}{2} \| f_h \|_{C_h^0} \left(2 - \left(p_j + 1\right) h\right). \tag{5.11}$$

Putting (5.10) and (5.11) together we have

$$\| B_j^h f_h \|_{C_h^0} \leq \| f_h \|_{C_h^0} \left(1 - \frac{\eta_j}{2}\right), j = 1, 2, ..., m. \tag{5.12}$$

Let

$$q_j = 1 - \frac{\eta_j}{2}$$

then the norm of

$$| B_j |< 1 - \frac{\eta_j}{2},\ 0 < q_j < 1. \tag{5.13}$$

We define

$$\widetilde{\varphi}_h = \sum_{j=1}^{m} \alpha_j \widetilde{\varphi}_j, h\left(x\right) \in C_h^0 \text{ on } x \in [0, 1]_h\,, j = 1, 2, ..., m. \tag{5.14}$$

But $\widetilde{\varphi}_j, h\left(x\right)$ is define in (5.7) . Now putting (4.15) , (5.8) into consideration its become

$$\| \widetilde{\varphi}_h - \varphi_h \|_{C_h^0} \leq C_3 h^2. \tag{5.15}$$

But φ_h is the trace of

$$\varphi = \sum_{k=1}^{m} \alpha_k V\left(x, \eta_k\right).$$

For any $\widetilde{f_h} \in C_h^0$ we defined $\widetilde{f_h}$ as ,

$$\widetilde{f_h} = \widetilde{\varphi}_h + \sum_{j=1}^{m} \alpha_j \widetilde{\psi}_{j,h} \text{ on } \gamma_h^4, \tag{5.16}$$

where $\widetilde{\psi}_{j,h} \in C_h^0, \;\; j = 1, 2, .., m$.

Let

$$\widetilde{\psi}_{j,h} = B_j^h \left(\widetilde{\varphi}_h + \sum_{k=1}^{m} \alpha_k \widetilde{\psi}_{k,h} \right), k = 1, 2, ..., m \tag{5.17}$$

where $\widetilde{\psi}_{j,h}$ is the solution of the system of equations (4.9). By virtue of chapter two, equations (3.8) is the system of equations which is fundamental. We can write directly the solution of (5.17) as a limit of the following sequences in C_h^0.

$$\widetilde{\psi}_{j,h}^0 = 0$$

$$\widetilde{\psi}_{j,h}^n = B_j^h \left(\widetilde{\varphi}_h + \sum_{k=1}^{m} \alpha_k \widetilde{\psi}_{k,h}^{n-1} \right), k = 1, 2, ..., m \text{ and } n = 1, 2.. \;\; . \tag{5.18}$$

Clearly the

$$\lim_{n\to\infty} \widetilde{\psi}_{j,h}^n = \lim_{n\to\infty} B_j^h \widetilde{\varphi}_h + B_j^h \sum_{k=1}^{m} \alpha_k \lim_{n\to\infty} \widetilde{\psi}_{j,h}^{n-1} \tag{5.19}$$

$$\widetilde{\psi}_{j,h} = B_j^h \widetilde{\varphi}_h + B_j^h \widetilde{\psi}_{j,h}$$

$$\widetilde{\psi}_{j,h} = B_j^h \left(\widetilde{\varphi}_h + \widetilde{\psi}_{j,h} \right). \tag{5.20}$$

We compare (5.12) , (5.16) and (5.20) we see that $\widetilde{f_h}$ is the trace of f_h on $[0, 1]_h$ since $\widetilde{f_h} = \widetilde{\psi}_{j,h}$. Then the inequality (5.12) is converges to a unique solution of equation (5.20) .

Now let equation (4.17) and (5.18) on $[0, 1]_h$ be compared. Consider $\psi_{j,h}^n$ to be the trace of $\psi_{j,h}^n$ on $[0, 1]_h$ also φ_h and $\left(B_j \varphi\right)_h$ to be the trace of φ and $\left(B_j \varphi\right)$ on $[0, 1]_h$.

Lemma 5.0.3. . *The inequality hold*

$$\| \, B_j^h \sum_{k=1}^{m} \alpha_k \psi_{k,h}^{n-1} - \left(B_j^h \sum_{k=1}^{m} \alpha_k \psi_{k,h}^{n-1} \right)_h \|_{C_h^0} \leq C_3^0 h^2 \tag{5.21}$$

where C_3^0 is a constant independent of n and h.

Proof. Let f say to be a non-zero, analogy f_h is also a non-zero hence problem (4.11) and (5.9) seen to has only a trivial solution. its follow from (5.21) , when $n = 1$, then (4.12) , (4.13) and (4.17) hold. □

Again when $n \geq 2$ we have

$$\| \psi_j^n \|_{C^0} \leq \| B_j \varphi \|_{C^0} + \| \left(B_j \sum_{k=1}^{m} \alpha_k \psi_k^{n-1} \right) \|_{C^0} \tag{5.22}$$

$$\leq \| B_j \|_{C^0} \| \varphi \|_{C^0} + \| B_j \|_{C^0} \max_{1\leq j\leq m} \| \psi_j^{n-1} \| \left(\sum_{k=1}^{m} | \alpha_k | \right) \tag{5.23}$$

$$\leq \left(1 - \frac{\eta_j}{2}\right) \| \varphi \|_{C^0} + \left(1 - \frac{\eta_j}{2}\right) \max_{1\leq j\leq m} \| \psi_j^{n-1} \| \left(\sum_{k=1}^{m} | \alpha_k | \right)$$

where

$$| B_j | < 1 - \frac{\eta_j}{2}. \tag{5.24}$$

Generally, for any $n, 1 \leq n < \infty$, recall that $q_j = 1 - \frac{\eta_j}{2}, 0 < q_j < 1$ we have

$$\max_{1\leq j\leq m} \| \psi_j^n \|_{C^0} \leq q \left(\| \varphi \|_{C^0} + \max_{1\leq j\leq m} \| \psi_j^{n-1} \| \right) \leq \frac{q}{1-q} \| \varphi \|_{C^0} \tag{5.25}$$

where

$$q = \max \left\{ \left(1 - \frac{\eta_1}{2}\right), \left(1 - \frac{\eta_j}{2}\right) \sum_{k=1}^{m} | \alpha_k | \right\} < 1. \tag{5.26}$$

Hence we have

$$\max_{0\leq n<\infty} \| \varphi + \sum_{k=1}^{m} \alpha_k \psi_k^n \|_{C^0} \leq \left(1 + \frac{q}{1-q} \sum_{k=1}^{m} | \alpha_k | \right) \| \varphi \|_{C^0} . \tag{5.27}$$

But the function

$$\psi_k^{n-1} = B_k \left(\varphi + \sum_{k=1}^{m} \alpha_k \psi_k^{n-2} \right), n \geq 2$$

is the trace of the problem.

$$\begin{aligned} L^n U^n &= g^n \text{ on } R \\ U^n &= 0 \text{ on } \gamma^d,\ d = 1, 2, 3 \\ U^n &= \varphi + \sum_{k=1}^{m} \alpha_k \psi_k^{n-2} \text{ on } \gamma^4. \end{aligned} \tag{5.28}$$

Since $Y_j \in \overline{R}_\delta$ and $1 \leq j \leq m$, where Y_j is the line segments.

Directly the estimation for derivative of $\psi_j^{n-1}(x), 0 \le x \le 1$ hold,

$$\max_{0\le n\le 1} \mid \frac{d^s\psi_j^{n-1}}{dx^s} \mid \le C_{s,j}^0 \parallel \varphi \parallel_{C^0}, n \ge 2, s \ge 4, j = 1,2,3,4$$

where $C_{s,j}^0$ is a constant independent of n and h. Since $V^n = 0$ on γ^d, $d = 1,3$ the derivative $\frac{d^{2\mu}\psi_j^{n-1}}{dx^{2\mu}} = 0, \mu = 0,1,2$ at a point $x = 0$and $x = 1$.

Directly from chapter 3 the problem Z_j^n , $j = 1,2,...,m$ defined as

$$\begin{aligned} L^n Z_j^n &= g^n \text{on } R \\ Z_j^n &= 0 \text{ on } \gamma^d,\ d = 1,2,3 \\ Z_j^n &= \psi_j^{n-1},\ j = 1,2,...,m \end{aligned} \tag{5.29}$$

belong to $C^{4,\lambda}\left(\overline{R}\right), 0 < \lambda < 1$.

Analogy we write

$$\max_{\overline{R}_h} \mid Z_{j,h}^n - Z_j^n \mid \le C_4^0 h^2,\ j = 1,2,3. \tag{5.30}$$

Recall that

$$B_i f(x) = W(x,\eta_i) \in C^0,\ i = 1,2,...,m.$$

From (4.12) and (5.10) we have

$$B_j^h f_h = \left(W_h\left(x, p_j h\right) \frac{\left(p_j+1\right)h - \eta_j}{h} + W_h\left(x, \left(p_j+1\right)h\right) \frac{\eta_j - p_j h}{h} \right) \in C_h^0 \tag{5.31}$$

on $x \in [0,1]_h$, $j = 1,2,...,m$.

We have

$$Z^n = f(x) = \psi_k^{n-1}$$

$$B_j \psi_k^{n-1} = Z_k^n\left(x,\eta_j\right), 0 \le x \le 1. \tag{5.32}$$

It follow that

$$B_j^h \psi_{k,h}^{n-1} = Z_{k,h}^n\left(x, p_j h\right) \frac{\left(p_j+1\right)h - \eta_j}{h} + Z_{k,h}^n\left(x, \left(p_j+1\right)h\right) \frac{\eta_j - p_j h}{h}$$

$$\text{on } [0,1]_h . \tag{5.33}$$

Clearly $B_j^h \psi_{k,h}^{n-1}$ is the trace of $B_j \psi_k^{n-1}$.

Now we compare (5.32) and (5.33) yield

$$\| B_j^h \psi_{k,h}^{n-1} - B_j \psi_{k.}^{n-1} \|_{C_h^0} \leq C_5^0 h^2, \tag{5.34}$$

where C_5^0 is a constant independent of n and h.

We now construct an inequality from (5.30) − (5.34) and since

$$\psi_j^n = B_j f(x) \tag{5.35}$$

$$f(x) = \varphi + \sum_{k=1}^{m} \alpha_k \psi_k^{n-1}.$$

Then by virtue of (5.21) we have

$$\| \ B_j^h \left(\sum_{k=1}^{m} \alpha_k \psi_{k,h}^{n-1} \right) - B_j \left(\sum_{k=1}^{m} \alpha_k \psi_k^{n-1} \right)_{h.} \|_{C_h^0}$$

$$\leq \sum_{k=1}^{m} | \alpha_k | B_j^h \psi_{k,h}^{n-1} - \left(B_j \psi_k^{n-1} \right)_{h.} \|_{C_h^0}, \tag{5.36}$$

where $B_j^h \psi_{k,h}^{n-1}$ and $B_j \psi_k^{n-1}$ are defined in (5.32) and (5.33) . We substitute (5.32) and (5.33) into (5.36) it become

$$\leq \sum_{k=1}^{m} | \alpha_k | \begin{pmatrix} \max\limits_{x \in [0,1]_h} | Z_{k,h}^n \left(x, p_j h \right) \frac{(p_j+1)h - \eta_j}{h} \\ + Z_{k,h}^n \left(x, \left(p_j + 1 \right) h \right) \frac{\eta_j - p_j h}{h} \ | \end{pmatrix} \tag{5.37}$$

$$+ \left(\max_{x \in [0,1]_h} | Z_{k,h}^n \left(x, \left(p_{j+1} \right) h \right) - Z_k^n \left(x, \left(p_j + 1 \right) h \right) \frac{\eta_j - p_j h}{h} \ | \right) + C_5^0 h^2.$$

But

$$\left(\max_{x \in [0,1]_h} | Z_{k,h}^n \left(x, \left(p_{j+1} \right) h \right) - Z_k^n \left(x, \left(p_j + 1 \right) h \right) \frac{\eta_j - p_j h}{h} \ | \right) \leq C_4^0 h^2.$$

Since from (5.30) we have

$$\max_{\overline{R}_h} | Z_{j,h}^n - Z_j^n | \leq C_4^0 h^2, \ j = 1, 2, 3.$$

It follow that

$$\| B_j^h \left(\sum_{k=1}^{m} \alpha_k \psi_{k,h}^{n-1} \right) - B_j \left(\sum_{k=1}^{m} \alpha_k \psi_k^{n-1} \right)_{h.} \|_{C_h^0} \leq \sum_{k=1}^{m} | \alpha_k | C_4^0 h^2 + C_5^0 h^2 \ \leq C_3^0 h^2,$$

where $C_3^0 = \sum\limits_{k=1}^{m} | \alpha_k | C_4^0 + C_5^0$.

Lemma 5.0.4. *We show that the inequality hold.*

$$\| \widetilde{\psi}^n_{j,h} - \psi^n_{j,h} \|_{C^0_h} \leq C^0 h^2, \; j = 1, 2, ..., m$$

where $\widetilde{\psi}^n_{j,h}$ *is the n-th element of sequence* (5.18) , $\psi^n_{j,h}$ *is the trace on* $[0,1]_h$ *of the n-th element in* (4.17) *and* C^0 *is a constant independent of n and h*

Proof. We compare (5.1) and (5.18) as $n = 0, 1$ then it follows

$$\| \widetilde{\psi}^0_{j,h} - \psi^0_{j,h} \|_{C^0_h} = 0 \tag{5.38}$$

$$\begin{aligned} &\| \widetilde{\psi}^1_{j,h} - \psi^1_{j,h} \|_{C^0_h} = \| B^h_j \widetilde{\varphi}_h - \left(B_j \varphi\right)_h \|_{C^0_h} \\ &\leq \| B^h_j (\widetilde{\varphi}_h - \varphi_h) \|_{C^0_h} + \| B^h_j \varphi_h - \left(B_j \varphi\right)_h \|_{C^0_h}. \end{aligned} \tag{5.39}$$

But

$$B^h_j < 1 - \frac{\eta_j}{2}.$$

Then it follow that

$$\| B^h_j (\widetilde{\varphi}_h - \varphi_h) \|_{C^0_h} \leq \left(1 - \frac{\eta_j}{2}\right) \| (\widetilde{\varphi}_h - \varphi_h) \|_{C^0}, \tag{5.40}$$

where

$$\| (\widetilde{\varphi}_h - \varphi_h) \|_{C^0} \leq C_3 h^2.$$

Hence

$$\| \widetilde{\psi}^1_{j,h} - \psi^1_{j,h} \|_{C^0_h} \leq \left(1 - \frac{\eta_j}{2}\right) C_3 h^2 \leq C_4 h^2, \; j = 1, 2, ..., m, \tag{5.41}$$

where $C_4 = \left(1 - \frac{\eta_j}{2}\right) C_3$. □

We take for $n \geq 2$

$$\begin{aligned} &\| \widetilde{\psi}^n_{j,h} - \psi^n_{j,h} \|_{C^0_h} \leq \| B^h_j \widetilde{\varphi}_h - \left(B_j \varphi\right)_h \|_{C^0_h} \\ &+ \| B^h_j \left(\sum_{k=1}^m \alpha_k \widetilde{\psi}^{n-1}_{k,h} - \sum_{k=1}^m \alpha_k \psi^{n-1}_{k,h} \right) \|_{C^0_h} \\ &+ \| B^h_j \left(\sum_{k=1}^m \alpha_k \psi^{n-1}_{k,h} \right) - \left(B_j \left(\sum_{k=1}^m \alpha_k \psi^{n-1}_{k,} \right)_h \right) \|_{C^0_h}, \; j = 1, 2, ..., m. \end{aligned} \tag{5.42}$$

But

$$\| B_j^h\widetilde{\varphi}_h - \left(B_j\varphi\right)_h \|_{C_h^0} \le \| B_j^h\left(\widetilde{\varphi}_h - \varphi_h\right) \|_{C_h^0} + \| B_j^h\varphi_h - \left(B_j\varphi\right)_h \|_{C_h^0} \tag{5.43}$$

$$\le \left(1 - \frac{\eta_j}{2}\right) C_3 h^2 \le C_4 h^2.$$

$$\| B_j^h\widetilde{\varphi}_h - \left(B_j\varphi\right)_h \|_{C_h^0} \le C_5 h^2.$$

We have

$$\| B_j^h\left(\sum_{k=1}^{m}\alpha_k\widetilde{\psi}_{k,h}^{n-1} - \sum_{k=1}^{m}\alpha_k\psi_{k,h}^{n-1}\right) \|_{C_h^0}$$

$$= \left(1 - \frac{\eta_j}{2}\right)\sum_{k=1}^{m} \mid \alpha_k \mid\| \left(\widetilde{\psi}_{k,h}^{n-1} - \psi_{k,h}^{n-1}\right) \|_{C_h^0} . \tag{5.44}$$

But

$$q = \left(1 - \frac{\eta_j}{2}\right)\sum_{k=1}^{m} \mid \alpha_k \mid < 1.$$

It follow that the

$$\| \widetilde{\psi}_{j,h}^n - \psi_{j,h}^n \|_{C_h^0} \| \widetilde{\psi}_{j,h}^n - \psi_{j,h}^n \|_{C_h^0} \le C_5 h^2$$

$$+q \| \widetilde{\psi}_{k,h}^{n-1} - \psi_{k,h}^{n-1} \|_{C_h^0}$$

$$\le C_5 h^2 + q \max_{0\le k\le m} \| \widetilde{\psi}_{k,h}^{n-1} - \psi_{k,h}^{n-1} \|_{C_h^0}, \tag{5.45}$$

where $C_5 = C_3^0 + C_4$. By virtue of (5.38) , (5.42) and (5.44) its become

$$\max_{0\le k\le m} \| \widetilde{\psi}_{k,h}^n - \psi_{k,h}^n \|_{C_h^0} \le C^0 h^2. \tag{5.46}$$

Without approximation we have from (4.17) and (5.27)

$$\max_{0\le j\le m} \| \psi_j^n - \psi_j \|_{C^0} \le \frac{q^{n+1}}{1-q} \| \varphi \|_{C^0}, n \ge 1, \tag{5.47}$$

where

$$\varphi = \sum_{k=1}^{m}\alpha_k V\left(x\eta_k\right).$$

And

$$q = \max\left\{\left(1 - \frac{\eta_1}{2}\right), \left(1 - \frac{\eta_j}{2}\right)\sum_{k=1}^{m} \mid \alpha_k \mid\right\} < 1, 0 < q < 1.$$

By virtue of Lemma (5.0.4) together with (5.47) it become

$$\max_{0\le j\le m} \| \widetilde{\psi}^n_{j,h} - \psi^n_{j,h} \|_{C^0_h} \le \max_{0\le j\le m} \| \widetilde{\psi}^n_{j,h} - \psi^n_{j,h} \|$$

$$+\max_{0\le j\le m} \| \psi^n_{j,h} - \psi_{j,h} \|_{C^0_h} \le C^0 h^2 + \frac{q^{n+1}}{1-q} \| \varphi \|_{C^0}, n \ge 2, \tag{5.48}$$

where $\widetilde{\psi}^n_{j,h}$ is the n-th term of the sequence (5.18)and $\psi_{j,h}$ is the trace of the function ψ_j on $[0,1]_h$.

Since

$$\widetilde{f}_h = \widetilde{\varphi}_h + \sum_{j=1}^{m} \alpha_j \widetilde{\psi}_{j,h} \text{ on } \gamma_h^4, \tag{5.49}$$

where $\widetilde{f}_h$ is the approximate function.

Then

$$f = \varphi + \sum_{k=1}^{m} \alpha_k \psi_k \text{ on } [0,1]$$

$$\widetilde{\varphi}_h = \sum_{j=1}^{m} \alpha_j \widetilde{\varphi}_{j,h(x)} \in C^0_h,\ x \in [0,1]_h$$

$$\widetilde{\psi}^n_{j,h} = B^h_j \left(\widetilde{\varphi}_h + \sum_{k=1}^{m} \alpha_k \widetilde{\psi}^{n-1}_{k,h} - \right),\ j = 1,2,...,m.\ \ n = 1,2,...$$

$\widetilde{\psi}^n_{j,h}$ is called the n-th element of sequences.

Directly by (5.15) , (5.48) and (5.49) we have

$$\| \widetilde{f}_h - f_h \|_{C^0_h} \le C^0_3 h^2 + \sum_{j=1}^{m} |\ \alpha_j\ | \frac{q^{n+1}}{1-q} \| \varphi \|_{C^0}, n \ge 2, \tag{5.50}$$

where f_h is the trace $[0,1]_h$ of f and C^0_3 is a constant independent of n and h.

Taking the Approximate solution of (4.6) - (4.8) by finite difference problem

$$\widetilde{U}^n_h = L\widetilde{U}^n_h \text{ on } R_h$$

$$\widetilde{U}^n_h = 0 \text{ on } \gamma^d,\ d = 1,2,3$$

$$\widetilde{U}^n_h = \tau_h \tag{5.51}$$

$$\widetilde{U}^n_h = \widetilde{f}_h \text{ on } \gamma^4 \tag{5.52}$$

where $\widetilde{f}_h$ is define in (5.49) .

Theorem 5.0.5. *Let the boundary function* $\tau(x)$ *in the nonlocal problem* (4.6) - (4.8) *belongs to a class of* $C^{2,\lambda}$, $0 < \lambda < 1$ *on* γ^2, *then the inequality hold.*

$$\max_{(x,y)\in\overline{R}_h} \mid \widetilde{U}_h^n - U_h \mid \leq C^0 h^2 + \sum_{j=1}^{m} \mid \alpha_j \mid \frac{q^{n+1}}{1-q} \parallel \varphi \parallel_{C^0}, n \geq 2, \tag{5.53}$$

where

$$q = \max\left\{\left(1 - \frac{\eta_1}{2}\right), \left(1 - \frac{\eta_j}{2}\right) \sum_{k=1}^{m} \mid \alpha_k \mid\right\} < 1,\ 0 < q < 1.$$

And

$$n = \max\left\{\left[\frac{\ln h^{-2}(1-q)^{-1}}{\ln q^{-1}}\right] + 1, 1\right\}$$

Proof. Since $f = \varphi + \sum_{k=1}^{m} \alpha_k \psi_k$ on $[0,1]$, clearly by virtue of lemma 1 and 2 seen that $f \in C^0\left(\gamma^4\right)$ but $C^0 \cap C^{2,\lambda}\left(\gamma^4\right) = C^0\left(\gamma^4\right)$. From (5.49) and (5.50) follow from theorem 1.1 on the method of composite meshes for Laplace's equation on polygon'. the inequality (5.53) hold. □

CHAPTER 6

NUMERICAL EXPERIMENTS

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 6.1

$$U_{xx} + U_{yy} + e^{y^2} U_x + \sin(\pi x) U_y - e^{x+y} U = 0 \text{ on R}$$

$$U = \varphi^m \text{ on } \gamma^m, m = 1, 2, 3.$$

$$U(x, 0) = U(x, 1), 0 \le x \le 1,$$

where

$$\varphi^1 = \frac{2 \sin \pi x}{1 + y^2},\ \varphi^2 = \tan^{-1} 2x,\ \varphi^3 = \tan^{-1}\left(y (y - 1)^2\right).$$

Problem 6.2

$$U_{xx} + U_{yy} + e^{y^2} U_x + \sin(\pi x) U_y - e^{x+y} U = 0 \text{ on R}$$

$$U(x, 2) = \sin \pi x, 0 \le x \le 1,$$

$$U(x, 0) = \frac{1}{4} U\left(x, \frac{1}{2}\right) + \frac{1}{8} U(x, 1) + \frac{1}{4} U\left(x, \frac{3}{2}\right)$$

Table 6.1: Solutions on the line $y = 0$ of Problem 6.1

h = 1/16	**h = 1/32**	**h = 1/64**	**h = 1/128**
$1.08143E-003$	$1.07733E-003$	$1.06995E-003$	$1.06878E-003$
$2.11043E-003$	$2.10093E-003$	$2.09985E-003$	$2.09723E-003$
$3.06437E-003$	$3.05995E-003$	$3.04753E-003$	$3.04516E-003$
$3.99917E-003$	$3.92006E-003$	$3.90146E-003$	$3.89853E-003$
$4.63414E-003$	$4.60194E-003$	$4.58118E-003$	$4.56626E-003$
$5.10924E-003$	$5.09734E-003$	$5.08458E-003$	$5.07102E-003$
$5.42355E-003$	$5.40011E-003$	$5.38575E-003$	$5.38023E-003$
$5.54481E-003$	$5.51245E-003$	$5.49120E-003$	$5.48215E-003$
$5.42355E-003$	$5.40011E-003$	$5.38575E-003$	$5.38023E-003$
$5.10924E-003$	$5.09734E-003$	$5.08458E-003$	$5.07102E-003$
$4.63414E-003$	$4.60194E-003$	$4.58118E-003$	$4.56626E-003$
$3.99917E-003$	$3.92006E-003$	$3.90146E-003$	$3.89853E-003$
$3.06437E-003$	$3.05995E-003$	$3.04753E-003$	$3.04516E-003$
$2.11043E-003$	$2.10093E-003$	$2.09985E-003$	$2.09723E-003$
$1.08143E-003$	$1.07733E-003$	$1.06995E-003$	$1.06878E-003$

Table 6.2: Solutions on the line $y = 0$ of Problem 6.2

h = 1/16	**h = 1/32**	**h = 1/64**	**h = 1/128**
$-1.24817E-006$	$-1.25923E-006$	$-1.26118E-006$	$-1.25801E-006$
$-2.61230E-006$	$-2.62049E-006$	$-2.62046E-006$	$-2.61320E-006$
$-4.06639E-006$	$-4.06610E-006$	$-4.06283E-006$	$-4.05144E-006$
$-5.54744E-005$	$-5.53623E-005$	$-5.52926E-005$	$-5.51406E-005$
$-6.97362E-005$	$-6.95075E-005$	$-6.94010E-005$	$-6.92162E-005$
$-8.25574E-005$	$-8.22155E-005$	$-8.20755E-005$	$-8.18652E-005$
$-9.30529E-005$	$-9.26090E-005$	$-9.24408E-005$	$-9.22131E-005$
$-1.00403E-005$	$-9.98743E-005$	$-9.96845E-005$	$-9.94483E-005$
$-1.03901E-005$	$-1.03307E-005$	$-1.03103E-005$	$-1.02868E-005$
$-1.02984E-005$	$-1.02350E-005$	$-1.02140E-005$	$-1.01914E-005$
$-9.72690E-005$	$-9.66213E-005$	$-9.64143E-005$	$-9.62074E-005$
$-8.65773E-005$	$-8.59469E-005$	$-8.57517E-005$	$-8.55715E-005$
$-7.09704E-005$	$-7.03932E-005$	$-7.02198E-005$	$-7.00737E-005$
$-5.07928E-005$	$-5.03143E-005$	$-5.01745E-005$	$-5.00691E-005$
$-2.67421E-005$	$-2.64308E-005$	$-2.63411E-005$	$-2.62829E-005$

In Problems 6.1 and 6.2 we see that the exact solution is unknown. This given rise to obtain the approximate values of Problems 6.1 and 6.2 on the line $y = 0$ by proposed method given in Tables 6.1 and 6.2, respectively. By repeated digits, from descending order mesh steps $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ we have the maximum error on these line decreases as $O\left(h^2\right)$ rate.

CHAPTER 7

CONCLUSIONS

The approximate solution of the nonlocal boundary value problem for general second order linear Elliptic equation on a rectangular domain is defined as a sum of 5-point solution of the two classical local Dirichlet problem. Nonlocal conditions is replaced by zero on the first classical local Dirichlet problem except where $W = f \in C^0$ on γ^4, also nonlocal boundary conditions of the original problem are replaced by non-homogeneous condition except where $W = 0$ on γ^4. The boundary values of a nonlocal condition problem were solved by using a local value of a special constructed function f, where $f \in C^0$. By simplicity $f = U(x, 0) = U(x, 1)$, its was assumed that $Bf = W(x, 1) \in C^0$ and $\sigma^0(x) = V(x, 1) \in C^0$. Original problem written as a non-linear equation $\varphi = \sigma^0 + \psi$ see in (2.39), the function ψ is defined as the $n - th$ iteration of the convergent fixed-point iterations. The uniform estimate of the error of the approximate solution of the nonlocal boundary value problem for general second order linear Elliptic equation for $n = \left(\left[\ln h^{-2}(1-q)^{-1} / \ln q^{-1}\right] + 1, 1\right)$ is of order $O\left(h^2\right)$, where h is the mesh step. In future I will like to generalize this particular problem with the height accuracy for fourth order and also seek for the exact and approximate solution of the problem.

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