DIFFERENCE METHOD FOR THE ELLIPTIC EQUATION WITH INTEGRAL NONLOCAL BOUNDARY CONDITION

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
RIFAT REIS

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

NICOSIA, 2020

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy of Science in Mathematics

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I declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

A constructive method for 5 and 9 point approximate solution of Laplace's and second order general linear elliptic equations with nonlocal integral boundary condition is proposed and justified. In this method, the approximate solution is defined as a solution of the classical Dirichlet problem by using special method to seek a function instead of the nonlocal boundary value.

Furthermore, a novel estimation for the convergence of the fourth order finite difference scheme for the second order general elliptic equation containing first order partial derivatives with variable coefficients is obtained.

The uniform estimate of the error of approximate solution for Laplace's equation and the second order general elliptic equations obtained by the proposed method is of order $O(h^2)$ and $O(h^4)$, when 5-point and 9-point scheme are used, respectively. These estimations are proved when the exact solutions are from the Hölder classes $C^{k,\lambda}, 0 < \lambda < 1$, on the closed solution domain. It is verified that the order $O(h^2)$ and $O(h^4)$ are obtained for Laplace's equation when $k = 2$ and $k = 4$, respectively. For the general elliptic equation the same estimations are obtained when $k = 4$ and $k = 6$, respectively. Numerical experiments are given to support the obtained theoretical analysis.

Keywords: Laplace's equation; second order linear elliptic equation; Dirichlet problem; nonlocal integral condition; finite difference scheme; uniform estimation

ÖZET

Yerel olmayan integral sınır şartlı Laplace ve ikinci mertebeden genel doğrusal elliptik denklemlerin 5 ve 9 nokta yaklaşık çözümleri için yapısal bir yöntem önerilir ve doğrulanır. Bu yöntemde yaklaşık çözüm, yerel olmayan sınır şartı yerine özel yöntemle bir fonksiyon bulunarak klasik Dirichlet probleminin bir çözümü olarak tanımlanır.

Ayrıca, değişken katsayılı birinci mertebeden kısmi türevleri içeren ikinci mertebeden genel elliptik denkleminin, dördüncü mertebeden sonlu farklar şemasının yakınsaklığı için yeni bir tahmin elde edilir.

5 nokta ve 9 nokta planı kullanılarak, Laplace ve ikinci mertebeden genel elliptik denklemleri için yaklaşık çözümün hatasının düzgün tahmini sırası ile $O(h^2)$ ve $O(h^4)$ mertebesindendir. Bu tahminler, kesin çözümler kapalı çözüm alanında $C^{k,\lambda}, 0 < \lambda < 1$, Hölder sınıfından olduğunda ispatlanır. Sırası ile $k = 2$ ve $k = 4$ olduğunda, Laplace denklemi için $O(h^2)$ ve $O(h^4)$ mertebelerin elde edildiği ispatanlanır. Ikinci mertebeden genel elliptik denklemleri için, sırası ile $k = 4$ ve $k = 6$ olduğunda aynı tahminler elde edilir. Elde edilen teorik sonuçları desteklemek için sayısal deneyimler verilir.

Anahtar Kelimeler: Laplace denklemi; ikinci mertebeden doğrusal elliptik denklem; Dirichlet problem; yerel olmayan integral şartı; sonlu farklar şeması; düzgün tahmin

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CHAPTER 1
INTRODUCTION

Bitsadze and Samarskii (1969) stated the nonlocal boundary value problem of finding a harmonic function on an open rectangle for the given continuous functions on three sides and on the fourth side of the rectangle is given by using the solution at as the middle of the rectangle which is parallel to this side (one level nonlocal boundary value problem).

The multilevel nonlocal boundary value problems which are the generalizations of the nonlocal Bitsadze-Samarskii type problem were studied by many authors (see in Gurbanov & Dosiyev, 1984; Il'in & Moiseev, 1990; Sapagovas, 2002; Gordeziani et all, 2005; Skubachevskii, 2008; Ashyralyev & Ozturk, 2012; Ashyralyev & Ozturk, 2013). Il'in and Moiseev (1990) verified that if the fourth derivatives of the solutions of the multilevel nonlocal boundary value problem are continuous on the closed rectangular domain, the error bound in the uniform metric and in the difference metric W_2^2 has a second order accuracy.

Another important generalization of the Bitsadze-Samarskii problem is the one with integral boundary condition. These type of problems have many applications in different engineering problems. (see Jack et all, 1975 and references given therein).

Different type of finite difference problem for Laplace's equation as an approximation of the nonlocal problem with integral boundary condition has been studied by many authors (see Sapagovas, 2008; Zhou et all, 2018 and references given therein). They all basically focused on the following two difficulties in the existence of the quadrature approximation of the integral condition on the side of the domain where nonlocal condition was given: (*i*) finding an approximate solution by solving the obtained system of equations which are non-band matrices, (*ii*) determining the rate of convergence of the approximate solution by appropriate smoothness conditions on the given data. In (Sapagovas, 2008), the system of finite difference equations in the case of integral boundary condition for Poisson equation has been studied for the spectrum of the matrix to apply an iterative method. Moreover, the author obtained some conditions for which this system has a unique solution. In

(Berikelashvili, 2001) and (Berikelashvili & Khomeriki, 2012), for the error of approximate solution, order of estimation $O(h^2)$ in the difference W_2^1 metric is obtained, where h is the mesh step. In (Zhou et all, 2018), a finite-difference approximation for the problem with integral boundary conditions is constructed by pre-reducing of the given problem to the problem with nonlocal conditions containing derivatives. The authors proved that when the fourth order partial derivatives of the exact solution are continuous on the closed solution domain, the uniform estimate is of order $O(h^2 |\ln h|)$.

Many researchers have been studied on the general elliptic equation with integral boundary condition (see Wang, 2002; Avalishvili et all, 2010; Sajavicius, 2014; Sapagovas et all, 2016 and given references therein). Wang (2002) investigated eigenvalue problems, existence and dynamic behavior of solutions of the elliptic equation with integral nonlocal condition by using comparison principle and a semigroup approach. Avalishvili and Gordeziani (2010) proved the uniqueness of the elliptic equation with two integral boundary conditions and obtained a new prior estimates. In (Sajavicius, 2014), the radial basis function collocation technique is used to find an approximate solution of elliptic equation with nonlocal integral boundary condition. Sapagovas, Stikoniene, Ciupalia and Joksiene (2016) focused on how convergence of iterative methods for the system of difference equations, approximating the elliptic two dimensional equation with integral nonlocal condition depends on the structure of spectrum for difference operator.

Research of the nonlocal boundary value problem for different type parabolic and hyperbolic equations with integral boundary condition and its finite difference scheme are conducted by numerous mathematician (see in Mesloub & Bouziani, 1999; Pul'kina, 2002; Dehghan & Tatari, 2007; Sapagovas & Jakubelience, 2011 and references given therein). Pul'kina (2002) proved the unique solvability of a hyperbolic equation with integral boundary condition in the function class W_2^2. Dehghan and Tatari (2007) used a radial basis function to find an approximation of the solution for the one-dimensional parabolic equation with integral boundary condition. They gave numerical results to show efficiency of the given method to compare with other type finite-difference method. Sapagovas and Jakubelience (2011) solved a two-dimensional parabolic equation with nonlocal integral condition by alternating direction method and they studied on the spectrum of the matrix obtained by the system of finite difference equations.

A new method for the solution of the Poisson equation with nonlocal boundary condition was given and the problem was defined as the sum of two classical local Dirichlet problems. (Volkov et all, 2013) By applying the contraction mapping principle, the uniqueness and existence of the classical solutions and approximate solutions of the multilevel nonlocal boundary value problem were proved with more general restriction for the coefficients in nonlocal condition. (Volkov, 2013; Volkov & Dosiyev, 2016).

In Chapter 2 at the first section, the 5-point approximation on a square grid with step size h of the nonlocal boundary value problem for Laplace's equation with integral boundary condition are proposed and justified by using the new constructive method given by Volkov and Dosiyev (2016). By applying trapezoidal rule for the integral boundary condition, the approximate problem is defined as the multilevel nonlocal boundary value problem that is given as the sum of two 5-point Dirichlet problem. In the first Dirichlet problem, the nonlocal condition is modified with zero. In the second Dirichlet problem, the local boundary condition is replaced by zeros values and the boundary value where nonlocal condition is given is defined as a function by using n-th iteration of the convergent fixed point iterations for nonlinear system of equations. It is verified that when the boundary functions are from the Hölder classes $C^{2,\lambda}, 0 < \lambda < 1$, continuous and vanish at the enpoints, the uniform estimate of the error of the approximate solution is order of $O(h^2)$, h is the step mesh.

At the second section in Chapter 2, we propose and justify the method given in Section 1 to solve the system of nonlocal 9-point finite difference problem for the Laplace equation with the integral boundary condition. The solution of this nonlocal difference problem is defined as a solution of the 9-point Dirichlet problem by constructing the approximate values of the solution on the side where the integral condition was given. Therefore, the approximate solution is obtained by solving a system with 9 diagonal matrices, for the realization of which proposed many fast algorithms. (see in Samarskii & Nikolaev, 1989, Vol 1-2). Moreover, the uniform estimate of the error of approximate solution is of order $O(h^4)$, when the given boundary functions on the sides belong to the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$, and $2m - th$ order of derivatives vanish at the endpoints for $m = 0, 1, 2$.

In Chapter 3 at the first section, the second order general elliptic operator containing first order partial derivatives with variable coefficients in 2-dimensions is introduced in the form

$$Lu = \Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu \tag{1.1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$, a, b and c are functions of (x, y). We construct the 5-point difference scheme for the approximation of the nonlocal problem with integral condition for the second order linear elliptic equation. The solution of the problem is defined as the sum of two 5-point Dirichlet problems which are given as multilevel problems by using the method given in Chapter 2. It is proved that when the boundary functions are from the class $C^{4,\lambda}, 0 < \lambda < 1$, the uniform estimate of the error of the approximate solution is order of $O(h^2)$.

At the second section in Chapter 3, the fourth order finite difference scheme for the solution the nonlocal boundary value problem of the second order elliptic equation with integral boundary condition is investigated. In (Dennis & Hudson, 1979;1980), the elliptic equation (1.1)is expressed as following two equations

$$\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu = r(x, y) \tag{1.2}$$

$$\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y} = -r(x, y) \tag{1.3}$$

to obtain a different type approximations which is diagonally dominant for certain significant cases by using difference correction method of Fox (1947). Gupta (1983) presented a fourth order finite difference scheme for a general class of second order elliptic equation on nine node points by using local power series representations. Karaa (2005) proposes a fourth-order difference scheme for the two dimensional elliptic equation on a regular hexagon over a seven point stencil. They all give the fourth order finite difference scheme deficiency from its convergence. However Dennis, Hudson (1979;1980) and the researcher (Gupta, 1983; Karaa, 2005 and references given therein) studying on fourth order finite difference scheme for the second order elliptic equation are focus on numerical results without proving the convergence of the finite difference scheme. In this section, we justified the fourth-order convergence of Dennis-Hudson's finite difference scheme under an assumption for the step size h as $hK \leq 2$, for some calculable positive constant K depending on the coefficients of the equation (1.1) . After demonstrating the convergence of the finite difference scheme, the method given in first section for the approximation of the second order elliptic equation with nonlocal integral condition is proposed and justified. The solution of this nonlocal difference problem is defined as a solution of the 9-point Dirichlet problem. The uniform estimate of the error of approximate solution is of order $O(h^4)$, when the given boundary functions are from the Hölder classes $C^{6,\lambda}$, $0 < \lambda < 1$.

In Chapter 4, numerical experiments are given to support the obtained theoretical results. Additionally, the CPU times are illustrated to show efficiency of proposed method.

The results of Chapter 2 in this dissertation are published in (Dosiyev & Reis, 2018;2019).

CHAPTER 2

DIFFERENCE DIRICHLET PROBLEM FOR THE APPROXIMATE SOLUTION OF LAPLACE'S EQUATION WITH NONLOCAL INTEGRAL CONDITION

2.1 SECOND ORDER ACCURACY FOR THE LAPLACE EQUATION WITH INTEGRAL BOUNDARY CONDITION

2.1.1 Overview

In this section, the 5-point approximation of the nonlocal boundary value problem of Laplace's equation with integral boundary condition is proposed and justified by using the new constructive method that Volkov and Dosiyev used (see in Volkov & Dosiyev, 2016). By applying trapezoidal rule for the integral boundary condition, the problem is defined as the multilevel nonlocal boundary value problem that is given as the sum of two 5-point Dirichlet problems. It is verified that when the boundary functions are from the class $C^{2,\lambda}, 0 < \lambda < 1$, the uniform estimate of the error of the approximate solution is order of $O(h^2)$, h is the step mesh.

2.1.2 Nonlocal boundary value problem

Let

$$R = \{(x, y) : 0 < x < a, 0 < y < b\} \tag{2.1}$$

be an open rectangle, $\gamma^m, m = 1, 2, 3, 4$, be its sides including the endpoints, numbered in the clockwise direction, beginning with the side lying on the y-axis and let $\gamma = \cup_{m=1}^{4} \gamma^m$ be the boundary of R.

Let C^0 denote the linear space of continuous functions of one variable x on the interval $[0, a]$ of x-axis, and vanish at the points $x = 0$ and $x = a$. For the function $f \in C^0$, we define the norm

$$\|f\|_{C^0} = \max_{0\leq x\leq a} |f(x)|\,. \tag{2.2}$$

It is clear that the space C^0 with this norm is complete.

Consider the following nonlocal boundary value problem

$$\triangle u = 0 \text{ on } R,\ u = 0 \text{ on } \gamma^1 \cup \gamma^3,\ u = \tau \text{ on } \gamma^2, \tag{2.3}$$

$$u(x,0) = \alpha \int_{\xi}^{b} u(x,y)dy + \mu(x), \quad 0 < x < a,\ 0 < \xi < b, \tag{2.4}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, $\tau = \tau(x) \in C^0$ and $\mu = \mu(x)$ are given functions and α is a given constant which holds $|\alpha| < \frac{1}{b-\xi}$. By replacing the integral condition (2.4) with its approximation using trapezoidal rule, we have

$$u(x_i,0) = \alpha \sum_{k=1}^{M} \rho_k u(x_i,\eta_k) + \mu_i,\ \ i = 1,2,...,N-1, \tag{2.5}$$

where $\rho_1 = \rho_M = \frac{h}{2}, \rho_j = h$ for $j = 1,2,...,M-1,\ \eta_j = \xi + (j-1)h,\ j = 1,2,...,M,\ h = \frac{a}{N}$, $(M-1)h + \xi = b$ and $\frac{\xi}{h}$ is an integer.
It follows that

$$|\alpha| \sum_{k=1}^{M} \rho_k = q_0 < 1. \tag{2.6}$$

We consider the following multilevel nonlocal boundary value problem on R :

$$\triangle U = 0 \text{ on } R,\ U = \tau \text{ on } \gamma^2,\ U = 0 \text{ on } \gamma^1 \cup \gamma^3\ , \tag{2.7}$$

$$U(x,0) = \alpha \sum_{k=1}^{M} \rho_k U(x,\eta_k) + \mu(x), \quad 0 \le x \le a. \tag{2.8}$$

Let V be a solution of the Dirichlet problem,

$$\triangle V = 0 \text{ on } R,\ V = \tau \text{ on } \gamma^2,\ V = 0 \text{ on } \gamma/\gamma^2. \tag{2.9}$$

We denote

$$\varphi_k(x) = V(x,\eta_k) \ \text{ for } k = 1,2,...,M, \tag{2.10}$$

and

$$\varphi = \alpha \sum_{k=1}^{M} \rho_k \varphi_k. \tag{2.11}$$

We consider the Dirichlet problem

$$\triangle W = 0 \text{ on } R,\ W = 0 \text{ on } \gamma/\gamma^4,\ W = f \ \text{ on } \gamma^4, \tag{2.12}$$

where f be an unknown function from C^0.
We define the operator $B_i : C^0 \to C^0$ as

$$B_i f(x) = W(x, \eta_i) \ \in \ C^0, \ \ i = 1, 2, ..., M. \tag{2.13}$$

Let

$$W_1(x, y) = \frac{1}{b} \|f\|_{C^0} (b - y), \quad (x, y) \in \overline{R}.$$

We put

$$\overset{+}{\underset{-}{\omega}} = W_1 \overset{+}{-} W \text{ on } \overline{R}.$$

Since W and W_1 are harmonic functions on R, we construct the following boundary value problem

$$\begin{aligned} &\triangle \overset{+}{\underset{-}{\omega}} = 0 \text{ on } R,\ \ \overset{+}{\underset{-}{\omega}} = \frac{1}{b} \|f\|_{C^0} (b - y) \ \text{ on } \gamma^m, m = 1, 3, \\ &\ \overset{+}{\underset{-}{\omega}} = 0 \ \text{ on } \gamma^2,\ \ \overset{+}{\underset{-}{\omega}} = \|f\|_{C^0} \overset{+}{-} f \ \text{ on } \gamma^4. \end{aligned} \tag{2.14}$$

The following estimate satisfies

$$\overset{+}{\underset{-}{\omega}} \geq 0 \text{ on } \gamma.$$

By maximum principle, it follows that

$$\overset{+}{\underset{-}{\omega}} \geq 0 \text{ on } \overline{R},$$

which yields

$$|W(x, y)| \leq \frac{1}{b} \|f\|_{C^0} (b - y) \ \text{ on } \overline{R}.$$

Therefore, we find that

$$|B_i| < 1 - \frac{\xi + (i-1)h}{b}, \quad i = 1, 2, ..., M, \tag{2.15}$$

and

$$0 < |B_M| < |B_{M-1}| < ... < |B_1| < 1. \tag{2.16}$$

Then the following inequality holds

$$|B_1|\, q_0 = q < 1, \tag{2.17}$$

where q_0 is defined in (2.6).

It is obvious that,

$$U(x,0) = f(x), \ \ 0 \le x \le a. \tag{2.18}$$

Since $U = V + W$, we have

$$U(x,\eta_k) = V(x,\eta_k) + W(x,\eta_k).$$

Then

$$f = \alpha \sum_{k=1}^{M} \rho_k \left(V(x,\eta_k) + W(x,\eta_k)\right) + \mu(x). \tag{2.19}$$

Relying on (2.10), (2.11), (2.13) and (2.19) , the function f satisfies the following relation

$$f = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k B_k f. \tag{2.20}$$

Existence of f :

Let

$$\psi_i^0 = 0, \ \psi_i^n = B_i \left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1} \right),$$

$$i = 1,2,...,M; \ \ n = 1,2,.... \tag{2.21}$$

Then, for the positive integers m and n with $m > n$, we write

$$\psi_i^m - \psi_i^n = B_i \left(\alpha \sum_{k=1}^{M} \rho_k \left(\psi_k^{m-1} - \psi_k^{n-1} \right) \right), \ \ i = 1,2,...,M.$$

By using the inequalites (2.16) and (2.17) , we get

$$\left\| \psi_i^m - \psi_i^n \right\|_{C^0} \le q \left\| \psi_i^{m-1} - \psi_i^{n-1} \right\|_{C^0}, \tag{2.22}$$

where q is defined by (2.17). In a similar way with (2.22), we reach

$$\left\|\psi_i^m - \psi_i^n\right\|_{C^0} \leq q^{n+1}\frac{1-q^{m-n}}{1-q}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right).$$

From this, we conclude that the sequences of functions (2.21) are fundamental. Therefore, there are limits

$$\lim_{n\to\infty}\psi_i^n = \psi_i \in C^0,\ i = 1, 2, ..., M. \tag{2.23}$$

The following limits also exist:

$$\lim_{n\to\infty} B_k\psi_i^n = B_k\psi_i \in C^0,\ i, k = 1, 2, ..., M. \tag{2.24}$$

By taking limit of (2.21) as $n \to \infty$, we obtain

$$\psi_i = B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k\right),\ i = 1, 2, ..., M. \tag{2.25}$$

Therefore we conclude that

$$\varphi + \mu + \alpha\sum_{i=1}^{M}\rho_i\psi_i = \varphi + \mu + \alpha\sum_{i=1}^{M}\rho_i B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k\right). \tag{2.26}$$

In the view of the relations (2.20) and (2.26), we obtain

$$f = \varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k. \tag{2.27}$$

Uniqueness of f :

Let $f^P \in C^0, p = 1, 2,$ be two functions satisfying the relation (2.20). That is

$$f^P = \varphi + \mu + \alpha\sum_{k=1}^{m}\rho_k B_k f^P, \quad p = 1, 2.$$

Then we reach the inequality

$$\left\|f^1 - f^2\right\|_{C^0} = \left\|\alpha\sum_{k=1}^{m}\rho_k B_k\left(f^1 - f^2\right)\right\|_{C^0} \leq q\left\|f^1 - f^2\right\|_{C^0},$$

which satisfies if $f^1 = f^2$.

2.1.3 Approximate solution of the nonlocal problem by the finite difference method

We say that $F \in C^{k,\lambda}(E)$, if F has k-th derivatives on E satisfying the Hölder condition with exponent λ.

We assume that $\tau(x) \in C^{2,\lambda}\left(\gamma^2\right)$, $\mu(x) \in C^{2,\lambda}\left(\gamma^4\right)$ in (2.3) and (2.4), respectively.

On the basis of Lemma 1 and Lemma 2 (Volkov & Dosiyev, 2016) it follows that the function φ defined by (2.11) and the functions $\psi_i, i = 1, 2, ..., M$, in (2.25) obtained as the limits of the sequences (2.21) belong to $C^{2,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq a$.

We define a square mesh with the mesh size $h = \frac{a}{N} = \frac{b}{M^*}$, $N, M^* > 2$ are integers, constructed with the lines $x, y = h, 2h, ...$Let D_h be the set of nodes of this square grid, $R_h = R \cap D_h$, and $\overline{R}_h = \overline{R} \cap D_h$, where R is the rectangle (2.1), and $\gamma_h^m = \gamma^m \cap D_h, m = 1, 2, 3, 4$.

Let

$$[0, a]_h = \left\{x = x_i,\ x_i = ih,\ i = 0, 1, ..., N,\ h = \frac{a}{N}\right\}$$

be the set of points divided by the step size h on $[0, a]$.

Let C_h^0 be the linear space of grid functions defined on $[0, a]_h$ that vanish at $x = 0$ and $x = a$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x\in[0,a]_h} |f_h| .$$

Let A_h be the operator as follows:

$$A_h u_h \equiv (u_h(x + h, y) + u_h(x - h, y) + u_h(x, y + h) + u_h(x, y - h)) /4.$$

Consider the system of grid equations

$$v_h = A_h v_h \text{ on } R_h, \quad v_h = \tau_h \text{ on } \gamma_h^2, \quad v_h = 0 \text{ on } \gamma_h/\gamma_h^2, \tag{2.28}$$

where τ_h is the trace of τ on γ_h^2 and we define

$$\widetilde{\varphi}_{i,h}(x) = v_h(x, \eta_i),\ i = 1, 2, ..., M. \tag{2.29}$$

Let w_h be a solution of the finite difference problem

$$w_h = A_h w_h \text{ on } R_h, \quad w_h = 0 \text{ on } \gamma_h/\gamma_h^4, \quad w_h = \widetilde{f_h} \text{ on } \gamma_h^4, \tag{2.30}$$

where $\widetilde{f_h} \in C_h^0$, is an arbitrary function.

Let B_i^h be a linear operator from C_h^0 to C_h^0 as follows:

$$B_i^h \widetilde{f_h}(x) = w_h(x, \eta_i), \; i = 1, 2, ..., M, \tag{2.31}$$

where w_h is the solution of the problem (2.30).

By Theorem 1.1 (Volkov, 1979), we have

$$\max_{(x,y)\in \overline{R}_h} |v_h - V_h| \leq c_1 h^2, \tag{2.32}$$

where v_h is a solution of the problem (2.28), V_h is the trace of the solution of (2.9) on $\overline{R}$ and c_1 is a constant independent of h.

Let

$$\overline{w}_h(x, y) = \frac{1}{b} \left\| \widetilde{f_h} \right\|_{C_h^0} (b - y), \quad (x, y) \in \overline{R}.$$

Then we have,

$$w_h \leq \overline{w}_h \text{ on } \gamma_h.$$

Additionally we get,

$$\triangle_h \overline{w}_h = 0.$$

It follows that,

$$\triangle_h (w_h - \overline{w}_h) = 0.$$

By maximum principle, it yields that

$$|w_h(x, y)| \leq \frac{1}{b} \left\| \widetilde{f_h} \right\|_{C_h^0} (b - y) \text{ on } \overline{R}.$$

Therefore, the following inequality holds in a similar thought of the estimate (2.15)

$$\left\| B_i^h \widetilde{f_h}(x) \right\|_{C_h^0} \leq \left\| \widetilde{f_h} \right\|_{C_h^0} \left(1 - \frac{\xi + (i-1)h}{b} \right), \; i = 1, 2, ..., M. \tag{2.33}$$

Define

$$\widetilde{\varphi}_h = \alpha \sum_{k=1}^{M} \rho_k \widetilde{\varphi}_{k,h}(x), \quad x \in [0, a]_h, \tag{2.34}$$

where $\widetilde{\varphi}_{k,h}$ is function (2.29).

By (2.11), (2.32) and (2.34), we obtain

$$\|\widetilde{\varphi}_h - \varphi_h\|_{C_h^0} \leq c_2 h^2, \tag{2.35}$$

where φ_h is the trace of the function φ defined by (2.11) on $[0,a]_h$ and c_2 is a constant independent of h.

In a similar thought with the relation (2.20) we have

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k B_k^h \widetilde{f}_h, \quad \text{on } \gamma_h^4, \tag{2.36}$$

where μ_h is the trace of the function μ defined by (2.4) on $[0,a]_h$.

Consider the following sequences in C_h^0 :

$$\widetilde{\psi}_{i,h}^0 = 0, \quad \widetilde{\psi}_{i,h}^n = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^{n-1} \right),$$

$$i = 1,2,...,M; \quad n = 1,2,.... \tag{2.37}$$

By using the inequality (2.33), the sequence $\left\{\widetilde{\psi}_{i,h}^n\right\}_{n=0}^{\infty}$ defined by (2.37) converges to the unique solution which is denoted by $\widetilde{\psi}_{i,h}, i = 1,2,...,M$. It follows that

$$\widetilde{\psi}_{i,h} = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right), \quad i = 1,2,...,M. \tag{2.38}$$

On the basis of (2.36) and (2.38), we have

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}. \tag{2.39}$$

Let $\psi_{i,h}^n$, φ_h and $(B_i\varphi)_h$ be the trace of ψ_i^n, φ and $B_i\varphi$ on $[0,a]_h$, respectively.

By using (2.21) and (2.37) we have, for all $i = 1,2,...,M$,

$$\left\|\widetilde{\psi}_{i,h}^0 - \psi_{i,h}^0\right\|_{C_h^0} = 0. \tag{2.40}$$

Then,

$$\left\|\widetilde{\psi}_{i,h}^1 - \psi_{i,h}^1\right\|_{C_h^0} \leq \left\|B_i^h\left(\widetilde{\varphi}_h - \varphi_h\right)\right\|_{C_h^0}$$

$$+ \left\|B_i^h\left(\varphi_h + \mu_h\right) - \left(B_i\left(\varphi + \mu\right)\right)_h\right\|_{C_h^0} . \tag{2.41}$$

Applying (2.33) and (2.35) it follows that

$$\left\|B_i^h\left(\widetilde{\varphi}_h-\varphi_h\right)\right\|_{C_h^0} \leq \left(1-\frac{\xi+(i-1)h}{b}\right)c_2h^2, \quad i=1,2,...,M. \tag{2.42}$$

Since φ and μ are in the class $C^{2,\lambda}, 0<\lambda<1$, on the interval $0 \leq x \leq a$, by Theorem 1.1 in (Volkov, 1979) and similarity to the estimate (2.35), the following inequality holds.

$$\left\|B_i^h\left(\varphi_h+\mu_h\right)-\left(B_i\left(\varphi+\mu\right)\right)_h\right\|_{C_h^0} \leq c_3h^2, \tag{2.43}$$

where c_3 is a constant independent of h.

From the relations (2.41)-(2.43), we have

$$\left\|\widetilde{\psi}_{i,h}^1-\psi_{i,h}^1\right\|_{C_h^0} \leq c_4h^2, \tag{2.44}$$

where c_4 is a constant independent of h.

For $n \geq 2$, we have

$$\begin{aligned}\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0} = \Bigg\| & B_i^h\left(\widetilde{\varphi}_h+\mu_h+\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^{n-1}\right) \\ & -\left(B_i\left(\varphi+\mu+\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\Bigg\|_{C_h^0}.\end{aligned} \tag{2.45}$$

Then,

$$\begin{aligned}\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0} \leq & \left\|B_i^h\left(\widetilde{\varphi}_h+\mu_h\right)-\left(B_i\left(\varphi+\mu\right)\right)_h\right\|_{C_h^0} \\ & +\left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^{n-1}-\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right\|_{C_h^0} \\ & +\left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)-\left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0} \\ i=1,2,...,M. &\end{aligned} \tag{2.46}$$

The difficulties of the inequality (2.46) comes from third term of the right side which is needed much effort to obtain an estimation.

By (2.13) , (2.15) and (2.21) we have

$$\begin{aligned}\left\|\psi_i^n\right\|_{C^0} \leq & \left\|B_i\left(\varphi+\mu\right)\right\|_{C^0}+\left\|B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right\|_{C^0} \\ \leq & \left(1-\frac{\xi+(i-1)h}{2}\right)\left\|\varphi+\mu\right\|_{C^0}+\left(1-\frac{\xi+(i-1)h}{2}\right)_{C^0}\max_{1\leq i\leq M}\left\|\psi_i^{n-1}\right\|\left(\alpha\sum_{k=1}^{M}|\rho_k|\right),\end{aligned} \tag{2.47}$$

for any n, $1 \leq n < \infty$,

$$\max_{1\leq i\leq M} \left\|\psi_i^n\right\|_{C^0} \leq q_1 \left(\|\varphi + \mu\|_{C^0} + \max_{1\leq i\leq M} \left\|\psi_i^{n-1}\right\| \right) \leq \frac{q_1}{1-q_1} \|\varphi + \mu\|_{C^0}, \tag{2.48}$$

where

$$q_1 = \left\{1 - \frac{\xi}{b}\right\} \tag{2.49}$$

Therefore,

$$\sup_{0\leq n<\infty} \left\| \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^n \right\| \leq \left(1 + \frac{q_1}{1-q_1} \alpha \sum_{k=1}^{M} |\rho_k| \right) \|\varphi + \mu\|_{C^0}. \tag{2.50}$$

The function $\psi_i^{n-1} = B_i \left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-2} \right)$, $n \geq 2$, is the trace of a solution of the following problem

$$\triangle V^n = 0 \text{ on } R, \; V^n = 0 \text{ on } \gamma^m, \; m = 1, 2, 3,$$
$$V^n = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-2} \text{ on } \gamma^4,$$

on the line segments $\eta_j = \xi + (j-1)h, \; j = 1, 2, ..., M$. In the view of (2.50), maximum principle and Lemma 3 in (Mikhailov, 1978), we have,for $0 \leq x \leq a$ and $i = 1, 2, ..., M$:

$$\max_{0\leq x\leq 1} \left| \frac{d^s \psi_i^{n-1}}{dx^s} \right| \leq c_{s,i}^0 \|\varphi + \mu\|_{C^0}, \; n \geq 2, \quad s \geq 4,$$

where $c_{s,i}^0$ are constants independent of n. Then $\psi_i^{n-1}(x) \in C^{4,\lambda}, 0 < \lambda < 1$, on $0 \leq x \leq a$. Since $V^n = 0$ on $\gamma^m, m = 1, 3$, the derivatives $d^{2r}\psi_i^{n-1}/dx^{2r} = 0, r = 0, 1, 2$, at $x = 0$ and $x = a$. Then, from Theorem 3.1 in (Volkov, 1965), the solution $z_i^n, i = 1, 2, ..., M$, of the following problems

$$\triangle z_i^n = 0 \text{ on } R, \; z_i^n = 0 \text{ on } \gamma^m, m = 1, 2, 3, \; z_i^n = \psi_i^{n-1}, \tag{2.51}$$

are from $C^{4,\lambda}\left(\overline{R}\right), 0 < \lambda < 1$. So, the following inequality holds from (Samarskii, 2001),

$$\max_{\overline{R}_h} \left| z_{i,h}^n - z_i^n \right| \leq c_5^0 h^2, \tag{2.52}$$

where $z_{i,h}^n$ is the 5-point finite difference solutions and c_5^0 is constant independent of h and n.

By (2.52), we have

$$\begin{aligned}
&\left\| B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)-\left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0}\\
&\leq \sum_{k=1}^{M}|\alpha\rho_k|\left\|B_i^h\psi_{k,h}^{n-1}-\left(B_i\psi_k^{n-1}\right)_h\right\|_{C_h^0}\\
&\leq c_5h^2,\quad i=1,2,...,M, \qquad (2.53)
\end{aligned}$$

where c_5 is a constant independent of h.

In the view of (2.6), (2.33), (2.44) and (2.53) yield

$$\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0}\leq c_6h^2+q_0\left\|\widetilde{\psi}_{i,h}^{n-1}-\psi_{i,h}^{n-1}\right\|_{C_h^0}, \qquad (2.54)$$

where q_0 is defined by (2.6) and $c_6=c_4+c_5$ is a constant independent of h. By induction, on the basis of (2.40) (2.44) and (2.54), we have

$$\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0}\leq c_6h^2 \qquad (2.55)$$

From (2.21) and by analogy of the estimation (48) in (Volkov and Dosiyev, 2016), we have

$$\left\|\psi_i^n-\psi_i\right\|_{C^0}\leq \frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right), \qquad (2.56)$$

where φ and μ are defined by (2.11) and (2.4), respectively and $q_1=1-\frac{\xi}{b}$.
According to estimates (2.55) and (2.56), we find that

$$\max_{1\leq i\leq m}\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}\right\|_{C_h^0}\leq c_6h^2+\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right), \qquad (2.57)$$

where $\psi_{i,h}$ is the trace of the function ψ_i on $[0,a]_h$.
Define

$$\widetilde{f}_h^n=\widetilde{\varphi}_h+\mu_h+\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^n, \qquad (2.58)$$

where $\widetilde{f}_h^n$ is an approximation of f defined by (2.27).
Combining estimates (2.35) and (2.57), we obtain

$$\left\|\widetilde{f}_h^n-f_h\right\|_{C^0}\leq c_7h^2+q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right), \qquad (2.59)$$

where $\widetilde{f}_h^n$ is computed function (2.58), f_h is the trace of f defined by (2.27), q_0 is the number given by (2.6), $c_7 = c_2 + q_0 c_6$ is a constant independent of h.

Let $U_h(x,t)$ be the solution of the system

$$U_h = A_h U_h \text{ on } R_h, \quad U_h = \tau \text{ on } \gamma_h^2, \quad U_h = 0 \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{2.60}$$

$$U_h = f \text{ on } \gamma_h^4, \tag{2.61}$$

which approximates the problem (2.7),(2.8) with f defined by (2.27).

Since τ, μ, φ and $\psi_i, i = 1, 2, ..., m$, belong to $C^{2,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq a$, By Theorem 1.1 in (Volkov, 1979), we have

$$\max_{(x,y)\in\overline{R}_h} |U_h - U| \leq c_8 h^2, \tag{2.62}$$

where U is the solution of the problem (2.7),(2.8) and c_8 is a constant independent of h.

Consider the actual finite difference problem

$$\widetilde{u}_h^n = A_h \widetilde{u}_h^n \text{ on } R_h, \quad \widetilde{u}_h^n = \tau_h \text{ on } \gamma_h^2, \quad \widetilde{u}_h^n = 0 \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{2.63}$$

$$\widetilde{u}_h^n = \widetilde{f}_h^n \text{ on } \gamma_h^4, \tag{2.64}$$

where $\widetilde{f}_h^n$ is computed function which approximates to f.

In the view of the inequality (2.59) and the grid maximum principle, we obtain

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U_h\right| \leq c_7 h^2 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right). \tag{2.65}$$

Consequently, according to estimates (2.62) and (2.65), the following inequality holds.

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U\right| \leq c_9 h^2 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{2.66}$$

where $\widetilde{u}_h^n$ is a solution of problem (2.63), (2.64), U is the solution of the problem (2.7),(2.8) and $c_9 = c_7 + c_8$ is a constant independent of h.

Using estimate (2.66) and by error estimate of trapezoidal rule, we derive final estimate

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - u\right| \leq c_{10} h^2 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{2.67}$$

where u is the solution of problem (2.3),(2.4) and c_{10} is a constant independent of h. Here right-hand side is $O\left(h^2\right)$ for

$$n = \max\left\{\left[\frac{\ln h^{-2}(1-q_1)^{-1}}{\ln q_1^{-1}}\right], 1\right\}, \tag{2.68}$$

where $[a]$ is the integer part of a.

Remark. The estimation (2.67) holds for nonlocal problem with integral boundary condition of Poisson's equation $\triangle u = F$ if $F \in C^{2,\lambda}(\overline{R}), 0 < \lambda < 1$ with $F(0,0) = F(a,0) = 0$, $F(0,b) = \tau''(0)$ and $F(a,b) = \tau''(a)$ by replacing the equations $\triangle v = F, \widetilde{v}_h^n = A_h\widetilde{v}_h^n - h^2F/4$ and $\widetilde{u}_h^n = A_h\widetilde{u}_h^n - h^2F/4$ instead of the equations $\triangle v = 0, \widetilde{v}_h^n = A_h\widetilde{v}_h^n$ and $\widetilde{u}_h^n = A_h\widetilde{u}_h^n$, respectively.

2.2 FOURTH ORDER ACCURACY FOR THE LAPLACE EQUATION WITH INTEGRAL BOUNDARY CONDITION

2.2.1 Overview

In this section, we propose and justify the method given in Section 2.1 to solve the system of nonlocal 9-point finite difference problem for the Laplace equation with the integral boundary condition. The solution of this nonlocal difference problem is defined as a solution of the 9-point Dirichlet problem by constructing the approximate values of the solution on the side where the integral condition was given. Therefore, the approximate solution is obtained by solving a system with 9 diagonal matrices, for the realization of which proposed many fast algorithms. (see in Samarskii, 1989). Moreover, the uniform estimate of the error of approximate solution is of order $O(h^4)$, when the given boundary functions on the sides belong to the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$.

2.2.2 Nonlocal boundary value problem

Let

$$R = \{(x, y) : 0 < x < a, 0 < y < b\}$$

be an open rectangle, γ^m, $m = 1, 2, 3, 4$, be its sides including the endpoints, numbered in the clockwise direction, beginning with the side lying on the y-axis and let $\gamma = \cup_{m=1}^{4}\gamma^m$ be the boundary of R and $\overline{R} = R \cup \gamma$. Let C^0 denote the linear space of continuous functions of one variable x on the interval $[0, a]$ of x-axis, and vanish at the points $x = 0$ and $x = a$. For the function $f \in C^0$ we define the norm

$$\|f\|_{C^0} = \max_{0\le x\le a} |f(x)| .$$

It is clear that the space C^0 which is defined with this norm is complete.

Consider the following nonlocal boundary value problem

$$\triangle u = 0 \text{ on } R, \;\; u = 0 \text{ on } \gamma^1 \cup \gamma^3, \; u = \tau \text{ on } \gamma^2, \tag{2.69}$$

$$u(x,0) = \alpha \int_{\xi}^{b} u(x,y)dy + \mu(x), \quad 0 < x < a, \; 0 < \xi < b, \tag{2.70}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, $\tau = \tau(x)$ and $\mu = \mu(x)$ are given functions which belong to C^0 and α is a given constant which holds the following inequality:

$$|\alpha| < \frac{1}{b - \xi}. \tag{2.71}$$

2.2.3 Nonlocal finite-difference problem and its reduction to the Dirichlet problem

We define a square mesh with the mesh size $h = \frac{a}{N} = \frac{b}{M^*}$, $N, M^* > 2$ are integers, constructed with the lines $x, y = h, 2h, \ldots$. Let D_h be the set of nodes of this square grid and let $R_h = R \cap D_h$, $\overline{R}_h = \overline{R} \cap D_h$. We put $\gamma_h^m = \gamma^m \cap D_h$, $m = 1, 2, 3, 4$, and $\gamma_h = \cup_{m=1}^{4} \gamma_h^m$.

Let

$$[0, a]_h = \left\{ x = x_i, \;\; x_i = ih, \;\; i = 0, 1, ..., N, \;\; h = \frac{a}{N} \right\}$$

be the set of points divided by the step size h on $[0, a]$.

Let C_h^0 be the linear space of grid functions defined on $[0, a]_h$ that vanish at $x = 0$ and $x = a$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x \in [0,a]_h} |f_h| \,.$$

We introduce the operator B_h,

$$Bu_h(x,y) \equiv (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/5 \tag{2.72}$$

$$+(u(x+h,y+h) + u(x+h,y-h) + \tag{2.73}$$

$$+u(x-h,y+h) + u(x-h,y-h))/20.$$

For the approximate solution of the nonlocal problem (2.69), (2.70), we consider a solution of the following system of the difference equations (see in Sapagovas, 2008)

$$u_h = Bu_h \text{ on } R_h, \;\; u_h = 0 \text{ on } \gamma_h^1 \cup \gamma_h^3, \quad u_h = \tau_h \text{ on } \gamma_h^2, \tag{2.74}$$

$$u_h(x,0) = \alpha \sum_{k=1}^{M} \rho_k u_h(x, \eta_k) + \mu_h \text{ on } \gamma_h^4, \tag{2.75}$$

where the equation (2.75) is obtained by approximating the integral in (2.70) and using Simpson's rule with $\rho_1 = \rho_M = \frac{h}{3}, \rho_j = \frac{h}{3}\left(3 + (-1)^j\right)$ for $j = 2, 3, ..., M-1$, $\eta_j = \xi + (j-1)h$, $j = 1, 2, ..., M$, $h = \frac{a}{N}$, $(M-1)h + \xi = b$, μ_h is the trace of μ on γ_h^4 and $\frac{\xi}{h}$ is an integer.

We reduce a solution of the nonlocal differential problem to the solution of the local Dirichlet problem.

Let v_h be the solution of the finite difference Dirichlet problem

$$v_h = Bv_h \text{ on } R_h, \quad v_h = \tau_h \text{ on } \gamma_h^2, \quad v_h = 0 \text{ on } \gamma_h/\gamma_h^2, \tag{2.76}$$

and we put

$$\widetilde{\varphi}_{i,h}(x) = v_h(x, \eta_i), \quad i = 1, 2, ..., M, \tag{2.77}$$

where τ_h is the trace of τ on γ_h^2.

Let w_h be a solution of the following finite difference Dirichlet problem

$$w_h = Bw_h \text{ on } R_h, \quad w_h = 0 \text{ on } \gamma_h/\gamma_h^4, \quad w_h = \widetilde{f_h} \text{ on } \gamma_h^4, \tag{2.78}$$

where $\widetilde{f_h} \in C_h^0$, is an arbitrary function.

We define a linear operator B_i^h from C_h^0 to C_h^0 as follows:

$$B_i^h \widetilde{f_h}(x) = w_h(x, \eta_i), \quad i = 1, 2, ..., M, \tag{2.79}$$

where w_h is the solution of the problem (2.78).

Let

$$w_h^*(x, y) = \frac{1}{b} \left\| \widetilde{f_h} \right\|_{C_h^0} (b - y) \text{ on } \overline{R}_h.$$

We have

$$|w_h(x, y)| \leq w_h^*(x, y), \quad (x, y) \in \gamma_h. \tag{2.80}$$

Since

$$L_h w_h^* = 0 \tag{2.81}$$

From (2.79)-(2.81) and by comparison theorem (see in Chapter 4 (Samarskii, 2001)), we have

$$\|w_h\|_{C_h^0} \leq \frac{1}{b}\left\|\widetilde{f_h}\right\|_{C_h^0}(b-y) \text{ on } \overline{R}_h.$$

Therefore,

$$\left\|B_i^h \widetilde{f_h}\right\|_{C_h^0} \leq \left\|\widetilde{f_h}\right\|_{C_h^0}\left(1-\frac{\xi+(i-1)h}{b}\right),\ i=1,2,...,M, \tag{2.82}$$

and then for the norm of operator B_i^h, we get

$$\left|B_i^h\right| < 1, \quad i=1,2,...,M. \tag{2.83}$$

Let

$$\widetilde{\varphi}_h = \alpha \sum_{k=1}^{M} \rho_k \widetilde{\varphi}_{k,h}(x), \quad x \in [0,a]_h\,, \tag{2.84}$$

where $\widetilde{\varphi}_{k,h}(x)$ is the function (2.77).

In the view of the inequality (2.71), we have

$$|\alpha| \sum_{k=1}^{M} \rho_k = q_0 < 1. \tag{2.85}$$

The inequalities (2.83) and (2.85) yield that

$$q_0 \left|B_1^h\right| = q < 1. \tag{2.86}$$

Lemma 2.2.1. A solution of the finite difference problem (2.74) , (2.75) can be represented as

$$u_h = v_h + w_h, \tag{2.87}$$

where v_h is the solution of problem (2.76) , w_h is the solution of problem (2.78) with $\widetilde{f_h}$ which is a solution of the following nonlinear equation

$$\widetilde{f_h} = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k B_k^h \widetilde{f_h}, \text{ on } \gamma_h^4. \tag{2.88}$$

Proof. According to (2.74) , (2.76) and (2.78) , the relation (2.87) holds on R_h and the boundary sides $\gamma_h^m,\ m = 1, 2, 3.$

From (2.84) and (2.88) , it follows that

$$\widetilde{f_h} = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \left[\widetilde{\varphi}_{k,h}(x) + B_k^h \widetilde{f_h}\right] \text{ on } \gamma_h^4.$$

Relying on (2.77) and (2.79), we have

$$\widetilde{f_h} = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \left[v_h(x, \eta_i) + w_h\,(x, \eta_i)\right] \text{ on } \gamma_h^4.$$

By virtue of (2.76) and (2.78) , we obtain

$$v_h(x, 0) + w_h\,(x, 0) = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \left[v_h(x, \eta_i) + w_h\,(x, \eta_i)\right] \text{ on } \gamma_h^4.$$

From (2.75) , it shows that the relation (2.87) is also satisfied on γ_h^4. □

Thus, the unknown function on γ_h^4 in problem (2.78) is a solution of the nonlinear equation (2.88).

Theorem 2.2.2. There exists a unique solution $\widetilde{f_h}$ of the nonlinear equation (2.88).

Proof. Consider the following sequences in C_h^0 :

$$\widetilde{\psi}_{i,h}^0 = 0, \quad \widetilde{\psi}_{i,h}^n = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^{n-1}\right),$$
$$i = 1, 2, ..., M; \quad n = 1, 2, ... \tag{2.89}$$

From this, for the positive integers m and n with $m > n$, we get

$$\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n = B_i^h \left(\alpha \sum_{k=1}^{M} \rho_k \left(\widetilde{\psi}_{k,h}^{m-1} - \widetilde{\psi}_{k,h}^{n-1}\right)\right), \ \ i = 1, 2, ..., M.$$

Applying the inequality (2.82), we reach

$$\left\|\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n\right\|_{C_h^0} \leq q \left\|\widetilde{\psi}_{i,h}^{m-1} - \widetilde{\psi}_{i,h}^{n-1}\right\|_{C_h^0} \tag{2.90}$$

where q is defined by (2.86). In a similar way with (2.90), we obtain

$$\left\|\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n\right\|_{C_h^0} \leq q^{n+1} \frac{1 - q^{m-n}}{1 - q} \left(\|\widetilde{\varphi}_h\|_{C_h^0} + \|\mu_h\|_{C_h^0}\right)$$

which shows that the sequences (2.89) are Cauchy sequences. Since C_h^0 is complete, there are limits

$$\lim_{n\to\infty} \widetilde{\psi}_{i,h}^n = \widetilde{\psi}_{i,h} \in C_h^0, \ \ i = 1, 2, ..., M.$$

By using (2.82) and (2.86),

$$\lim_{n\to\infty} B_k^h \widetilde{\psi}_{i,h}^n = B_k^h \widetilde{\psi}_{i,h} \in C_h^0, \ \ i, k = 1, 2, ..., M. \tag{2.91}$$

Using (2.91) and taking limit of (2.89) as $n \to \infty$, we have

$$\widetilde{\psi}_{i,h} = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right), \ \ i = 1, 2, ..., M. \tag{2.92}$$

We multiply both of side of the equation (2.92) by $\alpha\rho_i$ and summing for $i = 1, 2, ..., M$, we have

$$\widetilde{\varphi}_h + \mu_h + \alpha \sum_{i=1}^{M} \rho_i \widetilde{\psi}_{i,h} = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{i=1}^{M} \rho_i B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right) \tag{2.93}$$

In the view of the relations (2.88) and (2.93), we obtain a solution of the nonlinear equation (2.88) as

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}.$$

To show the uniqueness, let $\widetilde{f}_{h,p} \in C_h^0, \ p = 1, 2$, be two functions satisfying the relation (2.88). Then, we obtain the following inequality

$$\left\| \widetilde{f}_{h,1} - \widetilde{f}_{h,2} \right\|_{C_h^0} = \left\| \alpha \sum_{k=1}^{m} \rho_k B_k^h \left(\widetilde{f}_{h,1} - \widetilde{f}_{h,2} \right) \right\|_{C_h^0} \le q \left\| \widetilde{f}_{h,1} - \widetilde{f}_{h,2} \right\|_{C_h^0}$$

where $0 < q < 1$ is defined by (2.86). Hence $\widetilde{f}_{h,1} = \widetilde{f}_{h,2}$. □

2.2.4 Convergence of the finite difference problem

We say that $F \in C^{k,\lambda}(E)$, if F has k-th derivatives on E satisfying the Hölder condition with exponent λ. We assume that $\tau(x)$ and $\mu(x)$ in (2.69) and (2.70) are from $C^{4,\lambda}, 0 < \lambda < 1$, on

γ^2 and γ^4, respectively and $\tau^{(2m)}(0) = \tau^{(2m)}(a) = 0$, $\mu^{(2m)}(0) = \mu^{(2m)}(a) = 0$, $m = 0, 1, 2$. By using the $n - th$ iteration $\widetilde{\psi}^n_{i,h}$, $n \geq 1$ of (2.89), we define the function

$$\widetilde{f}^n_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}^n_{k,h}. \tag{2.94}$$

Hence, for the approximate solution of the nonlocal problem (2.69), (2.70), we define the following difference problem

$$\widetilde{u}^n_h = B_h \widetilde{u}^n_h \text{ on } R_h, \quad \widetilde{u}^n_h = \tau_h \text{ on } \gamma^2_h, \quad \widetilde{u}^n_h = 0 \text{ on } \gamma^1_h \cup \gamma^3_h, \tag{2.95}$$

$$\widetilde{u}^n_h = \widetilde{f}^n_h \text{ on } \gamma^4_h. \tag{2.96}$$

Theorem 2.2.3. The estimation holds

$$\max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}^n_h - u \right| \leq c_1 h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} c^*, \tag{2.97}$$

where $\widetilde{u}^n_h$ is a solution of problem (2.95), (2.96), u is the exact solution of nonlocal boundary value problem (2.69) , (2.70), c_1 and c^* are constants independent of h, q_0 is defined by (2.85) and $q_1 = 1 - \frac{\xi}{b}$.

Proof. Let U be the exact solution of system of the following problem

$$\triangle U = 0 \text{ on } R, \ U = \tau \text{ on } \gamma^2, \ U = 0 \text{ on } \gamma^1 \cup \gamma^3, \tag{2.98}$$

$$U(x, 0) = \alpha \sum_{k=1}^{M} \rho_k U(x, \eta_k) + \mu(x), \quad 0 \leq x \leq a. \tag{2.99}$$

Let V be a solution of the Dirichlet problem,

$$\triangle V = 0 \text{ on } R, \ V = \tau \text{ on } \gamma^2, \ V = 0 \text{ on } \gamma / \gamma^2, \tag{2.100}$$

and denote by

$$\varphi_k(x) = V(x, \eta_k) \text{ for } k = 1, 2, ..., M, \tag{2.101}$$

where $\eta_k = \xi + (k - 1)h$, $k = 1, 2, ..., M$. We define the function

$$\varphi = \alpha \sum_{k=1}^{M} \rho_k \varphi_k. \tag{2.102}$$

Consider the Dirichlet problem

$$\triangle W = 0 \ \text{ on } R, \ \ W = 0 \ \text{ on } \gamma/\gamma^4, \ W = f \ \text{ on } \ \gamma^4, \tag{2.103}$$

where f be an unknown function from C^0. The linear operator $B_i : C^0 \to C^0$ is defined as

$$B_i f(x) = W(x, \eta_i) \ \in \ C^0, \ \ i = 1, 2, ..., M. \tag{2.104}$$

Then following inequality holds for the norm $|B_i|$

$$|B_i| < \left(1 - \frac{\xi + (i-1)h}{b}\right), \ i = 1, 2, ..., M. \tag{2.105}$$

By analogy with the results in (Volkov, 2013), it is shown that a solution U of problem (2.98) , (2.99) can be represented as $U = V + W$ where V and W are the solutions of problem (2.100) and (2.103) respectively, when f defined by

$$f = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k. \tag{2.106}$$

Here the functions $\psi_1, \psi_2, ..., \psi_M$ are from C^0, and are defined as the solution of the nonlinear equations

$$\psi_i = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k\right), \ \ i = 1, 2, ..., M. \tag{2.107}$$

Therefore, the nonlocal problem (2.98), (2.99) is reduced to the following Dirichlet problem

$$\triangle U = 0 \ \text{ on } R, \ \ U = \tau \ \text{ on } \gamma^2, \ \ U = 0 \ \text{ on } \gamma^1 \cup \gamma^3, \tag{2.108}$$

$$U(x, 0) = f, \ 0 \leq x \leq a, \tag{2.109}$$

where f is defined by (2.106) . The solution ψ_i, $i = 1, 2, ..., M$, of system (2.107) is found as a limit of the infinite sequence of functions $\left\{\psi_i^n\right\}_{n=0}^{\infty}$ in C^0 defined by

$$\psi_i^0 = 0, \ \psi_i^n = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1}\right),$$

$$i = 1, 2, ..., M; \ \ n = 1, 2, ... \, . \tag{2.110}$$

Since $\tau(x)$ in (2.100) belongs to $C^{4,\lambda}\left(\gamma^2\right)$ and $\tau^{(2m)}(0) = \tau^{(2m)}(a) = 0$, $m = 0, 1, 2$, it follows from (Dosiyev, 2003) that

$$\max_{(x,y)\in\overline{R}_h} |v_h - V_h| \leq c_2 h^4, \tag{2.111}$$

where v_h is a solution of the problem (2.76), V_h is the trace of the solution of (2.100) on $\overline{R}_h$ and c_2 is a constant independent of h. Let φ_h, $\psi_{i,h}$ and $\psi_{i,h}^n$ be the trace of φ, ψ_i and ψ_i^n on $[0, a]_h$, respectively and let $(B_i(F))_h$ be the trace of $B_i(F)$ on $[0, a]_h$ for any function $F \in C^{4,\lambda}[0, a]$. By (2.77), (2.84), (2.101), (2.102) and (2.111), we obtain

$$\|\widetilde{\varphi}_h - \varphi_h\|_{C_h^0} \leq c_3 h^4, \tag{2.112}$$

where c_3 is a constant independent of h. By using (2.89) and (2.110), we have, for all $i = 1, 2, ..., M$,

$$\begin{aligned} \left\|\widetilde{\psi}_{i,h}^1 - \psi_{i,h}^1\right\|_{C_h^0} &\leq \left\|B_i^h(\widetilde{\varphi}_h - \varphi_h)\right\|_{C_h^0} \\ &\quad + \left\|B_i^h(\varphi_h + \mu_h) - (B_i(\varphi + \mu))_h\right\|_{C_h^0}. \end{aligned} \tag{2.113}$$

Applying (2.82) and (2.112), it follows that

$$\left\|B_i^h(\widetilde{\varphi}_h - \varphi_h)\right\|_{C_h^0} \leq c_4 h^4, \quad i = 1, 2, ..., M, \tag{2.114}$$

where c_4 is a constant independent of h. Similar to the inequality (2.111), we have

$$\left\|B_i^h(\varphi_h + \mu_h) - (B_i(\varphi + \mu))_h\right\|_{C_h^0} \leq c_5 h^4, \tag{2.115}$$

where c_5 is a constant independent of h. From the relations (2.113)-(2.115), we have

$$\left\|\widetilde{\psi}_{i,h}^1 - \psi_{i,h}^1\right\|_{C_h^0} \leq c_6 h^4, \tag{2.116}$$

where c_6 is a constant independent of h. For $n \geq 2$, we have

$$\begin{aligned} \left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\right\|_{C_h^0} = &\left\| B_i^h\left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^{n-1}\right)\right. \\ &\left. - \left(B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1}\right)\right)_h \right\|_{C_h^0}. \end{aligned}$$

Then,

$$\left\|\widetilde{\psi}^n_{i,h}-\psi^n_{i,h}\right\|_{C^0_h}\le\left\|B^h_i\left(\widetilde{\varphi}_h+\mu_h\right)-\left(B_i\left(\varphi+\mu\right)\right)_h\right\|_{C^0_h}$$
$$+\left\|B^h_i\left(\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}^{n-1}_{k,h}-\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right\|_{C^0_h}$$
$$+\left\|B^h_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)-\left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right)_h\right\|_{C^0_h},$$
$$i=1,2,...,M. \tag{2.117}$$

From (2.104) , (2.105) and (2.110)i it yields that

$$\left\|\psi^n_i\right\|_{C^0}\le\left\|B_i\left(\varphi+\mu\right)\right\|_{C^0}+\left\|B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right\|_{C^0} \tag{2.118}$$
$$\le\left(1-\frac{\xi+(i-1)h}{b}\right)\left\|\varphi+\mu\right\|_{C^0}+\left(1-\frac{\xi+(i-1)h}{b}\right)\max_{1\le i\le M}\left\|\psi^{n-1}_i\right\|\left(\alpha\sum_{k=1}^{M}|\rho_k|\right).$$

For any n, $1\le n<\infty$,

$$\max_{1\le i\le M}\left\|\psi^n_i\right\|_{C^0}\le q_1\left(\left\|\varphi+\mu\right\|_{C^0}+\max_{1\le i\le M}\left\|\psi^{n-1}_i\right\|\right)\le\frac{q_1}{1-q_1}\left\|\varphi+\mu\right\|_{C^0},$$

where

$$q_1=1-\frac{\xi}{b}.$$

So,

$$\sup_{0\le n<\infty}\left\|\varphi+\mu+\sum_{k=1}^{M}\rho_k\psi^n_k\right\|\le\left(1+\frac{q_1}{1-q_1}\sum_{k=1}^{M}|\rho_k|\right)\left\|\varphi+\mu\right\|_{C^0}. \tag{2.119}$$

The function $\psi^{n-1}_i=B_i\left(\varphi+\mu+\alpha\sum\limits_{k=1}^{M}\rho_k\psi^{n-2}_k\right)$, $n\ge 2$, is the trace of a solution of the following problem

$$\triangle V^n=0 \text{ on } R,\ V^n=0 \text{ on } \gamma^m,\ m=1,2,3, \tag{2.120}$$
$$V^n=\varphi+\mu+\alpha\sum_{k=1}^{M}\rho_k\psi^{n-2}_k \text{ on } \gamma^4,$$

on the line segments $\eta_j = \xi + (j-1)h,\ j = 1,2,...,M$. In the view of (2.119) , maximum principle and Lemma 3 in (Mikhailov, 1978), we have, for $0 \le x \le a$ and $i = 1,2,...,M$:

$$\max_{0\le x\le a}\left|\frac{d^s\psi_i^{n-1}}{dx^s}\right| \le c_{s,i}^0 \|\varphi+\mu\|_{C^0},\ n \ge 2,\quad s \ge 6,$$

where $c_{s,i}^0$ are constants independent of n.Then, $\psi_i^{n-1}(x) \in C^{6,\lambda}, 0 < \lambda < 1$, on $0 \le x \le a$, and $\frac{d^s\psi_i^{n-1}(0)}{dx^s} = \frac{d^s\psi_i^{n-1}(a)}{dx^s} = 0, s = 0,2,4,6$. Since $V^n = 0$ on $\gamma^m, m = 1,3$, the derivatives $d^{2r}\psi_i^{n-1}/dx^{2r} = 0, r = 0,1,2$, at $x = 0$ and $x = a$. Then, from Theorem 3.1 in (Volkov, 1965), the solution $z_i^n, i = 1,2,...,M$, of the following problems

$$\triangle z_i^n = 0 \text{ on } R,\ z_i^n = 0 \text{ on } \gamma^m, m = 1,2,3,\ z_i^n = \psi_i^{n-1}, \tag{2.121}$$

are from $C^{6,\lambda}\left(\overline{R}\right), 0 < \lambda < 1$. So, the following inequality holds from (Samarskii, 2001),

$$\max_{\overline{R}_h}\left|z_{i,h}^n - z_i^n\right| \le c_7 h^4, \tag{2.122}$$

where $z_{i,h}^n$ is the 9-point finite difference solutions and c_7 is constant independent of h and n. By (2.122), we have

$$\begin{aligned}&\left\|B_i^h\left(\alpha\sum_{k=1}^M \rho_k\psi_k^{n-1}\right) - \left(B_i\left(\alpha\sum_{k=1}^M \rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0}\\ &\le \sum_{k=1}^M |\alpha\rho_k| \left\|B_i^h\psi_{k,h}^{n-1} - \left(B_i\psi_k^{n-1}\right)_h\right\|_{C_h^0}\\ &\le c_8 h^4,\quad i = 1,2,...,M.\end{aligned} \tag{2.123}$$

where c_8 is a constant independent of h.

In the view of (2.82), (2.85), (2.115), (2.117) and (2.123), yields

$$\left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\right\|_{C_h^0} \le c_9 h^4 + q_0\left\|\widetilde{\psi}_{i,h}^{n-1} - \psi_{i,h}^{n-1}\right\|_{C_h^0}, \tag{2.124}$$

where q_0 is defined by (2.85) and c_9 is a constant independent of h. By virtue of (2.116), (2.124), we have

$$\left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\right\|_{C_h^0} \le c_{10}h^4\left(1 + q_0 + q_0^2 + ... + q_0^{n-1}\right) \le c_{11}h^4, \tag{2.125}$$

where c_{10}, c_{11} are constants independent of h. According to (2.110), it follows that

$$\left\|\psi_i^1\right\|_{C^0} \le \left(1 - \frac{\xi}{b}\right)\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{2.126}$$

$$\left\|\psi_i^n - \psi_i^{n-1}\right\|_{C^0} \leq |B_i|\,|\alpha| \sum_{k=1}^{M} |\rho_k| \left\|\psi_i^{n-1} - \psi_i^{n-2}\right\|_{C^0},\; i = 1, 2, ..., M, \tag{2.127}$$

where φ is defined by (2.102). From (2.126) and (2.127), we have

$$\left\|\psi_i^n - \psi_i^{n-1}\right\|_{C^0} \leq q_1^n \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right),\;\; i = 1, 2, ..., M,$$

where $q_1 = 1 - \frac{\xi}{b}$. Moreover, for any $m = 1, 2, ...,$ we obtain

$$\left\|\psi_i^{n+m} - \psi_i^n\right\|_{C^0} \leq q_1^{n+1} \left(\frac{1-q_1^m}{1 - q_1}\right)\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right),\;\; i = 1, 2, ..., M. \tag{2.128}$$

Since

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \left\|\psi_i^{n+m} - \psi_i^n\right\|_{C^0} + \left\|\psi_i^{n+m} - \psi_i\right\|_{C^0},\;\; i = 1, 2, ..., M, \tag{2.129}$$

by taking limit as $m \to \infty$, from (2.128) and (2.129), it follows that

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \frac{q_1^{n+1}}{1 - q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right),\;\; i = 1, 2, ..., M. \tag{2.130}$$

From (2.125) and (2.130), we have

$$\left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}\right\|_{C_h^0} \leq c_{11}h^4 + \frac{q_1^{n+1}}{1 - q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right),\;\; i = 1, 2, ..., M. \tag{2.131}$$

Let $U_h(x, y)$ be the solution of the system of grid equations

$$U_h = B_h U_h \;\text{ on } R_h, \quad U_h = \tau \text{ on } \gamma_h^2, \quad U_h = 0 \;\text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{2.132}$$

$$U_h = f_h \;\text{ on } \gamma_h^4, \tag{2.133}$$

which approximates problem (2.108), (2.109) when f_h is the trace of f on $[0, a]_h$. Since τ, μ, φ and $\psi_i, i = 1, 2, ..., M,$ belong to $C^{4,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq a$, and $2m - th$ order of derivatives vanish at the endpoints for $m = 0, 1, 2$ (see in Dosiyev, 2018), by (Dosiyev, 2003), we have

$$\max_{(x,y)\in\overline{R}_h} |U_h - U| \leq c_{12}h^4, \tag{2.134}$$

where U is the solution of problem (2.98),(2.99) and c_{12} is a constant independent of h. In the view of the inequalities (2.112) and (2.131), we obtain

$$\left\| \widetilde{f}_h^n - f_h \right\|_{C_h^0} \leq c_{13} h^4 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{2.135}$$

where q_0 is defined by (2.85) and c_{13} is a constant independent of h. By the grid maximum principle and from (2.135) we have

$$\max_{(x,y)\in\overline{R}_h} \left| \widetilde{u}_h^n - U_h \right| \leq c_{13} h^4 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{2.136}$$

where $\widetilde{u}_h^n$ is the solution of problem (2.95) , (2.96) and U_h is the solution of problem (2.132), (2.133). According to estimates (2.134) and (2.136), the following inequality holds.

$$\max_{(x,y)\in\overline{R}_h} \left| \widetilde{u}_h^n - U \right| \leq c_{14} h^4 + q_0 \frac{q_1^{n+1}}{1-q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{2.137}$$

where U is the solution of the problem (2.98), (2.99) and c_{14} is a constant independent of h. Using the estimate (2.137) and by the maximum principle for the Laplace equation with the truncation error of Simpson's rule which is order of $O(h^4)$, we obtain the final estimate

$$\begin{aligned} \max_{(x,y)\in\overline{R}_h} \left| \widetilde{u}_h^n - u \right| &\leq \max_{(x,y)\in\overline{R}_h} \left| \widetilde{u}_h^n - U \right| + \max_{(x,y)\in\overline{R}_h} |U - u| \\ &\leq c_1 h^4 + q_0 \frac{q_1^{n+1}}{1-q_1} c^*, \end{aligned} \tag{2.138}$$

where u is the solution of problem (2.69),(2.70) and c_1 is a constant independent of h and $c^* = \|\varphi\|_{C^0} + \|\mu\|_{C^0}$. □

Remark. In (2.138) the right-hand side is $O\left(h^4\right)$, when

$$\frac{q_1^{n+1}}{1-q_1} \approx h^4.$$

It follows that

$$n = \max\left\{ \left[\frac{\ln h^4(1-q_1)}{\ln q_1} \right], 1 \right\},$$

where $[a]$ is the integer part of a.

CHAPTER 3

DIFFERENCE DIRICHLET PROBLEM FOR THE APPROXIMATE SOLUTION OF THE GENERAL SECOND ORDER LINEAR ELLIPTIC EQUATION WITH NONLOCAL INTEGRAL BOUNDARY CONDITION

3.1 SECOND ORDER ACCURACY FOR THE SECOND ORDER ELLIPTIC EQUATION WITH INTEGRAL BOUNDARY CONDITION

3.1.1 Overview

In this section, the method given by Chapter 2 for the general second-order linear elliptic equation with nonlocal integral boundary condition is proposed and justifed. The solution of this nonlocal problem is defined as 5-point classical Dirichlet problem by finding a function isntead of boundary value where the nonlocal condition was given. The approximate solution is obtained by using 5-diagonal matrices which are determined from the system of finite difference equations. It is proved that the uniform estimate of the error of the approximate solution is order of $O(h^2)$, h is the step mesh, when the boundary functions have a fourth derivative satisfying a Hölder condition.

3.1.2 Nonlocal boundary value problem

Let

$$R = \{(x, y) : 0 < x < \beta_1, 0 < y < \beta_2\} \tag{3.1}$$

be an open rectangle, $\gamma^m, m = 1, 2, 3, 4$, be its sides including the endpoints, numbered in the clockwise direction, beginning with the side lying on the y-axis and let $\gamma = \cup_{m=1}^{4}\gamma^m$ be the boundary of R.

Let C^0 denote the linear space of continuos functions of one variable x on the interval $[0, \beta_1]$ of x-axis, and vanish at the points $x = 0$ and $x = \beta_1$. For the function $f \in C^0$ we define the norm

$$\|f\|_{C^0} = \max_{0 \le x \le \beta_1} |f(x)| . \tag{3.2}$$

It is obvious that, the space C^0 is complete by normed with this way.

The second order elliptic operator is given in the form

$$Lu = \Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu, \tag{3.3}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, a, b and c are functions of (x, y) which we suppose to be continuous with $c(x, y) \leq 0$.

Consider the following nonlocal boundary value problem

$$Lu = 0 \text{ on } R,\ u = 0 \text{ on } \gamma^1 \cup \gamma^3,\ u = \tau \text{ on } \gamma^2, \tag{3.4}$$

$$u(x, 0) = \alpha \int_{\xi}^{\beta_2} u(x, y)dy + \mu(x), \quad 0 < x < \beta_1,\ 0 < \xi < \beta_2, \tag{3.5}$$

where $\tau = \tau(x) \in C^0$ is a given function and α is a given constant satisfying the inequality $|\alpha| < \frac{1}{\beta_2 - \xi}$.

We consider the following multilevel nonlocal boundary value problem on R in the similar thought of replacing (2.4) to (2.5) in Chapter 2 by using trapezoidal rule,

$$LU = 0 \text{ on } R,\ U = \tau \text{ on } \gamma^2,\ U = 0 \text{ on } \gamma^1 \cup \gamma^3\ , \tag{3.6}$$

$$U(x, 0) = \alpha \sum_{k=1}^{M} \rho_k U(x, \eta_k) + \mu(x), \quad 0 \leq x \leq \beta_1. \tag{3.7}$$

where $\rho_1 = \rho_M = \frac{h}{2}, \rho_j = h$ for $j = 1, 2, ..., M - 1,\ \eta_j = \xi + (j - 1)h,\ j = 1, 2, ..., M,\ h = \frac{\beta_1}{N}$, $(M - 1)h + \xi = \beta_2$ and $\frac{\xi}{h}$ is an integer.

Therefore,

$$q_0 = |\alpha| \sum_{k=1}^{M} \rho_k < 1. \tag{3.8}$$

We consider the Dirichlet problem,

$$LV = 0 \text{ on } R,\ V = \tau \text{ on } \gamma^2,\ V = 0 \text{ on } \gamma/\gamma^2. \tag{3.9}$$

Let us put

$$\varphi_k(x) = V(x, \eta_k) \ \text{ for } k = 1, 2, ..., M, \tag{3.10}$$

and

$$\varphi = \alpha \sum_{k=1}^{M} \rho_k \varphi_k. \tag{3.11}$$

Let W be a solution of the Dirichlet problem,

$$LW = 0 \text{ on } R,\ W = 0 \text{ on } \gamma/\gamma^4,\ W = f \text{ on } \gamma^4, \tag{3.12}$$

where f be an unknown function from C^0.

Lemma 3.1.1. *Assume that*

$$b(x,y) \geq 0.$$

Then the following inequality holds

$$|W(x,y)| \leq \overline{W}(x,y) \text{ on } \overline{R},$$

where b is defined by (3.3) *and* $\overline{W}(x,y) = \frac{1}{\beta_2}\|f\|_{C^0}(\beta_2 - y)$, $(x,y) \in \overline{R}$

Proof. From (3.12), we have

$$|W(x,y)| \leq \overline{W}(x,y) \text{ on } \gamma.$$

Since $\Delta\overline{W} + a(x,y)\frac{\partial u}{\partial x} = 0$,we have

$$\begin{aligned} L\overline{W} &= -\frac{1}{\beta_2} b(x,y)\|f\|_{C^0} + c(x,y)\|f\|_{C^0}(\beta_2 - y) \\ &\leq -\frac{1}{\beta_2} b(x,y)\|f\|_{C^0}. \end{aligned} \tag{3.13}$$

By assumption, we get

$$L\overline{W} \leq 0.$$

Since $W - \overline{W} \leq 0$ on γ and $L(W - \overline{W}) = -L\overline{W} \geq 0$ on R, by the maximum principle (see Bers et all, 1964), the function $W - \overline{W}$ takes its positive maximum on γ. Then,

$$W \leq \overline{W} \text{ on } \overline{R}.$$

If we replace W with $-W$, we obtain in a similar way,

$$-W \leq \overline{W} \text{ on } \overline{R}.$$

Therefore,

$$|W| \leq \overline{W} \text{ on } \overline{R}.$$

□

We introduce the operator $B_i : C^0 \to C^0$ as

$$B_i f(x) = W(x, \eta_i) \in C^0, \quad i = 1, 2, ..., M. \tag{3.14}$$

By Lemma 3.1.1, it follows that

$$\|B_i f\|_{C^0} < \left(1 - \frac{\xi + (i-)h}{\beta_2}\right) \|f\|_{C^0}, \quad i = 1, 2, ..., M \tag{3.15}$$

and

$$0 < |B_M| < |B_{M-1}| < ... < |B_1| < 1. \tag{3.16}$$

Then the following inequality remains true

$$|B_1|\, q_0 = q < 1, \tag{3.17}$$

where q_0 is defined in (3.8).

By the facts $U(x, 0) = f(x), 0 \leq x \leq \beta_1$, and $U = V + W$ with combining (3.10),(3.11) and (3.14) we have,

$$f = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k B_k f. \tag{3.18}$$

Consider an infinite sequence of functions $\psi_i^n(x)$ on $0 \leq x \leq \beta_1$,

$$\psi_i^0 = 0, \ \psi_i^n = B_i \left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1}\right),$$
$$i = 1, 2, ..., M; \ \ n = 1, 2, ... \tag{3.19}$$

We assume that $\psi_k^n(x) \in C^{4,\lambda}, 0 < \lambda < 1, k = 1, 2, ..., M$, on $0 \leq x \leq \beta_1$.

The first aim is to prove existence and uniqueness of the function f. To achieve this, we prove next Lemma.

Lemma 3.1.2. *The infinite sequence of functions defined by* (3.19) *is fundamental.*

Proof. By virtue of (3.19) ,

$$\psi_i^m - \psi_i^n = B_i\left(\alpha\sum_{k=1}^{M}\rho_k\left(\psi_k^{m-1} - \psi_k^{n-1}\right)\right), \ \ i = 1, 2, ..., M.$$

By using the inequalities (3.15) and (3.17)we get

$$\left\|\psi_i^m - \psi_i^n\right\|_{C^0} \leq q\left\|\psi_i^{m-1} - \psi_i^{n-1}\right\|_{C^0} \tag{3.20}$$

where q is defined by (3.17). In a similar way with (3.20), we reach

$$\left\|\psi_i^m - \psi_i^n\right\|_{C^0} \leq q^{n+1}\frac{1 - q^{m-n}}{1 - q}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right)$$

From this, we conclude that the sequences of functions (3.19) are fundamental. □

By Lemma 3.2.1, there are limits

$$\lim_{n\to\infty}\psi_i^n = \psi_i \in C^0, \ i = 1, 2, ..., M. \tag{3.21}$$

Then

$$\lim_{n\to\infty}B_k\psi_i^n = B_k\psi_i \in C^0, \ i, k = 1, 2, ..., M. \tag{3.22}$$

By taking limit of (3.19) as $n \to \infty$, we het

$$\psi_i = B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k\right), \ i = 1, 2, ..., M. \tag{3.23}$$

We multiply the relation (3.23) by ρ_i, for each $i = 1, 2, ..., M$ and sum M number of equation. Then we multiply the existing relation by α and we add $\varphi + \mu$ to both sides. Then,

$$\varphi + \mu + \alpha\sum_{i=1}^{M}\rho_i\psi_i = \varphi + \mu + \alpha\sum_{i=1}^{M}\rho_i B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k\right) \tag{3.24}$$

From the relations (3.18) and (3.24), we obtain

$$f = \varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k. \tag{3.25}$$

Uniqueness of f :

Let $f^P \in C^0, p = 1, 2$, be two functions satisfying the relation (3.18). That is

$$f^P = \varphi + \mu + \alpha \sum_{k=1}^{m} \rho_k B_k f^P, \quad p = 1, 2.$$

Then we reach the inequality

$$\left\| f^1 - f^2 \right\|_{C^0} = \left\| \alpha \sum_{k=1}^{m} \rho_k B_k \left(f^1 - f^2 \right) \right\|_{C^0} \leq q \left\| f^1 - f^2 \right\|_{C^0}$$

which satisfies if $f^1 = f^2$.

3.1.3 Finite difference method for the approximate solution of the nonlocal boundary value problem

We say that $F \in C^{k,\lambda}(E)$, if F has k-th derivatives on E satisfying the Hölder condition with exponent λ.

We assume that $\tau(x) \in C^{4,\lambda}\left(\gamma^2\right), \mu(x) \in C^{4,\lambda}\left(\gamma^4\right)$ in (3.4) and (3.5), respectively.

On the basis of Lemma 1 and Lemma 2 (Dosiyev, 2018) it follows that the function φ defined by (3.11) and the functions $\psi_i, i = 1, 2, ..., m$, in (3.23) obtained as the limits of the sequences (3.19) belong to $C^{4,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq \beta_1$.

We define a square mesh with the mesh size $h = \frac{\beta_1}{N}$, $N > 2$ is an integer, constructed with the lines $x, y = h, 2h, ...$Let D_h be the set of nodes of this square grid, $R_h = R \cap D_h$, and $\overline{R}_h = \overline{R} \cap D_h$, where R is the rectangle (3.1), and $\gamma_h^m = \gamma^m \cap D_h, m = 1, 2, 3, 4$.

Let

$$[0, \beta_1]_h = \left\{ x = x_i, \ x_i = ih, \ i = 0, 1, ..., N, \ h = \frac{\beta_1}{N} \right\}$$

be the set of points divided by the step size h on $[0, \beta_1]$.

Let C_h^0 be the linear space of grid functions defined on $[0, \beta_1]_h$ that vanish at $x = 0$ and $x = \beta_1$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x \in [0, \beta_1]_h} |f_h|.$$

We approximate the operator L by finite difference operator as

$$L_h u_h \equiv \frac{1}{h^2}\left(u_h(x+h,y)+u_h(x-h,y)+u_h(x,y+h)+u_h(x,y-h)-4u_h(x,y)\right) \quad (3.26)$$

$$a(x,y)\frac{u_h(x+h,y)-u_h(x-h,y)}{2h}+b(x,y)\frac{u_h(x,y+h)-u_h(x,y-h)}{2h} \quad (3.27)$$

$$+c(x,y)u_h(x,y)=0$$

It is assumed that h is so small and then the maximum principle holds for functions u under the following assumption

$$hK \leq 1,$$

where $K = \max(|a|+|b|)$. This is achieved from the positivity of the coefficients of h-neighbors $u(x,y)$ (see Bers et all, 1964).

Let v_h be a solution of following the system of grid equations

$$L_h v_h = 0 \text{ on } R_h, \quad v_h = \tau_h \text{ on } \gamma_h^2, \quad v_h = 0 \text{ on } \gamma_h/\gamma_h^2, \quad (3.28)$$

where τ_h is the trace of τ on γ_h^2 and we define

$$\widetilde{\varphi}_{i,h}(x) = v_h(x,\eta_i), \ i=1,2,...,M. \quad (3.29)$$

Let w_h be a solution of the finite difference problem

$$L_h w_h = 0 \text{ on } R_h, \quad w_h = 0 \text{ on } \gamma_h/\gamma_h^4, \quad w_h = \widetilde{f}_h \text{ on } \gamma_h^4, \quad (3.30)$$

where $\widetilde{f}_h \in C_h^0$, is an arbitrary function.
Let B_i^h be a linear operator from C_h^0 to C_h^0 as follows:

$$B_i^h \widetilde{f}_h(x) = w_h(x,\eta_i), \ i=1,2,...,M \quad (3.31)$$

where w_h is the solution of the problem (3.30).
Let

$$\overline{W}_h(x_i,y_j) = \frac{1}{\beta_2}\left\|\widetilde{f}_h\right\|_{C_h^0}\left(\beta_2 - y_j\right), \ \ i=1,2,...,N, \ j=1,2,...,M.$$

Then we have,

$$L_h\overline{W}_h = \frac{1}{2}\left\|\widetilde{f}_h\right\|_{C_h^0}[-\beta_2 b + c(\beta_2 - y)] \leq 0$$

Therefore, the following inequality holds in a similar thought of the estimate (3.15)

$$\left\|B_i^h\widetilde{f}_h(x)\right\|_{C_h^0} \leq \left\|\widetilde{f}_h\right\|_{C_h^0}\left(1 - \frac{\xi + _{(i-1)h}}{\beta_2}\right),\ i = 1, 2, ..., M. \tag{3.32}$$

Define

$$\widetilde{\varphi}_h = \alpha\sum_{k=1}^{M}\rho_k\widetilde{\varphi}_{k,h}(x), \quad x \in [0, \beta_1]_h\,, \tag{3.33}$$

where $\widetilde{\varphi}_{k,h}$ is function (3.29).

By (see in Bers et all, 1964), we have

$$\max_{(x,y)\in\overline{R}_h}|v_h - V_h| \leq c_1 h^2, \tag{3.34}$$

where v_h is a solution of the problem (3.28), V_h is the trace of the solution of (3.9) on $\overline{R}$ and c_1 is a constant independent of h.

Combining (3.11), (3.33) and (3.34), the following inequality holds true

$$\|\widetilde{\varphi}_h - \varphi_h\|_{C_h^0} \leq c_2 h^2, \tag{3.35}$$

where φ_h is the trace of the function φ defined by (3.11) on $[0, \beta_1]_h$ and c_2 is a constant independent of h.

By analogy of (3.18) we get,

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k B_k^h\widetilde{f}_h, \quad \text{on } \gamma_h^4, \tag{3.36}$$

where μ_h is the trace of the function μ defined by (3.5) on $[0, \beta_1]_h$.

Consider the following sequences in C_h^0 :

$$\widetilde{\psi}_{i,h}^0 = 0, \quad \widetilde{\psi}_{i,h}^n = B_i^h\left(\widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^{n-1}\right),$$

$$i = 1, 2, ..., M; \quad n = 1, 2, ... \,. \tag{3.37}$$

For the following Lemma, let us denote $(B_i\varphi)_h$as the trace of $B_i\varphi$ on $[0, \beta_1]_h$.

Lemma 3.1.3. *The following inequality is true*

$$\left\|\widetilde{\psi}^n_{i,h} - \psi^n_{i,h}\right\|_{C^0_h} \leq c_3 h^2$$

where $\widetilde{\psi}^n_{i,h}$ *defined by* (3.37)*,* $\psi^n_{i,h}$ *be the trace of* ψ^n_i *on* $[0, \beta_1]_h$ *and* c_3 *is a constant independent of* h.

Proof. The sequence $\left\{\widetilde{\psi}^n_{i,h}\right\}_{n=0}^{\infty}$ defined by (3.37) is Cauchy sequence. Therefore, it converges to the unique solution $\widetilde{\psi}_{i,h} \in C^0_h, i = 1, 2, ..., M$. By taking limit of both side of (3.37) we have,

$$\widetilde{\psi}_{i,h} = B^h_i \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right), \quad i = 1, 2, ..., M. \tag{3.38}$$

From (3.36) and (3.38), it follows that

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}. \tag{3.39}$$

By virtue of (3.19) and (3.37) we have,

$$\left\|\widetilde{\psi}^0_{i,h} - \psi^0_{i,h}\right\|_{C^0_h} = 0, \text{ for all } i = 1, 2, ..., M, \tag{3.40}$$

The estimations (3.32) and (3.35) yields that

$$\left\|B^h_i \left(\widetilde{\varphi}_h - \varphi_h\right)\right\|_{C^0_h} \leq \left(1 - \frac{\xi + (i-1)h}{\beta_2}\right) c_2 h^2, \quad i = 1, 2, ..., M. \tag{3.41}$$

Since φ and μ are in the class $C^{4,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq \beta_1$, by (see Bers et all, 1964) and similarity to the estimate (3.35), we have

$$\left\|B^h_i \left(\varphi_h + \mu_h\right) - \left(B_i \left(\varphi + \mu\right)\right)_h\right\|_{C^0_h} \leq c_4 h^2 \tag{3.42}$$

where c_4 is a constant independent of h.

Then, by (3.41) and (3.42), we obtain

$$\begin{aligned} \left\|\widetilde{\psi}^1_{i,h} - \psi^1_{i,h}\right\|_{C^0_h} &\leq \left\|B^h_i \left(\widetilde{\varphi}_h - \varphi_h\right)\right\|_{C^0_h} + \left\|B^h_i \left(\varphi_h + \mu_h\right) - \left(B_i \left(\varphi + \mu\right)\right)_h\right\|_{C^0_h} \\ &\leq c_5 h^2, \end{aligned} \tag{3.43}$$

where c_5 is a constant independent of h.

For $n \geq 2$, we have

$$\left\|\widetilde{\psi}_{i,h}^{n} - \psi_{i,h}^{n}\right\|_{C_h^0} = \left\| B_i^h\left(\widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^{n-1}\right) - \left(B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0} \tag{3.44}$$

From this,

$$\begin{aligned}\left\|\widetilde{\psi}_{i,h}^{n} - \psi_{i,h}^{n}\right\|_{C_h^0} &\leq \left\|B_i^h\left(\widetilde{\varphi}_h + \mu_h\right) - \left(B_i\left(\varphi+\mu\right)\right)_h\right\|_{C_h^0} \\ &+ \left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^{n-1} - \alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right\|_{C_h^0} \\ &+ \left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right) - \left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0} \\ i &= 1,2,...,M.\end{aligned} \tag{3.45}$$

The difficulties of the inequality (3.45) is to achieve the estimation $O(h^2)$ are occurred by third term. By analogy of the estimation (32) in (Volkov & Dosiyev, 2016), we estimate the third term on the right side of (3.45) by (Bers et all, 1964):

$$\left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right) - \left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0} \leq c_5^0 h^2, \quad i = 1,2,...,M. \tag{3.46}$$

where c_5^0 is a constant independent of h.

By combining (3.8), (3.32), (3.42) and (3.46),

$$\left\|\widetilde{\psi}_{i,h}^{n} - \psi_{i,h}^{n}\right\|_{C_h^0} \leq c_6 h^2 + q_0\left\|\widetilde{\psi}_{i,h}^{n-1} - \psi_{i,h}^{n-1}\right\|_{C_h^0}, \tag{3.47}$$

where q_0 is defined by (3.8) and c_6 is a constant independent of h.

By induction, (3.40) (3.43) and (3.47) yield

$$\left\|\widetilde{\psi}_{i,h}^{n} - \psi_{i,h}^{n}\right\|_{C_h^0} \leq c_7 h^2, \tag{3.48}$$

where c_7 is a constant independent of h. □

Lemma 3.1.4. *The next estimation remains true*

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right) \tag{3.49}$$

where ψ_i^n, φ and μ are defined by (3.19) *,(3.11) and (3.5), respectively and* $q_1 = 1 - \frac{\xi}{\beta_2}$.

Proof. In the view of (3.19) , for positive integer $m, n \geq 0$,

$$\left\|\psi_i^{n+m} - \psi_i^n\right\|_{C^0} \leq q^{n+1}\frac{1-q^m}{1-q}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right). \tag{3.50}$$

By using triangle inequality,

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \left\|\psi_i^{n+m} - \psi_i^n\right\|_{C^0} + \left\|\psi_i^{n+m} - \psi_i\right\|_{C^0}. \tag{3.51}$$

Taking limit of (3.51) as $m \to \infty$, the second term of the right side vanishes. Therefore, by (3.50) , the proof is completed. □

According to the estimate (3.48) and Lemma 3.1.4, we find that

$$\max_{1\leq i\leq m}\left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}\right\|_{C_h^0} \leq c_7h^2 + \frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{3.52}$$

where $\psi_{i,h}$ is the trace of the function ψ_i on $[0,\beta_1]_h$.

Define

$$\widetilde{f}_h^n = \widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}_{k,h}^n. \tag{3.53}$$

where $\widetilde{f}_h^n$ is an approximation of f defined by (3.25).

By (3.35), (3.52) and (3.53)we reach

$$\left\|\widetilde{f}_h^n - f_h\right\|_{C^0} \leq c_8h^2 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{3.54}$$

where $\widetilde{f}_h^n$ is computed function (3.53), f_h is the trace of f defined by (3.25), q_0 is the number given by (3.8), c_8 is a constant independent of h.

Consider the actual finite difference problem

$$L\widetilde{u}_h^n = 0 \text{ on } R_h, \quad \widetilde{u}_h^n = \tau_h \text{ on } \gamma_h^2, \quad \widetilde{u}_h^n = 0 \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{3.55}$$

$$\widetilde{u}_h^n = \widetilde{f}_h^n \text{ on } \gamma_h^4, \tag{3.56}$$

where $\widetilde{f}_h^n$ is computed function which approximates to f and τ_h is the trace of τ on $[0,\beta_1]_h$.

Theorem 3.1.5. *The next estimation holds*

$$\max_{(x,y)\in\overline{R}_h}\left|\widetilde{u}_h^n - u\right| \leq c_9h^2 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right),$$

where u is the solution of problem (3.4),(3.5) and c_9 is a constant independent of h.

Proof. Let $U_h(x,t)$ be the solution of the system

$$LU_h = 0 \text{ on } R_h, \quad U_h = \tau \text{ on } \gamma_h^2, \quad U_h = 0 \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{3.57}$$

$$U_h = f \text{ on } \gamma_h^4, \tag{3.58}$$

which approximates the problem (3.6),(3.7) with f defined by (3.25).

Since τ, μ, φ and $\psi_i, i = 1, 2, ..., m$, are from $C^{4,\lambda}, 0 < \lambda < 1$, on $[0, \beta_1]$, By (see in Bers et al., 1964) we have

$$\max_{(x,y)\in\overline{R}_h} |U_h - U| \leq c_{10}h^2, \tag{3.59}$$

where U is the solution of the problem (3.6),(3.7) and c_{10} is a constant independent of h.

By virtue of (3.54) and the grid maximum principle, it follows that

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U_h\right| \leq c_8h^2 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right). \tag{3.60}$$

Then, in the view of (3.59) and (3.60), we obtain

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U\right| \leq c_{11}h^2 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{3.61}$$

where $\widetilde{u}_h^n$ is a solution of problem (3.55), (3.56), U is the solution of the problem (3.6),(3.7) and c_{11} is a constant independent of h.

The estimate (3.61) and error estimate of trapezoidal rule yield that

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - u\right| \leq c_9h^2 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right). \tag{3.62}$$

□

The right-hand side of (3.62) is order of $O\left(h^2\right)$ for

$$n = \max\left\{\left[\frac{\ln h^{-2}(1-q_1)^{-1}}{\ln q_1^{-1}}\right], 1\right\}. \tag{3.63}$$

where $[a]$ is the integer part of a.

3.2 FOURTH ORDER ACCURACY FOR THE SECOND ORDER ELLIPTIC EQUATION WITH INTEGRAL BOUNDARY CONDITION

3.2.1 Overview

In this section, the second-order elliptic equation with nonlocal integral condition and its finite difference scheme which is related to Dennis-Hudson finite difference scheme (Dennis & Hudson, 1979;1980) are justified and proposed.

The researcher studying fourth order difference scheme of second order elliptic equation for local problem focus on finite difference scheme and its numerical results without giving the convergence of the finite difference scheme. In this section, after analyzing Dennis-Hudson finite difference scheme, it s proved the convergence of the fourth order difference scheme of the general second order elliptic equation for local problem under some restriction on step size h, when the solution of the elliptic equation have a sixth derivative satisfying a Hölder condition.

Additionally, the constructive method for the approximation of the second order elliptic equation with integral condition is justified by using Dennis-Hudson finite difference scheme. It is verified that the uniform estimate of the error of the approximate solution is order of $O(h^4)$, h is the step mesh, when the boundary functions have a sixth derivative satisfying a Hölder condition.

3.2.2 Nonlocal boundary value problem

Let

$$R = \{(x, y) : 0 < x < \beta_1, 0 < y < \beta_2\}$$

be an open rectangle, γ^m, $m = 1, 2, 3, 4$, be its sides including the endpoints, numbered in the clockwise direction, beginning with the side lying on the y-axis and let $\gamma = \cup_{m=1}^{4} \gamma^m$ be the boundary of R and $\overline{R} = R \cup \gamma$. Let C^0 denote the linear space of continuos functions of one variable x on the interval $[0, \beta_1]$ of x-axis, and vanish at the points $x = 0$ and $x = \beta_1$. For the function $f \in C^0$ we define the norm

$$\|f\|_{C^0} = \max_{0 \le x \le \beta_1} |f(x)| .$$

Consider the following operator

$$Lu = \Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u, \tag{3.64}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, a, b and c are functions with $b(x,y) \geq 0$ and $c(x,y) \leq 0$.

Consider the nonlocal boundary value problem on R

$$Lu = 0 \text{ on } R,\ u = 0 \text{ on } \gamma^1 \cup \gamma^3,\ u = \tau \text{ on } \gamma^2 \tag{3.65}$$

$$u(x,0) = \alpha \int_{\xi}^{\beta_2} u(x,y)dy + \mu(x), \quad 0 < x < \beta_1, 0 < \xi < \beta_2, \tag{3.66}$$

where $\tau = \tau(x)$ and $\mu = \mu(x)$ are given functions from C^0 and α is a given constant with satisfy the inequality given below $|\alpha| < \frac{1}{\beta_2 - \xi}$.

We consider the following multilevel nonlocal boundary value problem to solve the problem (3.65) and (3.66) by using Simpson's rule for the boundary condition (3.66) ,

$$LU = 0 \text{ on } R,\ U = \tau \text{ on } \gamma^2,\ U = 0 \text{ on } \gamma^1 \cup \gamma^3 \tag{3.67}$$

$$U(x,0) = \alpha \sum_{k=1}^{M} \rho_k U(x,\eta_k) + \mu(x), \quad 0 \leq x \leq \beta_1, \tag{3.68}$$

where $\rho_1 = \rho_M = \frac{h}{3}, \rho_j = \frac{h}{3}\left(3 + (-1)^j\right)$ for $j = 2,3,...,M-1$, $\eta_j = \xi + (j-1)h$, $j = 1,2,...,M$, $h = \frac{\beta_1}{N}$, $(M-1)h + \xi = \beta_2$ and $\frac{\xi}{h}$ is an integer.

This yields

$$q_0 = |\alpha| \sum_{k=1}^{M} \rho_k < 1. \tag{3.69}$$

The solution U of problem (3.67), (3.68) is defined as a sum of two functions (see in Section 3.1)

$$U(x,y) = V(x,y) + W(x,y)$$

where V is the solution of the problem

$$LV = 0 \text{ on } R,\ V = \tau \text{ on } \gamma^2,\ V = 0 \text{ on } \gamma/\gamma^2, \tag{3.70}$$

and W is the solution of the problem

$$LW = 0 \text{ on } R,\ W = 0 \text{ on } \gamma/\gamma^4,\ W = f \ \text{ on } \gamma^4. \tag{3.71}$$

with f be an unknown function from C^0.

We define the operator B_i from C^0 to C^0 as follows

$$B_i f(x) = W(x, \eta_i),\ i = 1, 2, ..., M. \tag{3.72}$$

It is verified

$$\|B_i f\|_{C^0} < \left(1 - \frac{\xi + (i-1)h}{\beta_2}\right) \|f\|_{C^0}, \quad i = 1, 2, ..., M \tag{3.73}$$

and

$$0 < |B_M| < |B_{M-1}| < ... < |B_1| < 1. \tag{3.74}$$

Therefore, we put

$$q = |B_1|\, q_0 < 1. \tag{3.75}$$

We set

$$\varphi_k(x) = V(x, \eta_k) \ \text{ for } k = 1, 2, ..., M \tag{3.76}$$

and

$$\varphi = \alpha \sum_{k=1}^{M} \rho_k \varphi_k. \tag{3.77}$$

Consider the sequences in C^0

$$\psi_i^0 = 0,\ \psi_i^n = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1}\right)$$
$$i = 1, 2, ..., M;\ \ n = 1, 2, ... \tag{3.78}$$

where $\psi_i^n \in C^{6,\lambda}, 0 < \lambda < 1, i = 1, 2, ..., M,$ on $0 \le x \le \beta_1$.

The limit of the sequence is found as a solution of the following nonlinear equations

$$\psi_i = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k\right),\ i = 1, 2, ..., M. \tag{3.79}$$

Therefore, the function f in (3.72) is denoted by

$$f = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k \tag{3.80}$$

The existence and uniqueness of the functions f is proved in Section 3.1.

3.2.3 Convergence of Dennis-Hudson finite difference-scheme

We define a square mesh with the mesh size $h = \frac{\beta_1}{N} = \frac{\beta_2}{M^*}$, $N, M^* > 2$ are integers, constructed with the lines $x, y = h, 2h, ...$Let D_h be the set of nodes of this square grid, $R_h = R \cap D_h$, and $\overline{R}_h = \overline{R} \cap D_h$, and $\gamma_h^m = \gamma^m \cap D_h, m = 1, 2, 3, 4$.

Let

$$[0, \beta_1]_h = \left\{x = x_i,\ x_i = ih,\ i = 0, 1, ..., N,\ h = \frac{\beta_1}{N}\right\}$$

be the set of points divided by the step size h on $[0, \beta_1]$.

The values of $u(x, y)$ at (x_0, y_0), $(x_0 + h, y_0)$, $(x_0, y_0 + h)$, $(x_0 - h, y_0)$ and $(x_0, y_0 - h)$ are denoted by u_0, u_1, u_2, u_3 and u_4, respectively. For other type functions, the identical notations are utilized.

Special type approximations can be obtained by describing (3.64) as the two equations

$$\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu = r(x, y) \tag{3.81}$$

and

$$\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y} = -r(x, y). \tag{3.82}$$

The finite-difference approximations to (3.81) and (3.82) are

$$\begin{aligned}&\left(1 + \frac{1}{2}ha(x, y)\right)u(x + h, y) + \left(1 - \frac{1}{2}ha(x, y)\right)u(x - h, y)\\ &+ \left(2 - h^2 c(x, y)\right)u(x, y) = h^2 r(x, y),\end{aligned} \tag{3.83}$$

$$\left(1+\frac{1}{2}hb(x,y)\right)u(x,y+h)+\left(1-\frac{1}{2}hb(x,y)\right)u(x,y-h)$$
$$-2u(x,y)=h^2r(x,y). \tag{3.84}$$

The method (Dennis, 1960) is to reduced (3.83) and (3.84) along $y = y_0$ and $x = x_0$ by replacing

$$u=\phi e^{-h(x,y_0)} \text{ and } u=\varrho e^{-g(x_0,y)}, \tag{3.85}$$

where

$$h(x,y_0)=\frac{1}{2}\int_{x_0}^{x}a(z,y_0)dz \text{ and } g(x_0,y)=\frac{1}{2}\int_{y_0}^{y}b(x_0,z)dz. \tag{3.86}$$

Therefore, the equations (3.81) and (3.82) are written as

$$\frac{\partial^2\phi}{\partial x^2}-\left(\frac{1}{2}\frac{\partial a}{\partial x}+\frac{1}{4}a^2-c\right)\phi-re^{h}=0 \tag{3.87}$$

and

$$\frac{\partial^2\varrho}{\partial y^2}-\left(\frac{1}{2}\frac{\partial b}{\partial y}+\frac{1}{4}b^2\varrho\right)+re^{g}=0, \tag{3.88}$$

respectively.

By approximating the derivatives and putting $x = x_0, y = y_0$ in (3.87) and (3.88) , the following finite-difference equation at (x_0, y_0) is obtained (see in Dennis & Hudson, 1979)

$$u_1e^{h_1}+u_2e^{g_2}+u_3e^{h_3}+u_4e^{g_4}-\left(4+h^2\overline{\phi}_0\right)u_0+C\vartheta_0+C^{'}\delta_0=0, \tag{3.89}$$

where

$$C\vartheta_0=-\frac{1}{12}h^2\left[(F_1u_1+r_1)e^{h_1}-2(F_0u_0+r_0)\right.$$
$$\left.+(F_3u_3+r_3)e^{h_3}\right]+O(h^6), \tag{3.90}$$

$$C^{'}\delta_0=-\frac{1}{12}h^2\left[(G_2u_2-r_2)e^{h_{G2}}-2(G_0u_0-r_0)\right.$$
$$+(G_4u_4-r_4)e^{g_4}]+O(h^6) \tag{3.91}$$

and

$$F = \frac{1}{2}\frac{\partial a}{\partial x} + \frac{1}{4}a^2 - c \text{ and } G = \frac{1}{2}\frac{\partial b}{\partial y} + \frac{1}{4}b^2, \tag{3.92}$$

$$\overline{\phi}(x, y) = \frac{1}{2}\left[\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right] + \frac{1}{4}\left(a^2 + b^2\right) - c. \tag{3.93}$$

The last two terms of (3.89) represents finite difference correction and contains terms of $O\left(h^4\right)$ and higher (see in Dennis & Hudson, 1979). Using three point numerical method for integrations in (3.86) is recommended. By using Taylor expansion about the point (x_0, y_0) for the function $a(x, y_0)$, we have

$$h_1 = \frac{1}{2}ha_0 + \frac{1}{4}h^2\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{12}h^3\left(\frac{\partial^2 a}{\partial x^2}\right)_0 + \frac{1}{48}h^4\left(\frac{\partial^3 a}{\partial x^3}\right)_0 + \frac{1}{240}h^5\left(\frac{\partial^4 a}{\partial x^4}\right)_0 + ... \tag{3.94}$$

Then by putting (3.94) into the series expansion of e^{h_1}, we get

$$\begin{aligned} e^{h_1} = {} & 1 + \frac{1}{2}ha_0 + \frac{1}{4}h^2\left[\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{2}a_0^2\right] + \frac{1}{4}h^3\left[\frac{1}{12}a_0^3 + \frac{1}{2}a_0\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{3}\left(\frac{\partial^2 a}{\partial x^2}\right)_0\right] \\ & + \frac{1}{8}h^4\left[\frac{1}{48}a_0^4 + \frac{1}{4}a_0^2\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{4}\left(\frac{\partial a}{\partial x}\right)_0^2 + \frac{1}{3}a_0\left(\frac{\partial^2 a}{\partial x^2}\right)_0 + \frac{1}{6}\left(\frac{\partial^3 a}{\partial x^3}\right)_0\right] \\ & \frac{1}{16}h^5\left[\frac{1}{240}a_0^5 + \frac{1}{12}a_0^3\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{6}a_0^2\left(\frac{\partial^2 a}{\partial x^2}\right)_0 + \frac{1}{4}a_0\left(\frac{\partial a}{\partial x}\right)^2 \right. \\ & \left. + \frac{1}{6}a_0\left(\frac{\partial^3 a}{\partial x^3}\right)_0 + \frac{1}{3}\left(\frac{\partial a}{\partial x}\right)_0\left(\frac{\partial^2 a}{\partial x^2}\right)_0 + \frac{1}{15}\left(\frac{\partial^4 a}{\partial x^4}\right)_0\right] + O(h^6). \end{aligned} \tag{3.95}$$

Expanding e^{h_3} in Taylor series in a similar way with (3.95) yields that

$$e^{h_1} + e^{h_3} = 2 + h^2\left[\frac{1}{2}\left(\frac{\partial a}{\partial x}\right)_0 + \frac{1}{4}a_0^2\right] + h^4\overline{\lambda}_0 + O(h^6), \tag{3.96}$$

where

$$\overline{\lambda}(x, y) = \frac{1}{4}\left[\frac{1}{48}a^4 + \frac{1}{4}a^2\left(\frac{\partial a}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial a}{\partial x}\right)^2 + \frac{1}{3}a\left(\frac{\partial^2 a}{\partial x^2}\right) + \frac{1}{6}\left(\frac{\partial^3 a}{\partial x^3}\right)\right]. \tag{3.97}$$

Similarly, by using Taylor expansion about the point (x_0, y_0) for the function $b(x_0, y)$, we have

$$g_2 = \frac{1}{2}hb_0 + \frac{1}{4}h^2\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{12}h^3\left(\frac{\partial^2 b}{\partial y^2}\right)_0 + \frac{1}{48}h^4\left(\frac{\partial^3 b}{\partial y^3}\right)_0 + \frac{1}{240}h^5\left(\frac{\partial^4 b}{\partial y^4}\right)_0 + ... \tag{3.98}$$

Then,

$$e^{g_2} = 1 + \frac{1}{2}hb_0 + \frac{1}{4}h^2\left[\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{2}b_0^2\right] + \frac{1}{4}h^3\left[\frac{1}{12}b_0^3 + \frac{1}{2}b_0\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{3}\left(\frac{\partial^2 b}{\partial y^2}\right)_0\right]$$
$$+\frac{1}{8}h^4\left[\frac{1}{48}b_0^4 + \frac{1}{4}b_0^2\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{4}\left(\frac{\partial b}{\partial y}\right)_0^2 + \frac{1}{3}b_0\left(\frac{\partial^2 b}{\partial y^2}\right)_0 + \frac{1}{6}\left(\frac{\partial^3 b}{\partial y^3}\right)_0\right]$$
$$\frac{1}{16}h^5\left[\frac{1}{240}b_0^5 + \frac{1}{12}b_0^3\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{6}b_0^2\left(\frac{\partial^2 b}{\partial y^2}\right)_0 + \frac{1}{4}b_0\left(\frac{\partial b}{\partial y}\right)^2 \right. \tag{3.99}$$
$$\left. +\frac{1}{6}b_0\left(\frac{\partial^3 b}{\partial y^3}\right)_0 + \frac{1}{3}\left(\frac{\partial b}{\partial y}\right)_0\left(\frac{\partial^2 b}{\partial y^2}\right)_0 + \frac{1}{15}\left(\frac{\partial^4 b}{\partial y^4}\right)_0\right] + O(h^6). \tag{3.100}$$

If we use Taylor expansion for e^{g_4}, we obtain

$$e^{g_2} + e^{g_4} = 2 + h^2\left[\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_0 + \frac{1}{4}b_0^2\right] + h^4\overline{\mu}_0 + O(h^6), \tag{3.101}$$

where

$$\overline{\mu}(x,y) = \frac{1}{4}\left[\frac{1}{48}b^4 + \frac{1}{4}b^2\left(\frac{\partial b}{\partial y}\right) + \frac{1}{4}\left(\frac{\partial b}{\partial y}\right)^2 + \frac{1}{3}b\left(\frac{\partial^2 b}{\partial y^2}\right) + \frac{1}{6}\left(\frac{\partial^3 b}{\partial y^3}\right)\right]. \tag{3.102}$$

We add (3.96) and (3.101) .Therefore,

$$e^{h_1} + e^{g_2} + e^{h_3} + e^{g_4} = 4 + h^2\left(\overline{\phi}_0 + c_0\right) + h^4(\overline{\lambda} + \overline{\mu}) + O(h^6). \tag{3.103}$$

where $\overline{\phi}$, $\overline{\lambda}$ and $\overline{\mu}$ defined by (3.93), (3.97) and (3.102) .

From (3.89) and (3.103) , by eliminating $\overline{\phi}_0$ and ignoring all term of $O\left(h^4\right)$ and higher, the second orde finite difference is obtained as follows:

$$u_1e^{h_1} + u_2e^{g_2} + u_3e^{h_3} + u_4e^{g_4} - \left(e^{h_1} + e^{g_2} + e^{h_3} + e^{g_4} - h^2c_0\right)u_0 = 0. \tag{3.104}$$

We evaluate r_0, r_1, r_2, r_3, r_4 by (3.83) and (3.84) for (3.90) , (3.91). Then, for obtaining a similar finite difference correction, the equation (3.104) is expressed in the form

$$u_1e^{h_1} + u_2e^{g_2} + u_3e^{h_3} + u_4e^{g_4} - \left(e^{h_1} + e^{g_2} + e^{h_3} + e^{g_4} - h^2c_0\right)u_0 + D\vartheta_0 + D^{'}\delta_0 = 0, \tag{3.105}$$

where

$$D\vartheta_0 = C\vartheta_0 + h^4\overline{\lambda}_0u_0 + O(h^6) \tag{3.106}$$

and

$$D^{'}\delta_0 = C^{'}\delta_0 + h^4\overline{\mu}_0 u_0 + O(h^6). \tag{3.107}$$

We put

$$E(x,y) = F(x,y) + c(x,y). \tag{3.108}$$

Expanding $E(x,y)$ and using (3.95), it follows that

$$E_1 e^{h_1} + E_3 e^{h_3} - 2E_0 = h^2\left[\frac{1}{16}a_0^4 + \frac{3}{4}a_0^2\left(\frac{\partial a}{\partial x}\right)_0 + \frac{3}{4}\left(\frac{\partial a}{\partial x}\right)_0^2 + a_0\left(\frac{\partial^2 a}{\partial x^2}\right) + \frac{1}{2}\left(\frac{\partial^3 a}{\partial x^3}\right)_0\right] + O(h^4). \tag{3.109}$$

In a similar way, we obtain

$$G_2 e^{g_2} + G_4 e^{g_4} - 2G_0 = h^2\left[\frac{1}{16}b_0^4 + \frac{3}{4}a_0^2\left(\frac{\partial b}{\partial y}\right)_0 + \frac{3}{4}\left(\frac{\partial b}{\partial y}\right)_0^2 + a_0\left(\frac{\partial^2 b}{\partial y^2}\right) + \frac{1}{2}\left(\frac{\partial^3 b}{\partial y^3}\right)_0\right] + O(h^4). \tag{3.110}$$

In the view of (3.97) , (3.102) , (3.109) and (3.110) , we have,

$$h^4\overline{\lambda}_0 = \frac{h^2}{12}\left[E_1 e^{h_1} + E_3 e^{h_3} - 2E_0\right] + O(h^6) \tag{3.111}$$

and

$$h^4\overline{\mu}_0 = \frac{h^2}{12}\left[G_2 e^{g_2} + G_4 e^{g_4} - 2G_0\right] + O(h^6). \tag{3.112}$$

By ignoring the term of $O(h^6)$ and higher, we get the corrections (3.106) and (3.107) as follows:

$$D\vartheta_0 = -\frac{1}{12}h^2\left[(F_1 u_1 - E_1 u_0 + r_1)\, e^{h_1} - 2\,(F_0 u_0 - E_0 u_0 + r_0) + (F_3 u_3 - E_3 u_0 + r_3)\, e^{h_3}\right] \tag{3.113}$$

and

$$D^{'}\delta_0 = -\frac{1}{12}h^2\left[(G_2 u_2 - G_2 u_0 - r_2)\, e^{h_{G2}} + 2r_0 + (G_4 u_4 - G_4 u_0 - r_4)\, e^{g_4}\right]. \tag{3.114}$$

By combining (3.83), (3.84), (3.92), (3.105) , (3.108) (3.113) and (3.114) , the final finite-difference equation is written in the form

$$\begin{aligned}
L_h u \equiv\ & e^{f_1}\left[\frac{5}{6}-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial a}{\partial x}\right)_1+\frac{1}{4}a_1^2-c_1\right)\right]u(x+h,y)\\
&+e^{g_2}\left[\frac{5}{6}-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_2+\frac{1}{4}b_2^2-c_2\right)\right]u(x,y+h)\\
&+e^{f_3}\left[\frac{5}{6}-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial a}{\partial x}\right)_3+\frac{1}{4}a_3^2-c_3\right)\right]u(x-h,y)\\
&+e^{g_4}\left[\frac{5}{6}-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_4+\frac{1}{4}b_4^2-c_4\right)\right]u(x,y-h)\\
&+\frac{1}{12}\left[e^{f_1}\left(1-\frac{hb_1}{2}\right)+e^{g_4}\left(1+\frac{ha_4}{2}\right)\right]u(x+h,y-h)\\
&+\frac{1}{12}\left[e^{f_3}\left(1+\frac{hb_3}{2}\right)+e^{g_2}\left(1-\frac{ha_2}{2}\right)\right]u(x-h,y+h)\\
&+\frac{1}{12}\left[e^{f_3}\left(1-\frac{hb_3}{2}\right)+e^{g_4}\left(1-\frac{ha_4}{2}\right)\right]u(x-h,y-h)\\
&+\frac{1}{12}\left[e^{f_1}\left(1+\frac{hb_1}{2}\right)+e^{g_2}\left(1+\frac{ha_2}{2}\right)\right]u(x+h,y+h)\\
&-\left[e^{f_1}\left(1-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial a}{\partial x}\right)_1+\frac{1}{4}a_1^2\right)\right)\right.\\
&+e^{g_2}\left(1-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_2+\frac{1}{4}b_2^2\right)\right)\\
&+e^{f_3}\left(1-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial a}{\partial x}\right)_3+\frac{1}{4}a_3^2\right)\right)\\
&\left.+e^{g_4}\left(1-\frac{1}{12}h^2\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_4+\frac{1}{4}b_4^2\right)\right)-\frac{5}{6}h^2c_0\right]u(x,y)\\
=\ & 0. \qquad (3.115)
\end{aligned}$$

The restriction for the step size h is given by

$$hK \leq 2,$$

where $K=\max\left\{\max\left(|a|+|b|\right),\max\left(\sqrt{\frac{1}{2}\frac{\partial a}{\partial x}+a^2-c}+\sqrt{\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)+b^2-c}\right),\max\left(\frac{\frac{1}{2}\frac{\partial b}{\partial y}+b^2-c}{b}\right)\right\}$.
Everywhere, for all estimations, the all constants that we define are independent of h as $C_1, C_2, C_3, \ldots$.

The following all Lemmas and Theorems in present Section are proved whenever at least one of following conditions satisfy:

C1) If all a, b, c are arbitrary constants.

C2) If a and b are single variable functions depends on y and x, respectively.

Lemma 3.2.1. *If υ is any function defined on $\overline{R}_h$, then*

$$\max_{R_h} |\upsilon| \leq \max_{\gamma_h} |\upsilon| + \left(\beta_1^2 + \beta_2^2\right) \max_{R_h} \left|\overline{L}_h \upsilon\right| \tag{3.116}$$

where $\overline{L}_h \upsilon = h^{-2} L_h \upsilon$, $r^2 = \beta_1^2 + \beta_2^2$ and A is a constant with $A > 0$.

Proof. We define the function ϕ_{ij} as$\phi_{ij} = A\left(r^2 - y_i^2\right)$.

Then,

$$\begin{aligned}
L_h \phi_{ij} = & \frac{A}{12} h^2 (r^2 - y^2) \left(c_1 e^{h_1} + c_3 e^{h_3} + c_2 e^{g_2} + c_4 e^{g_4}\right) + A \frac{5}{6} h c_0 (r^2 - y^2) \\
& - A h^2 \left(\frac{5}{6} - \frac{1}{12} h^2 \left(\frac{1}{2} \left(\frac{\partial b}{\partial y}\right)_2 + \frac{1}{4} b_2^2 - c_2\right)\right) e^{g_2} \\
& - A h^2 \left(\frac{5}{6} - \frac{1}{12} h^2 \left(\frac{1}{2} \left(\frac{\partial b}{\partial y}\right)_4 + \frac{1}{4} b_4^2 - c_4\right)\right) e^{g_4} \\
& - A \frac{h^2}{6} \left(e^{h_1} + e^{h_3} + e^{g_2} + e^{g_4}\right) - A \frac{h}{3} y \left(e^{h_1} + e^{h_3}\right) \\
& + 2 y h A \left(\frac{5}{6} - \frac{1}{12} h^2 \left(\frac{1}{2} \left(\frac{\partial b}{\partial y}\right)_4 + \frac{1}{4} b_4^2 - c_4\right)\right) e^{g_4} \\
& - 2 y h A \left(\frac{5}{6} - \frac{1}{12} h^2 \left(\frac{1}{2} \left(\frac{\partial b}{\partial y}\right)_2 + \frac{1}{4} b_2^2 - c_2\right)\right) e^{g_2} \\
& - A \frac{h^2}{6} \left(b_1 e^{h_1} + b_3 e^{h_3}\right).
\end{aligned}$$

It follows that,

$$L_h \phi_{ij} \leq A h^2 \left[\overline{c}\left(r^2 - y^2\right) - \frac{1}{3}\right],$$

where $\overline{c} = \max\{c_0, c_1, c_2, c_3, c_4\}$.

Since $c(x, y) \leq 0$ and by picking $A = 3$, we have

$$L_h \phi_{ij} \leq -h^2.$$

From this,

$$\overline{L}_h \phi_{ij} \leq -1. \tag{3.117}$$

Now, we define the functions ω^+ and ω^- by

$$\omega^+ = \upsilon + \overline{N}\phi \text{ and } \omega^- = -\upsilon + \overline{N}\phi,$$

where $\overline{N} = \max\left|\overline{L}_h v\right|.$

Therefore,

$$\overline{L}_h \omega^{\pm} = \overline{L}_h\left[\pm\upsilon + \overline{N}\phi\right].$$

In the view of (3.117) ,

$$\overline{L}_h \omega^{\pm} \leq \pm\overline{L}_h[\upsilon] - N \leq 0.$$

By maximum principle, we have

$$\begin{aligned}\max_{R_h}(\omega^{\pm}) &\leq \max_{\gamma_h}\left(\omega^{\pm}\right) \\ &\leq \max_{\gamma_h}(\pm\upsilon) + 3\left(\beta_1^2 + \beta_2^2\right)N. \end{aligned} \tag{3.118}$$

Since $\omega_{ij}^{\pm} = \pm\upsilon_{ij} + \overline{N}\phi_{ij}$ and $\overline{N}\phi_{ij} \geq 0$, we get

$$\pm\upsilon_{ij} \leq \omega_{ij}^{\pm} \text{ for all } \left(x_i, y_j\right) \in R_h.$$

Hence, the inequality (3.118) yields that

$$\max_{R_h}(\pm\upsilon) \leq \max_{\gamma_h}(\pm\upsilon) + 3\left(\beta_1^2 + \beta_2^2\right)N,$$

which completes the proof □

We take

$$\upsilon = v_h - V_h$$

where v_h are solution of the problem (3.122), V_h is the trace of the solution of (3.70) on $\overline{R}_h$. Then from Lemma 3.2.1 we have,

$$\max_{R_h}|v_h - V_h| \leq \max_{\gamma_h}|v_h - V_h| + A\left(\beta_1^2 + \beta_2^2\right)\max_{R_h}\left|\overline{L}_h(v_h - V_h)\right|. \tag{3.119}$$

By (Dennis & Hudson, 1980) we have,

$$\max_{R_h} |L_h (v_h - V_h)| \le C_1 h^6.$$

It yields that

$$\max_{R_h} \left|\overline{L}_h (v_h - V_h)\right| \le C_1 h^4. \qquad (3.120)$$

By using (3.119) and (3.120), it follows that

$$\max_{(x,y)\in\overline{R}_h} |v_h - V_h| \le C_1 h^4. \qquad (3.121)$$

3.2.4 Approximate solution of the nonlocal boundary value problem by Dennis-Hudson's finite-difference scheme

We say that $F \in C^{k,\lambda}(E)$, if F has k-th derivatives on E satisfying the Hölder condition with exponent λ.

We assume that $\tau(x) \in C^{6,\lambda}\left(\gamma^2\right)$, $\mu(x) \in C^{6,\lambda}\left(\gamma^4\right)$ in (3.65) and (3.66), respectively.

Let C_h^0 be the linear space of grid functions defined on $[0, \beta_1]_h$ that vanish at $x = 0$ and $x = \beta_1$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x\in\left[0,\beta_1\right]_h} |f_h|$$

Let v_h be a solution of following the system of grid equations

$$L_h v_h = 0 \text{ on } R_h, \quad v_h = \tau_h \text{ on } \gamma_h^2, \quad v_h = 0 \text{ on } \gamma_h/\gamma_h^2, \qquad (3.122)$$

where τ_h is the trace of τ on γ_h^2 and we define

$$\widetilde{\varphi}_{i,h}(x) = v_h(x, \eta_i), \; i = 1, 2, ..., M. \qquad (3.123)$$

Let w_h be a solution of the finite difference problem

$$L_h w_h = 0 \text{ on } R_h, \quad w_h = 0 \text{ on } \gamma_h/\gamma_h^4, \quad w_h = \widetilde{f_h} \text{ on } \gamma_h^4, \qquad (3.124)$$

where $f_h \in C_h^0$, is an arbitrary function.

Let B_i^h be a linear operator from C_h^0 to C_h^0 as follows:

$$B_i^h f_h(x) = w_h\left(x, \eta_i\right), \; i = 1, 2, ..., M, \qquad (3.125)$$

where w_h is the solution of the problem (3.124).

Lemma 3.2.2. *The following estimate holds*

$$\left\|B_i^h f_h(x)\right\|_{C_h^0} \leq \|f_h\|_{C_h^0}\left(1-\frac{\xi+_{(i-1)h}}{\beta_2}\right),\ i=1,2,...,M. \tag{3.126}$$

Proof. We put

$$\overline{w}_h(x_i,y_i)=\frac{1}{\beta_2}\|f_h\|_{C_h^0}(\beta_2-y_i)\,,\ i=1,2,...,M,\ j=1,2,...,N.$$

It yields the following inequality on γ_h:

$$|w_h(x_i,y_i)| \leq \overline{w}_h(x_i,y_i)$$

By substitute $\overline{w}_h(x_i,y_i)$ in $L_h\overline{w}_h$ we have,

$$\begin{aligned} L_h\overline{w}_h = \frac{\|f_h\|_{C_h^0}}{2}\Bigg[&\frac{e^{f_1}}{12}\left(-h^2b_1+h^2c_1(2-y)\right)+\frac{e^{f_3}}{12}\left(-h^2b_3+h^2c_3(2-y)\right)\\ &+e^{g_2}\left(-h+\frac{h^3}{12}\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_2+\frac{1}{4}b_2^2-c_2\right)+\frac{1}{12}h^2c_2(2-y)\right)\\ &+e^{g_4}\left(h-\frac{h^3}{12}\left(\frac{1}{2}\left(\frac{\partial b}{\partial y}\right)_4+\frac{1}{4}b_4^2-c_4\right)+\frac{1}{12}h^2c_4(2-y)\right)\Bigg]\end{aligned}$$

Under the restriction $hK \leq 2$, it follows that

$$L_h\overline{w}_h \leq 0$$

Then we get,

$$w_h(x_i,y_i) \leq \overline{w}_h(x_i,y_i) \text{ on } \overline{R}_h.$$

Therefore we have the following inequalities

$$w_h-\overline{w}_h \leq 0 \text{ on } \gamma_h \text{ and } L(w_h-\overline{w}_h) \geq 0 \text{ on } R_h.$$

In the view of the maximum principle (see in Bers et all, 1964),

$$w_h \leq \overline{w}_h \text{ on } \overline{R}.$$

By replacing $-w_h$ with w_h, we obtain

$$-w_h \leq \overline{w}_h \text{ on } \overline{R}.$$

Consequently, we can reach the inequality

$$|w_h| \leq \overline{w}_h \text{ on } \overline{R}.$$

Hence, the proof is completed. □

Define

$$\widetilde{\varphi}_h = \alpha \sum_{k=1}^{M} \rho_k \widetilde{\varphi}_{k,h}(x), \quad x \in [0, \beta_1]_h\,, \tag{3.127}$$

where $\widetilde{\varphi}_{k,h}(x)$ is the function (3.123).

We define the function $\widetilde{f_h}$ with similar analogy of (3.39)

$$\widetilde{f_h} = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}. \tag{3.128}$$

where μ_h is the trace of μ defined in (3.66) on $[0, \beta_1]_h$ and $\widetilde{\psi}_{k,h} \in C_h^0, k = 1, 2, ..., M$, are the solution of the system of the equations

$$\widetilde{\psi}_{i,h} = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right),\ i = 1, 2, ..., M. \tag{3.129}$$

The solution of the system (3.129) are sought by using the fixed point iteration below:

$$\widetilde{\psi}_{i,h}^0 = 0, \quad \widetilde{\psi}_{i,h}^n = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^{n-1} \right),$$
$$i = 1, 2, ..., M; \quad n = 1, 2, ... \,. \tag{3.130}$$

By using the $n - th$ iteration $\widetilde{\psi}_{i,h}^n$, $n \geq 1$ of (3.130), we define the function

$$\widetilde{f}_h^n = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^n. \tag{3.131}$$

Let φ_h, $\psi_{i,h}$ and $\psi_{i,h}^n$ be the trace of φ, ψ_i and ψ_i^n on $[0, \beta_1]_h$, respectively and . In the view of (3.76) , (3.77), (3.123) , (3.121) and (3.127), we have

$$\|\widetilde{\varphi}_h - \varphi_h\|_{C_h^0} \leq C_2 h^4, \tag{3.132}$$

where φ_h be the trace of φ defined by (3.77) on $[0, \beta_1]_h$.

From (3.78) and (3.130), it follows that, for all $i = 1, 2, ..., M$,

$$\left\|\widetilde{\psi}^1_{i,h} - \psi^1_{i,h}\right\|_{C^0_h} \le \left\|B^h_i\left(\widetilde{\varphi}_h - \varphi_h\right)\right\|_{C^0_h} + \left\|B^h_i\left(\varphi_h + \mu_h\right) - \left(B_i\left(\varphi + \mu\right)\right)_h\right\|_{C^0_h}. \tag{3.133}$$

By using (3.126) and (3.132), we have

$$\left\|B^h_i\left(\widetilde{\varphi}_h - \varphi_h\right)\right\|_{C^0_h} \le C_3 h^4, \quad i = 1, 2, ..., M, \tag{3.134}$$

Let $(B_i(F))_h$ be the trace of $B_i(F)$ on $[0, \beta_1]_h$ for any function $F \in C^{6,\lambda}[0, \beta_1]$. Then,

$$\left\|B^h_i\left(\varphi_h + \mu_h\right) - \left(B_i\left(\varphi + \mu\right)\right)_h\right\|_{C^0_h} \le C_4 h^4, \tag{3.135}$$

By combining (3.133), (3.132) and (3.134), we reach

$$\left\|\widetilde{\psi}^1_{i,h} - \psi^1_{i,h}\right\|_{C^0_h} \le C_5 h^4, \tag{3.136}$$

For $n \ge 2$, we have

$$\left\|\widetilde{\psi}^n_{i,h} - \psi^n_{i,h}\right\|_{C^0_h} = \left\|B^h_i\left(\widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}^{n-1}_{k,h}\right) - \left(B_i\left(\varphi + \mu + \alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right)_h\right\|_{C^0_h}.$$

Then,

$$\begin{aligned}\left\|\widetilde{\psi}^n_{i,h} - \psi^n_{i,h}\right\|_{C^0_h} &\le \left\|B^h_i\left(\widetilde{\varphi}_h + \mu_h\right) - \left(B_i\left(\varphi + \mu\right)\right)_h\right\|_{C^0_h} \\ &+ \left\|B^h_i\left(\alpha\sum_{k=1}^{M}\rho_k\widetilde{\psi}^{n-1}_{k,h} - \alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right\|_{C^0_h} \\ &+ \left\|B^h_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right) - \left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi^{n-1}_k\right)\right)_h\right\|_{C^0_h}, \\ &i = 1, 2, ..., M.\end{aligned} \tag{3.137}$$

By analogy of (54) in (Dosiyev, 2018) and the convergence of Dennis-Hudson's finite difference scheme which is proved in (3.121) the following estimate holds

$$\max_{1\le k\le M}\left\|B_i^h\psi_k^{n-1}-\left(B_i\psi_k^{n-1}\right)_h\right\|_{C_h^0}\le C_6h^4,$$

Therefore, we get

$$\begin{aligned}&\left\|B_i^h\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)-\left(B_i\left(\alpha\sum_{k=1}^{M}\rho_k\psi_k^{n-1}\right)\right)_h\right\|_{C_h^0}\\&\le\sum_{k=1}^{M}|\alpha\rho_k|\left\|B_i^h\psi_k^{n-1}-\left(B_i\psi_k^{n-1}\right)_h\right\|_{C_h^0}\\&\le C_7h^4,\end{aligned}\tag{3.138}$$

By combining (3.126), (3.135), (3.137) and (3.138), we have

$$\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0}\le C_8h^4+q_0\left\|\widetilde{\psi}_{i,h}^{n-1}-\psi_{i,h}^{n-1}\right\|_{C_h^0},\tag{3.139}$$

From the estimations (3.136), (3.139), it yields

$$\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}^n\right\|_{C_h^0}\le C_9h^4\left(1+q_0+q_0^2+...+q_0^{n-1}\right)\le C_{10}h^4,\tag{3.140}$$

The relation (3.78) yields that

$$\left\|\psi_i^1\right\|_{C^0}\le\left(1-\frac{\xi}{\beta_2}\right)\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\tag{3.141}$$

and

$$\left\|\psi_i^n-\psi_i^{n-1}\right\|_{C^0}\le|B_i|\,|\alpha|\sum_{k=1}^{M}|\rho_k|\left\|\psi_i^{n-1}-\psi_i^{n-2}\right\|_{C^0},\ i=1,2,...,M,\tag{3.142}$$

where φ is defined by (3.77). By using (3.73), (3.75) , (3.141) and (3.142), we have

$$\left\|\psi_i^n-\psi_i^{n-1}\right\|_{C^0}\le q_1^n\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,...,M,$$

where $q_1=1-\frac{\xi}{\beta_2}$. Moreover, for any $m=1,2,...$, we find that

$$\left\|\psi_i^{n+m}-\psi_i^n\right\|_{C^0}\le q_1^{n+1}\left(\frac{1-q_1^m}{1-q_1}\right)\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,...,M.\tag{3.143}$$

Since

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \left\|\psi_i^{n+m} - \psi_i^n\right\|_{C^0} + \left\|\psi_i^{n+m} - \psi_i\right\|_{C^0}, \quad i = 1, 2, ..., M, \tag{3.144}$$

By taking limit as $m \to \infty$, from (3.143) and (3.144), it follows that

$$\left\|\psi_i^n - \psi_i\right\|_{C^0} \leq \frac{q_1^{n+1}}{1 - q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \quad i = 1, 2, ..., M. \tag{3.145}$$

From (3.140) and (3.145), we get

$$\left\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}\right\|_{C_h^0} \leq C_{10}h^4 + \frac{q_1^{n+1}}{1 - q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \quad i = 1, 2, ..., M. \tag{3.146}$$

Hence, for the approximate solution of the nonlocal problem (3.66), (3.67), we consider the following difference problem

$$L\widetilde{u}_h^n = 0 \ \text{ on } R_h, \quad \widetilde{u}_h^n = \tau_h \ \text{ on } \gamma_h^2, \quad \widetilde{u}_h^n = 0 \ \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{3.147}$$

$$\widetilde{u}_h^n = \widetilde{f}_h^n \ \text{ on } \gamma_h^4. \tag{3.148}$$

where $\widetilde{f}_h^n$ defined by (3.131) .

Theorem 3.2.3. The estimation holds

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - u\right| \leq C_{11}h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} C^*,$$

where $\widetilde{u}_h^n$ is a solution of problem (3.147), (3.148), u is the exact solution of nonlocal boundary value problem (3.66) , (3.67), $C^* = \|\varphi\|_{C^0} + \|\mu\|_{C^0}$, q_0 is defined by (3.69) and $q_1 = 1 - \frac{\xi}{b}$.

Proof. Let $U_h(x, y)$ be the solution of the system of grid equations

$$LU_h = 0 \ \text{ on } R_h, \quad U_h = \tau \text{ on } \gamma_h^2, \quad U_h = 0 \ \text{ on } \gamma_h^1 \cup \gamma_h^3, \tag{3.149}$$

$$U_h = f_h \ \text{ on } \gamma_h^4, \tag{3.150}$$

where f_h is the trace of f on $[0, \beta_1]_h$. Since τ, μ and $\psi_i, i = 1, 2, ..., M,$ belong to $C^{6,\lambda}, 0 < \lambda < 1$, on the interval $0 \leq x \leq 1$, by analogy of (3.121), we have

$$\max_{(x,y)\in\overline{R}_h} |U_h - U| \leq C_{13}h^4, \tag{3.151}$$

where U is the solution of problem (3.67),(3.68). The inequalities (3.126) and (3.146) yield that

$$\left\| \widetilde{f}_h^n - f_h \right\|_{C_h^0} \leq C_{14} h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{3.152}$$

where q_0 is defined by (3.69) . From the grid maximum principle and from (3.152) we have

$$\max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}_h^n - U_h \right| \leq C_{15} h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{3.153}$$

where $\widetilde{u}_h^n$ is the solution of problem (3.147) , (3.148) and U_h is the solution of problem (3.149), (3.150). In the view of the estimates (3.151) and (3.153), the following inequality remains true.

$$\max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}_h^n - U \right| \leq C_{16} h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} \left(\|\varphi\|_{C^0} + \|\mu\|_{C^0} \right), \tag{3.154}$$

where U is the solution of the problem (3.67), (3.68).

By using (3.154) and by the maximum principle for the second order elliptic equation (see in Bers et all, 1964) with the truncation error of Simpson's rule which is order of $O(h^4)$, we have the final estimate

$$\begin{aligned} \max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}_h^n - u \right| &\leq \max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}_h^n - U \right| + \max_{(x,y)\in \overline{R}_h} |U - u| \\ &\leq c_{11} h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} c_{12}, \end{aligned} \tag{3.155}$$

where u is the solution of problem (3.66),(3.67) and $c_{12} = \|\varphi\|_{C^0} + \|\mu\|_{C^0}$. □

Remark. In (3.155) the right-hand side is $O\left(h^4\right)$, when

$$\frac{q_1^{n+1}}{1 - q_1} \approx h^4. \tag{3.156}$$

By (3.156) we have

$$n = \max \left\{ \left[\frac{\ln h^4 (1 - q_1)}{\ln q_1} \right], 1 \right\},$$

where $[a]$ is the integer part of a.

CHAPTER 4
NUMERICAL EXPERIMENTS

4.1 NUMERICAL RESULTS FOR SECOND ORDER ACCURACY OF LAPLACE'S EQUATION

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 4.1

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \leq y \leq 2,$$
$$u(x, 2) = 100e^{-\pi} \sin \pi x, \quad 0 \leq x \leq 1,$$
$$u(x, 0) = \frac{1}{400} \int_{\frac{1}{8}}^{2} u(x, y)dy, \quad 0 < x < 1.$$

Problem 4.2

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \leq y \leq 2,$$
$$u(x, 2) = x^{\frac{61}{30}} \left(\tan^{-1} x - \frac{\pi}{4}\right), \quad 0 \leq x \leq 1,$$
$$u(x, 0) = \frac{1}{250} \int_{\frac{1}{4}}^{2} u(x, y)dy, \quad 0 < x < 1.$$

Table 4.1: Solutions on the line $y = 0$ of Problem 4.1

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$1.08143E-003$	$1.07733E-003$	$1.06995E-003$	$1.06878E-003$
$2.11043E-003$	$2.10093E-003$	$2.09985E-003$	$2.09723E-003$
$3.06437E-003$	$3.05995E-003$	$3.04753E-003$	$3.04516E-003$
$3.99917E-003$	$3.92006E-003$	$3.90146E-003$	$3.89853E-003$
$4.63414E-003$	$4.60194E-003$	$4.58118E-003$	$4.56626E-003$
$5.10924E-003$	$5.09734E-003$	$5.08458E-003$	$5.07102E-003$
$5.42355E-003$	$5.40011E-003$	$5.38575E-003$	$5.38023E-003$
$5.54481E-003$	$5.51245E-003$	$5.49120E-003$	$5.48215E-003$
$5.42355E-003$	$5.40011E-003$	$5.38575E-003$	$5.38023E-003$
$5.10924E-003$	$5.09734E-003$	$5.08458E-003$	$5.07102E-003$
$4.63414E-003$	$4.60194E-003$	$4.58118E-003$	$4.56626E-003$
$3.99917E-003$	$3.92006E-003$	$3.90146E-003$	$3.89853E-003$
$3.06437E-003$	$3.05995E-003$	$3.04753E-003$	$3.04516E-003$
$2.11043E-003$	$2.10093E-003$	$2.09985E-003$	$2.09723E-003$
$1.08143E-003$	$1.07733E-003$	$1.06995E-003$	$1.06878E-003$

Table 4.2: Solutions on the line $y = 0$ of Problem 4.2

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$-1.24817E-006$	$-1.25923E-006$	$-1.26118E-006$	$-1.25801E-006$
$-2.61230E-006$	$-2.62049E-006$	$-2.62046E-006$	$-2.61320E-006$
$-4.06639E-006$	$-4.06610E-006$	$-4.06283E-006$	$-4.05144E-006$
$-5.54744E-005$	$-5.53623E-005$	$-5.52926E-005$	$-5.51406E-005$
$-6.97362E-005$	$-6.95075E-005$	$-6.94010E-005$	$-6.92162E-005$
$-8.25574E-005$	$-8.22155E-005$	$-8.20755E-005$	$-8.18652E-005$
$-9.30529E-005$	$-9.26090E-005$	$-9.24408E-005$	$-9.22131E-005$
$-1.00403E-005$	$-9.98743E-005$	$-9.96845E-005$	$-9.94483E-005$
$-1.03901E-005$	$-1.03307E-005$	$-1.03103E-005$	$-1.02868E-005$
$-1.02984E-005$	$-1.02350E-005$	$-1.02140E-005$	$-1.01914E-005$
$-9.72690E-005$	$-9.66213E-005$	$-9.64143E-005$	$-9.62074E-005$
$-8.65773E-005$	$-8.59469E-005$	$-8.57517E-005$	$-8.55715E-005$
$-7.09704E-005$	$-7.03932E-005$	$-7.02198E-005$	$-7.00737E-005$
$-5.07928E-005$	$-5.03143E-005$	$-5.01745E-005$	$-5.00691E-005$
$-2.67421E-005$	$-2.64308E-005$	$-2.63411E-005$	$-2.62829E-005$

The exact solution of Problems 4.1 and 4.2 are unknown. The approximate values of Problems 4.1 and 4.2 on the line $y = 0$ obtained by proposed method are given in Tables 4.1 and 4.2, respectively. According to repeated digits, for the decreasing mesh steps $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ it follows that the maximum error on these line decreases as $O\left(h^2\right)$. To obtain these results, 6 iteration are applied for the construction of $\widetilde{f_h^n}$ with the successive error which is less than 10^{-16}.

Problem 4.3

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \leq y \leq 2,$$

$$u(x, 2) = e^{2\pi} \sin \pi x, \quad 0 \leq x \leq 1,$$

$$u(x, 0) = \frac{1}{100} \int_{\frac{1}{16}}^{2} u(x, y)dy + \mu(x), \quad 0 < x < 1,$$

where $u = e^{\pi y} \sin \pi x$ is the exact solution, $\mu(x) = \left[1 + \frac{\alpha}{\pi}\left(e^{\frac{\pi}{16}} - e^{2\pi}\right)\right] \sin \pi x$.

Table 4.3: Maximum errors for the solution of Problem 4.3

h	$Max\ error$	$Order\ of\ reduction$
1/16	$1.07964041 \times 10^{-3}$	
1/32	$2.70491138 \times 10^{-4}$	3.99140
1/64	$6.76593129 \times 10^{-5}$	3.99784
1/128	$1.69171146 \times 10^{-5}$	3.99945

Table 4.4: CPU times for Problem 4.1

h	$Discrete\ Fourier$	$Gauss\ Seidel$ $with\ reducing$	$Gauss\ Seidel$ $without\ reducing$
1/16	0.08625 s	0.10125 s	0.25200 s
1/32	0.91325 s	1.30225 s	4.00625 s
1/64	9.57500 s	12.77395 s	38.60125 s
1/128	148.11565 s	234.10215 s	636.12675 s

Table 4.5: CPU times for Problem 4.2

h	*Discrete Fourier*	*Gauss Seidel with reducing*	*Gauss Seidel without reducing*
1/16	0.10375 s	0.19525 s	0.39265 s
1/32	1.10225 s	2.10113 s	5.14255 s
1/64	12.7125 s	16.01345 s	41.26315 s
1/128	192.43125 s	310.26890 s	1003.13625 s

Table 4.6: CPU times for Problem 4.3

h	*Discrete Fourier*	*Gauss Seidel with reducing*	*Gauss Seidel without reducing*
1/16	0.06500 s	0.081250 s	0.21500 s
1/32	0.70312 s	1.21625 s	3.01856 s
1/64	7.79687 s	11.00625 s	34.11175 s
1/128	100.703125 s	192.23987 s	545.37500 s

In Table 4.3 for Problem 4.3, the maximum error for each step $h = \frac{1}{2^k}$, $k = 4, 5, 6, 7$ and the reduction orders are given. From the 3-nd column follows that the convergence order is $O\left(h^2\right)$.

In Tables 4.4, 4.5 and 4.6 the results of the CPU times in solving Problems 4.1, 4.2 and 4.3

are given, respectively. On columns 2 and 3 the CPU times for the realization of the proposed approaches by the discrete Fourier method and by the Gauss-Seidel method are given. For the construction of the local function $\widetilde{f_h^n}$ for Problems 4.1 and 4.2 just 6 iterations, are used. Problem 4.3 needs 4 iterations. In column 4, Gauss-Seidel method is used to solve the given problems without reducing to the Dirichlet problem. From these results follow that discrete Fourier method which can not be used to the problem without reducing to the Dirichlet problem is faster than others. The third and fourth columns show that for the method which is applicable for both approaches (as Gauss Seidel), the CPU times with reducing are less than the CPU times without reducing to the Dirichlet problem.

As it follows from Tables 4.4 – 4.6, the CPU times given in Tables 4.4 and 4.6 for Problems 4.1 and 4.3 are less than the results for Problem 4.2 given in Table 4.5. These take place because of low smoothness of the boundary function in Problem 4.2.

Problem 4.4

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = -3e^{-2\pi} \sin \pi x, \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{200} \int_{\frac{1}{2}}^{2} u(x, y)dy + \mu(x), \quad 0 < x < 1,$$

where $u = (e^{\pi x} - 1)(e^{\pi x} - e^{\pi}) \sin \pi y + (1 - 2y)e^{\pi y(1-y)} \sin \pi x$ is the exact solution, $g(x, y) = u_{xx}(x, y) + u_{yy}(x, y)$ and $\mu(x) = \left[1 + \frac{\alpha}{\pi}\left(e^{\frac{\pi}{4}} - e^{-2\pi}\right)\right] \sin \pi x + \frac{\alpha}{\pi}(e^{\pi x} - 1)(e^{\pi x} - e^{\pi})$.

Table 4.7: Solutions on the line $y = 0$ of Problem 4.4

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
0, 195064783	0, 195051561	0, 195050965	0, 195051024
0, 382669877	0, 382675485	0, 382680414	0, 382681929
0, 555545763	0, 555559813	0, 555568668	0, 555569489
0, 707116358	0, 707112791	0, 707109567	0, 707107659
0, 831446974	0, 831457026	0, 831459968	0, 831463480
0, 923895614	0, 923886902	0, 923882665	0, 923880204
0, 980766701	0, 980778955	0, 980782390	0, 980783963
0, 999969845	0, 999978899	0, 999982461	0, 999988475
0, 980766701	0, 980778955	0, 980782390	0, 980783963
0, 923895614	0, 923886902	0, 923882665	0, 923880204
0, 831446974	0, 831457026	0, 831459968	0, 831463480
0, 707116358	0, 707112791	0, 707109567	0, 707107659
0, 555545763	0, 555559813	0, 555568668	0, 555569489
0, 382669877	0, 382675485	0, 382680414	0, 382681929
0, 195064783	0, 195051561	0, 195050965	0, 195051024

Table 4.8: Maximum errors for the solution of Problem 4.4

h	*Max error*	*Order of reduction*
1/16	$4.59635288 \times 10^{-3}$	
1/32	$1.17381162 \times 10^{-3}$	3.91575
1/64	$2.95002406 \times 10^{-4}$	3.97899
1/128	$7.39382199 \times 10^{-5}$	3.98985

The exact solution of Poisson's equation given in Problem 4.4 is known. The approximate values of Problems 4.4 on the line y=0 demonstrated by proposed method are illustrated in Tables 4.7. According to order of reduction for the decreasing mesh steps in Table 4.8, the maximum error on these line decreases as $O\left(h^2\right)$.

4.2 NUMERICAL RESULTS FOR FOURTH ORDER ACCURACY OF LAPLACE'S EQUATION

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 4.5

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = 100e^{-\pi} \sin \pi x, \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{400} \int_{\frac{1}{8}}^{2} u(x, y) dy, \quad 0 < x < 1.$$

Problem 4.6

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = x^{\frac{181}{30}} \left(\tan^{-1} x - \frac{\pi}{4} \right), \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{250} \int_{\frac{1}{4}}^{2} u(x, y) dy, \quad 0 < x < 1.$$

Table 4.9: Solutions on the line $y = 0$ of Problem 4.5

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$1.06874E-003$	$1.06873E-003$	$1.06874E-003$	$1.06877E-003$
$2.09641E-003$	$2.09639E-003$	$2.09641E-003$	$2.09647E-003$
$3.04351E-003$	$3.04350E-003$	$3.04352E-003$	$3.04361E-003$
$3.87366E-003$	$3.87364E-003$	$3.87366E-003$	$3.87378E-003$
$4.55494E-003$	$4.55491E-003$	$4.55495E-003$	$4.55508E-003$
$5.06118E-003$	$5.06115E-003$	$5.06119E-003$	$5.06134E-003$
$5.37292E-003$	$5.37289E-003$	$5.37293E-003$	$5.37309E-003$
$5.47818E-003$	$5.47815E-003$	$5.47819E-003$	$5.47835E-003$
$5.37292E-003$	$5.37289E-003$	$5.37293E-003$	$5.37309E-003$
$5.06118E-003$	$5.06115E-003$	$5.06119E-003$	$5.06134E-003$
$4.55494E-003$	$4.55491E-003$	$4.55495E-003$	$4.55508E-003$
$3.87366E-003$	$3.87364E-003$	$3.87366E-003$	$3.87378E-003$
$3.04351E-003$	$3.04350E-003$	$3.04352E-003$	$3.04361E-003$
$2.09641E-003$	$2.09639E-003$	$2.09641E-003$	$2.09647E-003$
$1.06874E-003$	$1.06873E-003$	$1.06874E-003$	$1.06877E-003$

Table 4.10: Solutions on the line $y = 0$ of Problem 4.6

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$-2.69158E-006$	$-2.68953E-006$	$-2.68153E-006$	$-2.64961E-006$
$-5.51443E-006$	$-5.51067E-006$	$-5.49500E-006$	$-5.43245E-006$
$-8.61713E-006$	$-8.61191E-006$	$-8.58921E-006$	$-8.49847E-006$
$-1.21399E-005$	$-1.21335E-005$	$-1.21047E-005$	$-1.19893E-005$
$-1.61725E-005$	$-1.61651E-005$	$-1.61313E-005$	$-1.59957E-005$
$-2.07100E-005$	$-2.07019E-005$	$-2.06644E-005$	$-2.05138E-005$
$-2.56153E-005$	$-2.56069E-005$	$-2.55671E-005$	$-2.54074E-005$
$-3.05943E-005$	$-3.05858E-005$	$-3.05454E-005$	$-3.03827E-005$
$-3.51857E-005$	$-3.51775E-005$	$-3.51378E-005$	$-3.49784E-005$
$-3.87703E-005$	$-3.87626E-005$	$-3.87253E-005$	$-3.85752E-005$
$-4.06024E-005$	$-4.05955E-005$	$-4.05620E-005$	$-4.04271E-005$
$-3.98689E-005$	$-3.98629E-005$	$-3.98345E-005$	$-3.97198E-005$
$-3.57837E-005$	$-3.57787E-005$	$-3.57564E-005$	$-3.56664E-005$
$-2.77381E-005$	$-2.77337E-005$	$-2.77183E-005$	$-2.76563E-005$
$-1.55532E-005$	$-1.55474E-005$	$-1.55393E-005$	$-1.55078E-005$

The exact solution of Problems 4.5 and 4.6 are unknown. The approximate values of Problems 4.5 and 4.6 on the line $y = 0$ obtained by proposed method are given in Tables 4.9 and 4.10, respectively. According to repeated digits, for the decreasing mesh steps $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ it follows that the maximum error on these line decreases as $O\left(h^4\right)$. For getting this accuracy, 14 iteration are needed to obtain $\widetilde{f_h^n}$ with the successive error in absolute value 10^{-16} is taken.

Problem 4.7

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } R, \quad u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = e^{2\pi} \sin \pi x, \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{100} \int_{\frac{1}{16}}^{2} u(x, y) dy + \mu(x), \quad 0 < x < 1,$$

where $u = e^{\pi y} \sin \pi x$ is the exact solution, $\mu(x) = \left[1 + \frac{\alpha}{\pi}\left(e^{\frac{\pi}{16}} - e^{2\pi}\right)\right] \sin \pi x$.

Table 4.11: Maximum errors for the solution of Problem 4.7

h	$Max\ error$	$Order\ of\ reduction$
1/16	$1.40629393 \times 10^{-9}$	
1/32	$8.77042882 \times 10^{-11}$	16.03449
1/64	$5.47739631 \times 10^{-12}$	16.01203
1/128	$3.42279360 \times 10^{-13}$	16.00270

Table 4.12: CPU times for Problem 4.5

h	$Discrete\ Fourier$	$Gauss\ Seidel$ $with\ reducing$	$Gauss\ Seidel$ $without\ reducing$
1/16	0.10125 s	0.13325 s	0.65250 s
1/32	1.58375 s	2.27125 s	6.70625 s
1/64	19.87500 s	25.15375 s	81.11175 s
1/128	284.72625 s	467.22025 s	1325.14725 s

Table 4.13: CPU times for Problem 4.6

h	$Discrete\ Fourier$	$Gauss\ Seidel$ $with\ reducing$	$Gauss\ Seidel$ $without\ reducing$
1/16	0.19115 s	0.23565 s	0.71300 s
1/32	2.00135 s	3.97115 s	8.12375 s
1/64	26.6875 s	37.35625 s	90.72425 s
1/128	355.62775 s	580.22315 s	1798.54315 s

Table 4.14: CPU times for Problem 4.7

h	$Discrete\ Fourier$	$Gauss\ Seidel$ $with\ reducing$	$Gauss\ Seidel$ $without\ reducing$
1/16	0.11375 s	0.12125 s	0.62500 s
1/32	1.28437 s	2.18375 s	5.78125 s
1/64	17.96875 s	24.35625 s	79.23375 s
1/128	278.82815 s	443.0125 s	1243.84875 s

In Table 4.10 for Problem 4.7, the maximum error for each step $h = \frac{1}{2^k}$, $k = 4, 5, 6, 7$ and the reduction orders are given. From the 3-nd column follows that the convergence order is $O\left(h^4\right)$.

In Tables 4.11, 4.12 and 4.13 the results of the CPU times in solving Problems 4.5, 4.6 and 4.7

are given, respectively. On columns 2 and 3 the CPU times for the realization of the proposed approaches by the discrete Fourier method and by the Gauss-Seidel method are given. For the construction of the local function $\widetilde{f_h^n}$ for Problems 4.5 and 4.6 just 14 iterations, are used. Problem 4.7 needs 11 iterations. In column 4, Gauss-Seidel method is used to solve the given problems without reducing to the Dirichlet problem. From these results follow that discrete Fourier method which can not be used to the problem without reducing to the Dirichlet problem is faster than others. The third and fourth columns show that for the method which is applicable for both approaches (as Gauss Seidel), the CPU times with reducing are less than the CPU times without reducing to the Dirichlet problem.

As it follows from Tables 4.11 – 4.13, the CPU times given in Tables 4.11 and 4.13 for Problems 4.5 and 4.7 are less than the results for Problem 4.6 given in Table 4.12. These take place because of low smoothness of the boundary function in Problem 4.6.

4.3 NUMERICAL RESULTS FOR SECOND ORDER ACCURACY OF THE GENERAL SECOND ORDER ELLIPTIC EQUATION

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 4.8

$$u_{xx} + u_{yy} + e^{x^2 y} u_x + e^{x+y} u_y + \left(1 - e^{x+y}\right) u = g \text{ on } R,$$

$$u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$
$$u(x, 2) = e^{x+2} \sin \pi x, \quad 0 \le x \le 1,$$
$$u(x, 0) = \frac{1}{400} \int_{\frac{1}{4}}^{2} u(x, y) dy + \mu(x), \quad 0 < x < 1.$$

where $u(x, y) = e^{x+y} \sin \pi x$ is the exact solution, $\mu(x) = \frac{1}{400} e^x \sin \pi x (400 + e^{\frac{1}{4}} - e^2)$.

Problem 4.9

$$u_{xx} + u_{yy} + 10 y u_x + 10 x u_y - 10(x + y) u = g \text{ on } R,$$

$$u(0,y) = u(1,y) = 0, \quad 0 \le y \le 2,$$

$$u(x,2) = 2x(x-1), \quad 0 \le x \le 1,$$

$$u(x,0) = \frac{1}{300}\int_{\frac{1}{4}}^{2} u(x,y)dy + \mu(x), \quad 0 < x < 1.$$

where $u(x,y) = x(x-1)(y+1)^2$ is the exact solution, $\mu(x) = \frac{1667}{57600}x(1-x)$.

Table 4.15: Solutions on the line $y = 0$ of Problem 4.8

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
0.2076346734	0.2076535562	0.2076687923	0.2076699589
0.4336197832	0.4336283459	0.4336314544	0.4336354674
0.6701248856	0.6701398448	0.6701424531	0.6701429966
0.9079351178	0.9079402676	0.9079428901	0.9079427836
1.1364367811	1.1364389134	1.1364654330	1.1364659465
1.3442278558	1.3442300172	1.3442360241	1.3442365774
1.5190458799	1.5190547266	1.5190656739	1.5190671509
1.6487147567	1.6487209769	1.6487218770	1.6487216379
1.7213226345	1.7213298834	1.7213315642	1.7213313224
1.7260167756	1.7260211221	1.7260299877	1.7260326443
1.6535467253	1.6535586554	1.65356832445	1.6535690575
1.4969211386	1.4969345687	1.4969412133	1.4969447388
1.2519675367	1.2519781418	1.2519854427	1.2519894358
0.9180568793	0.9180249742	0.9180289882	0.9180283238
0.4981676325	0.4981776555	0.4981800113	0.4981821423

Table 4.16: Solutions on the line $y = 0$ of Problem 4.9

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
0.0585384954	0.0585584334	0.0585768987	0.0585783454
0.1093578543	0.1093698741	0.1093720085	0.1093738953
0.1523369392	0.1523398472	0.1523410479	0.1523418493
0.1875483921	0.1875384932	0.1875329182	0.1875326172
0.2148192832	0.2148273915	0.2148368218	0.2148391352
0.2343638921	0.2343659382	0.2343734679	0.2343748731
0.2460738219	0.2460898271	0.2460923731	0.2460934799
0.2532746392	0.2532584934	0.2532449384	0.2532443601
0.2460738219	0.2460898271	0.2460923731	0.2460934799
0.2343638921	0.2343659382	0.2343734679	0.2343748731
0.2148192832	0.2148273915	0.2148368218	0.2148391352
0.1875483921	0.1875384932	0.1875329182	0.1875326172
0.1523369392	0.1523398472	0.1523410479	0.1523418493
0.1093578543	0.1093698741	0.1093720085	0.1093738953
0.0585384954	0.0585584334	0.0585768987	0.0585783454

Table 4.17: Maximum errors for the solution of Problem 4.8

h	$Max\ error$	$Order\ of\ reduction$
1/16	3.71348976×10^{-4}	
1/32	9.36492001×10^{-5}	3.965321
1/64	2.35504136×10^{-5}	3.976535
1/128	5.90288162×10^{-6}	3.989653

The exact solution of Problems 4.8 and 4.9 are known. The approximate values of Problems 4.8 and 4.9 on the line $y = 0$ obtained by proposed method are given in Tables 4.14 and 4.15, respectively.

In Table 4.16 for Problem 4.9, the maximum error are illustrated for decreasing mesh $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. The order of reduction in 3-coulumn shows that the convergence order is

$O\left(h^2\right)$. To obtain this accuracy, 8 iteration are used on the way of construction $\widetilde{f}_h^n$ with the successive error in absolute value 10^{-16} is taken.

Problem 4.10

$$u_{xx} + u_{yy} + y^{\frac{181}{30}} u_x + x^{\frac{181}{30}} u_y - x^{\frac{181}{30}} yu = 0 \text{ on } R,$$

$$u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$
$$u(x, 2) = x^{\frac{361}{60}} \left(\tan^{-1} x - \frac{\pi}{4}\right), \quad 0 \le x \le 1,$$
$$u(x, 0) = \frac{1}{100} \int_{\frac{1}{16}}^{2} u(x, y) dy, \quad 0 < x < 1,$$

Table 4.18: Solutions on the line $y = 0$ of Problem 4.10

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$-3.24817E-004$	$-3.25923E-004$	$-3.26118E-004$	$-3.25801E-004$
$-4.61230E-004$	$-4.62049E-004$	$-4.62046E-004$	$-4.61320E-004$
$-6.06639E-004$	$-6.06610E-004$	$-6.06283E-004$	$-6.05144E-004$
$-7.54744E-004$	$-7.53623E-004$	$-7.52926E-004$	$-7.51406E-004$
$-8.97362E-004$	$-8.95075E-004$	$-8.94010E-004$	$-8.92162E-004$
$-9.25574E-004$	$-9.22155E-004$	$-9.20755E-004$	$-9.18652E-004$
$-1.03052E-004$	$-1.02609E-004$	$-1.02440E-004$	$-1.02213E-004$
$-3.00403E-004$	$-3.98743E-004$	$-3.96845E-004$	$-3.94483E-004$
$-3.03901E-004$	$-3.03307E-004$	$-3.03103E-004$	$-3.02868E-004$
$-3.02984E-004$	$-3.02350E-004$	$-3.02140E-004$	$-3.01914E-004$
$-1.07269E-004$	$-1.06621E-004$	$-1.06414E-004$	$-1.06207E-004$
$-9.65773E-004$	$-9.59469E-004$	$-9.57517E-004$	$-9.55715E-004$
$-6.09704E-004$	$-6.03932E-004$	$-6.02198E-004$	$-6.00737E-004$
$-3.07928E-004$	$-3.03143E-004$	$-3.01745E-004$	$-3.00691E-004$
$-1.67421E-004$	$-1.64308E-004$	$-1.63411E-004$	$-1.62829E-004$

Table 4.19: CPU times for Gauss Seidel with reducing for Problem 4.10

h	Problem 4.8	Problem 4.9	Problem 4.10
$1/16$	0.15 s	0.14915 s	0.20152 s
$1/32$	1.77465 s	1.81675 s	2.31325 s
$1/64$	21.12625 s	22.01125 s	27.22155 s
$1/128$	489.01125 s	494.01456 s	589.12372 s

The exact solution of Problem 4.10 is unknown. In Table 4.17 for Problem 4.10, the approximate values on the line $y = 0$, is given. According to repeated digits for the decreasing mesh $h = \frac{1}{2^k}, k = 4, 5, 6, 7$, it follows that the convergence order is $O\left(h^2\right)$.

In Table 4.18, the results of CPU times of the proposed approaches by the Gauss Seidel method are given for Problem 4.8, 4.9 and 4.10. To construct $\widetilde{f_h^n}$, 8 iterations are used for Problem 4.8 and 4.9. However for Problem 4.10, just 10 iterations are needed. To achieve all these, successive error is used as less than 10^{-16}.

In Problem 4.10, the smoothness of boundary functions are less than the smoothness of boundary functions in Problems 4.8 and 4.9. Therefore the CPU times in Problem 4.10 is higher than the CPU times in Problem 4.8 and 4.9.

4.4 NUMERICAL RESULTS FOR FOURTH ORDER ACCURACY OF GENERAL SECOND ORDER ELLIPTIC EQUATION

Let

$$R = \{(x, y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 4.11

$$u_{xx} + u_{yy} + y^{\frac{181}{30}} u_x + x^{\frac{181}{30}} u_y - x^{\frac{181}{30}} yu = 0 \text{ on } R,$$

$$u(0, y) = u(1, y) = 0, \quad 0 \leq y \leq 2,$$

$$u(x, 2) = x^{\frac{361}{60}} \left(\tan^{-1} x - \frac{\pi}{4}\right), \quad 0 \leq x \leq 1,$$

$$u(x, 0) = \frac{1}{100} \int_{\frac{1}{16}}^{2} u(x, y) dy, \quad 0 < x < 1,$$

Table 4.20: Solutions on the line $y = 0$ of Problem 4.11

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
$-3.25807E-004$	$-3.25804E-004$	$-3.25801E-004$	$-3.25801E-004$
$-4.61323E-004$	$-4.61323E-004$	$-4.61321E-004$	$-4.61321E-004$
$-6.05146E-004$	$-6.05145E-004$	$-6.05144E-004$	$-6.05144E-004$
$-7.51409E-004$	$-7.51408E-004$	$-7.51407E-004$	$-7.51406E-004$
$-8.92160E-004$	$-8.92162E-004$	$-8.92162E-004$	$-8.92162E-004$
$-9.18658E-004$	$-9.18655E-004$	$-9.18653E-004$	$-9.18652E-004$
$-1.02214E-004$	$-1.02214E-004$	$-1.02213E-004$	$-1.02213E-004$
$-3.94488E-004$	$-3.94486E-004$	$-3.94484E-004$	$-3.94484E-004$
$-3.02869E-004$	$-3.02868E-004$	$-3.02868E-004$	$-3.02868E-004$
$-3.01911E-004$	$-3.01913E-004$	$-3.01913E-004$	$-3.01914E-004$
$-1.06205E-004$	$-1.06205E-004$	$-1.06206E-004$	$-1.06207E-004$
$-9.55711E-004$	$-9.55713E-004$	$-9.55714E-004$	$-9.55715E-004$
$-6.00731E-004$	$-6.00734E-004$	$-6.00736E-004$	$-6.00736E-004$
$-3.00692E-004$	$-3.00691E-004$	$-3.00691E-004$	$-3.00691E-004$
$-1.62824E-004$	$-1.64305E-004$	$-1.62828E-004$	$-1.62829E-004$

Problem 4.12

$$u_{xx} + u_{yy} + e^{y^2} u_x + \sin(\pi x) u_y - e^{x+y} u = 0 \text{ on } R,$$

$$u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = e^2 \sin \pi x, \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{400} \int_{\frac{1}{4}}^{2} u(x, y) dy \quad 0 < x < 1,$$

Table 4.21: Solutions on the line $y = 0$ of Problem 4.12

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
0.3857382584	0.3857382492	0.3857382478	0.3857382466
0.4758495134	0.4758495189	0.4758495212	0.4758495231
0.6473929863	0.6473929798	0.6473929764	0.6473929714
0.8493612742	0.8493612879	0.8493612865	0.8493612823
1.1837489532	1.1837489603	1.1837489752	1.1837489801
1.3746380152	1.3746380199	1.3746380206	1.3746380215
1.4980766089	1.4980766093	1.4980766102	1.4980766113
1.5999742753	1.5999742689	1.5999742654	1.5999742641
1.4980766089	1.4980766093	1.4980766102	1.4980766113
1.3746380152	1.3746380199	1.3746380206	1.3746380215
1.1837489532	1.1837489603	1.1837489752	1.1837489801
0.8493612742	0.8493612879	0.8493612865	0.8493612823
0.6473929863	0.6473929798	0.6473929764	0.6473929714
0.4758495134	0.4758495189	0.4758495212	0.4758495231
0.3857382584	0.3857382492	0.3857382478	0.3857382466

In Problem 4.11 and 4.12, the exact solutions are not known. The approximate solutions of the problems are demonstrated on the line $y = 0$ in Table 4.19 and 4.20. According to repeated digits, for the decreasing mesh steps $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ it follows that the maximum error on these line decreases as $O\left(h^4\right)$. To construct $\widetilde{f_h^n}$, we need 17 iterations with successive error which is less than 10^{-16}.

Problem 4.13

$$u_{xx} + u_{yy} + u_y - \pi u = 0 \text{ on } R,$$

$$u(0, y) = u(1, y) = 0, \quad 0 \le y \le 2,$$

$$u(x, 2) = e^{2\pi} \sin(\pi x), \quad 0 \le x \le 1,$$

$$u(x, 0) = \frac{1}{600} \int_{\frac{1}{2}}^{2} u(x, y) dy + \mu(x), \quad 0 < x < 1,$$

where $u(x, y) = e^{\pi y}\sin(\pi x)$ is exact solution, $\mu(x) = \sin(\pi x)\left[\frac{1}{600\pi}\left(e^{\frac{1}{2}\pi} - e^{2\pi}\right) + 1\right]$.

Table 4.22: Maximum errors for the solution of Problem 4.13

h	$Max\ error$	$Order\ of\ reduction$
1/16	2.73648325×10^{-7}	
1/32	1.17039293×10^{-8}	16.05984
1/64	1.06417712×10^{-9}	16.01201
1/128	$6.65160015\times 10^{-11}$	15.99892

Table 4.23: CPU times for Gauss Seidel with reducing for Problem 4.13

h	Problem 4.11	Problem 4.12	Problem 4.13
1/16	0.29343 s	0.16743 s	0.14913 s
1/32	3.43721 s	2.46382 s	2.57218 s
1/64	36.83612 s	27.43587 s	28.34611 s
1/128	701.38212 s	537.38212 s	579.27362 s

In Table 4.21 for Problem 4.13, the maximum error are illustrated for decreasing mesh $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. The order of reduction in 3-coulumn shows that the convergence order is $O\left(h^4\right)$. To obtain this accuracy, 16 iteration are used on the way of construction $\widetilde{f_h^n}$ with the successive error in absolute value 10^{-16} is taken

In Table 4.22, the results of CPU times of the proposed approaches by the Gauss Seidel method are given for Problem 4.11, 4.12 and 4.13. To construct $\widetilde{f_h^n}$, 17 iterations are used for Problem 4.8 and 4.9. However for Problem 4.10, just 16 iterations are needed. To achieve all these, successive error is used as less than 10^{-16}.

CHAPTER 5
CONCLUSIONS

A constructive method for the exact and approximate solutions of the nonlocal boundary value problem for Laplace's and the general second order linear elliptic equations with nonlocal integral condition is proposed and justified. In the proposed method, the boundary values where the nonlocal condition was given, are constructed as a function by using the n-th term of the convergent fixed point iteration for the solution of the obtained nonlinear system of equations.

The second and fourth order finite difference schemes are constructed for Laplace's and second order linear elliptic equations. The convergence of all finite difference scheme are verified. As a novel error estimation of order $O(h^4)$, for the finite difference scheme constructed Dennis-Hudson (see in Dennis&Hudson, 1979;1980) for the second order general linear elliptic equations is obtained.

The uniform estimate of the error of approximate solution for Laplace's equation with integral boundary condition is of order $O(h^2)$ for 5-point scheme and $O(h^4)$ for 9-point scheme, when the given boundary functions on the sides belong to the Hölder classes $C^{2,\lambda}$ and $C^{4,\lambda}$, $0 < \lambda < 1$, respectively. For the second order elliptic equation, it is proved that when the boundary functions are from $C^{4,\lambda}, 0 < \lambda < 1$, the approximate solution by 5-point scheme of the nonlocal problem with integral boundary condition converges of order $O(h^2)$ and Dennis-Hudson's scheme converges of $O(h^4)$ when the exact solution belongs $C^{6,\lambda}, 0 < \lambda < 1$.

Finally, the proposed method can be used to get numerical solution of different nonlocal problems for other type partial differential equations. Also, it can be developed to obtain higher order as $O(h^p)$, $p > 4$, uniform estimate of the error of the approximate solution. Moreover, the existing fast algortihms (see in Samarskii, 1989 and the references therein) can be used for the realization of the obtained local finite-difference problems in the proposed approach

REFERENCES

Ashyralyev, A., & Ozturk, E. (2012). On Bitsadze-Samarskii type nonlocal boundary value problems for elliptic differential and difference equations. Well posedness, *Appl. Math. Comput.*, 219, 1093-1107.

Ashyralyev, A., & Ozturk, E. (2013). On a difference scheme of fourth order of accuracy for the Bitsadze-Samarskii type nonlocal boundary value problem. *Math. Methods Appl. Sci.*, 36, 936-955.

Avalishvili, G., Avalishvili, M., & Gordeziani, D. (2010). On a nonlocal problem with integral boundary conditions for a multidimensional elliptic equation. *App. Math. Letters.*, 24(2011), 566-571.

Avalishvili, G., Avalishvili, M., & Gordeziani, D. (2011). On nonlocal boundary value problems for some partial differential equations. *Bull. Georgian Natl. Acad. Sci.* (N.S.), 5, 31-37.

Bahvalov, N. S. (1959). Numerical solution of the Dirichlet problem for Laplace's equation.Vestnik Moskov University Series Matematka. Mechanika. Astronomiya, Fizika, Himiya, 3(5), 171-195.

Bakhalov, N.S., & Orekhov, M.Y. (1982). Fast methods for solving Poisson's equation.*Comput. Math. Math. Pyhs.*, 22(6), 107-114.

Berikelashvili, G.K. (2001). On the convergence of difference schemes for the third boundary value problem of elasticity theory. *Comput. Math. Math. Pyhs.*, 41(8), 1182-1189.

Berikelashvili, G.K., & Khomeriki, N. (2012). On the convergence of difference schemes for one nonlocal boundary value-problem. *Lithuanian Math. J.*, 52(4), 353-363.

Bitsadze, A.V., & Samarskii, A.A. (1969). On some simplest generalizations of linear elliptic problems. *Dokl. Akad. Nauk SSSR.*, 185(4), 739-740.

Dehgan, M., & Tatari, M. (2006). Solutions of a semilinear parabolic equation with an unknown control function using the decomposition procedure of Adomian. *Num. Methods for Partial Diff. Equ.*, https://doi.org/10.1002/num.20186.

Dennis, S.C.R., & Hudson, J.D. (1979). Accurate representations of partial differential equations by finite-difference methods. *J. Inst. Maths. Applics.*, 23, 43-51.

Dennis, S.C.R., & Hudson, J.D. (1980). Further accurate representations of partial differential equations by finite-difference methods. *J. Inst. Maths. Applics.*, 26, 369-379.

Dosiyev, A.A. (2003). On the maximum error in the solution of Laplace equation by finite difference method. Inter. *Journal of Pure and Appl. Math.*, 7, 223-235.

Dosiyev, A.A., & Reis, R. (2018). An approximate grid solution of a nonlocal boundary value problem with integral boundary condition for Laplace's equation. *ITM Web of Conference 22*.

Dosiyev, A.A., & Reis, R. (2019). A fourth-order accurate difference Dirichlet problem for the approximate solution of Laplace's equation with integral boundary condition. *Adv. Diff. Equ.*, https://doi.org/10.1186/s13662-019-2282-2.

Dosiyev, A.A. (2019). Difference method of fourth order accuracy for the Laplace equation with multilevel nonlocal conditions. *J. Comput. Appl. Math.*, 354, 587-596.

Fox, L. (1947). Some improvements in the use of relaxation methods for the solution of ordinary and partial differential equations. *Proc. Roy. Sos. London.14 Ser. A*, 190, 31-59.

Gordeziani, N., Natalini, P., & Ricci, P.E. (2005). Finite-difference methods for solution of nonlocal boundary value problems. *Comput. Math. Appl.*, 50, 1333-1344.

Gupta, M.M. 1983. A fourth order finite difference scheme for two-dimensional elliptic equations. *Sci. Comput.*, 147-154.

Gurbanov, I.A., & Dosiyev, A.A. (1984). On the numerical solution of nonlocal boundary problems for quasilinear elliptic equations. In: *Approximate Methods for Operator Equations*, Baku State University, pp. 64-74, Baku, Azerbaijan.

Il'in, V.A., & Moiseev, E.I. (1990). Two- dimensional nonlocal boundary value problems for Poisson's operator in differential and difference variants. *Mat. Mod.*, 2, 139-150.

Jack, J.J.B., Noble, D., & Tsien, R.W. (1975). Electric current flow in excitable cells.Clarendoni Oxford University.

Jakubelience, K., Cliupaila, R., & Sapagovas, M. (2017). Semi-implicit difference scheme for a two-dimensional parabolic equation with an integral boundary condition. *Math. Model. and Anal.*, 22(5), 617-633.

Karaa, S. (2006). High-Order Approximation of 2D convection-diffusion equation on hexagonal grids. *Numer. Methods Partial Diff. Eq.*, 22, 1238-1246.

Mesloub, S., & Bouziani, A. (1999). On a class of singular hyperbolic equation with a weighted integral condition. *Intern. J. Math. Math.* Sci., 22, 511–520.

Mikhailov, V. P. (1978). *Partial Differential Equations*. Mir., Moscow.

Pulkina, L.S. (1999). A nonlocal problem with integral conditions for hyperbolic equations, *Electronic J. Differential Eq.*, 45, 1–6.

Saadatmandi, A., & Dehgan, M. (2006). Numerical solution of the one-dimensional wave equation with an integral condition. Inc. *Numer Methods Partial Differential Eq.*, 23, 282–292.

Sajavicius, S. (2014) Radial basis function method for a multidimensional linear elliptic equation with nonlocal boundary conditions. *Comput. Math. Appl.*, 67(7), 1407-1420.

Samarskii, A.A., & Nikolaev, E.S. (1989). *Numerical Methods for Grid Equations*, vol. I, Direct Methods, Brikhöser Verlag, Basel-Boston-Berlin.

Samarskii, A.A., & Nikolaev, E.S. (1989). *Numerical Methods for Grid Equations*, vol. II, Iterative Methods, Brikhöser Verlag, Basel-Boston-Berlin.

Samarskii, A.A. (2001). *The Theory of Difference Schemes*, Dekker, New York.

Sapagovas, M.P. (2002). The eigenvalues of some problems with a nonlocal condition. *Diff. Equ.*, 38(7), 1020-1026.

Sapagovas, M.P. (2008). Difference method of increased order of accuracy for the Poisson equation with nonlocal conditions. *Differential Equations*, 44(7), 1018-1028.

Sapagovas, M. (2008). On the stability of a finite-difference scheme for nonlocal parabolic boundary-value problems. *Lith. Math. J.*, 48(3), 339-356.

Sapagovas, M., & Jakubelience, K. (2012). Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition. *Nonlin. Anal. Model. Contr.*, 17(1), 91-98.

Sapagovas, O., Cliupaila, R., Joksiene, Z., & Meskauskas T. (2013). Computational experiment for stability analysis of difference schemes with nonlocal conditions. *Informatica,* 24(2), 275-290.

Sapagovas, M., Stikoniene, O., Cliupaila, R., & Joksiene, Z. (2016). Convergence of iterative methods for elliptic equations with integral boundary conditions. *Electr. J. Diff. Equ.*, 118, 1-14.

Skubachevskii, A.L. (2008). On necessary conditions for the Fredholm solvability of nonlocal elliptic equations, *Proc. Steklov Ins. Math.*, 260(1), 238-253.

Smith, G. D. (1985). *Numerical solution of partial differential equations: Finite difference methods*. Oxford University Press.

Stikoniene, O., Sapagovas, M., & Cliupaila, R. (2014). On iterative methods for some elliptic equations with nonlocal conditions. *Nonlin. Anal. Model. Contr.*, 19(3), 517-535.

Volkov, E.A. (1965). On differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle. *Proceedings of the Steklov Institue of Mathematics Issue*. 77, 101-126.

Volkov, E.A. (2013). Approximate grid solution of a nonlocal boundary value problem for Laplace's equation on a rectangle. *Comput. Math. Math. Phys.*, 53(8), 1128-1138.

Volkov, E.A., Dosiyev, A.A., & Buranay, S.C. (2013). On the solution of a nonlocal problem. *Comput. Math. Appl.*, 66, 330-338.

Volkov, E.A. (2013). Solvability analysis of a nonlocal boundary value problem by applying the contraction mapping principle. *Comput. Math. Math. Phys.*, 53(10), 1494-1498.

Volkov, E.A., & Dosiyev, A.A. (2016). On the numerical solution of a multilevel nonlocal problem. *Mediterr. J. Math.*, 13, 3589-3604.

Wang, Y. (2002). Solutions to nonlinear elliptic equations with a nonlocal boundary condition. *Electr. J. Diff. Equat.*, 5, 1-16.

Wasov, W. (1952). On the turncation error in the solution of Laplace's equation by finite differences. *J. Res. Natl. Bur. Stand.*, 48, 345-348.

Zhou, L., & Yu, H. (2018). Error estimate of a high accuracy difference scheme for Poisson equation with two integral boundary conditions. *Adv. Differ. Equ.*, 2018-2025.

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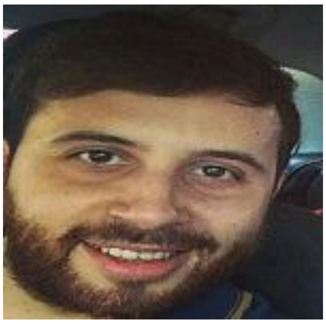

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CONFERENCES

- **Cmes'2018**, An approximate grid solution of a nonlocal boundary value problem with integral boundary condition for Laplace's equation.

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