THE SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC-TELEGRAPH EQUATIONS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
AHMAD MOHAMMAD SALEM AL-HAMMOURI

In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

NICOSIA, 2020

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Ahmad Mohammad Salem Al-Hammouri: THE SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC-TELEGRAPH EQUATIONS

Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy of Science in Mathematics

Examining Committee in Charge:

Prof. Dr. Evren Hinçal	Committee Chairman, Department of Mathematics, NEU
Prof. Dr. Allaberen Ashyralyev	Supervisor, Department of Mathematics, NEU
Prof. Dr. Deniz Ağırseven	Department of Mathematics, Trakya University
Assoc. Prof. Dr. Okan Gerçek	Department of Computer Engineering, Girne American University
Assist. Prof. Dr. Bilgen Kaymakamzade	Department of Mathematics, NEU

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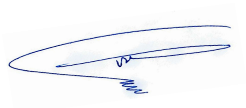
Signature: 

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To my parents…

ABSTRACT

In the present study, the source identification problem for the elliptic-telegraph differential equation in a Hilbert space with the self-adjoint positive definite operator is investigated. The main theorem on the stability of the space identification problem for the elliptic-telegraph differential equation is proved. In applications, theorems on the stability of three source identification problems for one dimensional differential equations with nonlocal conditions and for multidimensional elliptic-telegraph differential equations with local conditions are established. Furthermore, the main theorem on the stability of the difference scheme is established. In applications, theorems on the stability of difference schemes for three types of the space identification problems are proved. Numerical analysis is provided.

Keywords: Source identification problem; elliptic-telegraph differential equations; difference schemes; stability; accuracy

ÖZET

Bu çalışmada, kendine eşlenik pozitif tanımlı operatörlü bir Hilbert uzayında eliptik-telgraf diferansiyel denklemi için kaynak tanımlama problemi araştırılmıştır. Eliptik-telgraf diferansiyel denklemi için alan tanımlama probleminin kararlılığına ilişkin ana teorem kanıtlanmıştır. Uygulamalarda, yerel olmayan koşullara sahip tek boyutlu diferansiyel denklemler için ve yerel koşullu çok boyutlu eliptik-telgraf diferansiyel denklemler için üç kaynak tanımlama probleminin kararlılığına ilişkin teoremler oluşturulmuştur. Ayrıca, fark şemasının kararlılığı üzerine ana teorem ispatlanmıştır. Uygulamalarda, üç tip alan tanımlama problemi için fark şemalarının kararlılığına ilişkin teoremler kanıtlanmıştır. Sayısal analiz sağlanmıştır.

Anahtar Kelimeler: Kaynak Tanımlama Sorunu; Eliptik Telgraf Diferansiyel Denklemleri, Fark Şemaları, Kararlılık, Doğruluk

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SIP:	Source Identification Problem
BVP:	Boundary Value Problem
IVP:	Initial Value Problem
SAPDO:	Self-Adjoint Positive Definite Operator
PDE:	Partial Differential Equation
DE:	Difference Scheme
$\mathcal{L}$**:**	Laplace Transform
$\mathcal{F}$**:**	Fourier Transform

CHAPTER 1
INTRODUCTION

1.1 History

Source identification problems (SIPs) for partial differential equations (PDEs) are used to model biological, physical, system engineering and sociological processes and have been studied by many authors (Adavani and Biros, 2010; Ashyralyev and Ashyralyyev, 2014; Ashyralyev and Cekic, 2016; Ashyralyev and Emharab, 2017; Ashyralyev, et al., 2016; Ashyralyyev and Cay, 2018; Ashyralyyev, 2017; Avdonin and Nicaise, 2015; El Badia, et al., 2019; Katzourakis, 2019; Sabitov and Martem'yanova, 2012; Siskova and Slodicka, 2018).

The fast algorithms for the solution of the SIP with linear elliptic PDEs constraints was studied. The numerical techniques and the numerical experiments for the SIP with elliptic PDEs were constructed by (Adavani and Biros, 2010).

The authors investigated the boundary value problem (BVP) of determining the parameter of an elliptic equation in Banach space. Theorems of coercive stability estimates for the solution of BVP for multi-dimensional elliptic equations were proved (Ashyralyev and Ashyralyyev, 2014).

In particular, (Ashyralyev and Cekic, 2016) investigated the SIP for a telegraph equation with unknown parameter in a Hilbert space with the self-adjoint positive definite operator (SAPDO). Theorems of stability estimates for the solution of the telegraph equation were proved. In applications, three SIPs for telegraph equations were obtained. The well-posedness of Neumann-type elliptic overdetermined problem with integral condition has been well established.

The authors proved the various estimates for the solution of the identification problem of inverse problem for the elliptic type equation. The stability, almost coercive stability, and coercive stability inequalities for its solution have been obtained (Ashyralyyev and Cay, 2018; Ashyralyyev, 2017). The new methods of calculus of variations in L^{∞} to study the ill-posed inverse problem of identifying the source of a non-homogeneous linear elliptic equation for Dirichlet conditions was investigated (Katzourakis, 2019) .

The authors studied the inverse problem for an equation of elliptic-hyperbolic type with a nonlocal boundary condition. Theorems of the uniqueness criterion and the stability of solutions with respect to the BVP were proved (Sabitov and Martem'yanova, 2012).

The SIPs for the wave equation on graphs and the resolution of linear integral Volterra equations of the second kind for an interval was studied. Theorems of the uniqueness and existence of solutions were proved (Avdonin and Nicaise, 2015). The inverse source problem in time-fractional wave differential equation with dynamical boundary condition for Neumann boundary conditions was studied. Theorems of the uniqueness and existence of this solution were proved. The results of the numerical experiments were obtained (Siskova and Slodicka, 2018).

Various local and nonlocal BVPs for elliptic, hyperbolic, telegraph, hyperbolic- telegraph and elliptic- hyperbolic differential and difference equations and their applications have been investigated by many scientists (Ashyralyev and Modanli, 2015; Ashyralyev and Ozger, 2014; Ashyralyev and Sobolevskii, 2004; Ashyraliyev, 2012; Ashyraliyev, 2008; Biazar, et al., 2009; Dehghan and Shokri, 2008; Direk and Ashyraliyev, 2018; Gao and Chi, 2007; Gushchina, 2016; Ivanauskas, et al., 2013; Jator, 2015; De la Sen, 2013; Mansour, 2006; Saadatmandi and Dehghan, 2010; Sapagovas, et al., 2017; Sapagovas and Stikoniene, 2011; Stikoniene, et al., 2014; Novickij and Stikonas, 2014; Twizell, 1979; Tuan, et al., 2018; Kirane, et al., 2019; Sobolevskii, 1975; Ashyralyev and Al-Hammouri, 2019).

The nonlocal BVPs for hyperbolic-elliptic equation in a Hilbert space were studied, theorems on stability of this problem and the first and the second order of accuracy difference schemes (DSs) for approximate solutions of this problem were proved (Ashyralyev and Ozger, 2014).

The initial-value problem (IVP) for the integral-differential equation of the hyperbolic type in a Hilbert space was studied. Theorems of the uniqueness of solvability of this problem were proved. The convergence estimates for the solutions of difference schemes were obtained (Direk and Ashyraliyev, 2018). The equation of mixed elliptic-hyperbolic type in rectangular area with the conditions of periodicity and the nonlocal problem of A. A. Desin was studied. Theorems of convergence of the constructed series in the class of regular solutions and the stability of the solution were proved by (Gushchina, 2016).

The authors studied the stability of an explicit DS for linear hyperbolic equations with nonlocal integral boundary conditions. Theorem of the stability for linear hyperbolic equations with nonlocal integral boundary conditions was proved (Ivanauskas, et al., 2013). In particular, (Mansour, 2006) studied the existence of traveling wave solutions for a hyperbolic-elliptic system of PDEs and applied the geometric theory of singular perturbations. Theorem of the existence of the wave solution was proved.

The authors applied the standard method of finite DSs for nonlinear elliptic equations with integral condition. Theorems of the convergence of all methods for this solution were proved. In application, the results of convergence between iterative methods were applied for the first time to nonlinear system (Sapagovas, et al., 2017). The generalization of the alternating-direction implicit method for the two-dimensional nonlinear elliptic equation with integral boundary condition in one coordinate direction was analyzed. Theorem of the convergence of the iterative method was proved. Furthermore, the computational experiments of results were obtained (Sapagovas and Stikoniene, 2011). The iterative methods for the solution of the system of the difference equations derived from the elliptic equation with nonlocal conditions were applied. Theorems on the convergence of faster iterative methods were proven (Stikoniene, et al., 2014). The stability of a weighted finite DS for wave equation with nonlocal boundary conditions was studied . The linear hyperbolic equation with nonlocal integral boundary condition was investigated. The stability conditions in a special matrix norm were obtained (Novickij and Stikonas, 2014).

In this thesis, several identification problems for elliptic-telegraph equations can be reduced to the space SIP for the elliptic-telegraph equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t), 0 < t < 1, \\ -\frac{d^2u(t)}{dt^2} + Au(t) = p + g(t), -1 < t < 0, \\ u(0) = \varphi, u_t(0^+) = u_t(0^-), u(-1) = \psi, u(1) = \xi \end{cases} \tag{1.1}$$

in a Hilbert space H with the SAPDO $A \geq \delta I, \delta > 0$. Here p is the unknown parameter.

1.2 Methods of Solution of Source Identification Problem

It is known that SIPs for partial elliptic-telegraph differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. Now, let us illustrate these three different analytical methods by examples.

We consider Fourier series method for solution of SIPs for partial elliptic-telegraph differential equations.

First, we consider the Fourier series solution of the following SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + \sin x, \\ 0 < t < 1, 0 < x < \pi, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + \sin x, \\ -1 < t < 0, 0 < x < \pi, \\ u(0,x) = \sin x, u(\pm 1, x) = \sin x, 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, -1 \leq t \leq 1 \end{cases} \tag{1.2}$$

for a one dimensional elliptic-telegraph equation.

In order to solve the problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi) = 0 \tag{1.3}$$

generated by the space operator of problem (1.2). It is easy to see that the solution of this Sturm-Liouville problem is

$$u_k(x) = \sin kx, k = 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem (1.2) by formulas

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx, \tag{1.4}$$

$$p(x) = \sum_{k=1}^{\infty} p_k \sin kx, \tag{1.5}$$

where $A_k(t), k = 1, 2, ...$ are unknown functions and $p_k, k = 1, 2, ...$ are unknown parameters. Putting $u(t, x)$ and $p(x)$ functions into above equations and using given initial and boundary conditions, we obtain

$$\sum_{k=1}^{\infty} A_k''(t) \sin kx + \sum_{k=1}^{\infty} A_k'(t) \sin kx + \sum_{k=1}^{\infty} \left(k^2 + 1\right) A_k(t) \sin kx$$

$$= \sum_{k=1}^{\infty} p_k \sin kx + \sin x, 0 < t < 1,$$

$$-\sum_{k=1}^{\infty} A_k''(t) \sin kx + \sum_{k=1}^{\infty} \left(k^2 + 1\right) A_k(t) \sin kx$$

$$= \sum_{k=1}^{\infty} p_k \sin kx + \sin x, -1 < t < 0,$$

$$u(0, x) = \sum_{k=1}^{\infty} A_k(0) \sin kx = \sin x,$$

$$u(\pm 1, x) = \sum_{k=1}^{\infty} A_k(\pm 1) \sin kx = \sin x,$$

Equating coefficients of $\sin(kx)$, $k = 1, 2,...$ to zero, we get

$$\begin{cases} A_k''(t) + A_k'(t) + \left(k^2 + 1\right) A_k(t) = p_k, 0 < t < 1, \\ -A_k''(t) - \left(k^2 + 1\right) A_k(t) = p_k, -1 < t < 0, \\ A_k(0) = A_k(\pm 1) = 0, k \neq 1 \end{cases} \tag{1.6}$$

and

$$\begin{cases} A_1''(t) + A_1'(t) + 2A_1(t) = 1 + p_1, 0 < t < 1, \\ -A_1''(t) + 2A_1(t) = 1 + p_1, -1 < t < 0, \\ A_1(0) = A_1(\pm 1) = 1. \end{cases} \tag{1.7}$$

We obtain $A_k(t)$ and p_k for $k \neq 1$. Let $0 \leq t \leq 1$.Then, the auxilary equation is

$$q^2 + q + k^2 + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{k^2 + \frac{3}{4}}.$$

Therefore,

$$A_k^c(t) = e^{-\frac{t}{2}}\left[c_1 \cos\sqrt{k^2 + \frac{3}{4}}t + c_2 \sin\sqrt{k^2 + \frac{3}{4}}t\right]$$

is the solution of auxilary equation. Since $A_k^p(t) = \frac{p_k}{k^2+1}$ is the solution of nonhomegeonus equation

$$A_k''(t) + A_k'(t) + \left(k^2 + 1\right)A_k(t) = p_k,$$

we have that

$$A_k(t) = e^{-\frac{t}{2}}\left[c_1 \cos\sqrt{k^2 + \frac{3}{4}}t + c_2 \sin\sqrt{k^2 + \frac{3}{4}}t\right] + \frac{p_k}{k^2 + 1}.$$

Let $-1 \leq t \leq 0$.Then, the auxilary equation is

$$-q^2 + k^2 + 1 = 0.$$

We have two roots

$$q_1 = \sqrt{k^2 + 1}, q_2 = -\sqrt{k^2 + 1}.$$

Therefore,

$$A_k(t) = c_3 \cosh\sqrt{k^2 + 1}t + c_4 \sinh\sqrt{k^2 + 1}t + \frac{p_k}{k^2 + 1}.$$

Applying boundary conditions $A_k(0) = A_k(\pm 1) = 0, A_k'(0+) = A_k'(0-)$, we get

$$A_k(0) = c_3 + \frac{p_k}{k^2 + 1} = 0,$$

$$A_k(0) = c_1 + \frac{p_k}{k^2 + 1} = 0,$$

$$\sqrt{k^2+1}c_4 = -\frac{1}{2}c_1 + \sqrt{k^2+\frac{3}{4}}c_2,$$

$$A_k(1) = e^{-\frac{1}{2}}\left[c_1\cos\sqrt{k^2+\frac{3}{4}} + c_2\sin\sqrt{k^2+\frac{3}{4}}\right] + \frac{p_k}{k^2+1} = 0,$$

-

$$A_k(-1) = c_3\cosh\sqrt{k^2+1} - c_4\sinh\sqrt{k^2+1} + \frac{p_k}{k^2+1} = 0.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{k^2+1}c_4 + \frac{1}{2}c_1 - \sqrt{k^2+\frac{3}{4}}c_2 = 0, \\ e^{-\frac{1}{2}}\left[c_1\cos\sqrt{k^2+\frac{3}{4}} + c_2\sin\sqrt{k^2+\frac{3}{4}}\right] - c_1 = 0, \\ c_3\cosh\sqrt{k^2+1} - c_4\sinh\sqrt{k^2+1} - c_3 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\sqrt{k^2+\frac{3}{4}} & 0 & \sqrt{k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{k^2+\frac{3}{4}} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k^2+1} - 1 & -\sinh\sqrt{k^2+1} \end{vmatrix}$$

$$= \begin{vmatrix} -\sqrt{k^2+\frac{3}{4}} & 0 & \sqrt{k^2+1} \\ e^{-\frac{1}{2}}\sin\sqrt{k^2+\frac{3}{4}} & 0 & 0 \\ 0 & \cosh\sqrt{k^2+1} - 1 & -\sinh\sqrt{k^2+1} \end{vmatrix}$$

$$-(-1)^{1+3}\times\begin{vmatrix} \frac{1}{2} & -\sqrt{k^2+\frac{3}{4}} & \sqrt{k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{k^2+\frac{3}{4}} & 0 \\ 0 & 0 & -\sinh\sqrt{k^2+1} \end{vmatrix}$$

$$= \sqrt{k^2+1}e^{-\frac{1}{2}}\sin\sqrt{k^2+\frac{3}{4}}\left(-1+\cosh\sqrt{k^2+1}\right)$$

$$+\frac{1}{2}e^{-\frac{1}{2}}\sin\sqrt{k^2+\frac{3}{4}}\sinh\sqrt{k^2+1}$$

$$-\sqrt{k^2+\frac{3}{4}}\left(-1+e^{-\frac{1}{2}}\cos\sqrt{k^2+\frac{3}{4}}\right)\sinh\sqrt{k^2+1} \neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$.Then $p_k = 0$, and $A_k(t) = 0$.

Now, we obtain $A_1(t)$ and p_1. Let $0 \le t \le 1$.Then, the auxilary equation is

$$q^2 + q + 2 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{\frac{7}{4}}, q_2 = -\frac{1}{2} - i\sqrt{\frac{7}{4}}.$$

Therefore,

$$A_1^c(t) = e^{-\frac{t}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2}t + c_2 \sin\frac{\sqrt{7}}{2}t\right]$$

is the solution of auxilary equation. Since $A_1^p(t) = \frac{p_1+1}{2}$ is the solution of nonhomegeonus equation

$$A_1''(t) + A_1'(t) + 2A_1(t) = p_1 + 1,$$

we have that

$$A_1(t) = e^{-\frac{t}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2}t + c_2 \sin\frac{\sqrt{7}}{2}t\right] + \frac{p_1+1}{2}.$$

Let $-1 \le t \le 0$.Then, the auxilary equation is

$$-q^2 + 2 = 0.$$

We have two roots

$$q_1 = \sqrt{2}, q_2 = -\sqrt{2}.$$

Therefore,

$$A_1(t) = c_3 \cosh\sqrt{2}t + c_4 \sinh\sqrt{2}t + \frac{p_1+1}{2}.$$

Applying boundary conditions $A_1(0) = A_1(\pm 1) = 1, A_1'(0+) = A_1'(0-)$, we get

$$A_1(0) = c_3 + \frac{p_{1+1}}{2} = 1,$$

$$A_1(0) = c_1 + \frac{p_1+1}{2} = 1,$$

$$\sqrt{2}c_4 = -\frac{1}{2}c_1 + \frac{\sqrt{7}}{2}c_2,$$

$$A_1(1) = e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2} + c_2 \sin\frac{\sqrt{7}}{2}\right] + \frac{p_1+1}{2} = 1,$$

$$A_1(-1) = c_3 \cosh\sqrt{2} - c_4 \sinh\sqrt{2} + \frac{p_1+1}{2} = 1.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{2}c_4 + \frac{1}{2}c_1 - \frac{\sqrt{7}}{2}c_2 = 0, \\ e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2} + c_2 \sin\frac{\sqrt{7}}{2}\right] - c_1 = 0, \\ c_3 \cosh\sqrt{2} - c_4 \sinh\sqrt{2} - c_3 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2} - 1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{2} - 1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & \cosh\sqrt{2} - 1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$-(-1)^{1+3} \times \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{7}}{2} & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2} - 1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 \\ 0 & 0 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \sqrt{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\left(-1 + \cosh\sqrt{2}\right) + \frac{1}{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\sinh\sqrt{2}$$

$$-\frac{\sqrt{7}}{2}\left(-1 + e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}\right)\sinh\sqrt{2} \neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$.

Then $p_1 = 1$, and $A_1(t) = 1$. From that it follows $p(x) = \sin x$ and $A_1(t) = 1$ for all $-1 \leq t \leq 1$.

Therefore,

$$u(t,x) = A_1(t)\sin x = \sin x.$$

So, the exact solution of the problem (1.2) is

$$(u(t,x), p(x)) = (\sin x, \sin x).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \beta_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, -T < t < 0, \\ u(0,x) = \varphi(x), u(-T,x) = \psi(x), u(T,x) = \mu(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, -T \le t \le T \end{cases} \tag{1.8}$$

for the multidimensional elliptic-telegraph differential equations.
Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x)$, $\psi(x)$, $\mu(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with the boundary

$$S, \overline{\Omega} = \Omega \cup S.$$

However Fourier series method described in solving (1.8) can be used only in the case when (1.8) has constant coefficients.

Second, we consider the Fourier series solution of the following SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + 2e^{-t}\cos x - \cos x, \\ 0 < x < \pi, 0 < t < 1, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + e^{-t}\cos x - \cos x, \\ 0 < x < \pi, -1 < t < 0, \\ u(0,x) = \cos x, u(\pm 1, x) = e^{\mp 1}\cos x, 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0, -1 \le t \le 1 \end{cases} \tag{1.9}$$

for a one dimensional elliptic-telegraph equation.

In order to solve the problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi, u_x(0) = u_x(\pi) = 0$$

generated by the space operator of problem (1.9). It is easy to see that the solution of this Sturm-Liouville problem is

$$u_k(x) = \cos kx, k = 0, 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem (1.9) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos kx, \tag{1.10}$$

$$p(x) = \sum_{k=0}^{\infty} p_k \cos kx,$$

where $A_k(t), k = 0, 1, 2, ...$ are unknown functions and $p_k, k = 0, 1, 2, ...$ are unknown numbers. Putting $u(t,x)$ and $p(x)$ into main problem and using given initial and boundary conditions, we obtain

$$\sum_{k=0}^{\infty} A_k''(t)\cos kx + \sum_{k=0}^{\infty} A_k'(t)\cos kx + \sum_{k=0}^{\infty} \left(k^2+1\right) A_k(t)\cos kx$$

$$= \sum_{k=0}^{\infty} p_k \cos kx + 2e^{-t} \cos x - \cos x, 0 < t < 1,$$

$$- \sum_{k=0}^{\infty} A_k'' (t) \cos kx - \sum_{k=0}^{\infty} \left(k^2 + 1\right) A_k (t) \cos kx$$

$$= \sum_{k=0}^{\infty} p_k \cos kx + e^{-t} \cos x - \cos x, -1 < t < 0,$$

$$u (0, x) = \sum_{k=0}^{\infty} A_k (0) \cos kx = \cos x,$$

$$u (\pm 1, x) = \sum_{k=0}^{\infty} A_k (0) \cos kx = e^{\mp 1} \cos x, 0 \le x \le \pi. \tag{1.11}$$

Equating the coefficients of $\cos (kx), k = 0, 1, 2, ...$ to zero, we get

$$\begin{cases} A_k'' (t) + A_k' (t) + \left(k^2 + 1\right) A_k (t) = p_k, 0 < t < 1, \\ \\ -A_k'' (t) + \left(k^2 + 1\right) A_k (t) = p_k, k \ne 0, 1, -1 < t < 0, \\ \\ A_k (0) = A_k (\pm 1) = 0, k \ne 1, \end{cases} \tag{1.12}$$

$$\begin{cases} A_1'' (t) + A_1' (t) + 2A_1 (t) = p_1 + 2e^{-t} - 1, 0 < t < 1, \\ \\ -A_1'' (t) + 2A_1 (t) = p_1 + e^{-t} - 1, -1 < t < 0, \\ \\ A_1 (0) = 1, A_1 (\pm 1) = e^{\mp 1}. \end{cases} \tag{1.13}$$

We will obtain $A_k(t)$ for $k \ne 1$.

Let $0 \le t \le 1$.Then, the auxilary equation is

$$q^2 + q + k^2 + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i \sqrt{k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i \sqrt{k^2 + \frac{3}{4}}.$$

Therefore,

$$A_k^c(t) = e^{-\frac{t}{2}} \left[c_1 \cos \sqrt{k^2 + \frac{3}{4}} t + c_2 \sin \sqrt{k^2 + \frac{3}{4}} t \right].$$

is the solution of auxilary equation. Since $A_k^p(t) = \frac{p_k}{k^2+1}$ is the solution of nonhomegeonus equation

$$A_k''(t) + A_k'(t) + \left(k^2 + 1\right) A_k(t) = p_k.$$

Therefore,

$$A_k(t) = e^{-\frac{t}{2}} \left[c_1 \cos \sqrt{k^2 + \frac{3}{4}} t + c_2 \sin \sqrt{k^2 + \frac{3}{4}} t \right] + \frac{p_k}{k^2 + 1}.$$

Let $-1 \leq t \leq 0$.Then, the auxilary equation is

$$-q^2 + k^2 + 1 = 0.$$

We have two roots

$$q_1 = \sqrt{k^2 + 1}, q_2 = -\sqrt{k^2 + 1}.$$

Therefore,

$$A_k(t) = c_3 \cosh \sqrt{k^2 + 1} t + c_4 \sinh \sqrt{k^2 + 1} t + \frac{p_k}{k^2 + 1}.$$

Applying boundary conditions $A_k(0) = A_k(\pm 1) = 0$, we get

$$A_k(0) = c_1 + \frac{p_k}{k^2 + 1} = 0,$$
$$A_k(0) = c_3 + \frac{p_k}{k^2 + 1} = 0,$$
$$A_k(1) = e^{-\frac{1}{2}} \left[c_1 \cos \sqrt{k^2 + \frac{3}{4}} + c_2 \sin \sqrt{k^2 + \frac{3}{4}} \right] + \frac{p_k}{k^2 + 1} = 0,$$
$$A_k(-1) = c_3 \cosh \sqrt{k^2 + 1} - c_4 \sinh \sqrt{k^2 + 1} + \frac{p_k}{k^2 + 1} = 0.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{k^2 + 1} c_4 + \frac{1}{2} c_1 - \sqrt{k^2 + \frac{3}{4}} c_2 = 0, \\ e^{-\frac{1}{2}} \left[c_1 \cos \sqrt{k^2 + \frac{3}{4}} + c_2 \sin \sqrt{k^2 + \frac{3}{4}} \right] - c_1 = 0, \\ c_3 \cosh \sqrt{k^2 + 1} - c_4 \sinh \sqrt{k^2 + 1} - c_3 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}-1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{2}-1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & \cosh\sqrt{2}-1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$-(-1)^{1+3} \times \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{7}}{2} & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}-1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 \\ 0 & 0 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \sqrt{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\left(-1+\cosh\sqrt{2}\right)+\frac{1}{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\sinh\sqrt{2}$$

$$-\frac{\sqrt{7}}{2}\left(-1+e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}\right)\sinh\sqrt{2} \neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$.Then $p_k = 0$, and $A_k(t) = 0$.
Now, we obtain $A_1(t)$ and p_1. Let $0 \leq t \leq 1$.Then, the auxilary equation is

$$q^2 + q + 2 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{\frac{7}{4}}, q_2 = -\frac{1}{2} - i\sqrt{\frac{7}{4}}.$$

Therefore,

$$A_1^c(t) = e^{-\frac{t}{2}}\left[c_1\cos\frac{\sqrt{7}}{2}t + c_2\sin\frac{\sqrt{7}}{2}t\right]$$

is the solution of auxilary equation. Since $A_1^p(t) = \frac{p_1-1+2e^{-t}}{2}$ is the solution of nonhomegeonus equation

$$A_1''(t) + A_1'(t) + 2A_1(t) = p_1 + 2e^{-t} - 1,$$

we have that

$$A_1(t) = e^{-\frac{t}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2}t + c_2 \sin\frac{\sqrt{7}}{2}t\right] + \frac{p_1 + 2e^{-t} - 1}{2}.$$

Let $-1 \leq t \leq 0$.Then, the auxilary equation is

$$-q^2 + 2 = 0.$$

We have two roots

$$q_1 = \sqrt{2}, q_2 = -\sqrt{2}.$$

Therefore,

$$A_1(t) = c_3 \cosh\sqrt{2}t + c_4 \sinh\sqrt{2}t + \frac{p_1 + 2e^{-t} - 1}{2}.$$

Applying boundary conditions $A_1(0) = 1, A_1(\pm 1) = e^{\mp 1}, A_1'(0+) = A_1'(0-)$, we get

$$A_1(0) = c_3 + \frac{p_1 + 1}{2} = 1,$$

$$A_1(0) = c_1 + \frac{p_1 + 1}{2} = 1,$$

$$\sqrt{2}c_4 = -\frac{1}{2}c_1 + \frac{\sqrt{7}}{2}c_2,$$

$$A_1(1) = e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2} + c_2 \sin\frac{\sqrt{7}}{2}\right] + \frac{p_1 + 2e^{-1} - 1}{2} = e^{-1},$$

$$A_1(-1) = c_3 \cosh\sqrt{2} - c_4 \sinh\sqrt{2} + \frac{p_1 + 2e^{1} - 1}{2} = e.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{2}c_4 + \frac{1}{2}c_1 - \frac{\sqrt{7}}{2}c_2 = 0, \\ e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2} + c_2 \sin\frac{\sqrt{7}}{2}\right] - c_1 = 0, \\ c_3 \cosh\sqrt{2} - c_4 \sinh\sqrt{2} - c_3 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}-1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{2}-1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{7}}{2} & 0 & \sqrt{2} \\ e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 & 0 \\ 0 & \cosh\sqrt{2}-1 & -\sinh\sqrt{2} \end{vmatrix}$$

$$-(-1)^{1+3}\times \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{7}}{2} & \sqrt{2} \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}-1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2} & 0 \\ 0 & 0 & -\sinh\sqrt{2} \end{vmatrix}$$

$$= \sqrt{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\left(-1+\cosh\sqrt{2}\right)+\frac{1}{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{7}}{2}\sinh\sqrt{2}$$

$$-\frac{\sqrt{7}}{2}\left(-1+e^{-\frac{1}{2}}\cos\frac{\sqrt{7}}{2}\right)\sinh\sqrt{2}\neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$.

From that it follows that $p_1 = 1$ and $A_1(t) = e^{-t}$. Therefore,

$$p(x) = \cos x,$$

$$u(t,x) = A_0(t) + \sum_{k=1}^{\infty} A_k(t)\cos kx = e^{-t}\cos x.$$

So, the exact solution of the problem (1.9) is

$$(u(t,x), p(x)) = (e^{-t}\cos x, \cos x).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$
\begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\
-\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \beta_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + g(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega}, -T < t < 0, \\
u(0,x) = \varphi(x), u(-T,x) = \psi(x), u(T,x) = \mu(x), x \in \overline{\Omega}, \\
\frac{\partial u(t,x)}{\partial \overline{m}} = 0, x \in S, -T \leq t \leq T
\end{cases}
\tag{1.14}
$$

for the multidimensional elliptic-telegraph differential equations.

Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x)$, $\psi(x), \mu(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to boundary S.

However Fourier series method described in solving (1.14) can be used only in the case when (1.14) has constant coefficients.

Third, we consider the Fourier series solution of the following SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + e^{-t} - 1, \\ 0 < x < \pi, 0 < t < 1, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) - 1, \\ 0 < x < \pi, -1 < t < 0, \\ u(0,x) = 1, u(\pm 1, x) = e^{\mp 1}, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), -1 \le t \le 1 \end{cases} \tag{1.15}$$

for a one dimensional elliptic-telegraph equation.

In order to solve the problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi), u_x(0) = u_x(\pi)$$

generated by the space operator of problem (1.15). It is easy to see that the solution of this Sturm-Liouville problem is

$$u_k(x) = \cos 2kx, k = 0, 1, 2, ..., u_k(x) = \sin 2kx, k = 1, 2, ...$$

Then, we will obtain the Fourier series solution of problem (1.15) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) \cos 2kx + \sum_{k=1}^{\infty} B_k(t) \sin 2kx, \tag{1.16}$$

$$p(x) = \sum_{k=0}^{\infty} p_k \cos 2kx + \sum_{k=1}^{\infty} q_k \sin 2kx,$$

where $A_k(t), k = 0, 1, 2, ...,$ and $B_k(t), k = 1, 2, ...$ are unknown functions and $p_k, k = 0, 1, 2...,$ and $q_k, k = 1, 2, ...$ are unknown numbers. Putting $u(t,x)$ and $p(x)$ into main problem and using given initial and boundary conditions, we obtain

$$\sum_{k=0}^{\infty} A_k''(t) \cos 2kx + \sum_{k=1}^{\infty} B_k''(t) \sin 2kx + \sum_{k=0}^{\infty} A_k'(t) \cos 2kx$$

$$+\sum_{k=1}^{\infty} B_k'(t)\sin 2kx + \sum_{k=0}^{\infty}\left(4k^2+1\right)A_k(t)\cos 2kx + \sum_{k=1}^{\infty}\left(4k^2+1\right)B_k(t)\sin 2kx$$

$$= \sum_{k=0}^{\infty} p_k\cos 2kx + \sum_{k=1}^{\infty} q_k\sin 2kx + e^{-t} - 1, 0 < t < 1, 0 < x < \pi,$$

$$-\sum_{k=0}^{\infty} A_k''(t)\cos 2kx - \sum_{k=1}^{\infty} B_k''(t)\sin 2kx + \sum_{k=0}^{\infty}\left(4k^2+1\right)A_k(t)\cos 2kx$$

$$+\sum_{k=1}^{\infty}\left(4k^2+1\right)B_k(t)\sin 2kx$$

$$= \sum_{k=0}^{\infty} p_k\cos 2kx + \sum_{k=1}^{\infty} q_k\sin 2kx - 1, -1 < t < 0, 0 < x < \pi,$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos 2kx + \sum_{k=1}^{\infty} B_k(0)\sin 2kx = 1,$$

$$u(\pm 1,x) = \sum_{k=0}^{\infty} A_k(\pm 1)\cos 2kx + \sum_{k=1}^{\infty} B_k(\pm 1)\sin 2kx = e^{\mp 1}, 0 \le x \le \pi.$$

Equating the coefficients of $\cos kx, k = 0, 1, ...$and $\sin kx, k = 1, 2, ...$ to zero, we get

$$\begin{cases} A_k''(t) + A_k'(t) + \left(4k^2+1\right)A_k(t) = p_k, k \neq 0, 0 < t < 1, \\ -A_k''(t) + \left(4k^2+1\right)A_k(t) = p_k, k \neq 0, -1 < t < 0, \\ A_k(0) = A_k(\pm 1) = 0, k \neq 0, \end{cases} \tag{1.17}$$

$$\begin{cases} A_0''(t) + A_0'(t) + A_0(t) = p_0 + e^{-t} - 1, 0 < t < 1, \\ -A_0''(t) + A_0(t) = p_0 - 1, -1 < t < 0, \\ A_0(0) = 0, A_0(\pm 1) = e^{\mp 1}, \end{cases} \tag{1.18}$$

$$\begin{cases} B_k''(t) + B_k'(t) + \left(4k^2+1\right) B_k(t) = q_k, \ 0 < t < 1, \\ \\ -B_k''(t) + \left(4k^2+1\right) B_k(t) = q_k, -1 < t < 0, \\ \\ B_k(0) = B_k(\pm 1) = 0. \end{cases} \tag{1.19}$$

We obtain $A_k(t), k \neq 0$. It is clear that for $k \neq 0$, $A_k(t)$ the solution of the IVP (1.17). Let $-1 \leq t \leq 0$. Then, the auxilary equation is

$$-q^2 + 4k^2 + 1 = 0.$$

We have two roots

$$q_1 = \sqrt{4k^2+1}, q_2 = -\sqrt{4k^2+1}.$$

Therefore,

$$A_k^c(t) = c_1 \cosh \sqrt{4k^2+1}t + c_2 \sinh \sqrt{4k^2+1}t$$

is the solution of auxilary equation. Since $A_k^p(t) = \frac{p_k}{4k^2+1}$ is the solution of nonhomegeonus equation

$$A_k''(t) + A_k'(t) + \left(4k^2+1\right) A_k(t) = p_k.$$

Therefore,

$$A_k(t) = c_1 \cosh \sqrt{4k^2+1}t + c_2 \sinh \sqrt{4k^2+1}t + \frac{p_k}{4k^2+1}.$$

Let $0 \leq t \leq 1$. Then, the auxilary equation is

$$q^2 + q + 4k^2 + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{4k^2 + \frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{4k^2 + \frac{3}{4}}.$$

Therefore,

$$A_k(t) = e^{-\frac{t}{2}}\left[c_3 \cos\sqrt{4k^2+\frac{3}{4}}t + c_4 \sin\sqrt{4k^2+\frac{3}{4}}t\right] + \frac{p_k}{4k^2+1}.$$

Applying boundary conditions $A_k(0) = A_k(\pm 1) = 0, A_k'(0+) = A_k'(0-)$, we get

$$A_k(0) = c_1 + \frac{p_k}{4k^2+1} = 0,$$

$$A_k(0) = c_3 + \frac{p_k}{4k^2+1} = 0,$$

$$A_k(1) = e^{-\frac{1}{2}}\left[c_3 \cos\sqrt{4k^2+\frac{3}{4}} + c_4 \sin\sqrt{4k^2+\frac{3}{4}}\right] + \frac{p_k}{4k^2+1} = 0,$$

$$A_k(-1) = c_1 \cosh\sqrt{4k^2+1} - c_2 \sinh\sqrt{4k^2+1} + \frac{p_k}{4k^2+1} = 0.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{4k^2+1}c_2 + \frac{1}{2}c_1 - \sqrt{4k^2+\frac{3}{4}}c_4 = 0, \\ e^{-\frac{1}{2}}\left[c_3 \cos\sqrt{4k^2+\frac{3}{4}} + c_4 \sin\sqrt{4k^2+\frac{3}{4}}\right] - c_3 = 0, \\ c_1 \cosh\sqrt{4k^2+1} - c_2 \sinh\sqrt{4k^2+1} - c_1 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\sqrt{4k^2+\frac{3}{4}} & 0 & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{4k^2+1} - 1 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$= \begin{vmatrix} -\sqrt{4k^2+\frac{3}{4}} & 0 & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 & 0 \\ 0 & \cosh\sqrt{4k^2+1} - 1 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$-(-1)^{1+3} \times \begin{vmatrix} \frac{1}{2} & -\sqrt{4k^2+\frac{3}{4}} & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 \\ 0 & 0 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$= \sqrt{4k^2+1}e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}}\left(-1 + \cosh\sqrt{4k^2+1}\right)$$

$$+\frac{1}{2}e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}}\sinh\sqrt{4k^2+1}$$

$$-\sqrt{4k^2+\frac{3}{4}}\left(-1+e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}}\right)\sinh\sqrt{4k^2+1}\neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$. Then $p_k = 0$, and $A_k(t) = 0$ for $k \neq 0$.

Now, we obtain $A_0(t)$ and p_0. Let $0 \leq t \leq 1$. Then, the auxilary equation is

$$q^2 + q + 1 = 0.$$

We have two roots

$$q_1 = -\frac{1}{2} + i\sqrt{\frac{3}{4}}, q_2 = -\frac{1}{2} - i\sqrt{\frac{3}{4}}.$$

Therefore,

$$A_0^c(t) = e^{-\frac{t}{2}}\left[c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right]$$

is the solution of auxilary equation. Since $A_0^p(t) = p_0 + e^{-t} - 1$ is the solution of nonhomegeonus equation

$$-A_0''(t) + A_0'(t) + A_0(t) = p_0 + e^{-t} - 1,$$

we have that

$$A_0(t) = e^{-\frac{t}{2}}\left[c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right] + p_0 + e^{-t} - 1.$$

Let $-1 \leq t \leq 0$. Then, the auxilary equation is

$$-q^2 + 1 = 0.$$

We have two roots

$$q_1 = 1, q_2 = -1.$$

Therefore,

$$A_0(t) = c_3\cosh t + c_4\sinh t + p_0 + e^{-t} - 1.$$

Applying boundary conditions $A_0(0) = 1, A_0(\pm 1) = e^{\mp 1}, A_0'(0+) = A_0'(0-)$, we get

$$A_0(0) = c_3 + p_0 = 1,$$

$$A_0(0) = c_1 + p_0 = 1,$$

$$c_4 = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2,$$

$$A_0(1) = e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2} + c_2 \sin\frac{\sqrt{3}}{2}\right] + p_0 + e^{-1} - 1 = e^{-1},$$

$$A_0(-1) = c_3 \cosh 1 - c_4 \sinh 1 + p_0 + e^1 - 1 = e^1.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ c_4 + \frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2 = 0, \\ e^{-\frac{1}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2} + c_2 \sin\frac{\sqrt{3}}{2}\right] - c_1 = 0, \\ c_3 \cosh 1 - c_4 \sinh 1 - c_3 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 1 \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2} - 1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \cosh 1 - 1 & -\sinh 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\sqrt{3}}{2} & 0 & 1 \\ e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \cosh 1 - 1 & -\sinh 1 \end{vmatrix}$$

$$-(-1)^{1+3} \times \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \\ e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2} - 1 & e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\sinh 1 \end{vmatrix}$$

$$= e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}(-1 + \cosh 1) + \frac{1}{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}\sinh 1$$

$$-\frac{\sqrt{3}}{2}\left(-1 + e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}\right)\sinh 1 \neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$. Then $p_0 = 1$ and $A_0(t) = e^{-t}$. From that it follows that

$$p(x) = 1.$$

Finally, we obtain $B_k(t)$. It is clear that, $B_k(t)$ is the solution of the IVP (1.19).
Let $-1 \leq t \leq 0$. Then, the auxilary equation is

$$-m^2 + 4k^2 + 1 = 0.$$

We have two roots

$$m_1 = \sqrt{4k^2 + 1}, m_2 = -\sqrt{4k^2 + 1}.$$

Therefore,

$$B_k^c(t) = c_1 \cosh \sqrt{4k^2 + 1}t + c_2 \sinh \sqrt{4k^2 + 1}t.$$

is the solution of auxilary equation. Since $B_k^p(t) = \frac{q_k}{4k^2+1}$ is the solution of nonhomegeonus equation

$$B_k''(t) + B_k'(t) + \left(k^2 + 1\right) B_k(t) = q_k.$$

Therefore,

$$B_k(t) = c_1 \cosh \sqrt{4k^2 + 1}t + c_2 \sinh \sqrt{4k^2 + 1}t + \frac{q_k}{4k^2 + 1}.$$

Let $0 \leq t \leq 1$. Then, the auxilary equation is

$$m^2 + m + 4k^2 + 1 = 0.$$

We have two roots

$$m_1 = -\frac{1}{2} + i\sqrt{4k^2 + \frac{3}{4}}, m_2 = -\frac{1}{2} - i\sqrt{4k^2 + \frac{3}{4}}.$$

Therefore,

$$B_k(t) = e^{-\frac{t}{2}} \left[c_3 \cos \sqrt{4k^2 + \frac{3}{4}}t + c_4 \sin \sqrt{4k^2 + \frac{3}{4}}t \right] + \frac{q_k}{4k^2 + 1}.$$

Applying boundary conditions $B_k(0) = B_k(\pm 1) = 0, B_k'(0+) = B_k'(0-)$, we get

$$B_k(0) = c_1 + \frac{q_k}{4k^2+1} = 0,$$

$$B_k(0) = c_3 + \frac{q_k}{4k^2+1} = 0,$$

$$B_k(1) = e^{-\frac{1}{2}}\left[c_3 \cos\sqrt{4k^2+\frac{3}{4}} + c_4 \sin\sqrt{4k^2+\frac{3}{4}}\right] + \frac{q_k}{4k^2+1} = 0,$$

$$B_k(-1) = c_1 \cosh\sqrt{4k^2+1} - c_2 \sinh\sqrt{4k^2+1} + \frac{q_k}{4k^2+1} = 0.$$

Therefore,

$$\begin{cases} c_1 - c_3 = 0, \\ \sqrt{4k^2+1}c_2 + \frac{1}{2}c_1 - \sqrt{4k^2+\frac{3}{4}}c_4 = 0, \\ e^{-\frac{1}{2}}\left[c_3 \cos\sqrt{4k^2+\frac{3}{4}} + c_4 \sin\sqrt{4k^2+\frac{3}{4}}\right] - c_3 = 0, \\ c_1 \cosh\sqrt{4k^2+1} - c_2 \sinh\sqrt{4k^2+1} - c_1 = 0. \end{cases}$$

Since

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{2} & -\sqrt{4k^2+\frac{3}{4}} & 0 & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 & 0 \\ 0 & 0 & \cosh\sqrt{4k^2+1} - 1 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$= \begin{vmatrix} -\sqrt{4k^2+\frac{3}{4}} & 0 & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 & 0 \\ 0 & \cosh\sqrt{4k^2+1} - 1 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$-(-1)^{1+3} \times \begin{vmatrix} \frac{1}{2} & -\sqrt{4k^2+\frac{3}{4}} & \sqrt{4k^2+1} \\ e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}} - 1 & e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}} & 0 \\ 0 & 0 & -\sinh\sqrt{4k^2+1} \end{vmatrix}$$

$$= \sqrt{4k^2+1}e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}}\left(-1+\cosh\sqrt{4k^2+1}\right) + \frac{1}{2}e^{-\frac{1}{2}}\sin\sqrt{4k^2+\frac{3}{4}}\sinh\sqrt{4k^2+1}$$

$$-\sqrt{4k^2+\frac{3}{4}}\left(-1+e^{-\frac{1}{2}}\cos\sqrt{4k^2+\frac{3}{4}}\right)\sinh\sqrt{4k^2+1} \neq 0,$$

we have that $c_1 = c_2 = c_3 = c_4 = 0$. Then $q_k = 0$ and $B_k(t) = 0$, for all k.

Therefore,

$$u(t,x) = A_0(t) + \sum_{k=1}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx = e^{-t}.$$

So, the exact solution of the problem (1.15) is

$$(u(t,x), p(x)) = (e^{-t}, 1).$$

Note that using similar procedure one can obtain the solution of the following identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \beta_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, -T < t < 0, \\ u(0,x) = \varphi(x), u(-T,x) = \psi(x), u(T,x) = \mu(x), x \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \frac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u(t,x)}{\partial \overline{m}}\Big|_{S_2}, -T \leq t \leq T \end{cases} \tag{1.20}$$

for the multidimensional elliptic-telegraph differential equations.

Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x)$, $\psi(x), \mu(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions.

Here and in future $S = S_1 \cup S_2, S_1 \cap S_2 = \varnothing$. However Fourier series method described in solving (1.20) can be used only in the case when (1.20) has constant coefficients.

Now, we consider Laplace transform solution of SIPs for partial elliptic-telegraph differential equations.

Fourth, we consider the Laplace transform solution of the following SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(x) - e^{-t-x} - e^{-x}, \\ 0 < x < \infty, 0 < t < 1, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(x) - 2e^{-t-x} - e^{-x}, \\ 0 < x < \infty, -1 < t < 0, \\ u(0,x) = e^{-x}, u(\pm 1, x) = e^{\mp 1 - x}, 0 \le x < \infty, \\ u(t,0) = e^{-t}, u_x(t,0) = -e^{-t}, -1 \le t \le 1 \end{cases} \tag{1.21}$$

for a one dimensional elliptic-telegraph equation.

Here and in future, we will denote

$$\mathcal{L}\{u(t,x)\} = u(t,s).$$

Using formula

$$\mathcal{L}\{e^{-x}\} = \frac{1}{s+1} \tag{1.22}$$

and taking the Laplace transform of both sides of the differential equation and using conditions

$$u(t,0) = e^{-t},\ u_x(t,0) = -e^{-t},$$

we can write

$$\mathcal{L}\{u_{tt}(t,x)\} + \mathcal{L}\{u_t(t,x)\} - \mathcal{L}\{u_{xx}(t,x)\}$$

$$= \mathcal{L}\{p(x)\} - e^{-t}\mathcal{L}\{e^{-x}\} - \mathcal{L}\{e^{-x}\}, 0 < t < 1,$$

$$-\mathcal{L}\{u_{tt}(t,x)\} - \mathcal{L}\{u_{xx}(t,x)\}$$
$$= \mathcal{L}\{p(x)\} - 2e^{-t}\mathcal{L}\{e^{-x}\} - \mathcal{L}\{e^{-x}\}, -1 < t < 0,$$
$$\mathcal{L}\{u(0,x)\} = \mathcal{L}\{e^{-x}\}, \mathcal{L}\{u(\pm 1,x)\} = \mathcal{L}\left\{e^{\mp 1-x}\right\}$$

or

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) - s^2u(t,s) + su(t,0) + u_x(t,0) \\ = p(s) - e^{-t}\frac{1}{s+1} - \frac{1}{s+1}, 0 < t < 1, \\ -u_{tt}(t,s) - s^2u(t,s) + su(t,0) + u_x(t,0) \\ = p(s) - 2e^{-t}\frac{1}{s+1} - \frac{1}{s+1}, -1 < t < 0, \\ u(0,s) = \frac{1}{1+s}, u(\pm 1,s) = \frac{e^{\mp 1}}{1+s}. \end{cases}$$

Therefore, we get the following problem

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) - s^2u(t,s) + se^{-t} - e^{-t} \\ = p(s) - e^{-t}\frac{1}{s+1} - \frac{1}{s+1}, 0 < t < 1, \\ -u_{tt}(t,s) - s^2u(t,s) + se^{-t} - e^{-t} \\ = p(s) - 2e^{-t}\frac{1}{s+1} - \frac{1}{s+1}, -1 < t < 0, \\ u(0,s) = \frac{1}{1+s} \quad , u(\pm 1,s) = \frac{e^{\mp 1}}{1+s}. \end{cases} \tag{1.23}$$

Now we will obtain the solution of problem (1.23). Let $-1 \leq t \leq 0$. Then, we have the following BVP

$$\begin{cases} u_{tt}(t,s) + s^2u(t,s) = \frac{s^2+1}{s+1}e^{-t} - p(s) + \frac{1}{s+1}, -1 < t < 0, \\ u(0,s) = \frac{1}{1+s} \quad , u(-1,s) = \frac{e}{1+s}. \end{cases} \tag{1.24}$$

Applying the D'Alembert's formula, we obtain

$$u(t,s) = u(0,s)\cos st + \frac{1}{s}u_t(0,s)\sin st$$

$$+\frac{1}{s}\int_0^t \sin s(t-y)\frac{s^2+1}{s+1}e^{-y}dy + \frac{1}{s}\int_0^t \sin s(t-y)\left[\frac{1}{s+1} - p(s)\right]dy.$$

Applying the formula

$$\int_0^t \sin s(t-y)e^{-y}dy = \frac{1}{s^2+1}\left(-s\cos st + se^{-t} + \sin st\right)$$

and condition $u(0,s) = \frac{1}{s+1}$, we get

$$u(t,s) = \frac{1}{s}u_t(0,s)\sin st + \frac{1}{s}\frac{(se^{-t} + \sin st)}{s+1} \tag{1.25}$$

$$+\frac{1}{s}\int_0^t \sin s(t-y)\left[\frac{1}{s+1} - p(s)\right]dy.$$

Now, we will apply the condition $u(-1,s) = \frac{e}{1+s}$ and (1.25), we get

$$u(-1,s) = -\frac{1}{s}u_t(0,s)\sin s + \frac{1}{s}\frac{(se - \sin s)}{s+1}$$

$$-\frac{1}{s}\int_0^{-1} \sin s(1+y)\left(\frac{1}{s+1} - p(s)\right)dy = \frac{e}{1+s}.$$

Applying the formula

$$\int_0^{-1} \sin s\,(1+y)\,dy = \frac{\cos s - 1}{s}.$$

We get

$$u(-1,s) = -\frac{1}{s}u_t(0,s)\sin s + \frac{1}{s}\frac{(se - \sin s)}{s+1}$$

$$-\frac{1}{s}\frac{\cos s - 1}{s}\left(\frac{1}{s+1} - p(s)\right) = \frac{e}{1+s}.$$

Therefore

$$u_t(0,s) = -\frac{1}{s+1} - \left(\frac{\cos s - 1}{s\sin s}\right)\left(\frac{1}{s+1} - p(s)\right) \tag{1.26}$$

Applying the formula

$$\int_0^t \sin s\,(t-y)\,dy = \frac{1-\cos st}{s}.$$

Therefore, from (1.25) it follows that

$$u(t,s) = \frac{e^{-t}}{s+1} - \left[\frac{\sin st}{s\sin s}\left(\frac{\cos s - 1}{s}\right) + \frac{1-\cos st}{s^2}\right]\left(\frac{1}{s+1} - p(s)\right) \tag{1.27}$$

Now, let $0 \le t \le 1$. Applying (1.23), then we have the following BVP

$$\begin{cases} u_{tt}\,(t,s) + u_t\,(t,x) - s^2 u\,(t,s) = p\,(s) - \frac{1}{s+1} - e^{-t}\frac{s^2}{s+1}, 0 < t < 1, \\ u(0,s) = \frac{1}{s+1}, u(1,s) = \frac{e^{-1}}{s+1} \\ u_t(0,s) = -\frac{1}{s+1} - \left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1} - p(s)\right). \end{cases} \tag{1.28}$$

We denote

$$u(t,s) = v(t,s)e^{-\frac{t}{2}}. \tag{1.29}$$

Then

$$u_t\,(t,s) = -\frac{1}{2}e^{-\frac{t}{2}}v\,(t,s) + e^{-\frac{t}{2}}v_t\,(t,s)\,,$$

$$u_{tt}\ \,(t,s) = \frac{1}{4}e^{-\frac{t}{2}}v\,(t,s) - e^{-\frac{t}{2}}v_t\,(t,s) + e^{-\frac{t}{2}}v_{tt}\,(t,s)\,.$$

Using these formulas and (1.23), we get the following problem

$$\begin{cases} v_{tt}\,(t,s) - (\frac{1}{4} + s^2)v\,(t,s) = e^{\frac{t}{2}}\left(p\,(s) - \frac{1}{s+1}\right) - e^{-\frac{t}{2}}\frac{s^2}{s+1}, 0 < t < 1, \\ v(0,s) = \frac{1}{s+1}, v\,(1,s) = \frac{e^{-\frac{1}{2}}}{1+s}, \\ v_t(0,s) = \left(-\frac{1}{2(s+1)} - \left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1} - p(s)\right)\right). \end{cases}$$

Now, applying the D'Alembert's formula, we obtain

$$v(t,s) = \frac{1}{s+1}\cosh\sqrt{\frac{1}{4}+s^2 t} + \frac{1}{\sqrt{\frac{1}{4}+s^2}}\sinh\sqrt{\frac{1}{4}+s^2 t}$$

$$\times\left(-\frac{1}{2(s+1)}-\left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1}-p(s)\right)\right)$$

$$+\frac{1}{\sqrt{\frac{1}{4}+s^2}}\int_0^t \sinh\sqrt{\frac{1}{4}+s^2}\,(t-y)\left[e^{\frac{y}{2}}\left(p\,(s)-\frac{1}{s+1}\right)-\frac{s^2}{s+1}e^{-\frac{y}{2}}\right]dy. \tag{1.30}$$

Applying (1.26) and applying the formulas

$$\int_0^t \sinh\sqrt{\frac{1}{4}+s^2}\,(t-y)\,e^{\frac{y}{2}}dy$$

$$=\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s^2},$$

$$\int_0^t \sinh\sqrt{\frac{1}{4}+s^2}\,(t-y)\,e^{-\frac{y}{2}}dy \tag{1.31}$$

$$=\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{-\frac{t}{2}}\right)-\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s^2},$$

we get

$$v(t,s)=\frac{1}{s+1}\cosh\sqrt{\frac{1}{4}+s^2}t-\left(\frac{1}{2(s+1)}\right)\left(\frac{1}{\sqrt{\frac{1}{4}+s^2}}\sinh\sqrt{\frac{1}{4}+s^2t}\right)$$

$$+\left(\frac{1}{\sqrt{\frac{1}{4}+s^2}}\sinh\sqrt{\frac{1}{4}+s^2t}\right)\left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1}-p(s)\right)$$

$$-\left(\frac{1}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{-\frac{t}{2}}\right)-\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s+1}\right)$$

$$+\left(p\,(s)-\frac{1}{s+1}\right)\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s^2\sqrt{\frac{1}{4}+s^2}}\right).$$

Therefore

$$v(t,s)=\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}t}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1}-p(s)\right)+\frac{e^{-\frac{t}{2}}}{s+1}$$

$$+\left(p\left(s\right)-\frac{1}{s+1}\right)\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s^2\sqrt{\frac{1}{4}+s^2}}\right). \tag{1.32}$$

Applying the condition $v\left(1,s\right)=\frac{e^{-\frac{1}{2}}}{1+s}$, and applying (1.32), we get

$$\frac{e^{-\frac{1}{2}}}{1+s}=\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1}-p(s)\right)+\frac{e^{-\frac{1}{2}}}{s+1}$$

$$+\left(p\left(s\right)-\frac{1}{s+1}\right)\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}-e^{\frac{1}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right).$$

We have that

$$\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)\left(\frac{1}{s+1}-p(s)\right)$$

$$+\left(p\left(s\right)-\frac{1}{s+1}\right)\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}-e^{\frac{1}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right)=0.$$

Since

$$\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)$$

$$+\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}-e^{\frac{1}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right)\neq 0,$$

it follows that

$$p(s)=\frac{1}{s+1}, \tag{1.33}$$

and from the formula (1.26), we get

$$u_t(0,s)=-\frac{1}{s+1}.$$

Therefore, since we are taking the inverse of laplace transform, we have

$$p(x) = e^{-x}, u_t(0, s) = -e^{-x}, -1 \leq t \leq 1. \tag{1.34}$$

Now, to obtain $u(t, s)$, let $-1 \leq t \leq 0$. Using (1.26) and (1.27), and since $p(s) = \frac{1}{s+1}$, we have

$$u(t, s) = \frac{e^{-t}}{s+1}, -1 < t < 0, \tag{1.35}$$

Let $0 \leq t \leq 1$. Using (1.32) and since $p(s) = \frac{1}{s+1}$, we have

$$v(t, s) = \frac{e^{-\frac{t}{2}}}{s+1}, 0 < t < 1.$$

Therefore, from formula (1.29), it follows that

$$u(t, s) = v(t, s)e^{-\frac{t}{2}} = \frac{e^{-t}}{s+1}, 0 < t < 1. \tag{1.36}$$

Therefore, from formulas (1.35) and (1.36), we have

$$u(t, s) = \frac{e^{-t}}{s+1}, -1 \leq t \leq 1. \tag{1.37}$$

Using formula (1.37) and taking the inverse Laplace transform with respect to x, we get

$$u(t, x) = e^{-t-x}, -1 \leq t \leq 1.$$

Thus, the exact solution of problem (1.21) is

$$(u(t, x), p(x)) = \left(e^{-t-x}, e^{-x}\right).$$

Note that using similar procedure one can obtain the solution of the following identification problem for a one dimensional elliptic-telegraph equation.

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ -T < t < 0, \\ u(0,x) = \varphi(x), u(-T,x) = \psi(x), u(T,x) = \mu(x), x \in \overline{\Omega}^+, \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta(t,x), \\ 1 \le r \le n, -T \le t \le T, x \in S^+ \end{cases} \tag{1.38}$$

for the multidimensional elliptic-telegraph equations. Assume that $\alpha_r > \alpha > 0$ and $f(t,x)\left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \mu(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (1.38) can be used only in the case when (1.38) has $a_r(x)$ polynomials coefficients.

Fifth, we consider the Laplace transform solution of the following SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p\left(x\right) - 2e^{-x}, \\ 0 < x < \infty, 0 < t < 1, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p\left(x\right) - 2e^{-x}, \\ 0 < x < \infty, -1 < t < 0, \\ u\left(0, x\right) = e^{-x}, u_t\left(0, x\right) = 0, u\left(\pm 1, x\right) = e^{-x}, 0 \leq x < \infty, \\ u\left(t, 0\right) = 1, u\left(t, \infty\right) = 0, -1 \leq t \leq 1 \end{cases} \tag{1.39}$$

for a one dimensional elliptic-telegraph equation.

Using formula (1.22) and conditions $u\left(t, 0\right) = 1$, and taking the Laplace transform of both sides of the differential equation and initial conditions, we can write

$$\mathcal{L}\left\{u_{tt}\left(t, x\right)\right\} + \mathcal{L}\left\{u_t\left(t, x\right)\right\} - \mathcal{L}\left\{u_{xx}\left(t, x\right)\right\} = p\left(s\right) - 2\mathcal{L}\left\{e^{-x}\right\}, 0 < t < 1,$$

$$-\mathcal{L}\left\{u_{tt}\left(t, x\right)\right\} - \mathcal{L}\left\{u_{xx}\left(t, x\right)\right\} = p\left(s\right) - 2\mathcal{L}\left\{e^{-x}\right\}, -1 < t < 0,$$

$$\mathcal{L}\left\{u\left(0, x\right)\right\} = \mathcal{L}\left\{u\left(\pm 1, x\right)\right\} = \mathcal{L}\left\{e^{-x}\right\}, \mathcal{L}\left\{u_t\left(0, x\right)\right\} = 0.$$

or

$$\begin{cases} u_{tt}\left(t, s\right) + u_t\left(t, s\right) - s^2 u\left(t, s\right) + s + u_x\left(t, 0\right) = p\left(s\right) - 2\frac{1}{1+s}, 0 < t < 1, \\ -u_{tt}\left(t, s\right) - s^2 u\left(t, s\right) + s + u_x\left(t, 0\right) = p\left(s\right) - 2\frac{1}{1+s}, -1 < t < 0, \\ u\left(0, s\right) = \frac{1}{1+s}, u_t\left(0, s\right) = 0, u\left(\pm 1, s\right) = \frac{1}{1+s}. \end{cases} \tag{1.40}$$

We denote that

$$u_x\left(t, 0\right) = \beta(t). \tag{1.41}$$

Then, taking the Laplace transform with respect to t, we get

$$\begin{cases} \omega^2 u(\omega, s) - \omega\frac{1}{s+1} - u_t(0+, s) + \omega u(\omega, s) - \frac{1}{s+1} - s^2 u(\omega, s) \\ = \left(p(s) - \frac{s^2+s+2}{s+1}\right)\frac{1}{\omega} - \beta(\omega), 0 < t < 1, \\ -\omega^2 u(\omega, s) + \omega\frac{1}{s+1} + u_t(0-, s) - s^2 u(\omega, s) \\ = \left(p(s) - \frac{s^2+s+2}{s+1}\right)\frac{1}{\omega} - \beta(\omega), -1 < t < 0, \\ u(\pm 1, s) = \frac{1}{1+s}. \end{cases}$$

We denote that

$$u_t(0+, s) = u_t(0-, s) = \alpha(s).$$

Then

$$\begin{cases} \omega^2 u(\omega, s) - \omega\frac{1}{s+1} - \alpha(s) + \omega u(\omega, s) - \frac{1}{s+1} - s^2 u(\omega, s) \\ = \left(p(s) - \frac{s^2+s+2}{s+1}\right)\frac{1}{\omega} - \beta(\omega), 0 < t < 1, \\ -\omega^2 u(\omega, s) + \omega\frac{1}{s+1} + \alpha(s) - s^2 u(\omega, s) \\ = \left(p(s) - \frac{s^2+s+2}{s+1}\right)\frac{1}{\omega} - \beta(\omega), -1 < t < 0, \\ u(\pm 1, s) = \frac{1}{1+s}. \end{cases} \tag{1.42}$$

From that it follows

$$\begin{cases} \left(\omega^2 + \omega - s^2\right) u(\omega, s) \\ = \frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) + \alpha(s) - \beta(\omega) - \frac{-\omega^2-\omega+s^2+s+1}{(s+1)\omega}, 0 < t < 1, \\ \left(-\omega^2 - s^2\right) u(\omega, s) \\ = \frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) - \alpha(s) - \beta(\omega) - \frac{\omega^2+s^2+s+1}{(s+1)\omega}, -1 < t < 0, \\ u\left(\pm 1, s\right) = \frac{1}{1+s}. \end{cases}$$

or

$$\begin{cases} u(\omega, s) = \frac{1}{\omega^2+\omega-s^2}\left(\frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) + \alpha(s) - \beta(\omega) - \frac{-\omega^2-\omega+s^2+s+1}{(s+1)\omega}\right), 0 < t < 1, \\ u(\omega, s) = \frac{1}{-\omega^2-s^2}\left(\frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) - \alpha(s) - \beta(\omega) - \frac{\omega^2+s^2+s+1}{(s+1)\omega}\right), -1 < t < 0, \\ u\left(\pm 1, s\right) = \frac{1}{1+s}. \end{cases}$$

We have that

$$\begin{cases} u(\omega, s) = \frac{1}{2\sqrt{\omega^2+\omega}}\left(\frac{1}{s+\sqrt{\omega^2+\omega}} - \frac{1}{s-\sqrt{\omega^2+\omega}}\right) \\ \times\left(\frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) + \alpha(s) - \beta(\omega)\right) \\ +\frac{1}{(s+1)\omega} - \frac{1}{2\omega\sqrt{\omega^2+\omega}}\left(\frac{1}{s+\sqrt{\omega^2+\omega}} - \frac{1}{s-\sqrt{\omega^2+\omega}}\right), 0 < t < 1, \\ u(\omega, s) = -\frac{1}{\omega^2+s^2}\left(\frac{1}{\omega}\left(p\left(s\right) - \frac{1}{s+1}\right) - \alpha(s) - \beta(\omega)\right) \\ +\frac{1}{(s+1)\omega} + \frac{1}{\omega\left(\omega^2+s^2\right)}, -1 < t < 0, \\ u\left(\pm 1, s\right) = \frac{1}{1+s}. \end{cases} \tag{1.43}$$

Taking inverse of Laplace transform with respect to x in (1.43), we get

$$\begin{cases} u(\omega, x) = \frac{\beta(\omega)}{2\sqrt{\omega^2+\omega}} \sinh \sqrt{\omega^2+\omega} x \\ -\frac{1}{2\omega\sqrt{\omega^2+\omega}} \int_0^x \sinh \sqrt{\omega^2+\omega}\,(x-y)\,(p(y)-e^{-y})\,dy \\ +\frac{1}{2\sqrt{\omega^2+\omega}} \int_0^x \sinh \sqrt{\omega^2+\omega}\,(x-y)\,\alpha(y) dy \\ +\frac{1}{2\omega\sqrt{\omega^2+\omega}} \int_0^x \sinh \sqrt{\omega^2+\omega}\,(x-y)\,dy + \frac{1}{\omega}e^{-x}, 0 < t < 1, \\ u(\omega, x) = \frac{\beta(\omega)}{2\omega} \sin \omega x - \frac{1}{2\omega^2} \int_0^x \sin \omega\,(x-y)\,(p(y)-e^{-y})\,dy \\ +\frac{1}{2\omega} \int_0^x \sin \omega\,(x-y)\,\alpha(y) dy \\ +\frac{1}{2\omega^2} \int_0^x \sin \omega\,(x-y)\,dy + \frac{1}{\omega}e^{-x}, -1 < t < 0, \\ u\,(\pm 1, s) = \frac{1}{1+s}. \end{cases}$$

Passing limit when $x \to \infty$ and applying the condition

$$u(\omega, \infty) = 0,$$

we obtain

$$\begin{cases} \frac{\beta(\omega)}{2\sqrt{\omega^2+\omega}} \sinh\, \sqrt{\omega^2+\omega} x - \frac{1}{2\omega\sqrt{\omega^2+\omega}} \int_0^x \sinh\, \sqrt{\omega^2+\omega}\,(x-y)\,(p(y)-e^{-y})\,dy \\ \\ +\frac{1}{2\sqrt{\omega^2+\omega}} \int_0^x \sinh\, \sqrt{\omega^2+\omega}\,(x-y)\,\alpha(y)dy \\ \\ +\frac{1}{2\omega\sqrt{\omega^2+\omega}} \int_0^x \sinh\, \sqrt{\omega^2+\omega}\,(x-y)\,dy = 0, 0 < t < 1, \\ \\ \frac{\beta(\omega)}{2\omega} \sin\omega x - \frac{1}{2\omega^2} \int_0^x \sin\omega\,(x-y)\,(p(y)-e^{-y})\,dy \\ \\ +\frac{1}{2\omega} \int_0^x \sin\omega\,(x-y)\,\alpha(y)dy + \frac{1}{2\omega^2} \int_0^x \sin\omega\,(x-y)\,dy = 0, -1 < t < 0, \\ \\ u\,(\pm 1, s) = \frac{1}{1+s} \end{cases} \tag{1.44}$$

and

$$\begin{cases} u(\omega, x) = \frac{1}{\omega} e^{-x}, 0 < t < 1, \\ \\ u(\omega, x) = \frac{1}{\omega} e^{-x}, -1 < t < 0. \end{cases}$$

Taking inverse of Laplace transform with respect to t, we get

$$u(t, x) = e^{-x}, -1 \leq t \leq 1.$$

Now, from (1.44) taking the dirivative with respect to y, we obtain

$$-\frac{1}{2\omega^2} \sin\omega\,(x-y)\,(p(x)-e^{-x})$$

$$+\frac{1}{2\omega} \sin\omega\,(x-y)\,\alpha(x) + \frac{1}{2\omega^2} \sin\omega\,(x-y) = 0,$$

From that follows $\alpha(x) = 0$,$\beta(t) = -1$ and $\beta(\omega) = -\frac{1}{\omega}$ and using (1.43), we get

$$p\,(x) = e^{-x}, 0 \leq x < \infty.$$

Therefore, the exact solution of problem (1.42) is

$$(u\,(t, x)\,, p\,(x)) = (e^{-x}, e^{-x})\,.$$

Note that using similar procedure one can obtain the solution of the following identification problem for a one dimensional elliptic-telegraph equation.

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^{+}, \ 0 < t < T, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = p(x) + g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^{+}, \ -T < t < 0, \\ u(0,x) = \varphi(x), u(-T,x) = \psi(x), u(T,x) = \mu(x), x \in \overline{\Omega}^{+}, \\ u(t,x) = \alpha(t,x), u(t,\infty) = 0, -T \le t \le T, x \in S^{+} \end{cases} \tag{1.45}$$

for the multidimensional elliptic-telegraph equations. Assume that $\alpha_r > \alpha > 0$ and $f(t,x) \left(t \in (0,T), x \in \overline{\Omega}\right), g(t,x), \left(t \in (-T,0), x \in \overline{\Omega}\right), \varphi(x), \psi(x), \mu(x), \left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω^{+} is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^{+} and

$$\overline{\Omega}^{+} = \Omega^{+} \cup S^{+}.$$

However Laplace transform method described in solving (1.45) can be used only in the case when (1.45) has $a_r(x)$ polynomials coefficients.

Now, we consider Fourier transform solution of identification problems for elliptic-telegraph differential equations.

Sixth, we consider the Fourier trasform solution of the following problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(x) - e^{-x^2} + \left(-4x^2+2\right) e^{-t-x^2}, \\ 0 < t < 1, x \in \mathbb{R}^1, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = p(x) - e^{-x^2} + \left(-4x^2+1\right) e^{-t-x^2}, \\ -1 < t < 0, x \in \mathbb{R}^1, \\ u(0,x) = e^{-x^2},\ u(\pm 1, x) = e^{\mp 1 - x^2}, x \in \mathbb{R}^1 \end{cases} \tag{1.46}$$

for a one dimensional elliptic-telegraph differential equation.

We denote

$$\mathcal{F}\{u(t,x)\} = u(t,s).$$

Taking the Fourier transform of both sides of the differential equation (1.46) and boundary conditions, we can obtain

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} \\ -e^{-t}\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\left(e^{-x^2}\right)\right\}, 0 < t < 1, \\ -u_{tt}(t,s) + s^2 u(t,s) = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} \\ +e^{-t}\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\left(-e^{-x^2}\right)\right\} - e^{-t}\mathcal{F}\left\{e^{-x^2}\right\}, -1 < t < 0, \\ u(0,s) = \mathcal{F}\left\{e^{-x^2}\right\}, u(\pm 1, s) = \mathcal{F}\left\{e^{\mp 1 - x^2}\right\}. \end{cases} \tag{1.47}$$

Applying the formula

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}\left(e^{-x^2}\right)\right\} = -s^2 \mathcal{F}\left\{e^{-x^2}\right\},$$

we get

$$\begin{cases} u_{tt}(t,s) + u_t(t,s) + s^2 u(t,s) \\ = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} + s^2 e^{-t} \mathcal{F}\left\{e^{-x^2}\right\}, 0 < t < 1, \\ -u_{tt}(t,s) + s^2 u(t,s) \\ = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} + e^{-t}\left(s^2 - 1\right)\mathcal{F}\left\{e^{-x^2}\right\}, -1 < t < 0, \\ u(0,s) = \mathcal{F}\left\{e^{-x^2}\right\}, u(\pm 1, s) = e^{\mp 1}\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases} \tag{1.48}$$

Let $-1 \leq t \leq 0$. We consider the identification problem

$$\begin{cases} -u_{tt}(t,s) + s^2 u(t,s) \\ = p(s) - \mathcal{F}\left\{e^{-x^2}\right\} + e^{-t}\left(s^2 - 1\right)\mathcal{F}\left\{e^{-x^2}\right\}, -1 < t < 0, \\ u(0,s) = \mathcal{F}\left\{e^{-x^2}\right\}, u(-1, s) = e\mathcal{F}\left\{e^{-x^2}\right\}. \end{cases}$$

Applying Dalambert's formula, we get

$$u(t,s) = \cosh(st)\, u(0,s) + \frac{1}{s}\sinh(st)\, u_t(0,s)$$
$$-\frac{1}{s}\int_0^t \sinh(s(t-y))\left(p(s) - \mathcal{F}\left\{e^{-x^2}\right\} + e^{-t}\left(s^2 - 1\right)\mathcal{F}\left\{e^{-x^2}\right\}\right) dy$$
$$= \cosh(st)\,\mathcal{F}\left\{e^{-x^2}\right\} + \frac{1}{s}\sinh(st)\, u_t(0,s)$$
$$-\frac{1}{s}\int_0^t \sinh(s(t-y))\left(p(s) - \mathcal{F}\left\{e^{-x^2}\right\}\right) dy$$
$$-\frac{1}{s}\int_0^t \sinh(s(t-y))\left(e^{-y}\left(s^2 - 1\right)\right) dy\mathcal{F}\left\{e^{-x^2}\right\}.$$

Applying formulas

$$\mathcal{F}\left\{e^{-x^2}\right\} = \sqrt{\pi} e^{-\frac{s^2}{4}}, \tag{1.49}$$

$$\int_0^t \sinh s\,(t-y)\, e^{-y} dy = \frac{1}{s^2-1}\left(s\cosh st - se^{-t} + \sinh st\right),$$

$$\int_0^t \sinh\left(s\,(t-y)\right) dy = \frac{\cosh st - 1}{s},$$

we get

$$u\,(t,s) = \cosh\,(st)\ \sqrt{\pi}e^{-\frac{s^2}{4}} + \frac{1}{s}\sinh\,(st)\, u_t\,(0,s)$$

$$-\frac{1}{s}\frac{\cosh st - 1}{s}\left(p\,(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$-\frac{1}{s}\left(s^2-1\right)\frac{1}{s^2-1}\left(s\cosh st - se^{-t} + \sinh st\right)\sqrt{\pi}e^{-\frac{s^2}{4}}.$$

Therefore

$$u\,(t,s) = \frac{\sinh st}{s}\left(u_t\,(0,s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$-\left(\frac{\cosh st - 1}{s^2}\right)\left(p\,(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right) + e^{-t}\sqrt{\pi}e^{-\frac{s^2}{4}}. \qquad (1.50)$$

Therefore, using condition $u\,(-1,s) = e\mathcal{F}\left\{e^{-x^2}\right\}$, we get

$$e\sqrt{\pi}e^{-\frac{s^2}{4}} = \frac{\sinh s}{s}\left(u_t\,(0,s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$-\frac{\cosh s - 1}{s^2}\left(p\,(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right) + e\sqrt{\pi}e^{-\frac{s^2}{4}}.$$

From that it follows

$$u_t\,(0,s) = \sqrt{\pi}e^{-\frac{s^2}{4}}$$

$$-\left(\frac{s}{\sinh s}\right)\left(\frac{\cosh s - 1}{s^2}\right)\left(p\,(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right). \qquad (1.51)$$

Applying formulas (1.50) and (1.51), we get

$$u\,(t,s) = -\left(\frac{\sinh st}{\sinh s}\right)\left(\frac{\cosh s - 1}{s^2}\right)\left(p\,(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$-\left(\frac{\cosh st-1}{s^2}\right)\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right)+e^{-t}\sqrt{\pi}e^{-\frac{s^2}{4}}. \tag{1.52}$$

Now, let $0 \leq t \leq 1$. Applying (1.48), we have get following identification problem

$$\begin{cases} u_{tt}(t,s)+u_t(t,s)+s^2u(t,s) \\ =p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}+s^2e^{-t}\sqrt{\pi}e^{-\frac{s^2}{4}}, 0<t<1, \\ u(0,s)=\sqrt{\pi}e^{-\frac{s^2}{4}}, u(1,s)=e^{-1}\sqrt{\pi}e^{-\frac{s^2}{4}}. \end{cases} \tag{1.53}$$

We denote

$$u(t,s)=v(t,s)e^{-\frac{t}{2}}. \tag{1.54}$$

Then

$$u_t(t,s)=-\frac{1}{2}e^{-\frac{t}{2}}v(t,s)+e^{-\frac{t}{2}}v_t(t,s),$$

$$u_{tt}(t,s)=\frac{1}{4}e^{-\frac{t}{2}}v(t,s)-e^{-\frac{t}{2}}v_t(t,s)+e^{-\frac{t}{2}}v_{tt}(t,s).$$

Using these formulas and (1.48), we get the following problem

$$\begin{cases} v_{tt}(t,s)-(\frac{1}{4}+s^2)v(t,s)=e^{\frac{t}{2}}\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right) \\ +e^{-\frac{t}{2}}s^2\sqrt{\pi}e^{-\frac{s^2}{4}}, 0<t<1, \\ v(0,s)=\sqrt{\pi}e^{-\frac{s^2}{4}}, v(1,s)=e^{-\frac{1}{2}}\sqrt{\pi}e^{-\frac{s^2}{4}}, \\ v_t(0,s)=u_t(0,s)+\frac{1}{2}\sqrt{\pi}e^{-\frac{s^2}{4}}. \end{cases}$$

It is easy to see that

$$v_t(0,s)=-\frac{1}{2}\sqrt{\pi}e^{-\frac{s^2}{4}}-\left(\frac{s}{\sinh s}\right)\left(\frac{\cosh s-1}{s^2}\right)\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right). \tag{1.55}$$

Then, applying the D'Alembert's formula, we obtain

$$v(t,s) = \sqrt{\pi}e^{-\frac{s^2}{4}} \cosh \sqrt{\frac{1}{4}+s^2}t + \frac{1}{\sqrt{\frac{1}{4}+s^2}} \sinh \sqrt{\frac{1}{4}+s^2}t v_t(0,s) \tag{1.56}$$

$$+\frac{1}{\sqrt{\frac{1}{4}+s^2}} \int_0^t \sinh \sqrt{\frac{1}{4}+s^2}\,(t-y)$$

$$\times\left[e^{\frac{y}{2}}\left(p(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right) - s^2\sqrt{\pi}e^{-\frac{s^2}{4}}e^{-\frac{y}{2}}\right]dy.$$

Applying (1.55) and formulas

$$\int_0^t \sinh \sqrt{\frac{1}{4}+s^2}\,(t-y)\, e^{\frac{y}{2}} dy$$

$$= \frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh \sqrt{\frac{1}{4}+s^2}t - e^{\frac{t}{2}}\right) + \frac{1}{2}\sinh \sqrt{\frac{1}{4}+s^2}t}{s^2},$$

$$\int_0^t \sinh \sqrt{\frac{1}{4}+s^2}\,(t-y)\, e^{-\frac{y}{2}} dy$$

$$= \frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh \sqrt{\frac{1}{4}+s^2}t - e^{-\frac{t}{2}}\right) - \frac{1}{2}\sinh \sqrt{\frac{1}{4}+s^2}t}{s^2},$$

we get

$$v(t,s) = \sqrt{\pi}e^{-\frac{s^2}{4}} \cosh \sqrt{\frac{1}{4}+s^2}t$$

$$+\frac{\sinh \sqrt{\frac{1}{4}+s^2}t}{\sqrt{\frac{1}{4}+s^2}}\left(-\frac{1}{2}\sqrt{\pi}e^{-\frac{s^2}{4}} - \left(\frac{s}{\sinh s}\right)\left(\frac{\cosh s - 1}{s^2}\right)\left(p(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)\right)$$

$$+\frac{\left(p(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)}{\sqrt{\frac{1}{4}+s^2}}\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh \sqrt{\frac{1}{4}+s^2}t - e^{\frac{t}{2}}\right) + \frac{1}{2}\sinh \sqrt{\frac{1}{4}+s^2}t}{s^2}\right)$$

$$-\frac{\left(s^2\sqrt{\pi}e^{-\frac{s^2}{4}}\right)}{\sqrt{\frac{1}{4}+s^2}}\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh \sqrt{\frac{1}{4}+s^2}t - e^{-\frac{t}{2}}\right) - \frac{1}{2}\sinh \sqrt{\frac{1}{4}+s^2}t}{s^2}\right).$$

Then

$$v(t,s) = -\left(\frac{\sinh \sqrt{\frac{1}{4}+s^2}t}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cosh s - 1}{s \sinh s}\right)\left(p(s) - \sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$+e^{-\frac{t}{2}}\sqrt{\pi}e^{-\frac{s^2}{4}}+\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}t}{s^2\sqrt{\frac{1}{4}+s^2}}\right)$$

$$\times\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right). \tag{1.57}$$

Applying the condition $v(1,s)=e^{-\frac{1}{2}}\sqrt{\pi}e^{-\frac{s^2}{4}}$ and formula (1.57), we get

$$e^{-\frac{1}{2}}\sqrt{\pi}e^{-\frac{s^2}{4}}=-\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cosh s-1}{s\sinh s}\right)\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$+\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}-e^{\frac{1}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right)\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$+e^{-\frac{1}{2}}\sqrt{\pi}e^{-\frac{s^2}{4}}.$$

From that it follows

$$-\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}t}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right)$$

$$+\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right)$$

$$\times\left(p(s)-\sqrt{\pi}e^{-\frac{s^2}{4}}\right)=0.$$

Since

$$-\left(\frac{\sinh\sqrt{\frac{1}{4}+s^2}t}{\sqrt{\frac{1}{4}+s^2}}\right)\left(\frac{\cos s-1}{s\sin s}\right)$$

$$+\left(\frac{\sqrt{\frac{1}{4}+s^2}\left(\cosh\sqrt{\frac{1}{4}+s^2}t-e^{\frac{t}{2}}\right)+\frac{1}{2}\sinh\sqrt{\frac{1}{4}+s^2}}{s^2\sqrt{\frac{1}{4}+s^2}}\right)\neq 0,$$

it follows that

$$p(s)=\sqrt{\pi}e^{-\frac{s^2}{4}}=\mathcal{F}\left\{e^{-x^2}\right\}. \tag{1.58}$$

Therefore, taking the inverse Fourier transform with respect to x, we get

$$p(x) = e^{-x^2}. \tag{1.59}$$

Now, we will obtain $u(t, s)$. Applying formulas (1.52), (1.54), (1.57) and (1.58), we get

$$u(t, s) = e^{-t}\sqrt{\pi}e^{-\frac{s^2}{4}}, -1 \leq t \leq 1. \tag{1.60}$$

Therefore, taking the inverse Fourier transform with respect to x, we get

$$u(t, x) = e^{-t-x^2}.$$

Thus, the exact solution of problem (1.46) is

$$(u(t, x), p(x)) = \left(e^{-t-x^2}, e^{-x^2}\right).$$

Note that using the same manner one obtain the solution of the following BVP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{|r|=2m} \alpha_r \frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = p(x) + f(t, x), \\ 0 < t < T, x, r \in \mathbb{R}^n, |r| = r_1 + \ldots + r_n, \\ -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{|r|=2m} \alpha_r \frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = p(x) + g(t, x), \\ -T < t < 0, x, r \in \mathbb{R}^n, \\ u(0, x) = \varphi(x), u(-T, x) = \psi(x), u(T, x) = \mu(x), \\ x \in \mathbb{R}^n, -T \leq t \leq T \end{cases} \tag{1.61}$$

for a second order in t and $2m - th$ order in space variables multidimensional elliptic-telegraph differential equation.

Assume that $\alpha_r > \alpha > 0$ and $f(t, x)\left(t \in (0, T), x \in \overline{\Omega}\right)$, $g(t, x)\left(t \in (-T, 0), x \in \overline{\Omega}\right)$, $\varphi(x)$, $\psi(x)$, $\mu(x)$, $\left(x \in \overline{\Omega}\right)$ are given smooth functions.

However Fourier transform method described in solving (1.61) can be used only in the case when (1.61) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients or polynomial coefficients.

1.3 The Guide of the Thesis

In this section, let us briefly describe the contents of the various chapters of the thesis. It consists of five chapters.

First chpter is the introduction.

Second chpter is the theorem on stability of differential equations of SIP for the elliptic-telegraph equation is established. In applications, theorems on the stability of three SIPs for one dimensional differential equations with nonlocal conditions and multidimensional elliptic-telegraph differential equations with local conditions are established.

Third chpter is the theorem on stability of accuracy DSs for the numerical solution of SIP for the elliptic-telegraph equationis are established. In applications, theorems on the stability of DSs for three type of the space identification problems for elliptic-telegraph PDEs are proved.

Fourth chapter the first order of accuracy DSs for the numerical solution of SIP for a one-dimensional and two-dimensional elliptic-telegraph equation with dirichlet and neumann conditions are presented. Numerical results are provided.

Fifth chapter contains conclusion.

CHAPTER 2
STABILITY OF SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC-TELEGRAPH DIFFERENTIAL EQUATION

In this chapter, we consider the SIPs for elliptic-telegraph equations can be reduced to the space SIPs for the identification for elliptic-telegraph equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t), 0 < t < 1, \\ -\frac{d^2u(t)}{dt^2} + Au(t) = p + g(t), -1 < t < 0, \\ u(0) = \varphi, u_t(0^+) = u_t(0^-), u(-1) = \psi, u(1) = \xi \end{cases} \quad (2.1)$$

in a Hilbert space H with the SAPDO $A \geq \delta I, \delta > 0$. Here p is the unknown parameter. The rest of this chapter is organized as follows: In section 2.1, the main theorem on stability of problem (2.1) is established. In section 2.2, theorems on stability of three SIPs for elliptic-telegraph equations are proved.

Therefore, the main aim of this chapter is to investigate the space identification problem for the elliptic-telegraph equation with parameter p.

2.1 Stability of the Differential Problem

Denote that

$$u(t) = u(t; f(t), g(t), \varphi, \psi, \xi),\ \ p = p(f(t), g(t), \varphi, \psi, \xi).$$

By a solution of inverse problem (2.1) we mean a pair $(u(t), p)$ satisfying the following conditions:

1. The element $u(t)$ belongs to D for all $t \in [-1, 1]$, and the function $Au(t)$ is continuous on $[-1, 1]$, $p \in H$. Here, $D = D(A)$ is the domain of an operator A.

2. $u(t)$ is twice continuously differentiable on the interval $[-1, 1]$. The derivative at the endpoints of the interval are understood as the appropriate unilateral derivatives.

3. $(u(t), p)$ satisfies the evolution equation and local boundary conditions (2.1).

A solution of problem (2.1) defined in this manner will from now on be referred to as a solution of problem (2.1) in the space $C(H) \times H$. Here $C(H) = C([-1,1], H)$ is the space of continuous H-valued functions $u(t)$ defined on $[-1,1]$, equipped with the norm

$$\|u\|_{C(H)} = \max_{-1\leq t\leq 1} \|u(t)\|_H . \qquad (2.2)$$

In the present section, we will prove the main theorem on the stability of problem (2.1) in the space $C(H) \times H$.

To formulate our results we introduce the operator $G = A - \frac{\alpha^2}{4}I$. It is easy to see that for $\delta > \frac{\alpha^2}{4}$, G is the positive definite self-adjoint operator in the space H. Throughout, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$c(t) = \frac{e^{itG^{1/2}} + e^{-itG^{1/2}}}{2}.$$

Then from the definition of the sine operator-function $s(t)$

$$s(t)u = \int_0^t c(y)u dy$$

it follows that

$$s(t) = G^{-1/2}\frac{e^{itG^{1/2}} - e^{-itG^{1/2}}}{2i}.$$

Now, let us give four lemmas that will be needed in the sequel.

Lemma 2.1.1. Assume that

$$\delta > \frac{\alpha^2}{4}, \alpha > 0. \qquad (2.3)$$

Then for any $t \geq 0$, the estimates

$$\|c(t)\|_{H\to H} \leq 1, \left\|G^{1/2}s(t)\right\|_{H\to H} \leq 1, \qquad (2.4)$$

$$\left\|B^{\beta}\exp\{-Bt\}\right\|_{H\to H} \leq 1, 0 \leq \beta \leq 1, \left\|(I - \exp\{-2B\})^{-1}\right\|_{H\to H} \leq M(\delta)$$

are satisfied. Here $B = A^{1/2}$.

Proofs of these estimates are based on the spectral representation of the SAPDO in a Hilbert space.

Lemma 2.1.2. Assume that

$$\delta \geq \left(\frac{\alpha}{2}\right)^2 + 1, \alpha > 0.$$

Then, the operator

$$\left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)$$

has an inverse

$$E = \left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)^{-1}$$

and the following estimate holds

$$\|E\|_{H\to H} \leq \frac{1}{1 - \left(1 + \frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}}}. \tag{2.5}$$

Proof. The proof of the estimate (2.5) is based on the estimate

$$\left\|c(1) + \frac{\alpha}{2}s(1)\right\|_{H\to H} \leq 1 + \frac{\alpha}{2}. \tag{2.6}$$

Using the definitions of $c(t)$ and $s(t)$ and positivity and self-adjointness property of A, we obtain

$$\left\|c(1) + \frac{\alpha}{2}s(1)\right\|_{H\to H} \leq 1 + \frac{\alpha}{2}\sup_{\delta\leq\rho<\infty}\frac{1}{\left(\rho - \frac{\alpha^2}{4}\right)^{1/2}}$$

$$\leq 1 + \frac{\alpha}{2}\frac{1}{\left(\delta - \frac{\alpha^2}{4}\right)^{1/2}} \leq 1 + \frac{\alpha}{2}.$$

The proof of estimate (2.6) is completed. Lemma 2.1.2 is proved.

Lemma 2.1.3. Assume that

$$\left(\frac{\alpha}{2} + 1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1, \alpha \geq 4. \tag{2.7}$$

Then, the operator

$$I - B\left(I + e^{-B}\right)^{-1}\left(I - e^{-B}\right)Ee^{-\frac{\alpha}{2}}s(1)$$

has an inverse

$$Q = \left(I - B\left(I + e^{-B}\right)^{-1}\left(I - e^{-B}\right)Ee^{-\frac{\alpha}{2}}s(1)\right)^{-1}$$

and the following estimate holds

$$\|Q\|_{H\to H} \le M(\alpha,\delta), \tag{2.8}$$

where $M(\alpha,\delta) > 0$.

Proof. We have that

$$Q = \left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)$$
$$\times\left\{I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right) - B\left(I + e^{-B}\right)^{-1}\left(I - e^{-B}\right)e^{-\frac{\alpha}{2}}s(1)\right\}^{-1}.$$

First, we will prove the estimate

$$\left\|c(1) + \frac{\alpha}{2}s(1) + B\left(I - e^{-B}\right)\left(I + e^{-B}\right)^{-1} s(1)\right\|_{H\to H} \le 1 + \frac{\alpha}{2} + \delta^{1/2}. \tag{2.9}$$

Using the definition of $s(t)$ and positivity and self-adjointness property of A and the triangle inequality, we obtain

$$\left\|A^{1/2}G^{-1/2}\right\|_{H\to H} \le \sup_{\delta\le\rho<\infty}\left(\frac{\rho}{\rho - \frac{\alpha^2}{4}}\right)^{1/2} \le \delta^{1/2} \tag{2.10}$$

and

$$\left\|B\left(I + e^{-B}\right)^{-1} s(1)\right\|_{H\to H} \le \sup_{\delta\le\rho<\infty}\left(\frac{\rho}{\rho - \frac{\alpha^2}{4}}\right)^{1/2}\left(\frac{1 - e^{-\rho^{1/2}}}{1 + e^{-\rho^{1/2}}}\right) \le \delta^{1/2}.$$

From that and from the estimate (2.6), it follows estimate (2.9). Using $\delta \le \left(\frac{\alpha}{2} + 1\right)^2$, we get

$$\left(1 + \frac{\alpha}{2} + \delta^{1/2}\right)e^{-\frac{\alpha}{2}} \le 2\left(1 + \frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}}.$$

The proof of the estimate (2.8) is based on the estimate

$$2\sup_{4\le\alpha<\infty}\left(1 + \frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}} < 1.$$

Denote

$$g(\alpha) = \left(1 + \frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}}.$$

It is clear to see that

$$g'(\alpha) = -\frac{\alpha}{4}e^{-\frac{\alpha}{2}} < 0$$

for $\alpha > 0$. Therefore,

$$2\sup_{4\le\alpha<\infty}\left(1+\frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}} \le 2\left(1+\frac{4}{2}\right)e^{-2} = \frac{6}{e^2} < 1.$$

Lemma 2.1.3 is proved.

Lemma 2.1.4. For the solution of problem (2.1) we have the following formula

$$u(t) = v(t) + A^{-1}p, \tag{2.11}$$

$$p = A\xi - Av(1), \tag{2.12}$$

where

$$v(t) = \left(I - e^{-2B}\right)^{-1}\left[\left(e^{tB} - e^{-(2+t)B}\right)v_0 + \left(e^{-(1+t)B} - e^{-(1-t)B}\right)v_{-1}\right.$$

$$\left. - \left(e^{-(1+t)B} - e^{-(1-t)B}\right)(2B)^{-1}\int_{-1}^{0}\left(e^{-(1+y)B} - e^{-(1-y)B}\right)g(y)dy\right]$$

$$+ (2B)^{-1}\int_{-1}^{0}\left(e^{-|-t+y|B} - e^{(t+y)B}\right)g(y)dy, -1 \le t \le 0 \tag{2.13}$$

and

$$v(t) = e^{-\frac{\alpha}{2}t}\left(c(t) + \frac{\alpha}{2}s(t)\right)v_0 + e^{-\frac{\alpha}{2}t}s(t)v_0' \tag{2.14}$$

$$+ \int_{0}^{t} e^{-\frac{\alpha}{2}(t-y)}s(t-y)f(y)dy, 0 \le t \le 1,$$

$$v_{-1} = v_0 - \varphi + \psi, \tag{2.15}$$

$$v_0 = E\left\{e^{-\frac{\alpha}{2}}s(1)v_0' + \int_{0}^{1} e^{-\frac{\alpha}{2}(1-y)}s(1-y)f(y)dy + \varphi - \xi\right\}, \tag{2.16}$$

$$v_0' = Q\left(I - e^{-2B}\right)^{-1}\left[B\left(I - e^{-B}\right)^2\left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)^{-1}\right.$$

$$\times\left\{\int_0^1 e^{-\frac{\alpha}{2}(1-y)}s(1-y)f(y)dy + \varphi - \xi\right\}$$

$$\left. -2Be^{-B}(-\varphi + \psi) + \int_{-1}^{0}\left(e^{-(2+y)B} - e^{yB}\right)g(y)dy\right]. \tag{2.17}$$

Proof. We seek the solution of problem (2.1) by formula (2.11), where $v(t)$ is the solution of the following nonlocal BVP

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = f(t), 0 < t < 1, \\ -\frac{d^2v(t)}{dt^2} + Av(t) = g(t), -1 < t < 0, \\ v(0) - v(-1) = \varphi - \psi, v(0) - v(1) = \varphi - \xi, v_t(0^+) = v_t(0^-) \end{cases} \tag{2.18}$$

for the differential equation in a Hilbert space H with SAPDO A. Now, we will obtain the formula for the solution of nonlocal BVP (2.18). It is known (Ashyralyev and Sobolevskii, 2004) that for smooth data of the initial and boundary value problems

$$\begin{cases} v''(t) + \alpha v'(t) + Av(t) = f(t), 0 < t < 1, \\ v(0) = v_0, v'(0) = v_0', \end{cases} \tag{2.19}$$

$$\begin{cases} -v''(t) + Av(t) = g(t), -1 < t < 0, \\ v(0) = v_0, \quad v(-1) = v_{-1} \end{cases} \tag{2.20}$$

there are unique solutions of the IVP (2.19) and BVP (2.20) and formulas (2.13) and (2.14) hold. From nonlocal bondary condition $v_0 - v_{-1} = \varphi - \psi$ it follows (2.15). Now, we obtain v_0. Applying (2.14) and condition $v_0 - v(1) = \varphi - \xi$, we can write

$$e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)v_0 + e^{-\frac{\alpha}{2}}s(1)v_0' + \int_0^1 e^{-\frac{\alpha}{2}(1-y)}s(1-y)f(y)dy = v_0 - \varphi + \xi.$$

By Lemma 2.1.2, there exists the operator $E = \left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)^{-1}$ and the formula (2.16) holds. Now, we will obtain v_0'. Applying (2.13) and taking derivative at $t = 0$ and using the condition $v_t(0^+) = v_t(0^-)$, we get

$$v_0' = \left(I - e^{-2B}\right)^{-1}\left[B\left(I + e^{-2B}\right)v_0 - 2Be^{-B}v_{-1}\right.$$

$$\left. + \int_{-1}^{0}\left(e^{-(2+y)B} - e^{yB}\right)g(y)dy\right]. \tag{2.21}$$

From that and formulas (2.15), (2.16) and (2.21), it follows that

$$v_0' = \left(I - e^{-2B}\right)^{-1}\left[B\left(I + e^{-2B} - 2e^{-B}\right)\left(I - e^{-\frac{\alpha}{2}}\left(c(1) + \frac{\alpha}{2}s(1)\right)\right)^{-1}\right.$$

$$\times\left\{e^{-\frac{\alpha}{2}}s(1)v_0' + \int_0^1 e^{-\frac{\alpha}{2}(1-y)}s(1-y)f(y)dy + \varphi - \xi\right\}$$

$$\left. -2Be^{-B}\left(-\varphi + \psi\right) + \int_{-1}^{0}\left(e^{-(2+y)B} - e^{yB}\right)g(y)dy\right].$$

By Lemma 2.1.3, there exists the inverse operator

$$Q = \left(I - B\left(I + e^{-B}\right)^{-1}\left(I - e^{-B}\right)Ee^{-\frac{\alpha}{2}}s(1)\right)^{-1}$$

and the formula (2.17) holds. Therefore, for the formal solution of the problem (2.18) we have the formulas (2.13), (2.14), (2.15), (2.16) and (2.17). Formula for p follows from (2.11) and condition $u(1) = \xi$. Lemma 2.1.4 is proved.

Theorem 2.1.1. Suppose that $\varphi, \psi, \xi \in D(A)$, and $\alpha \geq 4, \left(\frac{\alpha}{2} + 1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Let $f(t)$ and $g(t)$ be continuously differentiable functions on $[0, 1]$ and $[-1, 0]$ respectively. Then there is a unique solution of the problem (2.1) and the stability inequalities

$$\max_{-1\leq t\leq 1}\|u(t)\|_H + \left\|A^{-1}p\right\|_H \leq M(\alpha, \delta)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H\right. \tag{2.22}$$

$$\left. + \max_{-1\leq t\leq 0}\left\|A^{-1/2}g(t)\right\|_H + \max_{0\leq t\leq 1}\left\|A^{-1/2}f(t)\right\|\right]_H,$$

$$\max_{-1\le t\le 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{-1\le t\le 1}\|Au(t)\|_H + \|p\|_H$$

$$\le M(\alpha,\delta)\left[\|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H + \|g(0)\|_H \right. \tag{2.23}$$

$$\left. + \max_{-1\le t\le 0}\|g'(t)\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\|f'(t)\|_H\right]$$

hold, where $M(\alpha,\delta)$ does not depend on $f(t), t\in[0,1], g(t), t\in[-1,0]$ and φ,ψ,ξ.

Proof. Applying formula (2.12), we can obtain estimates

$$\left\|A^{-1}p\right\|_H \le \|\xi\|_H + \|v(1)\|_H\,, \|p\|_H \le \|A\xi\|_H + \|Av(1)\|_H\,. \tag{2.24}$$

Therefore, by (2.11) we need to establish estimates for $\max\limits_{-1\le t\le 1}\|v(t)\|_H$, $\max\limits_{-1\le t\le 1}\|Av(t)\|_H$ and $\max\limits_{-1\le t\le 1}\left\|\frac{d^2v(t)}{dt^2}\right\|_H$. First, we obtain the estimate $\|v(t)\|_H$ for $-1\le t\le 1$ and the triangle inequality and estimates (2.3), (2.4), (2.5) and (2.8), we get (Ashyralyev and Sobolevskii, 2004)

$$\max_{-1\le t\le 0}\|v(t)\|_H \le M_1(\alpha,\delta)\left[\|v_0\|_H + \|v_{-1}\|_H + \max_{-1\le t\le 0}\left\|A^{-1/2}g(t)\right\|_H\right]. \tag{2.25}$$

Similarly, by (2.14) and the triangle inequality and estimates (2.3), (2.4), (2.5) and (2.8), we get (Ashyralyev and Sobolevskii, 2004)

$$\max_{0\le t\le 1}\|v(t)\|_H \le M_2(\alpha,\delta)\left[\|v_0\|_H + \left\|A^{-1/2}v_0'\right\|_H + \max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H\right]. \tag{2.26}$$

To end it we need estimates for $\|v_0\|_H$, $\|v_{-1}\|_H$ and $\left\|A^{-1/2}v_0'\right\|_H$. Using the triangle inequality and estimates (2.5), (2.8), (2.25) and (2.26), we get

$$\|v_0\|_H \le \|E\|_{H\to H}\left\{e^{-\frac{\alpha}{2}}\left\|A^{1/2}G^{-1/2}\right\|_{H\to H}\left\|G^{1/2}s(1)\right\|_{H\to H}\left\|A^{-1/2}v_0'\right\|_H\right.$$

$$+\int_0^1 e^{-\frac{\alpha}{2}(1-y)}\left\|A^{1/2}G^{-1/2}\right\|_{H\to H}\left\|G^{1/2}s(1-y)\right\|_{H\to H}\left\|A^{-1/2}f(y)\right\|_H dy$$

$$\left.+\|\varphi\|_H + \|\xi\|_H\right\} \le M_3(\alpha,\delta)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H\right.$$

$$\left.+\max_{-1\le t\le 0}\left\|A^{-1/2}g(t)\right\|_H + \max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H\right],$$

$$\|v_{-1}\|_H \leq \|v_0\|_H + \|\varphi\|_H + \|\psi\|_H$$

$$\leq M_4(\alpha, \delta) \left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H \right.$$

$$\left. + \max_{-1 \leq t \leq 0} \left\| A^{-1/2} g(t) \right\|_H + \max_{0 \leq t \leq 1} \left\| A^{-1/2} f(t) \right\|_H \right]$$

and

$$\left\| A^{-1/2} v_0' \right\|_H \leq \|Q\|_{H \to H} \left\| \left(I - e^{-2B}\right)^{-1} \right\|_{H \to H} \left[\left\| \left(I - e^{-B}\right)^2 \right\|_{H \to H} \|E\|_{H \to H} \right.$$

$$\times \left\{ \int_0^1 e^{-\frac{\alpha}{2}(1-y)} \left\| A^{1/2} G^{-1/2} \right\|_{H \to H} \left\| G^{1/2} s(1-y) \right\|_{H \to H} \left\| A^{-1/2} f(y) \right\|_H dy \right.$$

$$\left. + \|\varphi\|_H + \|\xi\|_H \right\} + 2 \left\| e^{-B} \right\|_{H \to H} \left(\|\varphi\|_H + \|\psi\|_H \right)$$

$$\left. + \int_{-1}^0 \left(\left\| e^{-(2+y)B} \right\|_{H \to H} + \left\| e^{yB} \right\|_{H \to H} \right) \left\| A^{-1/2} g(y) \right\|_H dy \right]$$

$$\leq M_5(\alpha, \delta) \left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H \right.$$

$$\left. + \max_{-1 \leq t \leq 0} \left\| A^{-1/2} g(t) \right\|_H + \max_{0 \leq t \leq 1} \left\| A^{-1/2} f(t) \right\| \right]_H .$$

Therefore

$$\max_{-1 \leq t \leq 1} \|v(t)\|_H \leq M_6(\alpha, \delta) \left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H \right.$$

$$\left. + \max_{-1 \leq t \leq 0} \left\| A^{-1/2} g(t) \right\|_H + \max_{0 \leq t \leq 1} \left\| A^{-1/2} f(t) \right\| \right]_H . \tag{2.27}$$

Applying formula (2.11) and estimates (2.24) and (2.27) and the triangle inequality, we obtain estimate (2.22).

Second, we obtain the estimate $\|Av(t)\|_H$, for $-1 \leq t \leq 1$. Using formulas (2.13) and (2.14) and integrating by parts, we can get formulas

$$Av(t) = \left(I - e^{-2B}\right)^{-1} \left[\left(e^{tB} - e^{-(2+t)B}\right) Av_0 \right.$$

$$+ \left(e^{-(1+t)B} - e^{-(1-t)B}\right) Av_{-1} - \frac{e^{-(1+t)B} - e^{-(1-t)B}}{2}$$

$$\times\left\{-2e^{-B}g(0)+\left(I+e^{-2B}\right)g(-1)+\int_{-1}^{0}\left(e^{-(1+y)B}+e^{-(1-y)B}\right)g'(y)dy\right\}\Bigg]$$

$$+(I+e^{2tB})g(t)-\frac{1}{2}(e^{-(1+t)B}-e^{-(1-t)B})g(-1) \tag{2.28}$$

$$-e^{tB}g(0)-\int_{-1}^{0}(e^{-(|y-t|B}+e^{(t+y)B})g'(y)dy, -1\leq t\leq 0,$$

and

$$Av(t)=e^{-\frac{\alpha}{2}t}\left(c(t)+\frac{\alpha}{2}s(t)\right)Av_0+e^{-\frac{\alpha}{2}t}s(t)Av_0'+AG^{-1}\left\{f(t)-e^{-\frac{\alpha}{2}t}c(t)f(0)\right.$$

$$\left.+\int_{0}^{t}e^{-\frac{\alpha}{2}(t-y)}c(t-y)\left(\frac{\alpha}{2}f(y)+f'(y)\right)dy\right\}, 0\leq t\leq 1. \tag{2.29}$$

Then, using (2.28) and estimates (2.3), (2.4), (2.5) and (2.8), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{-1\leq t\leq 0}\|Av(t)\|_H\leq M_7(\alpha,\delta)\left[\|Av_0\|_H+\|Av_{-1}\|_H\right. \tag{2.30}$$

$$\left.+\max_{-1\leq t\leq 0}\|g'(t)\|_H+\|g(0)\|_H\right].$$

Similarly, using (2.29) and estimates (2.3), (2.4), (2.5) and (2.8), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{0\leq t\leq 1}\|Av(t)\|_H\leq M_8(\alpha,\delta)\left[\|Av_0\|_H+\left\|A^{1/2}v_0'\right\|_H\right. \tag{2.31}$$

$$\left.+\|f(0)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right].$$

To end it we need estimates for $\|Av_0\|_H$, $\|Av_{-1}\|_H$ and $\left\|A^{1/2}v_0'\right\|_H$. Using formulas (2.16) and (2.17) and integrating by parts, we can write the formulas

$$Av_0=E\left\{e^{-\frac{\alpha}{2}}s(1)Av_0'+A\varphi-A\xi+AG^{-1}\right.$$

$$\left.\times\left(c(1)f(1)-c(1)f(0)-\int_{0}^{1}e^{-\frac{\alpha}{2}(1-y)}c(1-y)\left[\frac{\alpha}{2}f(y)+f'(y)\right]dy\right)\right\}$$

and

$$A^{1/2}v_0' = Q\left(I-e^{-2B}\right)^{-1}\left[AG^{-1}\left(I-e^{-B}\right)^2 E\right.$$

$$\times\left(c(1)f(1)-c(1)f(0)-\int_0^1 e^{-\frac{\alpha}{2}(1-y)}c(1-y)\left[\frac{\alpha}{2}f(y)+f'(y)\right]dy\right)$$

$$+A\varphi-A\xi\}-2Be^{-B}\left(-A\varphi+A\psi\right)$$

$$\left.\times\left\{2e^{-B}g(0)+\left(I+e^{-2B}\right)g(-1)+\int_{-1}^0\left(e^{-(2+y)B}-e^{yB}\right)g'(y)dy\right\}\right].$$

Then, using the estimate (2.5), (2.8), (2.30) and (2.31), we obtain

$$\|Av_0\|_H \le \|AE\|_{H\to H}\left\{e^{-\frac{\alpha}{2}}\left\|A^{3/2}G^{-1/2}\right\|_{H\to H}\left\|AG^{1/2}s(1)\right\|_{H\to H}\left\|A^{1/2}v_0'\right\|_H\right.$$

$$+\left\{\|f(1)\|_H+\|f(0)\|_H+\left(\|f(1)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H\right)\right\}$$

$$\left.+\|A\varphi\|_H+\|A\xi\|_H\right\}\le M_9(\alpha,\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H\right.$$

$$\left.+\max_{-1\le t\le 0}\|g'(t)\|_H+\|g(0)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H+\|f(0)\|_H\right],$$

$$\|Av_{-1}\|_H \le \|Av_0\|_H+\|A\varphi\|_H+\|A\psi\|_H$$

$$\le M_{10}(\alpha,\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\max_{-1\le t\le 0}\|g'(t)\|_H\right.$$

$$\left.+\|g(0)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H+\|f(0)\|_H\right]$$

and

$$\left\|A^{1/2}v_0'\right\|_H \le \|Q\|_{H\to H}\left\|\left(I-e^{-2B}\right)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|\left(I-e^{-B}\right)^2\right\|_{H\to H}\|E\|_{H\to H}\right.$$

$$\times\left\{\|f(t)\|_H+\|f(0)\|_H+\left(\|c(1)f(t)\|_H+\|c(1)\|_H\max_{0\le t\le 1}\|f'(t)\|_H\right)\right.$$

$$\left.+\|A\varphi\|_H+\|A\xi\|_H\right\}+2\left\|e^{-B}\right\|_{H\to H}\left(\|A\varphi\|_H+\|A\psi\|_H\right)$$

$$\left.+\left\{\left\|2e^{-B}\right\|_{H\to H}+\left\|A\left(I-e^{-2B}\right)^{-1}\right\|_{H\to H}\|g(-1)\|_H+\max_{-1\le t\le 0}\|g'(t)\|_H\right\}\right]$$

$$\le M_{11}(\alpha,\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\max_{-1\le t\le 0}\|g'(t)\|_H\right.$$

$$+ \|g(0)\|_H + \max_{0\le t\le 1} \|f'(t)\|_H + \|f(0)\|_H \Big].$$

From these estimates and formulas (2.30) and (2.31), it follows

$$\max_{-1\le t\le 1} \|Av(t)\|_H \le M_{12}(\alpha,\delta) \Big[\|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H$$

$$+ \|g(0)\|_H + \max_{0\le t\le 1} \|f'(t)\|_H + \|f(0)\|_H \Big]. \tag{2.32}$$

Using estimates (2.24), (2.32), we obtain

$$\|p\|_H \le \|A\xi\|_H + \|Av_1\|_H$$

$$\le M_{13}(\alpha,\delta) \Big[\|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H \tag{2.33}$$

$$+ \|g(0)\|_H + \max_{0\le t\le 1} \|f'(t)\|_H + \|f(0)\|_H \Big].$$

Finally, applying the triangle inequality and equations (2.19) and (2.20) and estimate (2.32), we get

$$\max_{-1\le t\le 1} \left\| \frac{d^2 v(t)}{dt^2} \right\|_H \le M_{14}(\alpha,\delta) \Big[\|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H$$

$$+ \|g(0)\|_H + \max_{0\le t\le 1} \|f'(t)\|_H + \|f(0)\|_H \Big]. \tag{2.34}$$

Estimate (2.23) follows from estimates (2.32), (2.33) and (2.34). Theorem 2.1.1 is proved.

2.2 Applications

In this section, we consider the applications of the Theorem 2.1.1.

First, we consider the equation

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) \\ = p(x) + f(t,x),\ 0 < t < 1, 0 < x < 1, \\ -u_{tt}(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) \\ = p(x) + g(t,x),\ -1 < t < 0, 0 < x < 1. \end{cases} \tag{2.35}$$

Let $D = (-1,1) \times (0,1), D_1 = D \bigcap (t > 0), D_2 = D \bigcap (t < 0), \Im = \{(t,x) : t = 0, 0 \le x \le 1\}$.

Problem. Find a pair of functions $(u(t,x), p(x))$ with the following properties:

1) $u(t,x) \in C(\overline{D}) \bigcap C^1 (D_1 \bigcup D_2 \bigcup \Im) \bigcap C^2 (D_1 \bigcup D_2)$,

2) $u(t,x)$ satisfies the equation (2.35) and the boundary conditions

$$\begin{cases} u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1), -1 \le t \le 1, \\ u(0,x) = \varphi(x), u_t(0^+,x) = u_t(0^-,x), \\ u(-1,x) = \psi(x), u(1,x) = \xi(x), 0 \le x \le 1. \end{cases} \tag{2.36}$$

Problem (2.35) and (2.36) has a unique solution $(u(t,x), p(x))$ for the smooth functions $a(x) \ge a > 0$, $a(1) = a(0), t \in (-1,1)$, $\delta, \alpha > 0$, $\varphi(x), \psi(x), \xi(x), x \in [0,1]$. This allows us to reduce the BVP (2.35) and (2.36) to the identification problem (2.1) in a Hilbert space $H = L_2[0,1]$ with a SAPDO A^x defined by formula

$$A^x u(x) = -(a(x)u_x)_x + \delta u(x) \tag{2.37}$$

with domain

$$D(A^x) = \{u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[0,1], u(1) = u(0), u_x(1) = u_x(0)\}.$$

Applying the symmetry property of the space operator A^x with the domain $D(A^x) \subset W_2^2[0,1]$ and estimates (2.22) and (2.23) in $H = L_2[0,1]$, we can obtain the following theorem on stability of problem (2.35) and (2.36).

Theorem 2.2.1. Suppose that $\varphi, \psi, \xi \in W_2^2[0,1]$, and $\alpha \geq 4, \left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Let $f(t,x)$ be continuously differentiable function in t on $[0,1]$ and $g(t,x)$ be continuously differentiable function in t on $[-1,0]$. Then the solutions of the identification problem (2.35) and (2.36) satisfy the stability estimates

$$\|u\|_{C([-1,1],L_2[0,1])} + \left\|(A^x)^{-1}p\right\|_{L_2[0,1]} \leq M_1(\alpha,\delta)\left[\|\varphi\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]}\right. \tag{2.38}$$

$$\left. + \|\xi\|_{L_2[0,1]} + \| f \|_{C([0,1],L_2[0,1])} + \| g \|_{C([-1,0],L_2[0,1])}\right],$$

$$\|u\|_{C^{(2)}([-1,1],L_2[0,1])} + \|u\|_{C\left([-1,1],W_2^2[0,1]\right)} + \|p\|_{L_2[0,1]}$$

$$\leq M_2(\alpha,\delta)\left[\|\varphi\|_{W_2^2[0,1]} + \|\psi\|_{W_2^2[0,1]} + \|\xi\|_{W_2^2[0,1]}\right. \tag{2.39}$$

$$\left. + \| f \|_{C^{(1)}([0,1],L_2[0,1])} + \|g\|_{C^{(1)}([-1,0],L_2[0,1])}\right].$$

Here $M_1(\alpha,\delta)$ and $M_2(\alpha,\delta)$ do not depend on $\varphi(x), \psi(x), \xi(x), f(t,x)$ and $g(t,x)$.
The Sobolev space $W_2^2[0,1]$ is defined as the set of all functions $u(x)$ defined on $[0,1]$ such that $u(x)$ and the second order derivative function $u''(x)$ are both locally integrable in $L_2[0,1]$, equipped with the norm

$$\|u(x)\|_{W_2^2[0,1]} = \left(\int_0^1 |u(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 |u_{xx}(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Proof. Problem (2.35) and (2.36) can be written as abstract problem (2.1) in a Hilbert space $H = L_2[0,1]$ with SAPDO $A = A^x$ defined by the formula (2.37). Here $f(t) = f(t,x), g(t) = g(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions with values in H and $p = p(x)$ is the unknown element of $L_2[0,1]$. Therefore, estimates (2.38) and (2.39) follow from estimates of Theorem 2.1.1. Theorem 2.2.1 is proved.

Second, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[-1,1] \times \Omega$, we consider the identification problem for elliptic-telegraph equations

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) - \sum\limits_{r=1}^{n} \left(a(x_r)u_{x_r}(t,x_r)\right)_{x_r} \\ = p(x) + f(t,x),\ 0 < t < 1, x = (x_1,...,x_n) \in \Omega, \\ -u_{tt}(t,x) - \sum\limits_{r=1}^{n} \left(a(x_r)u_{x_r}(t,x_r)\right)_{x_r} \\ = p(x) + g(t,x),\ \ -1 < t < 0, x = (x_1,...,x_n) \in \Omega, \\ u(0,x) = \varphi(x), u_t(0+,x) = u_t(0-,x), \\ u(-1,x) = \psi(x), u(1,x) = \xi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, -1 \le t \le 1. \end{cases} \tag{2.40}$$

Here $a_r(x) \geq a > 0$, $(x \in \Omega)$, $\varphi(x), \psi(x), \xi(x)(x \in \overline{\Omega})$ and $f(t,x)$, $(t \in (0,1)), g(t,x), (t \in (-1,0))(x \in \Omega)$ and $(\delta > 0)$ are given smooth functions.

We consider the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions $u(x)$ defined on $\overline{\Omega}$, equipped with the norm

$$\|u(x)\|_{L_2(\overline{\Omega})} = \left(\int \cdots \int_{x\in\overline{\Omega}} |u(x)|^2 dx_1 \cdots dx_n\right)^{\frac{1}{2}}.$$

Problem (2.40) has a unique solution $(u(t,x), p(x))$ for the smooth functions $\varphi(x), \psi(x), \xi(x)$ and $a_r(x)$. This allows us to reduce the problem (2.40) to the BVP (2.1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a SAPDO A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} \tag{2.41}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, u(x) = 0, x \in S\right\}.$$

Theorem 2.2.2. Suppose that $\varphi, \psi, \xi \in L_2(\overline{\Omega})$, and $\alpha \geq 4, \left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Let $f(t,x)$ be continuously differentiable function in t on $[0,1]$ and $g(t,x)$ be continuously differentiable function in t on $[-1,0]$. Then the solutions of the identification problem (2.40) satisfy the stability estimates

$$\|u\|_{C\left(L_2(\overline{\Omega})\right)} + \left\|(A^x)^{-1}p\right\|_{L_2(\overline{\Omega})} \leq M_3(\alpha,\delta)\left[\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} \right. \tag{2.42}$$

$$\left. + \| f \|_{C([0,1],L_2(\overline{\Omega}))} + \| g \|_{C([-1,0],L_2(\overline{\Omega}))}\right],$$

$$\|u\|_{C^{(2)}([-1,1],L_2[0,1])} + \|u\|_{C\left([-1,1],W_2^2[0,1]\right)} + \left\|(A^x)^{-1}p\right\|_{L_2(\overline{\Omega})}$$

$$\leq M_4(\alpha,\delta)\left[\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})}\right. \tag{2.43}$$

$$\left. + \| f \|_{C^{(1)}([0,1],L_2(\overline{\Omega}))} + \|g\|_{C^{(1)}([-1,0],L_2(\overline{\Omega}))} + \|f(0)\|_{L_2(\overline{\Omega})} + \|g(0)\|_{L_2(\overline{\Omega})}\right],$$

where $M_3(\alpha,\delta)$ and $M_4(\alpha,\delta)$ do not depend on $\varphi(x), \psi(x), \xi(x), f(t,x)$and $g(t,x)$.
Here and in the future, the Sobolev space $W_2^2(\overline{\Omega})$ is defined as the set of all functions u defined on $\overline{\Omega}$ such that u and all second order partial differential derivative functions $u_{x_r x_r}, r = 1,,...,n$ are both integrable in $L_2(\overline{\Omega})$, equipped with the norm

$$\|u\|_{W_2^2(\overline{\Omega})} = \|u\|_{L_2(\overline{\Omega})} + \left(\int \cdots \int_{x\in\overline{\Omega}} \sum_{r=1}^{n} \left|u_{x_r x_r}\right|^2 dx_1 \cdots dx_n\right)^{\frac{1}{2}}.$$

Proof. Problem (2.40) can be written as abstract problem (2.1) in a Hilbert space $H = L_2(\overline{\Omega})$ with SAPDO $A = A^x$ defined by the formula (2.41). Here $f(t) = f(t,x), g(t) = g(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $\overline{\Omega}$ with values in $H = L_2(\overline{\Omega})$ and $p = p(x)$ is the unknown element of $L_2(\overline{\Omega})$. Therefore, estimates (2.42) and (2.43) follow from estimates of Theorem 2.1.1 and the coercivity of the elliptic differential problem. Theorem 2.2.2 is proved.

Theorem 2.2.3. (Sobolevskii, 1975) For the solution of the elliptic differential problem

$$A^x u(x) = \mu(x), x \in \Omega, u(x) = 0, x \in S,$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \left\|u_{x_r x_r}\right\|_{L_2(\overline{\Omega})} \leq M_5 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here M_5 does not depend on $\mu(x)$.

Third, in $[-1,1]\times\Omega$, the identification problem for the elliptic-telegraph equation

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}\left(a(x_r)u_{x_r}(t,x_r)\right)_{x_r}+\delta u \\ =p(x)+f(t,x),\ 0<t<1, \\ -u_{tt}(t,x)-\sum\limits_{r=1}^{n}\left(a(x_r)u_{x_r}(t,x_r)\right)_{x_r}+\delta u \\ =p(x)+g(t,x),\ \ -1<t<0, \\ x=(x_1,...,x_n)\in\Omega, \\ u(0,x)=\varphi(x),u_t(0+,x)=u_t(0-,x), \\ u(-1,x)=\psi(x),u(1,x)=\xi(x),x\in\overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial\overrightarrow{m}}=0,x\in S,-1\le t\le 1. \end{cases} \tag{2.44}$$

is considered. Here, $\overrightarrow{m}$ is the normal vector to S, $a_r(x)\ge a>0$, $(x\in\Omega)$, $\varphi(x),\psi(x),\xi(x)(x\in\overline{\Omega})$ and $f(t,x)$, $(t\in(0,1))$, $g(t,x)$, $(t\in(-1,0))(x\in\Omega)$ and $(\delta>0)$ are given smooth functions. Problem (2.44) has a unique solution $(u(t,x),p(x))$ for the smooth functions $\varphi(x),\psi(x),\xi(x)$ and $a_r(x)$. This allows us to reduce the problem (2.40) to the BVP (2.1) in the Hilbert space $H=L_2(\overline{\Omega})$ with a SAPDO A^x defined by formula

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u \tag{2.45}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2(\overline{\Omega}),1\le r\le n,\frac{\partial u\,(x)}{\partial\overrightarrow{m}}=0,x\in S\right\}.$$

Theorem 2.2.4. For the solutions of problem (2.40), we have following stability estimates

$$\|u\|_{C\left(L_2(\overline{\Omega})\right)}+\left\|(A^x)^{-1}p\right\|_{L_2(\overline{\Omega})}$$

$$\leq M_6(\alpha,\delta)\left[\|\varphi\|_{L_2(\overline{\Omega})}+\|\psi\|_{L_2(\overline{\Omega})}+\|\xi\|_{L_2(\overline{\Omega})}\right. \tag{2.46}$$

$$\left.+\| f \|_{C([0,1],L_2(\overline{\Omega}))}+\| g \|_{C([-1,0],L_2(\overline{\Omega}))}\right],$$

$$\|u\|_{C^{(2)}([-1,1],L_2[0,1])}+\|u\|_{C\left([-1,1],W_2^2[0,1]\right)}+\left\|(A^x)^{-1}p\right\|_{L_2(\overline{\Omega})}$$

$$\leq M_7(\alpha,\delta)\left[\|\varphi\|_{L_2(\overline{\Omega})}+\|\psi\|_{L_2(\overline{\Omega})}+\|\xi\|_{L_2(\overline{\Omega})}\right. \tag{2.47}$$

$$\left.+\| f \|_{C^{(1)}([0,1],L_2(\overline{\Omega}))}+\|g\|_{C^{(1)}([-1,0],L_2(\overline{\Omega}))}+\|f(0)\|_{L_2(\overline{\Omega})}+\|g(0)\|_{L_2(\overline{\Omega})}\right].$$

where $M_6(\alpha,\delta)$ and $M_7(\alpha,\delta)$ do not depend on $\varphi(x),\psi(x),\xi(x),f(t,x)$and $g(t,x)$.

Proof. Problem (2.44) can be written in abstract form (2.1) in a Hilbert space $L_2(\overline{\Omega})$ with SAPDO $A = A^x$ defined by the formula (2.45). Here $f(t) = f(t,x), g(t) = g(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract function defined on $\overline{\Omega}$ with values of $H = L_2(\overline{\Omega})$ and $p = p(x)$ is the element of $L_2(\overline{\Omega})$. Therefore, estimates (2.46) and (2.47) follow from estimates of Theorem 2.1.1 and the coercivity of the elliptic differential problem. Furthermore, Theorem 2.2.4 is proved.

Theorem 2.2.5. (Sobolevskii, 1975) For the solution of the elliptic differential problem

$$\begin{cases} A^x u(x) = \mu(x), x \in \Omega, \\ \frac{\partial u(x)}{\partial \vec{m}} = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n}\left\|u_{x_r x_r}\right\|_{L_2(\overline{\Omega})} \leq M_8\|\mu\|_{L_2(\overline{\Omega})}.$$

Here M_8 does not depend on $\mu(x)$.

CHAPTER 3

STABILITY OF SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC-TELEGRAPH DIFFERENCE EQUATION

Note that absolute stable DSs for SIPs for elliptic-telegraph equations have not been well-investigated. Our goal in this chapter is to study the absolute stable DS for the approximate solution of the SIP (2.1). For the approximate solution of (2.1), applying the set of grid points

$$[-1,1]_\tau = \{t_k : t_k = k\tau, \ -N \le k \le N, \ N\tau = 1\},$$

we propose the first order of accuracy DS

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \alpha\tau^{-1}(u_{k+1} - u_k) + Au_{k+1} = p + f_k, \\ f_k = f(t_k), 1 \le k \le N-1, \\ -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = p + g_k, \\ g_k = g(t_k), -N+1 \le k \le -1, \\ u_1 - u_0 = u_0 - u_{-1}, u_0 = \xi, u_{-N} = \varphi, u_N = \psi. \end{cases} \tag{3.1}$$

The rest of this chapter is organized as follows: In section 3.1, the main theorem on the stability of the DS (3.1) is established. In section 3.2, theorems on stability of DSs for three SIPs for elliptic-telegraph equations are proved.

3.1 Stability of the Difference scheme

First of all, let us first give auxiliary statements from (Ashyralyev and Sobolevskii, 2004) and some lemmas which will be useful in the sequel.

Lemma 3.1.1. Assume that $\delta > \frac{\alpha^2}{4}, \alpha > 0$.Then the following estimates are satisfied

$$\begin{cases} \|R_1\|_{H\to H} \le \frac{1}{1+\frac{\alpha\tau}{2}}, \ \|R_2\|_{H\to H} \le \frac{1}{1+\frac{\alpha\tau}{2}}, \ \left\|\tau G^{1/2}R_1\right\|_{H\to H} \le 1, \\ \left\|\tau G^{1/2}R_2\right\|_{H\to H} \le 1, \left\|R_1R_2^{-1}\right\|_{H\to H} \le 1, \left\|R_2R_1^{-1}\right\|_{H\to H} \le 1. \end{cases} \tag{3.2}$$

Here, we denote

$$R_1 = \left(\left(1+\frac{\alpha\tau}{2}\right)I - i\tau G^{1/2}\right)^{-1}, R_2 = \left(\left(1+\frac{\alpha\tau}{2}\right)I + i\tau G^{1/2}\right)^{-1}, G = A - \frac{\alpha^2}{4}.$$

The proof of Lemma 3.1.1 is based on the spectral representation of the SAPDO in a Hilbert space.

Lemma 3.1.2. Assume that

$$\delta \geq \frac{\alpha^2}{4}, \alpha > 0. \tag{3.3}$$

Then, the operator

$$\left(I - (R_2 - R_1)^{-1}\left[R_1^N (R_2 - I) + R_2^N (I - R_1)\right]\right)$$

has an inverse

$$E_\tau = \left(I - (R_2 - R_1)^{-1}\left[R_1^N (R_2 - I) + R_2^N (I - R_1)\right]\right)^{-1}$$

and the following estimate holds:

$$\|E_\tau\|_{H \to H} \leq \left(1 + \frac{\alpha}{2}\right)\left(1 + \frac{\alpha\tau}{2}\right)^{-N}. \tag{3.4}$$

Proof. We have that

$$\begin{cases} I - R_2 = R_2\left(\frac{\alpha\tau}{2} + i\tau G^{1/2}\right), I - R_1 = R_1\left(\frac{\alpha\tau}{2} - i\tau G^{1/2}\right), \\ R_2 - R_1 = R_2 R_1\left(-2i\tau G^{1/2}\right). \end{cases}$$

Therefore, applying the spectral representation of SAPDO and estimate (3.2), we obtain

$$\begin{aligned} &\left\|(R_2 - R_1)^{-1}\left[R_1^N (R_2 - I) + R_2^N (I - R_1)\right]\right\|_{H \to H} \\ &\leq \left\|\frac{\alpha\tau}{2} R_2 R_1 (R_2 - R_1)^{-1}\left(R_2^{N-1} - R_1^{N-1}\right)\right\|_{H \to H} + \frac{1}{2}\left\|R_1^{N-1} + R_2^{N-1}\right\|_{H \to H} \\ &\leq \frac{\alpha\tau}{2}(N-1)\frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^N} + \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^{N-1}} = \left(1 + \frac{\alpha}{2}\right)\left(1 + \frac{\alpha\tau}{2}\right)^{-N}. \end{aligned}$$

Lemma 3.1.2 is proved.

Lemma 3.1.3. The following estimates hold (Ashyralyev and Sobolevskii, 2004):

$$\left\|R^N\right\|_{H \to H} \leq 1, \left\|A^{1/2} R^N\right\|_{H \to H} \leq 1, \left\|\left(I - R^{2N}\right)^{-1}\right\|_{H \to H} \leq M(\delta), \tag{3.5}$$

where

$$R = (I + \tau B)^{-1}, B = \frac{\tau A + A^{1/2}\sqrt{\tau^2 A + 4}}{2}.$$

Lemma 3.1.4. Assume that

$$\left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2+1, \alpha \geq 4. \tag{3.6}$$

Then, the operator

$$\left(I-\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)E_\tau\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right)$$

has an inverse

$$Q_\tau=\left(I-\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)E_\tau\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right)^{-1}$$

and the following estimate holds:

$$\|Q_\tau\|_{H\to H} \leq M(\alpha,\delta). \tag{3.7}$$

Proof. We have that

$$Q_\tau=E_\tau^{-1}\left(E_\tau^{-1}-\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right)^{-1}.$$

Then

$$Q_\tau=\left(I-\left(R_2-R_1\right)^{-1}\left[R_1^N\left(R_2-I\right)+R_2^N\left(I-R_1\right)\right]\right)$$

$$\times\left\{I-\left(R_2-R_1\right)^{-1}\left[R_1^N\left(R_2-I\right)+R_2^N\left(I-R_1\right)\right]\right.$$

$$\left.-\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right\}^{-1}.$$

First, we will proof the estimate

$$\left\|\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right\|_{H\to H}$$

$$\leq \sqrt{\delta}\left(1+\frac{\alpha\tau}{2}\right)^{-N+2}. \tag{3.8}$$

Using the positivity and self-adjointness property of A and the triangle inequality, we obtain

$$\left\|\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)\left(R_2-R_1\right)^{-1}\left(R_2^N-R_1^N\right)\right\|_{H\to H}$$

$$\leq \left\| \left(I + R^N\right)^{-1} \left(I - R^{N-1}\right) \right\|_{H \to H} \left\| BRG^{-1/2} \right\|_{H \to H}$$

$$\left\| \tau G^{1/2} \left(R_2 - R_1\right)^{-1} \left(R_2^N - R_1^N\right) \right\|_{H \to H}$$

It is clear to see that

$$\left\| \left(I + R^N\right)^{-1} \left(I - R^{N-1}\right) \right\|_{H \to H} \leq 1 \tag{3.9}$$

and

$$\left\| \tau G^{1/2} \left(R_2 - R_1\right)^{-1} \left(R_2^N - R_1^N\right) \right\|_{H \to H} \leq \left(1 + \frac{\alpha\tau}{2}\right)^{-N+2}. \tag{3.10}$$

Now, since $B^2RA^{-1} = I$, then

$$BR = AB^{-1}.$$

Therefore

$$\left\| BRG^{-1/2} \right\|_{H \to H} \leq \left\| A^{1/2}B^{-1} \right\|_{H \to H} \left\| A^{1/2}G^{-1/2} \right\|_{H \to H}$$

$$\leq . \sup_{\delta \leq \rho < \infty} \left| \frac{2}{\tau\rho^{1/2} + \sqrt{\tau^2\rho^2 + 4}} \right| \sup_{\delta \leq \rho < \infty} \left| \frac{\sqrt{\rho}}{\sqrt{\rho - \frac{\alpha^2}{4}}} \right| \leq \left| \frac{\sqrt{\delta}}{\sqrt{\delta - \frac{\alpha^2}{4}}} \right| \leq \sqrt{\delta}. \tag{3.11}$$

Using estimates (3.9), (3.10) and (3.11), we obtain

$$\left\| \left(I + R^N\right)^{-1} \tau BR \left(I - R^{N-1}\right) \left(R_2 - R_1\right)^{-1} \left(R_2^N - R_1^N\right) \right\|_{H \to H} \leq \delta^{1/2} \left(1 + \frac{\alpha\tau}{2}\right)^{-N+2}.$$

From that and estimate (3.4) it follows estimate (3.8). Lemma 3.1.4 is proved.

Lemma 3.1.5. The solution of problem (3.1) exists and the following formulas hold:

$$u_k = v_k + A^{-1}p, -N \leq k \leq N, p = A\psi - Av_N, \tag{3.12}$$

where

$$v_k = (I - R^{2N})^{-1} \Big\{ (R^{-k} - R^{2N+k})v_0 + (R^{N+k} - R^{N-k})v_{-N}$$

$$+(R^{N-k} - R^{N+k}) \sum_{r=-N+1}^{-1} B^{-1}(R^{N-r} - R^{N+r})R^{-1}(2 + \tau B)^{-1}\tau g_r \Big\}$$

$$+\sum_{r=-N+1}^{-1} B^{-1}(R^{-(k+r)} - R^{|r-k|})R^{-1}(2+\tau B)^{-1}\tau g_r, -N+1 \le k \le -1, \tag{3.13}$$

$$v_k = (R_2 - R_1)^{-1}\left\{\left[(I-R_1)R_2^k - (I-R_2)R_1^k\right]v_0\right.$$

$$\left.+(R_2^k - R_1^k)(v_1 - v_0) + \sum_{s=1}^{k-1} R_1R_2(R_2^{k-s} - R_1^{k-s})\tau^2 f_s\right\}, 2 \le k \le N, \tag{3.14}$$

$$v_1 = v_0 + \omega, v_{-N} = v_0 - \xi + \psi, \tag{3.15}$$

$$v_0 = E_\tau(R_2 - R_1)^{-1}\left((R_2^N - R_1^N)\omega\right.$$

$$\left.+\sum_{s=1}^{N-1} R_1R_2(R_2^{N-s} - R_1^{N-s})\tau^2 f_s\right) + \xi - \psi, \tag{3.16}$$

$$\omega = Q_\tau\left\{(I - R^{2N})^{-1}\left[(R^{N-1} - R^{N+1})(\varphi - \xi)\right.\right.$$

$$\left.+ (R^{N+1} - R^{N-1})\sum_{r=-N+1}^{-1} B^{-1}(R^{N-r} - R^{N+r})R^{-1}(2+\tau B)^{-1}\tau g_r\right]$$

$$- \sum_{r=-N+1}^{-1} B^{-1}(R^{(1-r)} - R^{|r+1|})R^{-1}(2+\tau B)^{-1}\tau g_r$$

$$\left.- E_\tau(R_2 - R_1)^{-1}\sum_{s=1}^{N-1} R_1R_2(R_2^{N-s} - R_1^{N-s})\tau^2 f_s + \xi - \psi\right\}. \tag{3.17}$$

Proof. It is clear that v_k is the solution of the following nonlocal BVP

$$\begin{cases} \tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + \alpha\tau^{-1}(v_{k+1} - v_k) + Av_{k+1} \\ = f_k, 1 \le k \le N-1, \\ -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k = g_k, \ -N+1 \le k \le -1, \\ v_1 - v_0 = \omega = v_0 - v_{-1}, \\ v_0 - v_{-N} = \xi - \varphi, v_0 - v_N = \xi - \psi \end{cases} \tag{3.18}$$

for the DS in a Hilbert space H with SAPDO A. Therefore, we obtain the formula for the solution of nonlocal BVP (3.18). There are a unique solutions of the initial value problem

$$\begin{cases} \tau^{-2}(v_{k+1}-2v_k+v_{k-1})+\alpha\tau^{-1}(v_{k+1}-v_k)+Av_{k+1} \\ =f_k,\ 1\le k\le N-1, v_0 \text{ and } v_1 \text{ are given,} \end{cases} \tag{3.19}$$

and the BVP

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k=g_k,\ -N+1\le k\le -1, \\ v_0 \text{ and } v_{-N} \text{ given} \end{cases} \tag{3.20}$$

and formulas (3.13) and (3.14) hold (Ashyralyev and Sobolevskii, 2004). From conditions $v_0-v_{-N}=\xi-\varphi$ and $v_1-v_0=\omega$ it follows (3.15).

Now, we obtain v_0. Applying (3.14) and conditions $v_0-v_N=\xi-\psi$, $\omega=v_1-v_0$, we can write

$$v_0-\xi+\psi=(R_2-R_1)^{-1}\Big\{\Big[R_1^N(R_2-I)+R_2^N(I-R_1)\Big]v_0$$

$$+(R_2^N-R_1^N)\omega+\sum_{s=1}^{N-1}R_1R_2(R_2^{N-s}-R_1^{N-s})\tau^2 f_s\Big\}.$$

By Lemma 3.1.2, there exists the operator

$$E_\tau=\Big(I-(R_2-R_1)^{-1}\Big[R_1^N(R_2-I)+R_2^N(I-R_1)\Big]\Big)^{-1}.$$

Therefore, from that it follows formula (3.16). Finally, we obtain ω. Applying the condition $v_{-1}=v_0-\omega$ and formula (3.13), we get

$$v_0-\omega=(I-R^{2N})^{-1}\Big\{(R-R^{2N-1})v_0+(R^{N-1}-R^{N+1})v_{-N}$$

$$+(R^{N+1}-R^{N-1})\sum_{r=-N+1}^{-1}B^{-1}(R^{N-r}-R^{N+r})R^{-1}(2+\tau B)^{-1}\tau g_r\Big\}$$

$$+\sum_{r=-N+1}^{-1}B^{-1}(R^{(1-r)}-R^{|r+1|})R^{-1}(2+\tau B)^{-1}\tau g_r.$$

From that and formulas (3.15) and (3.16) it follows that

$$\omega=\Big(I-(I-R^{2N})^{-1}\Big[R-R^{2N-1}+R^{N-1}-R^{N+1}\Big]\Big)$$

$$\times E_\tau(R_2-R_1)^{-1}(R_2^N-R_1^N)\omega$$

$$+\left\{(I-R^{2N})^{-1}\left[(R^{N-1}-R^{N+1})(\varphi-\xi)\right.\right.$$

$$\left.-(R^{N+1}-R^{N-1})\sum_{r=-N+1}^{-1}B^{-1}(R^{N-r}-R^{N+r})R^{-1}(2+\tau B)^{-1}\tau g_r\right]$$

$$-\sum_{r=-N+1}^{-1}B^{-1}(R^{(1-r)}-R^{|r+1|})R^{-1}(2+\tau B)^{-1}\tau g_r$$

$$\left.-E_\tau\left[(R_2-R_1)^{-1}\sum_{s=1}^{k-1}R_1R_2(R_2^{k-s}-R_1^{k-s})\tau^2 f_s+\xi-\psi\right]\right\}.$$

By Lemma 3.1.4, there exists the inverse operator

$$Q_\tau=\left(I-\left(I+R^N\right)^{-1}\tau BR\left(I-R^{N-1}\right)E_\tau(R_2-R_1)^{-1}\left(R_2^N-R_1^N\right)\right)^{-1}$$

and the formula (3.17) holds. Therefore, for the formal solution of the problem (3.18) we have the formulas (3.13), (3.14), (3.15), (3.16) and (3.17). Formula for p follows from (3.12) and condition $u_N=\psi$. Lemma 3.1.5 is proved.

Theorem 3.1.1. Assume that $\left(\frac{\alpha}{2}+1\right)^2\geq\delta\geq\left(\frac{\alpha}{2}\right)^2+1,\alpha\geq 4$ and $\varphi,\xi,\psi\in D(A)$. Then, for the solution of DS (3.1), the stability inequalities hold:

$$\max_{-N\leq k\leq N}\|u_k\|_H+\left\|A^{-1}p\right\|_H\leq M_1(\alpha,\delta)\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H\right. \tag{3.21}$$

$$\left.+\max_{-N+1\leq k\leq -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\leq k\leq N-1}\left\|A^{-1/2}f_k\right\|\right]_H,$$

$$\max_{-N+1\leq k\leq N-1}\left\|\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\right\|_H+\max_{-N\leq k\leq N}\|Au_k\|_H+\|p\|_H$$

$$\leq M_2(\alpha,\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H\right. \tag{3.22}$$

$$\left.+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\right]$$

hold, where $M_1(\alpha,\delta),M_2(\alpha,\delta)$ does not depend on $f_k,\ 1\leq k\leq N-1,\ g_k,\ -N+1\leq k\leq -1$, $\varphi,\ \psi$ and ξ.

Proof. Applying formula (3.12), we can obtain estimates

$$\left\|A^{-1}p\right\|_H\leq\|\psi\|_H+\|v_N\|_H,\ \ \|p\|_H\leq\|A\psi\|_H+\|Av_N\|_H. \tag{3.23}$$

Therefore, by (3.13) we need to establish estimates for $\max_{-N\le k\le N}\|v_k\|_H$, $\max_{-N\le k\le N}\|Av_k\|$ and $\max_{-N\le k\le N}\left\|\tau^{-2}\left(v_{k+1}-2v_k+v_{k-1}\right)\right\|_H$.

First, we obtain the estimate $\|v_k\|_H$, for $-N\le k\le N$ and the triangle inequality and estimates (3.2), (3.3), (3.4), (3.5) and (3.7), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{-N\le k\le 0}\|v_k\|_H\le M_3(\alpha,\delta)\left[\|v_0\|_H+\|v_{-N}\|_H+\max_{-N\le k\le 0}\left\|A^{-1/2}g_k\right\|_H\right]. \tag{3.24}$$

Similarly, by (3.14) and the triangle inequality and estimates (3.2), (3.3), (3.4), (3.5) and (3.7), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{0\le k\le N}\|v_k\|_H\le M_4(\alpha,\delta)\left[\|v_0\|_H+\left\|A^{-1/2}\omega\right\|_H+\max_{0\le k\le N}\left\|A^{-1/2}f_k\right\|_H\right]. \tag{3.25}$$

Now we need estimates for $\|v_0\|_H, \|v_{-N}\|_H$ and $\left\|A^{-1/2}\omega\right\|_H$. Using estimates (3.4), (3.7), (3.24),(3.25) and the triangle inequality, we get

$$\|v_0\|_H\le M_5(\alpha,\delta)\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H\right.$$
$$\left.+\max_{-N+1\le k\le -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}f_k\right\|_H\right],$$
$$\|v_{-N}\|_H\le\|v_0\|_H+\|\xi\|_H+\|\psi\|_H$$
$$+\max_{-N+1\le k\le -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}f_k\right\|_H\Big],$$
$$\left\|A^{-1/2}\omega\right\|_H\le M_6(\alpha,\delta)\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H\right.$$
$$\left.+\max_{-N+1\le k\le -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}f_k\right\|_H\right].$$

Therefore,

$$\begin{aligned}\max_{-N\le k\le N}\|v_k\|_H&\le M_7(\alpha,\delta)\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H\right.\\&\left.\max_{-N+1\le k\le -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}f_k\right\|_H\right].\end{aligned} \tag{3.26}$$

Applying formula (3.12) and estimates (3.23) and (3.26) and the triangle inequality, we obtain estimate (3.21).

Second, we obtain the estimate $\|Av_k\|_H$ for $-N \le k \le N$. Using formulas (3.13) and (3.14) and Abel's formula, we can get formulas

$$\begin{aligned} Av_k &= (I-R^{2N})^{-1}\Big\{(R^{-k}-R^{2N+k})Av_0+(R^{N+k}-R^{N-k})Av_{-N} \\ &+(R^{N-k}-R^{N+k})AB^{-1}(2+\tau B)^{-1}(I-R)^{-1}\Big[(R^N-R^{N-2})g_{-1} \\ &-(R^{2N}-R^{-1})g_{-N}-\sum_{r=-N+1}^{-2}R^{N-r-1}\tau(g_{r+1}-g_r) \\ &+\sum_{r=-N+1}^{-2}R^{N+r-1}\tau(g_{r+1}-g_r)\Big]\Big\} \\ &+AB^{-1}(2+\tau B)^{-1}(I-R)^{-1}\Big[(R^{-k}-R^{-2-k})g_{-1} \\ &+(R^{k+N-1}-R^{-k+N-1})g_{-N+1}+R^{-1}(g_{k+1}-g_k) \\ &-\sum_{r=-N+1}^{-2}R^{-k-r-1}\tau(g_{r+1}-g_r)+\sum_{r=-N+1}^{k-1}R^{r-k-1}\tau(g_{r+1}-g_r) \\ &+\sum_{r=k}^{-2}R^{r-k-1}\tau(g_{r+1}-g_r)\Big], -N\le k\le 0 \end{aligned} \quad . \tag{3.27}$$

and

$$\begin{aligned} Av_k &= (R_2-R_1)^{-1}\Big\{R_1R_2(R_1^{k-1}-R_2^{k-1})Av_0+(R_2^k-R_1^k)Av_0 \\ &+AR_1R_2\Big[(R_2-R_1)(I-R_1)^{-1}(I-R_2)^{-1}\tau^2 f_{k-1} \\ &+\Big((I-R_1)^{-1}R_1^k-(I-R_2)^{-1}R_2^k\Big)\tau^2 f_1 \\ &+\sum_{s=1}^{k-2}\Big[(I-R_2)^{-1}R_2^{k-s}-(I-R_1)^{-1}R_1^{k-s}\Big]\tau^2(f_{s+1}-f_s)\Big]\Big\}, \\ &0\le k\le N. \end{aligned} \quad . \tag{3.28}$$

It is clear that

$$\begin{cases} \max\limits_{-N\le k\le 0}\|g_k\|_H \le \|g_{-1}\|_H+\sum\limits_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H, \\ \max\limits_{0\le k\le N}\|f_k\|_H \le \|f_1\|_H+\sum\limits_{k=2}^{N-1}\|f_k-f_{k-1}\|_H. \end{cases} \tag{3.29}$$

Then, using (3.27) and (3.29) and estimates (3.2), (3.3), (3.4), (3.5) and (3.7), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{-N\le k\le 0}\|Av_k\|_H \le M_8(\alpha,\delta)\,[\|Av_0\|_H+\|Av_{-N}\|_H$$

$$+\|g_{-1}\|_H+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H\Big]. \tag{3.30}$$

Similarly, using (3.28) and (3.29) and estimates (3.2), (3.3), (3.4), (3.5) and (3.7), we obtain (Ashyralyev and Sobolevskii, 2004)

$$\max_{0\le k\le N}\|Av_k\|_H \le M_9(\alpha,\delta)\Big[\|Av_0\|_H + \left\|A^{1/2}\omega\right\|_H$$

$$+\|f_1\|_H + \sum_{k=2}^{N-1}\|f_k - f_{k-1}\|_H\Big]. \tag{3.31}$$

To end it we need estimates for $\|Av_0\|_H$, $\|Av_{-N}\|_H$ and $\left\|A^{1/2}\omega\right\|_H$. Using formulas (3.16) and (3.17) and Abel's formula, we can write the formulas

$$Av_0 = E_\tau(R_2-R_1)^{-1}\Big\{(R_2^N-R_1^N)A\omega$$

$$+AR_1R_2\Big[(R_2-R_1)(I-R_1)^{-1}(I-R_2)^{-1}\tau^2 f_{k-1}$$

$$+\Big((I-R_1)^{-1}R_1^k-(I-R_2)^{-1}R_2^k\Big)\tau^2 f_1$$

$$+\sum_{s=1}^{k-2}\Big[(I-R_2)^{-1}R_2^{k-s}-(I-R_1)^{-1}R_1^{k-s}\Big]\tau^2\left(f_{s+1}-f_s\right)\Big]\Big\}+A\xi-A\psi,$$

$$A^{1/2}\omega = Q_\tau\Big\{(I-R^{2N})^{-1}\Big[(R^{N-1}-R^{N+1})\left(A\varphi-A\xi\right)$$

$$+(R^{N+1}-R^{N-1})AB^{-1}(2+\tau B)^{-1}\left(I-R\right)^{-1}\Big[(R^N-R^{N-2})g_{-1}$$

$$-(R^{2N}-R^{-1})g_{-N}-\sum_{r=-N+1}^{-2}R^{N-r-1}\tau\left(g_{r+1}-g_r\right)$$

$$+\sum_{r=-N+1}^{-2}R^{N+r-1}\tau\left(g_{r+1}-g_r\right)\Big]\Big\}$$

$$+AB^{-1}(2+\tau B)^{-1}\left(I-R\right)^{-1}\Big[(R-R^{-1})g_{-1}+(R^{N-2}-R^N)g_{-N+1}$$

$$+R^{-1}\left(g_0-g_{-1}\right)-\sum_{r=-N+1}^{-2}R^{-r}\tau\left(g_{r+1}-g_r\right)+\sum_{r=-N+1}^{-2}R^{r}\tau\left(g_{r+1}-g_r\right)$$

$$+\sum_{r=-1}^{-2}R^{r}\tau\left(g_{r+1}-g_r\right)\Big]-A(R_2-R_1)^{-1}R_1R_2$$

$$\times\Big[(R_2-R_1)(I-R_1)^{-1}(I-R_2)^{-1}\tau^2 f_{k-1}$$

$$-\Big((I-R_1)^{-1}R_1^k-(I-R_2)^{-1}R_2^k\Big)\tau^2 f_1$$

$$-\sum_{s=1}^{k-2}\left[(I-R_2)^{-1}R_2^{k-s}-(I-R_1)^{-1}R_1^{k-s}\right]\tau^2\left(f_{s+1}-f_s\right)\Bigg]\Bigg\}+A\xi-A\psi.$$

Then, using estimates (3.4), (3.7), (3.30) and (3.31), we get

$$\|Av_0\|_H \leq M_{10}(\alpha,\delta)\,[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H$$
$$+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg],$$
$$\|Av_{-N}\|_H \leq \|Av_0\|_H+\|A\xi\|_H+\|A\psi\|_H$$
$$\leq M_{11}(\alpha,\delta)\,[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H$$
$$+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg],$$

$$\left\|A^{1/2}\omega\right\|_{H\to H} \leq M_{12}(\alpha,\delta)\,[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H$$
$$+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg].$$

From these estimates and formulas (3.30) and (3.31) it follows

$$\max_{-N\leq k\leq N}\|Av_k\|_H \leq M_{13}(\alpha,\delta\,[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H$$
$$+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg]. \qquad (3.32)$$

Using estimates (3.23) and (3.32), we obtain

$$\|p\|_H \leq \|A\psi\|_H+\|Av_N\|_H$$
$$\leq M_{14}(\alpha,\delta)\,[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H \qquad (3.33)$$
$$+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg].$$

Finally, applying the triangle inequality, we can obtain

$$\max_{-N\leq k\leq N}\left\|\tau^{-2}\left(v_{k+1}-2v_k+v_{k-1}\right)\right\|_H \leq M_{15}(\alpha,\delta)\,[\|A\varphi\|_H$$
$$+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H \qquad (3.34)$$
$$+\|f_1\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\Bigg].$$

Estimate (3.22) follows from estimates (3.32), (3.33) and (3.34). Theorem 3.1.1 is proved.

3.2 Applications

In this section, we study the first order of accuracy difference scheme in t for the approximate solution of identification problems.

First, we consider the identification problem (2.35). The discretization of SIP (2.35) is carried out in two stages. In the fist stage, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh,\ 0 \leq n \leq M,\ Mh = l\}.$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [0,l]_h} |\varphi(x)|^2 h \right)^{1/2}.$$

To the differential operator A^x defined by the formula (2.37) with its domain, we assign the difference operator A_h^x by the formula

$$A_h^x \phi(x) = \{-(a(x)\phi_{\bar{x}})_{x,n} + \delta\phi_n\}_1^{M-1} \tag{3.35}$$

acting in the space of grid functions $\phi^h(x) = \{\phi_n\}_0^M$ satisfying the following conditions $\phi_0 = \phi_M, \phi_1 - \phi_0 = \phi_M - \phi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the identification problem

$$\begin{cases} u_{tt}^h(t,x) + \alpha u_t^h(t,x) + A_h^x u^h(t,x) = p^h(x) + f^h(t,x),\ x \in [0,l]_h\,,\ 0 < t < 1, \\ -u_{tt}^h(t,x) + A_h^x u^h(t,x) = p^h(x) + g^h(t,x),\ x \in [0,l]_h\,,\ -1 < t < 0, \\ u^h(0,x) = \varphi^h(x), u_t^h(0^+,x) = u_t^h(0^-,x), \\ u^h(-1,x) = \psi^h(x), u^h(1,x) = \xi^h(x),\ x \in [0,l]_h\,. \end{cases} \tag{3.36}$$

In the second stage, we replace identification problem (3.36) with a first order of accuracy difference scheme

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \alpha \frac{u_{k+1}^h(x) - u_k^h(x)}{\tau} + A_h^x u_{k+1}^h(x) = p^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), \\ x \in [0,l]_h\,,\ 1 \leq k \leq N-1, \\ -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = p^h(x) + g_k^h, g_k^h(x) = g(t_k,x), \\ x \in [0,l]_h\,,\ -N+1 \leq k \leq -1, \\ u_1^h(x) - u_0^h(x) = u_0^h(x) - u_{-1}^h(x), u_0^h(x) = \xi^h(x), \\ u_{-N}^h(x) = \varphi^h(x), u_N^h(x) = \psi^h(x),\ x \in [0,l]_h\,. \end{cases} \tag{3.37}$$

Theorem 3.2.1. Suppose that $\alpha \geq 4, \left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Then, for the solution $\left\{\left\{u_k^h(x)\right\}_{-N}^{N}, p^h(x)\right\}$ of problem (3.37) the following stability estimates

$$\max_{-N\leq k\leq N} \|u_k\|_{L_{2h}} + \left\|\left(A_h^x\right)^{-1} p^h\right\|_{L_{2h}}$$

$$\leq M_1(\alpha,\delta)\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\xi^h\right\|_{L_{2h}}\right.$$

$$\left. + \max_{-N+1\leq k\leq -1}\left\|g_k^h\right\|_{L_{2h}} + \max_{1\leq k\leq N-1}\left\|f_k^h\right\|_{L_{2h}}\right],$$

$$\max_{-N+1\leq k\leq N-1}\left\|\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\|_{L_{2h}} + \max_{-N+1\leq k\leq N}\left\|u_k^h\right\|_{W_{2h}^2}$$

$$\leq M_2(\alpha,\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2} + \left\|\xi^h\right\|_{W_{2h}^2} + \left\|g_{-1}^h\right\|_{L_{2h}}\right.$$

$$\left. + \max_{-N+1\leq k\leq -2}\left\|\frac{1}{\tau}\left(g_k^h - g_{k-1}^h\right)\right\|_{L_{2h}} + \left\|f_1^h\right\|_{L_{2h}} + \max_{2\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k^h - f_{k-1}^h\right)\right\|_{L_{2h}}\right]$$

hold, where $M_1(\alpha,\delta)$ and $M_2(\alpha,\delta)$ do not depend on $\tau, h, f_k^h,\ 1\leq k\leq N-1,\ g_k^h,\ -N+1\leq k\leq -1,\ \varphi^h(x), \psi^h(x)$ and $\xi^h(x)$.

Proof. DS (3.37) can be written in the following abstract form

$$\begin{cases} \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} + \alpha\frac{u_{k+1}^h - u_k^h}{\tau} + A_h u_{k+1}^h = p^h + f_k^h, \\ 1\leq k\leq N-1, \\ -\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} + A_h u_k^h = p^h + g_k^h, \\ -N+1\leq k\leq -1, \\ u_1^h - u_0^h = u_0^h - u_{-1}^h, u_0^h = \xi^h, \\ u_{-N}^h = \varphi^h, u_N^h = \psi^h. \end{cases}$$

in a Hilbert space L_{2h} with operator $A_h = A_h^x$ by formula (3.35). Here, $f_k^h = f_k^h(x), g_k^h = g_k^h(x)$ are given abstract functions, $u_k^h = u_k^h(x)$ is unknown function and $p^h = p^h(x)$ is the element of L_{2h}. Therefore, the proof of Theorem 3.1.1 is based on the self- adjointness and positive definiteness of the space operator A_h in L_{2h}.

Second, we consider the SIP (2.40). The discretization of problem (2.40) is carried out in two stages.

In the first stage, we define the grid space

$$\overline{\Omega_h} = \{x = x_r = h_1 j_1, ..., h_n j_n, j = (j_1, ..., j_n)\, 0 \leq j_r \leq N_r,$$

$N_r h_r = 1, r = 1, ..., n\}, \Omega_h = \overline{\Omega_h} \cap \Omega, S_h = \overline{\Omega_h} \cap S$

and introduce the Hilbert space $L_{2h} = L_2\left(\overline{\Omega_h}\right)$ of the grid functions $\phi^h(x) = \{\phi(h_1 j_1, ..., h_n j_n)\}$ defined on $\overline{\Omega_h}$ equipped with the norm

$$\left\|\phi^h\right\|_{L_{2h}} = \left(\sum_{x\in\overline{\Omega_h}} \left|\phi^h(x)\right|^2 h_1 ... h_n\right)^{1/2}.$$

To the differential operator A^x defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} \tag{3.38}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, u(x) = 0, x \in S\right\}$$

we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} (\alpha_r(x) u_{x_r}^h)_{x_r, j_r} \tag{3.39}$$

where A_h^x is known as SAPDO in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of the difference operator A_h^x, we arrive at the following SIP

$$\begin{cases} u_{tt}^h(t,x) + \alpha u_t^h(t,x) + A_h^x u^h(t,x) \\ = p^h(x) + f^h(t,x),\ x \in \Omega_h,\ 0 < t < 1, \\ -u_{tt}^h(t,x) + A_h^x u^h(t,x) \\ = p^h(x) + g^h(t,x),\ x \in \Omega_h,\ -1 < t < 0, \\ u^h(0,x) = \varphi^h(x), u_t^h(0^+,x) = u_t^h(0^-,x), \\ u^h(-1,x) = \psi^h(x), u^h(1,x) = \xi^h(x), \Omega_h. \end{cases} \tag{3.40}$$

In the second stage, we replace (3.40) with DS (3.1)

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \alpha\tau^{-1}\left(u_{k+1}^h(x) - u_k^h(x)\right) + A_h^x u_{k+1}^h(x) \\ = p^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), 1 \le k \le N-1, x \in \Omega_h, \\ -\tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + A_h^x u_k^h(x) = p^h(x) + g_k^h, \\ g_k^h(x) = g(t_k,x), -N+1 \le k \le -1, x \in \Omega_h, \\ u_1^h(x) - u_0^h(x) = u_0^h(x) - u_{-1}^h(x), u_0^h(x) = \xi^h(x), \\ u_{-N}^h(x) = \varphi^h(x), u_N^h(x) = \psi^h(x), x \in \overline{\Omega}_h. \end{cases} \tag{3.41}$$

Theorem 3.2.2. Suppose that $\alpha \geq 4, \left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2+1$. Then, for the solution $\left\{\left\{u_k^h(x)\right\}_{-N}^{N}, p^h(x)\right\}$ of problem (3.41) the following stabilty estimates hold:

$$\max_{-N\leq k\leq N}\|u_k\|_{L_{2h}}+\left\|\left(A_h^x\right)^{-1}p^h\right\|_{L_{2h}}$$

$$\leq M_3(\alpha,\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}\right. \tag{3.42}$$

$$\left.+\max_{-N+1\leq k\leq -1}\left\|g_k^h\right\|_{L_{2h}}+\max_{1\leq k\leq N-1}\left\|f_k^h\right\|_{L_{2h}}\right]_H,$$

$$\max_{-N+1\leq k\leq N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{-N+1\leq k\leq N}\left\|u_k^h\right\|_{W_{2h}^2}$$

$$\leq M_4(\alpha,\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}+\left\|g_{-1}^h\right\|_{L_{2h}}\right. \tag{3.43}$$

$$\left.+\max_{-N+1\leq k\leq -2}\left\|\frac{1}{\tau}\left(g_k^h-g_{k-1}^h\right)\right\|_{L_{2h}}+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right].$$

hold, where $M_3(\alpha,\delta), M_4(\alpha,\delta)$ does not depend on $f_k^h,\ 1\leq k\leq N-1,\ g_k^h,\ -N+1\leq k\leq -1$, $\varphi^h(x),\ \psi^h(x)$ and $\xi^h(x)$.

Proof. DS (3.41) can be written in the form (3.1) in a Hilbert space $L_{2h}=L_2\left(\overline{\Omega_h}\right)$ with SAPDO $A_h=A_h^x$ by formula (3.38).

Here, $f_k^h=f_k^h(x), g_k^h=g_k^h(x)$ are given abstract mesh functions and $u_k^h=u_k^h(x)$ is unknown abstract mesh function defined on $\overline{\Omega_h}$ and $p^h=p^h(x)$ is the element of L_{2h}. Therefore, estimates (3.42) and (3.43) follow from SEs of Theorem 3.1.1 and the following theorem on the coercivity stabilty estimate for the solution of the elliptic difference problem generated by (3.39) in L_{2h}.

Theorem 3.2.3. (Sobolevskii, 1975) For the solution of the elliptic differential problem

$$\begin{cases} A_h^x u^h(x)=\mu^h(x), x\in\Omega_h, \\ u^h(x)=0, x\in S_h \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n}\left\|u^h_{x_r x_r}\right\|_{L_{2h}} \leq M_5\|\mu^h\|_{L_{2h}}.$$

Here M_5 does not depend on h and μ^h.

Third, we consider the SIP (2.44). The discretization of problem (2.44) is carried out in two stages. To the differential operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} + \delta u \tag{3.44}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, \frac{\partial u\,(x)}{\partial\overrightarrow{m}} = 0, x \in S\right\}$$

we assign the difference operator A^x_h by the formula

$$A^x_h u^h = -\sum_{r=1}^{n}(\alpha_r(x)u^h_{x_r})_{x_r, j_r} + \delta u^h, \tag{3.45}$$

where A^x_h is known as SAPDO in L_{2h}, acting in the space of grid functions $u^h\,(x)$ satisfying the conditions $D^h u^h\,(x) = 0$ for all $x \in S_h$. Here $D^h u^h\,(x)$ is the first order of approximation of $\frac{\partial u(x)}{\partial\overrightarrow{m}}$. With the help of the difference operator A^x_h, we arrive at the following BVP

$$\begin{cases} u^h_{tt}(t,x) + \alpha u^h_t(t,x) + A^x_h u^h(t,x) = p^h(x) + f^h(t,x), \\ 0 < t < 1, x \in \Omega_h, \\ -u^h_{tt}(t,x) + A^x_h u^h(t,x) = p^h(x) + g^h(t,x), \\ -1 < t < 0, x \in \Omega_h, \\ u^h(0,x) = \varphi^h(x), u^h_t(0^+,x) = u^h_t(0^-,x), \\ u^h(-1,x) = \psi^h(x), u^h(1,x) = \xi^h(x), x \in \overline{\Omega}_h. \end{cases} \tag{3.46}$$

In the second stage, we replace (3.46) with DS (3.1)

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\alpha\tau^{-1}\left(u_{k+1}^h(x)-u_k^h(x)\right)+A_h^x u_{k+1}^h(x) \\ =p^h(x)+f_k^h(x), f_k^h(x)=f^h(t_k,x), \\ 1\le k\le N-1, x\in\Omega_h \\ -\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+A_h^x u_k^h(x) \\ =p^h(x)+g_k^h, g_k^h(x)=g(t_k,x), \\ -N+1\le k\le -1, x\in\Omega_h \\ u_1^h(x)-u_0^h(x)=u_0^h(x)-u_{-1}^h(x), u_0^h(x)=\xi^h(x), \\ u_{-N}^h(x)=\varphi^h(x), u_N^h(x)=\psi^h(x), x\in\Omega_h. \end{cases} \tag{3.47}$$

Theorem 3.2.4 Assume that $\left(\frac{\alpha}{2}+1\right)^2 \ge \delta \ge \left(\frac{\alpha}{2}\right)^2+1$, $\alpha \ge 4$. Then, for the solution $\left\{\left\{u_k^h(x)\right\}_{-N}^N, p^h(x)\right\}$ of problem (3.47) the following stabilty estimates hold:

$$\max_{-N\le k\le N}\|u_k\|_{L_{2h}}+\left\|\left(A_h^x\right)^{-1}p^h\right\|_{L_{2h}}$$

$$\le M_6(\alpha,\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}\right. \tag{3.48}$$

$$\left.+\max_{-N+1\le k\le -1}\left\|g_k^h\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\right]_H,$$

$$\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{-N+1\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}$$

$$\le M_7(\alpha,\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}+\left\|g_{-1}^h\right\|_{L_{2h}}\right. \tag{3.49}$$

$$\left.+\max_{-N+1\le k\le -2}\left\|\frac{1}{\tau}\left(g_k^h-g_{k-1}^h\right)\right\|_{L_{2h}}+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right].$$

hold, where $M_6(\alpha,\delta), M_7(\alpha,\delta)$ does not depend on f_k^h, $1\le k\le N-1$, g_k^h, $-N+1\le k\le -1$, $\varphi^h(x)$, $\psi^h(x)$ and $\xi^h(x)$.

Proof. DS (3.47) can be written in abstract form (3.1) in a Hilbert space L_{2h} with SAPDO $A_h=A_h^x$ by formula (3.45). Here, $f_k^h=f_k^h(x)$, $g_k^h(x)$ are known mesh functions, $u_k^h=u_k^h(x)$ is unknown mesh function defined on $\overline{\Omega_h}$ and $p^h=p^h(x)$ is the element of L_{2h}. Therefore, estimates (3.48) and (3.49) follow from estimates of Theorem 3.1.1 and the following theorem on the coercivity stabilty estimates for the solution of the elliptic difference problem generated by (3.45) in L_{2h}.

Theorem 3.2.5. (Sobolevskii, 1975) For the solution of the elliptic differential problem

$$\begin{cases} A_h^x u^h(x) = \mu^h(x), x \in \Omega_h, \\ \\ D^h u^h(x) = 0, x \in S_h \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \left\| u_{x_r x_r} \right\|_{L_2(\overline{\Omega})} \leq M_8 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here M_8 does not depend on h and μ^h.

CHAPTER 4

NUMERICAL RESULTS

In this chapter, we propose the numerical solution of the first order of accuracy DS (3.1). The rest of this chapter is organized as follows: In section 4.1, the numerical analysis for one dimensional elliptic-telegraph equations is provided. In section 4.2, numerical analysis for two dimensional elliptic-telegraph equations is provided.
The solution of SIP (2.1) can be written as

$$u(t) = \omega(t) + q, \tag{4.1}$$

where element q is the solution of operator equation

$$Aq = p, \tag{4.2}$$

and the function $\omega(t)$ is the solution of the nonlocal BVP

$$\begin{cases} \omega_{tt}(t) + \alpha\omega_t(t) + A\omega(t) = f(t), 0 < t < 1, \\ -\omega_{tt}(t) + A\omega(t) = g(t), -1 < t < 0, \\ \omega(0) - \omega(-1) = \varphi - \psi, \omega(-1) - \omega(1) = \psi - \xi, \omega_t(0^+) = \omega_t(0^-). \end{cases} \tag{4.3}$$

Taking into account all of the above, the following numerical algorithm can be used for the approximate solutions of the identification problem (2.1):

1. Search the approximate solution of the nonlocal BVP (4.3).
2. Find approximate the source p by using formula

$$p = A\xi - A\omega(1) \tag{4.4}$$

3. Obtain the approximate value of q by formula (4.2).
4. Find the approximate solutions of SIP (2.1) by formula (4.1).

4.1 One dimensional case

First, the identification problem with the Dirichlet condition

$$\begin{cases} u_{tt}(t,x) + 2u_t(t,x) - u_{xx}(t,x) \\ = p(x) - \sin x, \;\; x \in (0,\pi), t \in (0,1), \\ -u_{tt}u(t,x) - u_{xx}(t,x) \\ = p(x) - \sin x, \;\;\; x \in (0,\pi), t \in (-1,0), \\ u(0,x) = \sin x, u(-1,x) = e\sin x, u(1,x) = e^{-1}\sin x, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, t \in [-1,1] \end{cases} \tag{4.5}$$

is studied. The exact solution pair of this problem is

$$(u(t,x), p(x)) = (e^{-t}\sin x, \sin x), 0 \le x \le \pi, -1 \le t \le 1.$$

Here and in future, we denote the set $[-1,1]_\tau \times [0,\pi]_h$ of all grid points

$$[-1,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_k = k\tau, -N \le k \le N,$$

$$N\tau = 1, x_n = nh, 0 \le n \le M, Mh = \pi\}.$$

For the numerical solution of SIP (4.5), we present the first order of accuracy DS in t

$$\begin{cases} \tau^{-2}\left(u_n^{k+1} - 2u_n^k + u_n^{k-1}\right) + 2\tau^{-1}\left(u_n^{k+1} - u_n^k\right) \\ -h^{-2}\left(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}\right) = p_n - \sin x_n, 1 \le k \le N-1, \\ -\tau^{-2}\left(u_n^{k+1} - 2u_n^k + u_n^{k-1}\right) - h^{-2}\left(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}\right) \\ = p_n - \sin x_n, -N+1 \le k \le -1, 1 \le n \le M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1}, u_n^0 = \sin x_n, \\ u_n^{-N} = e\sin x_n, u_n^N = e^{-1}\sin x_n, 0 \le n \le M, \\ u_0^k = u_M^k = 0, -N \le k \le N. \end{cases} \tag{4.6}$$

In the first step, we will obtain $\left\{\left\{\omega_n^k\right\}_{k=-N}^{N}\right\}_{n=0}^{M}$ as solution of nonlocal BVP

$$
\begin{cases}
\tau^{-2}\left(\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}\right)+2\tau^{-1}\left(\omega_n^{k+1}-\omega_n^k\right) \\
-h^{-2}\left(\omega_{n+1}^{k+1}-2\omega_n^{k+1}+\omega_{n-1}^{k+1}\right)=-\sin x_n,\ 1\le k\le N-1, \\
-\tau^{-2}\left(\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}\right)-h^{-2}\left(\omega_{n+1}^{k+1}-2\omega_n^{k+1}+\omega_{n-1}^{k+1}\right) \\
=-\sin x_n,-N+1\le k\le -1, 1\le n\le M-1, \\
\omega_n^1-\omega_n^0=\omega_n^0-\omega_n^{-1}, \omega_n^0-\omega_n^{-N}=(1-e)\sin x_n, \\
\omega_n^0-\omega_n^N=\left(1-e^{-1}\right)\sin x_n, 0\le n\le M, \\
\omega_0^k=\omega_M^k=0,-N\le k\le N.
\end{cases}
\tag{4.7}
$$

Here ω_k^n denotes the numerical approximation of $\omega(t;x)$ at (t_k,x_n). For obtaining the solution of DS (4.7), we can write it in the matrix form as

$$
\begin{cases}
A\omega_{n+1}+B\omega_n+C\omega_{n-1}=F_n,\ 1\le n\le M-1, \\
\omega_0=\overrightarrow{o}, \omega_M=\overrightarrow{o},
\end{cases}
\tag{4.8}
$$

where $\overrightarrow{o}$ is column matrix with $(2N+1)$ zero elements,

$$
A=C=\begin{bmatrix}
0 & 0 & 0 & \cdots & \vdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & \vdots & \cdots & 0 & 0 & 0 & b & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \vdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & b \\
0 & 0 & 0 & \cdots & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & b & 0 & \cdots & \vdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & b & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1
\end{bmatrix}_{(2N+1)\times(2N+1)}
\tag{4.9}
$$

$$
B = \begin{bmatrix}
-1 & 0 & 0 & \cdots & \vdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & \vdots & 0 & 0 & d & c & a & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & a \\
0 & 0 & 0 & \cdots & \vdots & 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
g & j & g & \cdots & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & g & j & g & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & -1
\end{bmatrix}_{(2N+1)\times(2N+1)}, \qquad (4.10)
$$

$w_s = n, n \pm 1, F_n$ are $(2N+1) \times 1$ column matrices

$$
F_n = \begin{bmatrix}
(1-e^1)\sin x_n \\
-\sin x_n \\
\vdots \\
-\sin x_n \\
0 \\
-\sin x_n \\
\vdots \\
-\sin x_n \\
(1-e^{-1})\sin x_n
\end{bmatrix}, \omega_s = \begin{bmatrix}
\omega_s^1 \\
\omega_s^2 \\
\vdots \\
\omega_s^N \\
\omega_s^{N+1} \\
\vdots \\
\omega_s^{2N} \\
\omega_s^{2N+1}
\end{bmatrix}
$$

for $s = n, n \pm 1$.

Here

$$
a = \tfrac{1}{\tau^2} + \tfrac{2}{\tau} + \tfrac{2}{h^2}, b = -\tfrac{1}{h^2}, c = -\tfrac{2}{\tau^2} - \tfrac{2}{\tau}, d = \tfrac{1}{\tau^2},
$$
$$
g = -\tfrac{1}{\tau^2}, j = \tfrac{2}{\tau^2} + \tfrac{2}{h^2}.
$$

For the solution of the matrix equation (4.8), we use the modified Gauss elimination method. We seek a solution of the matrix equation (4.8) by the following form

$$
\omega_n = \alpha_{n+1}\omega_{n+1} + \beta_{n+1}, n = M-1, ..., 2, 1, \omega_M = 0, \qquad (4.11)
$$

where α_n $(1 \le n \le M-1)$ are $(2N+1) \times (2N+1)$ square matrices and β_n $(1 \le n \le M-1)$

are $(2N+1)\times 1$ column vectors, calculated as,

$$\begin{cases} \alpha_{n+1} = -Q_n A, \beta_{n+1} = Q_n(DF_n - C\beta_n), \\ \\ Q_n = (B + C\alpha_n)^{-1}, n = 1, 2, ..., M-1, \end{cases} \quad (4.12)$$

where α_1 and β_1 and zero matrices.

In the second step, using (4.4), we get

$$p_n = \frac{\omega_{n+1}^N - 2\omega_n^N + \omega_{n-1}^N}{h^2}$$
$$-e^{-1}\frac{\sin x_{n+1} - 2\sin x_n + \sin x_{n-1}}{h^2}, 1 \le n \le M-1.$$

In the third step, using (4.2), we get

$$q_n = -\omega_n^N + e^{-1}\sin x_n, 0 \le n \le M. \quad (4.13)$$

In the fourth step, using (4.1), we obtain

$$u_n^k = \omega_n^k + q_n, n = 0, 1, ..., M, k = -N, ..., N. \quad (4.14)$$

Here and in future, we compute the error between the exact solution and numerical solution by

$$\begin{cases} \|E_\omega\|_\infty = \max\limits_{-N\le k\le N, 0\le n\le M} \left|\omega(t_k, x_n) - \omega_n^k\right|, \\ \|E_u\|_\infty = \max\limits_{-N\le k\le N, 0\le n\le M} \left|u(t_k, x_n) - u_n^k\right|, \\ \|E_p\|_\infty = \max\limits_{0<n<M} \left|p(x_n) - p_n\right|, \end{cases} \quad (4.15)$$

where $u(t,x), p(x)$ represent the exact solution, u_n^k represent the numerical solutions at (t_k, x_n) and p_n represent the numerical solutions at x_n. The numerical results are given in the Table 4.1.

Table 4.1.

Errors	$\|E_\omega\|_\infty$	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N = M = 20$	0.3123	0.2753	0.0344
$N = M = 40$	0.1306	0.1153	0.0148
$N = M = 80$	0.0600	0.0530	0.0069
$N = M = 160$	0.0288	0.0254	0.0033

As it is seen in Table (4.1), if N and M are doubled, the value of errors decrease by a factor of approximately $1/2$.

Second, the SIP with the Neumann condition

$$\begin{cases} u_{tt}(t,x) + 2u_t(t,x) - \frac{1}{2}u_{xx}(t,x) + \frac{1}{2}u(t,x) \\ = p(x) - \cos x, x \in (0,\pi), t \in (0,1), \\ -u_{tt}(t,x) - \frac{1}{2}u_{xx}(t,x) + \frac{1}{2}u(t,x) \\ = p(x) - \cos x, x \in (0,\pi), t \in (-1,0), \\ u(0,x) = \cos x, u(-1,x) = e\cos x, \ u(1,x) = e^{-1}\cos x, x \in [0,\pi], \\ u_x(t,0) = 0, u_x(t,\pi) = 0, t \in [-1,1] \end{cases} \tag{4.16}$$

is considered. The exact solution of the dentification problem (4.16) is

$$(u(t,x), p(x)) = (e^{-t}\cos x, \cos x), 0 \le x \le \pi, -1 \le t \le 1.$$

For the numerical solution of identification problem (4.16) we present the first order of accuracy DS in t

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1}-u_n^k}{\tau} - \frac{1}{2}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + \frac{1}{2}u_n^{k+1} \\ = p_n - \cos x_n, 1 \le k \le N-1, 1 \le n \le M-1, \\ -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{1}{2}\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \frac{1}{2}u_n^k \\ = p_n - \cos x_n, -N+1 \le k \le -1, 1 \le n \le M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1}, u_n^0 = \cos x_n, \\ u_n^{-N} = e\cos x_n, u_n^N = e^{-1}\cos x_n, 0 \le n \le M, \\ u_1^k - u_0^k = 0, u_M^k - u_{M-1}^k = 0, -N \le k \le N. \end{cases} \tag{4.17}$$

In the first step, we will obtain $\left\{\left\{\omega_n^k\right\}_{k=-N}^N\right\}_{n=0}^M$ as solution of nonlocal BVP

$$\begin{cases} \frac{\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}}{\tau^2} + 2\frac{\omega_n^{k+1}-\omega_n^k}{\tau} - \frac{1}{2}\frac{\omega_{n+1}^{k+1}-2\omega_n^{k+1}+\omega_{n-1}^{k+1}}{h^2} + \frac{1}{2}\omega_n^{k+1} \\ = -\cos x_n, 1 \le k \le N-1, 1 \le n \le M-1, \\ -\frac{\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}}{\tau^2} - \frac{1}{2}\frac{\omega_{n+1}^k-2\omega_n^k+\omega_{n-1}^k}{h^2} + \frac{1}{2}\omega_n^k \\ = -\cos x_n, -N+1 \le k \le -1, 1 \le n \le M-1, \\ \omega_n^1 - \omega_n^0 = \omega_n^0 - \omega_n^{-1}, \omega_n^0 - \omega_n^{-N} = (1-e)\cos x_n, \\ \omega_n^{-N} - \omega_n^N = (1-e^{-1})\cos x_n, 0 \le n \le M, \\ \omega_0^k - \omega_1^k = 0, \omega_M^k - \omega_{M-1}^k = 0, -N \le k \le N. \end{cases} \tag{4.18}$$

Here ω_k^n denotes the numerical approximation of $\omega(t;x)$ at (t_k, x_n). For obtaining the solution of DS (4.18), we can write it in the matrix form as

$$\begin{cases} A\omega_{n+1} + B\omega_n + C\omega_{n-1} = F_n, 1 \le n \le M-1, \\ \omega_0 = \omega_1, \omega_M = \omega_{M-1}, \end{cases} \tag{4.19}$$

where $F_n, w_s, s = n, n \pm 1$ are $(2N+1) \times 1$ column matrices

$$F_n = \begin{bmatrix} (1-e^1)\cos x_n \\ -\cos x_n \\ \vdots \\ -\cos x_n \\ 0 \\ -\cos x_n \\ \vdots \\ -\cos x_n \\ (1-e^{-1})\cos x_n \end{bmatrix}_{(2N+1)\times 1}, \omega_s = \begin{bmatrix} \omega_s^1 \\ \omega_s^2 \\ \vdots \\ \omega_s^N \\ \omega_s^{N+1} \\ \vdots \\ \omega_s^{2N} \\ \omega_s^{2N+1} \end{bmatrix}_{(2N+1)\times 1}$$

and A, B, C are $(2N+1)\times$ $(2N+1)$ square matrices in the form (4.9), (4.10) with the corresponding nonzero elements

$$a = \tfrac{1}{\tau^2} + \tfrac{2}{\tau} + \tfrac{1}{h^2} + \tfrac{1}{2}, b = -\tfrac{1}{2h^2}, c = -\tfrac{2}{\tau^2} - \tfrac{2}{\tau}, d = \tfrac{1}{\tau^2},$$
$$g = -\tfrac{1}{\tau^2}, j = \tfrac{2}{\tau^2} + \tfrac{1}{h^2} + \tfrac{1}{2}.$$

For the solution of the matrix equation (4.19), we use the modified Gauss elimination method. We seek a solution of the matrix equation (4.19) by the following form

$$\omega_n = \alpha_{n+1}\omega_{n+1} + \beta_{n+1}, \ n = M-1, ..., 1, 0, \tag{4.20}$$

where α_n $(1 \le n \le M-1)$ are $(2N+1)\times(2N+1)$ square matrices and β_n $(1 \le n \le M-1)$ are $(2N+1) \times 1$ column vectors, calculated as,

$$\begin{cases} \alpha_{n+1} = -Q_n A, \ \beta_{n+1} = Q_n(DF_n - C\beta_n), \\ Q_n = (B + C\alpha_n)^{-1}, \ n = 1, 2, ..., M-1, \end{cases} \tag{4.21}$$

α_1 is an identity $(2N+1)\times(2N+1)$ square matrix and β_1 is $(2N+1)\times 1$ column vector with zero elements,

$$\omega_M = \omega_{M-1} = (A + B + C\alpha_{M-1})^{-1}(F_M - C\beta_{M-1}).$$

In the second step, using (4.4), we get

$$p_n = \frac{\omega_{n+1}^N - 2\omega_n^N + \omega_{n-1}^N}{h^2} - \omega_n^N$$
$$-e^{-1}\left(\frac{\cos x_{n+1} - 2\cos x_n + \cos x_{n-1}}{h^2} - \cos x_n\right), 1 \le n \le M-1.$$

In the third step, using (4.2), we get

$$q_n = -\omega_n^N + e^{-1}\cos x_n, 0 \le n \le M. \tag{4.22}$$

In the fourth step, using (4.1), we obtain (4.14). The numerical results are given in the Table 4.2.

Table 4.2.

Errors	$\|E_\omega\|_\infty$	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N = M = 20$	0.1513	0.1870	0.0289
$N = M = 40$	0.0669	0.0839	0.0096
$N = M = 80$	0.0318	0.0414	0.0046
$N = M = 160$	0.01557	0.0208	0.0023

As it is seen in Table (4.2), if N and M are doubled, the value of errors decrease by a factor of approximately $1/2$.

4.2 Two dimensional case

First, SIP with the Dirichlet condition

$$\begin{cases} u_{tt}(t,x,y)+2u_t(t,x,y)-\frac{1}{2}u_{xx}(t,x,y)-\frac{1}{2}u_{yy}(t,x,y) \\ =p(x,y)-\sin x\sin y, x,y\in(0,\pi), t\in(0,1), \\ -u_{tt}(t,x,y)-\frac{1}{2}u_{xx}(t,x,y)-\frac{1}{2}u_{yy}(t,x,y) \\ =p(x,y)-\sin x\sin y, x,y\in(0,\pi), t\in(-1,0), \\ u(0,x,y)=\sin x\sin y, u(-1,x,y)=e\sin x\sin y, \\ u(1,x,y)=e^{-1}\sin x\sin y,\ x,y\in[0,\pi], \\ u(t,0,y)=0,\ u(t,\pi,y)=0,\ 0\le y\le\pi,\ t\in[-1,1], \\ u(t,x,0)=0,\ u(t,x,\pi)=0,\ 0\le x\le\pi,\ t\in[-1,1] \end{cases} \tag{4.23}$$

is studied. The exact solution of SIP (4.23) is

$$(u(t,x,y),p(x,y))=(e^{-t}\sin x\sin y,\sin x\sin y), 0\le x,y\le\pi, -1\le t\le 1.$$

Here and in future, we denote the set $[-1,1]_\tau\times[0,\pi]_h\times[0,\pi]_h$ of all grid points

$$[-1,1]_\tau\times[0,\pi]_h\times[0,\pi]_h=\{(t_k,x_n,y_m): t_k=k\tau, -N\le k\le N,$$

$$N\tau=1, x_n=nh, y_m=mh, 0\le n,m\le M, Mh=\pi\}.$$

For the numerical solution of SIP (4.23), we construct the first order of accuracy DS in t

$$\begin{cases} \frac{u_{n,m}^{k+1}-2u_{n,m}^{k}+u_{n,m}^{k-1}}{\tau^2}+2\frac{u_{n,m}^{k+1}-u_{n,m}^{k}}{\tau}-\frac{u_{n+1,m}^{k+1}-2u_{n,m}^{k+1}+u_{n-1,m}^{k+1}}{2h^2} \\ -\frac{u_{n,m+1}^{k+1}-2u_{n,m}^{k+1}+u_{n,m-1}^{k+1}}{2h^2}=p_{n,m}-\sin x_n\sin y_m, \\ 1\le k\le N-1, 1\le n,m\le M-1, \\ -\frac{u_{n,m}^{k+1}-2u_{n,m}^{k}+u_{n,m}^{k-1}}{\tau^2}-\frac{u_{n+1,m}^{k}-2u_{n,m}^{k}+u_{n-1,m}^{k}}{2h^2} \\ -\frac{u_{n,m+1}^{k}-2u_{n,m}^{k}+u_{n,m-1}^{k}}{2h^2}=p_{n,m}-\sin x_n\sin y_m, \\ -N+1\le k\le -1, 1\le n,m\le M-1, \\ u_{n,m}^{1}-u_{n,m}^{0}=u_{n,m}^{0}-u_{n,m}^{-1}, u_{n,m}^{0}=\sin x_n\sin y_m, 0\le n,m\le M, \\ u_{n,m}^{-N}=e\sin x_n\sin y_m, u_{n,m}^{N}=e^{-1}\sin x_n\sin y_m, 0\le n,m\le M, \\ u_{n,0}^{k}=0, u_{n,M}^{k}=0, u_{0,m}^{k}=0, u_{M,m}^{k}=0, \\ 0\le n,m\le M, -N\le k\le N. \end{cases} \tag{4.24}$$

In the first step, we will obtain $\left\{\left\{\omega^{k}_{n,m}\right\}_{k=-N}^{N}\right\}_{n,m=0}^{M}$ as solution of nonlocal BVP

$$\begin{cases} \frac{\omega^{k+1}_{n,m}-2\omega^{k}_{n,m}+\omega^{k-1}_{n,m}}{\tau^2}+2\frac{\omega^{k+1}_{n,m}-\omega^{k}_{n,m}}{\tau}-\frac{\omega^{k+1}_{n+1,m}-2\omega^{k+1}_{n,m}+\omega^{k+1}_{n-1,m}}{2h^2} \\ -\frac{\omega^{k+1}_{n,m+1}-2\omega^{k+1}_{n,m}+\omega^{k+1}_{n,m-1}}{2h^2}=-\sin x_n \sin y_m, \\ 1\le k\le N-1, 1\le n,m\le M-1, \\ -\frac{\omega^{k+1}_{n,m}-2\omega^{k}_{n,m}+\omega^{k-1}_{n,m}}{\tau^2}-\frac{\omega^{k}_{n+1,m}-2\omega^{k}_{n,m}+\omega^{k}_{n-1,m}}{2h^2} \\ -\frac{\omega^{k}_{n,m+1}-2\omega^{k}_{n,m}+\omega^{k}_{n,m-1}}{2h^2}=-\sin x_n \sin y_m, \\ -N+1\le k\le -1, 1\le n,m\le M-1, \\ \omega^{1}_{n,m}-\omega^{0}_{n,m}=\omega^{0}_{n,m}-\omega^{-1}_{n,m}, \omega^{0}_{n,m}-\omega^{-N}_{n,m}=(1-e)\sin x_n \sin y_m, \\ \omega^{0}_{n,m}-\omega^{N}_{n,m}=\left(1-e^{-1}\right)\sin x_n \sin y_m,\ 0\le n,m\le M, \\ \omega^{k}_{n,0}=0, \omega^{k}_{n,M}=0, \omega^{k}_{0,m}=0, \omega^{k}_{M,m}=0, \\ 0\le n,m\le M, -N\le k\le N. \end{cases} \tag{4.25}$$

where $\omega^{k}_{s,m}$ denotes the numerical approximation of $\omega(t,x,y)$ at (t_k,x_n,y_m). For obtaining the solution of DS (4.25), we can write it in the matrix form as

$$\begin{cases} A\omega_{n+1}+B\omega_n+C\omega_{n-1}=F_n, 1\le n\le M-1, \\ \omega_0=\vec{O}, \omega_M=\vec{O}, \end{cases} \tag{4.26}$$

where $\vec{O}$ is $(2N+1)(M+1)\times 1$ column matrix with zero elements, A, B, C are $(2N+1)(M+1)\times(2N+1)(M+1)$ square matrices:

$$A=C=\begin{bmatrix} O & O & \cdots & O & O \\ O & X & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & X & O \\ O & O & \cdots & O & O \end{bmatrix}, \tag{4.27}$$

$$B=\begin{bmatrix} I & O & O & \cdots & O & O & O \\ X & Y & X & \cdots & O & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ O & O & O & \cdots & X & Y & X \\ O & O & O & \cdots & O & O & O \end{bmatrix}, \tag{4.28}$$

$$X = Z = \begin{bmatrix} O_{N\times N} & O_{N\times 1} & P_{N\times N} \\ P_{N\times N} & O_{N\times 1} & O_{N\times N} \\ O_{1\times N} & 0 & O_{1\times N} \end{bmatrix}_{(2N+1)\times(2N+1)}, \tag{4.29}$$

$$Y = \begin{bmatrix} -1 & 0 & 0 & \cdots & \vdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \vdots & 0 & 0 & d & c & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & a \\ 0 & 0 & 0 & \cdots & \vdots & 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ g & j & g & \cdots & \vdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & g & j & g & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & -1 \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{4.30}$$

$w_s = n, n \pm 1, F_n$ are $(2N+1)(M+1) \times 1$ column matrices such as

$$F_n = \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix}_{(M+1)\times 1}, f_{n,m} = \begin{bmatrix} (1-e^{1}) \sin x_n \sin y_m \\ -\sin x_n \sin y_m \\ \vdots \\ -\sin x_n \sin y_m \\ 0 \\ -\sin x_n \sin y_m \\ \vdots \\ -\sin x_n \sin y_m \\ (1-e^{-1}) \sin x_n \sin y_m \end{bmatrix}_{(2N+1)\times 1}$$

,

$$\omega_s = \begin{bmatrix} \varpi_{s,0} \\ \varpi_{s,1} \\ \vdots \\ \varpi_{s,M-1} \\ \varpi_{s,M} \end{bmatrix}_{(M+1)\times 1}, \varpi_{s,m} = \begin{bmatrix} \omega^1_{s,m} \\ \omega^2_{s,m} \\ \vdots \\ \omega^N_{s,m} \\ \omega^{N+1}_{s,m} \\ \vdots \\ \omega^{2N}_{s,m} \\ \omega^{2N+1}_{s,m} \end{bmatrix}_{(2N+1)\times 1}$$

$$P_{N\times N} = diag\{0, b, ..., b\}, O = O_{(2N+1)\times(2N+1)}, I = I_{(2N+1)\times(2N+1)}$$
$$b = \tfrac{1}{\tau^2} + \tfrac{2}{\tau} + \tfrac{2}{h^2}, a = -\tfrac{1}{2h^2}, c = -\tfrac{2}{\tau^2} - \tfrac{2}{\tau}, d = \tfrac{1}{\tau^2},$$
$$g = -\tfrac{1}{\tau^2}, j = \tfrac{2}{\tau^2} + \tfrac{2}{h^2}.$$

For the solution of the matrix equation (4.26), we use the modified Gauss elimination method. We seek a solution of the matrix equation (4.26) by the following form

$$\omega_n = \alpha_{n+1}\omega_{n+1} + \beta_{n+1}, n = M-1, ..., 2, 1, \omega_M = \overrightarrow{O},$$

where α_n $(1 \leq n \leq M-1)$ are $(2N+1)\times(2N+1)$ square matrices and β_n $(1 \leq n \leq M-1)$ are $(2N+1)\times 1$ column vectors, calculated as,

$$\begin{cases} \alpha_{n+1} = -Q_n A, \beta_{n+1} = Q_n(DF_n - C\beta_n), \\ Q_n = (B + C\alpha_n)^{-1}, n = 1, 2, ..., M-1, \end{cases}$$

where α_1 is a zero matrix and β_1 is a zero matrix.

In the second step, using (4.2), we get

$$p_{n,m} = \frac{\omega^{2N+1}_{n+1,m} - 2\omega^{2N+1}_{n,m} + \omega^{2N+1}_{n-1,m}}{2h^2} + \frac{\omega^{2N+1}_{n,m+1} - 2\omega^{2N+1}_{n,m} + \omega^{2N+1}_{n,m-1}}{2h^2}$$
$$-e^{-1}\frac{\sin x_{n+1} - 2\sin x_n + \sin x_{n-1}}{2h^2}\sin y_m$$
$$-e^{-1}\frac{\sin y_{m+1} - 2\sin y_m + \sin y_{m-1}}{2h^2}\sin x_n, 1 \leq n, m \leq M-1$$

In the third step, using (4.2), we get

$$q_{n,m} = -\omega^N_{n,m} + e^{-1}\sin x_n \sin x_m,\ 0 \leq n, m \leq M. \tag{4.31}$$

In the fourth step, using (4.1), we obtain

$$u_{n,m}^{k} = \omega_{n,m}^{k} + q_{n,m}, n, m = 0, 1, ..., M, k = -N, ..., N.$$

We compute the error between the exact solution and numerical solution by

$$\begin{cases} \|E_u\|_\infty = \max\limits_{-N\le k\le N, 0<n,m<M} \left|u(t_k, x_n, y_m) - u_{n,m}^{k}\right|, \\ \|E_\omega\|_\infty = \max\limits_{-N\le k\le N, 0<n,m<M} \left|\omega(t_k, x_n, y_m) - \omega_{n,m}^{k}\right|, \\ \|E_p\|_\infty = \max\limits_{0<n,m<M} \left|p(x_n, y_m) - p_{n,m}\right|, \end{cases}$$

where $u(t, x, y), p(x, y)$ represent the exact solution, $u_{n,m}^{k}$ represent the numerical solutions at (t_k, x_n, y_m) and $p_{n,m}$ represent the numerical solutions at (t_k, x_n, y_m). The numerical results are given in the following table.

Table 4.3.

Errors	$\|E_\omega\|_\infty$	$\left\|E_p\right\|_\infty$	$\|E_u\|_\infty$
$N = M = 10$	0.9321	0.8196	0.0974
$N = M = 20$	0.3123	0.2753	0.0344
$N = M = 40$	0.1306	0.1153	0.0148

As it is seen in Table (4.3), if N and M are doubled, the value of errors decrease by a factor of approximately $1/2$.

Second, the SIP with the Neumann condition

$$\begin{cases} u_{tt}(t, x, y) + 2u_t(t, x, y) - \frac{1}{4}u_{xx}(t, x, y) - \frac{1}{4}u_{yy}(t, x, y) + \frac{1}{2}u(t, x, y) \\ = p(x, y) - \cos x \cos y + e^{-t} \cos x \cos y, x, y \in (0, \pi), t \in (0, 1), \\ -u_{tt}(t, x, y) - \frac{1}{4}u_{xx}(t, x, y) - \frac{1}{4}u_{yy}(t, x, y) + \frac{1}{2}u(t, x, y) \\ = p(x, y) - \cos x \cos y + e^{-t} \cos x \cos y, x, y \in (0, \pi), t \in (-1, 0), \\ u(0, x, y) = \cos x \cos y, u(-1, x, y) = e \cos x \cos y, \\ u(1, x, y) = e^{-1} \cos x \cos y, x, y \in [0, \pi], \\ u_x(t, 0, y) = u_x(t, \pi, y) = 0, t \in [-1, 1], 0 \le y \le \pi, \\ u_y(t, x, 0) = u_y(t, x, \pi) = 0, t \in [-1, 1], 0 \le x \le \pi \end{cases} \quad (4.32)$$

is investigated. The exact solution of SIP (4.32) is

$$(u(t,x,y),p(x,y)) = \left(e^{-t}\cos x\cos y, \cos x\cos y\right), 0\le x,y\le\pi, -1\le t\le 1.$$

For the numerical solution of SIP (4.32), we construct the first order of accuracy DS in t

$$\begin{cases} \frac{u_{n,m}^{k+1}-2u_{n,m}^{k}+u_{n,m}^{k-1}}{\tau^2} + 2\frac{u_{n,m}^{k+1}-u_{n,m}^{k}}{\tau} - \frac{u_{n+1,m}^{k+1}-2u_{n,m}^{k+1}+u_{n-1,m}^{k+1}}{4h^2} \\ -\frac{u_{n,m+1}^{k+1}-2u_{n,m}^{k+1}+u_{n,m-1}^{k+1}}{4h^2} + \frac{1}{2}u_{n,m}^{k+1} = p_{n,m} - \cos x_n\cos y_m, \\ 1\le k\le N-1, 1\le n,m\le M-1, \\ -\frac{u_{n,m}^{k+1}-2u_{n,m}^{k}+u_{n,m}^{k-1}}{\tau^2} - \frac{u_{n+1,m}^{k}-2u_{n,m}^{k}+u_{n-1,m}^{k}}{4h^2} \\ -\frac{u_{n,m+1}^{k}-2u_{n,m}^{k}+u_{n,m-1}^{k}}{4h^2} + \frac{1}{2}u_{n,m}^{k+1} = p_{n,m} - \cos x_n\cos y_m, \\ -N+1\le k\le -1, 1\le n,m\le M-1, \\ u_{n,m}^{1}-u_{n,m}^{0} = u_{n,m}^{0}-u_{n,m}^{-1}, u_{n,m}^{0} = \cos x_n\cos y_m, 0\le n,m\le M, \\ u_{n,m}^{-N} = e\cos x_n\cos y_m, 0\le n,m\le M, \\ u_{n,m}^{N} = e^{-1}\cos x_n\cos y_m, 0\le n,m\le M, \\ u_{n,0}^{k} = u_{n,1}^{k}, u_{n,M}^{k} = u_{n,M-1}^{k}, u_{0,m}^{k} = u_{1,m}^{k}, u_{M,m}^{k} = u_{M-1,m}^{k}, \\ 0\le n,m\le M, -N\le k\le N. \end{cases} \tag{4.33}$$

In the first step, we will obtain $\left\{\left\{\omega_{n,m}^{k}\right\}_{k=-N}^{N}\right\}_{n,m=0}^{M}$ as solution of nonlocal BVP

$$\begin{cases} \frac{\omega_{n,m}^{k+1}-2\omega_{n,m}^{k}+\omega_{n,m}^{k-1}}{\tau^2} + 2\frac{\omega_{n,m}^{k+1}-\omega_{n,m}^{k}}{\tau} - \frac{\omega_{n+1,m}^{k+1}-2\omega_{n,m}^{k+1}+\omega_{n-1,m}^{k+1}}{4h^2} \\ -\frac{\omega_{n,m+1}^{k+1}-2\omega_{n,m}^{k+1}+\omega_{n,m-1}^{k+1}}{4h^2} + \frac{1}{2}\omega_{n,m}^{k+1} = -\cos x_n\cos y_m, \\ 1\le k\le N-1, 1\le n,m\le M-1, \\ -\frac{\omega_{n,m}^{k+1}-2\omega_{n,m}^{k}+\omega_{n,m}^{k-1}}{\tau^2} - \frac{\omega_{n+1,m}^{k}-2\omega_{n,m}^{k}+\omega_{n-1,m}^{k}}{2h^2} \\ -\frac{\omega_{n,m+1}^{k}-2\omega_{n,m}^{k}+\omega_{n,m-1}^{k}}{2h^2} \frac{1}{2}\omega_{n,m}^{k+1} = -\cos x_n\cos y_m, \\ -N+1\le k\le -1, 1\le n,m\le M-1, \\ \omega_{n,m}^{1}-\omega_{n,m}^{0} = \omega_{n,m}^{0}-\omega_{n,m}^{-1}, \omega_{n,m}^{0}-\omega_{n,m}^{-N} = (1-e)\cos x_n\cos y_m, \\ \omega_{n,m}^{0}-\omega_{n,m}^{N} = \left(1-e^{-1}\right)\cos x_n\cos y_m, 0\le n,m\le M, \\ \omega_{n,0}^{k} = \omega_{n,1}^{k}, \omega_{n,M}^{k} = \omega_{n,M-1}^{k}, \omega_{0,m}^{k} = \omega_{1,m}^{k}, \omega_{M,m}^{k} = \omega_{M-1,m}^{k}, \\ 0\le n,m\le M, -N\le k\le N. \end{cases} \tag{4.34}$$

where $\omega_{s,m}^{k}$ denotes the numerical approximation of $\omega(t,x,y)$ at (t_k,x_n,y_m).

In the second step, using (4.34), we get

$$p_{n,m} = \frac{\omega^{2N+1}_{n+1,m} - 2\omega^{2N+1}_{n,m} + \omega^{2N+1}_{n-1,m}}{4h^2} + \frac{\omega^{2N+1}_{n,m+1} - 2\omega^{2N+1}_{n,m} + \omega^{2N+1}_{n,m-1}}{4h^2} - \frac{1}{2}\omega^{2N+1}_{n,m}$$
$$-e^{-1}\frac{\cos x_{n+1} - 2\cos x_n + \cos x_{n-1}}{4h^2}\cos y_m$$
$$-e^{-1}\frac{\cos y_{m+1} - 2\cos y_m + \cos y_{m-1}}{4h^2}\cos x_n,\ 1 \le n, m \le M-1.$$

In the third step, using (4.34), we get

$$q_{n,m} = -\omega^N_{n,m} + e^{-1}\cos x_n \cos x_m,\ \ 0 \le n, m \le M.$$

In the fourth step, using (4.1), we obtain

$$u^k_{n,m} = \omega^k_{n,m} + q_{n,m}, n, m = 0, 1, ..., M, k = -N, ..., N.$$

For obtaining the solution of DS (4.34), we will write it in the matrix form as

$$\begin{cases} A\omega_{n+1} + B\omega_n + C\omega_{n-1} = F_n, 1 \le n \le M-1, \\ \omega_0 = \omega_1, \omega_M = \omega_{M-1}, \end{cases}$$

where

$$B = \begin{bmatrix} I & -I & O & \cdots & O & O & O \\ X & Y & X & \cdots & O & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ O & O & O & \cdots & X & Y & X \\ O & O & O & \cdots & O & -I & I \end{bmatrix},$$

and A, C, X, Y are matrices defined by (4.27), (4.29), (4.30) with

$$P_{N\times N} = diag\{0, b, ..., b\}, O = O_{(2N+1)\times(2N+1)}, I = I_{(2N+1)\times(2N+1)}$$
$$b = \frac{1}{\tau^2} + \frac{2}{\tau} + \frac{1}{h^2} + \frac{1}{2}, a = -\frac{1}{4h^2}, c = -\frac{2}{\tau^2} - \frac{2}{\tau}, d = \frac{1}{\tau^2},$$
$$g = -\frac{1}{\tau^2}, j = \frac{2}{\tau^2} + \frac{1}{h^2} + \frac{1}{2}.$$

The numerical results are given in the next table.

Table 4.4.

Errors	$\|E_\omega\|_\infty$	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N = M = 10$	0.3278	0.3765	0.0408
$N = M = 20$	0.1334	0.1760	0.0181
$N = M = 40$	0.0610	0.0841	0.0088

As it is seen in Table (4.4), if N and M are doubled, the value of errors decrease by a factor of approximately $1/2$.

CHAPTER 5
CONCLUSIONS

This thesis is devoted to study the SIP for elliptic-telegraph differential equations with unknown parameter $p(x)$. The following results are obtained:

- Fourier series, Laplace transform and Fourier transform methods are applied for the solution of several identification problems for elliptic-telegraph differential equations.
- The main theorem on the stability estimate for the solution of the SIP for elliptic-telegraph equation is proved.
- Stability estimates for the solution of three SIPs for elliptic-telegraph equation are established.
- The first order of accuracy DSs for the approximate solution of the SIP for elliptic-telegraph equation for one Dimensional and two Dimensional are presented.
- The main theorem on the stability estimates for the solution of DSs for the approximate solution of identification problem for elliptic-telegraph equation are proved.
- Stability estimates for the solution of DSs for three SIPs for elliptic-telegraph equation are established.
- The Matlab implementation of these DSs is presented.
- The theoretical statements for the solution of these DSs are supported by the results of numerical examples.

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APPENDICES

APPENDIX 1

MATLAB PROGRAMMING

1. Matlab Implementation for One Dimensional first orders of Difference Schemes with Dirichlet Condition

```
function firstorder-de-d(N,M)
if nargin < 1;
end;
close;close;
M=20;
N=M;
tau=1/N;
h =pi/M;
a=-(1/(h^2));
d=(1/(tau^2));
c= -(2/(tau^2)) − (2/tau);
b=(1/(tau^2)) + (2/tau) + (2/(h^2));
g=-(1/(tau^2));
z=(2/(tau^2)) + (2/(h^2));
A=zeros(2*N+1,2*N+1);
B=zeros(2*N+1,2*N+1);
for i=2:N;
A(i,i+N+1)=a;
A(N+i,i)=a;
B(i,i+N-1)=d;
B(i,i+N)=c;
B(i,i+N+1)=b;
```

```
B(i+N,i-1)=g;
B(i+N,i)=z;
B(i+N,i+1)=g;
end;
B(1,1)=-1;
B(1,N+1)=1;
B(2*N+1,N+1)=1;
B(2*N+1,2*N+1)=-1;
B(N+1,N)=1;
B(N+1,N+1)=-2;
B(N+1,N+2)=1;
C=A;
D=eye(2*N+1,2*N+1);
for j=1:M+1;
fi(1,j)=(1-exp(1))*sin((j-1)*h);
fi(2*N+1,j)=(1-exp(-1))*sin((j-1)*h);
for k=2:N;
fi(k,j)=-sin((j-1)*h);
end;
fi(N+1,j)=0;
for k=N+2:2*N;
fi(k,j)=-sin((j-1)*h);
end;
end;
alpha1=zeros(2*N+1,2*N+1);
betha1=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alphaj-1);
alphaj=-Q*A;
```

```
bethaj=Q*(D*(fi(:,j))-C*bethaj-1);
end;
w=zeros(2*N+1,M+1);
for j=M:-1:1;
w(:,j)=alphaj*w(:,j+1)+bethaj;
end;
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
esw(k,j)=(exp(-(k-1-N)*tau)-1)*sin((j-1)*h);
end;
end;
for j=1:M+1;
for k=1:2*N+1;
esU(k,j)=(exp(-(k-1-N)*tau))*sin((j-1)*h);
end;
end;
for j=1:M+1;
ep(j)=sin((j-1)*h);
end;
for j=1:M+1;
q(j)=-w(2*N+1,j)+((exp(-1))*sin((j-1)*h));
end;
for j=2:M;
p(j)=-(((q(j+1))-(2*q(j))+(q(j-1)))/(h^2));
end;
p(1)=0;
p(M+1)=0;
for k=1:2*N+1;
```

```
for j=1:M+1;
U(k,j)=w(k,j)+q(j);
end;
end;
figure;
m(1,1)=min(min(w))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(esw) ; rotate3d ;axis tight;
figure;
surf(m);
hold;
surf(w); rotate3d; axis tight;
title('FIRST ORDER');
maxes=max(max(esw));
maxerrorw=max(max(abs(esw-w)));
maxerroru=max(max(abs(esU-U)));
maxerrorp=max(max(abs(ep-p)));
cevap1 = [maxerrorw,maxerroru,maxerrorp]
```

APPENDIX 2

MATLAB PROGRAMMING

2. Matlab Implementation for One Dimensional first orders of Difference Schemes with Neumann Condition

```
function firstorder-te-n(N,M)
if nargin <1;
end;
close;close;
M=40;
N=40;
tau=1/N;
h =pi/M;
b=-1/(2*h^2);
d=1/(tau^2);
c=-2/tau^2 - 2/tau;
a=1/(tau^2) + 2/tau + 1/h^2 + 1/2;
g=-1/(tau^2);
z=2/(tau^2) + 1/(h^2) + 1/2;
A = zeros(2*N+1,2*N+1);
B=zeros(2*N+1,2*N+1);
for i=2:N;
A(i,i+N+1)=b;
A(N+i,i)=b;
B(i,i+N-1)=d;
B(i,i+N)=c;
B(i,i+N+1)=a;
```

```
B(N+i,i-1)=g;
B(N+i,i)=z;
B(N+i,i+1)=g;
end;
C=A;
B(1,1)=-1;
B(1,N+1)=1;
B(2*N+1,N+1)=1;
B(2*N+1,2*N+1)=-1;
B(N+1,N)=1;
B(N+1,N+1)=-2;
B(N+1,N+2)=1;
D=eye(2*N+1,2*N+1);
fii=zeros(2*N+1,M+1);
for j=2:M;
fii(1,j)=(1-exp(1))*cos((j-1)*h);
fii(2*N+1,j)=(1-exp(-1))*cos((j-1)*h);
for k=2:N;
fii(k,j)=-cos((j-1)*h);
fii(N+k,j)=-cos((j-1)*h);
end;
fii(N+1,j)=0;
end;
alpha1=D;
betha1=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alphaj-1);
alphaj=-Q*A;
bethaj=Q*(D*(fii(:,j))-C*bethaj-1);
```

```
end;
w=zeros(2*N+1,M+1);
QZ=(A+B+C*alphaM-1);
Q=inv(QZ);
w(:,M+1)=Q*(D*fii(:,M)-C*bethaM-1);
w(:,M)=w(:,M+1);
for j=M-1:-1:1;
w(:,j)=alphaj*w(:,j+1)+bethaj;
end;
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
esw(k,j)=(exp(-(k-1-N)*tau)-1)*cos((j-1)*h);
esU(k,j)=exp(-(k-1-N)*tau)*cos((j-1)*h);
end;
end;
for j=1:M+1;
ep(j)=cos((j-1)*h);
end;
for j=1:M+1;
q(j)=-w(2*N+1,j)+exp(-1)*cos((j-1)*h);
end;
for j=2:M;
p(j)=-1/2*(q(j+1)-2*q(j)+q(j-1))/h^2 + 1/2 * q(j);
end;
p(1)=ep(1);p(M+1)=ep(M+1);
for k=1:2*N+1;
for j=1:M+1;
U(k,j)=w(k,j)+q(j);
```

```
end;
end;
figure;
m(1,1)=min(min(esw))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(esw); rotate3d ;axis tight; title('Exact');
figure;
surf(m);
hold;
surf(w); rotate3d; axis tight;
title('FIRST ORDER');
maxes=max(max(esw));
maxerrorw=max(max(abs(esw-w)));
maxerroru=max(max(abs(esU-U)));
maxerrorp=max(max(abs(ep-p)));
cevap1 = [maxerrorw,maxerroru,maxerrorp]
```

APPENDIX 3

MATLAB PROGRAMMING

3. Matlab Implementation for Two Dimensional first orders of Difference Scheme with Dirichlet Condition

```
function firstorder-te-d3(N,M)
if nargin < 1;
end;
close;close;
M=20;
N=M;
tau=1/N;
h =pi/M;
N21=2*N+1;
a=-1/(2*h^2); d = 1/tau^2;
c= -2/tau^2 - 2/tau;
b=1/tau^2 + 2/tau + 2/h^2;
g=-1/tau^2;
z=2/tau^2 + 2/h^2;
NK=N21*(M+1);
A=zeros(NK);
B=eye(NK);
for n=2:M;
ii=N21*(n-1);
for i=ii+2:ii+N;
A(i,i+N+1)=a;
A(N+i,i)=a;
```

```
B(i,i+N+1)=b;B(i,i+N)=c;B(i,i+N-1)=d;            B(i,i+N+1+N21)=a;B(i,i+N+1-N21)=a;
B(i,i)=0;
B(i+N,i-1)=g;B(i+N,i)=z;B(i+N,i+1)=g;
B(i+N,i+N21)=a;B(i+N,i-N21)=a;B(i+N,i+N)=0;
end;
B(ii+1,ii+1)=-1;B(ii+1,ii+N+1)=1;
B(ii+2*N+1,ii+N+1)=1;B(ii+2*N+1,ii+2*N+1)=-1;
B(ii+N+1,ii+N)=1;B(ii+N+1,ii+N+1)=-2;B(ii+N+1,ii+N+2)=1;
end;
C=A;
D=eye(NK);
fii=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h;ii=N21*(n-1);
for j=2:M;
y=(j-1)*h;
for k=2:N;
fii(ii+k,j)=-sin(x)*sin(y);
fii(ii+N+k,j)=-sin(x)*sin(y);
end;
fii(ii+1,j)=(1-exp(1))*sin(x)*sin(y);
fii(ii+N+1,j)=0;
fii(ii+2*N+1,j)=(1-exp(-1))*sin(x)*sin(y);
end;
end;
alpha1=zeros(NK);
betha1=zeros(NK,1);
for j=2:M
Q=inv(B+C*alphaj-1);
```

```
alphaj=-Q*A;
bethaj=Q*(D*(fii(:,j))-C*bethaj-1);
end;
w=zeros(NK,M+1);
for j=M:-1:1;
w(:,j)=alphaj*w(:,j+1)+bethaj;
end;
'EXACT SOLUTION OF THIS PROBLEM';
esw=zeros(NK,M+1);
esU=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h; ii=N21*(n-1);
for j=2:M;
y=(j-1)*h;
for k=1:2*N+1;
esw(ii+k,j)=(exp(-(k-1-N)*tau)-1)*sin(x)*sin(y);
esU(ii+k,j)=(exp(-(k-1-N)*tau))*sin(x)*sin(y);
end;
end;
end; ep=zeros(M+1,M+1);
q=zeros(M+1,M+1);
p=zeros(M+1,M+1);
for n=1:M+1;
x=(n-1)*h;ii=(2*N+1)*n;
for j=1:M+1; y=(j-1)*h;
ep(n,j)=sin(x)*sin(y);
q(n,j)=-w(ii,j)+exp(-1)*sin(x)*sin(y);
end;
end;
```

```
for n=2:M;
for j=2:M;
p(n,j)=-(q(n,j+1)-2*q(n,j)+q(n,j-1))/(2*h^2) - (q(n+1,j) - 2*q(n,j) + q(n-1,j))/(2*h^2);
end;
end;
for n=1:M+1;
ii=N21*(n-1);
for j=1:M+1;
for k=1:2*N+1;
U(ii+k,j)=w(ii+k,j)+q(n,j);
end;
end;
end;
maxes=max(max(esw));
maxerrorw=max(max(abs(esw-w)));
maxerroru=max(max(abs(esU-U)));
maxerrorp=max(max(abs(ep-p)));
cevap1 = [maxerrorw,maxerroru,maxerrorp]
```

APPENDIX 4

MATLAB PROGRAMMING

4. Matlab Implementation for Two Dimensional first orders of Difference Schemes with Neumann Condition

```
function firstorder-te-n3(N,M)
if nargin <1;
end;
close;close;
M=20;
N=M;
tau=1/N;
h =pi/M;
N21=2*N+1;
a=-1/(4*h^2); d = 1/tau^2;
c= -2/tau^2 - 2/tau;
b=1/tau^2 + 2/tau + 1/h^2 + 1/2;
g=-1/tau^2;
z=2/tau^2 + 1/h^2 + 1/2;
NK=N21*(M+1);
A=zeros(NK);
B=eye(NK);
for n=2:M;
ii=N21*(n-1);
for i=ii+2:ii+N;
A(i,i+N+1)=a;
A(N+i,i)=a;
```

```
B(i,i+N+1)=b;B(i,i+N)=c;B(i,i+N-1)=d;
B(i,i+N+1+N21)=a;B(i,i+N+1-N21)=a; B(i,i)=0;
B(i+N,i-1)=g;B(i+N,i)=z;B(i+N,i+1)=g;
B(i+N,i+N21)=a;B(i+N,i-N21)=a;B(i+N,i+N)=0;
end;
B(ii+1,ii+1)=-1;B(ii+1,ii+N+1)=1;
B(ii+2*N+1,ii+N+1)=1;B(ii+2*N+1,ii+2*N+1)=-1;
B(ii+N+1,ii+N)=1;B(ii+N+1,ii+N+1)=-2;B(ii+N+1,ii+N+2)=1;
end;
for i=1:N21;
B(i,i+N21)=-1;
B(NK+1-i,NK+1-i-N21)=-1;
end;
C=A;
D=eye(NK);
fii=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h;ii=N21*(n-1);
for j=2:M;
y=(j-1)*h;
for k=2:N;
fii(ii+k,j)=-cos(x)*cos(y);
fii(ii+N+k,j)=-cos(x)*cos(y);
end;
fii(ii+1,j)=(1-exp(1))*cos(x)*cos(y);
fii(ii+N+1,j)=0;
fii(ii+2*N+1,j)=(1-exp(-1))*cos(x)*cos(y);
end;
end;
```

```
alpha1=eye(NK);
betha1=zeros(NK,1);
for j=2:M
Q=inv(B+C*alphaj-1);
alphaj=-Q*A;
bethaj=Q*(D*(fii(:,j))-C*bethaj-1);
end;
w=zeros(NK,M+1);
QZ=A+B+C*alphaM-1;
QQ=inv(QZ);
w(:,M+1)=QQ*(D*fii(:,M)-C*bethaM-1);
w(:,M)=w(:,M+1);
for j=M-1:-1:1;
w(:,j)=alphaj*w(:,j+1)+bethaj;
end;
'EXACT SOLUTION OF THIS PROBLEM';
esw=zeros(NK,M+1);errw=zeros(NK,M+1);
esU=zeros(NK,M+1);
for n=2:M;
x=(n-1)*h; ii=N21*(n-1);
for j=2:M;
y=(j-1)*h;
for k=1:2*N+1;
esw(ii+k,j)=(exp(-(k-1-N)*tau)-1)*cos(x)*cos(y);
errw(ii+k,j)=esw(ii+k,j)-w(ii+k,j);
esU(ii+k,j)=(exp(-(k-1-N)*tau))*cos(x)*cos(y);
end;
end;
end;
```

```
ep=zeros(M+1,M+1);
q = zeros(M + 1, M + 1); p = zeros(M + 1, M + 1); errp = p;
for n=1:M+1;
x=(n-1)*h;ii=(2*N+1)*n;
for j=1:M+1;
y=(j-1)*h;
ep(n,j)=cos(x)*cos(y);
q(n,j)=-w(ii,j)+exp(-1)*cos(x)*cos(y);
end;
end;
for n=2:M;
for j=2:M;
p(n,j)=-(q(n,j+1)-2*q(n,j)+q(n,j-1))/(4*h^2) - (q(n + 1, j) - 2 * q(n, j) + q(n - 1, j))/(4 * h^2) +
q(n, j)/2;
errp(n,j)=p(n,j)-ep(n,j);
end;
end;
for n=1:M+1;
ii=N21*(n-1);
for j=1:M+1;
for k=1:2*N+1;
U(ii+k,j)=w(ii+k,j)+q(n,j);
end;
end;
end;
errU=zeros(NK,M+1);
for n=2:M;
ii=N21*(n-1);
for j=2:M;
```

```
for k=1:2*N+1;
errU(ii+k,j)=esU(ii+k,j)-U(ii+k,j);
end;
end;
end;
maxerrorw=max(max(abs(errw)));
maxerroru=max(max(abs(errU)));
maxerrorp=max(max(abs(errp)));
cevap1 = [maxerrorw,maxerroru,maxerrorp]
```

YAKIN DOĞU ÜNİVERSİTESİ

APPENDIX 5

ETHICAL APROVAL DOCUMENT

Date: 15/09/2020

To the **Graduate School of Applied Sciences**

The research project titled "**The Source Identification Problem for Elliptic-Telegraph Equation**" has been evaluated. Since the researcher(s) will not collect primary data from humans, animals, plants or earth, this project does not need to go through the ethics committee.

Title: Prof.Dr.

Name Surname: Allaberen Ashyralyev

Signature:

Role in the Research Project: Supervisor

APPENDIX 6

31.08.2020 Turnitin

Skip to Main Content
Ödevler
Öğrenciler
Not Defteri
Kütüphaneler
Takvim
Tartışma
Tercihler

Bu sayfa hakkında

Bu sizin ödev kutunuzdur. Bir yazılı ödevi görüntülemek için yazılı ödevin başlığını seçin. Bir Benzerlik Raporunu görüntülemek için yazılı ödevin benzerlik sütunundaki Benzerlik Raporu ikonunu seçin. Tıklanabilir durumda olmayan bir ikon Benzerlik Raporunun henüz oluşturulmadığını gösterir.

Ahmad Thesis

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

Sil İndir Şuraya taşı...

	Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
☐	Ahmad Al-hammouri	Abstract	%0 %0	0%	0%	0%	--	--	ödev indir	1376955667	31-Ağu-2020
☐	Ahmad Al-hammouri	Conclusion	%0 %0	0%	0%	0%	--	--	ödev indir	1376957325	31-Ağu-2020
☐	Ahmad Al-hammouri	Chapter 2	%10 %10	12%	15%	10%	--	--	ödev indir	1376956459	31-Ağu-2020
☐	Ahmad Al-hammouri	Chapter 3	%10 %10	13%	9%	12%	--	--	ödev indir	1376956780	31-Ağu-2020
☐	Ahmad Al-hammouri	Chapter 4	%12 %12	2%	13%	4%	--	--	ödev indir	1376957066	31-Ağu-2020
☐	Ahmad Al-hammouri	All Thesis	%13 %13	9%	15%	3%	--	--	ödev indir	1376958075	31-Ağu-2020
☐	Ahmad Al-hammouri	Chapter 1	%13 %13	0%	12%	13%	--	--	ödev indir	1376956049	31-Ağu-2020

CURRICULUM VITAE

Personal information:

First name: Ahmad Mohammad Salem Al-Hammouri

Nationality: Jordan

Data and Place of birth: Irbid – Jordan - 15-06-1986

Marital Status: Single

E-mailaddress: alhammouri.math@gmail.com

Education:

Degree	Institute	Year of Graduation
Ph.D. in Mathematics.	Near East University, Department of Mathematics	2020
M.Sc. in Mathematics.	Mutah University, Department of Mathematics	2016
B.Sc. in Mathematics.	Yarmouk University, Department of Mathematics	2009

WORK EXPERIENCE:

Years	Place	Enrollment
2009/2017	High School-Ministry of Education (Jordan)	Teacher

LANGUAGES:

- Arabic, (Speaking, Listing, Reading, Writing).
- English, (Speaking, Listing, Reading, Writing).
- Capable to communicate clearly and effectively with others.
- Capable to work under pressure.
- Capable of leading a workgroup and taking charge of responsibility positions.

- Analytical and creative in problem solving.
- Good computer skills.

INTERNATIONAL PUBLICATIONS:

Ashyralyev, Allaberen, and Ahmad Al-Hammouri. "Stability of the space identification problem for the elliptic-telegraph differential equation." Mathematical Methods in the Applied Sciences (2020).

Ashyralyev, Allaberen, and Ahmad Al-Hammouri and Charyyar Ashyralyyev. "On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation." Numerical Methods for Partial Differential Equations (2020).

Ashyralyev, Allaberen, and Ahmad Al-Hammouri. "A numerical algorithm for a source identification problem for the elliptic-telegraph equation." AIP Conference Proceedings (2019).

Ashyralyev, Allaberen, and Ahmad Al-Hammouri. A Note on the Elliptic-Telegraph Identification Problem with Non-Local Condition." AIP Conference Proceedings (2020).

Ashyralyev, Allaberen, and Ahmad Al-Hammouri. Numerical Solution for the Second Order of Accuracy Difference Scheme for the Source Identification Elliptic-Telegraph Problem." AIP Conference Proceedings (2020).

COURSES GIVE:

I can give any course in Mathematics as

Calculus, Linear algebra, Differential equations, Partial differential equations, Numerical analysis, Real analysis, Complex analysis, Differential geometry, Inequality, Abstract algebra and Functional analysis… etc