ILITY ANALYSIS OF FRACTIONAL ORDER MODEL ON CORONA TRANSMISSION DYNAMICS

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Sultan Hamed Alsaadi: STABILITY ANALYSIS OF FRACTIONAL ORDER MODEL ON CORONA TRANSMISSION DYNAMICS

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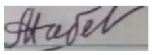
Prof. Dr. Allaberen Ashyralyev 	Committee Chairman, Department of Mathematics, NEU
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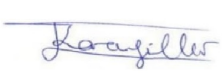
Prof. Dr. Tolgay Karanfiller 	Department of Mathematics, Trakya University
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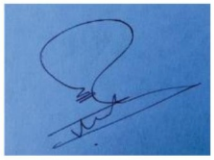
Signature: 

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To the soul of Sultan Qaboos…

ABSTRACT

In this thesis a fractional order mathematical model is constructed to study the dynamics of corona virus in Oman. The model consists of a system of eight non-linear fractional order differential equations in Caputo sense. Existence and uniqueness as well as the stability analysis of the solution of the model are given. The stability analysis is in the frame of Ulam-Hyers and generalized Ulam-Hyers criteria. Numerical simulations are given to support the theoretical results. Many informations on the dynamics of COVID -19 in Oman were obtained using this model. Also, many informations on the qualitative behaviour of the model were obtained.

***Keywords*:**Coronavirus; Existence; uniqueness; Ulam-Hyers stability; Mathematical Model.

ÖZET

Bu çalışmada, Umman'da corona virüs dinamiklerini incelemek için kesirli bir matematiksel model oluşturulmuştur. Model, Caputo anlamında sekiz doğrusal olmayan derece diferansiyel denklem sisteminden oluşur. Modelin çözümünün mevcudiyeti ve benzersizliği ile kararlılık analizi verilmiştir. Stabilite analizi, Ulam-Hyers ve genelleştirilmiş Ulam-Hyers kriterleri çerçevesindedir. Teorik sonuçları desteklemek için sayısal simülasyonlar verilmiştir. Bu model kullanılarak Umman'daki COVID-19 dinamikleri hakkında birçok bilgi elde edilmiştir. Ayrıca modelin nitel davranışı hakkında da birçok bilgi elde edilmiştir.

***Anahtar kelimeler*:** Koronavirüs, Varlık, Teklik, Ulam-Hyers Kararlılığı, Matematiksel Model.

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LIST OF ABBREVIATIONS

SIR:	Susceptible, Infected, Recovered
NP:	Nucleoprotein
$\boldsymbol{R_0}$:	Basic Reproduction Ratio
NGM:	Next Generation Matrix
IVP:	Initial Value Problem
$\mathbf{E(t)}$**:**	Exposed population
$\boldsymbol{I_A(t)}$:	Asymptomatic Infective population
$\boldsymbol{I_S(t)}$:	Symptomatic Infective Population
$\boldsymbol{I_I(t)}$:	Isolated Infective Population
$\boldsymbol{I_H(t)}$:	Hospitalized Infective Population
$\boldsymbol{R(t)}$:	Recovered Population
$\boldsymbol{D(t)}$:	Dead individuals

CHAPTER ONE
INTRODUCTION

Some epidemic diseases are capable of producing large number of infections starting from a fewer ones, an example of such diseases is COVID -19 (Hiroshi et al., 2020). Coronavirus disease is a respirational and zoonotic disease, caused by a virus of the coronaviridae family which originated in the city of Wuhan China on December 01, 2019 (Giuseppe and Mario, 2020). The Virus strain is severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), resulting in fever, coughing, breathing difficulties, fatigue, and myalgia. It may transform into pneumonia of high intensity.

Many scientists put their heads together in trying to find answers about the spread and infection of corona virus by examining virus samples (Ewen, 2020).Although the disease strain is known, but, the vaccine is still not available. Hence it is necessary to ensure strict mitigation actions in order to contain the virus. Many countries are taking different measures to cope with the virus.

Taking the China and South Korea cases into consideration, a lot of countries were able to contain the spread of the virus. It is proven that social distancing and testing are some of the key control measures. Another measure responsible for spreading the disease worldwide was regional and global travelers. Although, air travel is almost suspended now, but, this initial shock and countries (and people) not taking it too seriously has taken many countries in Europe and North America to a real bad situation.

Oman has taken various measures to avoid the spread of COVID - 19. From mid of March, Schools and Colleges/Universities were closed, consequently other non essential offices and services were closed. The situation is been monitored on daily basis by the government. However, there are many suspected cases which are not been tested so far. By the time of this report, most part of the country is partially locked down. Although, the spread of the disease is not that much as of today (June 17, 2020) but the new cases are appearing continuously.

We need modeling approach to understand the exact dynamics of the disease (Kumar et al., 2019; Kumar; 2020a; Kumar 2020b). This is what motivates this research.It is important to note that, in the classical order model, the state of epidemic does not depend on its history. However, in real life memory plays a vital role in studying the pattern of spread of any epidemic disease. It was found that the waiting times between doctor visits for a patient follow a power law model (Smethurst, 2001). It is worth to know that Caputo fractional time derivative is a consequence of power law (Meerschaertand Sikorskii, 2011). When dealing with real world problem Caputo fractional-order derivatives allows traditional initial and boundary conditions (Singh, 2020; Kumar, 2020). Furthermore, due to its non-local behaviour and its ability to change at every instant of time, Caputo fractional-order gives better result than the integer order (Ahmed et al., 2020; Baba et al., 2020; Qureshi et al., 2020).

In recent studies, Khan et al. studied a fractional-order model that describes the interaction among bats and unknown hosts, then among people and seafood market (Riley et al., 2003). To predict the trend of the Corona-Virus Yu et al. constructed a fractional time delay dynamic system that studied the local outbreak of COVID-19 (Chen et al., 2020). Also, to predict the possible outbreak of infectious diseases like COVID-19 and other diseases in the future Xu et al. proposed a generalized fractional-order SEIQRD model (Xu et al., 2020). Shaikh et al. used Bats-Hosts-Reservoir - People transmission fractional-order COVID-19 model to estimate the effectiveness of preventive measures and various mitigations, predicting future outbreaks and potential control strategies (Ali et al., 2019).

1.1 Epidemiology

This is the study of incidence, distribution, determinants as well as possible control measures of health – related occurrence in a specific population. The diseases causing epidemics can largely be divided into micro and macro parasites. Micro follows human to human transfer pattern examples are influenza, tuberculosis, gonorrhea etc. Macro follows humans to carrier to humans transfer pattern examples are malaria, black plague etc. It is also of significant importance to differentiate between these three important elements of epidemiology;

endemic epidemic, and pandemic. Endemic refers to continual occurrence of a disease in a population. Epidemic refers to an unexpected upsurge of a disease in a population. Pandemic refers to a global epidemic affecting inflated number of populations.

There are ample means in which epidemic disease spread. These means include but are not limited to population explosion, missing sanitation in underdeveloped countries, and modern transportation which enables international boarders cross. Another important means by which disease spread is due to loss of natural immunity, this is due to pills we take which contain synthesis vitamins instead of the vitamins we get from natural resources. There are many more means by which diseases spread which are not easy to be listed and studied in a single research.

1.1.1 History of mathematical epidemiology

Mathematical epidemiology can be traced to over three hundred years back. John Graunt was the first to publish a book in 1662 with a title " Natural and political observations made upon the bills of mortality" in 1662. It was problems concerning demography problems in Britain mainly in the seventeenth century. He calculated the risk of death of some certain diseases using the records of the death they caused. This analysis was the first to provide a systematic method of estimating the risk of death due to plague. This serves as the genesis of the theory of competing risks. Almost hundred years later that is in 1760 Daniel Bernoulli published a paper with the first epidemiological model. The aim of the model was to demonstrate the effect of inoculating patients with smallpox in reducing the progression. One year later, that is in 1761 D'Alembert developed another method of handling competing risks of death.

Hamer in the year 1906 was believed to be the inventor of modern mathematical biology. He applied mass action principle on deterministic epidemic model. Rose in 1911 proposed another simple epidemic model for malaria and in 1927 Kermack and Mckendrick proposed a generalized epidemic model (Ross, 1911). Nowadays these models are being modified by taking into account various epidemic units like chemotherapy, vaccination, migration, immunity, quarantine, resistance, non – monotonicity of incidence rates etc. Another

important modification was that by Anderson and May in 1982 (Anderson and May, 1982). In their model they considered non - homogeneous mixing in the population. This leads to another discovery by Liu et al. in 1987 in which he considered non – linear incidence rate instead of the usual bilinear incidence rates as in the previous models. Many models were formed and analyzed over the years for diseases such as influenza, AIDS, SARS, malaria, cholera, measles, smallpox, rubella, diphtheria, gonorrhea etc.

1.1.2 Stochastic and deterministic models

The most important models useful in the study of epidemiology are the stochastic and deterministic models.

Stochastic models are used especially in small or isolated populations when known heterogeneities are important. They consider minute population especially when every individual is significant in the model as such stochastic models are termed as individual level – models. These types of models can be arduous to make, sometimes need a lot of simulations to make predictions, and contribute very little in explaining disease dynamics.

Deterministic models sometimes referred to as compartmental models describe the dynamics of the disease at the population level. They categorize individuals into compartments or groups. For example SIR model has three compartments; susceptible, Infectious, and recovered. There are mathematical parameters between each compartment that describes the transition rate of individuals from one compartment to another. Deterministic models are easy to set up, and require less data, hence they are the most widely used epidemiological models. Nowadays complex deterministic models exist in literature which can integrate stochastic elements such as demographic etc.

1.1.3 SIR epidemic model

The most basic model that describes whether or not an epidemic will occur and how it occurs in a population is the SIR epidemic model. It was first developed by Kermack and Mckendrick in 1927 (Britton, 2003; Murray, 2004; Ellner and Guckenheimer, 2006). Modification of this model exist in literature, examples of this can be found in a book by

Hethcote (Hethcote, 2000), Dieckmann and Heesterbeek (Dieckmann and Heesterbeek, 2000), Anderson and May (Anderson and May, 1992), and Murray (Murray, 2004).

SIR model consists of three compartments; S, I, and R. S is the compartment of susceptibles, I is the compartment of infectives, and R is the recovered compartment. Figure 1, gives the description of the model and Table 1 provides the meaning of all the parameter as presented in equation (1.1).

Figure 1: Description of model 1

Table 1: Parameters of model 1.1

List of Parameter	**Meaning of the parameters**
S	Susceptibles
I	Infectives
R	Recovered
β	Rate at which susceptibles become infected
ɣ	Rate at which infected become recovered

The following system of equations represents the model

$$\frac{dS}{dt} = -\beta SI,$$

$$\frac{dI}{dt} = \beta SI - \gamma I, \qquad (1.1)$$

$$\frac{dR}{dt} = \gamma I.$$

1.1.4 The Basic reproduction number (R_0)

$R_0$is termed as the number of new infected individuals in a population that are infected as a result of one individual when everyone is susceptible and during the entire infection period. In simple models R_0 has the form given as

$$R_0 = (Number\ of\ contacts\ at\ a\ time)(Probability\ of\ infection\ in\ a\ contact)$$

$$(Period\ of\ epidemics)$$

In epidemiology, R_0 is considered as a threshold quantity, since $R_0 > 1$ implies epidemics and $R_0 < 1$ implies no epidemics. For many models $R_0 = 1$ implies transcritical bifurcation.

For complicated models that includes seasonality methods or heterogeneity the next generation matrix (NGM) method is used in computing the R_0. The NGM method was developed in 1990 by Diekmann et al. it was later after 12 years i.e in 2002 standardized by Van den Driessche and Watmough (Diekmann et al., 1900; Van Den Driessche and Watmough, 2002). It converts a system of ODE or PDE of an infectious disease model to an operator. The basic reproduction ratio here is defined to be the dominant eigenvalue (spectral radius) of this operator.

Consider the following deterministic model

$$\frac{dx_i(t)}{dt} = f_i(x), \qquad x(0) \in \overline{R_+^n}\,.$$

Where $x_i(t)$ stands for the number of individuals in compartment i at a given time t. Also let $x(0) \in \overline{R_+^n}$, where $\overline{R_+^n}$ is the nonnegative orthant of R^n. Let

$$X = \left\{x \in \overline{R_+^n} \backslash x_i = 0 \quad 1 \le i \le m\right\} \text{where} m \le n.$$

X: A set containing disease – free states.

Then

$$\frac{dx_i(t)}{dt} = F_i(x) - M_i(x),$$

where $F_i(x)$: Infection introduced into i compartment

$$M_i(x) = M_i^-(x) - M_i^+(x).$$

$M_i^+(x)$: Transfer into i compartment by other means.

$M_i^-(x)$: Transfer out of i compartment by other means.

The following assumptions hold true for these functions

i – The rate of movements are nonnegative, i.e if

$x \in \overline{R_+^n}$, then $F_i(x) > 0, M_i^+(x) > 0, M_i^-(x) > 0$ where $1 \leq i \leq n$.

ii – If a compartment is empty, then movement out of that compartment is not possible i.e if $x_i = 0$ then $M_i^-(x) = 0$.

iii – Movement of infection into non-infective class is not possible i.e

$$F_i(x) = 0 \quad for\ i > m.$$

iv – Disease free subspace is invariant, i.e when

$$x \in X\ both\ F_i(x) = 0, and\ M_i^+(x) = 0\ when\ \ 1 \leq i \leq m\ .$$

v – In the absence of new infection the disease free equilibrium is locally asymptotically stable i,e when $F(x) = 0$ all eigenvalues of the matrix at disease free equilibrium must be negative.

For disease free equilibrium of (1.1), we let square matrices $\overline{F}$ and $\overline{M}$ to be

$$\overline{F_{ij}} = \frac{\partial F_i}{\partial x_j} \quad when\ 1 \leq i, j \leq m,$$

$$\overline{M_{ij}} = \frac{\partial V_i}{\partial x_j} \quad when\ 1 \leq i, j \leq m.$$

Then $\overline{F}\overline{M}^{-1}$ is defined as NGM and $R_0 = \rho(\overline{F}\overline{M}^{-1})$ is the dominant eigenvalue (spectral radius) of the NGM.

1.2 Well – posedeness

A mathematical model is said to be well – posed if it satisfies the following three conditions

i. The model has a solution (existence of solution).
ii. The solution is unique (uniqueness of solution).
iii. The solution is stable (stability of solution).

If one of the above conditions failed, then the model is ill – posed.

1.2.1 Existence and uniqueness of solutions

Consider the initial value problem (IVP) defined as

$$\dot{x} = f(t,x),\ t \in R,$$

$$x(t_0) = a. \quad (1.2)$$

Here f is continuous and $domain(f)\ is\ contained\ in\ R \times R^n$, and $(t_0, a)$is constant.

Fundamental theorem of calculus implies that equation (1.2) is the same as

$$x(t) = a + \int_{t_0}^{t} f\big(s, x(s)\big)ds. \quad (1.3)$$

Now, our goal is to show that x(t) has a solution. We can do that by using either Picard iteration or Tonelli sequence.

For Picard iteration, an initial value is chosen for *x* and substituted in (1.3) and then the result is used to evaluate a new *x*. Setting $x_1(t) \coloneqq a$ then $x_{k+1}(t)$ is defined in terms of $x_k(t)$ where k >1 using equation

$x(t) = a + \int_{t_0}^{t} f\big(s, x(s)\big)ds,\ k \geq 1,$ $x_0(t)$is given.

If the absolute error $x_{k+1}(t)$ and $x_k(t)$ is less than the given tolerance value, take $x_k(t)$ as the result of the given problem.

For Tonelli sequence let $x_k(t)$ where $k \in N$ and for $t \geq t_0$ be defined by

$$x_k(t) = \begin{cases} a, & for\ t \in \left[t_0, t_0 + \frac{1}{k}\right], \\ a + \int_{t_0}^{t-\frac{1}{k}} f(s, x_k(s))dx, & for\ t \in \left[t_0 + \frac{1}{k}, \infty\right), \end{cases} \quad (1.4)$$

for$t \geq t_0, x_k(t)$ is defined in the same manner. Here Tonelli sequence will be used to prove the existence theorem and Picard iterates to prove the uniqueness theorem. See Theorem 1.1 and 1.2.

Theorem 1.1 (Cauchy – Peano).let$f: [t_0 - \alpha, t_0 + \alpha] \times \overline{B(a,\beta)} \to R^n$ be a continuous and bounded function, then the solution exists on the interval $[t_0 - b, t_0 + b]$, for $b = min\left\{\alpha, {}^{\beta}/_{M}\right\}$ and $M > 0$ is the upper bound for f.

Proof.(Murray and Miller, 1976).

Theorem 1.2 (Picard – Lindelof uniqueness theorem). Let M be the upper bound of the continuous function $f: [t_0 - \alpha, t_0 + \alpha] \times \overline{B(a,\beta)} \to R^n$. Moreover, let L be a Lipschitz constant for all $t \in [t_0 - \alpha, t_0 + \alpha]$ of a Lipschitz continuous function $f(t,.)$, then (1.2) has a unique solution on $[t_0 - b, t_0 + b]$, and $b = \min\{\alpha, \frac{\beta}{M}\}$.

Proof.(Coddington and Levinson, 1955).

1.2.2 Stability of solutions

If for all $\varepsilon > 0$ and $t_0 \in R$ there exists δ depending on $\varepsilon\ and\ t_0$such that if $\hat{x}(t)\ and\ x(t)$ are two solutions of (1.2) then

$$|x(t) - \hat{x}(t)| < \varepsilon, whenever\ |x(t_0) - \hat{x}(t_0)| < \delta \quad \forall t \geq t_0$$

then$\hat{x}(t)$is a stable solution of (1.2).

If $\hat{x}(t)$ is stable and $\forall t_0 \in R\ \exists \delta = \delta(t_0) > 0$ such that for another solution say $x(t)$, $|x(t_0) - \hat{x}(t_0)| < \delta$ then $x(t) \to \hat{x}(t)$as $t \to \infty$, then $\hat{x}(t)$ is asymptotically stable.

Since in this research we are more concerned with Lyapunov stability then what follow is a theorem and its proof on Lyapunov stability.

Theorem1.3 (Lyapunov stability). Let D be a neighbourhood of $x(t)$ and V be a continuously differentiable positive definite function $V: R \times D \to R$, whose orbital derivative $\dot{V}$ is negative semi – definite, then x(t) is a Lyapunov stable solution of (1.2).

Proof.(Parks, 1992).

Theorem 1.4 (Asymptotic stability). Let D be the neighbourhood of $x(t)$ a solution of (1.2) and $V: R \times D \to R$ be a positive definite function whose orbital derivative $\dot{V}$ is negative definite. Furthermore, let $\widehat{W}: D \to R$ such that $V(t,x) \leq \widehat{W}(x) \ \ \forall (t,x) \in R \times D$, then x(t) is anasymptotically stable solution of (1.2).

Proof.(Parks, 1992).

Theorem 1.5 (LaSalle's invariance principle).Let D be a neighbourhood of x(t) and let $V: D \to R$ be a continuously differentiable positive definite function whose orbital derivative is negative semi definite. Let I be the union of complete orbits contained in $\{x \in D | \dot{V}(x) = 0\}$. Then there is U, a neighbourhood of $x(t)$ such that $\forall x_0 \in U, W(x_0) \subseteq I$.

Proof.(Parks, 1992).

1.3 Research Aim and Objectives

1.3.1 Research Aim

The aim of this research is to study fractional - order epidemic model that investigates the dynamics of COVID-19 in Oman. Based on the memorability nature of Caputo fractional-order derivatives, this model can be fitted with data reasonably well. Then, based on the official data given by the Federal Ministry of Health Oman daily, numerical examples will be carried out.

1.3.2 Research Objectives

- To formulate the fractional order model for COVID-19 in Caputo sense.
- To show the existence and uniqueness of the solution of the model.

- To show the stability analysis of the solution of the model in the frame of Ulam – Hyers and generalized Ulam – Hyers.
- To carryout numerical simulations to illustrate the theoretical results

1.4 Outline of the Thesis

The thesis was divided into five chapters as follows:Chapter one gives introduction, Chapter two gives the literature review, chapter three discusses the construction and analysis of the model, chapter four gives the numerical simulation results, and lastly chapter five gives the summary and conclusion of the thesis.

CHAPTER 2
LITERATURE REVIEW

2.1 Fractional Calculus

The history of fractional calculus is almost as dated back as that of the ordinary calculus. It can be traced back to the 17th century, short after Newton and Leibniz formulated the ordinary integration and differentiation (Podlubny, 1999). L'Hopital sends letter to Leibniz (Leibniz, 1692), where he questioned what will happen if the order of the derivative were $\frac{1}{2}$, this eventually led to the birth of the theory of derivativesand integrals of arbitrary order.

Development of fractional calculus can take a similar analogies with the development in mathematics. It is a common knowledge that the history of mathematics is moved forwards by paradoxes and crisis as the advancement of civilization (Kline, 1990; Gu, 2003; Snapper, 1979). For example if we take the development of numbers, the integers on the number axis are very small portion just like isolated islands in the ocean, with the majority of water as fractionaland non-rational numbers. Moving a step forward, the discovery of complex number made people understand that the mighty real numbers are just like the planets in the big universe that only occupy a very tiny portion of the space.

By the same way, the advances of mathematical operations also experienced the development from some basic operations to complicated operations, that is, it starts from addition/subtraction to powers/roots, and then moves to integration/derivation and convolution. Also for the past 300 years, the development of fractional calculusmakes the theory of operation even more complete. Recent history of fractional calculus was given in (Machado et al., 2011). Fractional calculus is such an amazing tool that can be used to explain many phenomena of physics which the conventional math could not explain before. It is more especially good at depicting phenomenawith long memory, long range dependence, etc.

2.1.1 Preliminaries

Important definitions

There are more than 10 types of definitions for fractional order integrals and differentiations (Miller and Ross, 1993). For readers' convenience, some of the most commonly used definitions are briefly listed below. More details can be found in (Magin, 2006).

Definition1 (Qian and Wong, 2010): The fractional derivative of order $\alpha \in [n-1, n)$ of $f(x)$for Rieman-Liouville can be defined as:

$$ {}_{a}^{RL}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt, \quad n = [\alpha] + 1. $$

Definition2 (Qian and Wong, 2010): The fractional derivative of order $\alpha \in (n-1, n]$ of $f(x)$for Caputo can be defined as:

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t) dt, \quad n = [\alpha] + 1. $$

Definition3 (Ortiz et al., 2013): **(Linearity)**

If $f, g$are continuous and $b, c$are scalars, then

$$ {}_{a}^{RL}D_{x}^{\alpha} [bf(x) + dg(x)] = b\,{}_{a}^{RL}D_{x}^{\alpha} f(x) + d\,{}_{a}^{RL}D_{x}^{\alpha} g(x), $$

$$ {}_{a}^{C}D_{x}^{\alpha} [bf(x) + dg(x)] = b\,{}_{a}^{C}D_{x}^{\alpha} f(x) + d\,{}_{a}^{C}D_{x}^{\alpha} g(x). $$

Definition4:**(Contraction)**

For an operator $f: X \to X$which maps a metric space onto itself, it is contractive for $0 < q < 1$

$$ d\big(f(x), f(y)\big) = qd(x, y), \qquad \forall x, y \in X. $$

2.1.2 Important functions

Some important special functions which are frequently encountered in fractional calculus arelisted below. For more details, refer to (Magin, 2006).

Gamma function: Gamma function is one of the important functions because it is the fundamental element in almost all of the definitions in fractional integrals. It is usually considered as the factorial of non-integer numbers.
The integral form of gamma function can be written as:

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx \, , p > 0.$$

Some useful properties of gamma function to remember are:

$$\Gamma(1) = 1; \qquad \Gamma(n+1) = n! \;\; (n = 0,1,2 \ldots)$$

$$\Gamma\left({}^{1}/_{2}\right) = \sqrt{\pi}\,; \qquad \Gamma(x+1) \;\; = x\Gamma(x).$$

Similar to integer order calculus, the fractional order derivative of a variable with the same fractional order power is a constant,

$$\frac{d^{\alpha}}{dx^{\alpha}} x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)} x^{\alpha-\alpha} = \Gamma(\alpha+1).$$

Mittag-Leffer function: The Mittag-Leffer (M-L) function is a general form of the exponential function and it plays a very important role in solving the fractional differential equations just the same way exponential function does in ordinary differential equations. It has four forms Prajapati and Shukla, 2012; Chaurasia and Pandey, 2010), and the most commonly used forms are given below as 1-parameter and 2-parameter representation (Shukla and Prajapati, 2007):

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \qquad (\alpha > 0)$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} (\alpha > 0, \beta > 0).$$

Some of the beautiful properties of M-L function are as follows;

$$E_{1,1}(x) = e^x;$$

$$E_{1,2}(x) = \frac{e^x - 1}{x}.$$

- **Error function**

The error function is another special function for the "S" shape, and is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \qquad -\infty < x < \infty.$$

It has the following properties;

$$erf(0) = 0$$

$$erf(\infty) = 1$$

$$erf(x) + erfc(x) = 1$$

Where $erfc(x)$ *is called the complimentary error function*

Confluent hypergeometric function: A confluent hypergeometric function gives solution of a confluent hypergeometric equation, and is represented in the following form;

$$_1F_1(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!}, \quad -\infty < x < \infty$$

Where $(a)_n \; and \; (c)_n$ are the Pochhammer symbols.

$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a)}, \qquad and$$

$$(c)_n = \frac{\Gamma(c+1)}{\Gamma(c)}. n = 0,1,2 \ldots$$

Some of the most commonly used properties for the hypergeometric function are listed below;

$$_1F_1(1;1;x) = \; e^x,$$

$$\frac{1}{\Gamma(\alpha+1)} \; _1F_1(a;\alpha+1;at) = \; E_{1,2}(at).$$

2.1.3 Some important theorems

Theorem 1: (Banach contraction mapping principle)

For any contractive operator that maps a metric space onto itself has a unique fixed point. Moreover, if $f: X \to X$ is a contractive operator that maps a metric space onto itself with its fixed point a: $f(\boldsymbol{a}) = \boldsymbol{a}$; then for any iterative sequence

$$x_0, \qquad x_1 = f(x_0), \qquad x_2 = f(x_1), \ldots, x_{n+1} = f(x_n), \ldots,$$

It converges to $\boldsymbol{a}$.

For the continuous dynamical system we say that$\boldsymbol{a}$ is a solution or an equilibrium and for discrete dynamical system is a fixed point.

Theorem 2 (Matignon, 1996): The equilibriums solutions x^* of the any system say$(*)$ is locally asymptotically stable if all its eigenvalues λ_i of its Jacobian matrix $\frac{\partial f}{\partial x_i}$ evaluated at the equilibrium points satisfy the following:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \qquad 0 < \alpha < 1.$$

Theorem3 (Delvari et al., 2012): If$x = 0$ is an equilibrium solution of system$(*)$, and$\Omega \subseteq \mathbb{R}^n$ is a domain containing $x = 0$.

If$V(t,\ x)$: $[t_0,\ \infty] \times \Omega \to \mathbb{R}$ is continuously differentiable function such that:

i- $W_1(x) \leq V(t,x) \leq W_2(x)$ and

ii- $\quad {}_0^C D_t^\alpha V(t,x) \leq -W_3(x), \qquad fort \geq 0,\ x \in \Omega.$

where $W_1(x), W_2(x) and W_3(x)$ are continuous positive definite function on Ω and V is a Lyapunov candidate function, then $x = 0$ is globally asymptotically stable.

Theorem 4 (Vergas-De-Leon, 2015): Let $x(t) \in \mathbb{R}^+$be continuous and derivable function. Then, for any time instant $t \geq t_0$ and $\alpha \in (0,1)$

$${}_0^C D_t^\alpha [x(t) - x^* - x^* \ln\left(\frac{x(t)}{x^*}\right)] \leq \left(1 - \frac{x(t)}{x^*}\right) {}_0^C D_t^\alpha x(t), \qquad x^* \in \mathbb{R}^+.$$

2.2 Fractional Order Differential Equations

Fractional order differential equations (FODEs) are the basic tools used to describe FO dynamicsystems. Any type of fractional order system analysis, whether time domain, s-domain or even complex frequency domain, are all made up ofthe basis of FODEs. Hence, they are very importantand needs to be emphasized. In this section we give the two most common types of fractional order differential equations (linear and nonlinear fractional order differential equations) with example of each.

2.2.1 Linear FODE's

Linear FODE's are the most commonly used in fractional order controls because of their regularity and simplicity. The general expression of FODE's are in the following form:

$$a_1 {}_0^C D_t^{\alpha_1} y(t) + a_2 {}_0^C D_t^{\alpha_2} y(t) + \cdots + a_n {}_0^C D_t^{\alpha_n} y(t) = b_1 {}_0^C D_t^{\beta_1} u(t) + b_2 {}_0^C D_t^{\beta_2} u(t) + \cdots + b_n {}_0^C D_t^{\beta_n} u(t),$$

$where the orders, \alpha_i, \beta_j \quad (i, j = 1,2, \ldots)$can be arbitrary real numbers, i.e. $\alpha_i, \beta_j \quad \in \mathbb{R}$. If $\alpha_i$and β_j are integer multiples of a common factor, the equation is said to have a commensurateorder; and if there is no common factor exist it is said to be of non-commensurate order (Vinagre and Feliu, 2000). Example of linear FODE can be seen in the fractional Langevin equation.

The fractional Langevin equation: The original Langevin equation that describes the Brownianmotion of a particle in a fluid is:

$$m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} + \eta(t)$$

where x is the particle's position and $m$represents the particle's mass. The noise term $\eta(t)$denotes the effect of the collisions with the molecules of the fluid, which has a Gaussianprobability distribution with the correlation function as follows:

$$corr\left(\eta_i(t), \eta_j(t')\right) = 2\lambda k_b T \delta_{i,j} \delta(t - t'),$$

where k_b is Boltzmann's constant, T is the temperature, and δ is the Dirac's function.

However, the above equation of motion does not capture the hydrodynamics completely since it ignores the effects of the retarded viscous force due tothe acceleration of the particle and the added mass. Hence, the fractional Langevin equation is used tosupplement the missing dynamics, (Mainardi and Pironi, 1996),

$$m\frac{dx}{dt} = \frac{m}{\delta_e}\left[1 + \sqrt{T}\,{}_0^C D_t^{1/2}\right] x(t) + \eta(t).$$

where the explanation of the coefficients can be seen in (Mainardi and Pironi, 1996). But for the above fractional Langevin equation, the random force $\eta(t)$cannot be used for white noise uniquely.Instead, it can be used for a superposition of the white noise with a "fractional" noise. Therefore, the added mass and the fractional noise changed the velocity correlationfunction from exponential decay to algebraic or power law decay.

2.2.2 Nonlinear FODE'S

Example of nonlinear fractional differential equations can be seen in the Van der Pol fractional equation given below:

The fractional Van der Pol equation: The Van der Pol (VDP) equation was initially presented by Van der Pol around 1920s to mimic the self-sustaining oscillation in electricalcircuits using vacuum tubes, (Der Pol and Der Mark, 1927). It is among the first discovered instances of deterministicchaos, and it can originally be described by the following nonlinear ODE:

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x} + x = 0.$$

It can also be used to describe a different variety of phenomena, like a mass-spring-damper systemwith a nonlinear position-dependent damping coefficient, or an RLC electrical circuit with anegative-nonlinear resistor.

Later, a number of variant VDP equations were proposed, for example, Mickens et al.investigated the following two equations in (Mickens, 2002) ,which was termed as fractional VDP.

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x} + x^{1/3} = 0,$$

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x}^{1/3} + x = 0.$$

However, we can see that these dynamics only contains the fractionalpower of the state variables rather than fractional order derivatives, hence, they are not fractionalorder in the sense of calculus. In 2004, Pereira et al. considered the following fractionalderivative version VDP by substituting the capacitance with a "fractance" in a nonlinear RLC circuit model, (Pereira, 2004),

$$ {}_0^C D_t^{\alpha} x + \mu(x^2 - 1)\dot{x} + x = 0, \qquad 1 < \alpha < 2. $$

Also (Barbosa et al., 2004) presented the following fractional VDP with both derivatives being fractional order, as follows:

$$ {}_0^C D_t^{1+\alpha} x + \mu\,(x^2 - 1)\,{}_0^C D_t^{\alpha} x + x = 0, \qquad 0 < \alpha < 1. $$

CHAPTER 3

Model Formulation

Let the total population be N(t). The population is divided into eight compartments, namely; Susceptible populationS(t), Exposed populationE(t), Asymptomatic Infective population$I_A(t)$, Symptomatic Infective Population $I_S(t)$, Isolated Infective Population $I_I(t)$, Hospitalised Infective Population $I_H(t)$, Recovered Population $R(t)$ and then Dead Individuals $D(t)$.The dynamics of this population is represented by the following system of fractional order differential equations (FODE), and the meaning of parameters is given in Table 1.

$$\begin{cases} {}^{C}D_{0+}^{v}S(t) = -\beta S(\alpha I_A + \xi I_I + I_S), \\ {}^{C}D_{0+}^{v}E(t) = \beta S(\alpha I_A + \xi I_I + I_S) - kE, \\ {}^{C}D_{0+}^{v}I_A(t) = (1-p)kE - \gamma_A I_A, \\ {}^{C}D_{0+}^{v}I_S(t) = pkE - qI_S, \\ {}^{C}D_{0+}^{v}I_I(t) = q(1-\varphi)I_S - (\gamma_S + \mu_S)I_I, \\ {}^{C}D_{0+}^{v}I_H(t) = q\varphi I_S - (\gamma_H + \mu_H)I_H, \\ {}^{C}D_{0+}^{v}R(t) = \gamma_A I_A + \gamma_S I_I + \gamma_H I_H, \\ {}^{C}D_{0+}^{v}D(t) = \mu_S I_S + \mu_H I_H. \end{cases}$$

Table 2: Description of the parameters

Parameter	Description
β	effective contact rate
α	reduction of infectiousness in asymptomatic people
ξ	reduction of infectiousness in isolated people
k	progression from exposed to infectious class

p	proportion of asymptomatic cases
q	progression from asymptomatic unaware to self - isolated
φ	proportion of hospitalized people
$\gamma_A, \gamma_S, \gamma_H$	recovery rate for asymptomatic, symptomatic and hospitalised individuals respectively
μ_S, μ_H	mortality rate in isolated and hospitalized classes

3.1 Existence and Uniqueness results

The theory of existence and uniqueness of solutions is one of the most dominant fields in the theory of fractional-order differential equations. The theory has recently attracted the attention of many researchers, we are referring to [11,12] and the references therein for some of the recentgrowth. In this section, we discuss the existence and uniqueness of solution of the proposed model using fixed point theorems. Let us reformulate the proposed model (2) in the subsequent form.

$$\begin{cases} {}^{C}D_{0^+}^{v} S(t) = \theta_1(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} E(t) = \theta_2(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} I_A(t) = \theta_3(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} I_S(t) = \theta_4(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} I_I(t) = \theta_5(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} I_H(t) = \theta_6(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} R(t) = \theta_7(t, S, E, I_A, I_S, I_I, I_H, R, D), \\ {}^{C}D_{0^+}^{v} D(t) = \theta_8(t, S, E, I_A, I_S, I_I, I_H, R, D). \end{cases}$$

Where,

$$\begin{cases} \theta_1(t,S,E,I_A,I_S,I_I,I_H,R,D) = -\beta S(\alpha I_A + \xi I_I + I_S), \\ \theta_2(t,S,E,I_A,I_S,I_I,I_H,R,D) = \beta S(\alpha I_A + \xi I_I + I_S) - kE, \\ \theta_3(t,S,E,I_A,I_S,I_I,I_H,R,D) = (1-p)kE - \gamma_A I_A, \\ \theta_4(t,S,E,I_A,I_S,I_I,I_H,R,D) = pkE - qI_S, \\ \theta_5(t,S,E,I_A,I_S,I_I,I_H,R,D) = q(1-\varphi)I_S - (\gamma_S + \mu_S)I_I, \\ \theta_6(t,S,E,I_A,I_S,I_I,I_H,R,D) = q\varphi I_S - (\gamma_H + \mu_H)I_H, \\ \theta_7(t,S,E,I_A,I_S,I_I,I_H,R,D) = \gamma_A I_A + \gamma_S I_I + \gamma_H I_H, \\ \theta_8(t,S,E,I_A,I_S,I_I,I_H,R,D) = \mu_S I_S + \mu_H I_H. \end{cases}$$

Thus, the proposed model takes the form

$$\begin{cases} {}^{C}D_{0^+}^{v}\Phi(t) = \text{Ƕ}\big(t,\Phi(t)\big); \quad t \in J = [0,b], \quad 0 < \alpha \leq 1 \\ \Phi(0) = \Phi_0 \geq 0, \end{cases}$$

On condition that,

$$\begin{cases} \Phi(t) = (S,E,I_A,I_S,I_I,I_H,R,D)^T, \\ \Phi(0) = (S_0,E_0,I_{A0},I_{S0},I_{I0},I_{H0},R_0,D_0)^T, \\ \text{Ƕ}\big(t,\Phi(t)\big) = \big(\theta_i(t,S,E,I_A,I_S,I_I,I_H,R,D)\big)^T, \ \ i = 1,\dots,8, \end{cases}$$

where $(.)^T$ represents the transpose operation. In view of Theorem in [10], this problem is given by

$$\Phi(t) = \Phi_0 + J_{0^+}^{v}\text{Ƕ}\big(t,\Phi(t)\big)$$

$$= \Phi_0 + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\text{Ƕ}\big(\tau,\Phi(\tau)\big)d\tau.$$

Let $E = C([0,b];R)$ denotes the Banach space of all continuous functions from [0, b] to R endowed with the norm defined by

$$\|\Phi\| = \sup_{t\in J}|\Phi(t)|,$$

Where,

$$|\Phi(t)| = |S(t)| + |E(t)| + |I_A(t)| + |I_S(t)| + |I_I(t)| + |I_H(t)| + |R(t)| + |D(t)|$$

and $S, E, I_A, I_S, I_I, I_H, R, D \in C([0,b])$.

Theorem 3.1: Suppose that the function K ∈ C([J, R]) and maps bounded subset of J ×R^8 into relatively compact subsets of R. In addition, there exists constant LK > 0 such that (A1) |K(t, Φ1(t)) − K(t, Φ2(t))| ≤ LK|Φ1(t) − Φ2(t)|; for all t ∈ J and each Φ1, Φ2 ∈ C([J , R]). Then the integral equation which is equivalent with the proposed model (2) has a unique solution provided that ΩLK < 1, where

$$\Omega = \frac{b^{v}}{\Gamma(v+1)}$$

Proof: Consider the operator P: E → E defined by

$$(P\Phi)(t) = \Phi_0 + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1} ʁ\big(\tau, \Phi(\tau)\big)d\tau.$$

Obviously, the operator P is well defined and the unique solution of model (2) is just the fixed point of P. Indeed, let us take supt∈J kK(t, 0)k = M1 and κ ≥ kΦ0k + ΩM1. Thus, it is enough to show that PHκ ⊂ Hκ, where the set Hκ = {Φ ∈ E : kΦk ≤ κ}, is closed and convex. Now, for any Φ ∈ Hκ, yields

$$|(P\Phi)(t)| = |\Phi_0| + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\big|ʁ\big(\tau, \Phi(\tau)\big)\big|d\tau$$

$$\leq \Phi_0 + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\big[\big|ʁ\big(\tau, \Phi(\tau)\big)\big| + |ʁ(\tau,0)| - |ʁ(\tau,0)|\big]d\tau$$

$$\leq \Phi_0 + \frac{\big(L_{ʁ}ʁ + M_1\big)}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}d\tau$$

$$\leq \Phi_0 + \frac{\big(L_{ʁ}ʁ + M_1\big)}{\Gamma(v+1)}b^{v}$$

$$\leq \Phi_0 + \Omega\big(L_{ʁ}ʁ + M_1\big)$$

$$\leq k$$

Hence, the results follows. Also, given any Φ1, Φ2 ∈ E, we get

$$|(P\Phi_1)(t) - (P\Phi_2)(t)| \leq \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\left[\left|\text{Ꞑ}(\tau, \Phi_1(\tau))\right| - \left|\text{Ꞑ}(\tau, \Phi_2(\tau))\right|\right]d\tau$$

$$\leq \frac{L_{\text{Ꞑ}}}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}|\Phi_1(\tau) - \Phi_2(\tau)|d\tau$$

$$\leq \Omega L_{\text{Ꞑ}}|\Phi_1(t) - \Phi_2(t)|,$$

which implies that $|(P\Phi_1)(t) - (P\Phi_2)(t)| \leq \Omega L_{\text{Ꞑ}}|\Phi_1(t) - \Phi_2(t)|$. Therefore, as a consequence of Banach contraction principle, proposed model (2) possess a unique solution.

Next, we prove the existence of solutions of the proposed model (2) by employing the concept of well-known Krasnoselskii's fixed point theorem.

Theorem 3.2: Let$M \neq \emptyset$be a closed, bounded and convex subset of a Banach Space E. Let P_1, P_2be two operators that obey the given relations,

- $P_1\Phi_1 + P_2\Phi_2 \in M \quad whenever\ \Phi_1, \Phi_2 \in M$
- P_1 is compact and continuous
- P_2 is a contraction mapping

Then there exists $u \in M\ such\ that\ u = P_1u + P_2u.$

Theorem 3.3: Suppose that the function K : J × R^8 → R is continuous and satisfies condition (A1). In addition, assume that (A2) $|\text{Ꞑ}(t, \Phi)| \leq \varphi(t)\ \ for\ all\ (t, \Phi) \in$ J × R^8 and $\varphi \in C([0, b]; R_+)$, then the proposed model has at least one solution provided

$$L_{\text{Ꞑ}}\|\Phi_1(t) - \Phi_2(t)\| < 1.$$

Proof: Setting $\sup_{t\in J}|\varphi(t)| = \|\varphi(t)\|$ and $\eta \geq \|\Phi_0\| + \omega\|\varphi\|$, we consider $\beta_n = \{\Phi \in E: \|\Phi\| \leq \eta\}$.Consider the operators P_1, P_2 on B_η defined by

$$(P_1\Phi)(t) = \Phi_0 + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\text{Ꞑ}\big(\tau, \Phi(\tau)\big)d\tau,$$

and

$$(P_2\Phi)(t) = \Phi(t_0), \quad t \in J.$$

Thus, for any Φ_1, $\Phi_2 \in \beta_n$, yields

$$\|(P_1\Phi_1)(t) + (P_2\Phi_2)(t)\| \leq \|\Phi_0\| + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\left\|\text{Ƙ}\big(\tau, \Phi_1(\tau)\big)\right\|d\tau,$$

$$\leq \|\Phi_0\| + \Omega\|\varphi\|$$

$$\leq \boldsymbol{\eta} < \infty.$$

Hence, $P_1\Phi_1 + P_2\Phi_2 \in \beta_n$.

Next, we prove the contraction of the operator P_2. Obviously, given any t ∈ J and Φ_1, $\Phi_2 \in \beta_n$, gives

$$\|(P_2\Phi_1)(t) - (P_2\Phi_2)(t)\| \leq \|\Phi_1(t_0) - \Phi_2(t_0)\|$$

Since the function K is continuous, implies that the operator P_1 is continuous. Moreover, for any t ∈ J and $\Phi_1 \in \beta_n$,

$$\|P_1\Phi\| \leq \Omega\|\varphi\| < +\infty.$$

implies that P_1 is uniformly bounded. Finally, we show that the operator P1 is compact. Define

$$\sup_{(t,\Phi)\in J\times\beta_n} \text{Ƙ}\big(t, \Phi(t)\big) = \text{Ƙ}^*, gives$$

$$|(P_1\Phi)(t_2) - (P_2\Phi)(t_2)|$$

$$\leq \frac{1}{\Gamma(v)}\left|\int_0^{t_1}[(t_2-\tau)^{v-1} - (t_1-\tau)^{v-1}]\text{Ƙ}\big(\tau, \Phi(\tau)\big)\right.$$

$$\left. + \int_{t_1}^{t_2}[(t_2-\tau)^{v-1}]\text{Ƙ}\big(\tau, \Phi(\tau)\big)d\tau\right|$$

$$\leq \frac{\text{Ƙ}^*}{\Gamma(v)}[2(t_2-t_1)^v + (t_2^v - t_1^v)]$$

$$\to 0 \; as \; t_2 \to t_1.$$

Thus, P_1 is equicontinuous and so is relatively compact on β_n. Hence, as consequences of Arzel´a Ascoli theorem, P_1 is compact on β_n. Since all the hypotheses of Theorem [10] are true, proposed model (2) has at least one solution.

3.2 Stability results

In this section, we drive the stability of the proposed model (2) in the frame of Ulam-Hyers and generalized Ulam-Hyers stability. The concept of Ulam stability was introduced by Ulam [14,15]. Then, in several research papers on classical fractional derivatives, the aforementioned stability was investigated, see for example, [16,17,18,19]. Moreover, since stability is fundamental for approximate solution, we strive to use nonlinear functional analysis on Ulam-Hyers and generalized stability of the proposed model (2). Thus the following definitions are needed. Let $\varepsilon > 0$ and consider the inequality given below

$$\left| {}^{C}D_{0+}^{v}\bar{\Phi}(t) - ʁ\left(t, \bar{\Phi}(t)\right)\right| \leq \varepsilon \qquad t \in J.$$

where $\varepsilon = \max\left(\varepsilon_j\right)^T, j = 1, \ldots, 8.$

Definition 3.1: The proposed problem (2) is Ulam-Hyers stable if there exist $C_ʁ > 0$, such that for every $\varepsilon > 0$ and a solution $\bar{\Phi} \in$ E satisfying (5.1), there exists a unique solution $\Phi \in$ E of equation (2), with

$$|\bar{\Phi}(t) - \Phi(t)| \leq C_ʁ \varepsilon \qquad t \in J.$$

where $C_ʁ = max\left(C_{ʁj}\right)^T$.

Definition 3.2: Problem (2) is referred to generalized Ulam-Hyers stable if there exist a continuous function $\varphi_ʁ$: $R_+ \rightarrow R_+$, with $\varphi_ʁ(0) = 0$, such that for every solution $\bar{\Phi} \in$ E of the equation (14), there a solution $\Phi \in$ E of equation (2), such that

$$|\bar{\Phi}(t) - \Phi(t)| \leq \varphi_ʁ \varepsilon_1, \qquad t \in \text{J},$$

where $\varphi_ʁ = max\left(\varphi_{ʁj}\right)^T$.

Remark 3.1: A function $\bar{\Phi} \in$ E satisfy the inequality (14), if and only if there exists a function h ∈ E with the property below:

i. $|h(t)| \leq \varepsilon h = max\left(h_j\right)^T, t \in$ J

ii. ${}^{C}D_{0+}^{v}\bar{\Phi}(t) = \text{Ꙗ}\left(t, \bar{\Phi}(t)\right) + h(t), t \in$ J.

Theorem 3.1: Assume that $\bar{\Phi} \in$ E satisfies inequality (14), then $\bar{\Phi}$ satisfies the integral inequality describe by

$$\left|\bar{\Phi}(t) - \bar{\Phi}_0 - \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\text{Ꙗ}\left(\tau, \Phi(\tau)\right)d\tau\right| \leq \Omega_\varepsilon.$$

Proof: Thanks to (ii) of Remark 5.1

$${}^{C}D_{0+}^{v}\bar{\Phi}(t) = \text{Ꙗ}\left(t, \bar{\Phi}(t)\right) + h(t)$$

and theorem [10] gives

$$\bar{\Phi}(t) = \bar{\Phi}_0 + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\text{Ꙗ}\left(\tau, \Phi(\tau)\right)d\tau + \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}h(\tau)d\tau.$$

Using (i) of Remark 5.1 and (A2), we get

$$\left|\bar{\Phi}(t) - \bar{\Phi}_0 - \frac{1}{\Gamma(v)}\int_0^t (t-\tau)^{v-1}\text{Ꙗ}\left(\tau, \Phi(\tau)\right)d\tau\right| \leq \Omega_\varepsilon$$

Hence the desired result.

Theorem 3.2. Suppose that K : J × R^8 → R is continuous for every Φ ∈ E and hypotheses (A1) hold with $1 - \Omega L_{\text{Ꙗ}} > 0$. Thus, problem (2) is Ulam-Hyers and consequently, generalized Ulam-Hyers stable.

Proof. Suppose that $\bar{\Phi} \in$ E satisfies the inequality (14) and $\Phi \in$ E be a unique solution of (2). Thus, for any $\varepsilon > 0$, t ∈ J and Lemma 5.1, gives

$$|\bar{\Phi}(t) - \Phi(t)| = \max_{t\in J} \left| \bar{\Phi}(t) - \bar{\Phi}_0 - \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} Ку\big(\tau, \Phi(\tau)\big) d\tau \right|$$

$$\leq \max_{t\in J} \left| \bar{\Phi}(t) - \bar{\Phi}_0 - \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} Ку\big(\tau, \bar{\Phi}(\tau)\big) d\tau \right| +$$

$$\max_{t\in J} \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} \left| Ку\big(\tau, \bar{\Phi}(\tau)\big) - Ку\big(\tau, \Phi(\tau)\big) \right| d\tau$$

$$\leq \left| \bar{\Phi}(t) - \bar{\Phi}_0 - \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} Ку\big(\tau, \bar{\Phi}(\tau)\big) d\tau \right| + \frac{L_{Ку}}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} |\bar{\Phi}(\tau) - \Phi(\tau)| d\tau$$

$$\leq \Omega_\varepsilon + \Omega L_{Ку} |\bar{\Phi}(\tau) - \Phi(\tau)|$$

So,

$$|\bar{\Phi}(t) - \Phi(t)| \leq C_{Ку} \varepsilon,$$

where

$$C_{Ку} \varepsilon = \frac{\Omega_\varepsilon}{1 - \Omega L_{Ку}}.$$

So, setting $\varphi_{Ку}(\varepsilon) = C_{Ку}\varepsilon$ such that $\varphi_{Ку}(0) = 0$. We conclude that the proposed problem (2) is both Ulam-Hyers and generalized Ulam-Hyers stable.

CHAPTER 4

Numerical Simulations

Here we use COVID - 19 data obtained from Federal Ministry of Health Oman on 18th June 2020 for the Numerical simulations. The parameter values are given in Table 2. We can observe that the number of infection get to zero with time. This is true as per as any epidemic disease is concern. Herein, the fractional variant of the model under consideration via Caputo fractional operator is numerically simulated via first order convergent numerical techniques as proposed in[20–22]. These numerical techniques are accurate, conditionally stable, and convergent for solving fractional-order both linear and nonlinear system of ordinary differential equations. Now we discuss the obtained numerical outcomes of the governing model in respect of the approximate solutions. To this aim, we employed the effective Euler method under the Caputo fractional operator to do the job. The initial conditions are taken as $S(0) = 4, 602, 296$, $E(0) = 26818$, $I_A(0) = 300$, $I_S(0) = 169$, $I_I(0) = 113$, $I_H(0) = 56$, $R(0) = 13264$, $D(0) = 119$ and the parameters values are as in Table 2 below:

Table 3: Parameter values

Parameter	Value
β	calibration
α	0.5
ξ	calibration
k	0.2174
p	0.5
q	0.3448
φ	data

$\gamma_A, \gamma_S, \gamma_H$	0.1961
μ_S, μ_H	data

2.1 Simulation resutls

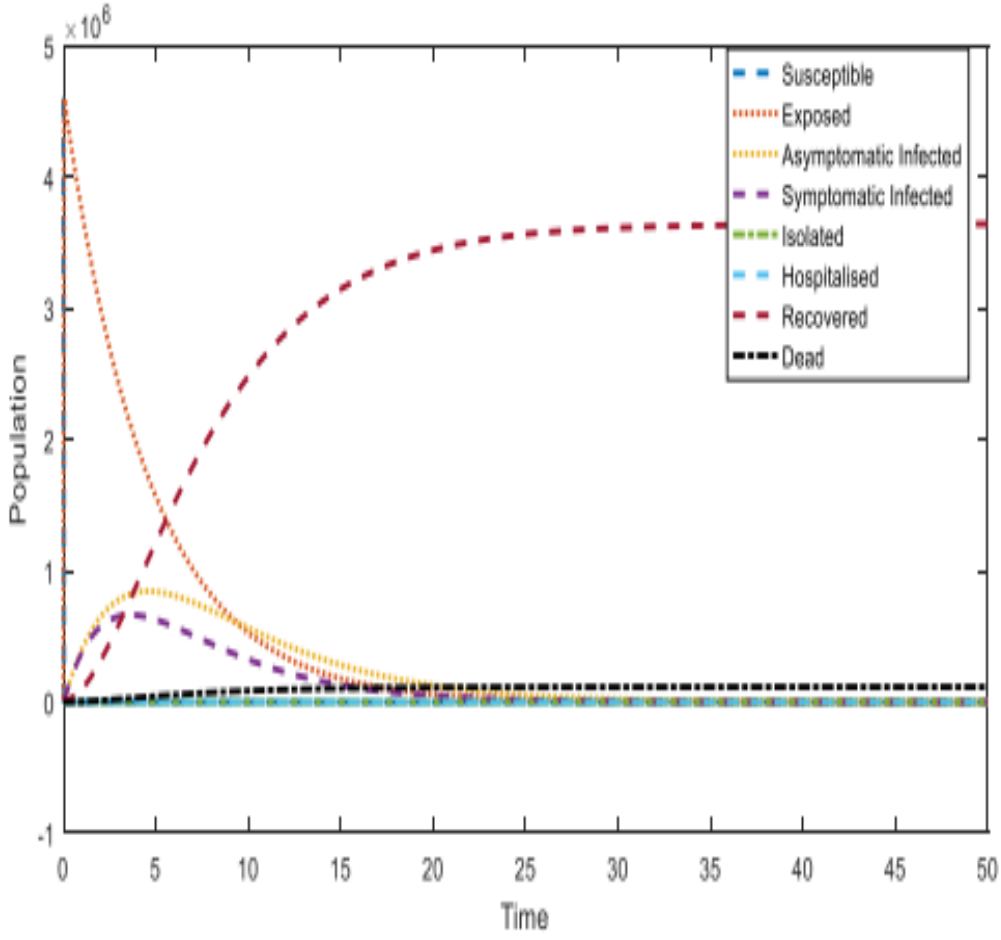


Figure 2: Dynamics of Different Populations in the Model

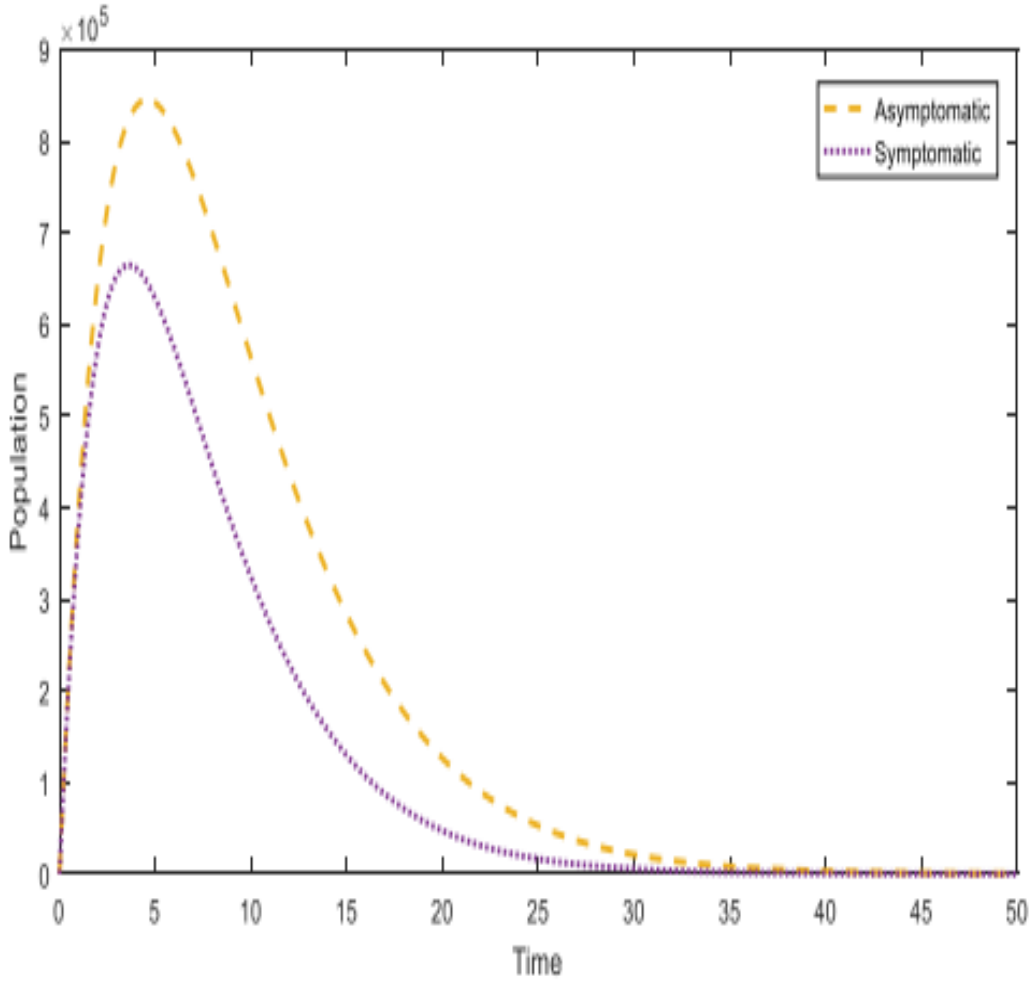


Figure 3: Asymptomatic versus Symptomatic

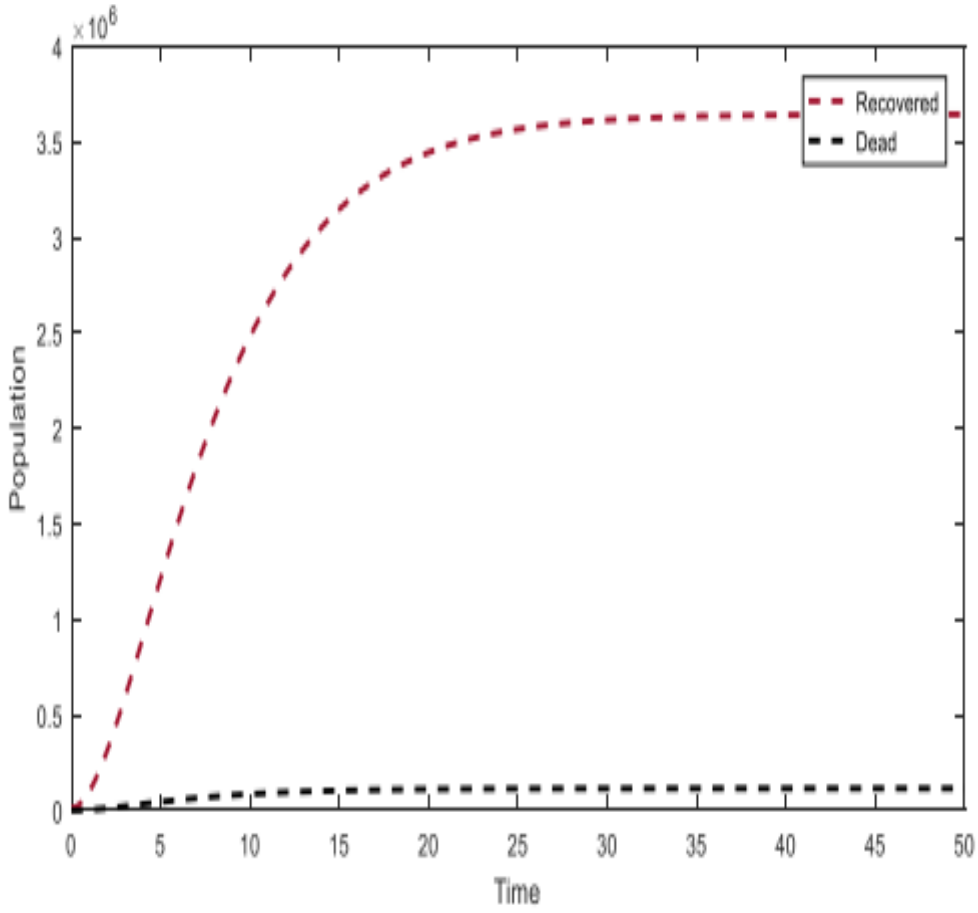


Figure 4: Recovered versus Dead

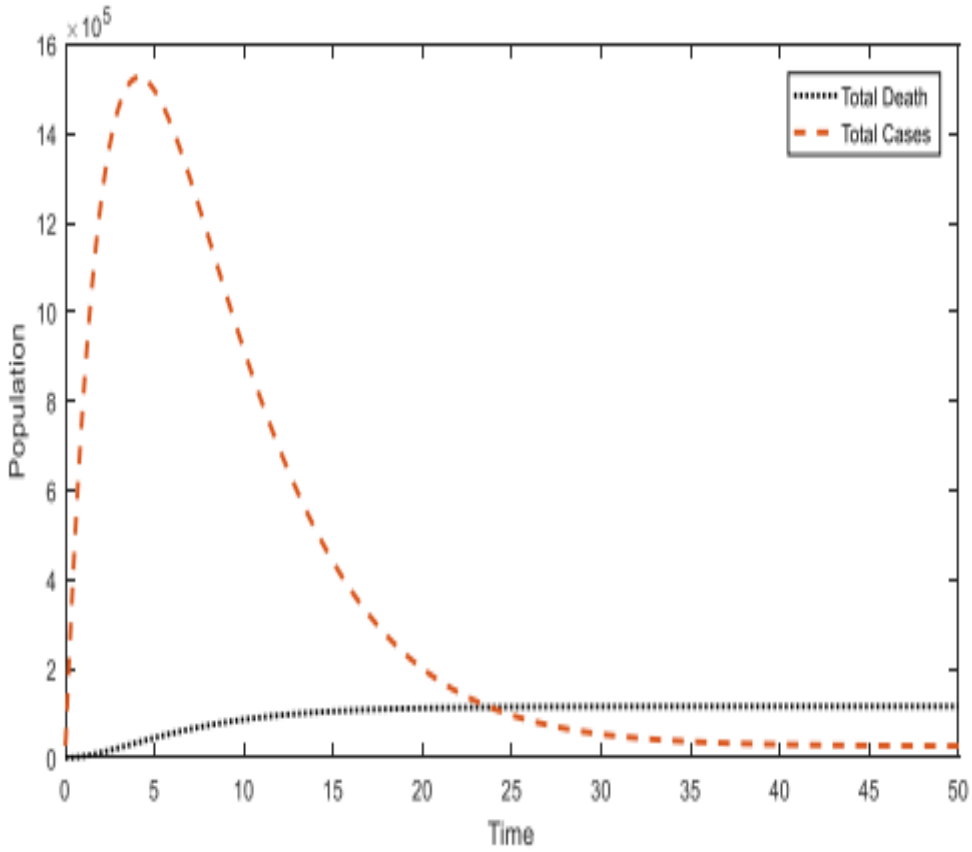


Figure 5: Total Infected Cases versus Total Death

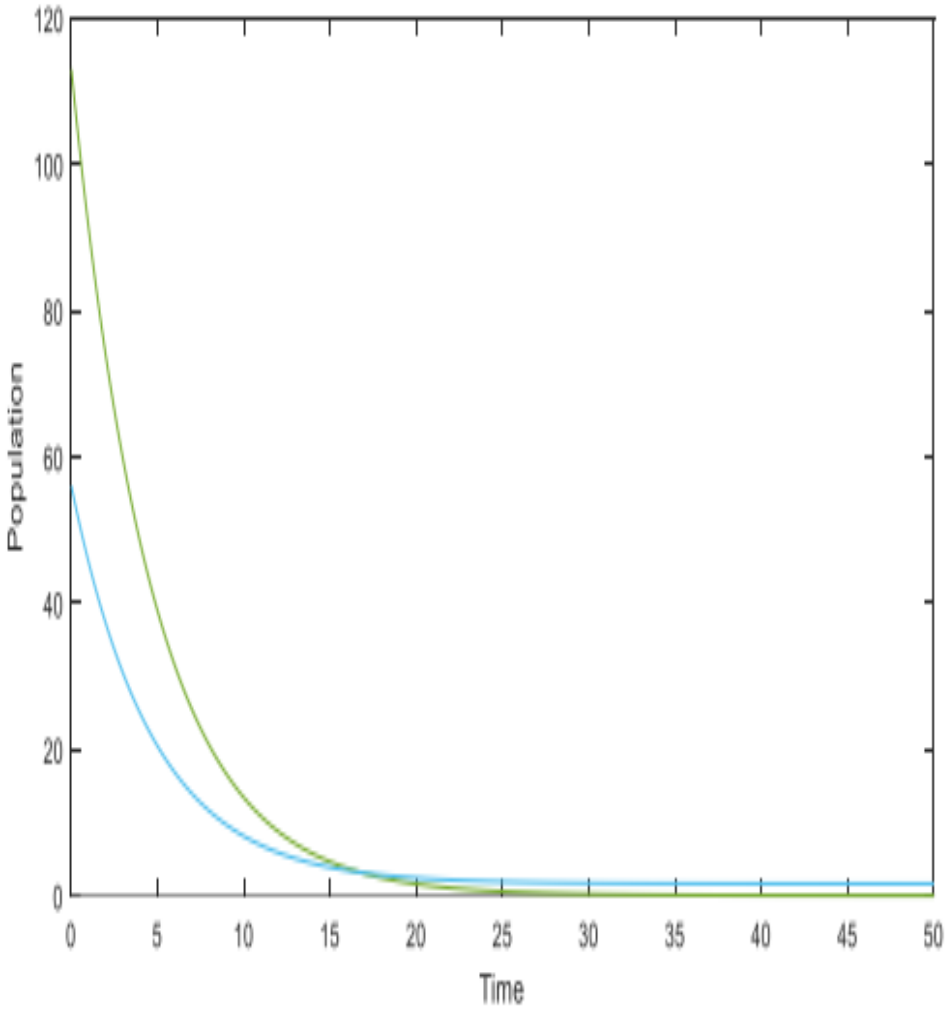


Figure 6: Isolated Infective cases versus Hospitalised Infective cases

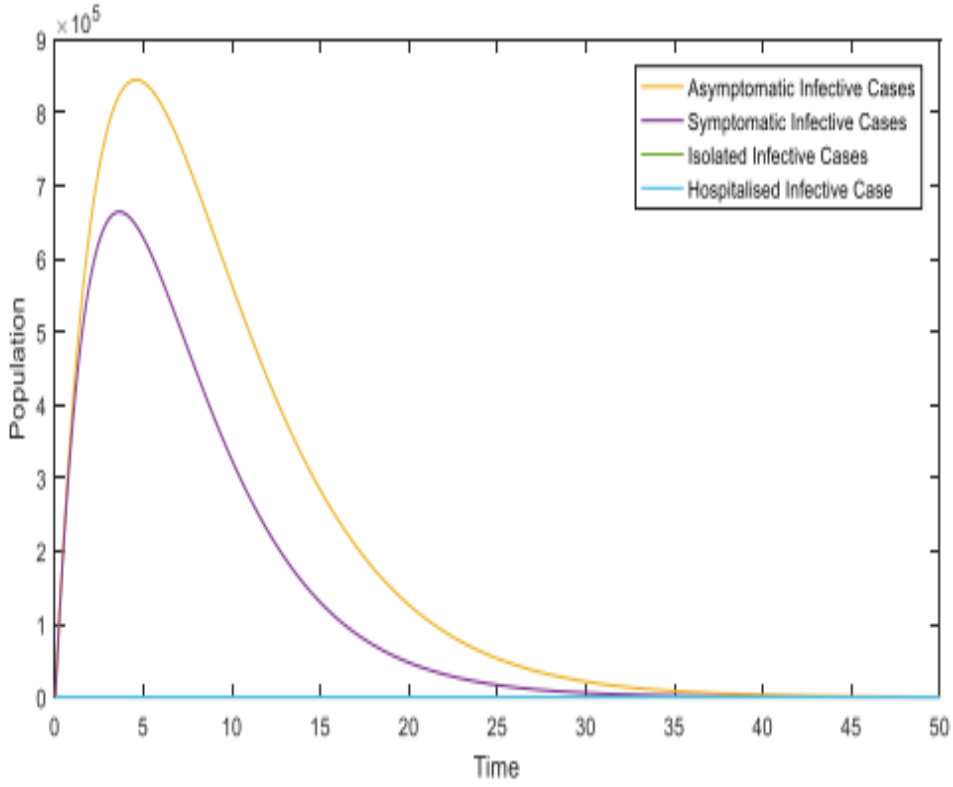


Figure 7: Various infection cases

From the graphs, we can see that FODEs have rich dynamics and are better descriptors of biological systems than traditional integer – order models. From Fig. 8 ($\alpha = 1$) to Fig. 12 ($\alpha = 0.2$), we can observe that the number of infection get to zero with time. This is true as per as any epidemic disease is concern. We note that that the solution of the model, with various values of a;continuously depends on the time – fractional derivative, but arrives to the equilibrium points. The displayed solution in Figs. 8 – 12 confirm that the fractional order plays the role of time – delay in the systems.

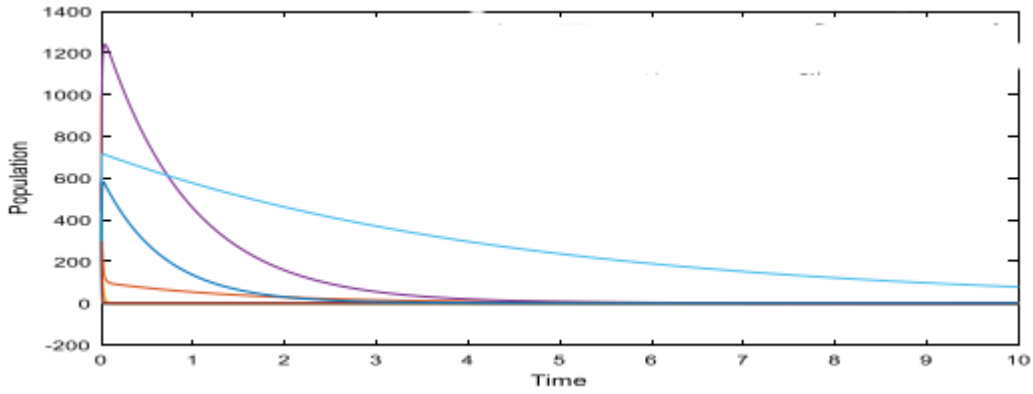


Figure 8 :Population dynamics for $\boldsymbol{\alpha = 1}$

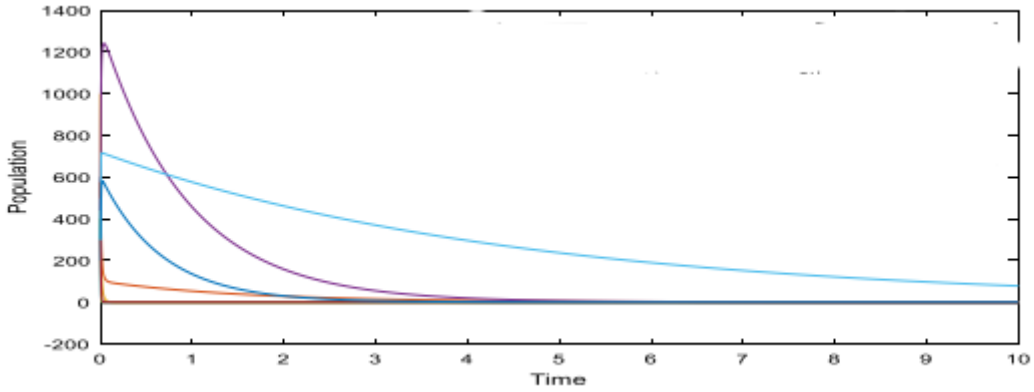


Figure 9 :Population dynamics for $\boldsymbol{\alpha = 0.8}$

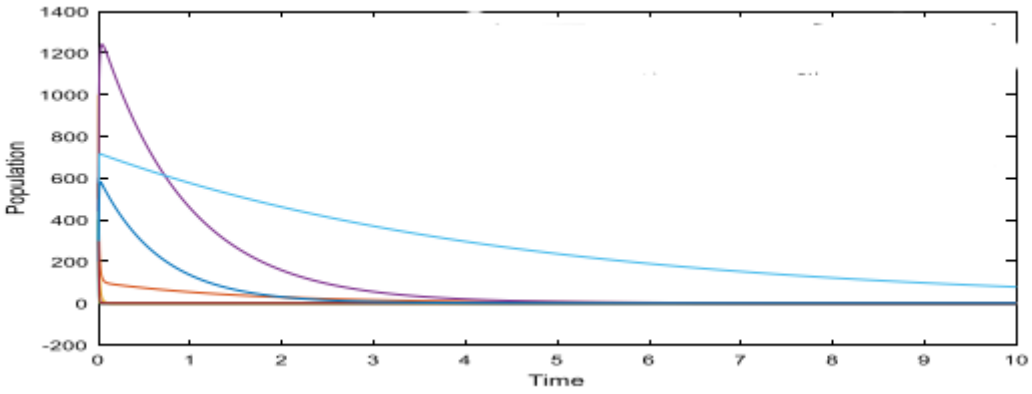
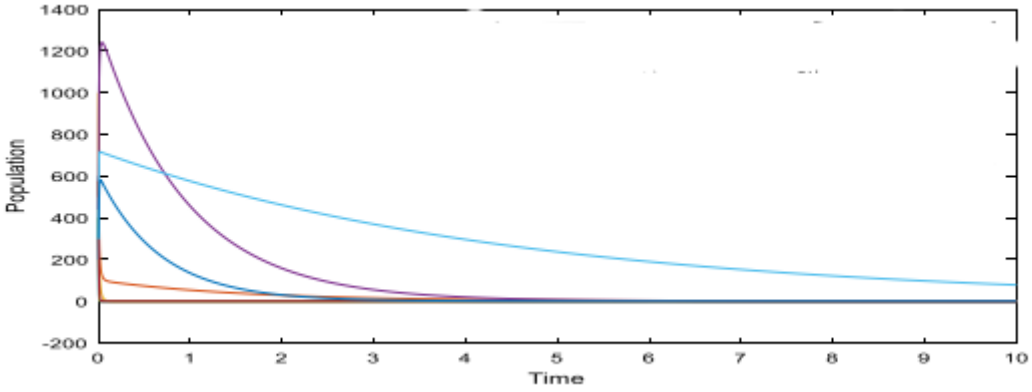


Figure 10: Population dynamics for $\boldsymbol{\alpha = 0.6}$



F**igure 11:** Population dynamics for $\boldsymbol{\alpha = 0.4}$

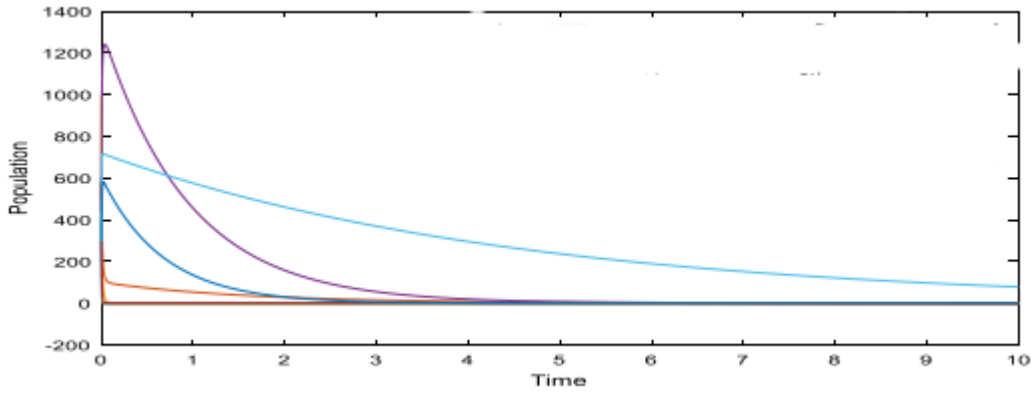


Figure 12 : Population dynamics for $\boldsymbol{\alpha = 0.2}$

CHAPTER 5

Discussion and Conclusion

Considering the values in the above table, we carry out the following numerical simulations with the fractional order value $\upsilon = 0.67$.; In Figure 1, dynamics of all the populations involved in the model are presented. It can be seen that with time, all the populations will tend to zero except the recovered and Dead populations. This requires more effort to be put in place in order to reduce the number of death and increase the recovery of the infected individuals. In Figure 2, Asymptomatic and Symptomatic cases are shown. It can be seen that there are more asymptomatic cases as compared to the symptomatic cases. This means, there are many positive individuals without any sign of the disease. Infection through this lane could only be stopped through contact trace. In Figure 3, Recovered case were depicted against Death cases. It can clearly be seen that there are more recovery cases than death cases. This is a good news and various means should be put in place to maintain the trend. Figure 4 presents the total Infected cases against total Death cases. It can be seen that indeed many infected individuals recovered. The death cases is very low, which shows that the disease is not fatal in Oman. Figure 5 shows the relationship between isolated infected cases and hospitalised infected cases. It is clear that as time goes on both populations will tend to zero. Finally Figure 6 shows the dynamics of various infection cases. It can be seen that isolated and hospitalised populations are very small compared to the asymptomatic and symptomatic cases. Hence there is need for more effort in tracing out the infected individuals. Many informations on the dynamics of COVID -19 in Oman were obtained using this model. Also many informations on the qualitative behaviour of the model were obtained.

5.1 Conclusion

In conclusion, this thesis consists of a system of eight non linear fractional order differential equations in Caputo sense. The existence and uniqueness of solution of the proposed model using fixed point theorems is discussed. Stability analysis in the frame of Ulam-Hyers and generalized Ulam-Hyers criteria is established. Numerical simulations were carried out

using real data from Federal ministry of health Oman. It was numerically shown that although the disease is not fatal in Oman, but there will be many death cases. Hence there is need for relevant authorities to take every available measure to curtail the spread of the disease.

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- Analytical and creative in problem solving.
- Good computer skills.

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I can give any course in Mathematics as:

Calculus, Linear algebra, Differential equations, Partial differential equations, Numerical analysis, Real analysis, Complex analysis, Differential geometry, Inequality, Abstract algebra and Functional analysis... etc