ABBA AUWALU

SOME FIXED POINT THEOREMS OF CONTRACTIVE MAPPINGS IN PENTAGONAL CONE METRIC SPACES

PhD THESIS

2021

NEAR EAST UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

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October, 2021

Approval

We certify that we have read the thesis submitted by Abba Auwalu titled "**Some Fixed Point Theorems of Contractive Mappings in Pentagonal Cone Metric Spaces**" and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

Abba Auwalu
20/10/2021

Acknowledgments

I am gratefully indebted to my supervisor, Prof. Dr. Evren Hınçal, for his suggestions, patience and advices during the course of this Dissertation work, may Allah (SWT) reward and bless him and his family, amen.

I whole heartily thank the Lecturers of Mathematics Department, Near East University, Cyprus. I would also like to express my whole gratitude to my family and friends.

Finally and specially I like to say thank you my wife, Ummi Sabo (Hajara), for being of much assistance to me.

Abba Auwalu

Abstract

Some Fixed Point Theorems of Contractive Mappings in Pentagonal Cone Metric Spaces

Auwalu, Abba

PhD, Department of Mathematics

October, 2021, 91 pages

In this thesis, we study some fixed points and common fixed points theorems of various contractive mappings in non-normal cone metric space, rectangular cone metric space and pentagonal cone metric space settings. Our results extend and improve many results obtained by many authors. We give some examples to elucidate our results.

Keywords: Cone metric space, cone pentagonal metric space, fixed point, contraction mapping principle, weakly compatible mappings

Özet

Some Fixed Point Theorems of Contractive Mappings in Cone Pentagonal Metric Spaces

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Bu tezde, normal olmayan koni beşgen metrik uzaylarda bir, iki, üç ve dört kendi eşlemesi için ortak sabit noktaların varlığını kanıtlıyoruz. Elde edilen sonuçlar, birçok yazarın elde ettiği yeni sonuçları genişletmekte ve iyileştirmektedir.

Keywords: Metrik uzay konisi, koni beşgen metrik uzay, sabit nokta, kasılma haritalama ilkesi, geniş haritalama

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List of Abbreviations

MS	Metric space
CMS	Cone metric space
RCMS	Rectangular cone metric space
PRCbMS-BA	Partial rectangular cone b - metric space over Banach algebra
PCMS	Pentagonal cone metric space
d(x,y)	Distance from x to y
‖x‖	Norm of x
Ø	Empty set
N	Field of natural numbers
R	Field of real numbers
C	Field of complex numbers
lim	limit

CHAPTER I

Introduction

Fixed point theory is one of the traditional theories in mathematics and has a large number of applications in it and many branches of nonlinear analysis. The starting point of metric fixed point theory is often associated with the renowned work which appeared in Banach's PhD thesis, known as the Banach contraction principle. Due to the wide applications of this principle, it is being investigated at a large in contemporary research and has been used and extended in many different directions (Saleh et al., 2014). Although the famous Banach contraction principle was proved in a metric space, but later on some modifications of the definition of a metric space appeared. One such modification was made by Liu and Xu (Liu & Xu, 2013). They replaced the set of real numbers, which forms the domain of distance function, with a Banach algebra and obtained cone metric spaces over Banach algebras and show that they are not equivalent to metric spaces in terms of existence of the fixed points of mappings. Further, they proved Banach contraction principle in such a space by replacing usual real contraction constant with a vector constant.

The study of existence and uniqueness of fixed point of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces; for example, see (Azam et al., 2009; Branciari, 2000; Garg & Agarwal, 2012; Huang & Zhang, 2007; Patil & Salunke, 2015).

Statement of the Problem

This research work concentrates on introducing a notion of new space and proving some new fixed point theorems in such a space.

Purpose of the Study

The purpose of this research work is to study and prove some new fixed point theorems of different contractive mappings in cone metric spaces and its generalizations.

Research Questions/Hypotheses

1. Can we extend and improve some fixed point theorems of contractive mappings in cone metric spaces to some more general one?

2. Can we introduce a notion of new space and prove some fixed point theorems of different contractive mappings in such a space?

Significance of the Study

This research work is important in the study of fixed point theorems of contractive mappings in the framework of cone metric spaces and its generalizations. Hence, this research work will serve as a resource document for researchers in the area of Fixed Point Theory.

Scope and Limitations

This research work focuses mainly on fixed point theorems for different contractive conditions in cone metric spaces. Thus, the research will be limited to a cone metric spaces and some of its generalizations.

Definition of Terms

In this section, we shall give definitions of some important concepts and some existing results required in the sequel. They can be found in (Kreyszig, 1978).

Definition 0.1. A **metric space** is a pair (K, η), where K is a non-empty set and η is a metric on K (or distance function on K), that is, a real - valued function $\eta : K \times K \to \mathbb{R}$ such that for all $x, y, z \in K$ we have the following:

$(M1)$ $\eta(x, y) \geq 0$ (Non-negativity);

$(M2)$ $\eta(x, y) = 0$ if and only if $x = y$ (Reflexive property);

$(M3)$ $\eta(x, y) = \eta(y, x)$ (Symmetric property);

$(M4)$ $\eta(x, y) \leq \eta(x, z) + \eta(z, y)$ (Triangle inequality).

We will sometimes, denote the metric space (K, η) simply by K.

Example 0.2. Consider the real line $\mathbb{R}$, the set of all real numbers, taken with the **usual metric** defined by

$$d(x, y) = |x - y|, \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Then $(\mathbb{R}, d)$ is a metric space.

Definition 0.3. Convergence sequence: A sequence $\{x_n\}$ of points of a metric space (K, η) is said to be convergent to $x \in K$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that

$$\eta(x_n, x) < \epsilon, \quad \text{for all } n \geq n_0. \tag{2}$$

This is denoted by $x_n \to x$ as $n \to \infty$ or $\eta(x_n, x) \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. The point x is called the limit of the sequence $\{x_n\}$.

Lemma 0.4. *The limit of a convergence sequence in a metric space is unique.*

Definition 0.5. Bounded sequence: A sequence $\{x_n\}$ in a metric space (K, η) is said to be bounded if there is a real number M and a point $x \in K$ such that

$$\eta(x_n, x) \leq M, \quad \text{for all } n. \tag{3}$$

Lemma 0.6. *Every convergence sequence in a metric space is bounded.*

Definition 0.7. Chauchy sequence: A sequence $\{x_n\}$ in a metric space (K, η) is said to be Cauchy sequence if $\eta(x_m, x_n) \to 0$ as $m, n \to \infty$.

Lemma 0.8. *Every convergence sequence in a metric space is a Cauchy sequence.*

Definition 0.9. Complete metric space: This is a metric space in which every Chauchy sequence is convergent.

Lemma 0.10. *The space $\mathbb{R}$ with usual metric is complete.*

Definition 0.11. Continuity in metric space: Let (X, η) and $(Y, \eta^{'})$ be metric spaces and f a function of X into Y, then f is continuous if and only if $x_n \to x$ implies $f(x_n) \to f(x)$.

Definition 0.12. Let (K, η) be a metric space and $T : K \to K$ be a mapping.

1. A point $x^* \in K$ is called a **fixed point** of the mapping T if and only if

$$T(x^*) = x^*. \tag{4}$$

2. The mapping T is called **Banach contraction** if there exists a real constant $\alpha \in [0, 1)$ such that

$$\eta(Tx, Ty) \leq \alpha\eta(x, y), \quad \text{for all } x, y \in K. \tag{5}$$

3. The mapping T is called **Kannan contraction** if there exists a real constant $\alpha \in [0, 1/2)$ such that

$$\eta(Tx, Ty) \leq \alpha\Big[\eta(x, Tx) + \eta(y, Ty)\Big], \quad \text{for all } x, y \in K. \tag{6}$$

4. The mapping T is called **Reich contraction** if there exists $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$ such that

$$\eta(Tx, Ty) \leq \alpha\eta(x, y) + \beta\eta(x, Tx) + \gamma\eta(y, Ty), \quad \text{for all } x, y \in K. \tag{7}$$

Definition 0.13. A **vector space** (or linear space) over a field F is a non-empty set K of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of F. Indeed, F is called a scalar field of the vector space K.

Definition 0.14. A **norm** on a (real or complex) vector space K is a real - valued function on K whose value at an $x \in K$ is denoted by $\|x\|$ (read "norm of x") and which has the properties for any x, y arbitrary vectors in K and α is any scalar:

$(N1)$ $\|x\| \geq 0$;

$(N2)$ $\|x\| = 0 \Longleftrightarrow x = 0$;

$(N3)$ $\|\alpha x\| = |\alpha|\|x\|$;

$(N4)$ $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

A norm on K defines a metric η on K which is given by

$$\eta(x, y) = \|x - y\|, \quad \text{for all } x, y \in K,$$

and is called the metric induced by the norm. The normed space just defined is denoted by $(K, \|.\|)$ or simply by K.

Definition 0.15. A **Normed linear space** say X, is a vector space with a norm defined on it. A large number of metric spaces in analysis can be regarded as normed linear spaces, so that a normed linear space is probably the most important kind of space in functional analysis, at least from the viewpoint of present-day applications. A complete normed space is called **Banach space**.

Definition 0.16. (Huang & Zhang, 2007).

Let E be a real Banach space and P a subset of E. P is called a **cone** if and only if:

1. P is closed, nonempty, and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}, \quad a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Example 0.17. (Deimling, 1985).

Let $E = \mathbb{R}^n$ with $P = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0,\ \forall i = 1, 2, \ldots, n\}$ then P is a cone.

Definition 0.18. (Huang & Zhang, 2007).

Given a cone $P \subset E$, we defined a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 0.19. (Huang & Zhang, 2007).

A cone P is called normal if there is a number $\lambda > 0$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq \lambda \|y\|, \tag{8}$$

The least positive number λ satisfying (8) is called the normal constant of P.

Example 0.20. (Rezapour & Hamlbarani, 2008).

Let $E = C^2_{\mathbb{R}}([0, 1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone $P = \{f \in E : f \geq 0\}$. For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then, $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$. Since $k\|f\| \leq \|g\|$, k is not normal constant of P. Therefore, P is a non-normal cone.

Definition 0.21. (Rudin, 1991).

Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, for all $y, z, x \in \mathcal{A}$ and $k \in \mathbb{R}$, the following are satisfy:

1. $y(zx) = (yz)x$;
2. $y(z + x) = yz + yx$ and $(y + z)x = yx + zx$;
3. $k(yz) = (ky)z = y(kz)$;
4. $\|yz\| \leq \|y\|\|z\|$.

A Banach algebra $\mathcal{A}$ is called unital if there exists a unit $e \in \mathcal{A}$ such that $ey = ye = y$, for any $y \in \mathcal{A}$.

Definition 0.22. (Liu & Xu, 2013). A subset $\mathcal{K}$ of $\mathcal{A}$ is called a cone if

1. $\mathcal{K}$ is nonempty, closed and $\{\theta, e\} \subset \mathcal{A}$, where θ is the zero of $\mathcal{A}$;
2. $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ for all non-negative real numbers α, β;
3. $\mathcal{K}^2 = \mathcal{K}\mathcal{K} \subset \mathcal{K}$;
4. $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$.

For a given cone $\mathcal{K} \subset \mathcal{A}$, we define a partial ordering $\preceq$ with respect to $\mathcal{K}$ by $y \preceq z$ if and only if $z - y \in \mathcal{K}$. The notation $y \ll z$ will stand for $z - y \in \mathcal{K}^\circ$, where $\mathcal{K}^\circ$ denotes the interior of $\mathcal{K}$. If $\mathcal{K}^\circ \neq \emptyset$ then $\mathcal{K}$ is called a solid cone.

Definition 0.23. (Xu & Radenović, 2014). Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{y_n\} \subset \mathcal{K}$ is said to be a c-sequence if for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $y_n \ll c$ for all $n > N$.

Lemma 0.24. (Shukla et al., 2016). *Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$.*

1. *If $\alpha, \beta \in \mathcal{A}$, $\gamma \in \mathcal{K}$ and $\alpha \preceq \beta$, then $\gamma\alpha \preceq \gamma\beta$.*
2. *If $\alpha \preceq \beta\alpha$, where $\alpha, \beta \in \mathcal{K}$ and $\rho(\beta) < 1$, then $\alpha = \theta$.*
3. *If $\alpha \in \mathcal{K}$ and $\rho(\alpha) < 1$, then $\rho(\alpha^q) < 1$ for any fixed $q \in \mathbb{N}$.*

Lemma 0.25. (Rudin, 1991; Huang & Radenović, 2015). *Let $\mathcal{A}$ be a unital Banach algebra and $\alpha \in \mathcal{A}$, then $\lim_{m\to\infty} \|\alpha^m\|^{\frac{1}{m}}$ exists and the spectral radius $\rho(\alpha)$ satisfies*

$$\rho(\alpha) = \lim_{m\to\infty} \|\alpha^m\|^{\frac{1}{m}} = inf_{m\geq 1}\|\alpha^m\|^{\frac{1}{m}}.$$

If $\rho(\alpha) < |\beta|$, then $(\beta e - \alpha)$ is invertible in $\mathcal{A}$. Moreover,

$$(\beta e - \alpha)^{-1} = \sum_{j=0}^{\infty} \frac{\alpha^j}{\beta^{j+1}} \text{ and } \rho\Big[(\beta e - \alpha)^{-1}\Big] \leq \frac{1}{|\beta| - \rho(\alpha)},$$

where β is a complex constant.

Lemma 0.26. (Rudin, 1991).

Let $\mathcal{A}$ be a unital Banach algebra and $\alpha, \beta \in \mathcal{A}$ such that α commutes with β. Then

$$\rho(\alpha + \beta) \leq \rho(\alpha) + \rho(\beta) \text{ and } \rho(\alpha\beta) \leq \rho(\alpha)\rho(\beta).$$

Lemma 0.27. (Huang & Radenović, 2016).

Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$, $\{y_n\}$ and $\{z_n\}$ be two c-sequences in $\mathcal{K}$. If $\alpha, \beta \in \mathcal{K}$ are two given vectors, then $\{\alpha y_n + \beta z_n\}$ is also a c-sequence in $\mathcal{K}$.

Lemma 0.28. (Huang & Radenović, 2016). *Let $\mathcal{A}$ be a unital Banach algebra. Let $\alpha \in \mathcal{A}$ and $\rho(\alpha) < 1$. Then $\{\alpha^n\}$ is a c-sequence in $\mathcal{A}$.*

Lemma 0.29. (Xu & Radenović, 2014). *Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$.*

1. *If $\alpha, \beta, \gamma \in \mathcal{K}$ and $\alpha \preceq \beta \ll \gamma$, then $\alpha \ll \gamma$.*
2. *If $\alpha \in \mathcal{A}$ and $\theta \preceq \alpha \ll \beta$ for each $\beta \in \mathcal{K}^\circ$, then $\alpha = \theta$.*
3. *$\{y_n\} \subset \mathcal{K}$ is a c-sequence provided that $\{y_n\} \to \theta$ as $n \to \infty$.*

Definition 0.30. (Xu & Radenović, 2014). Let $\mathcal{P}$ be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{y_i\} \subset \mathcal{P}$ is said to be a c-sequence if for each $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $y_i \ll c$ for all $i > n_0$.

Lemma 0.31. (Xu & Radenović, 2014). *Let $\mathcal{P}$ be a solid cone in a Banach algebra $\mathcal{A}$ and $\{y_i\} \subset \mathcal{P}$ be a sequence with $\|y_i\| \to 0$ $(i \to \infty)$, then for each $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $i > n_0$, we have $y_i \ll c$.*

Lemma 0.32. (Rudin, 1991). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $\tau \in \mathcal{A}$. If the spectral radius $\delta(\tau)$ of τ is less than one, i.e.*

$\delta(\tau) = \lim_{n\to\infty} \|\tau^n\|^{\frac{1}{n}} = inf_{n\in\mathbb{N}} \|\tau^n\|^{\frac{1}{n}} < 1$, *then $(e - \tau)$ is invertible in $\mathcal{A}$. Moreover,* $(e - \tau)^{-1} = \sum_{k=0}^{\infty} \tau^k$.

Remark 0.33. (Xu & Radenović, 2014).

If the spectral radius $\delta(\tau) < 1$, then $\|\tau^i\| \to 0$ $(i \to \infty)$

Lemma 0.34. (Xu & Radenović, 2014).

Let $\mathcal{A}$ be a real Banach algebra with a solid cone $\mathcal{P}$. For $a, b, c, \tau \in \mathcal{P}$, if

(1) *$a \preccurlyeq b \ll c$, then $a \ll c$.*

(2) *$a \preccurlyeq \tau a$ and $\delta(\tau) < 1$, then $a = \theta$.*

Definition 0.35. (Rashwan & Saleh, 2012). Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:

1. $0 < \varphi(t) < t$ for all $t \in P \setminus \{0\}$,
2. the series $\sum_{n \geq 0} \varphi^n(t)$ converge for all $t \in P \setminus \{0\}$.

From 1., we have $\varphi(0) = 0$, and from (2), we have $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in P \setminus \{0\}$.

Definition 0.36. (Abbas & Jungck, 2008). Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $w, x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Also, T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 0.37. (Abbas & Jungck, 2008). *Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

CHAPTER II

Literature Review

Metric Spaces

Fŕechet (1906) introduced the concept of a metric space as extension of the distance on the real line $\mathbb{R}$. Kreyszig (1978) considered a metric space as the generalization of real numbers which has been created in order to provide a basis for a unified treatment of important problems from various branches of Mathematical Analysis.

Metric Fixed Point Theory

The start of the general theory of fixed points of mappings in metric spaces is often associated with the classical principle of contractive mappings in Banach's 1922 Ph.D. thesis where it was used to establish the existence of a solution of an integral equation. Banach (1922) formulated the principle as an existence and uniqueness theorem for a fixed point of a contractive map of a complete metric space into itself.

Banach fixed theorem. Banach (1922) proved the following fixed point theorem also known as Banach Contraction Principle:

Theorem 2.1. (Banach Contraction Principle)

Let (K, η) be a complete metric space. Suppose that a mapping $J : K \to K$ satisfies the contractive condition

$$\eta(Jy, Jz) \leq \alpha\eta(y, z), \text{ for all } y, z \in K, \tag{9}$$

where $0 \leq \alpha < 1$ is a real constant. Then J has a unique fixed point in K.

The Banach Contraction Principle is one of the most important and useful results in the metric fixed point theory. It is perhaps one of the most widely used fixed point theorems in all analysis. This is because the contraction condition on the mapping is simple and easy to verify, because it requires only completeness assumption on the underlying metric space, and because it finds almost canonical applications especially in the theory of differential and integral equations (Saleh et al., 2014).

Later on, several successful attempts have been made to generalize or improve the Banach Contraction Principle by replacing the contractive condition (9) by some more general one as follows:

Kannan fixed theorem. Kannan (1968) proved the following fixed point theorem also known as Kannan Contraction Principle:

Theorem 2.2. (Kannan Contraction Principle)

Let (K, η) be a complete metric space. Suppose that a mapping $J : K \to K$ satisfies the contractive condition

$$\eta(Jy, Jz) \leq \alpha[\eta(y, Jy) + \eta(z, Jz)], \text{ for all } y, z \in K, \tag{10}$$

where $0 \leq \alpha < 1/2$ is a real constant. Then J has a unique fixed point in K.

Kannan further showed that the conditions (9) and (10) are independent of each other.

Reich fixed theorem. Reich (1971) proved the following fixed point theorem also known as Reich Contraction Principle:

Theorem 2.3. (Reich Contraction Principle)

Let (K, η) be a complete metric space. Suppose that a mapping $J : K \to K$ satisfies the contractive condition

$$\eta(Jy, Jz) \leq \alpha\eta(y, z) + \beta\eta(y, Jy) + \gamma\eta(z, Jz), \text{ for all } y, z \in K, \tag{11}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. Then J has a unique fixed point in K.

Reich further showed that the conditions (9) and (10) can be obatained from (11) by taking $\beta = \gamma = 0$ and $\alpha = 0, \beta = \gamma$, respectively.

Chatterjea fixed theorem. Chatterjea (1972) proved the following fixed point theorem also known as Chatterjea Contraction Principle:

Theorem 2.4. (Chatterjea Contraction Principle)

Let (K, η) be a complete metric space. Suppose that a mapping $J : K \to K$ satisfies the contractive condition

$$\eta(Jy, Jz) \leq \alpha[\eta(y, Jz) + \eta(z, Jy)], \text{ for all } y, z \in K, \tag{12}$$

where $0 \leq \alpha < 1/2$ is a real constant. Then J has a unique fixed point in K.

Wang fixed theorem. Wang et al. (1984) proved the following fixed point theorem also known as Wang Contraction Principle:

Theorem 2.5. (Wang Contraction Principle)

Let (K, η) be a complete metric space. Suppose that a mapping $J : K \to K$ satisfies the contractive condition

$$\eta(Jy, Jz) \geq \alpha\eta(y, z), \text{ for all } y, z \in K, \tag{13}$$

where $\alpha > 1$ is a real constant. Then J has a fixed point in K.

Cone Metric Spaces

Huang and Zhang (2007) introduced the notion of a cone metric space by replacing the set of real numbers $\mathbb{R}$ in metric by an ordered Banach space E as follows:

Definition 0.38. Let Y be a non-empty set and E an ordered Banach space. Suppose that $\rho : Y \times Y \to E$ is a mapping satisfying, $\forall y, z, x \in Y$, the following conditions:

(C_1) $\rho(y, z) \succ \theta$;

(C_2) $\rho(y, z) = \theta$ if and only if $y = z$;

(C_3) $\rho(y, z) = \rho(z, y)$;

(C_4) $\rho(y, z) \preccurlyeq \rho(y, x) + \rho(x, z)$.

Then ρ is called a cone metric on Y, and (Y, ρ) is a cone metric space (CMS).

Example 0.39. (Huang & Zhang, 2007).

Let $E = \mathbb{R}^2$, $P = \{(x, y) : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $\rho : X \times X \to E$ such that:

$$\rho(x, y) = (|x - y|, \alpha|x - y|), \text{ where } \alpha \geq 0 \text{ is a real constant.}$$

Then (X, ρ) is a cone metric space.

Remark 0.40. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}^2$ and $P = [0, \infty)$.

Huang and Zhang (2007) further proved some fixed point theorems for different contractive conditions in cone metric spaces. Later on, many authors have proved some fixed point theorems for different contractive types in cone metric spaces; for example, see (Abbas & Jungck, 2008; Ilić & Rakoćević, 2008; Rezapour & Hamlbarani, 2008).

Khamsi (2010) claimed that most of the cone metric fixed point results are merely copies of the classical ones and that any extension of known metric fixed point results to cone metric spaces is redundant; also that underlying Banach space and the associated cone subset are not necessary.

However, Radenović et al. (2011) proved that Khamsi's approach includes a small class of results and is very limited since it requires only normal cone metric spaces, so that all results with non-normal cones (which are proper extensions of the corresponding results for metric spaces) cannot be dealt with by his approach, for more details, see (Radenović et al., 2011; Suzana et al., 2019) and the references therein.

A rider to (Radenović et al., 2011) in overcoming the challenges raised by Khamsi, Liu and Xu (2013) introduced the notion of cone metric space over Banach algebras by replacing the Banach space E in cone metric with a Banach algebra $\mathcal{A}$. They proved that cone metric space over a Banach algebra is not equivalent to metric space in terms of existence of the fixed points of mappings. They further proved some fixed point theorems for different contractive conditions in cone metric space over a Banach algebra.

Rectangular Cone Metric Spaces

Azam et al. (2009) introduced the notion of rectangular cone metric space by replacing the triangle inequality in cone metric space with rectangular inequality:

Definition 0.41. Let X be a non-empty set and E be an ordered Banach space. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(RC_1) $\rho(x, y) \succ \theta$, for all $x, y \in X$;

(RC_2) $\rho(x, y) = \theta$ if and only if $x = y$, for all $x, y \in X$;

(RC_3) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;

(RC_4) $\rho(x, y) \preccurlyeq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ (Rectangular inequality).

Then ρ is called a rectangular cone metric on X, and (X, ρ) is called a rectangular cone metric space (RCMS).

Remark 0.42. Every cone metric space is rectangular cone metric space. The converse is not necessarily true.

Example 0.43. (Azam et al., 2009).
Let $X = \mathbb{N}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$. Define $\rho : X \times X \to E$ as follows:

$$\rho(x,y) = \begin{cases} (0,0), & \text{if } x = y; \\ (3,9), & \text{if } x, y \in \{1,2\},\ x \neq y; \\ (1,3), & \text{otherwise.} \end{cases}$$

Then (X, ρ) is a rectangular cone metric space, but (X, ρ) is not a cone metric space because it lacks the triangular property:

$$\begin{aligned} (3,9) = \rho(1,2) &> \rho(1,3) + \rho(3,2) \\ &= (1,3) + (1,3) \\ &= (2,6), \ \text{as } (3,9) - (2,6) = (1,3) \in P. \end{aligned}$$

Azam et al. (2009) further proved Banach contraction mapping principle in a normal rectangular cone metric space setting. Rashwan and Saleh (2012) extended and improved the result of (Azam et al., 2009) by omitting the assumption of normality condition.

Shukla et al. (2016) introduced the notion of a rectangular cone metric space over Banach algebras by replacing the Banach space E in rectangular cone metric with a Banach algebra $\mathcal{A}$ and proved Banach contraction principle in such a space.

Pentagonal Cone Metric Spaces

Garg and Agarwal (2012) introduced the notion of pentagonal cone metric space and proved Banach contraction mapping principle in a normal pentagonal cone metric space setting.

Definition 0.44. Let X be a non-empty set and E be an ordered Banach space. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(PC_1) $\rho(x, y) \succ \theta$, for all $x, y \in X$;

(PC_2) $\rho(x, y) = \theta$ if and only if $x = y$, for all $x, y \in X$;

(PC_3) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;

(PC_4) $\rho(x, y) \preccurlyeq \rho(x, z) + \rho(z, w) + \rho(w, u) + \rho(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ (Pentagonal property).

Then ρ is called a pentagonal cone metric on X, and (X, ρ) is called a pentagonal cone metric space (PCMS).

Remark 0.45. Every rectangular cone metric space and so cone metric space is pentagonal cone metric space. The converse is not necessarily true.

Example 0.46. (Garg & Agalwal, 2012).
Let $X = \mathbb{N}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x, y \geq 0\}$. Define $\rho : X \times X \to E$ as follows:

$$\rho(x, y) = \begin{cases} (0, 0), & \text{if } x = y; \\ (6, 12), & \text{if } x, y \in \{2, 3\},\ x \neq y; \\ (2, 4), & \text{otherwise.} \end{cases}$$

Then (X, ρ) is a pentagonal cone metric space, but (X, ρ) is not a cone metric space because it lacks the triangular property:

$$\begin{aligned} (6, 12) = \rho(2, 3) &> \rho(2, 4) + \rho(4, 3) \\ &= (2, 4) + (2, 4) \\ &= (4, 8),\ \text{as } (6, 12) - (4, 8) = (2, 4) \in P. \end{aligned}$$

Example 0.47. (Garg & Agalwal, 2012). Let $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a normal cone in E. Define $\rho : X \times X \to E$ as follows:

$$\rho(x, x) = 0, \forall x \in X;$$

$$\rho(1, 2) = \rho(2, 1) = (4, 8);$$

$$\rho(1, 3) = \rho(3, 1) = \rho(3, 4) = \rho(4, 3) = \rho(2, 4) = \rho(4, 2) = (1, 2);$$

$$\rho(1, 5) = \rho(5, 1) = \rho(2, 5) = \rho(5, 2) = \rho(3, 5) = \rho(5, 3) = \rho(4, 5) = \rho(5, 4) = (3, 6).$$

Then (X, ρ) is a pentagonal cone metric space, but (X, ρ) is not a rectangular cone metric space because it lacks the rectangular property:

$$
\begin{aligned}
(4,8) = \rho(1,2) &> \rho(1,3) + \rho(3,4) + \rho(4,2) \\
&= (1,2) + (1,2) + (1,2) \\
&= (3,6), \text{ as } (4,8) - (3,6) = (1,2) \in P.
\end{aligned}
$$

Lemma 0.48. (Jungck et al., 2009). *Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for $a, c, u, v, w \in E$, we have*

1. *If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$.*
2. *If $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.*
3. *If $u \leq v$ and $v \ll w$, then $u \ll w$.*
4. *If $c \in int(P)$ and $a_n \to 0$, then $\exists n_0 \in \mathbb{N} : \forall n > n_0,\ a_n \ll c$.*

Definition 0.49. (Garg & Agalwal, 2012). Let (X, d) be a pentagonal cone metric space. Let $\{x_n\}$ be a sequence in (X, ρ) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $\rho(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $\rho(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, ρ). If every Cauchy sequence is convergent in (X, ρ), then (X, ρ) is called a complete pentagonal cone metric space.

Lemma 0.50. (Garg & Agalwal, 2012). *Let (X, ρ) be a pentagonal cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $\|\rho(x_n, x)\| \to 0$ as $n \to \infty$.*

Lemma 0.51. (Garg & Agalwal, 2012). *Let (X, ρ) be a pentagonal cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $\|\rho(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.*

Definition 0.52. (Liu & Xu, 2013). Let (Y, ρ) be a cone metric space over Banach algebra $\mathcal{A}$, $y \in Y$ and $\{y_i\}$ be a sequence in (Y, ρ). Then we say

(1) $\{y_i\}$ converges to y if, for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number n_0 such that $\rho(y_i, y) \ll c$ for all $i \geq n_0$. We denote this by $y_i \to y$ $(i \to \infty)$.

(2) $\{y_i\}$ is a Cauchy sequence if, for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number n_0 which is independent of n such that $\rho(y_i, y_{i+n}) \ll c$ for all $i \geq n_0$.

(3) (Y, ρ) is said to be complete if every Cauchy sequence in (Y, ρ) is convergent.

Lemma 0.53. (Xu & Radenović, 2014). *Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$, $\mathcal{P}$ be the underlying solid cone and $\{y_i\}$ be a sequence in (Y, ρ). If $\{y_i\}$ converges to $y \in Y$, then*

(1) *$\{\rho(y_i, y)\}$ is a c-sequence.*

(2) *for any $j \in \mathbb{N}$, $\{\rho(y_i, y_{i+j})\}$ is a c-sequence.*

Lemma 0.54. (Xu & Radenović, 2014). *Let $\mathcal{A}$ be a Banach algebra with a solid cone $\mathcal{P}$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathcal{P}$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are c-sequences and $k_1, k_2 \in \mathcal{P}$ then $\{k_1\alpha_n + k_2\beta_n\}$ is also a c-sequence.*

The study of existence and uniqueness of fixed point of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction and Kannan contraction principles in various generalized metric spaces e.g., see (Azam et al., 2009; Branciari, 2000; Garg and Agarwal, 2012; George et al., 2015; Huang & Zhang, 2007; Huisheng et al., 2015; Jleli & Samet, 2009; Reddy & Rangamma, 2015b).

Jungck (1976) proved a common fixed point theorem for commuting mappings as a generalization of the Banach's fixed point theorem. The concept of the commutativity has been generalized in several ways. For instance, Sessa (1982) introduced the concept of weakly commuting mappings, Jungck (1986) extended this concept to compatible maps. Jungck and Rhoades (1998) introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true e.g., see (Pathak, 1995).

Motivated and inspired by the above results, it is our purpose in this research work to continue the study of the fixed point problems and prove some new fixed point theorems in the framework of a cone metric space over a Banach algebra and its generalization which are much more general than the metric space. In short, we intend to give affirmative answer to the following question: Can the above theorems hold for more general space, say, partial rectangular cone b-metric space over a Banach algebra?

CHAPTER III

Some Fixed Point Theorems in Cone Metric Spaces

Fixed Point Theorem for Generalized Expansive Mapping in Cone Metric Space over a Banach Algebra

In this section, we prove the existence of fixed points for generalized expansive mapping in cone metric space over a Banach algebra $\mathcal{A}$. The results obtained are significant extension and generalizations of recent results of (Jiang et al., 2016) and many well-known results in the literature. This section contains the results published in the American Institute of Physics (AIP) Conference Proceedings 1997, 020004 (2018). https://doi.org/10.1063/1.5048998

The following theorem is a generalization of Theorem 2.1 in (Jiang et al., 2016) and Theorem 2.5 in (Aage & Salunke, 2011).

Theorem 3.1. Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$ with a unit e and $\mathcal{P}$ be the underlying solid cone in $\mathcal{A}$. Let the mapping $\mathfrak{T} : Y \to Y$ be a surjective and satisfies the generalized expansive condition:

$$\rho(\mathfrak{T}y, \mathfrak{T}z) \succcurlyeq \vartheta_1\rho(y, z) + \vartheta_2\rho(\mathfrak{T}y, z) + \vartheta_3\rho(\mathfrak{T}z, y), \text{ for all } y, z \in Y, \tag{1}$$

where $\vartheta_k \in \mathcal{P}$ $(k = 1, 2, 3)$ such that $(e - \vartheta_2), (e - \vartheta_3), (\vartheta_1 - \vartheta_3)^{-1} \in \mathcal{P}$ and spectral radius $\delta[(\vartheta_1 - \vartheta_3)^{-1}(e - \vartheta_3)] < 1$. Then $\mathfrak{T}$ has a fixed point y_* in Y.

Proof. Let y_0 be arbitrary point in Y. Since $\mathfrak{T}$ is surjective, there exists $y_1 \in Y$ such that $\mathfrak{T}y_1 = y_0$. Again, we choose $y_2 \in Y$ such that $\mathfrak{T}y_2 = y_1$. Continuing this process, we construct a sequence $\{y_i\}$ in (Y, ρ) by

$$y_i = \mathfrak{T}y_{i+1}, \text{ for } i = 0, 1, 2, \ldots \tag{2}$$

Suppose $y_{j-1} = y_j$ for some $j \in \mathbb{N}$, then $y_* = y_j$ is a fixed point of $\mathfrak{T}$ and the result is proved. Hence, we assume that $y_{i-1} \neq y_i$ for all $i \in \mathbb{N}$. Observe that from the triangle inequality $\rho(y, z) \preccurlyeq \rho(y, x) + \rho(x, z)$, we have that

$$\rho(y, x) \succcurlyeq \rho(y, z) - \rho(x, z), \text{ for all } y, z, x \in Y. \tag{3}$$

Now, using (1), (2) and (3), we have

$$\begin{aligned}
\rho(y_i, y_{i-1}) &= \rho(\mathfrak{T}y_{i+1}, \mathfrak{T}y_i) \\
&\succcurlyeq \vartheta_1\rho(y_{i+1}, y_i) + \vartheta_2\rho(\mathfrak{T}y_{i+1}, y_i) + \vartheta_3\rho(\mathfrak{T}y_i, y_{i+1}) \\
&\succcurlyeq \vartheta_1\rho(y_{i+1}, y_i) + \vartheta_2\rho(y_i, y_i) + \vartheta_3\rho(y_{i-1}, y_{i+1}) \\
&\succcurlyeq \vartheta_1\rho(y_{i+1}, y_i) + \vartheta_3[\rho(y_{i-1}, y_i) - \rho(y_{i+1}, y_i)] \\
(e - \vartheta_3)\rho(y_i, y_{i-1}) &\succcurlyeq (\vartheta_1 - \vartheta_3)\rho(y_{i+1}, y_i) \\
\therefore \quad \rho(y_{i+1}, y_i) &\preccurlyeq \tau\rho(y_i, y_{i-1}),
\end{aligned} \tag{4}$$

where $\tau = (\vartheta_1 - \vartheta_3)^{-1}(e - \vartheta_3)$. Hence, from (4), we have

$$\rho(y_{i+1}, y_i) \preccurlyeq \tau\rho(y_i, y_{i-1}) \preccurlyeq \tau^2\rho(y_{i-1}, y_{i-2}) \preccurlyeq \cdots \preccurlyeq \tau^i\rho(y_1, y_0), \text{ for all } i \in \mathbb{N}. \tag{5}$$

Since $\delta(\tau) < 1$, it follows, by Lemma 0.32, that $(e - \tau)$ is invertible in $\mathcal{A}$. Moreover,

$$(e - \tau)^{-1} = \sum_{k=0}^{\infty} \tau^k. \tag{6}$$

Also by Remark 0.33, we obtain that

$$\|\tau^i\| \to 0 \ (i \to \infty). \tag{7}$$

Hence, for $i, j \in \mathbb{N}$ with $j > i$, using (5) and (6), we have

$$\begin{aligned}
\rho(y_j, y_i) &\preccurlyeq \rho(y_j, y_{j-1}) + \rho(y_{j-1}, y_i) \\
&\preccurlyeq \rho(y_j, y_{j-1}) + \rho(y_{j-1}, y_{j-2}) + \rho(y_{j-2}, y_i) \\
&\preccurlyeq \rho(y_j, y_{j-1}) + \rho(y_{j-1}, y_{j-2}) + \rho(y_{j-2}, y_{j-3}) + \cdots + \rho(y_{i+2}, y_{i+1}) + \rho(y_{i+1}, y_i) \\
&\preccurlyeq \tau^{j-1}\rho(y_1, y_0) + \tau^{j-2}\rho(y_1, y_0) + \tau^{j-3}\rho(y_1, y_0) + \cdots + \tau^{i+1}\rho(y_1, y_0) + \tau^i\rho(y_1, y_0) \\
&= \tau^i(e + \tau + \cdots + \tau^{j-i-3} + \tau^{j-i-2} + \tau^{j-i-1})\rho(y_1, y_0) \\
&\preccurlyeq \tau^i\left(\sum_{k=0}^{\infty} \tau^k\right)\rho(y_1, y_0) \preccurlyeq \tau^i(e - \tau)^{-1}\rho(y_1, y_0).
\end{aligned}$$

Therefore, using (7), we have that $\|\tau^i(e - \tau)^{-1}\rho(y_1, y_0)\| \to 0 \ (i \to \infty)$, and it follows, by Lemma 0.31, that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $j > i > n_0$, we have

$$\rho(y_j, y_i) \preccurlyeq \tau^i(e - \tau)^{-1}\rho(y_1, y_0) \ll c,$$

which implies, by Lemma 0.34(1) and Definition 0.52(2), that $\{y_i\}$ is a Cauchy sequence. Since (Y, ρ) is complete, there exists y_* in Y such that $y_i \to y_*$ $(i \to \infty)$. Since $\mathfrak{T}$ is a surjection mapping, there exists a point y_{**} in Y such that $\mathfrak{T}y_{**} = y_*$. Now, we claim that $y_{**} = y_*$. Indeed, using (1), (2) and (3), we have that

$$\begin{aligned}
\rho(y_*, y_i) &= \rho(\mathfrak{T}y_{**}, \mathfrak{T}y_{i+1}) \\
&\succcurlyeq \vartheta_1\rho(y_{**}, y_{i+1}) + \vartheta_2\rho(\mathfrak{T}y_{**}, y_{i+1}) + \vartheta_3\rho(\mathfrak{T}y_{i+1}, y_{**}) \\
&= \vartheta_1\rho(y_{**}, y_{i+1}) + \vartheta_2\rho(y_*, y_{i+1}) + \vartheta_3\rho(y_i, y_{**}) \\
\rho(y_*, y_{i+1}) + \rho(y_{i+1}, y_i) &\succcurlyeq \vartheta_1\rho(y_{**}, y_{i+1}) + \vartheta_2\rho(y_*, y_{i+1}) + \vartheta_3[\rho(y_i, y_{i+1}) - \rho(y_{**}, y_{i+1})] \\
(\vartheta_1 - \vartheta_3)\rho(y_{i+1}, y_{**}) &\preccurlyeq (e - \vartheta_2)\rho(y_{i+1}, y_*) + (e - \vartheta_3)\rho(y_i, y_{i+1}) \\
\rho(y_{i+1}, y_{**}) &\preccurlyeq (\vartheta_1 - \vartheta_3)^{-1}[(e - \vartheta_2)\rho(y_{i+1}, y_*) + (e - \vartheta_3)\rho(y_i, y_{i+1})] \\
&\preccurlyeq \alpha_1\rho(y_{i+1}, y_*) + \alpha_2\rho(y_i, y_{i+1}),
\end{aligned}$$

where $\alpha_1 = (\vartheta_1 - \vartheta_3)^{-1}(e - \vartheta_2), \alpha_2 = (\vartheta_1 - \vartheta_3)^{-1}(e - \vartheta_3) \in \mathcal{P}$. Using Lemma 0.53 and Lemma 0.54; $\{\rho(y_{i+1}, y_*)\}$, $\{\rho(y_i, y_{i+1})\}$ and $\{\alpha_1\rho(y_{i+1}, y_*) + \alpha_2\rho(y_i, y_{i+1})\}$ are c-sequences. Hence, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(y_{i+1}, y_{**}) \preccurlyeq \alpha_1\rho(y_{i+1}, y_*) + \alpha_2\rho(y_i, y_{i+1}) \ll c, \text{ for all } i > n_0, \tag{8}$$

which implies, by Lemma 0.34(1) and Definition 0.52(1), that $y_{i+1} \to y_{**}$. Since the limit of a convergent sequence in cone metric space over Banach algebras is unique, we have that $y_{**} = y_*$. Thus, $\mathfrak{T}y_* = y_*$. Hence, y_* is a fixed point of $\mathfrak{T}$.

Remark 0.55. Note that $\mathfrak{T}$ may have more than one fixed point see (Jiang et al., 2016).

The following theorem is a generalization of Theorems 2.1, 2.2, 2.6 and 3.1 in (Huang et al., 2012; Jiang et al., 2016; Aage & Salunke, 2011; Chouhan & Malviya, 2011), respectively.

Theorem 3.2. Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$ with a unit e and $\mathcal{P}$ be the underlying solid cone in $\mathcal{A}$. Let the mapping $\mathfrak{T} : Y \to Y$ be a surjective and satisfy the generalized expansive condition:

$$\rho(\mathfrak{T}y, \mathfrak{T}z) + \vartheta_1[\rho(y, \mathfrak{T}z) + \rho(z, \mathfrak{T}y)] \succcurlyeq \vartheta_2\rho(y, z) + \vartheta_3\rho(y, \mathfrak{T}y) + \vartheta_4\rho(z, \mathfrak{T}z), \tag{9}$$

for all $y, z \in Y$, where $\vartheta_k \in \mathcal{P}$ $(k = 1, 2, 3, 4)$ such that $(e + \vartheta_1 - \vartheta_4)$, $(e - \vartheta_1 - \vartheta_3)$, $(\vartheta_2 + \vartheta_3 - \vartheta_1)^{-1}$, $(\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1} \in \mathcal{P}$ and spectral radius $\delta[(\vartheta_2 + \vartheta_3 - \vartheta_1)^{-1}(e + \vartheta_1 - \vartheta_4)] < 1$. Then $\mathfrak{T}$ has a fixed point y_* in Y.

Proof. Define a sequence same as (2) in Theorem 3.1. Hence, using (9), we have

$$\begin{aligned}
\rho(\mathfrak{T}y_{i+1}, \mathfrak{T}y_i) + \vartheta_1[\rho(y_{i+1}, \mathfrak{T}y_i) + \rho(y_i, \mathfrak{T}y_{i+1})] &\succcurlyeq \vartheta_2\rho(y_{i+1}, y_i) + \vartheta_3\rho(y_{i+1}, \mathfrak{T}y_{i+1}) + \vartheta_4\rho(y_i, \mathfrak{T}y_i) \\
\rho(y_i, y_{i-1}) + \vartheta_1[\rho(y_{i+1}, y_{i-1}) + \rho(y_i, y_i)] &\succcurlyeq \vartheta_2\rho(y_{i+1}, y_i) + \vartheta_3\rho(y_{i+1}, y_i) + \vartheta_4\rho(y_i, y_{i-1}) \\
\rho(y_i, y_{i-1}) + \vartheta_1[\rho(y_{i+1}, y_i) + \rho(y_i, y_{i-1})] &\succcurlyeq (\vartheta_2 + \vartheta_3)\rho(y_{i+1}, y_i) + \vartheta_4\rho(y_i, y_{i-1}) \\
(e + \vartheta_1 - \vartheta_4)\rho(y_i, y_{i-1}) &\succcurlyeq (\vartheta_2 + \vartheta_3 - \vartheta_1)\rho(y_{i+1}, y_i) \\
(\vartheta_2 + \vartheta_3 - \vartheta_1)\rho(y_{i+1}, y_i) &\preccurlyeq (e + \vartheta_1 - \vartheta_4)\rho(y_i, y_{i-1}) \\
\rho(y_{i+1}, y_i) &\preccurlyeq \tau\rho(y_i, y_{i-1}), \qquad (10)
\end{aligned}$$

where $\tau = (\vartheta_2 + \vartheta_3 - \vartheta_1)^{-1}(e + \vartheta_1 - \vartheta_4)$. Hence, from (10), we get

$$\rho(y_{i+1}, y_i) \preccurlyeq \tau\rho(y_i, y_{i-1}) \preccurlyeq \tau^2\rho(y_{i-1}, y_{i-2}) \preccurlyeq \cdots \preccurlyeq \tau^i\rho(y_1, y_0), \text{ for all } i \in \mathbb{N}.$$

Using the same argument to the proof in Theorem 3.1, we get that $\{y_i\}$ is a Cauchy sequence. Since (Y, ρ) is complete, there exists y_* in Y such that $y_i \to y_*$ $(i \to \infty)$. Since $\mathfrak{T}$ is a surjection mapping, there exists a point z_* in Y such that $\mathfrak{T}z_* = y_*$. Next, we show that $z_* = y_*$. Using (2), (3) and (9), we have that

$$\begin{aligned}
\rho(y_i, y_*) &= \rho(\mathfrak{T}y_{i+1}, \mathfrak{T}z_*) \\
&\succcurlyeq -\vartheta_1[\rho(y_{i+1}, \mathfrak{T}z_*) + \rho(z_*, \mathfrak{T}y_{i+1})] + \vartheta_2\rho(y_{i+1}, z_*) \\
&\quad + \vartheta_3\rho(y_{i+1}, \mathfrak{T}y_{i+1}) + \vartheta_4\rho(z_*, \mathfrak{T}z_*) \\
&\succcurlyeq -\vartheta_1[\rho(y_{i+1}, y_*) + \rho(z_*, y_i)] + \vartheta_2\rho(y_{i+1}, z_*) \\
&\quad + \vartheta_3\rho(y_{i+1}, y_i) + \vartheta_4\rho(z_*, y_*) \\
\rho(y_i, y_{i+1}) + \rho(y_{i+1}, y_*) &\succcurlyeq -\vartheta_1\left[\rho(y_{i+1}, y_*) + (\rho(z_*, y_{i+1}) - \rho(y_i, y_{i+1}))\right] \\
&\quad + \vartheta_2\rho(y_{i+1}, z_*) + \vartheta_3\rho(y_{i+1}, y_i) + \vartheta_4[\rho(z_*, y_{i+1}) - \rho(y_*, y_{i+1})] \\
(\vartheta_2 - \vartheta_1 + \vartheta_4)\rho(y_{i+1}, z_*) &\preccurlyeq (e + \vartheta_1 + \vartheta_4)\rho(y_{i+1}, y_*) + (e - \vartheta_1 - \vartheta_3)\rho(y_i, y_{i+1}) \\
\rho(y_{i+1}, z_*) &\preccurlyeq \beta_1\rho(y_{i+1}, y_*) + \beta_2\rho(y_i, y_{i+1}),
\end{aligned}$$

where $\beta_1 = (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1}(e + \vartheta_1 + \vartheta_4), \beta_2 = (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1}(e - \vartheta_1 - \vartheta_3) \in \mathcal{P}$. By Lemma 0.53, Lemma 0.54; $\{\rho(y_{i+1}, y_*)\}$, $\{\rho(y_i, y_{i+1})\}$ and $\{\beta_1\rho(y_{i+1}, y_*) + \beta_2\rho(y_i, y_{i+1})\}$ are c-sequences. Hence, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(y_{i+1}, z_*) \preccurlyeq \beta_1\rho(y_{i+1}, y_*) + \beta_2\rho(y_i, y_{i+1}) \ll c, \text{ for all } i > n_0,$$

which implies that $y_{i+1} \to z_*$. Since the limit of a convergent sequence in cone metric space over Banach algebras is unique, we have that $z_* = y_*$. Hence, y_* is a fixed point of $\mathfrak{T}$. This completes the proof.

The following theorem is a generalization of Theorems 2.2 in (Huang et al., 2012).

Theorem 3.3. Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$ with a unit e and $\mathcal{P}$ be the underlying solid cone in $\mathcal{A}$. Let the mapping $\mathfrak{T} : Y \to Y$ be a continuous, surjection and satisfy the following condition:

$$\rho(\mathfrak{T}y, \mathfrak{T}z) \succcurlyeq \vartheta\Big\{\rho(y, z), \rho(y, \mathfrak{T}y), \rho(z, \mathfrak{T}z)\Big\}, \text{ for all } y, z \in Y, \tag{11}$$

where $\vartheta \in \mathcal{P}$ such that $\vartheta^{-1} \in \mathcal{P}$ and spectral radius $\delta(\vartheta^{-1}) < 1$. Then $\mathfrak{T}$ has a fixed point y_* in Y.

Proof. Define a sequence same as (2) in Theorem 3.2. Hence, using (11), we have

$$\begin{aligned}
\rho(y_i, y_{i-1}) &= \rho(\mathfrak{T}y_{i+1}, \mathfrak{T}y_i) \\
&\succcurlyeq \vartheta\Big\{\rho(y_{i+1}, y_i), \rho(y_{i+1}, \mathfrak{T}y_{i+1}), \rho(y_i, \mathfrak{T}y_i)\Big\} \\
&= \vartheta\Big\{\rho(y_i, y_{i+1}), \rho(y_i, y_{i-1})\Big\}.
\end{aligned}$$

We consider two cases as follows:

(1) If $\rho(y_i, y_{i-1}) \succcurlyeq \vartheta\rho(y_i, y_{i-1})$ then $\rho(y_i, y_{i-1}) \preccurlyeq \vartheta^{-1}\rho(y_i, y_{i-1})$. Since $\delta(\vartheta^{-1}) < 1$, by Lemma 0.34, we have $\rho(y_i, y_{i-1}) = \theta$, that is $y_i = y_{i-1}$. This is a contradiction (since we assumed that $y_i \neq y_{i-1}$).

(2) If $\rho(y_i, y_{i-1}) \succcurlyeq \vartheta\rho(y_i, y_{i+1})$ then $\rho(y_{i+1}, y_i) \preccurlyeq \vartheta^{-1}\rho(y_i, y_{i-1}) = \tau\rho(y_i, y_{i-1})$, where $\tau = \vartheta^{-1}$. Hence, we have

$$\rho(y_{i+1}, y_i) \preccurlyeq \tau\rho(y_i, y_{i-1}) \preccurlyeq \tau^2\rho(y_{i-1}, y_{i-2}) \preccurlyeq \cdots \preccurlyeq \tau^i\rho(y_1, y_0), \text{ for all } i \in \mathbb{N}.$$

Using the same argument to the proof in Theorem 3.1, we get that $\{y_i\}$ is a Cauchy sequence. Since (Y, ρ) is complete, there exists $y_* \in Y$ such that $y_i \to y_*$ $(i \to \infty)$. To show that y_* is a fixed point of $\mathfrak{T}$, since $\mathfrak{T}$ is continuous, so $\mathfrak{T}y_i \to \mathfrak{T}y_*$ $(i \to \infty)$, which implies that $y_{i-1} \to \mathfrak{T}y_*$ $(i \to \infty)$. Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we get $\mathfrak{T}y_* = y_*$. Thus, y_* is a fixed point $\mathfrak{T}$.

Common Fixed Point Theorem for Generalized Expansive Mappings in Cone Metric Spaces over Banach Algebras

In this section, we prove a common fixed point theorem for generalized expansive mapping in a cone metric space over a Banach algebra. Our results are significant extension and generalizations of recent results in the literature. This section contains the results published in the American Institute of Physics (AIP) Conference Proceedings 1997, 020004, (2018). https://doi.org/10.1063/1.5048998

The following theorem is a generalization of Theorems 2.1, 2.2 and 2.6 in (Huang et al., 2012; Jiang et al., 2016; Aage & Salunke, 2011), respectively.

Theorem 3.4. Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$ with a unit e and $\mathcal{P}$ be the underlying solid cone in $\mathcal{A}$. Let $\mathfrak{T}_1, \mathfrak{T}_2 : Y \to Y$ be two mappings such that $\mathfrak{T}_2 Y \subseteq \mathfrak{T}_1 Y$, either $\mathfrak{T}_1 Y$ or $\mathfrak{T}_2 Y$ is a complete subspace of Y and satisfy the following condition:

$$\rho(\mathfrak{T}_1 y, \mathfrak{T}_1 z) \succcurlyeq \vartheta_1 \rho(\mathfrak{T}_2 y, \mathfrak{T}_2 z) + \vartheta_2 \rho(\mathfrak{T}_1 y, \mathfrak{T}_2 y) + \vartheta_3 \rho(\mathfrak{T}_1 z, \mathfrak{T}_2 z), \tag{12}$$

for all $y, z \in Y$, where $\vartheta_k \in \mathcal{P}$ $(k = 1, 2, 3)$ such that $(e - \vartheta_2)$, $(e - \vartheta_3)$, ϑ_1^{-1}, $(\vartheta_1 - \vartheta_2)^{-1}$, $(\vartheta_1 + \vartheta_2)^{-1}$, $(\vartheta_1 + \vartheta_3)^{-1} \in \mathcal{P}$, spectral radius $\delta[(\vartheta_1 + \vartheta_3)^{-1}(e - \vartheta_2)] < 1$ and $\delta({\vartheta_1}^{-1}) < 1$. Then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have a unique point of coincidence in Y. Moreover, if $\mathfrak{T}_1$, $\mathfrak{T}_2$ are weakly compatible, then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have a unique common fixed point in Y.

Proof. Let y_0 be arbitrary point in Y. Since $\mathfrak{T}_2 Y \subseteq \mathfrak{T}_1 Y$, we choose $y_1 \in Y$ such that $\mathfrak{T}_1 y_1 = \mathfrak{T}_2 y_0$. Again, we choose $y_2 \in Y$ such that $\mathfrak{T}_1 y_2 = \mathfrak{T}_2 y_1$. Continuing this process, we construct a sequence $\{y_i\}$ in (Y, ρ) such that

$$\mathfrak{T}_1 y_i = \mathfrak{T}_2 y_{i-1}, \text{ for all } i = 0, 1, 2, \ldots. \tag{13}$$

If $y_{j-1} = y_j$ for some $j \geq 1$, then $\mathfrak{T}_1 y_j = \mathfrak{T}_2 y_j$ and y_j is a coincidence point of $\mathfrak{T}_1$, $\mathfrak{T}_2$, and the result is proved. Now, we assume that $y_{i-1} \neq y_i$ for all $i \in \mathbb{N}$. From (12) and (13), we get

$$\begin{aligned}
\rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i) &= \rho(\mathfrak{T}_1 y_i, \mathfrak{T}_1 y_{i+1}) \\
&\succcurlyeq \vartheta_1 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) + \vartheta_2 \rho(\mathfrak{T}_1 y_i, \mathfrak{T}_2 y_i) + \vartheta_3 \rho(\mathfrak{T}_1 y_{i+1}, \mathfrak{T}_2 y_{i+1}) \\
&= \vartheta_1 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) + \vartheta_2 \rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i) + \vartheta_3 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \\
(e - \vartheta_2)\rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i) &\succcurlyeq (\vartheta_1 + \vartheta_3)\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \\
\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) &\preccurlyeq (\vartheta_1 + \vartheta_3)^{-1}(e - \vartheta_2)\rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i) \preccurlyeq \tau \rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i),
\end{aligned}$$

where $\tau = (\vartheta_1 + \vartheta_3)^{-1}(e - \vartheta_2)$. Hence, we have

$$\begin{aligned}
\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) &\preccurlyeq \tau \rho(\mathfrak{T}_2 y_{i-1}, \mathfrak{T}_2 y_i) \\
&\preccurlyeq \tau^2 \rho(\mathfrak{T}_2 y_{i-2}, \mathfrak{T}_2 y_{i-1}) \preccurlyeq \cdots \\
&\preccurlyeq \tau^i \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1), \text{ for all } i \in \mathbb{N}.
\end{aligned} \tag{14}$$

For $i, j \in \mathbb{N}$ with $j > i$, using (6) and (14), we have

$$\begin{aligned}
\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_j) &\preccurlyeq \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) + \rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 y_j) \\
&\preccurlyeq \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) + \rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 y_{i+2}) + \rho(\mathfrak{T}_2 y_{i+2}, \mathfrak{T}_2 y_{i+3}) \\
&+ \cdots + \rho(\mathfrak{T}_2 y_{j-2}, \mathfrak{T}_2 y_{j-1}) + \rho(\mathfrak{T}_2 y_{j-1}, \mathfrak{T}_2 y_j) \\
&\preccurlyeq \tau^i \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) + \tau^{i+1} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) + \tau^{i+2} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) \\
&+ \cdots + \tau^{j-2} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) + \tau^{j-1} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) \\
&= \tau^i (e + \tau + \tau^2 + \cdots + \tau^{j-i-2} + \tau^{j-i-1}) \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) \\
&\preccurlyeq \tau^i \left(\sum_{k=0}^{\infty} \tau^k \right) \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) \preccurlyeq \tau^i (e - \tau)^{-1} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1).
\end{aligned}$$

Therefore, using (7), we have that $\|\tau^i (e - \tau)^{-1} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1)\| \to 0$ $(i \to \infty)$, and it follows, by Lemma 0.31, that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_j) \preccurlyeq \tau^i (e - \tau)^{-1} \rho(\mathfrak{T}_2 y_0, \mathfrak{T}_2 y_1) \ll c, \text{ for all } j > i > n_0,$$

which implies that $\{\mathfrak{T}_2 y_i\}$ is a Cauchy sequence. Suppose $\mathfrak{T}_2 Y$ is complete subspace of Y, then there exists $y_* \in \mathfrak{T}_2 Y \subseteq \mathfrak{T}_1 Y$ such that $\mathfrak{T}_2 y_i \to y_*$ $(i \to \infty)$ and also

$\mathfrak{T}_1 y_i \to y_*$ $(i \to \infty)$. If $\mathfrak{T}_1 Y$ is complete subspace of Y, then there exists $y_* \in \mathfrak{T}_1 Y$ such that $\mathfrak{T}_1 y_i = \mathfrak{T}_2 y_{i-1} \to y_*$ $(i \to \infty)$. Consequently, we can find z_* in $\mathfrak{T}_1 Y$ such that $\mathfrak{T}_1 z_* = y_*$. Now, we claim that $\mathfrak{T}_2 z_* = y_*$. Indeed, using (3), (12) and (13), we have that

$$\begin{aligned}
\rho(y_*, \mathfrak{T}_2 y_i) &= \rho(\mathfrak{T}_1 z_*, \mathfrak{T}_1 y_{i+1}) \\
&\succcurlyeq \vartheta_1 \rho(\mathfrak{T}_2 z_*, \mathfrak{T}_2 y_{i+1}) + \vartheta_2 \rho(\mathfrak{T}_1 z_*, \mathfrak{T}_2 z_*) + \vartheta_3 \rho(\mathfrak{T}_1 y_{i+1}, \mathfrak{T}_2 y_{i+1}) \\
&= \vartheta_1 \rho(\mathfrak{T}_2 z_*, \mathfrak{T}_2 y_{i+1}) + \vartheta_2 \rho(y_*, \mathfrak{T}_2 z_*) + \vartheta_3 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \\
\rho(y_*, \mathfrak{T}_2 y_{i+1}) + \rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 y_i) &\succcurlyeq \vartheta_1 \rho(\mathfrak{T}_2 z_*, \mathfrak{T}_2 y_{i+1}) + \vartheta_2 [\rho(y_*, \mathfrak{T}_2 y_{i+1}) \\
&\quad - \rho(\mathfrak{T}_2 z_*, \mathfrak{T}_2 y_{i+1})] + \vartheta_3 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \\
(\vartheta_1 - \vartheta_2) \rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 z_*) &\preccurlyeq (e - \vartheta_2) \rho(\mathfrak{T}_2 y_{i+1}, y_*) + (e - \vartheta_3) \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \\
\rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 z_*) &\preccurlyeq \gamma_1 \rho(\mathfrak{T}_2 y_{i+1}, y_*) + \gamma_2 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}),
\end{aligned}$$

where $\gamma_1 = (\vartheta_1 - \vartheta_2)^{-1}(e - \vartheta_2), \gamma_2 = (\vartheta_1 - \vartheta_2)^{-1}(e - \vartheta_3) \in \mathcal{P}$. Now, by Lemma 0.53, Lemma 0.54; $\{\rho(\mathfrak{T}_2 y_{i+1}, y_*)\}$, $\{\rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1})\}$ and $\{\gamma_1 \rho(\mathfrak{T}_2 y_{i+1}, y_*) + \gamma_2 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1})\}$ are c-sequences. Hence, for any $c \in \mathcal{P}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that

$$\rho(\mathfrak{T}_2 y_{i+1}, \mathfrak{T}_2 z_*) \preccurlyeq \gamma_1 \rho(\mathfrak{T}_2 y_{i+1}, y_*) + \gamma_2 \rho(\mathfrak{T}_2 y_i, \mathfrak{T}_2 y_{i+1}) \ll c, \text{ for all } i > N,$$

which implies that $\mathfrak{T}_2 y_{i+1} \to \mathfrak{T}_2 z_*$. Since the limit of a convergent sequence in a cone metric space over Banach algebras is unique, we have that $\mathfrak{T}_1 z_* = \mathfrak{T}_2 z_* = y_*$. Hence, y_* is a point of coincidence of $\mathfrak{T}_1$ and $\mathfrak{T}_2$. Next, we show that the point of coincidence of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ is unique. Suppose that y_{**} is another point of coincidence of $\mathfrak{T}_1$ and $\mathfrak{T}_2$. i.e. $\mathfrak{T}_1 z_{**} = \mathfrak{T}_2 z_{**} = y_{**}$ for some $z_{**} \in Y$. Then

$$\begin{aligned}
\rho(y_*, y_{**}) &= \rho(\mathfrak{T}_1 z_*, \mathfrak{T}_1 z_{**}) \\
&\succcurlyeq \vartheta_1 \rho(\mathfrak{T}_2 z_*, \mathfrak{T}_2 z_{**}) + \vartheta_2 \rho(\mathfrak{T}_1 z_*, \mathfrak{T}_2 z_*) + \vartheta_3 \rho(\mathfrak{T}_1 z_{**}, \mathfrak{T}_2 z_{**}) \\
&= \vartheta_1 \rho(y_*, y_{**}) + \vartheta_2 \rho(y_*, y_*) + \vartheta_3 \rho(y_{**}, y_{**}) = \vartheta_1 \rho(y_*, y_{**}) \\
\therefore \ \rho(y_*, y_{**}) &\preccurlyeq \vartheta_1^{-1} \rho(y_*, y_{**}).
\end{aligned}$$

Since $\delta(\vartheta_1^{-1}) < 1$, it follows, by Lemma 0.34, that $\rho(y_*, y_{**}) = \theta$, which implies that $y_* = y_{**}$. Thus, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have unique point of coincidence y_* in Y. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are weakly compatible then, by Lemma 0.37, we have that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have a unique common fixed point y_* in Y. This completes the proof.

The following Corollary is a generalization of Corollary 2.3 in (Aage & Salunke, 2011; Ahmad & Salunke, 2017; Huang et al., 2012; Jiang et al., 2016;), respectively.

Corollary 3.5. Let (Y, ρ) be a complete cone metric space over Banach algebra $\mathcal{A}$ with a unit e and $\mathcal{P}$ be the underlying solid cone in $\mathcal{A}$. Let $\mathfrak{T}_1 : Y \to Y$ be a surjective mapping and satisfy the following condition:

$$\rho(\mathfrak{T}_1 y, \mathfrak{T}_1 z) \succcurlyeq \vartheta\rho(y, z),$$

for all $y, z \in Y$, where $\vartheta, \vartheta^{-1} \in \mathcal{P}$ such that spectral radius $\delta(\vartheta^{-1}) < 1$. Then $\mathfrak{T}_1$ has a unique fixed point in Y.

Proof. Letting $\mathfrak{T}_2 = I$ (identity mapping), $\vartheta_1 = \vartheta$ and $\vartheta_2 = \vartheta_3 = \theta$ in Theorem 3.4, the result follows.

Example 0.56. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|y\| = \|y\|_\infty + \|y'\|_\infty$ for $y \in \mathcal{A}$, where multiplication in $\mathcal{A}$ is defined in the usual way. Then $\mathcal{A}$ is a Banach algebra with unit element $e = 1$ and the set $\mathcal{P} = \{y \in \mathcal{A} : y \geq 0, t \in [0, 1]\}$ is a non-normal cone in $\mathcal{A}$. Let $Y = \{1, 2, 3\}$. Consider a mapping $\rho : Y \times Y \to \mathcal{A}$ define by $\rho(y, y)(t) = \theta,$ for all $y \in Y$, $\rho(1, 2)(t) = \rho(2, 1)(t) = \rho(1, 3)(t) = \rho(3, 1)(t) = e^t$, and $\rho(2, 3)(t) = \rho(3, 2)(t) = \theta$. Then (Y, ρ) is a cone metric space over Banach algebra $\mathcal{A}$. Define mappings $\mathfrak{T}_1, \mathfrak{T}_2 : Y \to Y$ by $\mathfrak{T}_1(1) = 1$, $\mathfrak{T}_1(2) = 3$, $\mathfrak{T}_1(3) = 2$, $\mathfrak{T}_2(1) = 1$, $\mathfrak{T}_2(2) = 2$, $\mathfrak{T}_2(3) = 3$. Let $\vartheta_k \in \mathcal{P}$ $(k = 1, 2, 3)$ be defined by $\vartheta_1(t) = \frac{t+1}{8}$, $\vartheta_2(t) = \frac{t+1}{5}$, and $\vartheta_3(t) = \frac{t+1}{6}$. Some calculations show that all the conditions of Theorem 3.4 are satisfied and $y_* = 1$ is the unique coincidence and common fixed point of $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

Conclusion

In this section, we use the notion of generalized expansive mappings on cone metric space over Banach algebras and prove some new fixed point theorems for such mappings. Our results are actual generalization of the recent results in (Aage & Salunke, 2011; Ahmad & Salunke, 2017; Huang et al., 2012; Jiang et al., 2016) and others in the literature.

CHAPTER IV

Some Fixed Point Theorems in Partial Rectangular Cone b - Metric Spaces

In this section, we introduce the concept of a partial rectangular cone b - metric space over Banach algebras and prove some fixed point results under various contractive mappings in such a space. Some examples are given to elucidate the results. Our results extend and generalize many existing results in the literature. This section contains the results published in the Journal of Mathematics, Hindawi, Volume 2021, Article ID 8447435, 8 pages. https://doi.org/10.1155/2021/8447435

George et al. (2017) introduced the concept of a rectangular cone b-metric space over Banach algebras as a generalization of metric space and many of its generalizations. They proved some fixed point results in such a space. Very recently, Fernandez et al. (2020) introduced partial cone b - metric space over Banach algebras as a generalization of partial metric space and many of its generalizations. Motivated and inspired by these papers (George et al., 2017; Fernandez et al., 2020), we introduce the concept of a partial rectangular cone b - metric space over Banach algebras which generalized both rectangular cone b-metric space over Banach algebras and partial cone b - metric space over Banach algebras. Further, we prove some fixed point results under various contractive mappings in such a space. Examples are also given to elucidate our results. We start with definitions and some existing results required in the sequel.

Definition 0.57. (Fernandez et al., 2020) Let Y be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that, for all $y, z, x \in Y$, a mapping $\mathcal{P}_b \colon Y \times Y \to \mathcal{A}$ satisfies:

1. $y = z \Leftrightarrow \mathcal{P}_b(y, y) = \mathcal{P}_b(y, z) = \mathcal{P}_b(z, z)$;
2. $\theta \preceq \mathcal{P}_b(y, y) \preceq \mathcal{P}_b(y, z)$;
3. $\mathcal{P}_b(y, z) = \mathcal{P}_b(z, y)$;
4. $\mathcal{P}_b(y, z) \preceq s[\mathcal{P}_b(y, x) + \mathcal{P}_b(x, z)] - \mathcal{P}_b(x, x)$.

Then $(Y, \mathcal{P}_b)$ is called a partial cone b-metric space over $\mathcal{A}$ with coefficient $s \geq 1$.

Definition 0.58. (George et al., 2017) Let Y be a nonempty set and $\mathcal{K}$ a solid cone in a Banach algebra $\mathcal{A}$. Suppose that, for all $y, z \in Y$ and all distinct points

$x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_{rcb}\colon Y \times Y \to \mathcal{A}$ satisfies:

1. $\theta \preceq \mathcal{P}_{rcb}(y, z)$ and $\mathcal{P}_{rcb}(y, z) = \theta \Leftrightarrow y = z$;
2. $\mathcal{P}_{rcb}(y, z) = \mathcal{P}_{rcb}(z, y)$;
3. there exists $s \in \mathcal{K}$ with $e \preceq s$ such that

$$\mathcal{P}_{rcb}(y, z) \preceq s[\mathcal{P}_{rcb}(y, x_1) + \mathcal{P}_{rcb}(x_1, x_2) + \mathcal{P}_{rcb}(x_2, z)].$$

Then $\mathcal{P}_{rcb}$ is called a rectangular cone b-metric on Y, and $(Y, \mathcal{P}_{rcb})$ is called a rectangular cone b-metric space over $\mathcal{A}$ with coefficient s.

We now introduce the concept of a partial rectangular cone b-metric space ($\mathcal{P}_b^r$-cone metric space) over Banach algebras and give some of its topological property. Further, the notions of convergent sequence, θ-Cauchy sequence and θ-completeness in the setting of this new space are defined. Moreover, some fixed point theorems under various contractive mappings are proved in such a space.

Definition 0.59. Let Y be a nonempty set and $\mathcal{K}$ be a solid cone in a unital Banach algebra $\mathcal{A}$. Suppose that, for all $y, z \in Y$ and all distinct points $x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_b^r\colon Y \times Y \to \mathcal{A}$ satisfies:

(P1) $y = z \Leftrightarrow \mathcal{P}_b^r(y, y) = \mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, z)$;

(P2) $\theta \preceq \mathcal{P}_b^r(y, y) \preceq \mathcal{P}_b^r(y, z)$;

(P3) $\mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, y)$;

(P4) there exists $s \in \mathcal{K}$ with $e \preceq s$ such that

$$\mathcal{P}_b^r(y, z) \preceq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2).$$

Then $\mathcal{P}_b^r$ is called a partial rectangular cone b-metric on Y, and $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called a partial rectangular cone b-metric space over Banach algebra $\mathcal{A}$ with coefficient s (in short PRCbMS-BA).

Remark 0.60. In any PRCbMS-BA $(Y, \mathcal{P}_b^r, \mathcal{A})$ if $\mathcal{P}_b^r(y, z) = \theta$ for all $y, z \in Y$, then $y = z$, but the converse may not be true. Also every rectangular cone b-metric space over $\mathcal{A}$ is a $\mathcal{P}_b^r$-cone metric space over $\mathcal{A}$ with zero (θ) self distance, but there are $\mathcal{P}_b^r$-cone metric spaces over $\mathcal{A}$ which are not a rectangular cone b-metric space over $\mathcal{A}$.

Example 0.61. Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ with the norm

$$\|y\| = \|y\|_\infty + \|y'\|_\infty, \text{ for all } y \in \mathcal{A}.$$

Define multiplication pointwisely on $\mathcal{A}$. Then, $\mathcal{A}$ is a Banach algebra with unit $e(t) = 1, \forall t \in [0,1]$. Let $\mathcal{K} = \{y \in \mathcal{A} \colon y = y(t) \geq 0, t \in [0,1]\}$. Then $\mathcal{K}$ is a solid cone in $\mathcal{A}$. Let $Y = \{y_1, y_2, y_3, y_4\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}^r_b \colon Y \times Y \to \mathcal{K}$ by

$$\mathcal{P}^r_b(y,z)(t) = \begin{cases} \theta, & \text{if } y = z = y_1; \\ 2t, & \text{if } y, z \in \{y_1, y_2\}, y \neq z; \\ t, & \text{otherwise.} \end{cases}$$

Then $(Y, \mathcal{P}^r_b, \mathcal{A})$ is a PRCbMS-BA with coefficient $s = 4/3$ which is not a rectangular cone b-metric space over $\mathcal{A}$, because $\mathcal{P}^r_b(y_2, y_2)(t) \neq \theta$ and $\mathcal{P}^r_b(y_1, y_2)(t) = 2t > t = \mathcal{P}^r_b(y_1, y_3)(t) + \mathcal{P}^r_b(y_3, y_4)(t) + \mathcal{P}^r_b(y_4, y_2)(t) - \mathcal{P}^r_b(y_3, y_3)(t) - \mathcal{P}^r_b(y_4, y_4)(t)$.

Definition 0.62. Let $(Y, \mathcal{P}^r_b, \mathcal{A})$ be a PRCbMS-BA and $\mathcal{K}$ be a solid cone in $\mathcal{A}$. For each $y \in Y$ and each $c \in \mathcal{K}^\circ$, let

$$B_{\mathcal{P}^r_b}(y,c) = \{z \in Y \colon \mathcal{P}^r_b(y,z) \ll c + \mathcal{P}^r_b(y,y)\} \text{ and}$$
$$\mathcal{B} = \{B_{\mathcal{P}^r_b}(y,c) \colon y \in Y \text{ and } c \in \mathcal{K}^\circ\}. \text{ Then}$$

$$\tau_{\mathcal{P}} = \{\mathcal{U} \subset Y \colon \text{ for all } y \in \mathcal{U} \text{ there exists } B_{\mathcal{P}^r_b} \in \mathcal{B} \text{ and } y \in B_{\mathcal{P}^r_b} \subset \mathcal{U}\} \cup \emptyset,$$

is a topology on Y, $B_{\mathcal{P}^r_b}(y,c)$ is a $\mathcal{P}^r_b$-ball in $(Y, \mathcal{P}^r_b, \mathcal{A})$, $\mathcal{B}$ is a subbase for the topology $\tau_{\mathcal{P}}$ on Y, and $\mathcal{U}$ is a base generated by the subbase $\mathcal{B}$.

Definition 0.63. Let $(Y, \mathcal{P}^r_b, \mathcal{A})$ be a PRCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$, $y^* \in Y$ and $\{y_n\}$ be a sequence in Y. If for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}^r_b(y_n, y^*) \ll c + \mathcal{P}^r_b(y^*, y^*)$ for all $n > N$, then $\{y_n\}$ is said to be convergent in Y and converges to y^*. This fact is denoted by $y_n \to y^*$ as $n \to \infty$ or $\lim_{n\to\infty} y_n = y^*$.

Definition 0.64. Let $(Y, \mathcal{P}^r_b, \mathcal{A})$ be a PCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$ and $\{y_n\}$ be a sequence in Y. Then $\{y_n\}$ is called a θ-Cauchy sequence if $\{\mathcal{P}^r_b(y_n, y_m)\}$ is a c-sequence in $\mathcal{A}$. That is, if for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}^r_b(y_n, y_m) \ll c$ for all $n, m > N$.

Definition 0.65. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA, $\mathcal{K}$ be a solid cone in $\mathcal{A}$, $y^* \in Y$ and $\{y_n\}$ be a sequence in Y. Then $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called θ-complete if every θ-Cauchy sequence $\{y_n\}$ in Y converges to a point $y^* \in Y$. That is,

$$\lim_{n,m\to\infty} \mathcal{P}_b^r(y_n, y_m) = \lim_{n\to\infty} \mathcal{P}_b^r(y_n, y^*) = \mathcal{P}_b^r(y^*, y^*) = \theta.$$

Lemma 0.66. *Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA and $\{y_n\}$ be a sequence in Y. If $\{y_n\}$ converges to $y^* \in Y$, then*

(1) $\{\mathcal{P}_b^r(y_n, y^*)\}$ *is a c-sequence.*

(2) *for any $m \in \mathbb{N}$, $\{\mathcal{P}_b^r(y_n, y_{n+m})\}$ is a c-sequence.*

Proof. Follows from Definitions 0.23, 0.59 and 0.63.

Banach Contraction Principle on $\mathcal{P}_b^r$-Cone Metric Space over a Banach Algebra

Firstly, we present a variant of the Banach contraction principle on $\mathcal{P}_b^r$-cone metric space over Banach algebra $\mathcal{A}$ as follows:

Theorem 4.1. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ-complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \preceq s$. Suppose $F\colon Y \to Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \preceq \alpha \mathcal{P}_b^r(y, z) \text{ for all } y, z \in Y, \tag{1}$$

where $\alpha \in \mathcal{K}$ such that α commutes with s and $\rho(\alpha) < 1$. Then F has a unique fixed point.

Proof. Let y_0 be a point in Y. We define a sequence $\{y_n\}$ in Y by

$$y_n = Fy_{n-1} = F^n y_0 \ \text{ for all } n \geq 1. \tag{2}$$

If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $y^* = y_n = Fy_n$ is a fixed point of F, and the result is proved. Hence, we assume that $y_n \neq y_{n+1}$ for all $n \geq 0$. We will show that $y_n \neq y_{n+q}$ for all $n \geq 0$ and $q \geq 1$. Suppose that $y_n = y_{n+q}$ for some $n \geq 0, q \geq 1$, then $y_{n+1} = y_{n+q+1}$ and $Fy_n = Fy_{n+q}$. Then (1) implies that

$$\mathcal{P}_b^r(y_n, y_{n+1}) = \mathcal{P}_b^r(y_{n+q}, y_{n+q+1}) \preceq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{n+q}) \preceq \cdots \preceq \alpha^q \mathcal{P}_b^r(y_n, y_{n+1}).$$

Using Lemma 0.24, we obtain that $\mathcal{P}_b^r(y_n, y_{n+1}) = \theta$, that is $y_n = y_{n+1}$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Hence, from (1) and (2), we have that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_{n+1}) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \preceq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) \\ &\preceq \alpha^2 \mathcal{P}_b^r(y_{n-2}, y_{n-1}) \preceq \cdots \preceq \alpha^n \mathcal{P}_b^r(y_0, y_1) \\ \therefore \quad \mathcal{P}_b^r(y_n, y_{n+1}) &\preceq \alpha^n \mathcal{P}_b^r(y_0, y_1) \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{3}$$

Similarly, for all $n, m, q \in \mathbb{N}$, we obtain that

$$\begin{aligned} \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &= \mathcal{P}_b^r(Fy_{n+q-1}, Fy_{m+q-1}) \preceq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{m+q-1}) \\ &\preceq \alpha^2 \mathcal{P}_b^r(y_{n+q-2}, y_{m+q-2}) \preceq \cdots \preceq \alpha^q \mathcal{P}_b^r(y_n, y_m) \\ \therefore \quad \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &\preceq \alpha^q \mathcal{P}_b^r(y_n, y_m) \text{ for all } n, m, q \in \mathbb{N}. \end{aligned} \tag{4}$$

Observe that $\rho(s)$ exists because of Lemma 0.25, and since $\rho(\alpha) < 1$, there exists $q_1 \in \mathbb{N}$ such that $\rho(s)\rho(\alpha)^{q_1} < 1$ holds. Since α commutes with s, by Lemma 0.25 and Lemma 0.26, we have that

$$\rho(s\alpha^{q_1}) \leq \rho(s)\rho(\alpha)^{q_1} < 1 \text{ and } (e - s\alpha^{q_1}) \text{ is invertible in } \mathcal{A}. \tag{5}$$

Hence, by the condition (P4), for all $y, z, x_1, x_2 \in Y$ we have

$$\begin{aligned} \mathcal{P}_b^r(y, z) &\preceq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2) \\ \therefore \quad \mathcal{P}_b^r(y, z) &\preceq s\Big[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)\Big] \text{ for all } y, z, x_1, x_2 \in Y. \end{aligned}$$

This, using (4) and (5), implies that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &\preceq s\Big[\mathcal{P}_b^r(y_n, y_{n+q_1}) + \mathcal{P}_b^r(y_{n+q_1}, y_{m+q_1}) + \mathcal{P}_b^r(y_{m+q_1}, y_m)\Big] \\ &\preceq s\Big[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^{q_1} \mathcal{P}_b^r(y_n, y_m) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)\Big] \\ (e - s\alpha^{q_1})\mathcal{P}_b^r(y_n, y_m) &\preceq s\Big[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)\Big] \\ \therefore \quad \mathcal{P}_b^r(y_n, y_m) &\preceq (e - s\alpha^{q_1})^{-1} s\Big[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)\Big]. \end{aligned}$$

Using Lemma 0.33 and Lemma 0.52, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c-sequence in $\mathcal{A}$. Therefore, $\{y_n\}$ is a θ-Cauchy sequence in Y. From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is

θ-complete, hence there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^*. That is

$$\lim_{n\to\infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n,m\to\infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta.$$

Next, we will show that y^* is the unique fixed point of F.

$$\begin{aligned}
\mathcal{P}_b^r(y^*, Fy^*) &\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\
&\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\
&\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\
&\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha\mathcal{P}_b^r(y_n, y^*)] \\
\therefore \quad \mathcal{P}_b^r(y^*, Fy^*) &\preceq s[(e+\alpha)\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1})].
\end{aligned}$$

By Lemma 0.30 and Lemma 0.66, we have $\mathcal{P}_b^r(y^*, y_n) \to \theta$ as $n \to \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \to \theta$ as $n \to \infty$. Hence, we deduce that $\mathcal{P}_b^r(y^*, Fy^*) = \theta$. That is $y^* = Fy^*$. So, y^* is a fixed point of F. For uniqueness, we let z^* be another fixed point of F. Then, it follows from (1) that

$$\mathcal{P}_b^r(y^*, z^*) = \mathcal{P}_b^r(Fy^*, Fz^*) \preceq \alpha\mathcal{P}_b^r(y^*, z^*).$$

By Lemma 0.24, we get that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence $y^* = z^*$.

Kindly, observe that Theorem 4.1 extends and generalizes Theorem 3.5 in (George et al., 2017), Theorem 3.1 in (Jain & Chaubey, 2020), Theorem 2.1 in (George et al., 2015), Theorem 2.1 in (Liu & Xu, 2013) and Theorem 3.1 in (Xu & Radenovic, 2014).

Example 0.67. Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ with the norm

$$\|y\| = \|y\|_\infty + \|y'\|_\infty, \text{ for all } y \in \mathcal{A}.$$

Define multiplication pointwisely on $\mathcal{A}$. Then, $\mathcal{A}$ is a Banach algebra with unit $e(t) = 1, \forall t \in [0,1]$. Let $\mathcal{K} = \{y \in \mathcal{A} \colon y = y(t) \geq 0, t \in [0,1]\}$. Then $\mathcal{K}$ is a solid cone in $\mathcal{A}$. Let $Y = \{0,1,2,3\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}_b^r \colon Y \times Y \to \mathcal{K}$ by

$$\mathcal{P}_b^r(y,z)(t) = \begin{cases} y^2 t, & \text{if } y = z \neq 0; \\ 2(y^2+z^2)t, & \text{if } y, z \notin \{2,3\}, y \neq z; \\ (y^2+z^2)t, & \text{if } y, z \in \{2,3\}, y \neq z; \\ \frac{1}{2}t, & \text{if } y = z = 0. \end{cases}$$

Then $(Y, \mathcal{P}_b^r, \mathcal{A})$ is a θ-complete PRCbMS-BA with coefficient $s = 2$. Define a mapping $F : Y \to Y$ as follows:

$$Fy = \begin{cases} 0, & \text{if } y \in \{0,1\}; \\ 1, & \text{if } y \in \{2,3\}. \end{cases}$$

Hence, the mapping F satisfies all the conditions of Theorem 4.1 and $y^* = 0 \in Y$ is the unique fixed point of F.

Reich Contraction Principle on $\mathcal{P}_b^r$-Cone Metric Space over a Banach Algebra

Secondly, we present a variant of the Reich contraction principle on $\mathcal{P}_b^r$-cone metric space over Banach algebra $\mathcal{A}$ as follows:

Theorem 4.2. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ-complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \preceq s$. Suppose $F\colon Y \to Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \preceq \alpha \mathcal{P}_b^r(y, z) + \beta \mathcal{P}_b^r(y, Fy) + \gamma \mathcal{P}_b^r(z, Fz), \tag{6}$$

for all $y, z \in Y$, where $\alpha, \beta, \gamma \in \mathcal{K}$ commutes, $\rho(\alpha) + \rho(\beta + \gamma) < 1$ and $\min\{\rho(\beta), \rho(\gamma)\} < \frac{1}{\rho(s)}$. Then F has a unique fixed point.

Proof. Let y_0 be a point in Y. We define a sequence $\{y_n\}$ in Y by

$$y_{n+1} = Fy_n = F^{n+1}y_0 \text{ for all } n \geq 0. \tag{7}$$

From (6) and (7), we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) \\ &\preceq \alpha \mathcal{P}_b^r(y_n, y_{n-1}) + \beta \mathcal{P}_b^r(y_n, Fy_n) + \gamma \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) \\ \therefore \quad (e - \beta)\mathcal{P}_b^r(y_{n+1}, y_n) &\preceq (\alpha + \gamma)\mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \tag{8}$$

Similarly, on the other hand, we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) = \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \\ &\preceq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) + \beta \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma \mathcal{P}_b^r(y_n, Fy_n) \\ \therefore \quad (e - \gamma)\mathcal{P}_b^r(y_{n+1}, y_n) &\preceq (\alpha + \beta)\mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \tag{9}$$

Adding up (8) and (9), we have

$$(2e-\lambda)\mathcal{P}_b^r(y_{n+1},y_n) \preceq (2\alpha+\lambda)\mathcal{P}_b^r(y_n,y_{n-1}), \tag{10}$$

where $\lambda=(\beta+\gamma)\in\mathcal{K}$. Now, observe that

$$2\rho(\lambda) \le 2\rho(\alpha)+2\rho(\lambda) = 2[\rho(\alpha)+\rho(\beta+\gamma)] < 2.$$

This implies that $\rho(\lambda)<1<2$, then by Lemma 0.25 it follows that $(2e-\lambda)$ is invertible and $(2e-\lambda)^{-1}=\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}$. From (10), we get

$$\mathcal{P}_b^r(y_{n+1},y_n) \preceq (2e-\lambda)^{-1}(2\alpha+\lambda)\mathcal{P}_b^r(y_n,y_{n-1}) \preceq k\mathcal{P}_b^r(y_n,y_{n-1}), \tag{11}$$

where $k=(2e-\lambda)^{-1}(2\alpha+\lambda)\in\mathcal{K}$. Hence,

$$\begin{aligned}
&\mathcal{P}_b^r(y_{n+1},y_n) \preceq k\mathcal{P}_b^r(y_n,y_{n-1}) \preceq k^2\mathcal{P}_b^r(y_{n-1},y_{n-2}) \preceq \cdots \preceq k^n\mathcal{P}_b^r(y_1,y_0)\\
\therefore\ &\mathcal{P}_b^r(y_{n+1},y_n) \preceq k^n\mathcal{P}_b^r(y_1,y_0) \text{ for all } n\in\mathbb{N}.
\end{aligned} \tag{12}$$

We claim that $\rho(k)<1$. Indeed, since α commutes with $\lambda=\beta+\gamma$, it follows that

$$\begin{aligned}
(2e-\lambda)^{-1}(2\alpha+\lambda) &= \Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}\Big)(2\alpha+\lambda) = 2\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}\Big)\alpha + \sum_{j=0}^{\infty}\frac{\lambda^{j+1}}{2^{j+1}}\\
&= 2\alpha\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}\Big) + \lambda\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}\Big) = (2\alpha+\lambda)\Big(\sum_{j=0}^{\infty}\frac{\lambda^j}{2^{j+1}}\Big)\\
&= (2\alpha+\lambda)(2e-\lambda)^{-1}.
\end{aligned}$$

Therefore, $(2\alpha+\lambda)$ commutes with $(2e-\lambda)^{-1}$. Then, by Lemma 0.25 and Lemma 0.26, we obtain

$$\begin{aligned}
\rho(k) &= \rho\Big((2e-\lambda)^{-1}(2\alpha+\lambda)\Big) \le \rho\Big((2e-\lambda)^{-1}\Big)\rho(2\alpha+\lambda)\\
&\le \frac{1}{2-\rho(\lambda)}\Big[2\rho(\alpha)+\rho(\lambda)\Big] < 1 \quad (\text{since } \rho(\alpha)+\rho(\lambda)<1).
\end{aligned}$$

If $y_n=y_{n+1}$ for some $n\in\mathbb{N}$, then $y^*=y_n=Fy_n$ is a fixed point of F, and the result is proved. Hence, we assume that $y_n\neq y_{n+1}$ for all $n\ge 0$. We will show that $y_n\neq y_{n+q}$ for all $n\ge 0$ and $q\ge 1$. Suppose that $y_n=y_{n+q}$ for some $n\ge 0, q\ge 1$, then $y_{n+1}=y_{n+q+1}$ and $Fy_n=Fy_{n+q}$. Then (11) implies that

$$\mathcal{P}_b^r(y_{n+1},y_n) = \mathcal{P}_b^r(y_{n+q+1},y_{n+q}) \preceq k^q\mathcal{P}_b^r(y_{n+1},y_n).$$

Using Lemma 0.24, we obtain that $\mathcal{P}_b^r(y_{n+1}, y_n) = \theta$, that is $y_{n+1} = y_n$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Next, from (6), (7) and (12), we have

$$\begin{aligned}
\mathcal{P}_b^r(y_n, y_m) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_{m-1}) \\
&\preceq \alpha\mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta\mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma\mathcal{P}_b^r(y_{m-1}, Fy_{m-1}) \\
&= \alpha\mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta\mathcal{P}_b^r(y_{n-1}, y_n) + \gamma\mathcal{P}_b^r(y_{m-1}, y_m) \\
&\preceq \alpha\mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta k^{n-1}\mathcal{P}_b^r(y_0, y_1) + \gamma k^{m-1}\mathcal{P}_b^r(y_0, y_1) \\
\therefore \quad \mathcal{P}_b^r(y_n, y_m) &\preceq q\mathcal{P}_b^r(y_{n-1}, y_{m-1}) + (q^n + q^m)\mathcal{P}_b^r(y_0, y_1), \qquad (13)
\end{aligned}$$

where $q \in \{\alpha, \beta, \gamma, k\}$ such that $\rho(q) = \max\{\rho(\alpha), \rho(\beta), \rho(\gamma), \rho(k)\}$. Hence, from (13), we also obtain

$$\mathcal{P}_b^r(y_n, y_m) \preceq q^p\mathcal{P}_b^r(y_{n-p}, y_{m-p}) + p(q^n + q^m)\mathcal{P}_b^r(y_0, y_1), \qquad (14)$$

for all $p \in \{1, 2, \ldots, \min\{n, m\}\}$. Observe that $\rho(s)$ exists because of Lemma 0.25, and since $\rho(q) < 1$, there exists $q_0 \in \mathbb{N}$ such that $\rho(s)\rho(q)^{q_0} < 1$ holds. Further, since q commutes with s, by Lemma 0.25 and Lemma 0.26, we have that

$$\rho(sq^{q_0}) \leq \rho(s)\rho(q)^{q_0} < 1 \text{ and } (e - sq^{q_0}) \text{ is invertible in } \mathcal{A}. \qquad (15)$$

Therefore, from (14), we further obtain

$$\mathcal{P}_b^r(y_n, y_{n+q_0}) \preceq q^n\mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0})\mathcal{P}_b^r(y_0, y_1), \qquad (16)$$

$$\mathcal{P}_b^r(y_{m+q_0}, y_m) \preceq q^m\mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m)\mathcal{P}_b^r(y_0, y_1), \qquad (17)$$

$$\mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) \preceq q^{q_0}\mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0})\mathcal{P}_b^r(y_0, y_1). \qquad (18)$$

Hence, from (P4), (15), (16), (17) and (18), we have

$$\begin{aligned}
\mathcal{P}_b^r(y_n, y_m) &\preceq s[\mathcal{P}_b^r(y_n, y_{n+q_0}) + \mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) + \mathcal{P}_b^r(y_{m+q_0}, y_m)] \\
&\quad - \mathcal{P}_b^r(y_{n+q_0}, y_{n+q_0}) - \mathcal{P}_b^r(y_{m+q_0}, y_{m+q_0}) \\
&\preceq s[q^n \mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0})\mathcal{P}_b^r(y_0, y_1) \\
&\quad + q^{q_0}\mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0})\mathcal{P}_b^r(y_0, y_1) \\
&\quad + q^m \mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m)\mathcal{P}_b^r(y_0, y_1)] \\
(e - sq^{q_0})\mathcal{P}_b^r(y_n, y_m) &\preceq s\Big\{(q^n + q^m)\mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\
&\quad + q^m(m + (m + q_0)q^{q_0})]\mathcal{P}_b^r(y_0, y_1)\Big\} \\
\therefore \quad \mathcal{P}_b^r(y_n, y_m) &\preceq (e - sq^{q_0})^{-1} s\Big\{(q^n + q^m)\mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\
&\quad + q^m(m + (m + q_0)q^{q_0})]\mathcal{P}_b^r(y_0, y_1)\Big\}.
\end{aligned}$$

Using Lemma 0.33 and Lemma 0.52, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c-sequence in $\mathcal{A}$. Therefore, $\{y_n\}$ is a θ-Cauchy sequence in Y. From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is θ-complete, hence there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^*. That is

$$\lim_{n\to\infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n,m\to\infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta.$$

Next, we will show that y^* is the unique fixed point of F.

$$\begin{aligned}
\mathcal{P}_b^r(y^*, Fy^*) &\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\
&\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\
&\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\
&\preceq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha\mathcal{P}_b^r(y_n, y^*) \\
&\quad + \beta\mathcal{P}_b^r(y_n, Fy_n) + \gamma\mathcal{P}_b^r(y^*, Fy^*)] \\
\therefore \quad \mathcal{P}_b^r(y^*, Fy^*) &\preceq s[(e + \alpha)\mathcal{P}_b^r(y^*, y_n) + (e + \beta)\mathcal{P}_b^r(y_n, y_{n+1}) + \gamma\mathcal{P}_b^r(y^*, Fy^*)]. \qquad (19)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathcal{P}_b^r(Fy^*, y^*) &\preceq s[\mathcal{P}_b^r(Fy^*, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\ &\quad - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) - \mathcal{P}_b^r(y_n, y_n) \\ &\preceq s[\mathcal{P}_b^r(Fy^*, Fy_n) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\ &\preceq s[\alpha\mathcal{P}_b^r(y^*, y_n) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(y_n, Fy_n) \\ &\quad + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\ \therefore \quad \mathcal{P}_b^r(y^*, Fy^*) &\preceq s[(e+\alpha)\mathcal{P}_b^r(y^*, y_n) + (e+\gamma)\mathcal{P}_b^r(y_n, y_{n+1}) + \beta\mathcal{P}_b^r(y^*, Fy^*)]. \end{aligned} \tag{20}$$

By Lemma 0.30 and Lemma 0.66, we have $\mathcal{P}_b^r(y^*, y_n) \to \theta$ as $n \to \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \to \theta$ as $n \to \infty$. Hence, from (19) and (20), we deduce that $\mathcal{P}_b^r(y^*, Fy^*) \preceq s\gamma\mathcal{P}_b^r(y^*, Fy^*)$ and $\mathcal{P}_b^r(y^*, Fy^*) \preceq s\beta\mathcal{P}_b^r(y^*, Fy^*)$. Since $\min\{\rho(\beta), \rho(\gamma)\} < \frac{1}{\rho(s)}$, by Lemma 0.24, we have $\mathcal{P}_b^r(y^*, Fy^*) = \theta$. So that $y^* = Fy^*$. That is, y^* is a fixed point of F. For uniqueness, we let z^* be another fixed point of F. Then, it follows from (6) that

$$\begin{aligned}\mathcal{P}_b^r(y^*, z^*) &= \mathcal{P}_b^r(Fy^*, Fz^*) \\ &\preceq \alpha\mathcal{P}_b^r(y^*, z^*) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(z^*, Fz^*) \\ &= \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, y^*) + \gamma\mathcal{P}_b^r(z^*, z^*) \\ &\preceq \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, z^*) + \gamma\mathcal{P}_b^r(z^*, y^*) \\ \therefore \quad \mathcal{P}_b^r(y^*, z^*) &\preceq (\alpha+\beta+\gamma)\mathcal{P}_b^r(y^*, z^*).\end{aligned}$$

By Lemma 0.24 and Lemma 0.26, we have that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence $y^* = z^*$ i.e. the fixed point of F is unique.

Note that Theorem 4.2 extends and generalizes Theorem 3.1 in (George & Mitrovic, 2018).

Finally, we present a variant of the Kannan contraction principle on $\mathcal{P}_b^r$-cone metric space over Banach algebra $\mathcal{A}$ as follows:

Corollary 4.3. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ-complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \preceq s$. Suppose $F\colon Y \to Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \preceq \beta[\mathcal{P}_b^r(y, Fy) + \mathcal{P}_b^r(z, Fz)] \text{ for all } y, z \in Y,$$

where $\alpha \in \mathcal{K}$ such that $\rho(\beta) < 1/2$ and $\rho(s\beta) < 1$. Then F has a unique fixed point.

Proof. Put $\alpha = \theta$ and $\beta = \gamma$ in Theorem 4.2, the result follows.

Note that Corollary 4.3 generalizes Theorem 2.4 in (George et al., 2015), Theorem 2.3 in (Liu & Xu, 2013), and Theorem 3.3 in (Xu & Radenovic, 2014).

Conclusion

In this section, the concept of a partial rectangular cone b-metric space over Banach algebras was introduced and some new fixed point results under various contractive mappings were proved in such a space. Some examples were also given to elucidate the results. Our results extend and generalized many existing results in (George et al., 2015; George et al., 2017; George & Mitrovic, 2018; Jain & Chaubey, 2020; Liu & Xu, 2013; Xu & Radenovic, 2014).

CHAPTER V

Some Fixed Point Theorems in Pentagonal Cone Metric Spaces

In this chapter, we obtain some fixed point theorems of Banach type and Kannan type for self mappings in non-normal pentagonal cone metric spaces. We also give some examples to support the results.

Banach - Type Fixed Point Theorem in a Pentagonal Cone Metric Space

In this section, we prove Banach fixed point theorem for a self mapping in pentagonal cone metric spaces without assuming the normality condition. This section contains the results published in the Journal of Advanced Studies in Topology, 7(2), (2016), 60 - 67. https://doi.org/10.20454/jast.2016.1019

Theorem 5.1. Let (X, d) be a complete pentagonal cone metric space. Suppose the mapping $S : X \to X$ satisfy the following contractive condition:

$$d(Sx, Sy) \le \varphi\Big(d(x, y)\Big), \tag{1}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Sx_n, \text{ for all } n = 0, 1, 2, \ldots. \tag{2}$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (1) and (2), it follows that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Sx_{n-1}, Sx_n) \\ &\le \varphi\Big(d(x_{n-1}, x_n)\Big) = \varphi\Big(d(Sx_{n-2}, Sx_{n-1})\Big) \\ &\le \varphi^2\Big(d(x_{n-2}, x_{n-1})\Big) \le \cdots \le \varphi^n\Big(d(x_0, x_1)\Big). \end{aligned} \tag{3}$$

It again follows that

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Sx_{n-1}, Sx_{n+1}) \le \varphi\Big(d(x_{n-1}, x_{n+1})\Big) \\ &\le \varphi^2\Big(d(x_{n-2}, x_n)\Big) \le \cdots \le \varphi^n\Big(d(x_0, x_2)\Big). \end{aligned} \tag{4}$$

It further follows that

$$\begin{aligned} d(x_n, x_{n+3}) = d(Sx_{n-1}, Sx_{n+2}) &\leq \varphi\Big(d(x_{n-1}, x_{n+2})\Big) \\ &\leq \cdots \leq \varphi^n\Big(d(x_0, x_3)\Big). \end{aligned} \tag{5}$$

Similarly, for $k = 1, 2, 3, \dots$, we get

$$d(x_n, x_{n+3k+1}) \leq \varphi^n\Big(d(x_0, x_{3k+1})\Big), \tag{6}$$

$$d(x_n, x_{n+3k+2}) \leq \varphi^n\Big(d(x_0, x_{3k+2})\Big), \tag{7}$$

$$d(x_n, x_{n+3k+3}) \leq \varphi^n\Big(d(x_0, x_{3k+3})\Big). \tag{8}$$

By using (3) and pentagonal property, we have

$$\begin{aligned} d(x_0, x_4) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\leq \sum_{i=0}^{3} \varphi^i\Big(d(x_0, x_1)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_7) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) \\ &\qquad + d(x_4, x_5) + d(x_5, x_6) + d(x_6, x_7) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_1)\Big) \\ &\qquad + \varphi^4\Big(d(x_0, x_1)\Big) + \varphi^5\Big(d(x_0, x_1)\Big) + \varphi^6\Big(d(x_0, x_1)\Big) \\ &\leq \sum_{i=0}^{6} \varphi^i\Big(d(x_0, x_1)\Big). \end{aligned}$$

Hence, by induction, we obtain for each $k = 1, 2, 3, \dots$

$$d(x_0, x_{3k+1}) \leq \sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1)\Big). \tag{9}$$

Also, by using (3), (4), and pentagonal property, we have

$$\begin{aligned} d(x_0, x_5) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_5) \\ &\leq d(x_0, x_1) + \varphi\Big(d(x_0, x_1)\Big) + \varphi^2\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_2)\Big) \\ &\leq \sum_{i=0}^{2} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^3\Big(d(x_0, x_2)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned}
d(x_0,x_8) &\le d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)\\
&\quad + d(x_4,x_5)+d(x_5,x_6)+d(x_6,x_8)\\
&\le d(x_0,x_1)+\varphi\Big(d(x_0,x_1)\Big)+\varphi^2\Big(d(x_0,x_1)\Big)+\varphi^3\Big(d(x_0,x_1)\Big)\\
&\quad + \varphi^4\Big(d(x_0,x_1)\Big)+\varphi^5\Big(d(x_0,x_1)\Big)+\varphi^6\Big(d(x_0,x_2)\Big)\\
&\le \sum_{i=0}^{5}\varphi^i\Big(d(x_0,x_1)\Big)+\varphi^6\Big(d(x_0,x_2)\Big).
\end{aligned}$$

By induction, we obtain for each $k=1,2,3,\ldots$

$$d(x_0,x_{3k+2}) \le \sum_{i=0}^{3k-1}\varphi^i\Big(d(x_0,x_1)\Big)+\varphi^{3k}\Big(d(x_0,x_2)\Big). \tag{10}$$

Again, by using (3), (5), and pentagonal property, we have

$$\begin{aligned}
d(x_0,x_6) &\le d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_6)\\
&\le d(x_0,x_1)+\varphi\Big(d(x_0,x_1)\Big)+\varphi^2\Big(d(x_0,x_1)\Big)+\varphi^3\Big(d(x_0,x_3)\Big)\\
&\le \sum_{i=0}^{2}\varphi^i\Big(d(x_0,x_1)\Big)+\varphi^3\Big(d(x_0,x_3)\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(x_0,x_9) &\le d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)\\
&\quad + d(x_4,x_5)+d(x_5,x_6)+d(x_6,x_9)\\
&\le d(x_0,x_1)+\varphi\Big(d(x_0,x_1)\Big)+\varphi^2\Big(d(x_0,x_1)\Big)+\varphi^3\Big(d(x_0,x_1)\Big)\\
&\quad + \varphi^4\Big(d(x_0,x_1)\Big)+\varphi^5\Big(d(x_0,x_1)\Big)+\varphi^6\Big(d(x_0,x_3)\Big)\\
&\le \sum_{i=0}^{5}\varphi^i\Big(d(x_0,x_1)\Big)+\varphi^6\Big(d(x_0,x_3)\Big).
\end{aligned}$$

By induction, we obtain for each $k=1,2,3,\ldots$

$$d(x_0,x_{3k+3}) \le \sum_{i=0}^{3k-1}\varphi^i\Big(d(x_0,x_1)\Big)+\varphi^{3k}\Big(d(x_0,x_3)\Big). \tag{11}$$

Using inequalities (6) and (9), for $k = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(x_n, x_{n+3k+1}) &\leq \varphi^n\Big(d(x_0, x_{3k+1})\Big) \\ &\leq \varphi^n \sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1)\Big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{12}$$

Similarly for $k = 1, 2, 3, \ldots$, inequalities (7) and (10) implies that

$$\begin{aligned} d(x_n, x_{n+3k+2}) &\leq \varphi^n\Big(d(x_0, x_{3k+2})\Big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_2)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big) \\ &\qquad + \varphi^{3k}\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{13}$$

Again for $k = 1, 2, 3, \ldots$, inequalities (8) and (11) implies that

$$\begin{aligned} d(x_n, x_{n+3k+3}) &\leq \varphi^n\Big(d(x_0, x_{3k+3})\Big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1)\Big) + \varphi^{3k}\Big(d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k-1} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big) \\ &\qquad + \varphi^{3k}\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{3k} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \end{aligned} \tag{14}$$

Thus; by the inequalities (12), (13), and (14) we have, for each m,

$$d(x_n, x_{n+m}) \leq \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0, x_1) + d(x_0, x_2) + d(x_0, x_3)\Big)\Big]. \tag{15}$$

Since $\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)\Big)$ converges (by definition 0.35), where $d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3) \in P\setminus\{0\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)\Big) \in P\setminus\{0\}$. Hence

$$\lim_{n\to\infty} \varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)\Big)\Big] = 0.$$

Then, for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n\Big[\sum_{i=0}^{\infty} \varphi^i\Big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)\Big)\Big] \ll c, \quad \forall n \geq N_1. \tag{16}$$

Thus, from (15) and (16), we have

$$d(x_n, x_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_{n-1} = z$ as $n \to \infty$.

Now, we show that $Sz = z$. Given $c \gg 0$, we choose a natural numbers N_2, N_3, N_4 such that $d(z,x_n) \ll \frac{c}{4}, \quad \forall n \geq N_2$, $d(x_{n+1},x_n) \ll \frac{c}{4}, \quad \forall n \geq N_3$, and $d(x_{n-1},z) \ll \frac{c}{4}, \quad \forall n \geq N_4$.

Since $x_n \neq x_m$ for $n \neq m$, therefore by pentagonal property, we have

$$\begin{aligned} d(Sz,z) &\leq d(Sz,Sx_n)+d(Sx_n,Sx_{n-1})+d(Sx_{n-1},Sx_{n-2})+d(Sx_{n-2},z) \\ &\leq \varphi\Big(d(z,x_n)\Big)+d(x_{n+1},x_n)+d(x_n,x_{n-1})+d(x_{n-1},z) \\ &< d(z,x_n)+d(x_{n+1},x_n)+d(x_n,x_{n-1})+d(x_{n-1},z). \end{aligned} \tag{17}$$

Hence, from (17), we have

$$d(Sz,z) \ll \frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4} = c, \text{ for all } n \geq N,$$

where $N := \max\{N_2,N_3,N_4\}$. Since c is arbitrary we have $d(Sz,z) \ll \frac{c}{m}, \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(Sz,z) \to -d(Sz,z)$ as $m \to \infty$. Since P is closed, $-d(Sz,z) \in P$. Hence $d(Sz,z) \in P \cap -P$. By definition of cone, we get that $d(Sz,z) = 0$, and so $Sz = z$. Therefore, S has a fixed point that is z in X.

Next, we show that z is unique. For suppose z' be another fixed point of S such that $Sz' = z'$. Therefore,

$$d(z,z') = d(Sz,Sz') \leq \varphi\Big(d(z,z')\Big) < d(z,z').$$

Hence $z = z'$. This completes the proof of the theorem.

Corollary 5.2. Let (X, d) be a complete pentagonal cone metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(S^m x, S^m y) \leq \varphi d(x, y),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. From Theorem 5.1 we conclude that S^m has a fixed point say z, Hence

$$Sz = S(S^m z) = S^{m+1} z = S^m(Sz).$$

Then Sz is also a fixed point to S^m. By uniqueness of z, we have $Sz = z$.

Corollary 5.3. (Garg & Agarwal, 2012) Let (X, d) be a complete pentagonal cone metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 0.35. Hence the results follows from Theorem 5.1.

Corollary 5.4. (Rashwan & Saleh, 2012) Let (X, d) be a complete rectangular cone metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \varphi d(x, y),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 0.45 and Theorem 5.1.

Common Fixed Points of Four Maps in Pentagonal Cone Metric Spaces

In this section, we prove Banach - type fixed point theorem for four self mappings in non-normal pentagonal cone metric space. We give an example to illustrate the results. This section contains the results published in the Far East Journal of Mathematical Sciences, 100(7) (2016), 1141 - 1157.

https://doi.org/10.17654/ms100071141

Theorem 5.5. Let (X, d) be a pentagonal cone metric space. Suppose the mappings $f, g, U, V : X \to X$ satisfy the contractive conditions:

(C1) $d(fx, gy) \leq \alpha d(Ux, Vy)$;

(C2) $d(fx, fy) \leq \alpha d(Ux, Uy)$;

(C3) $d(gx, gy) \leq \alpha d(Vx, Vy)$;

for all $x, y \in X$, where $\alpha \in [0, 1)$. Suppose that $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$ and one of $f(X)$, $g(X)$, $U(X)$ or $V(X)$ is a complete subspace of X, then the pairs (f, U) and (g, V) have a unique point of coincidence in X. Moreover, if (f, U) and (g, V) are weakly compatible pairs, then f, g, U and V have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$, starting with x_0, we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Vx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Ux_{2n+2} \text{ for all } n = 0, 1, 2, \cdots .$$

Suppose that $y_k = y_{k+1}$ for some $k \in \mathbb{N}$. If $k = 2m,$ then $y_{2m} = y_{2m+1}$ for some $m \in \mathbb{N}$, then from (C1), we obtain

$$\begin{aligned} d(y_{2m+2}, y_{2m+1}) &= d(fx_{2m+2}, gx_{2m+1}) \\ &\leq \alpha d(Ux_{2m+2}, Vx_{2m+1}) \\ &= \alpha d(y_{2m+1}, y_{2m}) = 0. \end{aligned}$$

Therefore, $y_{2m+2} = y_{2m+1}$. In similar way, we can deduce that $y_{2m+2} = y_{2m+3} = y_{2m+4} = \cdots$. Hence, $y_n = y_k,$ for all $n \geq k$. Therefore, $\{y_n\}$ is a Cauchy sequence in X. Assuming that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Then from (C1), we have

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &= d(fx_{2m}, gx_{2m+1}) \\ &\leq \alpha d(Ux_{2m}, Vx_{2m+1}) \\ &= \alpha d(y_{2m-1}, y_{2m}). \end{aligned}$$

This implies that

$$\begin{aligned}
d(y_{2m}, y_{2m+1}) &\leq \alpha d(y_{2m-1}, y_{2m}) \\
&\leq \alpha^2 d(y_{2m-2}, y_{2m-1}) \\
&\;\vdots \\
&\leq \alpha^{2m} d(y_0, y_1), \ \forall m \geq 1. \qquad (18)
\end{aligned}$$

Also,

$$\begin{aligned}
d(y_{2m+1}, y_{2m+2}) &= d(fx_{2m+1}, gx_{2m+2}) \\
&\leq \alpha d(Ux_{2m+2}, Vx_{2m+1}) \\
&= \alpha d(y_{2m+1}, y_{2m}),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(y_{2m+1}, y_{2m+2}) &\leq \alpha d(y_{2m}, y_{2m+1}) \\
&\leq \alpha^2 d(y_{2m-1}, y_{2m}) \\
&\;\vdots \\
&\leq \alpha^{2m+1} d(y_0, y_1), \ \forall m \geq 1. \qquad (19)
\end{aligned}$$

Hence, from (18) and (19), we have

$$d(y_n, y_{n+1}) \leq \alpha^n d(y_0, y_1), \ \forall n \geq 1. \qquad (20)$$

From (C2), (20), pentagonal property, and the fact that $0 \leq \alpha < 1$, we obtain

$$\begin{aligned}
d(y_{2m}, y_{2m+2}) &= d(fx_{2m}, fx_{2m+2}) \\
&\leq \alpha d(Ux_{2m}, Ux_{2m+2}) \\
&= \alpha d(y_{2m-1}, y_{2m+1}) \\
&\leq \alpha\Big(d(y_{2m-1}, y_{2m}) + d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}) + d(y_{2m+2}, y_{2m+1})\Big) \\
&\leq \alpha\Big(\alpha^{2m-1} d(y_0, y_1) + \alpha^{2m} d(y_0, y_1) + \alpha^{2m+1} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1)\Big) \\
&\leq \alpha^{2m} d(y_0, y_1) + \alpha^{2m+1} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1) + \alpha^{2m+3} d(y_0, y_1) \\
&\leq \frac{\alpha^{2m}}{1-\alpha} d(y_0, y_1), \ \forall m \geq 1. \qquad (21)
\end{aligned}$$

From (C3), (21), pentagonal property, and the fact that $0 \leq \alpha < 1$, we obtain

$$\begin{aligned} d(y_{2m+1}, y_{2m+3}) &= d(gx_{2m+1}, gx_{2m+3}) \\ &\leq \alpha d(Vx_{2m+1}, Vx_{2m+3}) \\ &= \alpha d(y_{2m}, y_{2m+2}) \\ &\leq \frac{\alpha^{2m+1}}{1-\alpha} d(y_0, y_1), \ \forall m \geq 1. \end{aligned} \tag{22}$$

Hence, from (21) and (22), we have

$$d(y_n, y_{n+2}) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n \geq 1. \tag{23}$$

For the sequence $\{y_n\}$, we consider $d(y_n, y_{n+p})$ in two cases as follows:

If p is odd say $p = 2k+1$, where $k \geq 1$, then by pentagonal property and (20), we have

$$\begin{aligned} d(y_n, y_{n+2k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+2k+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots \\ &\quad + d(y_{n+2k-1}, y_{n+2k}) + d(y_{n+2k}, y_{n+2k+1}) \\ &\leq \alpha^n d(y_0, y_1) + \alpha^{n+1} d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \cdots \\ &\quad + \alpha^{n+2k-1} d(y_0, y_1) + \alpha^{n+2k} d(y_0, y_1) \\ &\leq \frac{2\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n \geq 1. \end{aligned}$$

If p is even say $p = 2k$, where $k \geq 1$, then by pentagonal property, (20) and (23), we have

$$\begin{aligned} d(y_n, y_{n+2k}) &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+2k}) \\ &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + \cdots \\ &\quad + d(y_{n+2k-2}, y_{n+2k-1}) + d(y_{n+2k-1}, y_{n+2k}) \\ &\leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \alpha^{n+3} d(y_0, y_1) + \cdots \\ &\quad + \alpha^{n+2k-2} d(y_0, y_1) + \alpha^{n+2k-1} d(y_0, y_1) \\ &\leq \frac{2\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n \geq 1. \end{aligned}$$

Therefore, combining the above two cases, we get

$$d(y_n, y_{n+p}) \leq \frac{2\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n, p \in \mathbb{N}. \tag{24}$$

Since $\alpha \in [0, 1)$, we get, as $n \to \infty$, $\frac{2\alpha^n}{1-\alpha} \to 0$. Hence, for every $c \in E$ with $c \gg 0$, $\exists n_0 \in \mathbb{N}$ such that

$$d(y_n, y_{n+p}) \ll c, \text{ for all } n \geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in X.

Suppose $U(X)$ is a complete subspace of X, there exists a points $p, q \in U(X)$ such that $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} U_{2n+2} = q = Up$.

Now, we show that $Up = fp$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(y_{2n+2}, q) \ll \frac{c}{4}, \ \ \forall n \geq M_1$, $d(y_{2n-1}, q) \ll \frac{c}{4\lambda}, \ \ \forall n \geq M_2$ and $d(y_{2n}, y_{2n+1}) \ll \frac{c}{4}, \ \ \forall n \geq M_3$. Since $y_n \neq y_m$ for $n \neq m$, by pentagonal property and (C2), we have that

$$\begin{aligned}
d(fp, q) &\leq d(fp, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&= d(fp, fx_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\leq \lambda d(Up, Ux_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&= \lambda d(q, y_{2n-1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(fp, q) &\leq \lambda d(y_{2n-1}, q) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq M,
\end{aligned}$$

where $M := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(fp, q) \ll \frac{c}{m}, \ \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(fp, q) \to -d(fp, q)$ as $m \to \infty$. Since P is closed, $-d(fp, q) \in P$. Hence $d(fp, q) \in P \cap -P$. By definition of cone we get that $d(fp, q) = 0$, and so $Up = fp = q$. Hence, q is a point of coincidence of f and U.

Since $q = fp \in f(X)$ and $f(X) \subseteq V(X)$, there exists $r \in X$ such that $q = Vr$. Now, we show that $Vr = gr$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(y_{2n+2}, q) \ll \frac{c}{4}, \ \ \forall n \geq M_1$, $d(y_{2n-1}, y_{2n}) \ll \frac{c}{4\lambda}, \ \ \forall n \geq M_2$ and

$d(y_{2n}, y_{2n+1}) \ll \frac{c}{4}, \quad \forall n \geq M_3$. Since $y_n \neq y_m$ for $n \neq m$, by pentagonal property and (C1), we have that

$$\begin{aligned} d(gr, q) &\leq d(gr, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\ &= d(gr, fx_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\ &\leq \lambda d(Ux_{2n}, Vr) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\ &\leq \lambda d(y_{2n-1}, q) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q). \end{aligned}$$

Therefore,

$$\begin{aligned} d(gr, q) &\leq \lambda d(y_{2n-1}, q) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq M, \end{aligned}$$

where $M := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(gr, q) \ll \frac{c}{m}, \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gr, q) \to -d(gr, q)$ as $m \to \infty$. Since P is closed, $-d(gr, q) \in P$. Hence $d(gr, q) \in P \cap -P$. By definition of cone we get that $d(gr, q) = 0$, and so $Vr = gr = q$. Hence, q is a point of coincidence point of g and V.

Thus, the pairs (f, U) and (g, V) have common point of coincidence q in X. Now, suppose the pairs (f, U) and (g, V) are weakly compatible mappings. Then

$$fq = fUp = Ufp = Uq = q_1,$$

and

$$gq = gVr = Vgr = Vq = q_2.$$

Hence, from (C1), we have

$$\begin{aligned} d(q_1, q_2) &= d(fq, gq) \\ &\leq \lambda d(Uq, Vq) \\ &= \lambda d(q_1, q_2), \end{aligned}$$

which implies that

$$d(q_1, q_2) = 0.$$

That is, $q_1 = q_2$.

Therefore
$$fq = gq = Uq = Vq.$$
Also,
$$\begin{aligned} d(q, gq) &= d(fp, gq) \\ &\leq \lambda d(Up, Vq) \\ &= \lambda(d(q, gq), \end{aligned}$$
which implies that
$$d(q, gq) = 0.$$
Hence $gq = q$, or $fq = gq = Uq = Vq = q$.

Thus, q is the common fixed point of f, g, U, and V.

Next, we show that q is unique. For suppose q' be another common fixed point of f, g, U, and V. That is,
$$fq' = gq' = Uq' = Vq' = q',$$
for some $q' \in X$. Then from (C1), we have
$$\begin{aligned} d(q, q') &= d(fq, Gq') \\ &\leq \lambda d(Uq, Vq') \\ &= \lambda d(q, q'), \end{aligned}$$
which implies that
$$d(q, q') = 0.$$
Hence, $q = q'$.

Therefore, the mappings f, g, U, and V have a unique common fixed point in X.

Similarly, if $f(X)$, $g(X)$, or $V(X)$ is a complete subspace of X, then we can easily prove that f, g, U, and V have unique common fixed point in X. This completes the proof of the theorem.

Remark 0.68. If P is a normal cone, and (X, d) a rectangular cone metric space in in the above Theorem 5.5, then we get the Theorem 2.1 in (Reddy & Rangamma, 2015a).

The following example illustrates the result of Theorem 5.5.

Example 0.69. Let $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$d(x, x) = 0, \forall x \in X;$$

$$d(1, 2) = d(2, 1) = (4, 8);$$

$$d(1, 3) = d(3, 1) = d(3, 4) = d(4, 3) = d(2, 4) = d(4, 2) = (1, 2);$$

$$d(1, 5) = d(5, 1) = d(2, 5) = d(5, 2) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (3, 6).$$

Then (X, d) is a complete pentagonal cone metric space, but (X, d) is not a rectangular cone metric space because it lacks the rectangular property:

$$\begin{aligned} (4, 8) = d(1, 2) &> d(1, 3) + d(3, 4) + d(4, 2) \\ &= (1, 2) + (1, 2) + (1, 2) \\ &= (3, 6) \text{ as } (4, 8) - (3, 6) = (1, 2) \in P. \end{aligned}$$

Define a mapping $f, g, U, V : X \to X$ as follows:

$$f(x) = 4, \ \forall x \in X.$$

$$g(x) = \begin{cases} 4, & \text{if } x \neq 5; \\ 2, & \text{if } x = 5. \end{cases}$$

$$U(x) = \begin{cases} 3, & \text{if } x = 1; \\ 1, & \text{if } x = 2; \\ 2, & \text{if } x = 3; \\ 4, & \text{if } x = 4; \\ 5, & \text{if } x = 5. \end{cases}$$

$$V(x) = x, \ \forall x \in X.$$

Clearly $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$, and the pairs (f, U) and (g, V) are weakly compatible mappings. The conditions of Theorem 5.5 holds for all $x, y \in X$, where $\lambda = \frac{1}{3}$, and 4 is the unique common fixed point of the mappings f, g, U and V.

Corollary 5.8. Let (X, d) be a pentagonal cone metric space. Suppose the mappings $f, g, U : X \to X$ satisfies the contractive conditions:

(C1) $d(fx, gy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

(C2) $d(fx, fy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

(C3) $d(gx, gy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $f(X) \cup g(X) \subseteq U(X)$, and if $U(X)$, or $f(X) \cup g(X)$ is a complete subspace of X, then the pairs (f, U) and (g, U) have a unique point of coincidence in X. Moreover, if (f, U) and (g, U) are weakly compatible pairs then f, g and U have a unique common fixed point in X.

Proof. Putting $V = U$ in Theorem 5.5. This completes the proof.

Corollary 5.10. (Garg & Agarwal, 2012). Let (X, d) be a pentagonal cone metric space and P be a normal cone with normal constant k. Suppose the mapping $f : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then f has a unique fixed point in X.

Proof. Putting $g = f$, $V = U = I$, and P is a normal cone in Theorem 5.5. This completes the proof.

Corollary 5.11. (Reddy & Rangamma, 2015a). Let (X, d) be a rectangular cone metric space and P be a normal cone with normal constant k. Suppose the mappings $f, g : X \to X$ satisfies the contractive condition:

(C1) $d(fx, gy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

(C2) $d(fx, fy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

(C3) $d(gx, gy) \leq \lambda\Big(d(Ux, Uy)\Big)$;

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $f(X) \cup g(X) \subseteq U(X)$, and if $U(X)$, or $f(X) \cup g(X)$ is a complete subspace of X, then the pairs (f, U) and (g, U) have a unique point of coincidence in X. Moreover, if (f, U) and (g, U) are weakly compatible pairs then f, g and U have a unique common fixed point in X.

Proof. This follows from the Remark 0.45, putting $V = U$, and P is a normal cone in Theorem 5.5.

Corollary 5.12. (Azam et al., 2009). Let (X, d) be a rectangular cone metric space and P be a normal cone with normal constant k. Suppose the mapping $f : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then f has a unique fixed point in X.

Proof. Using Remark 0.45, putting $g = f$, $V = U = I$, and P is a normal cone in Theorem 5.5. This completes the proof.

Corollary 5.13. (Huang & Zhang, 2007). Let (X, d) be a cone metric space and P be a normal cone with normal constant k. Suppose the mapping $f : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then f has a unique fixed point in X.

Proof. Using Remark 0.45, putting $g = f$, $V = U = I$, and P is a normal cone in Theorem 5.5. This completes the proof.

Kannan - Type Fixed Point Theorem for Two Maps in Pentagonal Cone Metric Spaces

In this section, we prove Kannan - type contraction principle in pentagonal cone metric spaces for two self mappings. We give an example to illustrate the results. This section contains the results published in the International Journal of Pure and Applied Mathematics, 108 (1) (2016), 29 - 38. https://doi.org/10.12732/ijpam.v108i1.5

Theorem 5.14. Let (X, d) be a pentagonal cone metric space. Suppose the mappings $f, g : X \to X$ satisfy the contractive condition:

$$d(fx, fy) \leq \lambda\Big(d(gx, fx) + d(gy, fy)\Big), \tag{25}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \subseteq g(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then the mappings f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_n = gx_{n+1} \text{ for all } n = 0, 1, 2, \cdots .$$

Now, we define a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n = 0, 1, 2, \cdots$. If $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then $y_k = fx_{k+1} = gx_{k+1}$. That is, f and g have a point of coincidence y_k in X. We assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Then, from (25), we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\leq \lambda\Big(d(gx_n, fx_n) + d(gx_{n+1}, fx_{n+1})\Big) \\ &= \lambda\Big(d(y_{n-1}, y_n) + d(y_n, y_{n+1})\Big). \end{aligned}$$

So that,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(y_{n-1}, y_n) \\ &\leq r d(y_{n-1}, y_n), \text{ where } r = \frac{\lambda}{1-\lambda} \in [0, 1) \\ &\leq r^2 d(y_{n-2}, y_{n-1}) \\ &\vdots \\ &\leq r^n\Big(d(y_0, y_1)\Big), \quad \forall n \geq 1. \end{aligned} \tag{26}$$

Also from (25) and (26), we obtain

$$\begin{aligned} d(y_n, y_{n+2}) &= d(fx_n, fx_{n+2}) \\ &\leq \lambda\Big(d(gx_n, fx_n) + d(gx_{n+2}, fx_{n+2})\Big) \\ &\leq \lambda\Big(d(y_{n-1}, y_n) + d(y_{n+1}, y_{n+2})\Big) \\ &\leq \lambda\Big(r^{n-1} d(y_0, y_1) + r^{n+1} d(y_0, y_1)\Big) \\ &\leq \lambda r^{n-1}(1 + r^2) d(y_0, y_1). \end{aligned}$$

That is,

$$d(y_n, y_{n+2}) \leq \alpha r^{n-1} d(y_0, y_1), \quad \forall n \geq 1, \tag{27}$$

where $\alpha = \lambda(1 + r^2) > 0$.

For the sequence $\{y_n\}$, we consider $d(y_n, y_{n+p})$ in two cases as follows:

If p is odd say $p = 2m+1$, where $m \geq 1$, then by pentagonal property and (26), we have

$$\begin{aligned}
d(y_n, y_{n+2m+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+2m+1}) \\
&\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots \\
&\quad + d(y_{n+2m-1}, y_{n+2m}) + d(y_{n+2m}, y_{n+2m+1}) \\
&\leq r^n d(y_0, y_1) + r^{n+1} d(y_0, y_1) + r^{n+2} d(y_0, y_1) + \cdots \\
&\quad + r^{n+2m-1} d(y_0, y_1) + r^{n+2m} d(y_0, y_1) \\
&\leq \frac{r^n}{1-r} d(y_0, y_1), \quad \forall n \geq 1.
\end{aligned}$$

If p is even say $p = 2m$, where $m \geq 2$, then by pentagonal property, (26) and (27), we have

$$\begin{aligned}
d(y_n, y_{n+2m}) &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+2m}) \\
&\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + \cdots \\
&\quad + d(y_{n+2m-2}, y_{n+2m-1}) + d(y_{n+2m-1}, y_{n+2m}) \\
&\leq \alpha r^{n-1} d(y_0, y_1) + r^{n+2} d(y_0, y_1) + r^{n+3} d(y_0, y_1) + \cdots \\
&\quad + r^{n+2m-2} d(y_0, y_1) + r^{n+2m-1} d(y_0, y_1) \\
&\leq \alpha r^{n-1} d(y_0, y_1) + \frac{r^n}{1-r} d(y_0, y_1).
\end{aligned}$$

Since $r \in [0, 1)$, we get, as $n \to \infty$, $\frac{r^n}{1-r} \to 0$ and $\alpha r^{n-1} \to 0$. Hence, for every $c \in E$ with $c \gg 0$, $\exists n_0 \in \mathbb{N}$ such that

$$d(y_n, y_{n+p}) \ll c, \text{ for all } n \geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d). Since $g(X)$ is a complete subspace of X, there exists a points $u, v \in g(X)$ such that $\lim_{n\to\infty} y_n = v = gu$.

Now, we show that $gu = fu$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(v, y_n) \ll \frac{c(1-\lambda)}{3}, \ \forall n \geq M_1$, $d(y_n, y_{n+1}) \ll \frac{c(1-\lambda)}{3}, \ \forall n \geq M_2$ and $d(y_{n+1}, y_{n+2}) \ll \frac{c(1-\lambda)}{3(1+\lambda)}, \ \forall n \geq M_3$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$\begin{aligned}
d(gu, fu) &\leq d(gu, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, fu) \\
&\leq d(v, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(fx_{n+2}, fu) \\
&\leq d(v, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \lambda\Big(d(gu, fu) + d(gx_{n+2}, fx_{n+2})\Big) \\
&\leq d(v, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \lambda\Big(d(gu, fu) + d(y_{n+1}, y_{n+2})\Big) \\
d(gu, fu) &\leq \frac{1}{1-\lambda}\Big(d(v, y_n) + d(y_n, y_{n+1}) + (1+\lambda)d(y_{n+1}, y_{n+2})\Big) \\
&\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c, \text{ for all } n \geq M,
\end{aligned}$$

where $M := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(gu, fu) \ll \frac{c}{m}, \ \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gu, fu) \to -d(gu, fu)$ as $m \to \infty$. Since P is closed, $-d(gu, fu) \in P$. Hence $d(gu, fu) \in P \cap -P$. By definition of cone we get that $d(gu, fu) = 0$, and so $gu = fu = v$. Hence, v is a point of coincidence of f and g. Similarly, if $f(X)$ is a complete subspace of X the result holds.

Next, we show that v is unique. For suppose v' be another point of coincidence of f and g, that is $gu' = fu' = v'$, for some $u' \in X$, then

$$d(v, v') = d(fu, fu') \leq \lambda\Big(d(gu, fu) + d(gu', fu')\Big) \leq \lambda\Big(d(v, v) + d(v', v')\Big).$$

Hence, $v = v'$. Since (f, g) is weakly compatible, by Lemma 0.37, v is the unique common fixed point of f and g. This completes the proof of the theorem.

To illustrate Theorem 5.14, we give the following example.

Example 0.70. Let $X = \{a, b, c, d, e\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $\rho : X \times X \to E$ as follows:

$$\rho(x, x) = 0, \forall x \in X;$$

$$\rho(a, b) = \rho(b, a) = (4, 16);$$

$$\begin{aligned}
\rho(a, c) = \rho(c, a) = \rho(c, d) = \rho(d, c) = \rho(b, c) = \rho(c, b) = \rho(b, d) \\
= \rho(d, b) = \rho(a, d) = \rho(d, a) = (1, 4);
\end{aligned}$$

$$\rho(a, e) = \rho(e, a) = \rho(b, e) = \rho(e, b) = \rho(c, e) = \rho(e, c) = \rho(d, e) = \rho(e, d) = (5, 20).$$

Then (X,ρ) is a complete cone pentagonal metric space, but (X,ρ) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned}(4,16) = \rho(a,b) &> \rho(a,c)+\rho(c,d)+\rho(d,b)\\ &= (1,4)+(1,4)+(1,4)\\ &= (3,12),\ \text{as } (4,16)-(3,12)=(1,4)\in P.\end{aligned}$$

Define a mapping $f,g: X \to X$ as follows:

$$f(x) = \begin{cases} d, & \text{if } x \neq e;\\ b, & \text{if } x = e.\end{cases}$$

$$g(x) = \begin{cases} c, & \text{if } x = a;\\ a, & \text{if } x = b;\\ b, & \text{if } x = c;\\ d, & \text{if } x = d;\\ e, & \text{if } x = e.\end{cases}$$

Clearly $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also f and g are weakly compatible mappings. Thus, the conditions of Theorem 5.14 holds for all $x, y \in X$, where $\lambda = \frac{1}{5}$ and $d \in X$ is the unique common fixed point of the mappings f and g.

Corollary 5.15. (Auwalu, 2016b). Let (X,d) be a complete pentagonal cone metric space and P be a normal cone with normal constant k. Suppose the mapping $S: X \to X$ satisfies the contractive condition:

$$d(Sx,Sy) \leq \lambda\Big(d(x,Sx)+d(y,Sy)\Big), \tag{14}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. S has a unique fixed point in X.

2. For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Take $g = I$ and P be a normal cone in Theorem 5.14. This completes the proof.

Corollary 5.16. (Reddy & Rangamma, 2015b) Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:

$$d(Sx, Sy) \leq \lambda\Big(d(gx, Sx) + d(gy, Sy)\Big),$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \subseteq g(X)$ and $S(X)$ or $g(X)$ is a complete subspace of X, then the mappings S and g have a unique coincidence point in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. This follows from Remark 0.45 and Theorem 5.14, where P is a normal cone.

Corollary 5.17. (Jleli & Samet, 2009). Let (X, d) be a complete cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:

$$d(Sx, Sy) \leq \lambda\Big(d(x, Sx) + d(y, Sy)\Big), \tag{15}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. S has a unique fixed point in X.

2. For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Take $g = I$ and P be a normal cone in Theorem 5.14 and Remark 0.45. This completes the proof.

Conclusion

In this section, we prove Banach fixed point theorem for a self mapping in cone pentagonal metric spaces without assuming the normality condition, Banach - type fixed point theorem for four self mappings in non-normal cone pentagonal metric space and Kannan - type contraction principle in cone pentagonal metric spaces for two self mappings. We give some examples to illustrate the results.

CHAPTER VI

Conclusion and Recommendations

This chapter presents conclusions based on the research findings according to the objective of the research and gives recommendations accordingly.

Conclusion

In this research work, we study some fixed points and common fixed points theorems of self mappings in non-normal cone metric space, rectangular cone metric space and pentagonal cone metric space settings. Our results extend and improve the results in (Azam et al., 2009; Garg & Agarwal, 2012; George et al., 2015; George et al., 2017; George & Mitrovic, 2018; Huang & Zhang, 2007; Jain & Chaubey, 2020; Jleli & Samet, 2009; Liu & Xu, 2013; Patil & Salunke, 2015; Rashwan & Saleh, 2012; Reddy & Rangamma, 2015b; Xu & Radenovic, 2014), and many others in the literature.

The thesis is structured in six (6) chapters as follows:

In the first chapter, we give some definitions of terms, some examples, and limitations of the thesis. We also give a collection of some significant results and notions in the setting of metric spaces.

In the second chapter, we review some related literature, study some metric fixed points theorems. We also give a collection of some significant results and notions in the area of cone metric spaces and its generalizations.

In the third chapter, we prove some fixed point theorems for generalized expansive mappings in cone metric space over Banach algebra. We further give an example to elucidate the results.

In the fourth chapter, we introduced a new space and prove some fixed point theorems for different contractive condition mappings in such a space. We also give an example to illustrate the results.

In the fifth chapter, we obtain some fixed point theorems of Banach type for one and four self mappings in non-normal cone pentagonal metric spaces, Kannan type for two self mappings in non-normal cone pentagonal metric spaces. We also give some examples to support the results.

Finally, in the sixth chapter, we summarize and conclude the research work.

Recommendations According to Findings

The presented results come from single or joint papers of the author and/or his supervisor as coauthor. We have published at lease fourteen (14) articles in the following Journals and Conference proceedings:

1. Journal of Mathematics, Hindawi (2021).
https://doi.org/10.1155/2021/8447435

2. Functional Analysis in Interdisciplinary Applications - II, Springer Proceedings in Mathematics & Statistics 351 (2021).
https://doi.org/10.1007/978-3-030-69292-6-7

3. AIP Conference Proceedings 1997, 020004 (2018).
https://doi.org/10.1063/1.5048998

4. ITM Web of Conferences 22, 01003 (2018).
https://doi.org/10.1051/itmconf/20182201003

5. AIP Conference Proceedings 2325, 020060 (2021).
https://doi.org/10.1063/5.0040595

6. Journal of Mathematics and Applications, no 42, pp 21-33, (2019). Rzeszow University of Technology, Poland,

7. University Thought Publication in Natural Sciences, 8(2), pp. 54-60, (2018).
https://doi.org/10.5937/univtho8-18216

8. International Journal of Pure and Applied Mathematics.
https://doi.org/10.12732/ijpam.v108i1.5

9. Journal of Informatics and Mathematical Sciences.

10. Journal of Mathematics and Computational Sciences.

11. Far East Journal of Mathematical Sciences,
https://doi.org/10.17654/ms100071141

12. British Journal of Mathematics and Computer Science,

https://doi.org/10.9734/bjmcs/2016/25172

13. Journal of Advanced Studies in Topology,
https://doi.org/10.20454/jast.2016.1019

14. European Journal of Pure and Applied Mathematics. http://www.ejpam.com

We hope these results will be useful in the area of Fixed Point Theory and may be generalized in further spaces with efficient conditions.

Recommendations for Further Research

We strongly recommend other researchers in this area to see if they can extend/or improve this work, say n-polygonal cone metric spaces. We highly recommend that, other Mathematicians shall embrace research in the area of Fixed Point Theory as it has very wide coverage of importance in applications.

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Surname, Name	: Auwalu, Abba
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EDUCATION

Degree	Institution	Year of Graduation
M.Sc.	Bayero University, Department of Mathematics	2013
B.Sc.	Bayero University, Department of Mathematics	2008

WORK EXPERIENCE

Year	Place	Enrollment
2018-present	SLU, Department of Mathematics	Lecturer I
2016-2018	SLU, Department of Mathematics	Lecturer II
2014-2016	JSU, Department of Mathematics	Asst. Lecturer
2012-2014	JICORAS, Department of Mathematics	Lecturer I
2002-2012	MOES & T. Dutse – Jigawa State	Maths Teacher

FOREIGN LANGUAGES

English, fluently spoken and written

HONOURS AND AWARDS

- NEU Scholarship award (PhD Mathematics), NEU, Turkey, 2014 - 2018.
- Best Mathematics & Chemistry Student, F.C.E. Kano, Nigeria, 2003.
- Appreciation letter for selfless service, Rumfa College, Kano, Nigeria, 2002.

MEMBERSHIP OF PROFESSIONAL ORGANIZATIONS

- *Member*, Nigerian Mathematical Society (NMS), Nigeria.
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PUBLICATIONS IN INTERNATIONAL REFERED JOURNALS (IN COVERAGE OF SSCI/SCI-EXPANDED, ESCI, AND AHCI):

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- Ali, B. and Auwalu, A. (2012). Synchronal and Cyclic algorithms for Fixed point problems and Variational inequality problems in Banach spaces. A paper presented at the 31st Annual Conference of the Nigerian Mathematical Society (NMS) held at Ahmadu Bello University, Kaduna State, Nigeria, from 2nd to 5th October, 2012.

- Auwalu, A. (2011). Application of finite Markov chain to a model of Schooling: A case study of Govt. College, Kano. A paper presented at the

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REVIEWER:

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Appendices

Appendix A

Turnitin Similarity Report

Abstract

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SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

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Chapter 1

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PRIMARY SOURCES

1	books.ifmo.ru Internet Source	%3
2	Wang, LiMin Wang, ShuangCheng Li, XiongF. "Extracting credible dependencies for averaged one-dependence estimator analysis.(Research Article)(R", Mathematical Problems in Engineering, Annual 2014 Issue Publication	%1
3	www.pmf.untz.ba Internet Source	%1
4	arizona.openrepository.com Internet Source	<%1
5	Submitted to Higher Education Commission Pakistan Student Paper	<%1
6	thales.doa.fmph.uniba.sk Internet Source	<%1
7	en.wikipedia.org Internet Source	<%1

Chapter 2

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PRIMARY SOURCES

1	Arqub, Omar Abu Momani, Shaher Mezel, Sa. "Existence, uniqueness, and characterization theorems for nonlinear fuzzy integrodifferential equatio", Mathematical Problems in Engineering, Annual 2015 Issue Publication	%2
2	www.maths.mq.edu.au Internet Source	%1
3	users.metu.edu.tr Internet Source	%1

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Chapter 3

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1	Submitted to University of Birmingham Student Paper	%1
2	users.metu.edu.tr Internet Source	%1
3	mathrefresher.blogspot.com Internet Source	<%1
4	www.math.brown.edu Internet Source	<%1
5	Submitted to Coventry University Student Paper	<%1
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PRIMARY SOURCES

1	Submitted to University of Strathclyde Student Paper	%1
2	www.math.kth.se Internet Source	%1
3	Submitted to University of Southampton Student Paper	<%1
4	arizona.openrepository.com Internet Source	<%1
5	www.scribd.com Internet Source	<%1
6	mitpress.mit.edu Internet Source	<%1
7	Matevosyan, O.A.. "Solutions of the robin problem for the system of elastic theory in external domains.", Journal of Mathematical Sciences, March 1 2014 Issue Publication	<%1

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Chapter 5

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1	arizona.openrepository.com Internet Source	%1
2	www.math.kent.edu Internet Source	<%1
3	proiectidsrc.acadiasi.ro Internet Source	<%1

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Chapter 6

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