COMPLETE ASYMPTOTIC EXPANSION FOR SOME POSITIVE LINEAR OPERATORS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
IBRAHIM SULEIMAN

In Partial Fulfilment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2016

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I hereby declare that, all the information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my parents…

ABSTRACT

Many mathematicians of recent have contributed on the complete asymptotic expansion for several operators. In this work, some positive linear operators are presented and their complete asymptotic expansions is also obtained, where all the coefficients in the expansions turn out to be in terms of the Stirling numbers of the first and second kind.

Keywords: Meyer-König and Zeller operators (MKZ), Chlodovsky operators, Stirling numbers, complete asymptotic expansion

ÖZET

Sonzamanlarda bir çok matematikci, bazı operatörler için tam asimtotik açılımlar üzerinde çalışmalarda bulundular. Bu çalışmada, bazı linear operatörler verilmiş ve bu operatörlerin tam asimtotik genişlemeleri verilmiştir. Genişlemelerdeki tüm bu katsayılar, birinci ve ikinci tipteki Stirling sayılarına dönüştürülür.

Anahtar Kelimeler: Meyer- König and Zeller operatörleri, Chlodovsky operatörleri, Stirling sayıları, tam asimtotik genişlemesi

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CHAPTER 1

INTRODUCTION

In mathematical analysis, operators are basically extension of functions where the domain and range are vector spaces.

1.1 Linear Operators

Definition 1.1

For an operator $T: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ to be called a linear operator it must satisfy the following conditions.

i) The domain of T, $D(T)$ is a vector space and the range $R(T)$lies in a vector space over the same field k

ii) For all $\boldsymbol{x}, \boldsymbol{y} \in D(T)$and scalars $\alpha, \beta \in k$

$$T(\boldsymbol{x}+\boldsymbol{y}) = T\boldsymbol{x} + T\boldsymbol{y}$$

$$T(\alpha\boldsymbol{x}) = \alpha T\boldsymbol{x}$$

That is $T(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \alpha T\boldsymbol{x} + \beta T\boldsymbol{y}$

Examples of linear operators.

i) Identity operators: Let $\boldsymbol{X}$ be a vector space over a field K.

$I_X: \boldsymbol{X} \rightarrow \boldsymbol{X},$ such that

$I_X(\boldsymbol{x}) = \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \boldsymbol{X}$

Verification: $I_X(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \alpha\boldsymbol{x}+\beta\boldsymbol{y} = \alpha I_X(\boldsymbol{x}) + \beta I_X(\boldsymbol{y})$

Where $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\alpha, \beta \in K$.

ii) Zero operators: Let $\boldsymbol{X}, \boldsymbol{Y}$ be vector spaces over the same field K.

$O: \boldsymbol{X} \to \boldsymbol{Y}$such that

$$O(\boldsymbol{x}) = \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \boldsymbol{X}$$

Verification: $O(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{0} = \alpha O(\boldsymbol{x}) + \beta O(\boldsymbol{y})$

Where $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\alpha, \beta \in K$

1.2 Positive Linear Operators

Definition 1.2

A function $\theta: M_n \to M_m$ is said to be a positive linear function if $\theta(A) \geq \theta(B)$ whenever $A \geq B$.

1.3 Asymptotic Expansion

Erdelyi(1956) gave a detail description of asymptotic expansions (also known as asymptotic series or Poincare' expansion (after Henri Poincare'))of a function $f(x)$ as an expansion of that function in terms of a series, the partial sum of which do not necessarily converge, but such that taking an initial partial sum provide an asymptotic formula for f. Asymptotic formula means a statement of equality between two functions which is not a true equality but which mean the ratio of the two functions approaches 1 as the variable approaches some value usually infinity.

1.3.1 Asymptotic formula

Definition 1.3

Let $f(n)$ be a quantity of functions depending on a natural number n. A function $p(n)$ of n is an asymptotic formula for $f(n)$ if

$$\lim_{n \to \infty} \frac{f(n)}{p(n)} = 1$$

Denoted by $f(n) \sim p(n)$ as $n \to \infty$

Definition 1.4

Let $f(n)$ be a function defined for all sufficiently large n and let $\phi_k(n)$ $(k = 0,1,2 \ldots)$ be a sequence of functions satisfying

$$\phi_{k+1}(n) = o\big(\phi_k(n)\big)(n \to \infty) \qquad \text{for each k}$$

(ϕ_k is called the gauge function). A (formal) series of the form

$$\sum_{k=0}^{\infty} a_k \phi_k(n)$$

is called an asymptotic series for a function $f(n)$ as $n \to \infty$ if for each k,

$$f(n) = \textstyle\sum_{k=0}^{m} a_k \phi_k(n) + o\big(\phi_m(n)\big)(n \to \infty)(1.0)$$

An equivalent property to 1.0 is

$$f(n) = \sum_{k=0}^{m-1} a_k \phi_k(n) + O\big(\phi_m(n)\big)$$

If this holds, we write

$$f(n) \sim \sum_{k=0}^{\infty} a_k \phi_k(n)\ (n \to \infty) \quad (1.1)$$

Equation 1.1 is called the asymptotic expansion of f with respect to $\{\phi_k\}$ as $n \to \infty$.

The symbols O and o called 'big O' and 'little o' respectively are known as Landau symbols. If f and g are two functions defined on some subset of the real numbers, one writes

$$f(n) = O\big(g(n)\big) \text{as} (n \to \infty)$$

If and only if $\exists\ M \in \mathbb{R}$ and $n_0 \in \mathbb{R}$ such that $|f(n)| \leq M|g(n)|, \quad \forall\ n \geq n_0$

And $f(n) = o\big(g(n)\big)$ if for every $\varepsilon > 0, \exists\ n_0$ such that $\frac{f(n)}{g(n)} < \varepsilon\ \ \forall\ n \geq n_0$

Thus,

$$h(n) \in o(p(n)) \Longrightarrow h(n) \in O(p(n))$$

1.4 Stirling Numbers

Stirling numbers plays an important role in analytic and combinatorics problems. They were introduced in the eighteenth century by James Stirling to whom they are named after. These numbers are classified into two kinds namely: Stirling numbers of the first kind and Stirling numbers of the second kinds (Toufik and Matthias, 2015).

1.4.1 Stirlingnumbers of the first kind

The Stirling numbers of the first kind are represented by
$S_n^k = s(n,k) = (-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}$. They are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k$$

$(x)_n$is the falling factorial given by

$$(x)_n = x(x-1)(x-2)\ldots(x-n+1), \quad (x)_0 = 1$$

Thus,

$$(x)_5 = x(x-1)(x-2)(x-3)(x-4)(x-5)$$

$$= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$$

Therefore,

$$s(5,0) = 0,\ \ s(5,1) = 24,\ \ s(5,2) = -50,\ \ s(5,3) = 35,\ \ s(5,4) = -10,$$

$$s(5,5) = 1$$

1.4.2 Stirling numbers of the second kind

The Stirling numbers for the second kind are given by

$$\sigma_n^k = s(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

with $\binom{k}{j}$ a binomial coefficient

They determine the number of ways to partition a set of n labeled objects into k non-empty unlabeled subsets.

Thus, $s(3,1) = 1, s(3,2) = 3, \quad s(3,3) = 1$

the value $S_n^k = \sigma_n^k = 0$ for $k > n$

1.5 Scope and Limitations

This work focuses on the complete asymptotic expansions of the Meyer-König and Zeller operators (Mayer-Konig and Zeller, 1960) and that of Chlodovsky operators (Chlodovsky, 1937).

CHAPTER 2

LITERATURE REVIEW

2.1 Results on Meyer-König and Zeller Operators and that of ChlodovskyOperators

In this a brief history of some results on Meyer-König and Zeller operators and that of Chlodovsky operators is presented.

Meyer-König and Zeller operators (Meyer-König and Zeller, 1960) in the slight modification of Cheney and Sharma (Cheney and Sharma, 1964) which associates to each function f defined on $[0,1]$, the power series

$$M_n(f;x) = \begin{cases} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k (1-x)^{n+1} & x \in [0,1) \\ f(1) & x = 1 \end{cases} \quad (2.0)$$

While the Chlodovsky operatorsC_n were introduced by Chlodovsky (1937) as a generalization of the Bernstein operators $B_n$on an infinite interval.

$$(C_n f)(x) := \begin{cases} \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), & 0 \le x \le b_n \\ f(x) & x > b_n \end{cases} \quad (2.1)$$

Where f is a function defined on $[0,\infty)$ and bounded on every finite interval$[0,b] \subset [0,\infty)$, with $P_{n,k}$ defined by

$$P_{n,k}(y) = \binom{n}{k} y^k (1-y)^{n-k} \quad , \ 0 \le y \le 1.$$

and$(b_n)_{n=1}^{\infty}$is a positive increasing sequence of real with the condition that

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

In approximation theory these operators are the focus of several investigations.

Alkemade (1984) succeeded in obtaining the second moment of Meyer-König and Zeller operators in terms of hypergeometric series.

$$(M_n e_2)(x) = x^2 + \frac{x(1-x)^2}{n+1} F_1(1,2; n+2: x) \quad (x \in [0,1) \qquad (2.2)$$

He also proved the asymptotic expansion

$$(M_n e_2)(x) - x^2 = \frac{x(1-x)^2}{n} + \frac{x(1-x)^2(2x-1)}{n^2} + O(n^{-3})(n \to \infty)(2.3)$$

Where the function e_2 is defined by $e_2: x \to x^2$. The result about the second moment for the Meyer-König and Zeller operators was extended by Ulrich (1995) to higher order moments, that is $M_n e_r$ $(r = 0,1,2, \dots)$ Where the function e_r is defined by $e_r: x \to x^r$.

He also went forward to derive the complete asymptotic expansion in the form $M_n(t^r; x) - x^r$ as $n \to \infty$ in the form

$$M_n(t^r; x) \sim x^r + \sum_{k=1}^{\infty} C_k^{[r]}(x) n^{-k} (n \to \infty)(2.4)$$

The coefficients $C_k^{[r]}$ $(k = 1,2,3, \dots, r \in \mathbb{N})$ are given in terms of Stirling numbers of the first and second kind

Ibikli and Karsli (2005) introduced a Chlodovsky type Durrmeyer operator as follows:

$$D_n: BV[0,\infty) \to \mathcal{P}$$

$$(D_n f)(x) = \frac{(n+1)}{b_n}\sum_{k=0}^{n} P_{k,n}\left(\frac{x}{b_n}\right)\int_0^{b_n} f(t)P_{k,n}\left(\frac{t}{b_n}\right)dt, \qquad 0 \le x \le b_n (2.5)$$

Where $\mathcal{P}: \{P: [0,\infty) \to \mathbb{R}\}$is a polynomial functions set, and $P_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is the Bernstein basis. The authors estimated the rate of convergence of the operator D_n , for functions of bounded variation on the interval $[0,\infty)$, by means of the technique of probability theory.

The rate of convergence of the Chlodovsky-Bernstein operators $(C_n f)(x)$ was estimated by Karsli and Ibikli (2007) for functions defined on the interval$[0, b_n]$, for $b_n \to \infty$, which are of bounded variation on $[0,\infty)$

Karsli (2008) define a new kind of operator Meyer-König and Zeller Durrmeyer operators (MKZD) for functions defined on $[0, b_n]$, named Chlodovsky-type MKZD operators as

$$(M_n^* f)(x) = \sum_{k=0}^{\infty}\frac{n+k}{b_n}M_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} f(t)b_{n,k}\left(\frac{t}{b_n}\right)dt, \qquad 0 \le x \le b_n (2.6)$$

Where

$$M_{n,k}(x) = \binom{n+k-1}{k}x^k(1-x)^n \text{and} b_{n,k}(t) = n\binom{n+k}{k}t^k(1-t)^{n-1}$$

The authors studied the behavior of the M_n^* operators for functions of bounded variation and estimate by means of techniques of probability theory the rate of convergence of the operators on the on the interval $[0, b_n]$, $(n \to \infty)$.

Psych-Taberska (2009) estimated the rate of convergence Chlodovsky- Kantorovich polynomials in classes of locally integrable functions. Namely,

If

$$f \in L_{loc}[0,\infty) \quad \text{and if} \quad \lim_{n\to\infty} \int_0^{b_n} |f(u)|du \exp\left(-\sigma \frac{n}{b_n}\right) = 0 \quad \text{foreach } \sigma > 0$$

Then

$\lim_{n\to\infty} (K_n f)(x) = f(x)$ almost every where on $[0,\infty)$

CHAPTER 3

MEYER-KÖNIG AND ZELLER OPERATORS

3.1 Complete Asymptotic Expansion for theMeyer-König and Zeller Operators

Meyer-König and Zeller operators (Meyer-König and Zeller, 1960) in the slight modification of Cheney and Sharma (Cheney and Sharma, 1964) which associates to each function f defined on $[0,1]$, the power series

$$M_n(f;x) = \begin{cases} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k (1-x)^{n+1} & x \in [0,1) \\ f(1) & x = 1 \end{cases} \quad (3.0)$$

Letting $B^*[0,1]$to be the set of all functions $f(t)$ which are bounded on $[0,1]$ and continuous to the left at $t = 1$, the operators $(M_n(n \in \mathbb{N}))$ are obviously seen to be linear operators define on $B^*[0,1]$ since it follows from definition 1.1 that

$$M_n(\alpha f + \beta g; x) = \begin{cases} \sum_{k=0}^{\infty} (\alpha f + \beta g)\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k (1-x)^{n+1}, & x \in [0,1) \\ (\alpha f + \beta g)(1) & x = 1 \end{cases}$$

$$= \begin{cases} \sum_{k=0}^{\infty} (\alpha f)\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k (1-x)^{n+1} \\ (\alpha f)(1) \end{cases}$$

$$+\begin{cases}\sum_{k=0}^{\infty}(\beta g)\left(\frac{k}{k+n}\right)\binom{k+n}{k}x^k(1-x)^{n+1}, & x\in[0,1)\\ (\beta g)(1) & x=1\end{cases}$$

$$=\alpha\begin{cases}\sum_{k=0}^{\infty}f\left(\frac{k}{k+n}\right)\binom{k+n}{k}x^k(1-x)^{n+1}\\ (f)(1)\end{cases}$$

$$+\beta\begin{cases}\sum_{k=0}^{\infty}g\left(\frac{k}{k+n}\right)\binom{k+n}{k}x^k(1-x)^{n+1}, & x\in[0,1)\\ (g)(1) & x=1\end{cases}$$

$$=\alpha M_n(f;x)\ +\beta M_n(g;x).$$

for all f and g define on $[0,1]$, $\alpha,\beta\epsilon\mathbb{R}$.

To show that the operators $M_n(n\in\mathbb{N})$ are positive operators, definition 1.2 is applied on $M_n(n\in\mathbb{N})$ as follows:

Let f and g be functions define on $[0,1]$ such that

$f(x)\geq g(x),\ \ x\epsilon[0,1]$, then this implies

$$\begin{cases}f\left(\frac{k}{k+n}\right)\binom{k+n}{k}x^k(1-x)^{n-k}, & x\in[0,1)\\ (f)(1) & ,\quad x=1\end{cases},\qquad k=0,1,2,\dots$$

$$\geq\begin{cases}g\left(\frac{k}{k+n}\right)\binom{k+n}{k}x^k(1-x)^{n-k}, & x\in[0,1)\\ (g)(1) & ,\quad x=1\end{cases},\qquad k=0,1,2,\dots$$

$$\Rightarrow \begin{cases} \sum_{k}^{\infty} f\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k(1-x)^{n-k} \ , & x \in [0,1) \\ (f)(1) \qquad\qquad , & x = 1 \end{cases}$$

$$\geq \begin{cases} \sum_{k}^{\infty} g\left(\frac{k}{k+n}\right)\binom{k+n}{k} x^k(1-x)^{n-k} \ , & x \in [0,1) \\ (g)(1) \qquad\qquad , & x = 1 \end{cases}$$

$\Rightarrow \quad M_n(f;x) \geq M_n(g;x).$

Thus, M_n are positive operators.

The work of Ulrich (1997) on the complete asymptotic expansion for the operators M_n in the form

$$M_n(f(t);x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f;x)n^{-k}, \qquad (n \to \infty). \qquad (3.1)$$

Provided f possesses derivatives of sufficiently high order at $x \in [0,1]$is studied in detail in this chapter. The latter formula means that

$$M_n(f(t);x) \sim f(x) + \sum_{k=1}^{q} a_k(f;x)n^{-k} + o(n^{-q}), \quad (n \to \infty). \qquad (3.2)$$

for every positive integer q. Where $a_k(f;x)$ $(k \in \mathbb{N})$ are coefficients.

Let $K^q(x)$ be the class of all functions $f(t) \in B^*[0,1]$ which are q times differentiable at $x \in [0,1]$. The following general approximation theorem proved by Sikkema (1970) will be useful in establishingEquation3.1

Theorem 3.0

For even $q \geq 2$ and fixed $x \in [0,1]$ let $L_n: K^q(x) \to C[0,1]$be a sequence of positive linear operators. If

$$L_n((t-x)^p; x) = O\left(n^{-\left[\frac{p+1}{2}\right]}\right)(n \to \infty)(p = 0,1, \dots, q+2))(3.3)$$

then for each $f \in K^q(x)$

$$L_n(f(t); x) = \sum_{p=0}^{q} \frac{1}{p!} L_n((t-x)^p; x) f^{(p)}(x) + o\left(n^{-\frac{q}{2}}\right), \quad (n \to \infty)(3.4)$$

Furthermore,

if $f \in K^{(q+2)}(x)$,

the term $o(n^{-q/2})$, in (3.14) can be replaced by $O(n^{-(q/2+1)})$,

Also the complete asymptotic expansion for the moment $M_n(t^r; x)(r \in \mathbb{N})$ by Ulrich (1995) given below plays an important role in establishing 3.1

Theorem 3.1

For the function $f(t) = t^r (r \in \mathbb{N})$ we have for every $x \in [0,1]$ the asymptotic expansion

$$M_n(t^r;x) \sim x^r + \sum_{k=1}^{\infty} C_k^{[r]}(x) n^{-k} (n \to \infty) \qquad (3.5)$$

The coefficients are given by

$$C_k^{[r]}(x) = \sum_{j=1}^{r} \binom{r}{j} (-1)^j H(j-1, k+j-1, x), \qquad (3.6)$$

Where $H(j,m,x)$is define as

$$H(j,m,x) = \sum_{i=j}^{m} S_i^j \sigma_m^i (1-x)^{i+1} \, (0 \le j \le m). \qquad (3.17)$$

The quantities S_j^i and σ_j^idenote the Stirling numbers of the first and second kind respectively presented in chapter 1.

$x^{(j)} = \sum_{i=0}^{j} S_j^i x^i$ and$x^j = \sum_{i=0}^{j} \sigma_j^i x^i \, (j \in \mathbb{N})$.

$$x^{(j)} = x(x-1) \ldots (x-j+1)$$

is the falling factorial.

The following steps are followed to arrive at the result in Equation3.1

Step 1:Theorem 3.0 is applied on the operator M_n.

Step 2: simplification of the result established in step 1 to arrive at Equation3.1

To apply Theorem 3.0 on M_n the condition

$$M_n((t-x)^p;x) = O\left(n^{-\left[\frac{p+1}{2}\right]}\right)(n\to\infty)(p=0,1,\dots,q+2)$$

must be satisfied, and to establish it, the procedure is as follows:

$$M_n((t-x)^p;x) = M_n\left(\sum_{r=0}^{p}\binom{p}{r}t^r(-x)^{p-r};x\right) \quad \text{(by application of binomial theorem)}$$

$$=\sum_{r=0}^{p}\binom{p}{r}(-x)^{p-r}M_n(t^r;x)$$

$$\sim\sum_{r=0}^{p}\binom{p}{r}(-x)^{p-r}\left(x^r+\sum_{k=1}^{\infty}C_k^{[r]}(x)n^{-k}\right) \qquad (n\to\infty)$$

$$=\sum_{r=0}^{p}\binom{p}{r}(-x)^{p-r}x^r+\sum_{r=0}^{p}\binom{p}{r}(-x)^{p-r}\sum_{k=1}^{\infty}C_k^{[r]}(x)n^{-k} \qquad (n\to\infty)$$

$$=(x-x)^p+\sum_{r=0}^{p}\binom{p}{r}(-x)^{p-r}\sum_{k=1}^{\infty}C_k^{[r]}(x)n^{-k} \qquad (n\to\infty)$$

$$=\sum_{k=1}^{\infty}n^{-k}\sum_{r=1}^{p}\binom{p}{r}(-x)^{p-r}C_k^{[r]}(x) \qquad (n\to\infty)$$

$$= \sum_{k=1}^{\infty} n^{-k} \sum_{r=1}^{p} \binom{p}{r} (-x)^{p-r} \sum_{j=1}^{r} \binom{r}{j} (-1)^j H(j-1, k+j-1, x) \quad (n \to \infty)$$

$$= \sum_{k=1}^{\infty} n^{-k} \sum_{r=1}^{p} \binom{p}{r} (-x)^{p-r} \sum_{j=1}^{r} \binom{r}{j} (-1)^j \sum_{i=j-1}^{k+j-1} S_i^{j-1} \sigma_{k+j-1}^{i} (1-x)^{i+1}$$

$$(n \to \infty)$$

$$= \sum_{k=1}^{\infty} n^{-k} \sum_{j=1}^{p} \binom{p}{j} (-1)^j \sum_{r=j}^{p} \binom{p-j}{r-j} (-x)^{p-r} \sum_{i=0}^{k} S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1} (1-x)^{i+j}$$

$$(n \to \infty)$$

Where $\binom{p}{r}\binom{r}{j} = \binom{p}{j}\binom{p-j}{r-j}$

$$= \sum_{k=1}^{\infty} n^{-k} \sum_{j=1}^{p} \binom{p}{j} (-1)^j \sum_{r=0}^{p-j} \binom{p-j}{r} (-x)^{p-j-r} \sum_{i=0}^{k} S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1} (1-x)^{i+j}$$

$$(n \to \infty)$$

$$= \sum_{k=1}^{\infty} n^{-k} \sum_{j=1}^{p} \binom{p}{j} (-1)^j (1-x)^{p-j+j} \sum_{i=0}^{k} S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1} (1-x)^{i} \quad (n \to \infty)$$

$$= \sum_{k=1}^{\infty} n^{-k} (1-x)^p \sum_{i=0}^{k} (1-x)^i \sum_{j=1}^{p} \binom{p}{j} (-1)^j S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1} \quad (n \to \infty)$$

Letting

$$S(p,k,i) = \sum_{j=1}^{p} \binom{p}{j} (-1)^j S_{i+j-1}^{j-1} \sigma_{k+j-1}^{i+j-1}$$

leads to the following Lemma

LEMMA 3.0

For every positive integer p and fixed $x \in [0,1]$ the asymptotic expansion holds

$$M_n((t-x)^p; x) \sim (1-x)^p \sum_{k=1}^{\infty} n^{-k} \sum_{i=0}^{k} (1-x)^i S(p,k,i) \quad (3.8)$$

Observations:

At $p = 1$, $S(1,k,i) = (-1)^1 S_i^0 \sigma_k^i = (-1)^1 (0) \sigma_k^i = 0.$

for $i = 0,1,\ldots,k$ and $k \in \mathbb{N}$

Some properties of Stirling numbers

$$S_n^{n-k} = C_{k,0}\binom{n}{2k} + \cdots + C_{k,k-1}\binom{n}{k+1} \quad (3.9)$$

$$\sigma_n^{n-k} = \bar{C}_{k,0}\binom{n}{2k} + \cdots + \bar{C}_{k,k-1}\binom{n}{k+1} \quad (3.10)$$

Where

$$\begin{cases} C_{k,0} = (-1)^k \bar{C}_{k,0} \\ \bar{C}_{k,0} = 1 \cdot 3 \cdot 5 \cdots (2k-1) \end{cases} \quad (3.11)$$

For Equation3.9and Equation 3.10 the coefficients $C_{k,l}$ and $\bar{C}_{k,l}$ are independent of n and they satisfy certain differential equations whose general solutions are unknown (Jordan, 1950). Equation 3.12 and Equation 3.13 were obtained by Reif (Ulrich, 1997)

$$C_{k,1} = \frac{k-1}{3} C_{k,0} C_{k,2} = \frac{(k-1)(k-2)(4k-3)}{9(2k-1)} C_{k,0} (3.12)$$

$$\bar{C}_{k,1} = \frac{k-1}{3} \bar{C}_{k,0} \bar{C}_{k,2} = \frac{(k-1)(k-2)(2k-3)}{18(2k-1)} \bar{C}_{k,0} (3.13)$$

the above properties Equation3.9 and Equation3.10 leads to

$$S_{i+j-1}^{j-1} = C_{i,0} \binom{j+i-1}{2i} + \cdots + C_{i,i-1} \binom{j+i-1}{i+1} \qquad (i,j = 1,2,\ldots) \qquad (3.14)$$

$$\sigma_{k+j-1}^{i+j-1} = \bar{C}_{k-i,0} \binom{j+k-1}{2(k-i)} + \cdots + \bar{C}_{k-i,k-i-1} \binom{j+k-1}{k-i+1} \qquad (3.15)$$

$$(i = 0,1,\ldots k-1;\ j,k = 1,2,\ldots)$$

The well-known expression below (which appear in chapter 1)

$$\sigma_n^p = \frac{(-1)^p}{p!} \sum_{j=o}^{p} \binom{p}{j} (-1)^j j^n.$$

Leads to, for all $p \geq 2$

$$\sum_{j=o}^{p} \binom{p}{j} (-1)^j j^n. = (-1)^j p! \, \sigma_n^p = 0, \qquad (n = 1, \dots, p-1)(3.16)$$

Now for $i = 0$

$$S(p,k,0) = \sum_{j=1}^{p} \binom{p}{j} (-1)^j S_{j-1}^{j-1} \sigma_{k+j-1}^{j-1} = \sum_{j=1}^{p} \binom{p}{j} (-1)^j \sigma_{k+j-1}^{j-1}, \qquad (3.17)$$

$$(k = 1,2, \dots)$$

Where for fixed k the Stirling number σ_{k+j-1}^{j-1} is a polynomial in j of degree $2k$ without constant summand. For the case $1 \leq i \leq k$.TheS_{i+j-1}^{j-1} is a polynomial in j of degree $2i$ without constant summand and σ_{k+j-1}^{i+j-1} is a polynomial in j of degree $2(k-i)$

Thus 3.16 leads to

$$S(p,k,i) = 0 \qquad (i = 0, \dots, k)\text{for}2k < p. \; (3.18)$$

For $p = 2k$ and $p = 2k - 1$

Reif (Ulrich, 1997) established that

$$S(2k,k,i) = (-1)^i \bar{C}_{k,0} \binom{k}{i} (i = 0, \dots, k)$$

$$S(2k-1,k,i) = (-1)^{i+1} \bar{C}_{k,1} \binom{k}{i} \left\{ 4 \left(\binom{k-1}{i} \right) + 5 \binom{k-1}{i-1} \right\} ,$$

$$(i = 0, \dots, k; k = 1,2, \dots)$$

Where $\binom{k-1}{i-1}$ is to be read as 0for $i = 0$.

Thus $M_n((t-x)^{2k};x) \sim (1-x)^{2k} n^{-k} \sum_{i=0}^{k} (1-x)^i \, S(2k,k,i)$

$$=(1-x)^{2k} n^{-k} \sum_{i=0}^{k} (1-x)^i \, (-1)^i \bar{C}_{k,0} \binom{k}{i}$$

$$=\bar{C}_{k,0} (1-x)^{2k} n^{-k} \sum_{i=0}^{k} \binom{k}{i} (x-1)^i$$

$$= \bar{C}_{k,0} (1-x)^{2k} x^k n^{-k}$$

Hence,

$$M_n((t-x)^{2k};x) = \bar{C}_{k,0} (1-x)^{2k} x^k n^{-k} + O\left(n^{-(k+1)}\right), \qquad (3.19)$$

Also,

$$M_n((t-x)^{2k-1};x) \sim (1-x)^{2k-1} n^{-k} \sum_{i=0}^{k} (1-x)^i \, S(2k-1,k,i)$$

$$= (1-x)^{2k-1} n^{-k} \sum_{i=0}^{k} (1-x)^i \, (-1)^{i+1} \bar{C}_{k,1} \left\{ 4\left(\binom{k-1}{i}\right) + 5\binom{k-1}{i-1} \right\}$$

$$= \bar{C}_{k,1} (1-x)^{2k-1} n^{-k} \left[-4 \sum_{i=0}^{k} \binom{k-1}{i} (x-1)^i - 5 \sum_{i=0}^{k} \binom{k-1}{i-1} (x-1)^i \right]$$

$$= \bar{C}_{k,1} (1-x)^{2k-1} n^{-k} \left[-4 \sum_{i=0}^{k-1} \binom{k-1}{i} (x-1)^i - 5 \sum_{i=0}^{k-1} \binom{k-1}{i} (x-1)^{i+1} \right]$$

$$= \bar{C}_{k,1}(1-x)^{2k-1}n^{-k}\left[-4\sum_{i=0}^{k-1}\binom{k-1}{i}(x-1)^i - 5(x-1)\sum_{i=0}^{k-1}\binom{k-1}{i}(x-1)^i\right]$$

$$= \bar{C}_{k,1}(1-x)^{2k-1}n^{-k}[-4(x)^{k-1} - 5(x-1)(x)^{k-1}]$$

$$= \bar{C}_{k,1}(1-x)^{2k-1}x^{k-1}(1-5x)n^{-k}$$

Thus,

$$M_n((t-x)^{2k-1};x) = \bar{C}_{k,1}(1-x)^{2k-1}x^{k-1}(1-5x)n^{-k} + O\left(n^{-(k+1)}\right)(3.20)$$

Hence the corollary follows

Corollary 3.0 For all integer p and each $x \in [0,1]$ it holds

$$M_n((t-x)^p;x) = O\left(n^{-\left[\frac{p+1}{2}\right]}\right)(n \to \infty).$$

Now the condition of Theorem 3.0 is satisfied thus, the Theorem below follows

Theorem 3.2

For $q \geq 2$ even, $x \in [0,1]$ and $f \in K^{(q)}(x)$ the asymptotic relation follows

$$M_n(f(t);x) = \sum_{p=0}^{q}\frac{1}{p!}M_n((t-x)^p;x)f^p(x) + o(n^{-q/2}), \qquad (n \to \infty)$$

$$= f(x) + \sum_{p=2}^{q} \frac{1}{p!} f^p(x)(1-x)^p \sum_{k=\left[\frac{p+1}{2}\right]}^{\frac{q}{2}} n^{-k} \sum_{i=0}^{k} (1-x)^i S(p,k,i) \ + o\left(n^{-\frac{q}{2}}\right)$$

$$(n \to \infty)$$

$$= f(x) + \sum_{p=2}^{q} \frac{1}{p!} f^p(x)(1-x)^p b(n,p,q;x) \ + o\left(n^{-\frac{q}{2}}\right), \quad (n \to \infty)(3.21)$$

Where

$b(n,p,q;x) = \sum_{k=\left[\frac{p+1}{2}\right]}^{\frac{q}{2}} n^{-k} \sum_{i=0}^{k} (1-x)^i S(p,k,i).$

Furthermore,

if $f \in K^{(q+2)}(x)$, the term $o\left(n^{-\frac{q}{2}}\right)$ in 3.21 can be replaced by $O\left(n^{-q/(2+1)}\right)$,

Letting $K^{(\infty)}(x) = \cap_{q\in\mathbb{N}} K^{(q)}(x)$ be the class of all functions $f(t) \in B^*[0,1]$ which are infinitely often differentiable at $\in [0,1]$. By reformulating Theorem 3 leads to the complete asymptotic expansion for the operators M_n

Theorem 3.3: let $x \in [0,1]$and $f \in K^{(\infty)}(x)$. Then

$$M_n(f(t);x) \sim f(x) + \sum_{k=1}^{\infty} n^{-k} \sum_{p=2}^{2k} \frac{f^p(x)}{p!} (1-x)^p \sum_{i=0}^{k} (1-x)^i S(p,k,i)$$

$$= f(x) + \sum_{k=1}^{\infty} a_k(f;x)n^{-k} \quad , \qquad (k \in \mathbb{N})$$

Where

$$a_k(f;x) = \sum_{p=2}^{2k} \frac{f^p(x)}{p!}(1-x)^p \sum_{i=0}^{k}(1-x)^i S(p,k,i)$$

Establishing Equation 3.1 which gives the complete asymptotic expansion for the operators M_n.

CHAPTER 4

CHLODOVSKY OPERATORS

4.1 Complete Asymptotic Expansion for the Chlodovsky Operators

Chlodovsky operators C_n were introduced by Chlodovsky (1937) as a generalization of the Bernstein operators B_n(Vijay and Ravi, 2014) on an infinite interval.

$$(C_n f)(x) := \begin{cases} \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), & 0 \le x \le b_n \\ f(x) & x > b_n \end{cases} \qquad (4.0)$$

Where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b] \subset [0,\infty)$, with $P_{n,k}$ defined by

$$P_{n,k}(y) = \binom{n}{k} y^k (1-y)^{n-k} \quad , \ 0 \le y \le 1.$$

and $(b_n)_{n=1}^{\infty}$ is a positive increasing sequence of real with the condition that

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

In addition if $M(b;f) := sup_{0\le x\le b}|f(x)|$,then Chlodovsky established that if

$$\lim_{n\to\infty} \exp(-\alpha \frac{n}{b_n}) M(b;f) = 0 \tag{4.1}$$

For every $\alpha > 0$, then $(C_n f)(x)$ converges to $f(x)$ at each point of continuity of f.

The linearity of these operators is established as follows:

$$C_n(\alpha f + \beta g)(x): = \begin{cases} \sum_{k=0}^{n} (\alpha f + \beta g)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), & 0 \le x \le b_n \\ (\alpha f + \beta g)(x) & x > b_n \end{cases}$$

$$= \begin{cases} \sum_{k=0}^{n} (\alpha f)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right) \\ (\alpha f)(x) \end{cases} + \begin{cases} \sum_{k=0}^{n} (\beta g)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), & 0 \le x \le b_n \\ (\beta g)(x) & , x > b_n \end{cases}$$

$$= \alpha \begin{cases} \sum_{k=0}^{n} (f)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right) \\ (f)(x) \end{cases} + \beta \begin{cases} \sum_{k=0}^{n} (g)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right) & , \quad 0 \le x \le b_n \\ (g)(x) & , x > b_n \end{cases}$$

$$= \alpha C_n(f)(x) + \beta C_n(g)(x).$$

For all f and g defined on $[0, \infty)$ and $\alpha, \beta \epsilon \mathbb{R}$.

To show that the operators $C_n (n \in \mathbb{N})$are positive operators, definition 1.2 is applied on $C_n (n \in \mathbb{N})$as follows:

Let f and g be functions define on $[0, \infty)$ such that

$f(x) \ge g(x), \quad x \epsilon [0, \infty)$, then this implies

$$\begin{cases} f\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right) \\ (f)(x) & , \end{cases} \ge \begin{cases} g\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), 0 \le x \le b_n \\ (g)(x) \qquad x > b_n \end{cases} \quad k = 0,1,2, \ldots . n$$

$$\Rightarrow \begin{cases} \sum_{k=0}^{n} (f)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right) \\ (g)(x) \end{cases} \ge \begin{cases} \sum_{k=0}^{n} (g)\left(\frac{b_n}{n}k\right) P_{n,k}\left(\frac{x}{b_n}\right), & 0 \le x \le b_n \\ (g)(x) & , \quad x \ge b_n \end{cases}$$

$$\Rightarrow \quad C_n(f)(x) \ge C_n(g)(x).$$

Thus, C_n are positive operators.

The work of Karsli (2013) on the complete asymptotic expansion for the operators C_n in the form

$$(C_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} C_k(f; n, x) n^{-k}, \qquad (n \to \infty) \,. (4.2)$$

Where f defined on $[0, \infty)$, satisfying condition (4.11) (for every $\alpha > 0$) and f has derivatives of sufficiently high order at x ($x \in (0, \infty)$ exists is studied.

$C_k(f; n, x)$ are in terms of the Stirling numbers of the first and second kind.

The establishment of (4.12) follows as a corollary of the Theorem below which appeared in the work of Karsli (2013)

Theorem 4.0

If f defined on $[0, \infty)$ the closed interval satisfies condition (4.11) for every $\alpha > 0$ and if $f^{2s}(x)$ exists at a given $x \geq 0$, then the Chlodovsky operators satisfy the asymptotic relation

$$(C_n f)(x) = f(x) + \sum_{k=1}^{s} C_k(f; n, x) n^{-k} + o\left(\frac{n}{b_n}\right)^{-s}, \qquad (n \to \infty) \qquad (4.3)$$

The coefficients $C_k(f; n, x)$ are given in terms of Stirling numbers of the first and second kind.

There are given by

$$C_k(f; n, x) = \sum_{m=k+1}^{2k} b_n^m \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k)$$

Furthermore,

$$\lim_{n\to\infty}\left(\frac{n}{b_n}\right)^s [\ (C_nf)(x) - f(x) - \sum_{k=1}^{2s-1}\frac{f^{(k)}(x)}{k!}\ (C_n(t-x)^k)(x)]$$

$$= \lim_{n\to\infty}\left(\frac{n}{b_n}\right)^s [\ (C_nf)(x) - f(x) - \sum_{k=1}^{2s-1}\frac{f^{(k)}(x)}{k!} T_{n,k}(x)n^{-k}]$$

$$\equiv \lim_{n\to\infty}\left(\frac{n}{b_n}\right)^s [R_{n,2s}(x) + \frac{f^{(2s)}(x)}{(2s)!} T_{n,2s}(x)n^{-2s}]$$

$$= \frac{x^2}{2^2 s!} f^{(2s)}(x) \qquad (4.4)$$

$where\ T_{n,k}(x) = (C_n(t-x)^k)(x)n^k$

To be able to establish Theorem 4.0, Stirling numbers of the first and second kind defined which appears in chapter 1,

$[x]_j = \sum_{i=1}^{j} S(j,i)x^i$ and$x^j = \sum_{i=1}^{j} \sigma(j,i)[x]_i$, $j \in \mathbb{N}_0$

plays an important role

Where $[x]_p = x(x-1)(x-2)\ldots(x-p+1), [x]_0 = 1, \qquad x \in \mathbb{R}$

is the falling difference polynomial.

Observations:

$$S(0,0) = \sigma(0,0) = 1$$

$$S(j,0) = \sigma(j,0) = 0, \qquad \forall\, j \in \mathbb{N}_0$$

$$\frac{[n]_j}{n^j} = \sum_{k=0}^{j} S(j,i) n^{k-j}$$

Also the following results are required:

Lemma 4.0

For $(C_n t^s)(x), \quad s = 0,1,2$ one has for $0 \le x \le b_n$.

$$(C_n 1)(x) = 1, \qquad (C_n t)(x) = x, \qquad (C_n t^2)(x) = x^2 + \frac{x(b_n - x)}{n}$$

Thus,

$$(C_n(t-x))(x) = 0\,, \qquad (C_n(t-x)^2)(x) = \frac{x(b_n - x)}{n}$$

Butzer-Karsli (2009) gave the proof of Lemma 4.1.

Lemma 4.1: The central moment of order $m \in \mathbb{N}_0$ any fixed $x \in [0,\infty)$.

$$T^*_{n,m}(x) := \sum_{k=0}^{n} \left(\frac{b_n}{n} k - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right),$$

Satisfy the inequality

$$\left|T_{n,m}^{*}(x)\right| \leq A_m(x)\frac{x(b_n - x)}{b_n}\left(\frac{b_n}{n}\right)^{[\frac{m+1}{2}]}, (n \in \mathbb{N},\ n > b_n),$$

Where $A_m(x)$ denotes a polynomial in x, of degree $\left[\frac{m}{2}\right] - 1$, with non negative coefficient independent of n or

$$\left|T_{n,m}^{*}(x)\right| \leq P_m(x)\left(\frac{b_n}{n}\right)^{[\frac{m+1}{2}]}, (n \in \mathbb{N},\ n > b_n),$$

Where $P_m(x)$ denotes a polynomial in x, of degree $\left[\frac{m}{2}\right]$, with non-negative coefficient independent of n , and $[a]$denotes the integral part of a

The first part of the Lemma below is due to Chlodovsky (1937)

Lemma 4.2: For $t \in [0,1]$ the inequality

$$0 \leq z \leq \tfrac{3}{2}\sqrt{nt(1-t)} \quad ,$$

implies

$$\sum_{|k-nt|\geq 2z\sqrt{nt(1-t)}} P_{n,k}(t) \leq 2\exp(-z^2),$$

In particular, for $0 \leq \delta \leq x < b_n$ and sufficiently large n

$$\sum_{1}^{*} := \sum_{\left|\frac{b_n}{n}-x\right|\geq\delta} P_{n,k}(\frac{x}{b_n}) \leq 2\exp\left(-\frac{\delta^2}{4x}\frac{n}{b_n}\right) (4.5)$$

The proof of 4.5 was given by Albrycht and Radecki (1960).

Also for $r, n \in \mathbb{N}, \quad r \leq n,$ one has

$$(C_n t^r)(x) = \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{kb_n}{n}\right)^r$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) k^r$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \sum_{v=0}^{r} \sigma(r,v)[k]_v$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x}{b_n}\right)^v \left(1-\frac{x}{b_n}\right)^{n-k} \sum_{v=0}^{r} b_n^v \sigma(r,v)\frac{d^v}{dx^v}\left(\frac{x}{b_n}\right)^k$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{v=0}^{r} b_n^v \sigma(r,v)\left(\frac{x}{b_n}\right)^v \frac{[n]_v}{b_n^v}$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{v=0}^{r} \sigma(r,v)\left(\frac{x}{b_n}\right)^v [n]_v$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{v=0}^{r} \sigma(r,v) \left(\frac{x}{b_n}\right)^v \sum_{k=0}^{v} S(v,k) n^k$$

$$= \left(\frac{b_n}{n}\right)^r \sum_{v=0}^{r} \sum_{k=0}^{v} \sigma(r,v) S(v,k) \left(\frac{x}{b_n}\right)^v n^k$$

$$= \sum_{k=0}^{r} n^k \left(\frac{b_n}{n}\right)^r \sum_{v=k}^{r} \sigma(r,v) S(v,k) \left(\frac{x}{b_n}\right)^v$$

$$= b_n^r \sum_{k=0}^{r} n^{k-r} \sum_{v=k}^{r} \sigma(r,v) S(v,k) \left(\frac{x}{b_n}\right)^v$$

$$= b_n^r \sum_{k=0}^{r} n^{-k} \sum_{v=r-k}^{r} \sigma(r,v) S(v,r-k) \left(\frac{x}{b_n}\right)^v$$

Thus, the Lemma below follows

Lemma 4.3: For $r, n \in \mathbb{N}, \quad r \leq n,$ one has

$$(C_n t^r)(x) = b_n^r \sum_{k=0}^{r} n^{-k} \sum_{v=r-k}^{r} \sigma(r,v) S(v,r-k) \left(\frac{x}{b_n}\right)^v$$

From the above lemma follows; for $m, n \in \mathbb{N}$,

$$(C_n(t-x)^m)(x) = (C_n(\sum_{p=0}^{m}\binom{m}{p}t^p(-x)^{m-p}))(x)$$

(by applying Binomial Theorem)

$$= \sum_{p=0}^{m}\binom{m}{p}(-x)^{m-p}(C_n t^p)(x)$$

$$= \sum_{p=0}^{m}\binom{m}{p}(-x)^{m-p}\left[b_n^p\sum_{k=0}^{p}n^{-k}\sum_{i=p-k}^{p}\sigma(p,i)S(i,p-k)\left(\frac{x}{b_n}\right)^i\right]$$

$$= \sum_{k=0}^{m}n^{-k}\sum_{p=k}^{m}\binom{m}{p}b_n^p(-1)^{m-p}x^{m-p}\sum_{i=0}^{k}\sigma(p,p-i)S(p-i,p-k)\left(\frac{x}{b_n}\right)^{p-i}$$

$$= \sum_{k=0}^{m}n^{-k}\sum_{p=k}^{m}\binom{m}{p}b_n^m(-1)^{m-p}\sum_{i=0}^{k}\sigma(p,p-i)S(p-i,p-k)\left(\frac{x}{b_n}\right)^{m-i}$$

Thus, Lemma below follows.

Lemma 4.4: For $m,n \in \mathbb{N}$,

$$(C_n(t-x)^m)(x)$$

$$= \sum_{k=0}^{m}n^{-k}\sum_{p=k}^{m}\binom{m}{p}b_n^m(-1)^{m-p}\sum_{i=0}^{k}\sigma(p,p-i)S(p-i,p-k)\left(\frac{x}{b_n}\right)^{m-i}$$

All tools for proving Theorem 4.0 are now in place; hence, the proof goes as follows.

Proof of Theorem 4.0

Case 1:

For $x = 0$, (4.13) is valid since by its hypothesis $f^{(2s)}$exists since $(C_n)(0) = f(0)$

Case 2:

$x > 0$,

$$(C_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) f\left(\frac{kb_n}{n}\right) \text{ (4.6)}$$

Taylor's theorem gives

$$f\left(\frac{kb_n}{n}\right) = \sum_{i=0}^{2m} \frac{f^{(i)}(x)}{i!}\left(\frac{kb_n}{n} - x\right)^{i} + \left(\frac{kb_n}{n} - x\right)^{2m} h\left(\frac{kb_n}{n} - x\right)$$

Where $h(y)$ converges to zero with y, into the representation 4.6,

Thus,

$$(C_n f)(x) = \sum_{k=0}^{n}\left[\sum_{m=0}^{2s} \frac{f^{(m)}(x)}{m!}\left(\frac{kb_n}{n} - x\right)^{m} + \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right)\right] P_{n,k}\left(\frac{x}{b_n}\right)$$

$$= \sum_{k=0}^{n}\left[f(x) + \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!}\left(\frac{kb_n}{n} - x\right)^{m} + \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right)\right] P_{n,k}\left(\frac{x}{b_n}\right)$$

$$= f(x) + \sum_{k=0}^{n} \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!} \left(\frac{kb_n}{n} - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right) + R_{n,2s}(x)$$

Where $R_{n,2s}(x) \coloneqq \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)$

At this point, the following is put into consideration.

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h(y)| < \varepsilon$ for $|y| \leq \delta$.

Choosing δ so small that $\delta \leq x$. This allows the sum $R_{n,2s}(x)$ to be split into two parts as follows:

$$R_{n,2s}(x) \coloneqq \sum_{\left|\frac{kb_n}{n} - x\right| < \delta} \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)$$

$$+ \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)$$

$$\coloneqq R_{n,2s,1}(x) + R_{n,2s,2}(x)$$

Observations:

$$R_{n,2s,1}(x) = \sum_{\left|\frac{kb_n}{n} - x\right| < \delta} \left(\frac{kb_n}{n} - x\right)^{2s} h\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)$$

$$\leq \varepsilon \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{kb_n}{n}-x\right)^{2s} P_{n,k}\left(\frac{x}{b_n}\right)$$

$$< \varepsilon T^*_{n,2s}$$

$$\leq \varepsilon P_{2s}(x)\left(\frac{b_n}{n}\right)^{s}$$

$$\Rightarrow R_{n,2s,1}(x) = o\left(\frac{n}{b_n}\right)^{-s}$$

$$\left|R_{n,2s,2}(x)\right| = \left|\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{kb_n}{n}-x\right)^{2s} h\left(\frac{kb_n}{n}-x\right) P_{n,k}\left(\frac{x}{b_n}\right)\right|$$

$$= \left|\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left[f\left(\frac{kb_n}{n}\right) - \sum_{m=0}^{2s} \frac{f^{(m)}(x)}{m!}\left(\frac{kb_n}{n}-x\right)^{m}\right] P_{n,k}\left(\frac{x}{b_n}\right)\right|$$

$$\leq \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|f\left(\frac{kb_n}{n}\right)\right| P_{n,k}\left(\frac{x}{b_n}\right) + |f(x)| \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} P_{n,k}\left(\frac{x}{b_n}\right)$$

$$+|f'(x)| \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right| P_{n,k}\left(\frac{x}{b_n}\right)$$

$$+\frac{|f''(x)|}{2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right|^2 P_{n,k}\left(\frac{x}{b_n}\right)$$

$$+\frac{|f'''(x)|}{6} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right|^3 P_{n,k}\left(\frac{x}{b_n}\right)$$

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$$+\frac{\left|f^{(2s)}(x)\right|}{(2s)!} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right|^{2s} P_{n,k}\left(\frac{x}{b_n}\right)$$

$$\coloneqq \sum_1^*(n) + \sum_2^*(n) + \cdots + \sum_{2s}^*(n)$$

By Cauchy- Schwartz inequality

$$\sum_1^*(n) = \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|f\left(\frac{kb_n}{n}\right)\right| P_{n,k}\left(\frac{x}{b_n}\right)$$

$$\leq \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \frac{\left|\frac{kb_n}{n}-x\right|^{2s}}{\delta^{2s}} \left\{\left|f\left(\frac{kb_n}{n}\right)\right| \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$= \frac{1}{\delta^{2s}} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right|^{2s} \left\{\left|f\left(\frac{kb_n}{n}\right)\right| \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq \frac{1}{\delta^{2s}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|f\left(\frac{kb_n}{n}\right)\right|^2 \left|\frac{kb_n}{n}-x\right|^{4s} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}}$$

but $= \sqrt{sup_{0\leq x\leq a}|f(x)|^2} = M(a;f)$,

$$\leq \frac{M(b_n;f)}{\delta^{2s}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n}-x\right|^{4s} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\leq \frac{M(b_n;f)}{\delta^{2s}} \left\{T^*_{n,4s}(x)\right\}^{\frac{1}{2}} \left\{2\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\Rightarrow \sum_{1}^{*}(n) = o\left(\frac{n}{b_n}\right)^{-s}$$

By the same approach

$$\sum_{2}^{*}(n) = o\left(\frac{n}{b_n}\right)^{-s}$$

For $i = 3$

$$\sum_{3}^{*}(n) = |f'(x)| \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right| \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq |f'(x)| \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{\left|\frac{kb_n}{n} - x\right|}{\delta}\right)^{2s-1} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right| \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq |f'(x)| \frac{1}{\delta^{2s-1}} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right|^{2s} \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq |f'(x)| \frac{1}{\delta^{2s-1}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n} - x\right|^{4s} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\leq |f'(x)| \frac{1}{\delta^{2s-1}} \{T_{n,4s}^{*}(x)\}^{\frac{1}{2}} \left\{2\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\Rightarrow \sum_{3}^{*}(n) = o\left(\frac{n}{b_n}\right)^{-s}$$

For $i = 4$

$$\sum_{4}^{*}(n) = \frac{|f''(x)|}{2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right|^2 \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq \frac{|f''(x)|}{2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{\left|\frac{kb_n}{n} - x\right|}{\delta}\right)^{2s-2} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right|^2 \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq \frac{|f''(x)|}{2} \frac{1}{\delta^{2s-2}} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left\{\sqrt{P_{n,k}\left(\frac{x}{b_n}\right)} \left|\frac{kb_n}{n} - x\right|^{2s} \sqrt{P_{n,k}\left(\frac{x}{b_n}\right)}\right\}$$

$$\leq \frac{|f''(x)|}{2} \frac{1}{\delta^{2s-2}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{kb_n}{n} - x\right|^{4s} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}} \left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} P_{n,k}\left(\frac{x}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\leq \frac{|f''(x)|}{2} \frac{1}{\delta^{2s-1}} M(b_n; f) \{T_{n,4s}^{*}(x)\}^{\frac{1}{2}} \left\{2 \exp\left(-\frac{\delta^2}{8x} \frac{n}{b_n}\right)\right\}^{\frac{1}{2}}$$

$$\Rightarrow \sum_{4}^{*}(n) = o\left(\frac{n}{b_n}\right)^{-s}$$

$i = 5,6 \ldots 2s$ follows in a similar way

Thus, $R_{n,2s,2}(x) = o\left(\frac{n}{b_n}\right)^{-s}$

Therefore,

$$(C_n f)(x) = f(x) + \sum_{k=0}^{n} \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!} \left(\frac{kb_n}{n} - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right) + R_{n,2s}(x)$$

$$= f(x) + \sum_{k=0}^{n} \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!} \left(\frac{kb_n}{n} - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right) + o\left(\frac{x}{b_n}\right)^{-s}$$

All that is needed to establish 4.3 for $x > 0$ is in place thus, letting

$$T_{n,m}(x)n^{-m} = \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right) (4.7)$$

and proceeding to obtain

$$(C_n f)(x) = f(x) + \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!} \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^m P_{n,k}\left(\frac{x}{b_n}\right) + o\left(\frac{x}{b_n}\right)^{-s}$$

$$= f(x) + \sum_{m=1}^{2s} \frac{f^{(m)}(x)}{m!} T_{n,m}(x)n^{-m} + o\left(\frac{x}{b_n}\right)^{-s}$$

$$= f(x) + \sum_{m=2}^{2s} \frac{f^{(m)}(x)}{m!} T_{n,m}(x)n^{-m} + o\left(\frac{x}{b_n}\right)^{-s}$$

Since $T_{n,1}(x) = 0$

$$= f(x) + \sum_{m=2}^{2s} \frac{f^{(m)}(x)}{m!} n^{-m} [b_n^m \sum_{k=\left[\frac{[m+1)}{2}\right]}^{m} n^{m-k} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k) \,] + o\left(\frac{x}{b_n}\right)^{-s}$$

$$= f(x) + \sum_{m=2}^{2s} \frac{f^{(m)}(x)}{m!} \,[b_n^m \sum_{k=\left[\frac{[m+1)}{2}\right]}^{m} n^{-k} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k) \,] + o\left(\frac{x}{b_n}\right)^{-s}$$

$$= f(x) + \sum_{k=1}^{s} n^{-k} \,[\sum_{m=k+1}^{2k} b_n^m \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k) \,] + o\left(\frac{x}{b_n}\right)^{-s}$$

since

$$S(j, 0) = \sigma(j, 0) \qquad \forall\, j \in \mathbb{N},$$

this yields

$$(C_n f)(x) \; = f(x) + \sum_{k=1}^{s} n^{-k} \,[\sum_{m=k+1}^{2k} b_n^m \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k) \,] + o\left(\frac{x}{b_n}\right)^{-s}$$

$$= f(x) + \sum_{k=1}^{s} C_k(f; n, x) n^{-k} + o\left(\frac{n}{b_n}\right)^{-s}, \qquad (n \to \infty)$$

Where

$$C_k(f; n, x) = \sum_{m=k+1}^{2k} b_n^m \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{k} \left(\frac{x}{b_n}\right)^{m-i}$$

$$* \sum_{p=k}^{m} \binom{m}{p} (-1)^{m-p} \sigma(p, p-i) S(p-i, p-k)$$

Which is the required 4.3.

To establish 4.4, the procedure goes as follows:

Case 1:

For $x = 0$, 4.4 is valid since by its hypothesis $f^{(2s)}$exists since $(C_n)(0) = f(0)$

Case 2:

$x > 0$, using prove by induction yields

for $s = 1$,

$$\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^1 [\, (C_n f)(x) - f(x) - \sum_{k=1}^{2-1} \frac{f^{(k)}(x)}{k!} \, (C_n(t-x)^k)(x) n^{-k}]$$

$$= \lim_{n\to\infty} \frac{n}{b_n} [\, (C_n f)(x) - f(x) - f^{(')}(x) \, (C_n(t-x))]$$

$$= \lim_{n\to\infty} \frac{n}{b_n}[\ (C_n f)(x) - f(x) - f^{(\prime)}(x) T_{n,1}(x) n^{-1}]$$

$$= \lim_{n\to\infty} \frac{n}{b_n}[\ (C_n f)(x) - f(x) - f^{(\prime)}(x)(0) n^{-1}]$$

$$= \lim_{n\to\infty} \frac{n}{b_n}[\ (C_n f)(x) - f(x)]$$

$$\equiv \lim_{n\to\infty} \frac{n}{b_n}[R_{n,2}(x) + \frac{f^{(2)}(x)}{(2)!} T_{n,2}(x) n^{-2}]$$

$$= \frac{f^{(2)}(x)}{(2)!} x$$

Suppose it is true for $s = l > 1$

That is,

$$\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^l [\ (C_n f)(x) - f(x) - \sum_{k=1}^{2l-1} \frac{f^{(k)}(x)}{k!}\ (C_n (t-x)^k)(x)]$$

$$= \lim_{n\to\infty} \left(\frac{n}{b_n}\right)^l [\ (C_n f)(x) - f(x) - \sum_{k=1}^{2l-1} \frac{f^{(k)}(x)}{k!} T_{n,k}(x) n^{-k}$$

$$\equiv \lim_{n\to\infty} \left(\frac{n}{b_n}\right)^l [R_{n,2l}(x) + \frac{f^{(2l)}(x)}{(2l)!} T_{n,2l}(x) n^{-2l}]$$

$$= \frac{x^l}{2^l l!} f^{(2l)}(x)$$

Next is to establish that it holds for $s = l + 1$ using the relation presented by Karsli (2013) that

$$T_{n,2p}(x) = \frac{n^{2p} b_n^{2p} (2p)!}{2^p p n^p} \left[\frac{x}{b_n} \left(1 - \frac{x}{b_n} \right) \right]^p + O(n^{-p})$$

Leads to

$$\lim_{n \to \infty} \left(\frac{n}{b_n} \right)^{l+1} [\ (C_n f)(x) - f(x) - \sum_{k=1}^{2l+1} \frac{f^{(k)}(x)}{k!}\ (C_n (t-x)^k)(x)]$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n} \right)^{l+1} [\ (C_n f)(x) - f(x) - \sum_{k=1}^{2l+1} \frac{f^{(k)}(x)}{k!} T_{n,k}(x) n^{-k}$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n} \right)^{l+1} [\sum_{k=1}^{2l+2} \frac{f^{(k)}(x)}{k!} T_{n,k}(x) n^{-k} - \sum_{k=1}^{2l+1} \frac{f^{(k)}(x)}{k!} T_{n,k}(x) n^{-k} + o\left(\frac{x}{b_n} \right)^{-(l+1)}]$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n} \right)^{l+1} \left[\frac{f^{(2s+2)}(x)}{(2s+2)!} T_{n,2l+2}(x) n^{-(2l+2)} + o\left(\frac{x}{b_n} \right)^{-(l+1)} \right]$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n}\right)^{l+1} \frac{f^{(2l+2)}(x)}{(2l+2)!} T_{n,2l+2}(x) n^{-(2l+2)} + \lim_{n \to \infty} \left(\frac{n}{b_n}\right)^{l+1} o\left(\frac{x}{b_n}\right)^{-(l+1)}$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n}\right)^{l+1} \frac{f^{(2l+2)}(x)}{(2l+2)!} T_{n,2l+2}(x) n^{-(2l+2)}$$

$$= \lim_{n \to \infty} \left(\frac{n}{b_n}\right)^{l+1} \frac{f^{(2l+2)}(x)}{(2l+2)!} \left[\frac{n^{2l+2} b_n^{2l+2} (2l+2)!}{2^{l+1}(l+1) n^{l+1}} \left[\frac{x}{b_n}\left(1 - \frac{x}{b_n}\right)\right]^{l+1} + O(n^{-k})\right] n^{-(2l+2)}$$

$$= \frac{f^{(2l+2)}(x)}{2^{l+1}(l+1)} \lim_{n \to \infty} \left[x\left(1 - \frac{x}{b_n}\right)\right]^{l+1}$$

$$= \frac{f^{(2l+2)}(x)}{2^{l+1}(l+1)} x^{m+1}$$

Thus, establishing 4.4

Theorem 4.0 has been established hence, the corollary below on the complete asymptotic follows:

Corollary 4.0

If f defined on $[0, \infty)$ satisfies condition (4.11) for every $\alpha > 0$, and all derivatives of f in x exists. Then the operators $(C_n f)$ have the complete asymptotic expansion

$$(C_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} C_k(f; n, x) n^{-k}, \qquad (n \to \infty).$$

$C_n(f; n, x)$are coefficient as given in theorem 4.0.

CHAPTER 5

CONCLUSION

5.1 Conclusion

The complete asymptotic expansions for the Mayer- Konigand Zeller operators and that of the Chlodovsky operator was investigated and established. Using a similar procedure The complete asymptotic expansions for the Modified Gamma operator (Karsli, 2011) can be obtain which is stated as follows: If $f \in W_\gamma[0,\infty)$ $(W_\gamma[0,\infty)(\gamma \geq 0)$ is the space of all locally bounded and integrable functions defined on $[0,\infty)$ such that the growth condition $|f(t)| \leq Mt^\gamma$ fore very $t > 0$ and for some constant $M > 0)$ and if $f^{(2s)}(x)$ exists at a given $x \geq 0$, then the Gamma operators $(M_{n,k}f)$have the complete asymptotic expansion

$$(M_{n,k}f)(x) \sim f(x) + \sum_{m=1}^{\infty} C_{n,k,m}(f;x)n^{-m} , \qquad (n \to \infty)$$

Where $C_{n,k,m}(f;x)$ are coefficients given in terms of Stirling numbers.

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