OPTIMAL CONTROL ANALYSIS FOR FRACTIONAL ORDER MODELS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
BASHIR ABDULLAHI BABA

In Partial Fulfillment of the Requirements for the Degree ofDoctor of Philosophy in Electrical Engineering

NICOSIA, 2021

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Approval of Director of Institute of Graduate Studies

Prof. Dr. K. Hüsnü Can BAŞER

We certify this thesis is satisfactory for the award of the degree of Doctor of Philosophy in Electrical and Electronics Engineering

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my parents…

ABSTRACT

We studied the use of an optimal control as applied to fractional order models. Two mathematical models were developed based on fractional order differential equations in Caputo sense and optimal control laws were incorporated in the models, we showed that the fractional order models were best descriptors of the dynamics underlying infectious disease and other biologically related real life problems (like illegal drug usage), this is because of their memory dependents property.

For the first model, optimal control problem of COVID-19 pandemic outbreak was formulated in which we gave the state equations as well as the co-state equations and we also found the optimal policy that can be used in tackling the COVID-19 infections by placing two control laws, $u_1(t)$ that stands for the methods used in educating people about the infection disease, mask usage, restricting movement, and all control methods taking to prevents people in susceptible human population from acquiring the disease and $u_2(t)$ that stands for quarantine, treatment and monitoring of those that are already infected. The effectiveness of the control program was shown from the result of the numerical simulation that were carried out using RK-4. It can be noted that the control laws caused the reduction of vulnerable people in the susceptible population by $(1 - u_1(t))$ because of the campaign in educating people that were carried out and all the methods used in place. And also, because of the treatment and monitoring that were provided in the quarantine, number of those that are affected is also reduced by $(1 - u_2(t))$.

For our second model, we designed the optimal control problem on the fractional order model for illegal drug usage, we use two control laws, $u_1(t)$ which is of two parts, government part (that consists of awareness campaign, proper monitoring and guidance, severe punishment to the culprits when caught) and parents part (that consists of taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$ which stands for catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users. We also carried out the numerical simulations in order to show the effectiveness of our control programs used. From the simulation results, it can be seen that, the optimal methods that can reduces the number of illicit drug users remarkably is when the two control laws were applied together, hence the best strategy of curtailing the problem is to apply both two control measures at the same time.

Keywords: Optimal control; mathematical model; fractional order model; fractional optimal control; COVID-19; illicit drugs

ÖZET

Bu tezde kesirli mertebeden modellere uygulanan optimal kontrol araştırılmıştır. Araştırma sonunda tesirli olacak iki matamatiksel model geliştirildi. Yaratılan model kontrol kurallarını tamamen kullanarak bulaşıcı hastalıkların ve biyolojik olarak ilişkili diğer gerçek yaşam problemlerinin altında yatan dinamiklerin en iyi tanımlayıcısı olduğu kanıtlanmıştır.

İlk model için COVID-19 salgınını etkileyen parametreler listelenip kontrol parametrelerini oluşturacak şekilde formüle dahil edilmiştir. İnsanları enfeksiyon yayılmasında, maske kullanımı, hareketi kısıtlama yöntemleri anlamına gelen $u_1(t)$ ve hastalığın yayılmasını önlemek için alınan tüm kontrol yöntemleri u_2 (t) parametresi ile tanımlanmıştır. Buna göre u_2 (t) enfekte olanların karantina, tedavi ve izlenmesini ifade eder. Yaratılan modelin etkinliği, RK-4 metodu kullanılarak gerçekleştirilen sayısal simülasyon sonucunda gösterilmiştir. İnsanları eğitmek için yürütülen kampanya ve uygulanan tüm yöntemler nedeniyle, nüfustaki savunmasız kişilerin (1- u_1 (t)) kadar azalmasına neden olduğu vurgulanabilir. Ayrıca karantinada sağlanan tedavi ve izleme sayesinde etkilenenlerin sayısı da (1- u_2 (t)) kadar azalmış olur.

İkinci model, yasadışı uyuşturucu kullanımı için kesirli sıra modelinde optimal kontrol problemi tasarlanmıştır. İki bölümden oluşan model devlet kanadını belirleyen parametre u_1 (t), farkındalık kampanyası ve izleme amaçlı kullanılmıştır. İkinci bölüm, yasadışı uyuşturucu kullanıcılarının izlenmesi ve tedavisi için rehabilitasyon merkezlerini kullanarak, yasadışı uyuşturucu kullanıcılarını yakalamak ve cezalandırmayı ifade eden u_2 (t) parametresi kullanılmıştır. Ayrıca kullanılan kontrol programlarımızın etkinliğini göstermek için sayısal simülasyonlar gerçekleştirilmiştir. Simülasyon sonuçlarından, yasadışı uyuşturucu kullanıcılarının sayısını önemli ölçüde azaltabilecek en uygun yöntemlerin, iki kontrol yasasının birlikte uygulandığı model olduğu kanıtlanmıştır. Bu öngörüye göre sorunu azaltmak için en iyi strateji her iki kontrol önlemini de uygulamak olduğu kanıtlanmıştır.

Anahtar Kelimeler: Optimal kontrol; matematiksel model; kesirli mertebe modeli; kesirli optimal kontrol; COVID-19; yasadışı ilaçlar

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LIST OF ABBREVIATIONS

BVP:	Boundary Value Problem
FDE:	Fractional Differential Equation
FO:	Fractional Order
FOC:	Fractional Optimal Control
FOCP:	Fractional Optimal Control Problem
FOMPC:	Fractional Order Model Predictive Controller
FOPID:	Fractional Order Proportional Integral Differential
FOSMC:	Fractional Order Sliding Mode Controller
FOTF:	Fractional Order Transfer Function
IO:	Integer Order
IVP:	Initial Value Problem
NLP:	Non-linear Optimization Problem
OC:	Optimal Control
OCP:	Optimal Control Problem
ODE:	Ordinary Differential Equation

PMP: Pontryangin’s Maximum Principle

RK-4: Runge-Kutta Fourth Order

CHAPTER 1

INTRODUCTION

Mathematical model is a method used to describe a given system by using mathematical equations and notations so that studying the effects of different components of the system will be easier or the system's proper explanation will be become very easy and also mathematical modeling can make it easy to know the pattern of the behavior of a system (Abramowitz and Stegun, 1968). The method used for developing a mathematical model is known as mathematical modeling (Press et al., 1987). Mathematical modeling becomes one of the most important tools used in studying and designing solution to almost all our todays' problems, especially in engineering, biological sciences and medicine, economics and other social sciences. Mathematical modeling has been continuously used in the public health community as one of the most important research tool used in analyzing and controlling infectious disease. It has been used over the years to studied and analyzed many diseases dynamics, such as Tuberculosis, HIV/AIDS, and Malaria, this is because it is very useful in knowing the pattern to use in controlling an infectious disease, due to the fact that the mathematical models provide the long term as well as short term forecasts for the occurrences of the diseases (Anderson et al., 1986; Nikolas et al., 1997; Christopher and Jorge, 2000; Guihua and Zhen, 2005; Tripathi et al., 2007; Mukandavire et al., 2009a; Hernandez-Vargasa and Middleton, 2013; Mastroberardino et al.,2015; Jabbari et al., 2016; Cai et al., 2009; Zhao et al., 2012; Okosun et al., 2013).

Another important tool used for analyzing and solving problems especially infectious diseases is an optimal control theory, it is used to study a mathematical models, optimal control theory was first presented in the year 1986, after developing the popularly known Pontryagin Maximum Principle (PMP) (Pontryagin and Boltyanskii, 1986), optimal control theory tells us more about the dynamics of a given disease and helps in providing the most convenient methods of controlling and tackling the issue (Okyere et al., 2016). In these days, it is almost necessary to use optimal control theory for all the infectious disease models (Okyere et al.;

Mojaver and Kheiri, 2016; Karrakchou et al., 2006; Adams et al., 2004; Gul et al., 2008; Mukandavire et al., 2009b; Makinde and Okosun, 2011; Yusuf and Benyah, 2012; Mwanga et al., 2014; Choi et al., 2015; Rihan et al., 2014).

Since most of the physical phenomena like biological systems, possess an after-effect or persistent memory in their nature, they could be more appropriately described by fractional differential equations because the fractional differential equations have the after effect memory too. This is the motivation behind consideration of a fractional model in this thesis.

When optimal control theory is used on fractional order models (models formed using fractional calculus) then it is known to be fractional optimal control problems (FOCPs), this type of optimal control problems are the general form of the traditional optimal control problems (OCPs). Fractional differential equations (FDEs) are used in FOCP instead of normal integer order differential equations, and the performance index used is given in form of fractional integration operator (Ali et al., 2016). Many researchers works on the fundamental basis of FOCPs in theoretical perceptions in which many academic articles most of which studied the methods of formulating the FOCPs and the way of deriving their optimal conditions using specified control and state variables analytically and numerically were published in literature (Agrawal, 2004; Agrawal and Baleanu, 2007; Agrawal, 2008; Jelicic and Petrovacki, 2009; Agrawal et al., 2010; Odzijewicz et al., 2012; Kamocki, 2014; Chinnathambi et al., 2019; Al-Mdallal and Abu Omer, 2018; Al-Mdallal and Hajji, 2015; Hajji and Al-Mdallal, 2018). These days, most of infectious disease models use the FOCPs because it is faster and more accurate in controlling the diseases, the reason behind that is the memory dependents property of the fractional-order model. Therefore, we can say that the FOCPs possess all the potentials of becoming the most acceptable tool that can be used to model infectious diseases and other biological related systems that possess memory in their nature.

Many researchers use FOC on infectious disease models like (Ding et al., 2012) who presented fractional OC for the model of HIV-Immune system and they solved the problem using a forward backward algorithm. Basir et al. (2015) gave a FOC for an enzyme kinetic model and also provide its numerical solution. Kheiri et al. (2018) developed a fractional order model for

HIV/AIDS with treatment in which they incorporated three control laws to the model which are ART treatment, Proper use of condoms, and Change in behavior control, purposely to curtail the spread of the disease (HIV/AIDS epidemic). Sweilam et al. (2017) present FOC for the model of the new West Nile virus and provide the solution of the problem numerically by the use of two simple techniques. Ali and Ameen (2020) use FOCP to formulate and investigate the model of pine wilt disease transmission dynamics with three controls laws that can be used to curtail the spread of the disease.

1.1 Research Aim and Objectives

1.1.1 Research Aim

The main aim of this research is to investigate the use of optimal control theory as applied to the fractional order models, fractional order mathematical models will be used in developing and solving optimal control problems. A mathematical models will be developed based on fractional order differential equations that can represent the underlying dynamics of a particular disease or any given real life problem. Then by the use of the optimal control technique, the optimal strategies for the given control measures employed to control the disease/problem and their individual effect on either reducing or eradicating the disease/problem at whole in the time of the disease outbreak in the society at the same time by considering a bio economic approach will be developed and analyzed. Then we will perform the numerical simulations in order to show the effectiveness of the control laws used.

1.1.2 Research Objectives

a. General Objectives

The general objectives of this work is to find the best strategy which can be used for a given infectious disease/problem that can significantly reduce/eradicate the spread of the disease/problem by the use of optimal control theory applied on fractional order models of infectious diseases/any real life problem.

b. Specific Objectives

To know the nature of the disease/problem.

To formulate the integer order model of the disease/problem.

To formulate the fractional order model of the disease/problem.

To find the uniqueness and existence of the Solutions of the fractional order model.

To formulate the optimal control problem on the fractional order model.

To perform the numerical simulation.

1.2 Contributions

The main contributions of this thesis are:

Formulating and developing the optimal control problem with fractional order models of two different problems, the COVID 19 infection and the Illegal Drug Usage, by using their respective fractional order mathematical models in the Caputo sense as follows:

I. We successfully developed an optimal control problem of COVID-19 pandemic outbreak in which we gave the state equations as well as the co-state equations and we also found the optimal policy that can be used in tackling the COVID-19 infections by placing two control laws, $u_1(t)$ that stands for the methods used in educating people about the infection disease, mask usage, restricting movement, and all control methods taking to prevents people in susceptible human population from acquiring the disease and $u_2(t)$ that stands for quarantine, treatment and monitoring of those that are already infected. The effectiveness of the control program was shown from the result of the numerical simulation that were carried out using RK-4. It can be noted that the control laws caused the reduction of vulnerable people in the susceptible population by $(1 - u_1(t))$ because of the campaign in educating people that were carried out and all the methods used in place. And also, because of the treatment and monitoring that were provided in the quarantine, number of those that are affected is also reduced by $(1 - u_2(t))$.

II. We also formulated fractional order optimal control problem for Illegal Drug usage, we also gave its state equations as well as its co-state equations, and also two time dependents control measures $u_1(t)$(awareness campaign from government side, proper monitoring and guidance, severe punishment to the culprits when caught and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$(catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users) were used to found the best strategy to be followed in order to curtail the problem.

1.3 Outline of the Thesis

This thesis comprises of the reports of two articles written on optimal control for two different fractional order models, one of the models is of COVID-19 infections but in form of fractional order and the other is of Illicit Drug Usage also in fractional order form.

The thesis was divided into five chapters as given below:

Chapter one talks about the general introduction, Chapter two brings the literature review, chapter three gives our first article which was published in an SCI journal (Chaos solitons and fractal) “Optimal Control Problem for Fractional Order Model of COVID-19 Pandemic”, chapter four gives our second article which is under review “Fractional Optimal Control of Illicit Drug usage”, and lastly chapter five gives the summary and conclusion.

CHAPTER 2

LITERATURE REVIEW

Control is a method used in trading off among stability, transient and steady state performance, fuel or time optimality, etc., it is a similar to a way of finding a balance among various performance indices under some factors that are mutually constrains. It is one of the most essential field in modern technology.

Control is a very old subject that is still advancing greatly over the time. It is just like a very long history of ancestors owing young blood, it has been always receiving an intensive attention in the field of research. Its history started as far back to 2000 years ago, where people in Arabs, Greeks and Ancient Rome made some important cognition based on the principle of feedbacks, from which many fantastic projects of control systems were built such as the shower systems in the imperial palace, the float valve level regulator for water clocks and the automatic gates in the temples (Lewis, 1990). Zhang Heng, a Chinese polymath use the principle of "suspended pendulum" to invent the seismograph in 132 A.D. (Han Dynasty), which is one of the most famous applications containing the thoughts of control. The first officially adopted automatic control system in the modern sense was the speed regulator of the steam engine invented by James in 1788 (Nof, 2009).

Considering a long history, control has an immense application almost across all aspects of our daily life, right from healthcare to military, from agriculture to industry, etc. The execution of controls can be very simple just like an auto-flush toilet or very complicated like launching a rocket.

2.1 Optimal Control

Optimal Control (OC) is a method of finding a strategy, policy, or scheme of reaching an optimal outcome in a given system, it can also be defined as the method used to control some of the parameters in a model to get an optimized output by finding the control and the state of trajectory in a dynamic system after minimizing a performance index within a given time frame (Bryson Jr., 1996).

The history of OC started as an augmentation of the calculus of variations. As far back as seventeenth century, the first official result for calculus of variations was made public as a result of the challenge made by Johann Bernoulli to the rest of the then known mathematicians like Newton, L'Hopital, Jacob Bernoulli, Leibniz and Von Tschirnhaus on the Brachistochrone problem,which says that "in case of a very small body which is travelling under the effect of gravity, what will happen? Which of the sides between the two fixed sidesof the body will allows it to travels in the smallest possible time?''

Solution of some specific problems in calculus of variations were found and a general form of mathematical theories were formulated by Lagrange and Euler. Theoretical physics is one of the most important area where calculus of variations can be applied,especially in the area connected to Hamiltonian's Principle or theLeast Action's Principle. Later, in late 1920s and early 1930s Hotelling, Evans, Ross, and Ramsey, discovered its applications to economics, and thereafter, more and more applications were published occasionally (Sussmann and Willems, 1997).

Calculus of variations becomes the generalization in optimal control theory which was highly inspired by its applications in military and it has been growing quickly since 1950. The first most notable result was reached by Lev S. Pontryagin a Russian mathematician (1908-1988) and his team members (V. G. Boltyanskii, R. V. Gamkrelidz and E. F. Misshchenko) by presenting and demonstrating the most popular Pontryagin Maximum Principle (Pontryagin and Boltyanskii, 1986). This principle gave researchers a proper conditions needed for optimizing problems that have restrictions of differential equations and alsomake generalization in the problems of variations to such a way that, the control and state variables will be separated and control constraints will be admitted. In this type of problems, OC theory providesmost equivalents results, like the one which is more expected. Although, the two methods differ, the OC theorymost of the timesgives an insight into a problem which may probably have not be easy by the calculus of variations. This is why OC can be applied insome problems that do not warrant the application of calculus of variations, like in those problems that are associated with the constraints at the functions' derivatives (Leitmann, 1997).

OC theory also provides a new techniques of Dynamic Programming in mathematics this comes after R. E. Bellman discovered that, Dynamic Programming uses the optimal control's principle and it is the most proper tool that can be used to solve the discrete problems, because it allows a momentous reduction in the time and complexity involves infinding the optimal controls (Kirk, 1998). Optimal control's principleallows the possibility of obtaining different technique for continuous problem this results in discovering a most popularly known Hamilton-Jacobi-Bellman equation which can be used to solve a partial differential equation. This finding brings about a new relationship that links the OCP and the Lyapunov's stability theory.

During early days, before the generation of computers, not all OCP can be solved, rather only some simple problems can be solved, and hence its applications is limited to simple problems. But the arrival of the computer makes it possible to apply the OC theory to more complex problems. Below are some selected examples of area of applications:

_ Application in physical systems, like robotics, motors and machineries' performance, etc. (Goh, 2008; Molavi and Khaburi, 2008).

_ Application in aerospace, like driven problems, satellite launchers' development, etc. (Bonnard and Janin, 2008; Hermant, 2010).

_ Application in economics and management, like optimal investment of production strategies, optimal exploitation of natural resources, energy policies (Munteanu et al., 2008; Sun and Li, 2008).

_ Application in biology and medicine, like radiotherapy, infectious disease, physiological functions' regulation etc. (Joshi, 2002; Joshi et al., 2006; Lenhert and Workman, 2007; Nanda et al., 2007).

Nowadays, the OC theory is sustained with many techniques. It was made possible to change controls of a system in order to get a desired result, in which the system in question may be: ordinary differential, partial differential, stochastic, differential, discrete, integral-differential, combination of discrete and continuous systems or even fractional order differential equations.

In this research our goal is to study the use of OC theory on fractional order ordinary differential equations.

2.1.1 Optimal Control Problem

A traditional optimal control problem must have a performance index or cost function ($J[x(t), u(t)]$), a set of state variables ($x(t) \in X$), a set of control variables ($u(t) \in U$)within a time t, while $t_0 \leq t \leq t_f$. Its main aim is to find a continuous piecewise control $u(t)$ together with the associated state variable $x(t)$ that maximize the given objective functional. Below is an example of a typical OCP in Lagrange formulation.

Definition 2.1 **(**Lagrange formulation)**:** The optimal control problem in Lagrange form is given in the form:

$$\max_u J[x(t), u(t)] = \int_{t_0}^{t_f} f(t, x(t), u(t))\, d\ ,$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \tag{2.1}$$

$$x(t_0) = x_0.$$

$x(t_f)$is not restricted, this means, it can take any value, and it can also be fixed as$x(t_f) = x_f$.

For this purposes, the functions f and g have to be continuous and differentiable in all the three cases above. It is also assumed that the control set U is a measurable function of

Lebesgue, that will makes the control(s) to maintain been continuous and piecewise, and the related states variables also have be differentiable and piecewise too.

In most of the OCP notes emphasis have been given on maximizing a function because it is always possible to swap either back or forth between either maximizing or minimizing a given function by directly reversing the cost function as follows:

$$m \quad \{J\} = -m \quad \{-J\}. \tag{2.2}$$

2.1.2 Optimal Control Formulation

The most popular equivalent formulation methods that can be used in describing an OCP are three, namely Lagrange (which its example was already presented in the previous section above), Bolza and Mayer forms (Chachuat, 2007).

Definition 2.2 **(**Bolza formulation)**:** The formulation of optimal control problem in Bolza form is given as:

$$\max_u J[x(t), u(t)] = \emptyset\left(t_0, x(t_0), t_f, x(t_f)\right) + \int_{t_0}^{t_f} f(t, x(t), u(t))\, d \ ,$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \tag{2.3}$$

$$x(t_0) = x_0.$$

$$whe \quad \emptyset \ i \ u \ c \qquad d \qquad f \qquad .$$

Definition 2.3 **(**Mayer formulation)**:** The formulation of optimal control problem in Mayer form is given as:

$$\max_u J[x(t), u(t)] = \emptyset\left(t_0, x(t_0, t_f, x\left(t_f\right))\right)$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \tag{2.4}$$

$$x(t_0) = x_0.$$

2.1.3 Pontryagin's Maximum Principle (PMP)

In 20th century, Pontryagin and his team members construct one of the most useful principle in OCP which is also the first fundamental circumstancesneeded in finding the optimal control. This finding was termed as the greatest mathematical achievementat that time. The team developed the theory of how to use an adjoint functions of a differential equation to fix to the objective function. Adjoint functions are considered in terms of purpose just like Lagrange multipliers are considered in multivariable calculus which attached constraints to the function of multiple variables that will either be maximized or minimized.

Definition 2.4 **(**Hamiltonian Equation)**:** Given an optimal control problem in Lagrange form presented in the section above, then the function:

$$H\big(t, x(t), u(t), \lambda(\mathrm{t})\big) = f\big(t, x(t), u(t)\big) + \lambda(\mathrm{t}) g\big(t, x(t), u(t)\big). \qquad (2.5)$$

Is called Hamiltonian function while $\lambda(\mathrm{t})$above is called adjoint variable.

Theorem 2.1 **(**Pontryagin's Maximum Principle (PMP))**:** If $u^*(t)$ and $x^*(t)$ are optimal values for a given OCP (like the one in definition 5) then, there must be piecewise and differentiable adjoint variable $\lambda(\mathrm{t})$ with:

$$H\big(t, x^*(t), u(t), \lambda(\mathrm{t})\big) \leq H\big(t, x^*(t), u^*(t), \lambda(\mathrm{t})\big). \qquad (2.6)$$

for all controls u at each time t.

where H is the Hamiltonian which was defined previously and $\lambda'(t)$ given as follows:

$$\lambda'(t) = \frac{\partial \quad \big(t, x^*(t), u^*(t), \lambda(\mathrm{t})\big)}{\partial}, \qquad (2.7)$$

$$\lambda\big(t_f\big) = 0.$$

Proof. Proofing the above theorem is very complex, hence we will not provide it here, but it can be found from the original Pontryagin's text (Pontryagin and Boltyanskii, 1986).

Remark: The above condition,$\lambda(t_f) = 0$ is known as transversality condition, whichis only been used in the situation where the OCP does not possesssome values called terminal in its state variables, that means $x(t_f)$ can take any value.

One of the biggest contribution of PMP is changing the entire problem of trying to find a control value which either minimizes or maximizes the objective function within the state ODE and specified initial conditions to become a problem of optimization of Hamiltonian pointwise. As consequence, taking the Hamiltonian and the adjoint equation together, we get:

$$\frac{\partial}{\partial} = 0. \tag{2.8}$$

The Hamiltonian has a captious point foru^*at every assigned t,which is popularlyknown asthe optimal condition. Hence, we can getour necessary conditions by using the Hamiltonian alone, without calculating the integral within the objective function.

2.1.4 Optimal Control along Payoff Terms

There are some cases, in which we need to minimize or maximize the terms throughout our time interval, and there are some cases in which we need to minimize or maximize our function in a specified time interval, in most cases, at the interval's end. In some scenarios, the time period of the state values has to be taken into consideration by the objective function, for example, the number of people that are infected by the disease at the end of the epidemic in a given model (Lenhert and Workmann, 2007).

Definition 2.5 **(**OCP for payoff term)**:** An OCP alongside a payoff term can be given in the following form:

$$\max_u J[x(t), u(t)] = \emptyset\left(x(t_f)\right) + \int_{t_0}^{t_f} f(t, x(t), u(t))\, d \ ,$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \tag{2.9}$$

$$x(t_0) = x_0.$$

whe $\emptyset\big(x(t_f)\big)$is the desired result with respect to the level of the population $x(t_f)$ and it is known to be the payoff or salvage.

Applying PMP, we can derived the necessary conditions of the OCP with payoff as:

Proposition 2.1(The conditions needed): If $u^*(t)$ and $x^*(t)$ are the optimal values of a given OCP (like the one in definition 5) then, there must be a differentiable and piecewise adjoint variable λ(t) with:

$$H\big(t, x^*(t), u(t), \lambda(t)\big) \leq H\big(t, x^*(t), u^*(t), \lambda(t)\big). \qquad (2.10)$$

for all controls u at every time t,

where H is the Hamiltonian which was defined previously and

$$\lambda'(t) = \frac{\partial \quad \big(t, x^*(t), u^*(t), \lambda(t)\big)}{\partial} \, (a \qquad c \qquad),$$

$$\frac{\partial}{\partial} = 0 \qquad (O \qquad c \qquad), \qquad (2.11)$$

$$\lambda\big(t_f\big) = \emptyset'\big(x(t_f)\big)(T \qquad c \qquad).$$

2.1.5 Optimal Control for Bounded Control

Definition 2.6 (OCP for bounded control). An OCPalongside bounded control can be expressed as follows:

$$\max_u J[x(t), u(t)] = \int_{t_0}^{t_f} f(t, x(t), u(t))\, d \;,$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \qquad (2.12)$$

$$x(t_0) = x_0,$$

$$a \leq u(t) \leq b.$$

$$whe \quad a, b\ a \quad f \quad r \quad co \quad a \quad a < b.$$

Alternative necessary conditions are required in order to solve problems with bounds on their controls.

Proposition 2.2 (The conditions needed). If $u^*(t)$ and $x^*(t)$ are the optimal values of a given OC problem (like the one in definition 5) then, there must be a piecewise and differentiable adjoint variable $\lambda(t)$ with:

$$H\big(t, x^*(t), u(t), \lambda(t)\big) \leq H\big(t, x^*(t), u^*(t), \lambda(t)\big). \qquad (2.13)$$

For all controls u at each time t,

where H is the Hamiltonian which was defined previously and

$$\lambda'(t) = \frac{\partial \quad \big(t, x^*(t), u^*(t), \lambda(t)\big)}{\partial} \;(a \qquad c \qquad), \qquad (2.14)$$

$$\lambda\big(t_f\big) = 0 \qquad (t \qquad c \qquad),$$

By using PMP, the optimal control must satisfy the following optimality condition:

$$u^* = \begin{cases} a, & i \quad \frac{\partial}{\partial} < 0, \\ a < \tilde{u} < b & i \quad \frac{\partial}{\partial} = 0, \qquad (o \qquad c \qquad) \\ b, & i \quad \frac{\partial}{\partial} > 0. \end{cases}$$

It means that, the maximization may occur at all the allowed controls, and $\tilde{u}$ is can be found by using the following:

$$\frac{\partial}{\partial} = 0$$

(2.15)

Precisely, the control u^* maximizes H pointwise optimally with regards to $a \leq u \leq b$.

Proof:

Proofing for the above was given in (Kamien and Schwartz, 1991).

Note; if the problem is for minimization, then $u^{\square}$ will be selected in such a way to minimizes H optimally pointwise. This will result in inter changing < to > as came in first and third lines forthe optimality condition.

2.2 Methods of Solving Optimal Control Problems

The world has witnessed an amazing development in the field of computational mathematics within the last decades, not only in terms of issues concerning hardware like speed, efficiency and capacity of the memory, but also in terms of robustness of softwares used. The accomplishments recorded in the area of the numerical solution methodsin both integral and differentialequations gave rise to methods of simulating the most highly complex real world scenarios. In the same way, the optimal control problems also get along with these improvements and the numerical methods as well as algorithms for solving OC have been advanced significantly.

2.2.1 Dynamic Systems' Numerical Solutions

Mathematically, dynamic system can be characterized by a set of ordinary differential equations (ODEs). Categorically, dynamics of a system are presented asa system of n-ODEs for$t_0 \leq t \leq t_f$, as:

$$\dot{y} = \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \\ . \\ . \\ . \\ \dot{y_n} \end{bmatrix} = \begin{bmatrix} f_1(y_1(t), \ldots y_n(t), t) \\ f_2(y_1(t), \ldots y_n(t), t) \\ . \\ . \\ . \\ f_n(y_1(t), \ldots y_2(t), t) \end{bmatrix} \tag{2.16}$$

The problems in solving ODE can be divided into two, namely: Initial Value Problems (IVP) and Boundary Value Problems (BVP), this classification depends on the specification of the conditions at the marginin the domain. For Initial Value Problems (IVP)all the specification of the conditions are made at the initial state. While in the Boundary Value Problems (BVP) the specifications of the conditions are made at both initial and final point.

There are various numerical techniques used in solving the Initial Value Problems (IVP) in literature like, Euler method and Runge-Kutta method while to solve Boundary Value Problems (BVP), there are some techniques like shooting methods.

a. Euler method

Euler technique is a single-step techniqueand one of the most popular techniques used for the numerical solution of dynamic systems. In this discretization method, if a differential equation is given as in the following form:

$\dot{x} = f(x(t), t)$, thenis possible to make an appropriate approximation as this:

$$x_{n+1} \approx x_n + hf(x(t_n), t_n) \quad (2.17)$$

The above approximation x_{n+1} of $x(t)$ at timet_{n+1}possess an error with h^2order. This shows that, the trade-off among the accuracy of the calculation and its complexity depends solely on the selected value of h . In general, when value of h is decreased the calculation will be more accurate but takes longer time.

Is very difficult for Euler approximation method to be effective for systems with many higher order. Hence more exact and complex methods were presented. One of those techniquesdeveloped is Runge-Kutta method.

b. Runge-Kutta technique

Runge-Kutta technique is a multiple-step method, in which at a given time t_{k+1} the solution can be found from a specified set of preceding values $t_{j-k}, \dots,\ t_k$with $j$been the steps number. In this method if the differential equation is in this form$\dot{x} = f(x(t), t)$, then we can make the following appropriate approximation using:

Runge-Kutta second order

$$x_{n+1} \approx x_n + \frac{h}{2}[f(x_n(t), t_n) + f(x_{n+1}, t_{n+1})], \qquad (2.18)$$

Or Runge-Kutta fourth order

$$x_{n+1} \approx x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \qquad (2.19)$$

where

$$k_1 = f(x(t), t),$$

$$k_2 = f\left(x(t) + \frac{h}{2}k_1, t + \frac{h}{2}\right),$$

$$k_3 = f\left(x(t) + \frac{h}{2}k_2, t + \frac{h}{2}\right),$$

$$k_4 = f(x(t) + hk_3, t + h).$$

The above approximation x_{n+1} of $x(t)$ at the point t_{n+1} has an error that depends on h^3in case of Runge-Kuttasecond order technique andh^5in case of the Runge-Kutta fourth order technique.

2.2.2 Numerical Solution of Optimal Control Problems

Solution of OCP using numerical methods was started as far back as 1950s by Bellman findings. Since that time up till today, there are a lot of complex techniques and different corresponding applications for the complexities have been substantially developed (Rao, 2009).

Numerical methods for solving OCP can be categories to two main divisions, namely: indirect techniques and direct techniques. For the first division, the OCP is indirectly be solved by converting it to a BVP, by the help of the PMP. While in second division, the optimal problem’s solution can be found directly by simply duplicating the optimization problem with an infinite dimension to a problem with finite dimension.

a. Indirect techniques

For an indirect technique, the optimal conditions in the first-order original OCP can be found by using PMP. This techniquedirects to a BVP with multiple point which can be solved in order to find a prospect of the optimal trajectory known as extremals.

It is necessary to have control equations notably stated as well as the transversality conditions and all the adjoint equations provided they exist in case of an indirect method. Note that the formation the problem and the technique employed in solving the problem do not have any direct relationship between the two. Is possible to consider any method of solving OCP to solve a problem formulated directly or indirectly. Example of numerical approach using indirect method of solving OC problems is Forward-Backward sweep method.

Forward – Backward sweep method

The technique was described in a book authored by Lenhart and Workman (Lenhart and Workmann, 2007) and it is popularly called forward-backward sweep method. The method starts by guessing initial value of the control variable. Then, the equations of state are solved simultaneously in a forward time while the equations of adjoint are solved in a backward time. After which control is updated by putting the calculated values of adjoints and states in their corresponding equations, and thismethod is repeated until it reached convergence.

For example, consider $\vec{x} = (x_1, \ldots, x_{N+1})$ a $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{N+1})$ to be the vector equationsof the adjoint and thestate for a given OCP. Then, we can demonstrate the whole algorithm of Forward-Backward as follows:

1st Step. Guess the initial value of$\vec{u}$ in the range ($\vec{u}$ = 0 is mostly used here);

2^nd^Step. By the use of $x_1 = x(t_0) = a$ (the initial condition) and the values of $\vec{u}$(guessed), solve $\vec{x}$in amanner of time forwardbased on its differential equation of the system's optimality;

3^rd^ Step. By the use of $\lambda_{N+1} = \lambda(t_f) = 0$ (the transversality condition) and the values of $\vec{u}$ and $\vec{x}$, solve $\vec{\lambda}$ in a manner of time backward based on its differential equation of the system's optimality;

4**^th^**Step. By using the calculated values of $\vec{x}$ and $\vec{\lambda}$ in their corresponding equations of the optimal control update $\vec{u}$

5**^th^**Step. Check if the value of the variables found are very close to the one found at previous iteration (i.e. convergence), then the current values are the solutions, otherwise go back to Step 2.

b. Direct methods

Another class of numerical technique for optimization of dynamic systems has evolved, and it was called direct methods.

This development was as a result of the demand of solvingcomplex problems in optimization, the technique becomes popular by the help of the rapid increase in the computational world.

In this technique an array of points $x_1, x_2, \dots, x^{*}$ is constructed in such a way that the objective function is minimized, and typically, $F(x_1) > F(x_2) > \cdots > F(x^{*})$. In this method approximation of the state variables and/or control variables are done by a suitable function of approximation (like piecewise constant parameterization or polynomial approximation). At the same time, the approximation cost function is done by function of approximation. Then, the problem will be formulated again in a normal nonlinear optimization problem (NLP) form by treating the coefficients of the function of approximations as variables of optimization as follows:

$$\min_{x,u} F(x)$$

$$s.t. \quad C_i(x) = 0, \qquad i \in E \qquad (2.20)$$

$$C_j(x) \geq 0, \qquad j \in I$$

where $C_i, i \in E$ is the set of equality constraint and $C_j, j \in I$ is the set of inequality constraint.

The NLP is much simpler to solve compared to the BVP, because of its sparsity and the availability of too many notable software programs designed to deals withits features. Hence due to this, the number of different kind of problems that may be solved by the use of direct methods is far more than those that may be solved by the use of indirect methods. Therefore the direct methods becomes more famous these days and many research also written a highly developed software programs for the usage of these methods.

2.2.3 Specific Optimal Control software

Some of the software programs developed specifically for this purpose are given below, most of the softwares are specifically the solution providersof OCP and standard NLP that can be used after the process of discretization.

a. OC-ODE

The Optimal Control of Ordinary-Differential Equations (OC-ODE) (Gerdts, 2009) was presented in 2009, it is a combination of the routines of OCP in FORTRAN 77together with ordinary differential equations (ODEs). It transforms OCP to NLP (finite-dimensional) directly by the use of automatic direct discretization method. OC-ODE contains some procedures that can analyzed the estimation for numerical adjoint and sensitivity.

b.DOTcvp

The Dynamic Optimization Toolbox with Vector Control Parametrization (DOTcvp) (Hirmajer et al., 2009) is a MATLAB tool box for dynamic optimization. It gives a space to the compiler of a FORTRAN to buildits ‘‘.dll” files of the ODE, Jacobian, and sensitivities. Though, a FORTRAN compiler must to be installed in the space of the MATLAB. It also uses method of vector parametrization of the control in calculating the profiles of the optimal control, when a solution to the control is provided in piecewise sense. Note that to use this software the OCP must be given in Mayer form. DOTcvp graphical interface (GUI) is very user friendly.

SUNDIALS tool (Hindmash et al., 2005) this is the modified form of DOTcvp that can be used to solve the IVP and Jacobian automatic generation and also for the gradients. In addition, Newton or Functional iteration module together with the Adams or BDF linear multistep method can also be used in solving the IVP.

c. Muscod-II

One of the recent solver is Multiple Shooting CODe for Optimal Control (Muscod-II) (Kuhl et al., 2007) which is an advanced version of AMPL that can be used in solving a combination of integer nonlinear ODE and DAE constrained OCP.

AMPL (Fourer et al., 2002) is a mathematical programminglanguage for modeling whichFourer, Gay and Kernighan produced in 2002. This modeling language was used in coordinating and automating the modeling work, it has an ability of handlinglarge amountof data and it can also be used as in machines solvers and independent solvers, this can allow users to focus solely on the model without worrying on the techniques to be used in solving. But, the AMPL modeling language on its own does not support any method of formation of the differential equations. Therefore, to use AMPL easily and properly, TACO Toolkit was developedand implementedas a small set of extensions without any need for explicit encoding of discretization schemes.

2.3 Fractional Calculus

The history of fractional calculus is almost as dated back as that of the ordinary calculus. It can be traced back to the 17th century, short after Newton and Leibniz formulated the ordinary integration and differentiation (Podlubny, 1999). L'Hopital sends letter to Leibniz (Leibniz, 1692), where he questioned what will happen if the order of the derivative were $\frac{1}{2}$, this eventually led to the birth of the theory of derivatives and integrals of arbitrary order.

Development of fractional calculus can take a similar analogies with the development in mathematics. It is a common knowledge that the history of mathematics is moved forwards by paradoxes and crisis as the advancement of civilization (Kline, 1990; Gu, 2003; Snapper, 1979). For example if we take the development of numbers, the integers on the number axis are very small portion just like isolated islands in the ocean, with the majority of water as fractional and non-rational numbers. Moving a step forward, the discovery of complex number made people understand that the mighty real numbers are just like the planets in the big universe that only occupy a very tiny portion of the space.

By the same way, the advances of mathematical operations also experienced the development from some basic operations to complicated operations, that is, it starts from addition/subtraction to powers/roots, and then moves to integration/derivation and convolution. Also for the past 300 years, the development of fractional calculus makes the theory of operation even more complete. Recent history of fractional calculus was given in (Machado et al., 2011). Fractional calculus is such an amazing tool that can be used to explain many phenomena of physics which the conventional math could not explain before. It is more especially good at depicting phenomena with long memory, long range dependence, etc.

2.3.1 Preliminaries

a. Important definitions

There are more than 10 types of definitions for fractional order integrals and differentiations (Miller and Ross, 1993). For readers' convenience, some of the most commonly used definitions are briefly listed below. More details can be found in (Magin, 2006).

Definition 2.7 (Qian and Wong, 2010): The fractional derivative of order $\alpha \in [n-1, n)$ of $f(x)$ for Rieman-Liouville can be defined as:

$$ {}^{R}_{a}D^{\alpha}_{x} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)d \quad , \quad n = [\alpha] + 1. \quad (2.21)$$

Definition2.8(Qian and Wong, 2010): The fractional derivative of order $\alpha \in (n-1, n]$ of $f(x)$ for Caputo is defined as:

$$ {}^{C}_{a}D^{\alpha}_{x} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{n}(t)d \quad , \quad n = [\alpha] + 1. \quad (2.22)$$

Definition 2.9 (Ortiz et al., 2013): **(Linearity)**

If f, g are continuous and b, c are scalars, then

$$ {}^{K}_{a}D^{\alpha}_{x}[b\ (x) + d\ (x)] = b\,{}^{K}_{a}D^{\alpha}_{x}f(x) + d\,{}^{K}_{a}D^{\alpha}_{x}g(x), \qquad (2.23)$$

$$ {}^{C}_{a}D^{\alpha}_{x}[b\ (x) + d\ (x)] = b\,{}^{C}_{a}D^{\alpha}_{x}f(x) + d\,{}^{C}_{a}D^{\alpha}_{x}g(x).$$

Definition 2.10(Baba, 2019):**(Contraction)**

For an operator $f: X \rightarrow X$ which mapped a metric space onto itself, it is contractive for $0 < q < 1$

$$d(f(x), f(y)) = q\ (x, y), \qquad \forall\, x, y \in X. \qquad (2.24)$$

b. Important functions

Some important special functions which are frequently encountered in fractional calculus are listed below. For more details, refer to (Magin, 2006).

- ***Gamma function***

Gamma function is one of the important functions because it is the fundamental element in almost all of the definitions in fractional integrals. It is usually considered as the factorial of non-integer numbers.

The integral form of gamma function can be written as:

$$\Gamma(p) = \int_{0}^{\infty} x^{p-1}e^{-x}d\ , p > 0. \qquad (2.25)$$

Some useful properties of gamma function to remember are:

$$\Gamma(1) = 1; \qquad \Gamma(n+1) = n!\ \ (n = 0,1,2 \ldots)$$

$$\Gamma(1/2) = \sqrt{\pi}; \qquad \Gamma(x+1) = x\Gamma(x). \qquad (2.26)$$

Similar to integer derivative, the fractional order derivative of a variable with the same fractional order power is a constant,

$$\frac{d^{\alpha}}{dx^{\alpha}}x^{\alpha} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \alpha + 1)}x^{\alpha-\alpha} = \Gamma(\alpha + 1). \tag{2.27}$$

) ***Mittag-Leffer function***

The Mittag-Leffer function (M-L) is general form of exponential function that plays a very important role during solving the fractional differential equations just the same way exponential function does in ordinary differential equations. It has four forms (Prajapati and Shukla, 2012; Chaurasia and Pandey, 2010), and the most commonly used forms are given below as 1-parameter and 2-parameter representation (Shukla and Prajapati, 2007):

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(\alpha \quad + 1)} \, (\alpha > 0) \tag{2.28}$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(\alpha \quad + \beta)} \, (\alpha > 0, \beta > 0). \tag{2.29}$$

Some of the beautiful properties of M-L function are as follows;

$$E_{1,1}(x) = e^{x};$$

$$E_{1,2}(x) = \frac{e^{x} - 1}{x}. \tag{2.30}$$

) ***Error function***

The error function is another special function for the “S” shape, and is defined as:

$$e \quad (x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^{2}} d \quad, \qquad -\infty < x < \infty. \tag{2.31}$$

It has the following properties;

$$e \quad (0) = 0$$

$$e \quad (\infty) = 1 \tag{2.32}$$

$$e \quad (x) + e\ f \quad (x) = 1$$

Where $e \quad (x)$ i: c the cc $e \quad f$

) ***Confluent hypergeometric function***

This function is used to get the solution for the equations of confluent hypergeometric, and is represented as follows;

$$ {}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!}, \quad -\infty < x < \infty \tag{2.33}$$

where $(a)_n$ a $(c)_n$ are the Pochhammer symbols.

$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a)}, \qquad a \tag{2.34}$$

$$(c)_n = \frac{\Gamma(c+1)}{\Gamma(c)} . n = 0,1,2 \ldots$$

Some of the most commonly used properties for the hypergeometric function are listed below;

$$ {}_1F_1(1;1;x) = e^x, \tag{2.35}$$

$$\frac{1}{\Gamma(a+1)}\ {}_1F_1(a; a+1; a\ ) = E_{1,2}(a\ ).$$

c. Some important theorems

Theorem 2.2 (Baba, 2019)**:** ***(Principle of Banach contraction mapping)***

For any operator of contractive that mapped a metric space onto itself has a unique fixed point. Moreover, when $f: X \rightarrow X$ is an operator of contractive that mapped a metric space onto itself with its fixed point a: $f(\boldsymbol{a}) = \boldsymbol{a}$; then for any continual sequence:

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1), \ldots, x_{n+1} = f(x_n), \ldots, \tag{2.36}$$

that converges to u.

Then we said that u is a solution or an equilibrium for the continuous dynamical system andfor the discrete dynamical system is a fixed point.

Theorem2.3 (Matignon, 1996): For the equilibrium solutions x^{*} of a given system say (□)to be asymptotically stablelocally, then all its eigenvalues λ_i in its Jacobian matrix $\frac{\partial}{\partial x_i}$which is evaluated for the equilibrium points must satisfy the following:

$$|\arg(\lambda_i)| > \frac{\alpha}{2}, \qquad 0 < \alpha < 1. \tag{2.37}$$

Theorem 2.4 (Delvari et al., 2012): If $x = 0$ is an equilibrium solution of system (□), and $\Omega \subset \mathbb{R}^n$ is a domain containing $x = 0$.

If $V(t, x): [t_0, \infty] \times \Omega \rightarrow \mathbb{R}$ is continuously differentiable function given as:

$$W_1(x) \le V(t,x) \le W_2(x) \quad \text{and} \tag{2.38}$$

$${}_{0}^{C}D_t^{\alpha}V(t,x) \le -W_3(x), \qquad f \quad t \ge 0, \quad x \in \Omega.$$

where,$W_1(x)$, $W_2(x)$ a $W_3(x)$ are definite function that are continuous and positive on Ω and V is a Lyapunov candidate function, then $x = 0$ is globally asymptotically stable.

Theorem 2.5 (Vergas-De-Leon, 2015): Let $x(t) \in \mathbb{R}^{+}$be continuous and derivable function. Then, for any time instant $t \ge t_0$ and $\alpha \in (0,1)$

$${}_{0}^{C}D_t^{\alpha}\left[x(t) - x^{*} - x^{*}\ln\left(\frac{x(t)}{x^{*}}\right)\right] \le \left(1 - \frac{x(t)}{x^{*}}\right){}_{0}^{C}D_t^{\alpha}x(t), \qquad x^{*} \in \mathbb{R}^{+}. \tag{2.39}$$

2.3.2 Fractional Order Differential Equations (FODEs)

Fractional order differential equations (FODEs) are the basic tools used to describe fractional order dynamic systems. Any type of fractional order system analysis, whether time domain, s-domain or even complex frequency domain, are all made up of the basis of FODEs. Hence, they are very important and needs to be emphasized. In this section we give the two most common types of fractional order differential equations (linear and nonlinear FODEs) with example of each.

a. Linear FODE's

Linear FODE's are the most commonly used in fractional order controls because of their regularity and simplicity. The general expression of FODE's are as follows:

$$a_1 {}_C^C D_t^{\alpha_1} y(t) + a_2 {}_C^C D_t^{\alpha_2} y(t) + \cdots + a_n {}_C^C D_t^{\alpha_n} y(t) = b_1 {}_C^C D_t^{\beta_1} u(t) +$$

$$b_2 {}_C^C D_t^{\beta_2} u(t) + \cdots + b_n {}_C^C D_t^{\beta_n} u(t), \qquad (2.40)$$

$whe \quad he \quad e$, α_i, β_j $(i, j = 1,2, \dots)$can be arbitrary real numbers, i.e. α_i, β_j □ ℝ. If α_i and β_j are integer multiples of a common factor, the equation is said to have a commensurate order; and if there is no common factor exist it is said to be of non-commensurate order (Vinagre and Feliu, 2000). Example of linear FODE can be seen in the fractional Langevin equation.

The fractional Langevin equation: The Langevin equation that defines the Brownian movementof particles in a fluid is:

$$m\frac{d^2x}{dt^2} = \lambda\frac{d}{d} + \eta(t) \qquad (2.41)$$

where x represent the particle's position and m represents the particle's mass. The noise term $\eta(t)$denotes the impact's effect with the molecules of the fluid, which has a Gaussian probability distribution with the correlation function as follows:

$$c \quad \left(\eta_i(t), \eta_j(t')\right) = 2\lambda k_b T \delta_{i,j} \delta(t - t'), \qquad (2.42)$$

where k_b is Boltzmann's constant, T is the temperature, and δ is the Dirac's function.

However, the above equation of motion does not capture the hydrodynamics completely since it ignores the effects of the force of viscosity due to the acceleration of the particle and the added mass. Hence, the fractional Langevin equation is used to supplement the missing dynamics, (Mainardi and Pironi, 1996),

$$m\frac{d}{d} = \frac{m}{\delta_e}\left[1 + \sqrt{T}\,{}_{c}^{t}D_t^{1/2}\right]x(t) + \eta(t). \quad (2.43)$$

where the explanation of the coefficients can be seen in (Mainardi and Pironi, 1996). But for the above fractional Langevin equation, the random force $\eta(t)$cannot be used for white noise uniquely. Instead, it can be used for a superposition of the white noise with a "fractional" noise. Therefore, the added mass and the fractional noise changed the velocity correlation function from exponential decay to algebraic or power law decay.

b. Nonlinear FODE'S

Example of nonlinear fractional differential equations can be seen in the Van der Pol fractional equation given below:

***The fractional Van der Pol equation*:** The Van der Pol (VDP) equation was initially presented by Van der Pol around 1920s to mimic the self-sustaining oscillation in electrical circuits using vacuum tubes, (Der Pol and Der Mark, 1927). It is among the first discovered instances of deterministic chaos, and it can originally be defined by the use of the nonlinear ODEthat follows:

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x} + x = 0. \quad (2.44)$$

The above can also be used to describe a different variety of phenomena, like a mass-spring-damper system which has a damping coefficient that is nonlinear, or an RLC circuit with a resistor that is negative and nonlinear.

Later, a number of variant VDP equations were proposed, for example, Mickens et al. investigated the following two equations in (Mickens, 2002) ,which was termed as fractional VDP.

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x} + x^{1/3} = 0, \qquad (2.45)$$

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x}^{1/3} + x = 0.$$

However, we can see that these dynamics only contains the fractional power of the state variables rather than fractional order derivatives, hence, they are not fractional order in the sense of calculus. In 2004, Pereira et al. considered the following fractional derivative version VDP by substituting the capacitance with a “fractance" in a nonlinear RLC circuit model, (Pereira, 2004),

$${}_{C}^{C}D_t^{\alpha}x + \mu\,(x^2 - 1)\dot{x} + x = 0, \qquad 1 < \alpha < 2. \qquad (2.46)$$

Also (Barbosa et al., 2004) presented the following fractional VDP with both derivatives being fractional order, as follows:

$${}_{C}^{C}D_t^{1+\alpha}x + \mu\,(x^2 - 1)\,{}_{C}^{C}D_t^{\alpha}x + x = 0, \qquad 0 < \alpha < 1. \qquad (2.47)$$

2.4 Fractional Order Controllers

As stated in the previous section, fractional calculus was started since the time when the integer order calculus starts. The area of application of fractional calculus has been increasing rapidly. Fractional calculus allows us to define a real object more precisely than the traditional integer order calculus, this is because generally real objects wereoriginally fractional (Nakagava and Sorimachi, 1992;Podlubny, 1999a; Westerlund and Ekstam, 1994), but however, many of them have a very low fractionality. An example for a fractional order systems can be seen in a semi-infinite lousy transmission line of the voltage-current relation (Wang, 1987) and in the semi-infinite solid heat for diffusion, in which the flow of the heat is half (0.5) -derivative of its temperature (Podlubny, 1999b).

The frequent usage of integer order models over the fractional order models was due to the nonexistence of the solution techniquesof the FDE in those days. But nowadays there are many techniques that can be used for approximation of fractional order derivatives and/or integrals hence in the present days the fractional calculus can easily be used in different areas of applications like, control theory (it can be used in new fractional controllers and system modeling), electrical circuits theory (can be used in fractances, capacitor theory, etc.).

There are four situations for a fractional control of closed-loop control systems, as pointed in (Chen, 2006). They are

- Integer order (IO) model with Integer order (IO) controller.
- Integer order (IO) model with Fractional order (FO) controller.
- Fractional order (FO) model with Integer order (IO) controller.
- Fractional order (FO) model with Fractional order (FO) controller.

One of the major concern of any control engineer is, how to do the work better. Hence to get a better way of doing the work is one of the essence of control engineering. There are many evidences that showed that the controller designed from best fractional order modelwill perform better than the one designed from the best integer order model for a given system. Also many researchers in literature has given a reason of why is better to use a controller from fractional order models even in a place where its integer order counterpart can works approximately well (Monje, 2006; Monje et al., 2008). Fractional order control can be found more useful in situations where the dynamic of the system is of the nature of distributed parameter, this will make fractional order controllers more popular and will increase its area of application and worldwiderecognition.

Moreover, it was discovered that using fractional order controllers on integer order systems gives higher chance of adjustability in changing both gaincharacteristics as well as the phase characteristics of the controller if compared to the one designed from integer order (IO) models. This ability of flexibilities causes the fractional order (FO) controller to become one of the mighty tools used to design a robust control system that have fewer control tuning parameters. This means that a fractional order controller design witha few tuning knobs can have almost the same robustness with that of integer order (IO) design with a very high tuning knobs.

There are many different types of controllers designed with fractional order model in literature, in the following sections we give some of those controllers such as fractional order proportional integral derivative controllers(FO PID), fractional order model predictive controllers(FO MPC), fractional order sliding mode controllers(FO SMC), fractional order optimal controllers(FOD OC).

2.4.1 Fractional Order PID Controllers

PID controllers can be seen in many applications at industrial process control. It was estimated that 95% of controllers that are used as the closed-loop controllers in controlling the industrial applications are PID controllers. The term PID means Proportional Integral and Derivative. Hence the PID controllers are the combination of the mentioned three different controllers in a logical methods that it gives a single controlled output.

PID controller were started in 1911, where Elmer Sperry the pioneer of PID controller invented the basic Proportional controller. Later, in 1933 the Taylor Instrumental Company (TIC) developed a first Pneumatic controller which is fully tunable. Years later, some control engineers think of the way of removing the error that was in the steady-state of the Proportional controllers by bringing back some of the false values until the mistake is not zero. This method of removing the error gives what we now called Proportional Integral (PI) controller. The pioneer Pneumatic PID controller was invented in 1940 by using a derivative action in order to decrease the problems of overshooting.

Fractional order PID controllers are PID controllers designed from a fractional order models. Fractional controllers $PI^{\lambda}D^{\mu}$ also known as $PI^{\lambda}D^{\sigma}$controllers, were studied by (Podlubny, 1999a) in time domain and by (Petras, 1999) in frequency domain. The general way of presenting transfer function for$PI^{\lambda}D^{\sigma}$ can be given as:

$$C(S) = \frac{U(S)}{E(S)} = K_p + T_i S^{-\lambda} + T_d S^{\sigma}. \qquad (2.48)$$

with λ and σ been real numbers (positive), K_p is the gain of proportional, T_i *is* the constant of integration while T_d *is* the constant of differentiation. It can be clearly seen that by taking $\lambda = 1$ and $\sigma = 1$, we have the traditional (IO) P controller, when $\lambda = 0$ ($T_i = 0$) we have the PD^{σ} controller, and when $\sigma = 0$ ($T_d = 0$) we have the PI^{λ} controller etc.

All the aboveclasses of controllers are distinct classes of the $PI^{\lambda}D^{\sigma}$ controller that have output formula given as:

$$U(t) = K_p e(t) + T_i D_t^{-\lambda} e(t) + T_d D_t^{\sigma} e(t). \tag{2.49}$$

It can be noted that $PI^{\lambda}D^{\sigma}$ controller above can increase the performance of systems control because of its increasing number of tuning knobs introduced. But in theory, the $PI^{\lambda}D^{\sigma}$ it self is a linear filter which is dimensionally infinite because of the availability of the fractional order unit in either the differential or integral part or both. For literature in tuning methods of controllers, see (Monje et al., 2004; Chen et al., 2004).

In the same way that, a lot of papers on PID controllers have been published, every year, we expect many papers on FO PID to be published in the coming years. Some of the already published work are (Axtell and Bise, 1990; Blas et al., 2002; Manabe, 1961; Mehaut et al., 2004; Monje et al., 2009; Monje et al., 2008; Oustaloup, 2006; Xue and Chen, 2002; Xue et al., 2007).

2.4.2 Fractional Order Model Predictive Controllers

An optimal control theory using numerical optimization method is called Model predictive control (MPC). We can predict the future plant responses and the future control efforts of any system by the use of a system model predictive control where we optimized the system at given intervals of time with regards to a given performance function. MPC is a Computational method use to improve the performance control of applications and systems mostly at either chemical industries. Nowadays the predictive control is one among the popular progressive control techniques that are presently been used especially at industries (Rawlings, 2000; Muske and Rawlings, 1993; Bemporad, 2006; Morari and Lee, 1999; Garcia etal., 1989).

The basic principle of MPC is described as: For MPC, systems' models are used in order to predict what will be the output and also the control efforts that is needed to give the earmarked trajectory. Therefore, in case of MPC the model's accuracy always gives the control as well asthe exact prospect trajectory of the input which will be following the earmarked signal. This in short gives the elementary principle of operation of MPC. We can see that MPC is not a single technique but more of a methodology and can be called by various names like Model Predictive Control (MPC), Receding Horizon Control (RHC), Model Based Predictive Control (MBPC), Internal Model Control (IMC), Moving Horizon Control (MHC), etc.

For those systems whose are fractional in nature and were also described by a fractional order models instead of integer order, then their MPC controllers can also be fractional order MPC.

One of the key requirement of designing MPC is getting a state space model. For a fractional order model, approximation technique is used in obtaining this state space model, because we cannot convert the Fractional Order Transfer Function (FOTF) directly to a state space model by the use of simulation softwares such as MATLAB. Hence, we need to get its approximation that is, the transfer function of its identical in integer form, which can represent the FOTF and this equivalent integer transfer function will be converted easily to state space model using the simulation softwares.

The general form of the FOTFis given as:

$$G(s) = \frac{a}{S^{\alpha} + b}, \quad a, b \in \mathbb{R}. \tag{2.50}$$

with, $\alpha < 1$.

2.4.3 Fractional Order Sliding Mode Controllers

Sliding mode control (SMC) is one of the control techniques that are not linearand which changes the system's dynamics by using a non-continuous control signal which compels the whole system to slide through a transection of the original behavior of the systems. SMC is popularly known as a particular type of Variable structure control system. The main property of SMC is activating the control law which forces the systems' states to change from its initial states to a new predefined sliding surface. SMC, is among the mostsuccessful robust control methods and it has been applied to a different types of complex systems in engineering and science problems. By the introduction of fractional calculus in the early 17th century, which consists of integration and differentiations of fractional orders, it results in increase in the applications of FO controllers in almost all engineering applications. In fact, FO controllers are now very useful especially in the practical world. The SMC methodology has also been designed for FO based systems in many published works (Hosseinnia et al., 2010; Tavazoei and Haeri, 2008). Some of the applications of FO SMC has been given in (Yin et al., 2013; Yin et al., 2014).

A fractional order sliding mode control (FOSMC) is a freshly developed class in SMC family which show an additional benefits more than the integer order SMC because of the additional degrees of freedom to the controllers as a result of the flexibility of fractional orders in the derivative and integration. Hence these benefits are always expected when the traditional SMCenhances to become a Fractional Order SMC.

2.4.4 Fractional Order Optimal Controllers

Fractional Optimal Control Problems (FOCPs) are those problems of optimal control that contains fractional order models, they can be termed as the universal form of traditional optimal control problems (OCPs). The differential equations in FOCP are of fractional order that is FDEs, and its performance index are represented with fractional operator of integration (Choi et al., 2015).There are Several research works in literature give basic theories and essential foundationfor FOCPs, many of them studied in details the procedure of designing FOCPs and found the conditions of the optimal control for different states variables by the use ofboth numerical technique and analytical techniques (Agrawal, 2004; Agrawal and Baleanu, 2007; Agrawal, 2008; Jelicic and Petrovacki, 2009; Agrawal et al., 2010; Odzijewicz et al., 2012; Kamocki, 2014; Chinnathambi et al., 2019; Al-Mdallal and Abu Omer, 2018; Al-Mdallal and Hajji, 2015; Hajji and Al-Mdallal, 2018).

The optimal control of fractional order systems is studied relatively more than any other fractional control method. Nowadays, this control technique has been applied to the models of infectious diseases because it is more rapid and incurtailing the diseases, because of the property of the FO modelsthat makes it depends on the memory. Therefore, it has high possibility of becoming the outmost convenient tool used to modelinfectious and all other systems that are related to memory. Ding et al. (2012) used OC on fractional order model of HIV-Immune system and find the solution to the problem by applying an algorithm known as forward backward algorithm. Basir et al. (2015) gave an OC for fractional order kinetic model of enzyme with its numerical solution. Kheiri et al. (2018) created fractional model of HIV/AIDS with treatment, they incorporated three control techniques (effective use of condoms, ART treatment, and behavioral change control) together with their model. Sweilam et al. (2017) gave OC of the fractional order model for novel West Nile virus and use two numerical techniques to find the solution. Ali and Ameen (2020) studied and created the OC on the fractional order model for the transmission dynamics of pine wilt disease and proposed three s controls measures for controlling the disease.

The general method of forming and solving the problem of fractional optimal control (FOCP) is given by (Agrawal et al., 2004). In their formulation, they uses the left and right R-L definition of FO derivatives as in the form expressed below:

$$J(u) = \int_{0}^{T} F(x, u, t)\, d \ . \qquad (2.51)$$

Based on the following constraints of the dynamic system.

$${}_{a}^{C}D_{t}^{\alpha} x = G(x, u, t\ )(0 < \alpha < 1), \qquad (2.52)$$

and the initial conditions:

$$x(0) = x_0$$

where $x(t)$is the state variable.

The cost criteria for the integral in quadratic form is given below:

$$J(u) = \frac{1}{2}\int_{0}^{1}[q(t)x^2(t) + r(t)u^2]\, d\ . \quad (2.53)$$

Subject to the following:

$$ {}_{a}^{C}D_{t}^{\alpha}x = a(t)x + b(t)u. \quad (2.54)$$

Using the derivation given in (Agrawal et al., 2004), then the Euler-Lagrange equations for the above FOCP can be obtained as:

$$ {}_{a}^{C}D_{t}^{\alpha}x = G(x, u, t\ ),$$

$$ {}_{t}^{C}D_{1}^{\alpha}\lambda = \frac{\partial}{\partial} + \lambda\frac{G}{\partial} \quad (2.55)$$

$$0 = \frac{\partial}{\partial} + \lambda\frac{G}{\partial}$$

with $x(0) = x_0$ and $\lambda(1) = 0$.

The solution of the fractional Euler-Lagrange equations and many more methods of FOCPs with their results were all available in the literature.

CHAPTER 3

FRACTIONAL OPTIMAL CONTROL FOR COVID – 19 PANDEMIC MODEL

This chaptergives anoptimal control problem designed for the fractional order model of COVID-19. A Caputo fractional order derivative was used in formulating the model. We gave the state equations as well as the co-state equations and we also found the optimal policy that can be used in tackling the COVID-19 infections by placing two control laws, $u_1(t)$ that stands for the methods used in educating people about the infection disease, mask usage, restricting movement, and all control methods taking to prevents people in susceptible human population from acquiring the disease and $u_2(t)$ that stands for quarantine, treatment and monitoring of those that are already infected. The effectiveness of the control program was shown from the result of the numerical simulation that were carried out using RK-4. It can be noted that the control laws caused the reduction of vulnerable people in the susceptible population by $(1 - u_1(t))$ because of the campaign in educating people that were carried out and all the methods used in place. And also, because of the treatment and monitoring that were provided in the quarantine, number of those that are affected is also reduced by $(1 - u_2(t))$.

3.1 Introduction

COVID-19 is a new family of corona virusthat was detected in December 2019,at Wuhan, Chinafrom few patients that were diagnosed for bunch of intense respiratory illness from pneumonia cases (ECDC, 2019). It was found that, thisdisease was originally in relation with Severe Acute Respiratory Syndrome (SARS-COV).SARS-COV and Middle-East Respiratory Syndrome (MERS-COV) were among the classes of the Corona Viruses whose infect animals, they were emerged in the year 2002 and becomes an epidemic for humans (ECDC, 2019). Both SARS-COV and MERS-(COV) were categories as Corona Viruses fromzoonotic and their origin were said to possibly be Bats, and were firstlydetectedsince 1960.

This pandemic, COVID-19 escalate very fast and within short period of time it became global pandemic, presently the outbreak affects more than 150 countries, with infected people of more than 700,000 andmortality of almost 30,000. The disease can be spread out from an infected people by means of droplets produced as a result of cough, sneeze, and contacting a contaminated place, objects, or componentsthat are related to personal usage of infected people to the healthy people through their eye, nose and mouth (Img. Corona Ebook, 2019). Respiratory symptoms, shortness or difficulties in breathing, cough and fever, are among the main signs of the infection. The disease can result in creating failure in kidney, pneumonia, SARS, or in worth casesresult to death (WHO, 2019).

Tahir et al. (2019), created a mathematical model for MERS by using a differential equations with nonlinear system, for their model they considered camel as the source of the disease which transmits the virus to the human population that are infected, from there human to human transmission occur, after that patient to the clinic center transmission occur and then patient to care center transmission.

Optimal Control theory is one of the most powerful mathematical tools that can be utilized effectively to control a transmissionof infections during epidemic. In most cases OC is adoptedin the control of transmissionfor most of theepidemic diseases that their dose or medication is well-known. After the emergence of COVID – 19 pandemic, a lot of peopleinvestigate it by modeling the disease together withOCalone (Baba et al., 2020; Abioye et al., 2020; Peter et al., 2020a) while some of them considered the use of Fractional Order Model alone (Baba and Nasidi, 2020a; Baba and Nasidi, 2020b; Ahmed et al., 2020; Peter et al., 2020b). But in this work we considered the use of both, in which we use Fractional Order Model and incorporateour model with Optimal Control.

In this work we assumed the bat to be the source of the outbreak of COVID-19, then the (Tahir et al, 2019) model was modified by incorporating the vulnerable population of human and vulnerable population of Bats.Caputo Fractional Order derivative was then used to formulate the model because the accuracy of the result from Fractional Order exceeds that of integer order.We then gave the state equations as well as the co-state equations and we also found the optimal policy that can be used in tacklingthe COVID-19 infections

3.2 Formation of the Model

Our model was divided into the following separates compartments; Population for susceptible Bats, Population forinfective Bats, Population for susceptible Humans,Population for infective Humans, Population for Human to Human transmission, Population forinfective Individuals to Family Members transmission, Population for Patient to Clinic Center transmission and Population for Patient to Care Center transmission, the notations of the compartments are as follows:$S_b(t)$, $I_b(t)$, $S_h(t)$, $I_h(t)$, $H_h(t)$, $F_m(t)$, $P_c(t)$, $C_c(t)$respectively. By considering Bats to be the source of the disease, we considered that new born Bats belong to the Population for susceptible Bats S_b, with λ_b rate. This new born can later become members of the Population for infective Bats $I_b(t)$, at the rateβ_1. We also consideredthat the new born Humans belong to the Population for susceptible Humans$S_h$by $\lambda_h$rate. This new born can later become members of the Population for infective Humans $I_h$because of their association with already infected Bats with β_2 rate.Thedisease continue to transmit from the population for infected Human to the population for Human to HumanH_h, at rate β_3, population for infected topopulation for family member transmissionF_m, at rate β_4 then to population for patient to clinic center transmissionP_c at rate , β_5 and population for patient to care center transmissionC_c at the rate β_6. The description of variables used are given in Table 3.1 while that of parameters used are given in Table 3.2.

Table 3.1: Model variables used

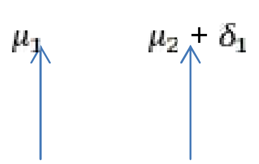
Variable	**Notation**
Population for susceptible Bats	$S_b(t)$
population for infective Bats	$I_b(t)$
Population for susceptible Humans	$S_h(t)$
Population forinfective Humans	$I_h(t)$
Population for Human to Human transmission	$H_h(t)$
Population for infective individual to Family members transmission	$F_m(t)$
Population for Patient to Clinic Center transmission	$P_c(t)$
Population for Patient to Care Center transmission	$C_c(t)$

Table 3.1 Continued

Table 3.2: Model parameters and their descriptions

Parameters	**Explanation**
$\mu_i, i = 1, 2, \ldots, 8$	Death rates in $S_b, I_b, S_h, I_h, H_h, F_m, P_c$, a $C_c$populations due to natural cause
$\delta_i, \; i = 1,2,3, \ldots, 6$	Death in I_h, H_h, F_m, P_c, a $C_c$populations due to disease cause
λ_b	Rate of birth in Bats population
λ_h	Rate of birth in Human population
$\beta_i, \; i = 1,2, \ldots, 6$	Transmission rates

The dynamics of the transmission is shown in Figure 3.1 and it was presented with a Nonlinear Caputo Fractional Order Differential Equations (FODE) nonlinear system of equations 3.1.



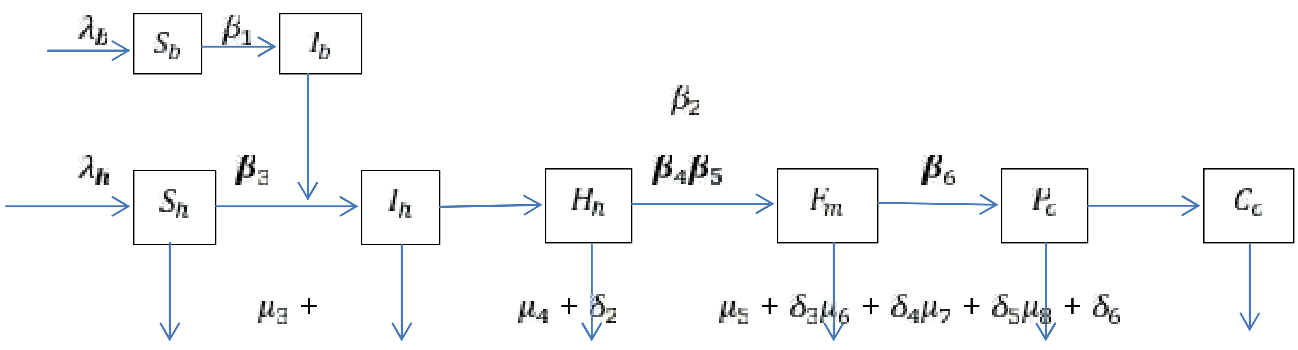


Figure 3.1: Transfer diagram for the transmission of the dynamics of COVID-19

$$
\begin{aligned}
{}^{C}_{0}D_t^{\alpha} S_b(t) &= \lambda_b^{\alpha} - \mu_1^{\alpha} S_b - \beta_1^{\alpha} S_b I_b \\
{}^{C}_{0}D_t^{\alpha} I_b(t) &= \beta_1^{\alpha} S_b I_b - (\mu_2^{\alpha} + \delta_1^{\alpha}) I_b - \beta_2^{\alpha} S_h I_b, \\
{}^{C}_{0}D_t^{\alpha} S_h(t) &= \lambda_h^{\alpha} - \mu_3^{\alpha} S_h - \beta_2^{\alpha} S_h I_b, \\
{}^{C}_{0}D_t^{\alpha} I_h(t) &= \beta_2^{\alpha} S_h I_b - (\mu_4^{\alpha} + \delta_2^{\alpha}) I_h - \beta_3^{\alpha} I_h H_h, \\
{}^{C}_{0}D_t^{\alpha} H_h(t) &= \beta_3^{\alpha} I_h H_h - (\mu_5^{\alpha} + \delta_3^{\alpha}) H_h - \beta_4^{\alpha} H_h F_m, \\
{}^{C}_{0}D_t^{\alpha} F_m(t) &= \beta_4^{\alpha} H_h F_m - (\mu_6^{\alpha} + \delta_4^{\alpha}) F_m - \beta_5^{\alpha} P_c F_m, \\
{}^{C}_{0}D_t^{\alpha} P_c(t) &= \beta_5^{\alpha} P_c F_m - (\mu_7^{\alpha} + \delta_5^{\alpha}) P_c - \beta_6^{\alpha} P_c C_c\,, \\
{}^{C}_{0}D_t^{\alpha} C_c(t) &= \beta_6^{\alpha} P_c C_c - (\mu_8^{\alpha} + \delta_6^{\alpha}) C_c,
\end{aligned}
\tag{3.1}
$$

with

$$S_b(0), I_b(0), S_h(0), I_h(0), H_h(0), F_m(0), P_c(0), C_c(0)\ a \quad a \quad \geq 0.$$

3.3 Basic Reproduction Number and Stability Analysis

At this section, we solved for the equilibriumof the model and performed the analysis of the local stability for the solutions, and also we found the Basic Reproduction number by using the stability conditions.

3.3.1 Equilibriums solutions

First we start by simultaneously solving the system of equations (3.2) which was found after equating the system in (3.1) to be zero, solving (3.2) gave us the solution at equilibrium.

$$
\begin{aligned}
&\lambda_b^\alpha - \mu_1^\alpha S_b - \beta_1^\alpha S_b I_b = 0,\\
&\beta_1^\alpha S_b I_b - (\mu_2^\alpha + \delta_1^\alpha) I_b - \beta_2^\alpha S_h I_b = 0,\\
&\lambda_h^\alpha - \mu_3^\alpha S_h - \beta_2^\alpha S_h I_b = 0,\\
&\beta_2^\alpha S_h I_b - (\mu_4^\alpha + \delta_2^\alpha) I_h - \beta_3^\alpha I_h H_h = 0,\\
&\beta_3^\alpha I_h H_h - (\mu_5^\alpha + \delta_3^\alpha) H_h - \beta_4^\alpha H_h F_m = 0,\\
&\beta_4^\alpha H_h F_m - (\mu_6^\alpha + \delta_4^\alpha) F_m - \beta_5^\alpha P_c F_m = 0,\\
&\beta_5^\alpha P_c F_m - (\mu_7^\alpha + \delta_5^\alpha) P_c - \beta_6^\alpha P_c C_c = 0,\\
&\beta_6^\alpha P_c C_c - (\mu_8^\alpha + \delta_6^\alpha) C_c = 0.
\end{aligned}
\tag{3.2}
$$

After solving (3.2) we got two equilibriums, namely E_0 (disease free equilibrium) and E_1 (endemic equilibrium).

where;

$$
E_0 = (S_b^0, \quad I_b^0, \quad S_h^0, \quad I_h^0, \quad H_h^0, \quad F_m^0, \quad P_c^0, \quad C_c^0) = \left(\frac{\lambda_b^\alpha}{\mu_1^\alpha}, 0, \frac{\lambda_h^\alpha}{\mu_3^\alpha}, 0, 0, 0, 0, 0\right),
$$

and

$$
E_1 = (S_b^1, \quad I_b^1, \quad S_h^1, \quad I_h^1, \quad H_h^1, \quad F_m^1, \quad P_c^1, \quad C_c^1),
$$

with

$$S_b^1 = \frac{\lambda_b^u}{\mu_1^u + \beta_1^u I_b^1},$$

$$I_b^1 = \frac{1}{2}\left[\frac{\lambda_b^u - \lambda_h^u}{\mu_2^u + \delta_1^u} + \frac{\mu_1^u}{\beta_1^u} + \frac{\mu_1^u}{\beta_2^u} \pm \sqrt{\left(\frac{\lambda_b^u - \lambda_h^u}{\mu_2^u + \delta_1^u} + \frac{\mu_1^u}{\beta_1^u} + \frac{\mu_1^u}{\beta_2^u}\right)^2 + \frac{4\lambda_b^u \mu_3^u}{\beta_2^u(\mu_2^u + \delta_1^u)}\left(1 - \frac{1}{R_0}\right)}\right],$$

$$S_h^1 = \frac{\lambda_h^u}{\mu_3^u + \beta_2^u I_b^1},$$

$$I_h^1 = \frac{\beta_2^u \beta_4^u \beta_6^u \lambda_h^u I_b^1}{[\beta_4^u \beta_6^u(\mu_4^u + \delta_2^u) + \beta_3^u \beta_6^u(\mu_6^u + \delta_4^u) + \beta_3^u \beta_5^u(\mu_8^u + \delta_6^u)](\mu_3^u + \beta_2^u I_b^1)},$$

$$H_h^1 = \frac{1}{\beta_4^u}\left[\mu_6^u + \delta_4^u + \frac{\beta_5^u(\mu_8^u + \delta_6^u)}{\beta_6^u}\right],$$

$$F_m^1 = \frac{\beta_2^u \beta_3^u \beta_6^u \lambda_h^u I_b^1}{[\beta_4^u \beta_6^u(\mu_4^u + \delta_2^u) + \beta_3^u \beta_6^u(\mu_6^u + \delta_4^u) + \beta_3^u \beta_5^u(\mu_8^u + \delta_6^u)](\mu_3^u + \beta_2^u I_b^1)} - \frac{1}{\beta_4^u}(\mu_5^u + \delta_3^u),$$

$$F_c^1 = \frac{(\mu_8^u + \delta_6^u)}{\beta_6^u},$$

$$C_c^1 = \frac{\beta_2^u \beta_3^u \beta_5^u \lambda_h^u I_b^1}{[\beta_4^u \beta_6^u(\mu_4^u + \delta_2^u) + \beta_3^u \beta_6^u(\mu_6^u + \delta_4^u) + \beta_3^u \beta_5^u(\mu_8^u + \delta_6^u)](\mu_3^u + \beta_2^u I_b^1)} - \frac{\beta_5^u}{\beta_4^u \beta_6^u}(\mu_5^u + \delta_3^u) - \frac{1}{\beta_6^u}(\mu_7^u + \delta_5^u).$$

In the disease free equilibrium only susceptible human population and susceptible bird population are non-zero but all the other populations are zero. Disease free equilibrium is presents all the time. While in the endemic equilibrium every population is present in other word there is no population that is zero at endemic equilibrium and this equilibrium point is more useful as far as mathematical analysisis concern.

3.3.2 Analysis of the Stability

In order to perform the stability analysis, we need to first create the Jacobian matrix of (3.1) as follows: by letting

$$J = \begin{bmatrix} N & \beta_1^\alpha S_b & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1^\alpha I_b & A & -\beta_2^\alpha I_b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_2^\alpha S_h & Q & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2^\alpha S_h & \beta_2^\alpha I_b & B & -\beta_3^\alpha I_h & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_3^\alpha H_h & C & -\beta_4^\alpha H_h & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_4^\alpha F_m & D & -\beta_4^\alpha F_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_5^\alpha P_c & E & -\beta_6^\alpha C_c \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_6^\alpha C_c & G \end{bmatrix}. \tag{3.3}$$

whe:

$$N = -\mu_1^\alpha - \beta_1^\alpha I_b, \qquad A = \beta_1^\alpha S_b - (\mu_2^\alpha + \delta_1^\alpha) - \beta_2^\alpha S_h,$$

$$B = -(\mu_4^\alpha + \delta_2^\alpha) - \beta_3^\alpha H_h, \ C = \beta_3^\alpha I_h - (\mu_5^\alpha + \delta_3^\alpha) - \beta_4^\alpha F_m,$$

$$D = \beta_4^\alpha H_h - (\mu_6^\alpha + \delta_4^\alpha) - \beta_5^\alpha P_c, E = \beta_5^\alpha F_m - (\mu_7^\alpha + \delta_5^\alpha) - \beta_6^\alpha C_c,$$

$$G = \beta_6^\alpha P_c - (\mu_8^\alpha + \delta_6^\alpha), Q = \mu_3^\alpha - \beta_2^\alpha I_b,$$

Theorem 3.1: The E_C is stable locally and asymptotically.

Proof: We evaluate our Jacobian matrix using E_C, and letting the following:

$$X_1 = \beta_1^\alpha \frac{\lambda_b^\alpha}{\mu_1^\alpha} - (\mu_2^\alpha + \delta_1^\alpha) - \beta_2^\alpha \frac{\lambda_h^\alpha}{\mu_3^\alpha}, \quad X_2 = -(\mu_4^\alpha + \delta_2^\alpha),$$

$$X_3 = -(\mu_5^\alpha + \delta_3^\alpha), \ X_4 = -(\mu_6^\alpha + \delta_4^\alpha),$$

$$X_5 = -(\mu_7^u + \delta_5^u), \qquad X_6 = -(\mu_8^u + \delta_6^u),$$

From where we get;

$$J(E_0) = \begin{bmatrix} -\mu_1^u & -\beta_1^u \lambda_b^u/\mu_1^u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_2^u \lambda_h^u/\mu_3^u & -\mu_3^u & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2^u \lambda_h^u/\mu_3^u & 0 & X_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_6 \end{bmatrix}.$$

By evaluating we have:

$$d \quad |J(E_0) - K \ | = 0,$$

From where we get the following eigenvalues as

$$K_1 = -\mu_1^u,$$

$$K_2 = \beta_1^u \frac{\lambda_b^u}{\mu_1^u} - (\mu_2^u + \delta_1^u) - \beta_2^u \frac{\lambda_h^u}{\mu_3^u},$$

$$K_3 = -\mu_3^u,$$

$$K_4 = -(\mu_4^u + \delta_2^u),$$

$$K_5 = -(\mu_5^u + \delta_3^u),$$

$$K_6 = -(\mu_6^u + \delta_4^u),$$

$$K_7 = -(\mu_7^u + \delta_5^u),$$

$$K_8 = -(\mu_8^u + \delta_6^u).$$

Since the $R(K_i) = 0, \quad i = 1,2,\ldots,8$ then clearly

$$|A \quad (K_1)| = \pi > \frac{\alpha}{2}, \qquad 0 < \alpha < 1.$$

Then by using the*theorem of stability, we can see that* E_C was proved to be stable locally and asymptotically.

Note: If the Eigen value$K_2 < 0$, DFE is then stable, that is:

$$\beta_1^{\alpha}\frac{\lambda_b^{\alpha}}{\mu_1^{\alpha}} - (\mu_2^{\alpha} + \delta_1^{\alpha}) - \beta_2^{\alpha}\frac{\lambda_h^{\alpha}}{\mu_3^{\alpha}} < 0,$$

By simplifying, we get;

$$\frac{\beta_1^{\alpha}\lambda_b^{\alpha}\mu_3^{\alpha}}{\mu_1^{\alpha}(\mu_3^{\alpha}(\mu_2^{\alpha} + \delta_1^{\alpha}) + \beta_2^{\alpha}\lambda_h^{\alpha})} < 1.$$

The basic reproduction number (R_C) can be defined as:

$$R_C = \frac{\beta_1^{\alpha}\lambda_b^{\alpha}\mu_3^{\alpha}}{\mu_1^{\alpha}(\mu_3^{\alpha}(\mu_2^{\alpha} + \delta_1^{\alpha}) + \beta_2^{\alpha}\lambda_h^{\alpha})}.$$

Theorem 3.2: For Endemic Equilibrium E_1 to be stable, $R_C > 1$.

Proof:

Since all the populations that are present in the $E_1$is related to I_b^1 then is enough to show the stability of I_b^1 alone,

Therefore, $I_b^1 > 0$, then

Either

$$\frac{\lambda_b^{\alpha} - \lambda_h^{\alpha}}{\mu_2^{\alpha} + \delta_1^{\alpha}} + \frac{\mu_1^{\alpha}}{\beta_1^{\alpha}} + \frac{\mu_1^{\alpha}}{\beta_2^{\alpha}} > \sqrt{\left(\frac{\lambda_b^{\alpha} - \lambda_h^{\alpha}}{\mu_2^{\alpha} + \delta_1^{\alpha}} + \frac{\mu_1^{\alpha}}{\beta_1^{\alpha}} + \frac{\mu_1^{\alpha}}{\beta_2^{\alpha}}\right)^2 + \frac{4\lambda_b^{\alpha}\mu_3^{\alpha}}{\beta_2^{\alpha}(\mu_2^{\alpha} + \delta_1^{\alpha})}\left(1 - \frac{1}{R_C}\right)}$$

$$0 > \frac{4\lambda_b^{\alpha}\mu_3^{\alpha}}{\beta_2^{\alpha}(\mu_2^{\alpha} + \delta_1^{\alpha})}\left(1 - \frac{1}{R_C}\right)$$

$$0 > \left(1 - \frac{1}{R_c}\right)$$

$$R_c < 1.$$

This can result in making I_b^1 to be complex.

Or

$$\sqrt{\left(\frac{\lambda_b^a - \lambda_h^a}{\mu_2^a + \delta_1^a} + \frac{\mu_1^a}{\beta_1^a} + \frac{\mu_1^a}{\beta_2^a}\right)^2 + \frac{4\lambda_b^a \mu_3^a}{\beta_2^a(\mu_2^a + \delta_1^a)}\left(1 - \frac{1}{R_c}\right)} > -\left(\frac{\lambda_b^a - \lambda_h^a}{\mu_2^a + \delta_1^a} + \frac{\mu_1^a}{\beta_1^a} + \frac{\mu_1^a}{\beta_2^a}\right)$$

$$\frac{4\lambda_b^a \mu_3^a}{\beta_2^a(\mu_2^a + \delta_1^a)}\left(1 - \frac{1}{R_c}\right) > 0$$

$$\left(1 - \frac{1}{R_c}\right) > 0$$

$$R_c > 1.$$

3.4. Formulation of the Optimal Controller

For this sub section, we incorporated the system model (3.1) with two control programs which are time-dependent namely, $u_1(t)$ that stands for the methods used in educating people about the infection disease, mask usage, restricting movement, and all control methods taking to prevents people in susceptible human population from acquiring the disease and $u_2(t)$ that stands for quarantine, treatment and monitoring of those that are already infected. It is expected that the control laws caused the reduction of vulnerable people in the susceptible population by $(1 - u_1(t))$ because of the campaign in educating people that were carried out and all the methods used in place. And also, because of the treatment and monitoring that were provided in the quarantine, number of those that are affected is also reduced by $(1 - u_2(t))$.

Therefore, our model inthe system equations (3.1) turn into:

$$
\begin{aligned}
{}^{C}_{0}D_t^{\alpha} S_b(t) &= \lambda_b^{\alpha} - \mu_1^{\alpha} S_b - \beta_1^{\alpha} S_b I_b, \\
{}^{C}_{0}D_t^{\alpha} I_b(t) &= \beta_1^{\alpha} S_b I_b - (\mu_2^{\alpha} + \delta_1^{\alpha}) I_b - \beta_2^{\alpha} S_h I_b, \\
{}^{C}_{0}D_t^{\alpha} S_h(t) &= \lambda_h^{\alpha} - \mu_3^{\alpha} S_h - \beta_2^{\alpha} (1 - u_1) S_h I_b, \\
{}^{C}_{0}D_t^{\alpha} I_h(t) &= \beta_2^{\alpha} (1 - u_1) S_h I_b - (\mu_4^{\alpha} + \delta_2^{\alpha}) I_h - \beta_3^{\alpha} (1 - u_2) I_h H_h, \qquad (3.4) \\
{}^{C}_{0}D_t^{\alpha} H_h(t) &= \beta_3^{\alpha} (1 - u_2) I_h H_h - (\mu_5^{\alpha} + \delta_3^{\alpha}) H_h - \beta_4^{\alpha} H_h K_m, \\
{}^{C}_{0}D_t^{\alpha} K_m(t) &= \beta_4^{\alpha} H_h K_m - (\mu_6^{\alpha} + \delta_4^{\alpha}) K_m - \beta_5^{\alpha} P_c K_m, \\
{}^{C}_{0}D_t^{\alpha} P_c(t) &= \beta_5^{\alpha} P_c K_m - (\mu_7^{\alpha} + \delta_5^{\alpha}) P_c - \beta_6^{\alpha} P_c C_c, \\
{}^{C}_{0}D_t^{\alpha} C_c(t) &= \beta_6^{\alpha} P_c C_c - (\mu_8^{\alpha} + \delta_6^{\alpha}) C_c.
\end{aligned}
$$

and the objective function can be given as:

$$J(u_1, u_2) = \int_0^{t_f} (aS_h + bI_h + cu_1^2 + d\,u_2^2)\, d \quad , \qquad (3.5)$$

With S_h been the population for susceptible Humans and I_h been the population of infective Humans. t_f been the ending time while the constants*a, b, c, d*standsfor weights and they are always positive. The objective here is minimizing the number of populations of susceptible Humans and infective Humansat the same time to minimize the expense of the two controls u_1, u_2. Hence, we need to getthe optimal control u_1^{*}, u_2^{*}, such that:

$$J(u_1^{*}, u_2^{*}) = \min_{u_1, u_2} \{ J(u_1, u_2) \mid u_1, u_2 \in \Omega \}. \qquad (3.6)$$

With the set of control as:

$$\Omega$$

The expenses of minimizing the number of population of susceptible Humans is represented by the term $aS_h$and that of minimizing the number of population of infective Humans is represented by bI_h , all the expenses associated with the control $u_1$like the expense for awareness, etc. is represented by cu_1^2 and also, all the expenses associated with the control $u_1$like the expenses for quarantine, etc. are represented by du_2^2. The sufficient conditions required forthe optimal control to fulfil can be found the most popularPMP. The said principle can be used to turn Equations (3.3) and (3.1) into a point-wise minimizingproblem of the Hamiltonian *M* with respect to (u_1, u_2) stated asfollows:

$$M = aS_h + bI_h + cu_1^2 + du_2^2 + \gamma_{S_b}\{\lambda_b^a - \mu_1^a S_b - \beta_1^a S_b I_b\} + \gamma_{I_b}\{\beta_1^a S_b I_b - (\mu_2^a + \delta_1^a) I_b - \beta_2^a S_h I_b\} + \gamma_{S_h}\{\lambda_h^a - \mu_3^a S_h - \beta_2^a (1-u_1) S_h I_b\} + \gamma_{I_h}\{\beta_2^a(1-u_1)S_h I_b - (\mu_4^a + \delta_2^a) I_h - \beta_3^a(1-u_2) I_h H_h\} + \gamma_{H_h}\{\beta_3^a(1-u_2) I_h H_h - (\mu_5^a + \delta_3^a) H_h - \beta_4^a H_h F_m\} + \gamma_{F_m}\{\beta_4^a H_h F_m - (\mu_6^a + \delta_4^a) F_m - \beta_5^a P_c F_m\} + \gamma_{P_c}\{\beta_5^a P_c F_m - (\mu_7^a + \delta_5^a) P_c - \beta_6^a P_c C_c\} + \gamma_{C_c}\{\beta_6^a P_c C_c - (\mu_8^a + \delta_6^a) C_c\}. \quad (3.7)$$

where,γ_{S_b} , γ_{I_b} , γ_{S_h} , γ_{I_h} , γ_{H_h} , γ_{F_m} , γ_{P_c} a γ_{C_c} are the adjoint variables

$$-\frac{d\gamma_{S_b}}{d} = \frac{\partial}{\partial S_b} = \gamma_{S_b}(-\mu_1^a - \beta_1^a I_b) + \gamma_{I_b}\beta_1^a I_b,$$

$$-\frac{d\gamma_{I_b}}{d} = \frac{\partial}{\partial I_b} = \gamma_{I_b}(\beta_1^a S_b - (\mu_2^a + \delta_1^a) - \beta_2^a S_h) - \gamma_{S_b}\beta_1^a S_b + \gamma_{S_h}\beta_2^a(1-u_1)S_h,$$

$$-\frac{d\gamma_{S_h}}{d} = \frac{\partial}{\partial S_h} = a + \gamma_{S_h}(-\mu_3^a - \beta_2^a(1-u_1)I_b) - \gamma_{I_b}\beta_2^a I_b + \gamma_{I_h}\beta_2^a(1-u_1)I_b,$$

$$-\frac{d\gamma_{I_h}}{d} = \frac{\partial}{\partial I_h} = b + \gamma_{I_h}(-(\mu_4^a + \delta_2^a) - \beta_3^a(1-u_2)H_h) - \gamma_{H_h}\beta_3^a(1-u_2)H_h,$$

$$-\frac{d\gamma_{H_h}}{d} = \frac{\partial}{\partial H_h} = \gamma_{H_h}(\beta_3^a(1-u_2)I_h - (\mu_5^a + \delta_3^a) - \beta_4^a F_m) + \gamma_{I_h}\beta_3^a(1-u_2)I_h,$$

$$-\frac{d\gamma_{F_m}}{d} = \frac{\partial}{\partial F_m} = \gamma_{F_m}(\beta_4^a H_h - (\mu_6^a + \delta_4^a) - \beta_5^a P_c) + \gamma_{H_h}\beta_4^a H_h, \quad (3.8)$$

$$-\frac{d\gamma_{P_c}}{d} = \frac{\partial}{\partial P_c} = \gamma_{P_c}(\beta_5^a F_m - (\mu_7^a + \delta_5^a) - \beta_6^a C_c) + \gamma_{F_m}\beta_5^a F_m,$$

$$-\frac{d\gamma_{C_c}}{d} = \frac{\partial}{\partial C_c} = \gamma_{C_c}(\beta_6^a P_c - (\mu_8^a + \delta_6^a)) + \gamma_{P_c}\beta_6^a P_c.$$

The transversality conditions are $\gamma_{S_E}(t_f) = \gamma_{I_b}(t_f) = \gamma_{S_h}(t_f) = \gamma_{I_h}(t_f) = \gamma_{H_h}(t_f) = \gamma_{F_m}(t_f) = \gamma_{P_c}(t_f) = \gamma_{L_c}(t_f) = 0$

For $0 < u_i < 1$, for $i = 1, 2$, from the interior of the controls, we have:

$$\frac{\partial}{\partial u_1} = 2cu_1 + \gamma_{S_h}\beta_2^{\alpha} S_h I_b - \gamma_{I_h}\beta_2^{\alpha} S_h I_b = 0,$$

$$\frac{\partial}{\partial u_2} = 2du_2 + \gamma_{I_h}\beta_3^{\alpha} I_h H_h - \gamma_{H_h}\beta_3^{\alpha} I_h H_h = 0.$$

from where;

$$u_1 = \frac{1}{2c}\left[\beta_2^{\alpha} S_h I_b (\gamma_{I_h} - \gamma_{S_h})\right], \tag{3.9}$$

$$u_2 = \frac{1}{2d}\left[\beta_3^{\alpha} I_h H_h (\gamma_{H_h} - \gamma_{I_h})\right].$$

Theorem 3.3: The *control values (u_1^{*}, u_2^{*}) which can minimize $J(u_1, u_2)$ over U are given by:*

$$u_1^{*} = m \left\{0, m \left[1, \frac{1}{2c}\left[\beta_2^{\alpha} S_h I_b (\gamma_{I_h} - \gamma_{S_h})\right]\right]\right\}, \tag{3.10}$$

$$u_2^{*} = m \left\{0, m \left[1, \frac{1}{2d}\left[\beta_3^{\alpha} I_h H_h (\gamma_{H_h} - \gamma_{I_h})\right]\right]\right\}.$$

Where $\gamma_{S_E}, \gamma_{I_E}, \gamma_{S_h}, \gamma_{I_h}, \gamma_{H_h}, \gamma_{F_m}, \gamma_{P_c}$ a γ_{L_c}, are, co-state variables that satisfy(3.1-3.8) *as well as the transversality conditions that follows:* $\gamma_{S_E}(t_f) = \gamma_{I_E}(t_f) = \gamma_{S_h}(t_f) = \gamma_{I_h}(t_f) = \gamma_{H_h}(t_f) = \gamma_{F_m}(t_f) = \gamma_{P_c}(t_f) = \gamma_{L_c}(t_f) = 0$ *and*

$$u_1^{*} = \begin{cases} 0, & i \quad u_1 \leq 0, \\ u_1, & i \quad 0 < u_1 < 1, \\ 1, & i \quad u_1 \geq 0, \end{cases} \tag{3.11}$$

$$u_2^{*} = \begin{cases} 0, & \text{if} \quad u_2 \leq 0, \\ u_2, & \text{if} \quad 0 < u_2 < 1, \\ 1, & \text{if} \quad u_2 \geq 0. \end{cases}$$

Proof:

To prove the existence of the optimal control solution we use the convexity of the integrand of J with respect to control u_1 and u_2, for the boundedness of the solutions of the stateand the Lipschitz property of the system of the state with respect to the variables of the state. Hence, we applyPMP and get the following:

$$ {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{S_b}(t) = \frac{\partial}{\partial S_b}; \quad {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{I_b}(t) = \frac{\partial}{\partial I_b}; \quad {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{S_h}(t) = \frac{\partial}{\partial S_h}$$

$$ {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{I_h}(t) = \frac{\partial}{\partial I_h}; \quad {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{H_h}(t) = \frac{\partial}{\partial H_h}; \quad {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{F_m}(t) = \frac{\partial}{\partial F_m} \qquad (3.12)$$

$$ {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{P_c}(t) = \frac{\partial}{\partial P_c}; \quad {}_{\mathrm{C}}^{\mathrm{t}}D_{t_f}^{\alpha}\gamma_{C_c}(t) = \frac{\partial}{\partial C_c}.$$

with, $\gamma_{S_b}(t_f) = \gamma_{I_b}(t_f) = \gamma_{S_h}(t_f) = \gamma_{I_h}(t_f) = \gamma_{H_h}(t_f) = \gamma_{F_m}(t_f) = \gamma_{P_c}(t_f) = \gamma_{C_c}(t_f) = 0$

The conditions for the optimality can be gotten after differentiating the Hamiltonian M with respect to u_1 and u_2:

$$\frac{\partial}{\partial u_1} = 0; \quad \frac{\partial}{\partial u_2} = 0 \qquad (3.13)$$

The adjoint system (3.7) and (3.8) comes from the solution of (3.9), and the pair of the optimal control (3.8) can be gotten from (3.10). The optimal system is comprises of the controlled system (3.2) and its initial conditions, system of adjoint (3.5) and conditions for transversality (3.6).

3.5. Numerical Simulation

Numerical simulations were performedat this sub section by the use of the values of the variables and parameter as given in (Tahir et al., 2019), Table 3.3 and Table 3.4 provides the variables and parameters used for the simulation respectively.

Table 3.3: Variables' value used

Notation	Variable	Value
S_b	Population for susceptible Bats	0.00-600.00
I_b	Population for infective Bats	200.00-500.00
S_h	Population for susceptible Humans	10,000,000.00
I_h	Populations for infective Humans	240.00-440.00
H_h	Population for Human to Human transmission	$100.00 - 400.00$
F_m	Population for infected individual to family members transmission	$40.00 - 200.00$
P_c	Population forPatient to Clinic Center transmission	$0.00 - 300.00$
C_c	Population for Patient to Care Center transmission	$0.00 - 300.00$

Table 3.4: Parameters' value used

Notation	Parameter	Value
β_1	Rate oftransmission from population of susceptible Bats toinfective Bats	01.2300
β_2	Rate of transmission frompopulation of infective Bats to infective Humans	00.1000
β_3	Rate of transmission from population of infectiveHumans to Healthy Humans	00.0060

β_4	Rate of transmission from population of infectiveHumans to own family member	01.0090
β_5	Rate of transmission from population of patient to clinic center	00.0040
β_6	Rate of transmission from population of patient to care center	00.0900
λ_b	Rate of given birth to Bats	01.5000
λ_h	Rate of given birth to Humans	01.2500
μ_1	Rate of natural death in the population of susceptible Bats	01.7000
μ_2	Rate of natural death in the population of infective Bats	00.1340
μ_3	Rate of natural death in the population of susceptible Humans	00.5000
μ_4	Rate of natural death in the population of infective Humans	00.1343
μ_5	Rate of natural death in the population of infective Humans to Healthy Humans	00.0024
μ_6	Rate of natural death in the population of infective Humans to own Family	00.0074
μ_7	Rate of natural death in the population of Patient to Clinic Center	00.3440
μ_8	Rate of natural death in the population of Patient to Care Center	00.5410
δ_1	Disease induced death in infected bats population	0.0143
δ_2	Death as a result of the disease in population of infective Humans	00.3002
δ_3	Death as a result of the disease in population of infective Humans to Healthy	00.0054
δ_4	Death as a result of the disease in population of infective Humans to own family	0.0019
δ_5	Death as a result of the disease in population of Patient to Clinic Center	0.0640
δ_6	Death as a result of the disease in population of Patient to Care Center	0.4400

Figures 3.2-3.6 give the simulation results, in which Figure 3.2 shows the all populations' dynamicagainst time. Figure 3.3 presents the population for the susceptible humans against that of infective humans. Figure 3.4 gives the population of infective humansagainst time atvariousα values. While Figures 3.5 and 3.6 give the effectiveness of our control lawsused to curtail the infection.

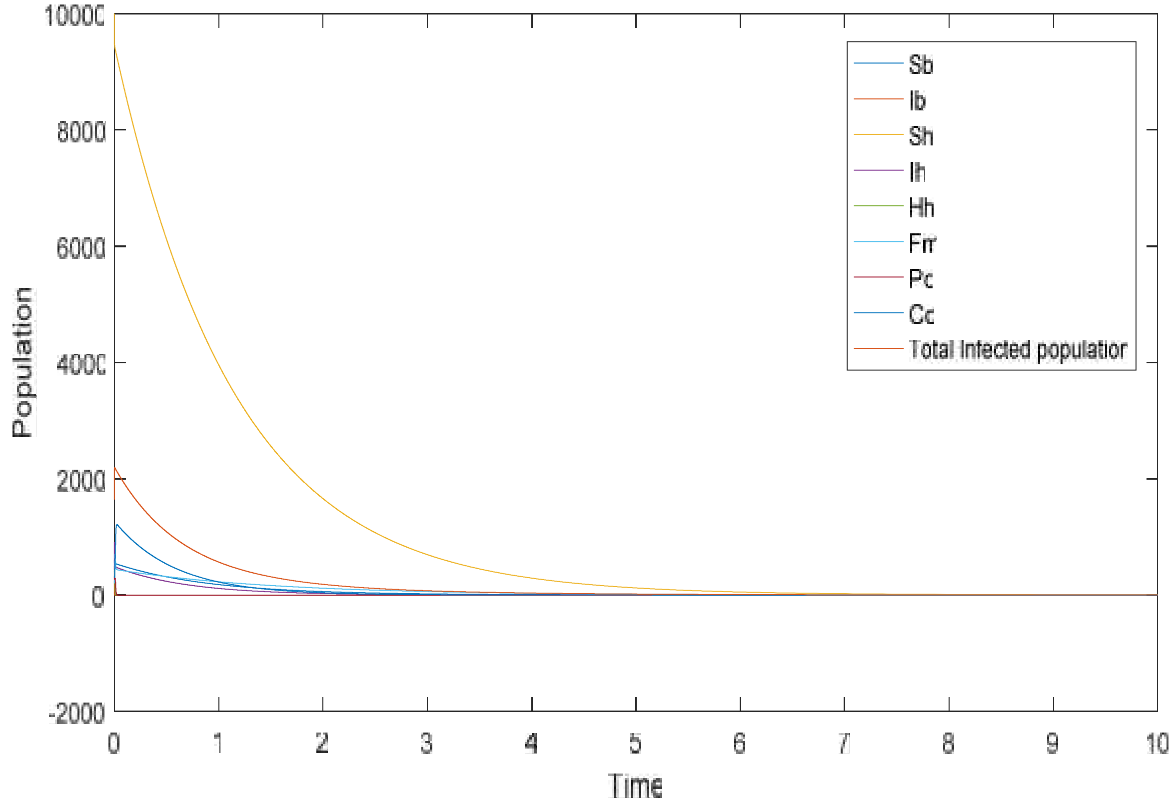


Figure 3.2: All populations' dynamicagainst time

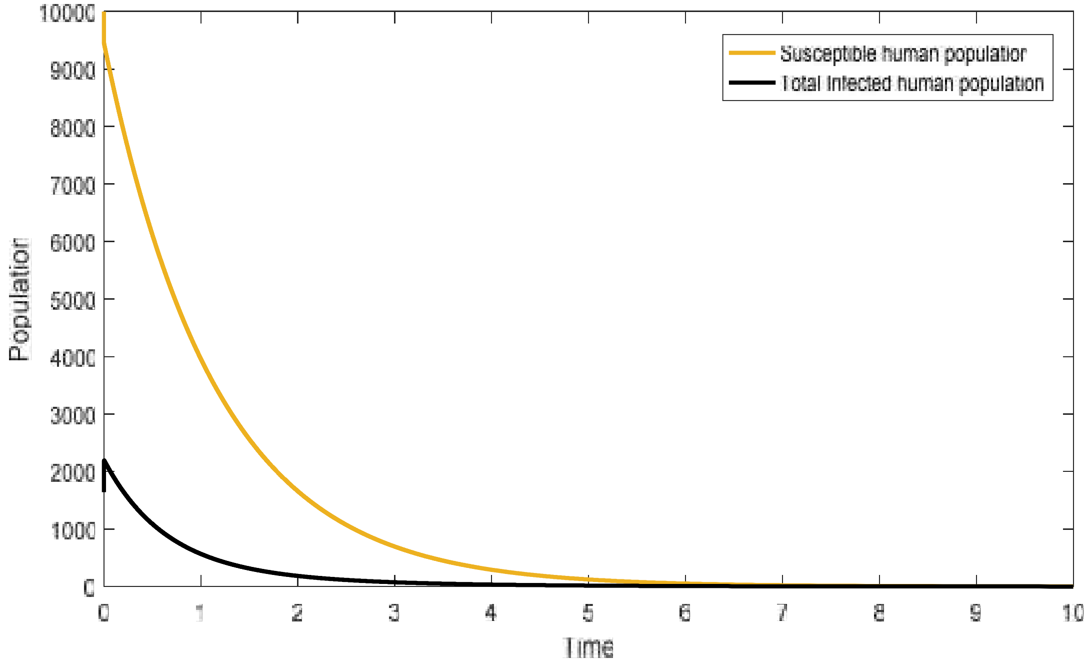


Figure 3.3: Population of susceptible humansagainst Population of infective humans

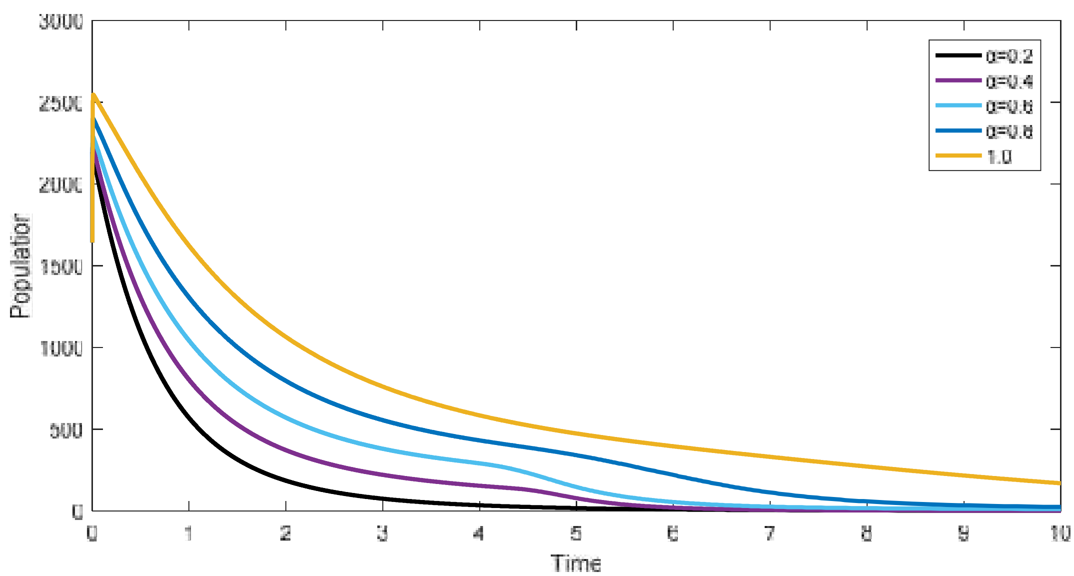


Figure 3.4: Population of infective humans for differentαvalues

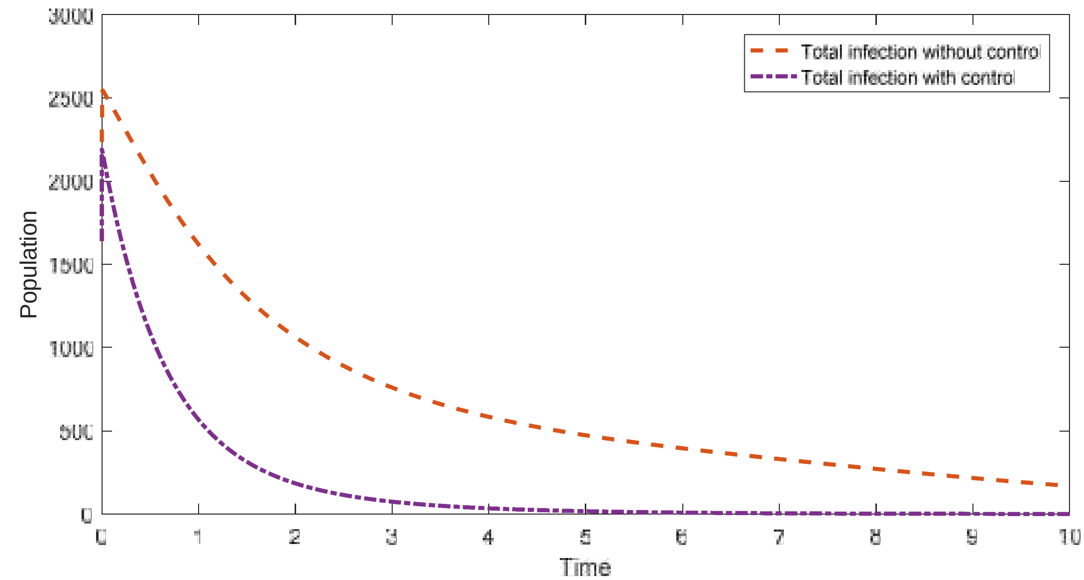


Figure 3.5: Population of infective humans with control against that without control

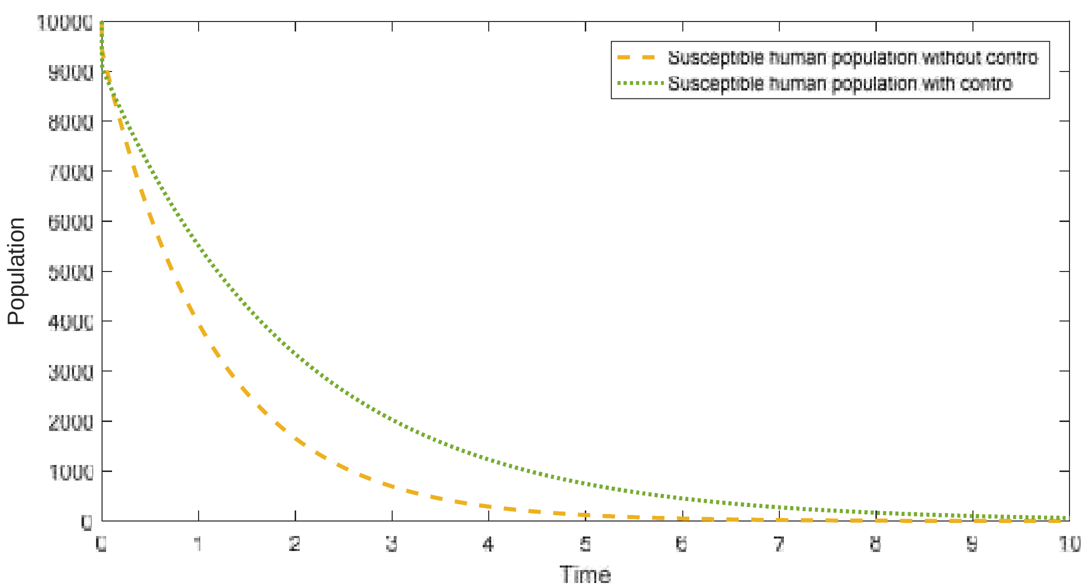


Figure 3.6: Population of susceptible humans with control against that without control

3.5 Results and Discussion

We can see from Figure3.3 that, there are too much people that may be infected to the disease over a period of time, and Figure 3.4 shows that biological systems are describes better with FODEs because they have richer dynamics than conventional integer models. It can also be seen that the model's solution, depends continuously on the time fractional derivative for various values of α,, but it can decays back to equilibrium. Figure 3.5 presents the comparison between population of infective people with the present of controllers and that without controller in place, as it can be seen, the population of infective people when there is a present of controllersdecreased notably because of the effect of the controllers. Also from Figure 3.6 which gives the comparison between the population ofthe susceptible people with the present of controllers and that without the controllers, we also noted here that the population of susceptible people when there is a present of controllersincreased notably because of the effect of the controllers that reduced the number of exposed people.

3.6. Summary and Conclusion

In this chapter we have given the formation of an FOCP for COVID-19 pandemic. A Caputo fractional order derivative was used in formulating the model. We gave the state equations as well as the co-state equations and we also found the optimal policy that can be used in tackling the COVID-19 infections by placing two control laws, $u_1(t)$ that stands for the methods used in educating people about the infection disease, mask usage, restricting movement, and all control methods taking to prevents people in susceptible human population from acquiring the disease and $u_2(t)$ that stands for quarantine, treatment and monitoring of those that are already infected. The effectiveness of the control program was shown from the result of the numerical simulation that were carried out using RK-4. We also found all the equilibrium solutions and theirstability analysis were given locally. The result of the simulation clearly shows that by optimally using the control measures the population of the infective will reduce and the population of the susceptible will increases. Then we can conclude that the control measures will significantly contribute in curtailing the disease when optimally used.

CHAPTER 4

FRACTIONAL OPTIMAL CONTROL MODEL TO STUDY ILLICIT DRUG USAGE

One of the serious problems worldwide is the increase in illicit drug usage, which was mainly due to improper maintenance and/or lack of proper principles governing the situation. This chapter presents a Fractional Optimal Control problem (FOCP) was formulated for the illicit drug usage with a mathematical model of Caputo form fractional order derivative. State as well as co-state equations were found and the best strategy that will notably reduce the increase in illicit drug usage is presented by the help of two control laws that are both depends on time, $u_1(t)$ (awareness campaign against illicit drug, proper monitoring and guidance, severe punishment to the culprits when caught from government side and from the parents side, taking responsibility of their wards, proper monitoring and all other measures that can be taken to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$(catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users). The effectiveness of the controller was shown by carrying out the numerical simulations.

4.1 Introduction

Illicit drug can simply be define as using the drugs which are banned by law in non-medical way (Degenhardt et al., 2004). In 2012 it was reported that, there were betweenone hundred and sixty two million and three hundred and twenty four million people within the age bracket of 15 and 64 globally that had used an illicit drug (UNODC, 2014). The risk of premature morbidity and mortality that results from illicit drug usage largely depend on frequency, route of administration, and dose (Degenhardt et al., 2004). Moreover, the risks of the mortality increase by increasing both the amount being consumed and frequency (Frischer et al., 1994). Donoghoe estimated that, in 1990, there were 10, 000 death globally as a result of the illicit drug usage and almost 60% of them are from developing countries (Murry et al., 2007). In addition 50% of the reported cases of mental illness are caused by illicit drug usage (CMHA, 2005).

There are different types of illicit drug usage pattern that was categories base on the type of the drugs are in use. Drinking, smoking, and injection are the most rampant (Ibrahim, 2016). Many consumers take a mixture of alcohol with cannabis, Lacasera with lizard dung, Tramadol and Codeine or inhale latrine and paint (WACD, 2014). Though, it is still not clear which of these drugs is being used more frequently by the consumers, United Nation Office on Drugs and Crime (UNODC, 2011) claimed that those that are smoking cannabis are the most possibly be the highest used than any other drugs.

Illicit drugs usage has many negative side effects among which are: It creates contortion in the consciousness for the consumers' sense (Inciardi et al., 1993). It causes shrink in the brain and damage the cells of the brain, which leads to damaging the whole brain (Ejikeme, 2010). It beclouds the consumers' sense of judgment (Maguire and Pastore, 1999). Some drugs like stimulant results in nervousness, restlessness, causes the consumers to become more aggressive and anxietywhich is beyond the consumers' control by activating the central nervous system (Lahey, 2010). Many research reported that breakdown of emotionand lack of controlling one's self that causes inner-city crises, crime, and youth violence are all due to drug abuse (Klantschnig, 2014). Illicit drug use is still one of the notable worldwide threat to the public health despite the number of theoretical and clinical studies and educational campaigns from both governmental and nongovernmental organization (UNODC, 2014; CMHA, 2005; UNODC, 2012; Mushayabasa et al., 2012; Mushayabasa and Bhunu, 2011).

Mathematical models play an important roles in studying the dynamics of drug usage (Nyabadza et al., 2013; Samanta, 2011; White and Comiskey, 2007). For example, (White and Comiskey, 2007) created a model for evaluating the duty of treatment and relapsing in heroin's dynamic. Nyabadza et al. (2013) proposed a model for assessing the abuse in crystal meth ”Tik” in the availability of supply chain of the drugs.

An optimal control theory provides us with more detailsabout the dynamics for any mathematical model and it is yet another tool for model's analysis. There are someOCPthatcontains fractional calculus, those are referred to Fractional Optimal Control Problems (FOCPs), this type of OCP are the general form of the Classical Optimal Control Problems (OCPs), the Differential Equations (DEs) used for FOCP were alwaysgiven as Fractional Differential Equations (FDEs), and also the performance function is always provided as Fractional Integration Operator (Ali et al., 2016).

Fractional Optimal Control problem for mathematical model of Illicit Drug Usage was designed in this chapter with Caputo format of Fractional Order Derivative (FOD). The equations for the state as well as that for co-state were provided and the optimal technique that cannotably reduce the increase in illicit drug usage was found.

4.2 Formulation of the Model

We divide the total number of the people $N(t)$ to five classes. The classes are $S(t)$,the class of those that are not drug users but mingles with drug users,$L(t)$, the class oflight or occasional drug users,$H(t)$, class of heavy drug users, $M(t)$, class of mentally ill due to drug usageα $D(t)$,class of detected illicit drug users. Table 4.1 gave the meaning of the variables and parameters which areall positive.

Table 4.1: Meaning of parameters used in the model

Notation	Parameter
Λ	Recruitment of individuals.
β	Interaction strengthamong susceptible people and illicit drug consumers
k	Modification factor.
Ψ	Rate of converting the Light drug consumers to become Heavy consumers.
γ	Rate of identifying and rehabilitating Light drug consumers
ρ	Rate of identifying and rehabilitating Heavy drug consumers
ϵ	Rate of identifying and rehabilitating Mentally ill drug consumers

Table 4.1 continued

o	Rate of developingMental illness by Light drug consumers
□	Rate of developingMental illness by Heavy drug consumers
ψ	Rate of forever quittingdue to ceasing or death as a result of the drugs by Light drug consumers.
d	Rate of forever quittingdue to ceasing or death as a result of the drugs by Heavy drug consumers.
ω	Rate of recovering from the rehabilitation center.
δ	Rate of dying of mentally ill people due to the illegal drug usage.

The model can be seen from the followingsystem of Fractional Order Differential Equations (FODEs).

$$
\begin{aligned}
{}_{C}^{C}D_t^{\alpha}S(t) &= \Lambda^{\alpha} - \beta^{\alpha}(L + k^{\alpha}H)S - \mu^{\alpha}, \\
{}_{C}^{C}D_t^{\alpha}L(t) &= \beta^{\alpha}(L + k^{\alpha}H)S - (\Psi^{\alpha} + \gamma^{\alpha} + o^{\alpha} + \mu^{\alpha} + \psi^{\alpha})L, \\
{}_{C}^{C}D_t^{\alpha}H(t) &= \Psi^{\alpha}L - (\rho^{\alpha} + □^{\alpha} + \mu^{\alpha} + d^{\alpha})H \qquad (4.1) \\
{}_{C}^{C}D_t^{\alpha}M(t) &= o^{\alpha}L + □^{\alpha}H - (\epsilon^{\alpha} + \mu^{\alpha} + \delta^{\alpha})M, \\
{}_{0}^{C}D_t^{\alpha}D(t) &= \gamma^{\alpha}L + \rho^{\alpha}H + \epsilon^{\alpha}M - (\mu^{\alpha} + \omega^{\alpha})D, \\
S(0) \geq 0,\ L(0) &\geq 0,\ H(0) \geq 0,\ M(0) \geq 0,\ a \quad D(0) \geq 0.
\end{aligned}
$$

4.3 Optimal Control

In this sub section, optimal control problem was formulated and incorporated to our model by the use of two time-dependent control measures, $u_1(t)$(awareness campaign from government side, proper monitoring and guidance, severe punishment to the culprits when caught and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$(catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users). The vulnerable people in susceptible class was assumed to be reduced by $(1 - u_1(t))$ because of the enlightenment and the rest of relatedprecautions taken. Also, the class of light illicit drug consumers will be reduced by $(1 - u_2(t))$ because of the identifying, treating and punishing the consumers when caught by the authority.

Therefore, ourmodel in the system of equations (1) change to:

$$
\begin{aligned}
{}_0^C D_t^\alpha S(t) &= \Lambda^\alpha - \beta^\alpha(1 - u_1)(L + k^\alpha H)S - \mu^\alpha, \\
{}_0^C D_t^\alpha L(t) &= \beta^\alpha (1 - u_1)(L + k^\alpha H)S - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)(1 - u_2)L, \\
{}_0^C D_t^\alpha H(t) &= \Psi^\alpha(1 - u_2)L - (\rho^\alpha + \square^\alpha + \mu^\alpha + d^\alpha)H, \qquad (4.2)\\
{}_0^C D_t^\alpha M(t) &= \sigma^\alpha(1 - u_2)L + \square^\alpha H - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)M, \\
{}_0^C D_t^\alpha D(t) &= \gamma^\alpha(1 - u_2)L + \rho^\alpha H + \epsilon^\alpha M - (\mu^\alpha + \omega^\alpha)D.
\end{aligned}
$$

with the objective function given as:

$$J(u_1, u_2) = \int_0^{t_f} (a \quad + b \quad + c u_1^2 + d u_2^2)\, d \quad , \qquad (4.3)$$

where S is the susceptible population and L is the light illicit drug users' population, t_f is the ending time while the constant *a, b, c, and d*were the positive weights. The goal here is minimizing the number of people in susceptible and light illicit drug consumers'classesand at the same timealso minimizing the expense of the controls u_1, u_2. Therefore, we search for optimal controls $u_1^{\square}, u_2^{\square}$, such that

$$J(u_1^{\square}, u_2^{\square}) = \min_{u_1, u_2} \{ J(u_1, u_2) \mid u_1, u_2 \square \Omega \}, \qquad (4.4)$$

where the control set is

$$\Omega = \{ (u_1, u_2) \mid u_i : [0, t_f] \to [0, \infty)\ L \qquad m \qquad , \quad i = 1,2, \}$$

The term a standsfor the expense of reducing the number of people in susceptible class and b stands for the expense of reducing the number of people in the light drug consumers' class, cu_1^2 is the cost of awareness campaign, proper monitoring and guidance, severe punishment to the culprits when caught all from the government side and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population and also, du_2^2 is the cost of catching the illicit drug users and punishing them, and also the cost of using rehabilitation centers for monitoring and treatment of light illicit drug users. The sufficient conditions required by the OC can be found usingthe PMP. The principle converts Equations (4.2) - (4.4) to the point-wise problem for minimizing *T*(the Hamiltonian that follows)with respect to(u_1, u_2).

$$T = a \quad + b \quad + cu_1^2 + du_2^2 + \lambda_S[\Lambda^\alpha - \beta^\alpha (1 - u_1)(L + k^\alpha H)S - \mu^\alpha] + \lambda_L[\beta^\alpha (1 - u_1)(L + k^\alpha H)S - (\Psi^\alpha + \gamma^\alpha + o^\alpha + \mu^\alpha + \psi^\alpha)(1 - u_2)L] + \lambda_H[\Psi^\alpha(1 - u_2)L - (\rho^\alpha + \square^\alpha + \mu^\alpha + d^\alpha)H] + \lambda_M[o^\alpha(1 - u_2)L + \square^\alpha H - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)M] + \lambda_D[\gamma^\alpha(1 - u_2)L + \rho^\alpha H + \epsilon^\alpha M - (\mu^\alpha + \omega^\alpha)D]. \tag{4.5}$$

where, $\lambda_S, \lambda_L, \lambda_H, \lambda_M\ a \quad \lambda_D$ are the variable of the co-state.

$$-\frac{d\lambda_S}{d} = \frac{\partial}{\partial} = a + \beta^\alpha(1 - u_1)(L + k^\alpha H)(\lambda_L - \lambda_S),$$

$$-\frac{d\lambda_L}{d} = \frac{\partial}{\partial} = b + \beta^\alpha(1 - u_1)S(\lambda_L - \lambda_S) + (1 - u_2)[\lambda_H\Psi^\alpha + \lambda_D\gamma^\alpha + \lambda_M o^\alpha - (\Psi^\alpha + \gamma^\alpha + o^\alpha + \mu^\alpha + \psi^\alpha)\lambda_L], \tag{4.6}$$

$$-\frac{d\lambda_H}{d} = \frac{\partial}{\partial} = \beta^\alpha(1 - u_1)k^\alpha S(\lambda_L - \lambda_S) - (\rho^\alpha + \square^\alpha + \mu^\alpha + d^\alpha)\lambda_H + \lambda_M\square^\alpha + \lambda_D\rho^\alpha,$$

$$-\frac{d\lambda_M}{d} = \frac{\partial}{\partial} = \lambda_D\epsilon^\alpha - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)\lambda_M,$$

$$-\frac{d\lambda_D}{d} = \frac{\partial}{\partial} = -\lambda_D(\mu^{\alpha} + \omega^{\alpha}).$$

The transversality conditions are:

$$\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0 \qquad (4.7)$$

for the control set, when 0 < u_i <1, for i = 1, 2 we have:

$$\frac{\partial}{\partial u_1} = 2cu_1 + \beta^{\alpha} S(L + k^{\alpha} H)(\lambda_S - \lambda_L) = 0, \qquad (4.8)$$

$$\frac{\partial}{\partial u_2} = 2du_2 + L(\Psi^{\alpha} + \gamma^{\alpha} + \sigma^{\alpha} + \mu^{\alpha} + \psi^{\alpha})\lambda_L - (\lambda_H \Psi^{\alpha} + \lambda_D \gamma^{\alpha} + \lambda_M \sigma^{\alpha})L = 0.$$

from where;

$$u_1 = \frac{S}{2c}[-\beta^{\alpha}(L + k^{\alpha} H)(\lambda_S - \lambda_L)], \qquad (4.9)$$

$$u_2 = \frac{L}{2d}[(\lambda_H \Psi^{\alpha} + \lambda_D \gamma^{\alpha} + \lambda_M \sigma^{\alpha}) - (\Psi^{\alpha} + \gamma^{\alpha} + \sigma^{\alpha} + \mu^{\alpha} + \psi^{\alpha})\lambda_L].$$

Theorem 4.1: The *control parameters (* u_1^{*}, u_2^{*} *) that minimizes* $J(u_1, u_2)$ *over* U *are given by:*

$$u_1^{*} = m \left\{0, m \left[1, \frac{S}{2c}[-\beta^{\alpha}(L + k^{\alpha} H)(\lambda_S - \lambda_L)]\right]\right\}, \qquad (4.10)$$

$$u_2^{*} = m \left\{0, m \left[1, \frac{L}{2d}[(\lambda_H \Psi^{\alpha} + \lambda_D \gamma^{\alpha} + \lambda_M \sigma^{\alpha}) - (\Psi^{\alpha} + \gamma^{\alpha} + \sigma^{\alpha} + \mu^{\alpha} + \psi^{\alpha})\lambda_L]\right]\right\}$$

Where $\lambda_S, \lambda_L, \lambda_H, \lambda_M$ a λ_D *are the adjoin variables satisfying* (4.1-4.10) *and the following transversality conditions:* $\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0$ *and*

$$u_1^{*} = \begin{cases} 0, & i \quad u_1 \leq 0, \\ u_1, & i \quad 0 < u_1 < 1, \\ 1, & i \quad u_1 \geq 0, \end{cases} \qquad (4.11)$$

$$u_2^{*} = \begin{cases} 0, & \text{if} \quad u_2 \leq 0, \\ u_2, & \text{if} \quad 0 < u_2 < 1, \\ 1, & \text{if} \quad u_2 \geq 0. \end{cases}$$

***Proof*:**

To prove the existence of the optimal control solution we use the convexity of the integrand of J with respect to control u_1 and u_2, for the boundedness of the solutions of the stateand the Lipschitz property of the system of the state with respect to the variables of the state. Hence, we applyPMP and get the following:

$${}^{C}_{C}D^{\alpha}_{t_f} S\,(t) = \frac{\partial}{\partial}; \quad {}^{C}_{C}D^{\alpha}_{t_f} L\,(t) = \frac{\partial}{\partial}; \quad {}^{C}_{C}D^{\alpha}_{t_f} H\,(t) = \frac{\partial}{\partial}$$

$${}^{C}_{0}D^{\alpha}_{t_f} M\,(t) = \frac{\partial}{\partial}; \quad {}^{C}_{C}D^{\alpha}_{t_f} D\,(t) = \frac{\partial}{\partial}. \tag{4.12}$$

with, $\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0$.

The conditions for the optimality can be gotten after differentiating the Hamiltonian M with respect to u_1 and u_2:

$$\frac{\partial}{\partial u_1} = 0\,; \qquad \frac{\partial}{\partial u_2} = 0. \tag{4.13}$$

The adjoint system (4.4) and (4.7) comes from the solution of (4.10), and the pair of the optimal control (4.9) can be gotten from (4.12). The optimal system is comprises of the controlled system (4.2) and its initial conditions, system of adjoint (4.4) and conditions for transversality (4.6).

4.4 Numerical Simulations

In this section the numerical simulations are carried out to show the effectiveness of the controllers and to also support the analytic results, by considering a total population of 10,000 people. The parameter values used are given in Table 4.2.

Table 4.2: Values of parameters used

Notation	Parameter	Value
Λ	Recruitment of individuals.	200.0
β	Interaction strengthamong susceptible people and illicit drug consumers	0.000041
k	Modification factor.	4.000
Ψ	Rate of converting the Light drug consumers to become Heavy consumers.	0.009
γ	Rate of identifying and rehabilitating Light drug consumers	0.035
ρ	Rate of identifying and rehabilitating Heavy drug consumers	0.123
ϵ	Rate of identifying and rehabilitating mentally ill drug consumers	0.003
σ	Rate of developingMental illness by Light drug consumers	0.001
□	Rate of developingMental illness by Heavy drug consumers	0.014
ψ	Rate of forever quittingdue to ceasing or death as a result of the drugs by Light drug consumers.	0.120
d	Rate of forever quittingdue to ceasing or death as a result of the drugs by Heavy drug consumers.	0.002
ω	Rate of recovering from the rehabilitation center.	0.002
δ	Rate of dying of mentally ill people due to the illegal drug usage.	0.003

Figures 4.1-4.6 show the simulation results, Figure 4.1 gives the dynamic of different population without any control measure applied. Figure 4.2 gives the dynamic of different

population when only control measure u_1 is applied. Figure 4.3 gives the dynamic of different population when only control measure u_2 is applied. Figure 4.4 gives the dynamic of the population with both controls u_1 andu_2. Figure 4.5 gives dynamic for different population when $R_C > 0$ and Figure 4.6 gives dynamics for different population when $R_C < 0$.

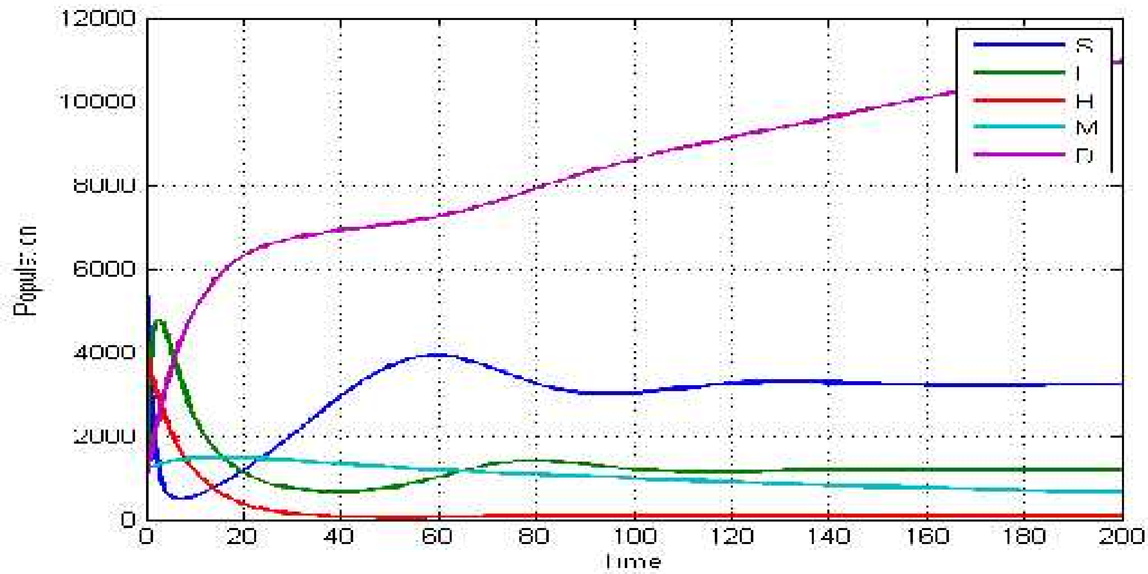


Figure 4.1: Massive increase in the population of illicit drug users without control

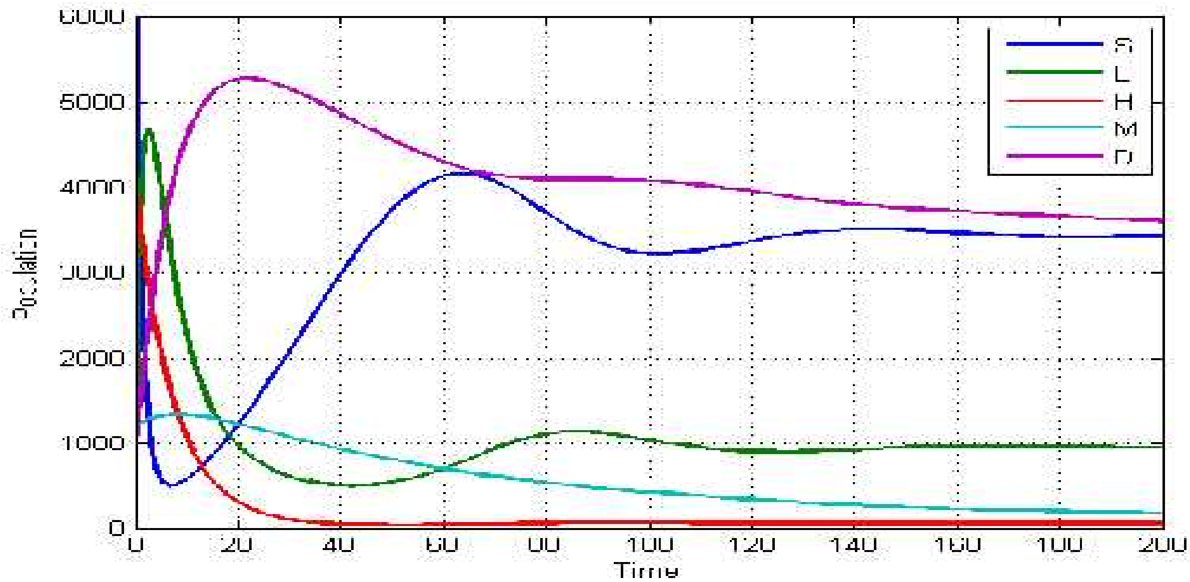


Figure 4.2: Dynamics of different populations when only u_1 is applied

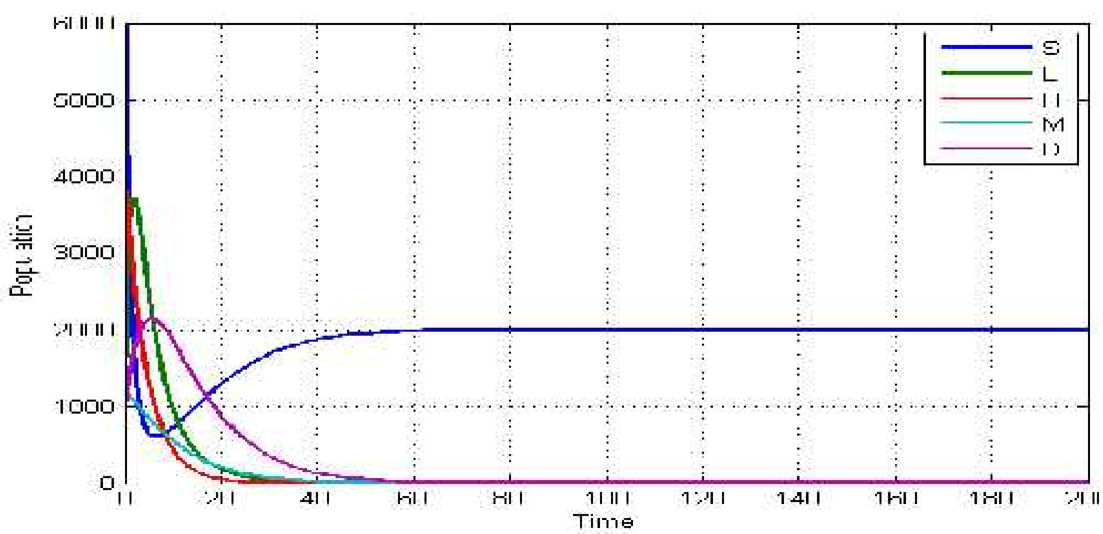


Figure 4.3: Dynamics of different populations when only u_2 is applied

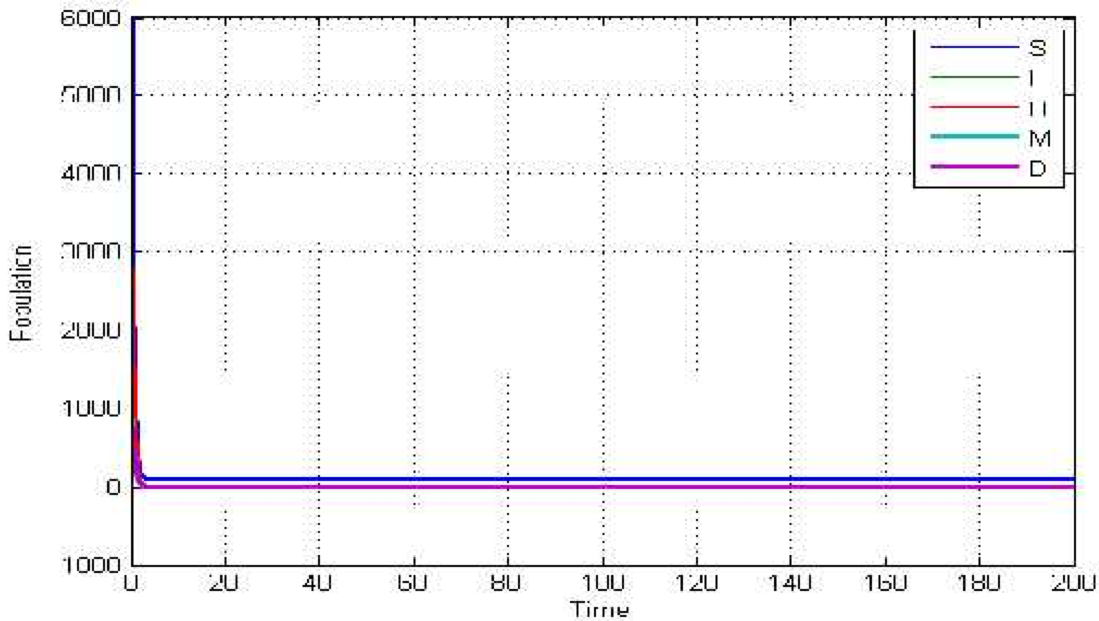


Figure 4.4: Populations' dynamics when both u_1 and u_2 are successfully applied

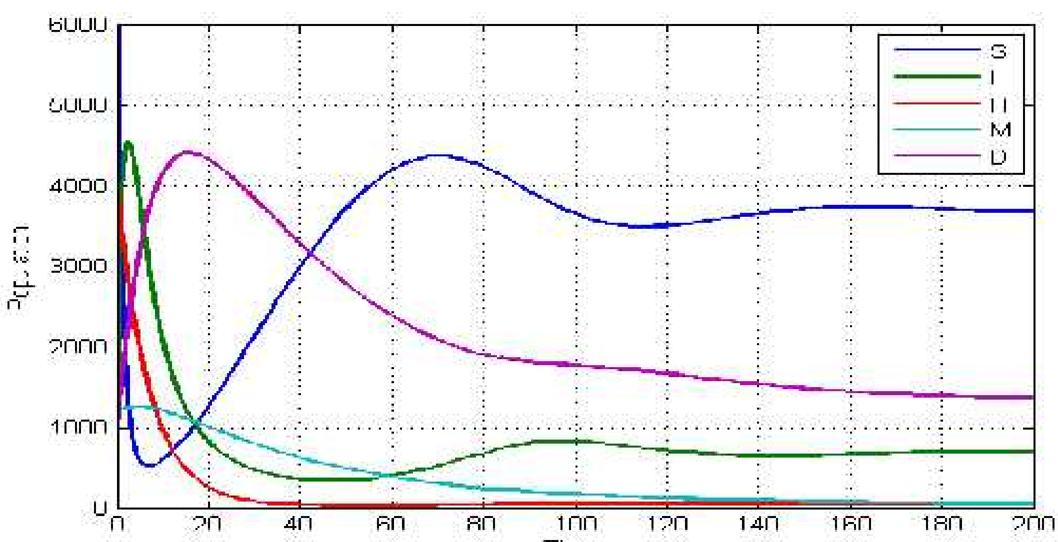


Figure 4.5: Dynamics of different populations when $R_C > 0$

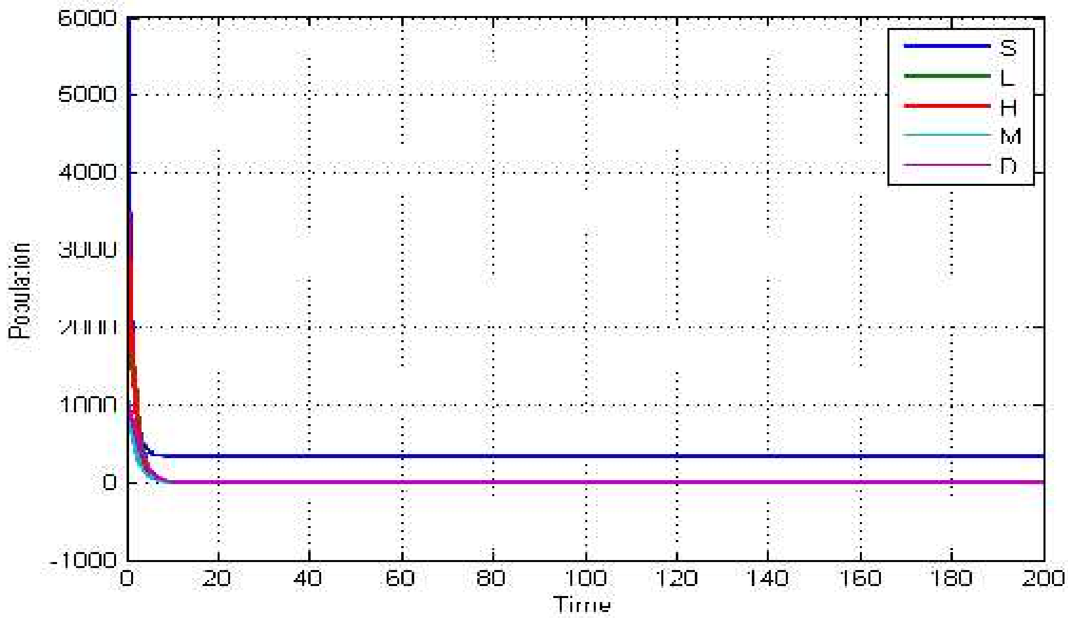


Figure 4.6: Dynamics of different populations when $R_C < 0$

4.5 Discussion

As we can see from the above Figures, Figure 4.1 gives dynamicsfor the total population without any control measure applied, it can be seen that there is a massive increase in the population of illicit drug users when there is no control. Figure 4.2 gives the dynamics of different populations when only u_1 is applied, it is observed that the strategy alone does not have much influence in curtailing the problem of illicit drug, though it reduce the number of population of the illicit drug. Figure 4.3 gives the dynamics of different populations when only u_2 is applied, it is cleared that u_2 control strategy reduce the number of the illicit drug consumers more than u_1 but still it does not curtail the problem significantly. Figure 4.4 shows the dynamic of different population when both u_1 a u_2 are applied, we can see that by successfully applying the two control measures the population of illicit drug users is drastically reduced, hence the best strategy for curtailing the problem is to applied both control measures.

4.6 Summary and Conclusion

FOCP model of the dynamics of illicit drug users was proposed at this chapter, in which the state equations as well as co-state equations were provided. The optimal control problems consists of two time dependents controls measures, $u_1(t)$(awareness campaign from government side, proper monitoring and guidance, severe punishment to the culprits when caught and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$(catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users). We also performed the numerical simulations to show the effectiveness of our control laws. It was noted that when both the two control laws were used in an optimal way the illicit drug consumers' population reduces notably, hence the best strategy of curtailing the problem is to apply both two control measures simultaneously.

CHAPTER 5

CONCLUSION

Optimal Control Problems (OCP) adapted to Fractional Order (FO) models was investigated in this thesis, we considered two different fractional order models in Caputo sense. The first model is based on the recent outbreak of COVID-19 infections in which two time dependents controls measures,$u_1(t)$and $u_2(t)$were used in the model and the state as well as co-state equations were given. We also found two equilibrium solutions of the model and the local stability analyses of the solutions were performed. By using the numerical simulation we showed that the control measures used can significantly contribute in curtailing the disease when optimally used. The second model is based on the Illegal drug usage, we also used two time dependents controls measures on the model again and the equations for the state as well as that of co-state were also given. By carrying out the numerical simulation we conclude that the best method to be adopted tocurtail the problem is to optimally apply both the two control measures at the same time.

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ID: 969425, 18 pages.

APPENDIX 1

ETHICAL APPROVAL LETTER

TO GRADUATE SCHOOL OF APPLIED SCIENCES

REFERENCE: BASHIR ABDULLAHI BABA (20178164)

I would like to inform you that the above candidate is one of our postgraduate students in Electrical and Electronics Engineering department he is taking thesis under my supervision and the thesis entailed: **OPTIMAL CONTROL ANALYSIS FOR FRACTIONAL ORDER MODELS.**The data used in his thesis does not require any ethical report.

Please do not hesitate to contact me if you have any further queries or questions.

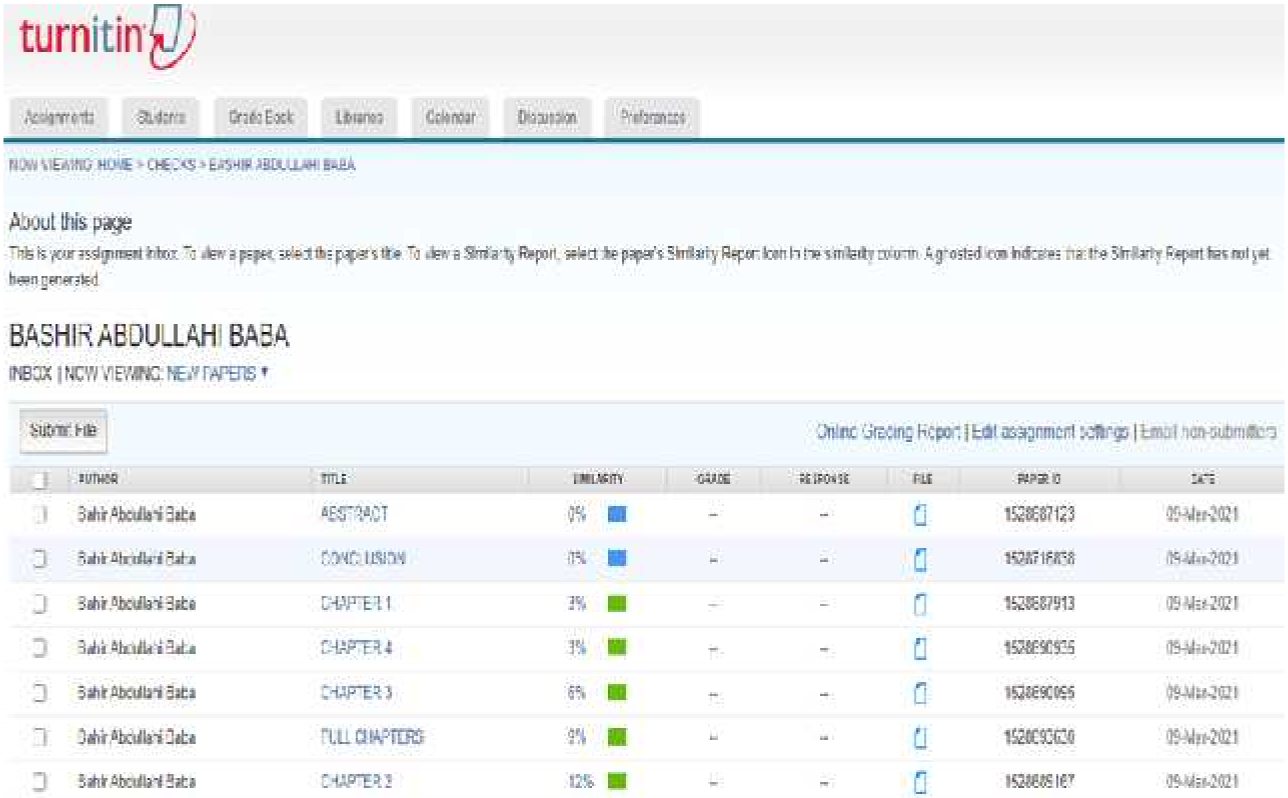
Thank you very much indeed.

Best Regards,

Prof. Dr. Bulent Bilgehan
Near East University,
Dean of the Faculty of Engineering,
Chairman of Electrical and Electronics Engineering Department,
Near East Boulevard, ZIP: 99138
Nicosia / TRNC, North Cyprus,
Mersin 10 – Turkey.
Email: Bulent.bilgehan@neu.edu.tr

APPENDIX 2

SIMILARITY REPORT



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Bahir Abdullahi Baba	CONCLUSION	0%	--	--		1528716858	09-Mar-2021
Bahir Abdullahi Baba	CHAPTER 1	3%	--	--		1528687913	09-Mar-2021
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Bahir Abdullahi Baba	FULL CHAPTERS	9%	--	--		1528693630	09-Mar-2021
Bahir Abdullahi Baba	CHAPTER 2	12%	--	--		1528689167	09-Mar-2021

APPENDIX 3

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name : Baba, Bashir Abdullahi

Nationality : Nigeria

Date and Place of Birth : 12 January 1983, Kano

Marital Status : Married

EDUCATION

Degree	Institution	Year of Graduation
M.Sc.	Meliksa University Kayseri, Turkey, Department of Electrical and Computer Engineering	2014
B.Sc.	Bayero University Kano, Nigeria, Department of Electrical Engineering	2006

WORK EXPERIENCE

Year	Place	Enrollment
2019 – Present	Department of Electrical and Electronics Engineering, NEU	Teaching Assistant
2015 – Present	Department of Computer Science, Sule Lamido University, Jigawa, Nigeria	Assistant Lecturer
2008 – 2012	Cash and Teller Unit, Zenith Bank Plc, Kafanchan Branch, Nigeria	Teller Service officer
2007 – 2008	Brains College, Kano, Nigeria	Teacher

FOREIGN LANGUAGES

English, spoken and written fluently

PUBLICATION IN INTERNATIONAL REFREED JOURNALS (IN COVERAGE OF SSCI AND SCI-EXPANDED):

- Baba, B. A., and Bilgehan, B. (2021). Optimal control of a fractional order model for COVID-19 pandemic.*Chaos, solitons & Fractals*, 144(5):110678, DOI: 10.1016/j.chaos.2021.110678

- Baba, I. A., Baba, B. A., and Esmaili, P. (2020). A mathematical model to study the effectiveness of some of the strategies adopted in curtailing the spread of COVID-19.*Computational and Mathematical Methods in Medicine*, 2020(1), DOI: 10.1155/2020/5248569

BULLETIN PRESENTED IN INTERNATIONAL ACADEMIC MEETINGS AND PUBLISHED IN PROCEEDING BOOKS:

- Baba, B.A., and Esmaili, P. (2021). Design of full state feedback controller for controlling depth of underwater robot. 4th International Conference of Mathematical Sciences (ICMS 2020), 2334, 060018, DOI: 10.1063/5.0042107

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- Baba, B. A., Ismaili, P., and Baba, I. A. (2019). Optimal control approach to study two strain malaria model. 3rd International Conference of Mathematical Sciences (ICMS 2019), 2183(1), 070004, DOI: 10.1063/1.5136166

COURSES GIVEN (*from 2015 to 2021*)

Undergraduate:

- Mechatronic Components and Instruments
- Marine Electro technology I
- Marine Electrotechnology II
- Electronics Laboratory I
- Electronics Laboratory II

- Mathematics for Technicians I
- Mathematics for Technicians II
- Electrical safety
- Refrigeration and Air conditioning
- Electrical Home Appliances
- Microprocessor and Micro computer
- Computer Architecture

HOBBIES

- Football, Readings, Travel, Music

OTHER INTERESTS

- Machines Learning, Robotics, Web-Design, Database, Data Structures and Programming Logics.