AMNA-ALZADJALI

DYNAMICAL MODEL TO OPTIMIZE STUDENT' S ACADEMIC PERFORMANCE USING SYSTEM OF ODEs

PhD THESIS

2022

**NEAR EAST UNIVERSITY**

**INSTITUTE OF GRADUATE STUDIES**

**DEPARTMENT OF MATHEMATICS**

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**Supervisor**

**Prof.Dr. Evren Hınçal**

**Nicosia**

**February, 2022**

# Approval

We certify that we have read the thesis submitted by **Amna Hashim Alzadjali** titled **"Dynamical Model to Optimize Student's Academic Performance Using System of ODE**" and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Mathematics.

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## Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

Amna Hashim Alzadjali

8/2/2022

## Acknowledgements

I would like to express my gratitude to my primary supervisor, Prof.Dr. Evren Hınçal, who guided me throughout this project and the staff in the mathematics department of the University of Near East. I would also like to thank my family and friends who supported me.

**Amna Hashim Alzadjali**

## Abstract

**Dynamical Model to Optimize Student's Academic Performance Using System of ODEs**


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**PhD, Department of Mathematics**


**February, 2022, 68 pages**


Excellent student's academic performance is the uppermost priority and goal of educators and facilitators. The dubious marginal rate between admission and graduation rates unveils the rates of dropout and withdrawal from school. To improve the academic performance of students, we optimize the performance indices to the dynamics describing the academic performance in the form of nonlinear system ordinary differential equations (ODE). We established the uniform boundedness of the model and the existence and uniqueness result. The independence and interdependence equilibria were found to be both locally and globally asymptotically stable. Subsequently, the basic reproduction number $R_0$ describing the progress from one academic categorical performance to another was obtained. $R_0 < 1$ implies an unprogressive, while $R_0 > 1$ implies the progressive. The optimal control analysis was carried out, and lastly, numerical method and numerical simulation was run to visualize the impact of performance index in optimizing the academic performance.



***Key Words**:* Optimization, academic performance, education, ODEs

# Özet

## ODE Sistemini Kullanarak Öğrencinin Akademik Performansını Optimize Etmek İçin Dinamik Model

**Alzadjali, Amna Hashim**

**PhD, Department of Mathematics**

**February, 2022, 68 pages**

Mükemmel öğrencinin akademik performansı, eğitimcilerin ve kolaylaştırıcıların en yüksek önceliği ve hedefidir. Kabul ve mezuniyet oranları arasındaki şüpheli marjinal oran, okulu bırakma ve okulu bırakma oranlarını ortaya çıkarmaktadır. Öğrencilerin akademik performansını iyileştirmek için, performans endekslerini, doğrusal olmayan sistem adi diferansiyel denklemler (ODE) biçiminde akademik performansı tanımlayan dinamiklere göre optimize ediyoruz. Modelin tekdüze sınırlılığını varlık ve teklik sonucunu belirledik. Bağımsızlık ve karşılıklı bağımlılık dengesinin hem yerel hem de küresel olarak asimptotik olarak kararlı olduğu bulundu. Bunun neticesinde , bir akademik kategorik performanstan diğerine ilerlemeyi tanımlayan temel yeniden üretim sayısı R_0 elde edildi. R_0<1 ilerleyici olmayan anlamına gelirken, R_0>1 ilerleyici anlamına gelir. Optimal kontrol analizi yapılmış ve son olarak, akademik performansı optimize etmede performans indeksinin etkisini görselleştirmek için sayısal yöntem ve sayısal simülasyon çalıştırılmıştır.

***Keywords*****:** Optimizasyon, akademik performans, eğitim, ODE'ler.

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## List of Abbreviations

**COVID-19:** Coronavirus Disease 2019

**MATLAB:** Matrix Laboratory

**NGM:** Next Generation Matrix

**ODE:** Ordinary Differential Equations

**RK:** Runge-Kutta

# CHAPTER I

## Introduction

The human capital theory perceives education and learning activities as an investment in people to increase the productivity of goods and services (Marginson, S. ., 2017). The industrial and technological development of a country depends on literacy as a requirement for its success, especially when a literate member of society engages in an active and effective role in the development process. There is no doubt that combining the skills of improving income generation with knowledge of sustainable development will assist mankind to improve his material condition of living through the use of resources available to him.

In fact, it was asserted that financial literacy that is blended with knowledge, skill and sound moral values is essential for individual to attain financial well-being. Hence, building the financial literacy of citizens will reduce poverty and improve personal financial management and welfare of the adult learner and the community (Matewos, K. R. , et al., 2016).

Academic performance of student is very crucial that makes a student improve himself intellectually, and admitted him into scientific and technological professions. Thus, it should be seen as a continuous process that requires regular study and sustained life-long learning. Poor academic performance steadies the scientific research and technological development which further result an underdeveloped economy.

Incentive theory was established in the 1940s and 1950s, by psychologists such as Clark Hull, who upheld the views that, the anticipation of reward by learners can influence positive performance which can lead to persistence in learning and achieving a high-grade score. While, the anticipation for punishment may result to withdrawal, less participation and less success reckoned in learning by the same learner who is disposed to two different situations of reward and punishment (Kendra, C. ., 2018). It is on the basis of the Incentive Theory, that the study was set to explore and optimize measures leading to effective academic performance of student.

The thesis outline to start with the introduction, definitions and theorems in chapter one, followed by the literature review in chapter two. Chapter three is the model formulation with uniform boundedness, existence and uniqueness inclusively, while chapter four gives stability analysis. Chapter five is the detail formation and analysis of optimal control and lastly, chapter six gives numerical method and numerical simulation, result discussion and conclusion.

**Definition of Terms**

**Definition 1 (Contraction)**

Let $(X,\ d)$ be a complete metric space, an operator $F: X \to X$ is said to be contractive if there is $\alpha \in (0,\ 1)$ such that (Nemer, M., 2015)

$$d(Fx, Fy) \leq \alpha\, d(x, y), \qquad \forall x, y\ \in X \tag{1.1}$$

**Definition 2 (Lipschitz Continuity)**

A function$f: X \subseteq \mathbb{R} \to\ \mathbb{R}$ is said to be Lipschitz continuous if $\exists!\, L > 0$ such that (Nemer, M., 2015)

$$d(fx, fy) \leq L\, d(x, y), \qquad \forall x, y\ \in X \tag{1.2}$$

**Remark 1:** one can observe that:

i. Every contractive operator is Lipschitz continuous
ii. Every Lipschitz is uniformly continuous. i.e
 given$\epsilon > 0,\ \exists\ \delta < \frac{\epsilon}{L}$: with $d(x, y) < \delta$ such that

$$d(Fx, Fy) \leq L\delta < \epsilon \tag{1.3}$$

**Definition 3 (Equilibrium)**

let

$$\begin{cases} x' = f(t, x), & \\ & f: \mathbb{R} \times \mathbb{R}^n\ \to \mathbb{R}^n \end{cases} \tag{1.4}$$

be a non-autonomous dynamical system.$x_e \in \mathbb{R}^n$is said to be the equilibrium of (1.4) if

$$f(t, x_e) = 0 \tag{1.5}$$

Meanwhile the equilibrium (or synonymously fixed point, steady state, stationary point, resting point) is the constant solution of the dynamical system (Dahleh, M. , et al., 2004).

**Definition 4 (Stability)**

The equilibrium solution$x_e$ of (1.4) is said to be (Dahleh, M. , et al., 2004)

i. **Stable:**if given $\epsilon > 0$, there is a $\delta > 0$such that every solution that start $x(0) = x_0$ satisfies

$$||x_0 - x_e|| < \delta \quad (1.6)$$

Exist for all $t > 0$ and satisfies

$$||x(t) - x_e|| < \epsilon, \quad \forall\ t \geq 0 \quad (1.7)$$

ii. **Asymptotically Stable:**if it is stable and if $\exists!\ \delta_0 > 0$ satisfies

$$||x_0 - x_e|| < \delta_0 \quad (1.8)$$

Then

$$\lim_{t\to\infty} x(t) = x_e \quad (1.9)$$

**Remark 2:** above statements means that all the solutions that starts sufficiently close (within the distance $\delta$) to $x_e$ stay close (within the distance $\epsilon$) to $x_e$. The trajectories that start sufficiently close to $x_e$ must not only stay close, but must eventually converge to $x_e$as $t \to \infty$.

**Definition 5 (Optimal Control)**

A fairly general continuous time optimal control problem can be defined as follows (Becerra, V. M. ., 2008):

**Problem i**: To find the control vector trajectory $u : [t_0,\ t_f] \in \mathbb{R} \to \mathbb{R}^n$ minimize the performance index:

$$J(\boldsymbol{u}) = \varphi\left(\boldsymbol{x}(t_f)\right) + \int_{t_0}^{t_f} L(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ t)dt \quad (1.10)$$

Subject to:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ t)\ , \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \qquad (1.11)$$

Where

$$\boldsymbol{x} = (x_{\boldsymbol{1}}, \quad x_{\boldsymbol{2}}, \quad x_{\boldsymbol{3}}, \dots, x_{\boldsymbol{n}})^{\boldsymbol{T}},\ \boldsymbol{f} = (f_{\boldsymbol{1}}, \quad f_{\boldsymbol{2}}, \quad f_{\boldsymbol{3}}, \dots, f_{\boldsymbol{n}})^{\boldsymbol{T}}$$

And

$\varphi :\ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a terminal cost function.

**Problem ii:** find$t_f$ and $\boldsymbol{u}(t)$ to minimize:

$$J = \int_{t_0}^{t_f} 1dt = t_f - t_0 \qquad (1.12)$$

Subject to:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ t)\ , \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \qquad (1.13)$$

This special type of optimal control problem is called the minimum time Problem.

**Remark 3:** It is often assumed that $x(t)$ and$u(t)$ are piecewise continuous and that the functions $\boldsymbol{f}$ meet certain differentiability conditions. These conditions are required to implement certain solution algorithms.

**Definition 6 (Hamiltonian)**

With a time varying Largrange's multiplier function$\lambda :\ [t_0,\ t_f] \to \mathbb{R}$, also known as the co-state define Hamiltonian function $H$ as (Becerra, V. M. ., 2008):

$$H(\boldsymbol{x}(t), \boldsymbol{u}(t), \lambda(t), t) = L(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ t) + \lambda(t)^T \boldsymbol{f}(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ t) \qquad (1.14)$$

Such that

$$J(\boldsymbol{u}) = \varphi\left(\boldsymbol{x}(t_f)\right) + \int_{t_0}^{t_f} \{H(\boldsymbol{x}(t), \boldsymbol{u}(t), \lambda(t), t) - \lambda^T(t)\boldsymbol{x}\}dt \qquad (1.15)$$

**Theorem 1 (Banach's Fixed Point Theorem) (**Banach, S. ., 1922**):**

Let $(X, d)$ be a complete metric space and let $T: X \to X$ be contractive operator. i.e there is a constant $\alpha \in (0,1)$ such that

$$\left||Tx - Ty|\right| \le \alpha\left||x - y|\right|, \qquad \forall\, x, y \in X \tag{1.16}$$

Then there exist a unique fixed point $x \in X$ such that$Tx = x$.

**Theorem 2 (Local Stability) ((**Dahleh, M. , et al., 2004**):**

The equilibrium solution$x_e$ of non-autonomous system is locally asymptotically stable if all the eigenvalues of Jacobian matrix$\frac{\partial f}{\partial x}$ at equilibrium satisfy

$$Re(\lambda_i) < 0, \quad i = 1,2, \dots, n$$

**Theorem 3 (**Lyapunov, A. M. ., 1892**): (Lyapunov Function Theorem)**

Let $x^*$ be an equilibrium point and $D \subseteq \mathbb{R}^n$ be a neighborhood of $x^*$. Let $V: D \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable function such that

i- $V(x^*) = 0 \ \ and\ V(x) > 0 \ \ for\ x \in D - \{x^*\}$

ii- $\frac{dV(x(t))}{dt} \le 0, \ \ x \in D$then the equilibrium point $x^*$ is stable.

Further, $x^*$ is asymptotically stable if

$$\frac{dV\big(x(t)\big)}{dt} < 0, \ \ x \in D - \{x^*\}$$

**Remark 4:** The idea behind Lyapunov's theorem is to establish properties of the equilibrium point (or, more generally, of the nonlinear system) by studying how certain carefully selected scalar functions of the state evolve as the system state evolves. This approach contrasts from theorem 2, which attempts to establish properties of the equilibrium point by studying the behavior of the linearized system at that point.

The continuous scalar positive definite function $V(x)$may be thought of as an energy function. If the time derivative of $V(x)$ along any trajectory of the system (1.4) is negative throughout the region (except at the origin), then this implies that the energy is strictly decreasing over time. In this case, because the energy is lower bounded by 0, the energy must go to 0, which implies that all trajectories converge to the zero state.

**Lemma 1 (Arithmetic-Geometric Inequality) (**Carlson, B. C. ., 1966**):**

If $x_1,\ x_2,\ x_3, \dots, x_n$is a sequence of non-negative real numbers, then

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \sqrt{x_1 \,.\, x_2 .\ x_3, \dots, x_n} \tag{1.17}$$

**Proof**

By applying the Jensen's inequality which states that value of a concave function of an arithmetic mean is greater than or equal to the arithmetic mean of the function values.

Since the logarithm function is concave, we have

$$\log\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \sum_{i=1}^{n} \frac{1}{n} \log x_i$$

$$= \sum_{i=1}^{n} \log x_i^{\frac{1}{n}}$$

$$= \log \prod_{i=1}^{n} x_i^{\frac{1}{n}}$$

$$\log\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \log \prod_{i=1}^{n} x_i^{\frac{1}{n}} \tag{1.18}$$

Taking the antilog of (1.18) gives the required (1.17)

**Theorem 4 (Pontryagin Maximum Principle) (**Abaidoo, A. ., 2018**):**

If $\boldsymbol{u}^*(t), \boldsymbol{x}^*(t)\big(t \in [t_0,\ t_f]\big)$ is a solution of the optimal control problem (1.10), (1.11) then there exist a non-zero absolutely continuous function $\lambda(t)$ such that $\lambda(t), \boldsymbol{x}^*(t), \boldsymbol{u}^*(t)$ satisfy the system

$$\frac{dx}{dt} = \frac{\partial H}{\partial \lambda}, \qquad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{1.19}$$

Such that, for almost all $t \in [t_0,\ t_f]$ the function in equation (2.4) attains its maximum:

$$H\big(\lambda(t), \boldsymbol{x}^*(t), \boldsymbol{u}^*(t)\big) = M(\lambda, x),$$

$$M\left(\lambda(t_f), \boldsymbol{x}^*(t_f)\right) = \sup\{H(\lambda, x, u) : u \in \mathcal{U}\} \tag{1.20}$$

And such that at terminal time $t_f$ the conditions

$M\left(\lambda(t_f), \boldsymbol{x}^*(t_f)\right) = 0, \qquad \lambda_0(t_f) \leq 0$ are satisfied.

If the functions $\lambda(t), x(t), u(t)$ satisfy the relation (1.19), (1.20) (i.e$x(t), u(t)$ are Portryagin extremals), then the condition

$\mathcal{M}(t) = M\big(\lambda(t), x(t)\big) = const., \qquad \lambda_0(t) = const$ hold.

**Remark 5** (Crosnoe, et al., 2004): for a minimum, it is necessary for the stationary (optimality) condition to give:

$$\frac{\partial H^T}{\partial \boldsymbol{u}} = 0 \tag{1.21}$$

**Aims and Objectives**

This research is aimed at optimizing the academic performance of the student through the following objectives:

- i- To construct mathematical model for the dynamics of categorical performance of student.
- ii- To establish the existence, uniqueness and stability results.
- iii- To formulate optimal control problem that optimize the academic performance of students.
- iv- To carryout control analysis
- v- To perform numerical simulation results.

**Scope and Limitation**

Although there is impediment to academic performance of student, but they are not of main interest of this research. This research only focuses on optimizing the academic performance of student.

**Methodology**

In this research:

i- We use information and assumptions to construct mathematical model

ii- We apply Banach's fixed point theorem and Picard-Lindelop theorem to establish the existence and uniqueness results.

iii- We carryout stability analysis (local and global) using both the linearization and Lyapunov theorems.

iv- With the use of Hamiltonian and the Pontryaging maximum principle we carryout control analysis.

v- Using Matlab software the numerical result will be performed.

# CHAPTER II

## Literature Review

Research related conceptual definitions, descriptions and information related to the subject that already exists in the literature are given in this chapter.

### Mathematical Model

Models of systems have become part of our everyday lives: In particular, many processes can be described with mathematical equations, that is, by mathematical models. Such models have use in a diverse range of disciplines.

According to Encyclopedia Britannica, a mathematical model is defined as ‘‘either a physical representation of mathematical concepts or a mathematical representation of reality.’’ Physical mathematical models, such as graphs of curves or surfaces defined by analytic equations or three-dimensional replicas of cylinders, pyramids, and spheres, are used to visualize mathematical terms and concepts. Such models present realistic depictions of abstract mathematical definitions. In contrast, a mathematical representation of reality uses mathematics to describe a phenomenon of nature. There are many mathematical tools that can be used in this process, including statistics, calculus, probability, and differential equations. Different methods may provide insights to different aspects of the problem, and there is often much debate about what approach is preferable (Robeva, R. S. , et al., 2008).

Mathematical modeling occurs in many natural phenomena and has a diversity of applications and thus a range of possible approaches. In a more practical and analytical mode there is a plethora of applications. Mathematical models are used extensively in biology and ecology to examine population fluctuations, water catchments, erosion and the spread of pollutants, to name just a few. Fluid mechanics is another extensive area of research, with applications ranging from the modelling of evolving tsunamis across the ocean, to the flow of lolly mixture into moulds (Robeva, R. S. , et al., 2008).

Modeling is an extremely powerful tool, a framework for research, debate and planning, which provides a valuable source of information for decision-making. The use of information from modeling process to reach decisions is now very much in the public view, this trend is likely to continue, as such modeling results in an efficient and economical way of understanding, analyzing and designing processes. Currently, mathematical modeling provides a means by which many political and management planning decisions are made, both locally and globally.

There are many modelling approaches that can be taken when formulating a mathematical model. The empirical approach is the most basic, but also the least useful. The idea is to fit a curve through a set of data and then to use this curve in order to predict outcomes where there are no data. The stochastic approach is the probabilistic that try to estimate the probability of certain outcomes based on the available data. These models can by extremely complicated, although this is not necessarily the case. They do have the advantage of incorporating a degree of uncertainty within them and ideally should be used when there is a high degree of variability in the problem. This method is typically used for models of small populations when reproduction rates need to be predicted over a time interval. They also have valuable application in many other areas such as economic fluctuations, insurance problems, telecommunications and traffic theory, and biological models (Barnes, B. , & Fulford, G. ., 2011).

In a simulation model one writes a computer program that applies a set of rules, or possibly even physically builds a scale model. It is intended to produce a set of data that mimic a real outcome including extreme events. Typically, such models are used in engineering applications as an aid to identifying problems that may arise during use or construction. Statistical models concern the testing (referred to as hypothesis testing) of whether a set of empirical data is from one or another category. Statistical testing is used widely in psychology, paleontology and the biological sciences (Barnes, B. , & Fulford, G. ., 2011). The deterministic approach formulates mathematical equations describing the basic fundamental relationships between the variables of the problem. This process is widely used and can be extremely accurate (Barnes, B. ., & Fulford, G. ., 2011).

With the emergence of communicable diseases and the pioneer work of Kermack and McKendrick (Kermack, W O, & McKendrick, A. G. ., 1926a; Kermack, W O, & McKendrick, A. G. ., 1932b; Kermack et al., 1933), Epidemiological model continue to receive significant attention. In the difficult moment of disease outbreaks and in particular the current global pandemic COVID-19, epidemiologist use mathematical model to understand the causes of a disease, then to predict its course, and finally to develop ways of controlling it, including comparisons of different possible approaches. The first step is obtaining and analyzing observed data.

The analytic approaches to models for endemic diseases and epidemics are quite different. The analysis of a model for an endemic disease begins with the search for equilibria, which are constant solutions of the model. Usually there is a disease free equilibrium and there are one or more endemic equilibria, with a positive number of infected individuals (Brauer, F. ., 2017).

The next step is to linearize about each equilibrium and determine its stability. The basic reproduction $R_0$ number is a sensitive parameter describing the number of secondary infected generated by the single infected pathogen in a susceptible population (McDonald, G. ., 1957). Usually, if $R_0 < 1$ the only equilibrium is the disease-free equilibrium and this equilibrium is asymptotically stable. If$R_0 > 1$, the usual situation is that the disease-free equilibrium is unstable and there is a unique endemic equilibrium which is asymptotically stable (Brauer, F. ., 2017).

**Academic Performance of Student**

Academic performance of a student serve as bedrock for knowledge acquisition and the development of skills which have direct impact on socio-economic development of a country (Farooq, M. , et al., 2011), it determines the success or failure of any academic institution (Blevins, B. ., 2009). There are many factors enhance and impede student's academic performance as attributed to students, parents, teachers and environments. The student's factors include self- motivation, interest in a subject, punctuality in class, regular studying and access to learning materials. Class attendance and students attitudes toward their learning have impact on academic performance (Ma, X. ., & Klinger, D. ., 2000). Peng, S. S., and Hall, S. T. (1995) Confirmed that in the case of mathematics, student's attitude towards the subject has a direct impact on their academic performance.

Qualified teachers and facilitators render effective facilitation which enhances academic performance. However, performance target, completion of syllabus, paying attention to weak students, assignment and student evaluation have significant impact too (Abubakar, A. , et al., 2018).

Parental background and status have significant impact on student's academic performance. Educated parent provide home school tutorial to their ward and are more encouraging. Jeynes, W. H. (2002) Found that students with high level of parental involvement in their academic excel their counterparts with no such involvement.

Environmental factors are schools factors involve the enabling environment, infrastructure, adequate facilities and learning materials, well-equipped laboratories. Crosnoe, R. et al. (2004) Revealed that the availability of physical resources such as library, textbooks, adequacy of classroom and spacious playing ground affect the performance of the students. Mccoy, L. P. (2005) Emphasized the use of instructional equipment facilitates effective service delivery and enhances teaching and learning. Distanced school also affects student's performance; Baliyan, S. P., and Khama, D. (2020), Eamon, M. K. (2005) opined that the more school distance the more tire students become. Also fairly disciplined schools perform better than the less or no disciplined school. Effective disciplined is used to control student's behavior, which has direct impact to their academic performance (Ehiane, S. ., 2014). Furthermore, student to teacher ratio or class size also affects the performance. Effective teaching in a moderate class ratio enhances the performance (Lopez, O. S. ., 1995).

Age has significant impact on academic performance. Older students are likely to dropout than the younger ones (Marginson, S. ., 2017). Found that there is a significant positive impact of age on academic performance in mathematics and science but the degree of the association is weak. However, mathematical model is an important tool used to optimize a real life problem for quickest and effective resolution. With regards to the subject matter we proposed dynamical model to optimize student's academic performance.

# CHAPTER III

## Model Formulation

The model is constructed based on the assumption that the newly intake are admitted into average class at the rate $\lambda$. The average student $A$ may then become weak, excellent or graduate at the rates $\phi, \beta$ and $\theta_2$ respectively. Below average student$B$ may become weak, average or graduate at the rates $\gamma, \alpha$ and $\theta_1$ respectively. Excellent student $E$ graduate at rate $\theta_3$. But Upon mingling with weak student $W$, the excellent student $E$ may be influenced to become weak at the rate$\eta$.

The rate at which student leaves school either through death or expelled is assumed to be the same in all the compartments.

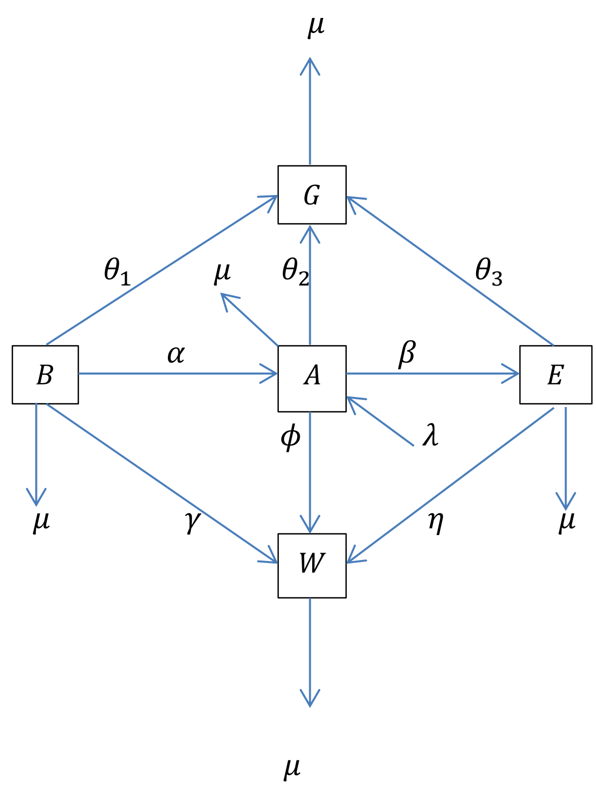


**Figure 1.** Schematic Diagram describing the Dynamics of Student Academic Performance

The performance dynamics is described by the nonlinear system of ODEs

$$\frac{dB}{dt} = p\eta WE - \theta_1 B - \alpha BA - \mu B - \gamma BW \tag{3.1}$$

$$\frac{dA}{dt} = \alpha BA + \lambda - \theta_2 A - \mu A + (1-p)WE - \phi WA - \beta AE \tag{3.2}$$

$$\frac{dE}{dt} = \beta AE - \eta WE - \theta_3 E - \mu E \tag{3.3}$$

$$\frac{dW}{dt} = \gamma BW + \phi WA - \mu W \tag{3.4}$$

$$\frac{dG}{dt} = \theta_1 B + \theta_2 A + \theta_3 E - \mu G \tag{3.5}$$

**Uniform Boundedness**

**Theorem 5:** all the solutions of model are confined within bounded subset of $\mathbb{R}^5$

$$\Psi = \left\{(\mathrm{B}, \mathrm{A}, \mathrm{E}, \mathrm{W}, \mathrm{G}\ ) \in \mathbb{R}^5 \colon \mathrm{B}, \mathrm{A}, \mathrm{E}, \mathrm{W}, \mathrm{G} \leq \frac{\lambda}{\mu}\right\}$$

**Proof**

Let the population size be

$$N = B + A + E + W + G \tag{3.6}$$

Then

$$\frac{dN}{dt} = \frac{dB}{dt} + \frac{dA}{dt} + \frac{dE}{dt} + \frac{dW}{dt} + \frac{dG}{dt}$$

Substituting (3.1) – (3.5) gives

$$\frac{dN}{dt} = \lambda - \mu N \tag{3.7}$$

Solving the first order linear ODE (3.7) using integration factor method

Define the integration factor

$$I = e^{\int \mu dt} = e^{\mu t}$$

$$I^{-1} = e^{-\mu t}$$

The solution to (3.7) takes the form

$$N(t) = I^{-1} \int \lambda\, I\, dt + kI^{-1}$$

$$= e^{-\mu t} \int \lambda\, e^{\mu t}\, dt + ke^{-\mu t}$$

$$\therefore N(t) = \frac{\lambda}{\mu} + ke^{-\mu t} \tag{3.8}$$

The long-term behavior of the solution (3.8) yield

$$\lim_{t \to \infty} N(t) = \frac{\lambda}{\mu} \tag{3.9}$$

Meanwhile, as time increases without bound all the solutions converge to the equilibrium $N_e = \frac{\lambda}{\mu}$ of (3.7), hence the equilibrium$N_e$is globally asymptotically stable.

**Existence and Uniqueness**

**Theorem 6:** the system (3.1) – (3.5) is Lipschitz continuous

**Proof**

Let the system (3.1) - (3.5) be of the form

$$f_1(t, B) = p\eta WE - \theta_1 B - \alpha BA - \mu B - \gamma BW \tag{3.1 *}$$

$$f_2(t, A) = \alpha BA + \lambda - \theta_2 A - \mu A + (1 - p)WE - \phi WA - \beta AE \tag{3.2 *}$$

$$f_3(t, E) = \beta AE - \eta WE - \theta_3 E - \mu E \tag{3.3 *}$$

$$f_4(t, W) = \gamma BW + \phi WA - \mu W \tag{3.4 *}$$

$$f_5(t,G)=\theta_1 B+\theta_2 A+\theta_3 E-\mu G \qquad (3.5*)$$

$$|f_1(t,B)-f_1(t,B^*)|=|\theta_1+\alpha A+\mu+\gamma W|\big||B-B^*|\big|$$

$$\leq\left(|\theta_1|+|\alpha|\max_{t\in[0,T]}\big||A|\big|+|\mu|+|\gamma|\max_{t\in[0,T]}\big||W|\big|\right)\big||B-B^*|\big|$$

$$\therefore \big||f_1(t,B)-f_1(t,B^*)|\big|\leq L_1\big||B-B^*|\big|,\qquad L_1$$
$$=|\theta_1|+|\alpha|\max_{t\in[0,T]}\big||A|\big|+|\mu|+|\gamma|\max_{t\in[0,T]}\big||W|\big| \quad (3.10)$$

Analogously,

$$|f_2(t,A)-f_2(t,A^*)|$$
$$=|(\alpha B-\theta_2-\mu+-\phi W-\beta E)A-(\alpha B-\theta_2-\mu+-\phi W-\beta E)A^*|$$

$$=|\alpha B-\theta_2-\mu+-\phi W-\beta E|\big||A-A^*|\big|$$

$$\leq\left(|\alpha|\max_{t\in[0,T]}\big||B|\big|+|\theta_2|+|\mu|+|\phi|\max_{t\in[0,T]}\big||W|\big|+|\beta|\max_{t\in[0,T]}\big||E|\big|\right)\big||A-A^*|\big|$$

$$\therefore \big||f_2(t,A)-f_2(t,A^*)|\big|\leq L_2\big||A-A^*|\big|, \qquad (3.11)$$

$$L_2=|\alpha|\max_{t\in[0,T]}\big||B|\big|+|\theta_2|+|\mu|+|\phi|\max_{t\in[0,T]}\big||W|\big|+|\beta|\max_{t\in[0,T]}\big||E|\big|<\infty$$

$$|f_3(t,E)-f_3(t,E^*)|=|(\beta A-\eta W-\theta_3-\mu)E-(\beta A-\eta W-\theta_3-\mu)E^*|$$

$$=|\beta A-\eta W-\theta_3-\mu|\big||E-E^*|\big|$$

$$\leq\left(|\beta|\max_{t\in[0,T]}\big||A|\big|+|\eta|\max_{t\in[0,T]}\big||W|\big|+|\theta_3|+|\mu|\right)\big||E-E^*|\big|$$

$$\therefore \big||f_3(t,E)-f_3(t,E^*)|\big|\leq L_3\big||E-E^*|\big| \qquad (3.12)$$

$$L_3=|\beta|\max_{t\in[0,T]}\big||A|\big|+|\eta|\max_{t\in[0,T]}\big||W|\big|+|\theta_3|+|\mu|<\infty,$$

$$|f_4(t,W)-f_4(t,W^*)|=|(\gamma B+\phi A-\mu)W-(\gamma B+\phi A-\mu)W^*|$$

$$= |\gamma B + \phi A - \mu|\left||W - W^*|\right|$$

$$\leq \left(|\gamma| \max_{t\in[0,T]}\left||B|\right| + |\phi| \max_{t\in[0,T]}\left||A|\right| + |\mu|\right)\left||W - W^*|\right|$$

$$\therefore \left||f_4(t,W) - f_4(t,W^*)|\right| \leq L_4\left||W - W^*|\right| (3.13)$$

$$L_4 = |\gamma| \max_{t\in[0,T]}\left||B|\right| + |\phi| \max_{t\in[0,T]}\left||A|\right| + |\mu| < \infty$$

$$|f_5(t,G) - f_5(t,G^*)| = |-\mu G + \mu G^*|$$

$$= |\mu|\left||G - G^*|\right|$$

$$\therefore \left||f_5(t,G) - f_5(t,G^*)|\right| \leq L_5\left||G - G^*|\right| \qquad (3.14)$$

$$L_5 = |\mu| < \infty$$

To rewrite the system (3.1) – (3.5) in compact form, let the system be written in matrix form as

$$\begin{bmatrix} B' \\ A' \\ E' \\ W' \\ G' \end{bmatrix} = \begin{bmatrix} -\theta_1 - \mu & 0 & 0 & 0 & 0 \\ 0 & -\theta_2 - \mu & 0 & 0 & 0 \\ 0 & 0 & -\theta_3 - \mu & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 \\ \theta_1 & \theta_2 & \theta_3 & 0 & -\mu \end{bmatrix} \begin{bmatrix} B \\ A \\ E \\ W \\ G \end{bmatrix} + \begin{bmatrix} p\eta WE - \alpha BA - \gamma BW \\ \alpha BA + (1-p)\eta WE - \phi WA - \beta AE \\ \beta AE - \eta WE \\ \gamma BW + \phi WA \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently,

$$x' = f(t,x), \qquad x(t_0) = x_0 \qquad (3.15)$$

$$f(t,x) = Px + g(x) + \Lambda,$$

$$x = (B,A,E,W,G)^T, \qquad \Lambda = (\lambda,0,0,0,0)^T,$$

$$P = \begin{bmatrix} -\theta_1 - \mu & 0 & 0 & 0 & 0 \\ 0 & -\theta_2 - \mu & 0 & 0 & 0 \\ 0 & 0 & -\theta_3 - \mu & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 \\ \theta_1 & \theta_2 & \theta_3 & 0 & -\mu \end{bmatrix},$$

$$g(x) = \begin{bmatrix} p\eta WE - \alpha BA - \gamma BW \\ \alpha BA + (1-p)\eta WE - \phi WA - \beta AE \\ \beta AE - \eta WE \\ \gamma BW + \phi WA \\ 0 \end{bmatrix}$$

Since$f \in C([t_0 - \epsilon,\ t_0 + \epsilon] \times U \subseteq \mathbb{R}^5)$, then (3.15) has the equivalent integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau)\big)\, d\tau \qquad (3.16)$$

Also,

$$y(t) = x_0 + \int_{t_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \qquad (3.17)$$

**Theorem 7:** let $f: [t_0 - a,\ t_0 + a] \times U \subseteq \mathbb{R}^5 \to \mathbb{R}^5$ be uniformly continuous in $t$ and Lipschitz in the second variable. Suppose $f$ is bounded, i.e$M = \max_{t \in [t_0 - a, t_0 + a]} ||f(t,x)||$ such that $||f(t,x)|| \leq M$. If $L_0 a < 1$ then there exist unique $x$ that solves (3.15).$\delta < \min\left(\frac{r}{M}, \frac{1}{L_0}\right)$

**Proof**

Let $X = \left\{x \in C[t_0 - a,\ t_0 + a] :\ \big||x(t) - x_0|\big| \leq r\right\}$

Define

$$d: d(x, \quad y) = \max_{t\in[t_0-a,t_0+a]} \left||x(t) - y(t)|\right|$$

Given $\epsilon > 0, \exists!\ n \equiv n(\epsilon) \in \mathbb{N}$:

$$d(x_n, \quad x_m) < \epsilon$$

$$\Longrightarrow \max_{t\in[t_0-a,t_0+a]} \left||x_n(t) - x_m(t)|\right| < \epsilon$$

$$\Leftrightarrow \left||x_n(t) - x_m(t)|\right| < \epsilon, \qquad \forall\ t \in [t_0 - a, t_0 + a] \tag{3.18}$$

Hence $(x_n)_{n\geq 1}$ is Cauchy sequence in $\mathbb{R}^5$ (with respect to $||.||$ ).

Since $(\mathbb{R}^5,\ ||.||)$ is complete, then $\exists!\ \ x^* \in \mathbb{R}^5 :\ x_n \to x^*$as $n \to \infty$.

Fix $n$ and allow $m \to \infty$ in (3.17) to get

$$\left||x_n(t) - x^*(t)|\right| \leq \epsilon \qquad \forall\ t \in [t_0 - a, \quad t_0 + a]$$

So,

$$Sup_{t\in[t_0-a,t_0+a]} \left||x_n(t) - x^*(t)|\right| \leq \epsilon$$

$$\Longrightarrow x_n \to x^*\ \ uniformly$$

Since, $x_n$ is continuous, $x^*$ is also continuous

$$d(x_n, \quad x^*) < \epsilon$$

Hence $X$ is complete metric space induced by norm.

Define the fixed point operator $T\!:\ X \to X$ by

$$Tx(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau)\big)\, d\tau \tag{3.19}$$

$$Ty(t) = x_0 + \int_{t_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \tag{3.20}$$

To show that $T$ is well defined

$$\big||Tx(t) - x_0|\big| = \left\|\int_{t_0}^{t} f\big(\tau, x(\tau)\big)\, d\tau\right\|$$

$$\leq \int_{t_0}^{t} \big||f\big(\tau, x(\tau)\big)|\big|\, d\tau$$

$$\leq M \int_{t_0}^{t} d\tau$$

$$= M|t - t_0|$$

$$\leq Ma$$

$$\big||Tx(t) - x_0|\big| \leq r, \qquad a \leq \frac{r}{M} \tag{3.21}$$

So,

$$Tx(t) \in \overline{B_r}(x_0) \quad \forall\, t \in \; [t_0 - a,\; t_0 + a].$$

To show contraction

$$\big||Tx(t) - Ty(t)|\big| = \left\|\int_{t_0}^{t} \big[f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big)\big]\, d\tau\right\|$$

$$\leq \int_{t_0}^{t} \big||f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big)|\big|\, d\tau$$

$$\leq \int_{t_0}^{t} L(\tau)\left||x(\tau) - y(\tau)|\right| d\tau$$

$$\leq \int_{t_0}^{t} L(\tau) \max_{s\in[t_0-a,t_0+a]} \left||x(s) - y(s)|\right| d\tau$$

$$\leq L_0\left||x - y|\right||t - t_0|$$

$$\left||Tx(t) - Ty(t)|\right| \leq L_0 a\left||x - y|\right| \tag{3.22}$$

Since $L_0 a < 1$ then $T$ is contraction.

# CHAPTER IV

## Stability Analysis

Since the state equation (3.5) depends on the previous states, it suffices to analyze (3.1) – (3.4).

### Existence of Equilibria

To find the equilibria, let the system (3.1) – (3.4) be

$$f_1(t,B) = p\eta WE - \theta_1 B - \alpha BA - \mu B - \gamma BW = 0 \tag{4.1}$$

$$f_2(t,A) = \alpha BA + \lambda - \theta_2 A - \mu A + (1-p)WE - \phi WA - \beta AE = 0 \tag{4.2}$$

$$f_3(t,E) = \beta AE - \eta WE - \theta_3 E - \mu E = 0 \tag{4.3}$$

$$f_4(t,W) = \gamma BW + \phi WA - \mu W = 0 \tag{4.4}$$

On solving (4.1) – (4.4), the independence equilibrium is found to be

$$E_* = (B^0, A^0, E^0, W^0) = \left(0, \frac{\lambda}{\mu+\theta_2}, 0, 0\right)$$

And the interdependence equilibrium$E_{**} = (B^*, A^*, E^*, W^*)$

$$B^* = \frac{\mu - \phi A^*}{\gamma} \tag{4.5}$$

$$W^* = \frac{\beta A^* - \theta_3 - \mu}{\eta} \tag{4.6}$$

$$E^* = \frac{[\eta(\theta_1+\mu) - \gamma(\theta_3+\mu) + (\eta\alpha+\gamma\beta)A^*](\mu - \phi A^*)}{\gamma p \eta(\beta A^* - \theta_3 - \mu)} \tag{4.7}$$

Which$A^*$ depend on

$$a_0 A^3 + a_1 A^2 + a_2 A + a_3 = 0 \tag{4.8}$$

Where,

$$a_0 = p\beta\phi(\eta\alpha + \gamma\beta)$$

$$a_1 = p\phi\beta(\gamma\beta - \eta\alpha) - (\eta\alpha + \gamma\beta)[p\beta\mu - \phi(1-p)(\theta_3 + \mu)] + p\beta\phi[\eta(\theta_1 + \mu) - \gamma(\theta_3 + \mu)]$$

$$a_2 = p\eta\alpha[\mu\beta + \phi(\phi\beta + \mu)] + p\gamma\beta[\eta(\lambda - \theta_2 - \mu) - 2\phi(\theta_3 + \mu)] - \mu(1-p)(\theta_3 + \mu)(\eta\alpha + \gamma\beta) - [p\beta\mu - \phi(1-p)(\theta_3 + \mu)][\eta(\theta_1 + \mu) - \gamma(\theta_3 + \mu)]$$

$$a_3 = -p\eta[\alpha\mu(\phi\beta + \mu) + \gamma(\theta_3 + \mu)(\lambda - \theta_2 - \mu)] + p\phi\gamma(\theta_3 + \mu)^2 - \mu(1-p)(\theta_3 + \mu)[\eta(\theta_1 + \mu) - \gamma(\theta_3 + \mu)]$$

**Reproduction Number**

A parameter describing the progressive student performance in the dynamics can be obtain using the Next generation matrix (NGM) approach introduced by (Diekmann, O. , et al., 2009)

As in the literature, the system (3.1) – (3.4) can be split into

Transition part

$$V = \begin{bmatrix} -\theta_1 - \mu & 0 & 0 & 0 \\ 0 & -\theta_2 - \mu & 0 & 0 \\ 0 & 0 & -\theta_3 - \mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix}$$

And intermingling part

$$g = \begin{bmatrix} p\eta WE - \alpha BA - \gamma BW \\ \alpha BA + (1-p)\eta WE - \phi WA - \beta AE \\ \beta AE - \eta WE \\ \gamma BW + \phi WA \\ 0 \end{bmatrix}$$

Linearizing $g$ around the independence equilibrium

$$F = \begin{bmatrix} \frac{\partial g_1}{\partial B} & \frac{\partial g_1}{\partial A} & \frac{\partial g_1}{\partial E} & \frac{\partial g_1}{\partial G} \\ \frac{\partial g_2}{\partial B} & \frac{\partial g_2}{\partial A} & \frac{\partial g_2}{\partial E} & \frac{\partial g_2}{\partial G} \\ \frac{\partial g_3}{\partial B} & \frac{\partial g_3}{\partial A} & \frac{\partial g_3}{\partial E} & \frac{\partial g_3}{\partial G} \\ \frac{\partial g_4}{\partial B} & \frac{\partial g_4}{\partial A} & \frac{\partial g_4}{\partial E} & \frac{\partial g_4}{\partial G} \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha A - \gamma W & -\alpha B & p\eta W & p\eta E \\ \alpha A & \alpha B - \phi W - \beta E & (1-p)\eta W - \beta A & (1-p)\eta E - \phi A \\ 0 & \beta E & \beta A - \eta W & -\eta E \\ \gamma W & \phi W & 0 & \phi A + \gamma B \end{bmatrix}$$

$$F_0 = \begin{bmatrix} -\frac{\alpha\lambda}{\theta_2+\mu} & 0 & 0 & 0 \\ \frac{\alpha\lambda}{\theta_2+\mu} & 0 & -\frac{\beta\lambda}{\theta_2+\mu} & -\frac{\phi\lambda}{\theta_2+\mu} \\ 0 & 0 & \frac{\beta\lambda}{\theta_2+\mu} & 0 \\ 0 & 0 & 0 & \frac{\phi\lambda}{\theta_2+\mu} \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -\frac{1}{\theta_1+\mu} & 0 & 0 & 0 \\ 0 & -\frac{1}{\theta_2+\mu} & 0 & 0 \\ 0 & 0 & -\frac{1}{\theta_3+\mu} & 0 \\ 0 & 0 & 0 & -\frac{1}{\mu} \end{bmatrix}$$

$$-F_0V^{-1} = \begin{bmatrix} \frac{\alpha\lambda}{(\theta_2+\mu)(\theta_1+\mu)} & 0 & 0 & 0 \\ -\frac{\alpha\lambda}{(\theta_2+\mu)(\theta_1+\mu)} & 0 & \frac{\beta\lambda}{(\theta_2+\mu)(\theta_3+\mu)} & -\frac{\phi\lambda}{\mu(\theta_2+\mu)} \\ 0 & 0 & -\frac{\beta\lambda}{(\theta_2+\mu)(\theta_3+\mu)} & 0 \\ 0 & 0 & 0 & \frac{\phi\lambda}{\mu(\theta_2+\mu)} \end{bmatrix}$$

The characteristics equation

$$|-F_0V^{-1} - KI| = 0 \tag{4.9}$$

The eigenvalues

$$K_1 = -\frac{\alpha\lambda}{(\theta_2+\mu)(\theta_1+\mu)}$$

$$K_2 = 0$$

$$K_3 = \frac{\beta\lambda}{(\theta_2 + \mu)(\theta_3 + \mu)}$$

$$K_4 = \frac{\phi\lambda}{\mu(\theta_2 + \mu)}$$

By (Diekmann, O, et al., 2009), the dominant eigenvalue is the reproduction number, hence

$$R_0 = \frac{\phi\lambda}{\mu(\theta_2 + \mu)} \quad (4.10\ )$$

Provided,

$$\phi > \frac{\beta\mu}{(\theta_3 + \mu)} \quad (4.11)$$

**Local Stability**

We formed the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial B} \frac{\partial f_1}{\partial A} \frac{\partial f_1}{\partial E} \frac{\partial f_1}{\partial G} \\ \frac{\partial f_2}{\partial B} \frac{\partial f_2}{\partial A} \frac{\partial f_2}{\partial E} \frac{\partial f_2}{\partial G} \\ \frac{\partial f_3}{\partial B} \frac{\partial f_3}{\partial A} \frac{\partial f_3}{\partial E} \frac{\partial f_3}{\partial G} \\ \frac{\partial f_4}{\partial B} \frac{\partial f_4}{\partial A} \frac{\partial f_4}{\partial E} \frac{\partial f_4}{\partial G} \end{bmatrix}$$

$$J = \begin{bmatrix} \theta_1 - \alpha A - \mu - \gamma W & -\alpha B & p\eta W & p\eta E \\ \alpha A & \alpha B - \theta_2 - \mu - \phi W - \beta E & (1-p)\eta W - \beta A & (1-p)\eta E - \phi A \\ 0 & \beta E & \beta A - \eta W - \theta_3 - \mu & -\eta E \\ \gamma W & \phi W & 0 & \phi A + \gamma B - \mu \end{bmatrix}$$

**Theorem 8:** the independence$E^0$ equilibrium is locally asymptotically stable

**Proof**

The Jacobian matrix at $E_* = \left(0, \frac{\lambda}{\mu+\theta_2}, 0, 0\right)$gives

$$J_{E_*} = \begin{bmatrix} -\theta_1 - \frac{\alpha\lambda}{\theta_2+\mu} - \mu & 0 & 0 & 0 \\ \frac{\alpha\lambda}{\theta_2+\mu} & -\theta_2 - \mu & -\frac{\beta\lambda}{\theta_2+\mu} & -\frac{\phi\lambda}{\theta_2+\mu} \\ 0 & 0 & \frac{\beta\lambda}{\theta_2+\mu} - \theta_3 - \mu & 0 \\ 0 & 0 & 0 & \frac{\phi\lambda}{\theta_2+\mu} - \mu \end{bmatrix}$$

The characteristic polynomial

$$\left|J_{E_*} - KI\right| = 0 \tag{4.12}$$

Where

$$I = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}$$

Hence

$$det \begin{vmatrix} -\theta_1 - \frac{\alpha\lambda}{\theta_2+\mu} - \mu - K & 0 & 0 & 0 \\ \frac{\alpha\lambda}{\theta_2+\mu} & -\theta_2 - \mu - K & -\frac{\beta\lambda}{\theta_2+\mu} & -\frac{\phi\lambda}{\theta_2+\mu} \\ 0 & 0 & \frac{\beta\lambda}{\theta_2+\mu} - \theta_3 - \mu - K & 0 \\ 0 & 0 & 0 & \frac{\phi\lambda}{\theta_2+\mu} - \mu - K \end{vmatrix} = 0$$

The eigenvalues obtained are

$$K_1 = -\theta_1 - \frac{\alpha\lambda}{\theta_2+\mu} - \mu \tag{4.13}$$

$$K_2 = -\theta_2 - \mu \tag{4.14}$$

$$K_3 = \frac{\beta\lambda}{\theta_2 + \mu} - \theta_3 - \mu$$

$$= \frac{\beta\mu R_0}{\phi} - (\theta_3 + \mu)$$

Since from (4.11)

$$\theta_3 + \mu > \frac{\beta\mu}{\phi}$$

$$K_3 = \frac{\beta\mu}{\phi}(R_0 - 1) \tag{4.15}$$

$$K_4 = \frac{\phi\lambda}{\theta_2 + \mu} - \mu$$

$$= \mu(R_0 - 1) \tag{4.16}$$

For $E_*$ to be stable we require$K_3, K_4 < 0$, which implies that$R_0 < 1$.

## Global Stability

**Theorem 9:** the independence equilibrium $E_*$ is globally asymptotically stable in the interior of $\Psi$

**Proof**

Define $L: \{(B, A, E, W) \in \Psi : \text{ A} > 0\} \rightarrow \mathbb{R}$ by

$$L = \frac{1}{2}[B + A - A^0 + E + W]^2 \tag{4.17}$$

Since,

$$L(B^0, A^0, E^0, W^0) = 0$$

$$L(B,A,E,W) > 0, \qquad B \neq B^0, \qquad A \neq A^0, \qquad E \neq E^0, \qquad W \neq W^0$$

Then $L$ is positive definite.

The time derivative of $L \in C'$

$$\frac{dL}{dt} = [B + A - A^0 + E + W]\left[\frac{dB}{dt} + \frac{dA}{dt} + \frac{dE}{dt} + \frac{dW}{dt}\right] \tag{4.18}$$

Substituting (3.1) – (3.4) into (4.18) gives

$$\frac{dL}{dt} = [B + A - A^0 + E + W][-(\theta_1 + \mu)B + (\theta_2 + \mu)A - (\theta_3 + \mu)E - \mu W + \lambda]$$

$$\begin{aligned} = -(\theta_1 + \mu)B^2 - (\theta_3 + \mu)E^2 - \mu W^2 - (\theta_2 + \mu)AB - (\theta_3 + \mu)BE - \mu WE \\ - (\theta_1 + \mu)(A - A^0)B - (\theta_2 + \mu)(A - A^0)A - (\theta_3 + \mu)(A - A^0)E \\ - \mu W(A - A^0) - (\theta_1 + \mu)BE - (\theta_2 + \mu)AE - \mu WE - (\theta_1 + \mu)BW \\ - (\theta_2 + \mu)AW - (\theta_3 + \mu)EW \\ + \lambda(B + A - A^0 + E + W) \end{aligned} \tag{4.19}$$

Applying the relation between arithmetic and geometric means: $\forall\, x, y \in \mathbb{R},\ xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, (4.19) becomes

$$\frac{dL}{dt} \le -(\theta_1+\mu)B^2 - (\theta_3+\mu)E^2 - \mu W^2 - (\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}B^2\right)$$
$$- (\theta_3+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}E^2\right) - \mu\left(\frac{1}{2}W^2+\frac{1}{2}B^2\right)$$
$$- (\theta_1+\mu)\left(\frac{1}{2}(A-A^0)^2+\frac{1}{2}B^2\right) - (\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}(A-A^0)^2\right)$$
$$- (\theta_3+\mu)\left(\frac{1}{2}(A-A^0)^2+\frac{1}{2}E^2\right) - \mu\left(\frac{1}{2}W^2+\frac{1}{2}(A-A^0)^2\right)$$
$$- (\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}E^2\right) - (\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}E^2\right) - \mu\left(\frac{1}{2}W^2+\frac{1}{2}E^2\right)$$
$$- (\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}W^2\right) - (\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}W^2\right)$$
$$- (\theta_3+\mu)\left(\frac{1}{2}E^2+\frac{1}{2}W^2\right) + \lambda(B+A-A^0+E+W)$$

$$= -\Bigg[\left(4\mu+\frac{5}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)B^2 + \left(4\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{5}{2}\theta_3\right)E^2 + 2(\theta_2+\mu)A^2$$
$$+ \left(4\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)W^2$$
$$+ \left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(A-A^0)^2$$
$$- \lambda(B+A-A^0+E+W)\Bigg] \tag{4.20}$$

Since

$$\left(4\mu+\frac{5}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)B^2 + \left(4\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{5}{2}\theta_3\right)E^2 + 2(\theta_2+\mu)A^2$$
$$+ \left(4\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)W^2$$
$$+ \left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(A-A^0)^2 > \lambda(B+A-A^0+E+W)$$

Then

$$\frac{dL}{dt} < 0 \tag{4.21}$$

Hence $E_*$ is globally asymptotically stable.

**Theorem 10:** the interdependence equilibrium $E_{**}$ is globally asymptotically stable in the interior of $\Psi$

**Proof**

Define $L$: $\{(B, A, E, W) \in \Psi : \text{B, A, E, W} > 0\} \to \mathbb{R}$ by

$$L = \frac{1}{2}[B - B^* + A - A^* + E - E^* + W - W^*]^2 \tag{4.22}$$

Since,

$$L(B^*, A^*, E^*, W^*) = 0$$

$$L(B, A, E, R) > 0, \qquad B \neq B^*, \qquad A \neq A^*, \qquad E \neq E^*, \qquad W \neq W^*$$

$L$is positive definite.

The time derivative of $L \in C'$

$$\frac{dL}{dt} = [B - B^* + A - A^* + E - E^* + W - W^*]\left[\frac{dB}{dt} + \frac{dA}{dt} + \frac{dE}{dt} + \frac{dW}{dt}\right] \tag{4.23}$$

Substituting (3.1) – (3.4) into (4.23) gives

$$\begin{aligned}\frac{dL}{dt} = -(\theta_1 + \mu)(B - B^*)B - (\theta_2 + \mu)(B - B^*)A - (\theta_3 + \mu)(B - B^*)E \\ - \mu W(B - B^*) - (\theta_1 + \mu)(A - A^*)B - (\theta_2 + \mu)(A - A^*)A \\ - (\theta_3 + \mu)(A - A^*)E - \mu W(A - A^*) - (\theta_1 + \mu)(E - E^*)B \\ - (\theta_2 + \mu)(E - E^*)A - (\theta_3 + \mu)(E - E^*)E - \mu W(E - E^*) \\ - (\theta_1 + \mu)(W - W^*)B - (\theta_2 + \mu)(W - W^*)A - (\theta_3 + \mu)(W - W^*)E \\ - \mu(W - W^*)W \\ + \lambda(B - B^* + A - A^* + E - E^* + W - W^*)\end{aligned} \tag{4.25}$$

Applying the relation between arithmetic and geometric means: $\forall\, x, y \in \mathbb{R},\ xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, (4.25) becomes

$$
\begin{aligned}
\frac{dL}{dt} \leq -(\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}(B-B^*)^2\right)-(\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}(B-B^*)^2\right) \\
-(\theta_3+\mu)\left(\frac{1}{2}E^2+\frac{1}{2}(B-B^*)^2\right)-\mu\left(\frac{1}{2}W^2+\frac{1}{2}(B-B^*)^2\right) \\
-(\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}(A-A^*)^2\right)-(\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}(A-A^*)^2\right) \\
-(\theta_3+\mu)\left(\frac{1}{2}E^2+\frac{1}{2}(A-A^*)^2\right)-\mu\left(\frac{1}{2}W^2+\frac{1}{2}(A-A^*)^2\right) \\
-(\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}(E-E^*)^2\right)-(\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}(E-E^*)^2\right) \\
-(\theta_3+\mu)\left(\frac{1}{2}E^2+\frac{1}{2}(E-E^*)^2\right)-\mu\left(\frac{1}{2}W^2+\frac{1}{2}(E-E^*)^2\right) \\
-(\theta_1+\mu)\left(\frac{1}{2}B^2+\frac{1}{2}(W-W^*)^2\right)-(\theta_2+\mu)\left(\frac{1}{2}A^2+\frac{1}{2}(W-W^*)^2\right) \\
-(\theta_3+\mu)\left(\frac{1}{2}E^2+\frac{1}{2}(W-W^*)^2\right)-\mu\left(\frac{1}{2}W^2+\frac{1}{2}(W-W^*)^2\right) \\
+\lambda(B-B^*+A-A^*+E-E^*+W-W^*)
\end{aligned}
$$

$$
\begin{aligned}
= -\Big[2(\theta_1+\mu)B^2+2(\theta_2+\mu)A^2+2(\theta_3+\mu)E^2+2\mu W^2 \\
+\left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(B-B^*)^2 \\
+\left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(A-A^*)^2 \\
+\left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(E-E^*)^2 \\
+\left(2\mu+\frac{1}{2}\theta_1+\frac{1}{2}\theta_2+\frac{1}{2}\theta_3\right)(W-W^*)^2 \\
-\lambda(B-B^*+A-A^*+E-E^*+W-W^*)\Big]
\end{aligned}
$$

Since

$$
\begin{aligned}
2(\theta_1+\mu)B^2 + 2(\theta_2+\mu)A^2 + 2(\theta_3+\mu)E^2 + 2\mu W^2 \\
&+ \left(2\mu + \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{2}\theta_3\right)(B-B^*)^2 \\
&+ \left(2\mu + \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{2}\theta_3\right)(A-A^*)^2 \\
&+ \left(2\mu + \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{2}\theta_3\right)(E-E^*)^2 \\
&+ \left(2\mu + \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{2}\theta_3\right)(W-W^*)^2 \\
&> \lambda(B-B^* + A - A^* + E - E^* + W - W^*)
\end{aligned}
$$

Then

$$\frac{dL}{dt} < 0 \tag{4.26}$$

Hence $E_{**}$ is globally asymptotically stable.

# CHAPTER V

## Formulation of Optimal Control

The optimal control strategy is aimed at optimizing student's academic performance which reflects in the increase of number of graduating students.

Let the control rates:

$u_1(t) \in [0,\ u_1(t)_{max}]$ be the self-motivation that makes weak student to become below average student.

$u_2(t) \in [0,\ u_2(t)_{max}]$ be punctuality in class that makes weak student to become average.

$u_3(t) \in [0,\ u_3(t)_{max}]$ be the interest in the subject that makes below average student to become average.

$u_4(t) \in [0,\ u_4(t)_{max}]$ be regular studying that makes average student to become excellent.

$u_5(t) \in [0,\ u_5(t)_{max}]$ be examination performance and character that make below average, average and excellent students to graduate.

Then the control dynamics is described by the nonlinear system of ODEs below

$$\frac{dB}{dt} = p\eta WE - (\theta_1 + u_5)B - (\alpha + u_3)BA - \mu B - (\gamma + u_1)BW \qquad (5.1)$$

$$\frac{dA}{dt} = (\alpha + u_3)BA + \lambda - (\theta_2 + u_5)A - \mu A + (1 - p)WE - (\phi + u_2)WA - (\beta + u_4)AE \quad (5.2)$$

$$\frac{dE}{dt} = (\beta + u_4)AE - \eta WE - (\theta_3 + u_5)E - \mu E \qquad (5.3)$$

$$\frac{dW}{dt} = (\gamma + u_1)BW + (\phi + u_2)WA - \mu W \quad (5.4)$$

$$\frac{dG}{dt} = (\theta_1 + u_5)B + (\theta_2 + u_5)A + (\theta_3 + u_5)E - \mu G \quad (5.5)$$

Subject to the objective functional

$$J(u(t)) = \int_{t_0}^{t} [c_1W + c_2W + c_3B + c_4A + c_5(B + A + E) + \frac{c_6}{2}u_1{}^2(t) + \frac{c_7}{2}u_2{}^2(t) + \frac{c_8}{2}u_3{}^2(t) + \frac{c_9}{2}u_4{}^2(t) + \frac{c_{10}}{2}u_5{}^2(t)]dt \quad (5.6)$$

Where $c_i \geq 0,\ i = 1,2, \dots ,10$ are the weights parameters that balanced the size of the terms.

As in (Fleming, W. & Rishel, R., 1975) we seek for optimal control $u^*$ such that

$$J(u^*) = \min\{J(u) : u \in \boldsymbol{\mathcal{U}}\} \quad (5.7)$$

Where

$\boldsymbol{\mathcal{U}}$is the set of admissible controls defined by

$$\boldsymbol{\mathcal{U}} = \{u_i(t)\ : 0 \leq u_i(t) \leq 1, \quad i = 1,2, \dots 10, \quad u_i(t)\ is\ Lebesgue\ measurable\}$$

**Characterization of Optimal Control**

To derive the optimal academic performance of student, define Hamiltonian

$$H = c_1W + c_2W + c_3B + c_4A + c_5(B + A + E) + \frac{c_6}{2}u_1{}^2(t) + \frac{c_7}{2}u_2{}^2(t) + \frac{c_8}{2}u_3{}^2(t) + \frac{c_9}{2}u_4{}^2(t) + \frac{c_{10}}{2}u_5{}^2(t) + \sum_{i=1}^{5} \Lambda_i f_i$$

$$= c_1W + c_2W + c_3B + c_4A + c_5(B + A + E) + \frac{c_6}{2}{u_1}^2(t) + \frac{c_7}{2}{u_2}^2(t) + \frac{c_8}{2}{u_3}^2(t)$$
$$+ \frac{c_9}{2}{u_4}^2(t) + \frac{c_{10}}{2}{u_5}^2(t)$$
$$+ \Lambda_1[p\eta WE - (\theta_1 + u_5)B - (\alpha + u_3)BA - \mu B - (\gamma + u_1)BW]$$
$$+ \Lambda_2[(\alpha + u_3)BA + \lambda - (\theta_2 + u_5)A - \mu A + (1 - p)WE$$
$$- (\phi + u_2)WA - (\beta + u_4)AE]$$
$$+ \Lambda_3[(\beta + u_4)AE - \eta WE - (\theta_3 + u_5)E - \mu E]$$
$$+ \Lambda_4[(\gamma + u_1)BW + (\phi + u_2)WA - \mu W]$$
$$+ \Lambda_5[(\theta_1 + u_5)B + (\theta_2 + u_5)A + (\theta_3 + u_5)E$$
$$- \mu G] \tag{5.8}$$

**Theorem 11**: let $\boldsymbol{x} = (B,\ A,\ E,\ W,\ G)$ with associated optimal control variales $u_1, u_2, u_3, u_4, u_5$ then there exist a co-state variable satisfying

$$\frac{d\Lambda_i}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}}, \quad i = 1,2,3,4,5 \tag{5.9}$$

**Proof**

Applying (2.7)

$$\frac{d\Lambda_1}{dt} = -\frac{\partial H}{\partial B}$$

$$\frac{d\Lambda_1}{dt} = c_3 + c_5 - [\theta_1 + u_5 + (\alpha + u_3)A + \mu + (\gamma + u_1)W]\Lambda_1 + (\alpha + u_3)A\Lambda_2$$
$$+ (\gamma + u_1)W\Lambda_4 + (\theta_1 + u_5)\Lambda_5 \tag{5.10}$$

Analogously,

$$\frac{d\Lambda_2}{dt} = c_4 + c_5 - (\alpha + u_3)B\Lambda_1 + [(\alpha + u_3)B - \theta_2 - u_5 - \mu - (\phi + u_2)W - (\beta + u_4)E]\Lambda_2 + (\beta + u_4)E\Lambda_3 + (\phi + u_2)W\Lambda_4 + (\theta_2 + u_5)\Lambda_5 \quad (5.11)$$

$$\frac{d\Lambda_3}{dt} = c_5 + p\eta W\Lambda_1 + [(1 - P)W - (\beta + u_4)A]\Lambda_2 + [(\beta + u_4)A - \eta W - \theta_3 - u_5 - \mu]\Lambda_3 + (\theta_3 + u_5)\Lambda_5 \quad (5.12)$$

$$\frac{d\Lambda_4}{dt} = c_1 + c_2 + [p\eta E - (\gamma + u_1)B]\Lambda_1 + [(1 - P)E - (\phi + u_2)A]\Lambda_2 - \eta E\Lambda_3 + [(\gamma + u_1)B + (\phi + u_2)A - \mu]\Lambda_4 \quad (5.13)$$

$$\frac{d\Lambda_5}{dt} = -\mu\Lambda_5 \quad (5.14)$$

Subject to transversality condition as in (Mccoy, L. P. ., 2005; McDonald, G. ., 1957)

$$\Lambda_1(t) = \Lambda_2(t) = \Lambda_3(t) = \Lambda_4(t) = \Lambda_5(t) = 0 \quad (5.15)$$

Applying the optimality condition $\frac{\partial H}{\partial u_i} = 0,\ \ i = 1,2,3,4,5$ implies that

$$u_1 = \frac{BW(\Lambda_1 - \Lambda_4)}{c_6} \quad (5.16)$$

$$u_2 = \frac{AW(\Lambda_2 - \Lambda_4)}{c_7} \quad (5.17)$$

$$u_3 = \frac{BA(\Lambda_1 - \Lambda_2)}{c_8} \quad (5.18)$$

$$u_4 = \frac{AE(\Lambda_2 - \Lambda_3)}{c_9} \quad (5.19)$$

$$u_5 = \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}} \qquad (5.20)$$

Hence

$$u_1^*(t) = \min\left\{1, \quad max\left(0, \quad \frac{BW(\Lambda_1 - \Lambda_4)}{c_6}\right)\right\} \qquad (5.21)$$

$$u_2^*(t) = \min\left\{1, \quad max\left(0, \quad \frac{AW(\Lambda_2 - \Lambda_4)}{c_7}\right)\right\} \qquad (5.22)$$

$$u_3^*(t) = \min\left\{1, \quad max\left(0, \quad \frac{BA(\Lambda_1 - \Lambda_2)}{c_8}\right)\right\} \qquad (5.23)$$

$$u_4^*(t) = \min\left\{1, \quad max\left(0, \quad \frac{AE(\Lambda_2 - \Lambda_3)}{c_9}\right)\right\} \qquad (5.24)$$

$$u_5^*(t) = \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\} \qquad (5.25)$$

By substituting values $u_1{}^*, u_2{}^*, u_3{}^*, u_4{}^*$ and $u_5{}^*$ into the ontrol system (5.1) – (5.5) we obtained

$$\begin{aligned}\frac{dB}{dt} = p\eta WE - \Bigg(\theta_1 &+ \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\Bigg)B \\ &- \left(\alpha + \min\left\{1, \quad max\left(0, \quad \frac{BA(\Lambda_1 - \Lambda_2)}{c_8}\right)\right\}\right)BA - \mu B \\ &- \left(\gamma + \min\left\{1, \quad max\left(0, \quad \frac{BW(\Lambda_1 - \Lambda_4)}{c_6}\right)\right\}\right)BW \qquad (5.26)\end{aligned}$$

$$\frac{dA}{dt} = \left(\alpha + \min\left\{1, \quad max\left(0, \quad \frac{BA(\Lambda_1 - \Lambda_2)}{c_8}\right)\right\}\right) BA + \lambda$$
$$- \left(\theta_2\right.$$
$$\left. + min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\right) A$$
$$- \mu A + (1 - p)WE$$
$$- \left(\phi + \left\{1, \quad max\left(0, \quad \frac{AW(\Lambda_2 - \Lambda_4)}{c_7}\right)\right\}\right) WA$$
$$- \left(\beta + \min\left\{1, \quad max\left(0, \quad \frac{AE(\Lambda_2 - \Lambda_3)}{c_9}\right)\right\}\right) AE \quad (5.27)$$

$$\frac{dE}{dt} = \left(\beta + \min\left\{1, \quad max\left(0, \quad \frac{AE(\Lambda_2 - \Lambda_3)}{c_9}\right)\right\}\right) AE - \eta WE$$
$$- \left(\theta_3\right.$$
$$\left. + \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\right) E$$
$$- \mu E \quad (5.28)$$

$$\frac{dW}{dt} = \left(\gamma + \min\left\{1, \quad max\left(0, \quad \frac{BW(\Lambda_1 - \Lambda_4)}{c_6}\right)\right\}\right) BW$$
$$+ \left(\phi + \min\left\{1, \quad max\left(0, \quad \frac{AW(\Lambda_2 - \Lambda_4)}{c_7}\right)\right\}\right) WA$$
$$- \mu W \quad (5.29)$$

$$\frac{dG}{dt} = \left(\theta_1 + \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\right)B$$
$$+ \left(\theta_2\right.$$
$$\left. + \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\right)A$$
$$+ \left(\theta_3\right.$$
$$\left. + \min\left\{1, \quad max\left(0, \quad \frac{B\Lambda_1 + A\Lambda_2 + E\Lambda_3 - (B + A + E)\Lambda_5}{c_{10}}\right)\right\}\right)E$$
$$- \mu G \tag{5.30}$$

# CHAPTER VI

## Numerical Method and Simulation

In order to solve the system (3.1) – (3.5) numerically, here we recalled the ODE 45 algorithm which is based on an explicit Runge – Kutta (4, 5) formula, the Dormand – Prince pair (Baliyan, S. P., & Khama, D. ., 2020).

Consider the problem of solving numerically the system of first order ODEs

$$\begin{cases} x'(t) = f\big(t, x(t)\big) \\ x(t_0) = x_0 \end{cases} \tag{6.1}$$

Under suitable continuity and differentiability conditions approximation $x_n$ to the true solution of $x(t_n)$ at point $t_n$,

Where

$$t_{n+1} = t_n + h_n,$$

$$h_n = \theta(t_n)h, \quad 0 < \theta(t_n) \leq 1, \quad n = 0,1,2, \ldots$$

Can be obtained using an explicit Runge – Kutta (RK) formula given by

$$x_{n+1} = x_n + h_n \Phi(x_n, t_n)$$

$$x_{n+1} = x_n + \sum_{i=1}^{s} b_i k_i \tag{6.2}$$

Where

$$k_1 = h_n f(x_n)$$

$$k_i = h_n f\left(x_n + \sum_{j=1}^{i-1} a_{ij} k_j\right), \qquad i = 2,3, \ldots., s$$

And usually $x(t_0) = x_0$.

The local truncation error $\tau_{n+1}$, of this method at $t_{n+1}$ is given by

$$\tau_{n+1} = x(t_n) + h_n \Phi(x(t_n), \qquad h_n) - x(t_{n+1})$$

$$= h_n \Phi(x(t_n), \qquad h_n) - [x(t_{n+1}) - x(t_n)]$$

Which using Taylor series expansion about $t_n$ gives

$$\tau_{n+1} = h_n \Phi(x(t_n), \qquad h_n) - h_n \Delta[x(t_n), h_n] \tag{6.3}$$

Where

$$\Delta[x(t), h] = \sum_{r=1}^{\infty} \frac{h^{r-1}}{r!} x^r(t)$$

If $\phi$ and $\Delta$ agree to $O(h^p)$ then the process is said to be a $p^{th}$ order RK formula (RKP) and $\tau_{n+1}$ can then be written

$$\tau_{n+1} = \sum_{j=1}^{\infty} h_n^{p+j} \phi_{p+j-1}[x(t_n)], \tag{6.4}$$

Where

$$\phi_r[x(t_n)] = \sum_{i=1}^{n_{r+1}} a_i^{(r+1)} F_i^{(r+1)}[x(t_n)], \qquad r = 1,2, \dots ..,$$

Are termed error functions, $F_i^{(r+1)}$, $i = 1,2, \dots ., n_{r+1}$. Being elementary differentials of order $r + 1$of $f$.

Note that if the formula is of order $p$then $\phi_r \equiv 0$, $r = 1,2, \dots, p-1$. This implies that

$$a_i^{(r+1)} = 0, \quad i = 1,2, \dots, n_{r+1}, \quad r = 1,2, \dots, p-1 \tag{6.5}$$

For consistency (Kendra, C. ., 2018) the following equation must be satisfied:

$$a_1^{(1)} = \sum_{i=1}^{s} b_i - 1 = 0, \quad (n_1 = 1).$$

These equations together with (6.5) are termed equations of conditions for the $RK_P$ formula.

Applying (6.2), the system (3.1) – (3.5) can be discretized as

$$B_{n+1} = B_n + \sum_{i=1}^{s} b_i k_i \tag{6.6}$$

$$A_{n+1} = A_n + \sum_{i=1}^{s} b_i k_i \tag{6.7}$$

$$E_{n+1} = E_n + \sum_{i=1}^{s} b_i k_i \tag{6.8}$$

$$W_{n+1} = W_n + \sum_{i=1}^{s} b_i k_i \tag{6.9}$$

$$G_{n+1} = G_n + \sum_{i=1}^{s} b_i k_i \tag{6.10}$$

Subject to:

$$a_i^{(r+1)} = 0, \quad i = 1,2,\dots,n_{r+1}, \quad r = 1,2,\dots,p-1$$

$$a_1^{(1)} = \sum_{i=1}^{s} b_i - 1 = 0, \quad (n_1 = 1).$$

Where

$$k_1 = h_n f(x_n)$$

$$k_i = h_n f\left(x_n + \sum_{j=1}^{i-1} a_{ij} k_j\right), \qquad i = 2,3, \dots, s$$

**Numerical Simulation**

Here we used the following values of variables and parameters

Table 1.

Model Variables and their Values

| Variable | Initial Value |
|---|---|
| $B = B(t)$ | $B(0) =$1000 |
| $A = A(t)$ | $A(0) =$5000 |
| $E = E(t)$ | $E(0) =$50 |
| $W = W(t)$ | $W(0) =$2000 |
| $G = G(t)$ | $G(0) =$5000 |
| $N = N(t)$ | $N(0) =$13050 |

Table 2.

Model Parameters and their Interpretation

| Parameter | Interpretation | Value |
|---|---|---|
| $\boldsymbol{\alpha}$ | Rate at which below average becomes average | **0.71** |
| $\boldsymbol{\beta}$ | Rate at which average becomes excellent | **0.3** |
| $\boldsymbol{\gamma}$ | Rate at which weak becomes below average | **0.51** |
| $\boldsymbol{\theta_1}$ | Rate at which below average graduates | **0.15** |
| $\boldsymbol{\theta_2}$ | Rate at which average graduates | **0.25** |
| $\boldsymbol{\theta_3}$ | Rate at which excellent graduates | **0.6** |

| | | |
|---|---|---|
| $\mu$ | Rate at which student leave school either through death or expelled | **0.7** |
| $\lambda$ | Admission rate | **0.7** |
| $\phi$ | Rate at whichaverage becomes weak | **0.55** |
| $\eta$ | Rate at which excellent becomes weak | **0.2** |

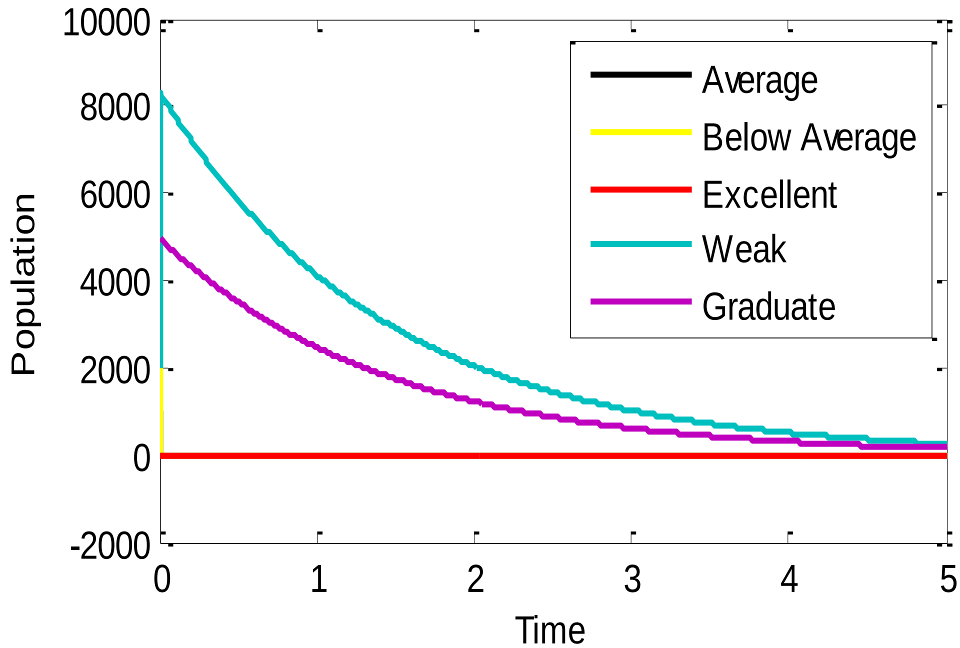


**Figure 2.** Dynamics of Different Populations in the Model

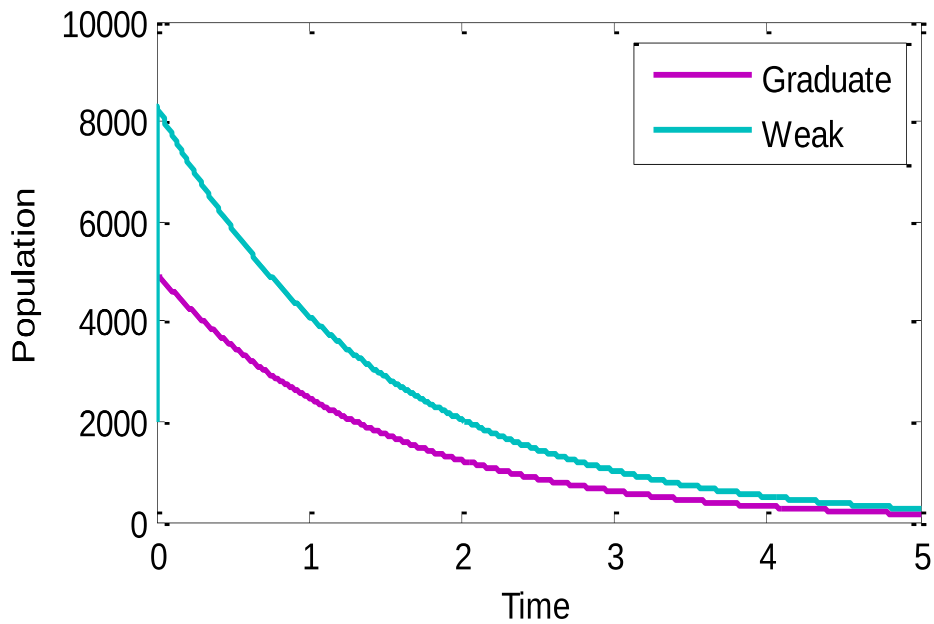


**Figure 3.** Comparison between Dynamics of Weak and Graduate Students

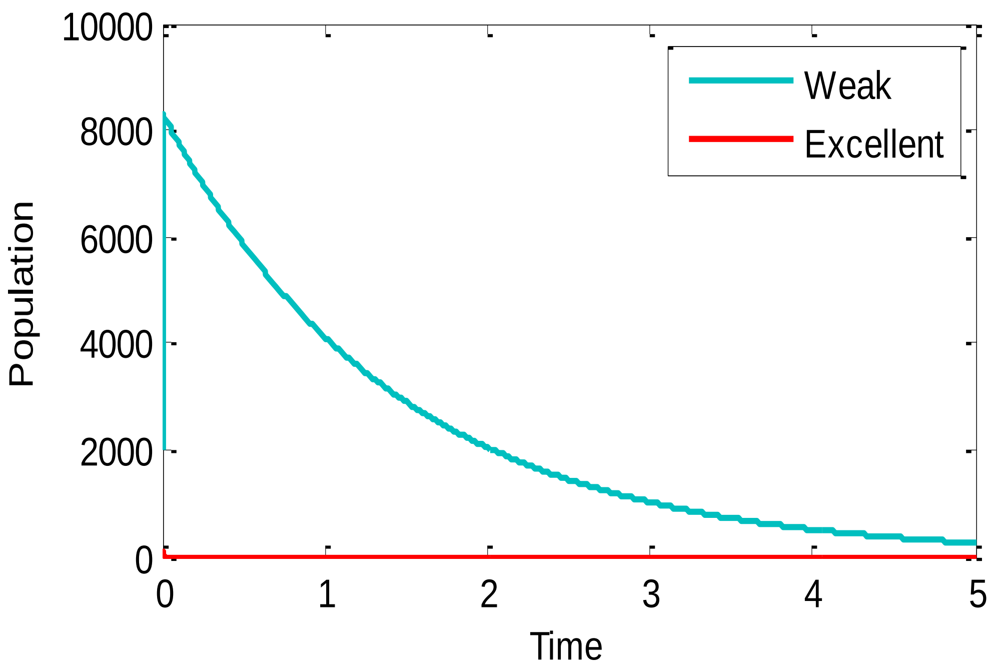


**Figure 4.** Comparison between Dynamics of Weak and Excellent Students

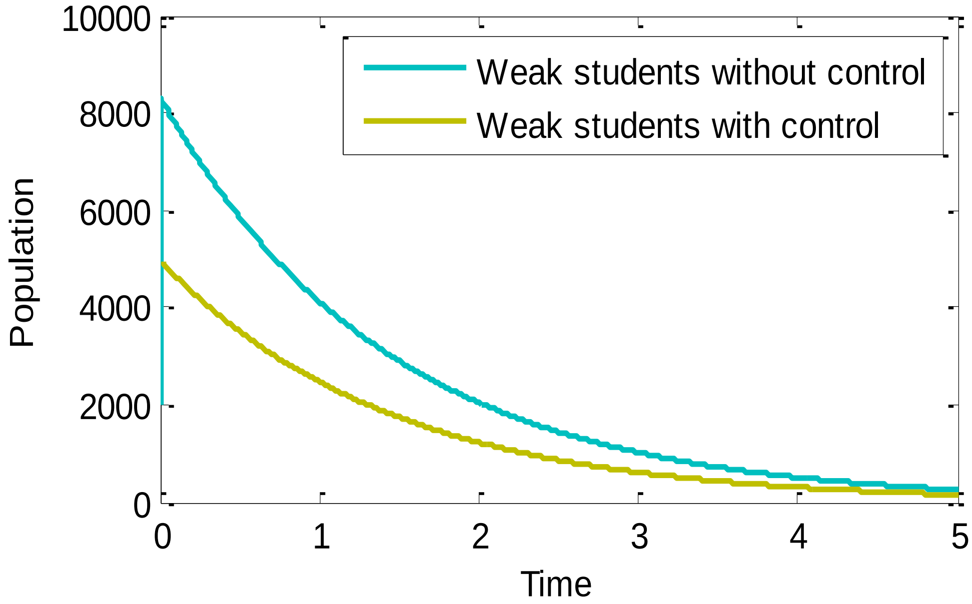


**Figure 5.** Comparison between Dynamics of Weak Students with and without Control

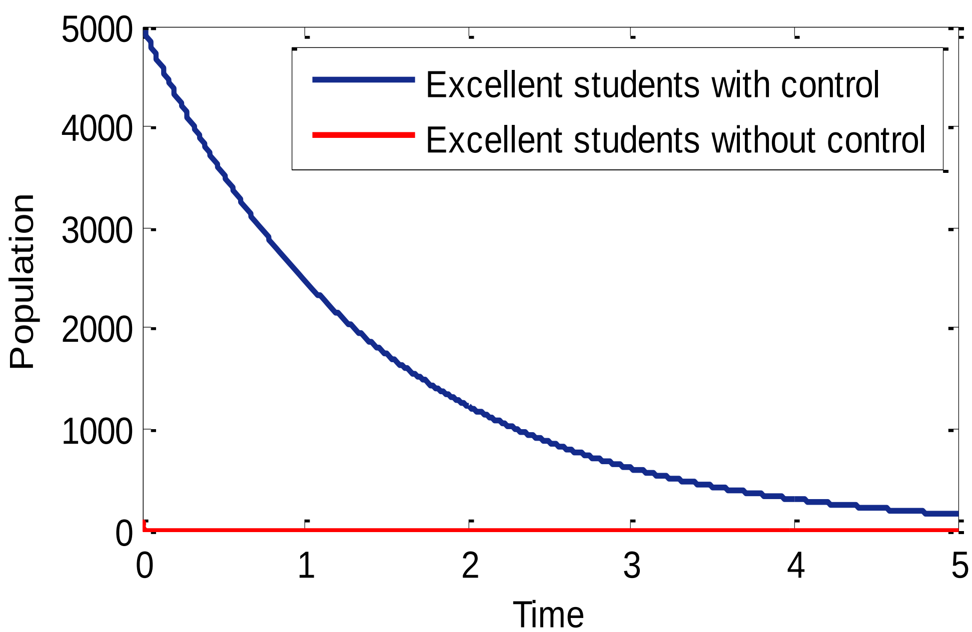


**Figure 6.** Comparison between Dynamics of Excellent Students with and without Control

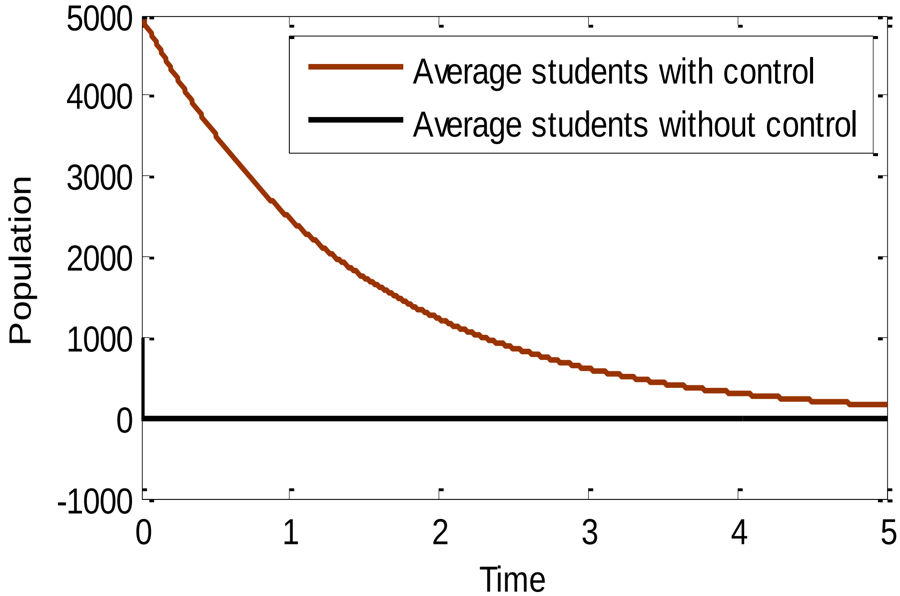


**Figure 7.** Comparison between Dynamics of Average Students with and without Control

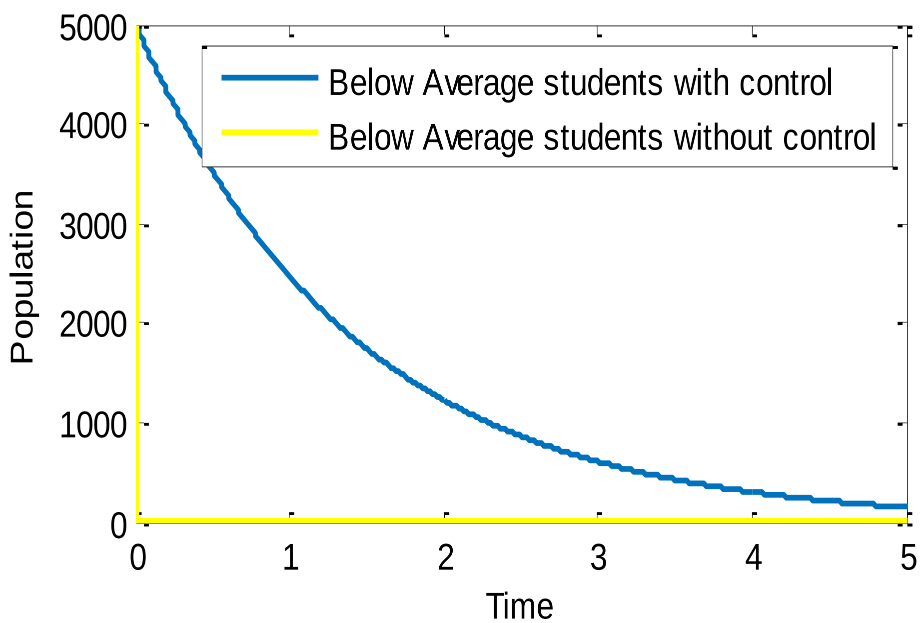


**Figure 8.** Comparison between Dynamics of below Average Students with and without Control

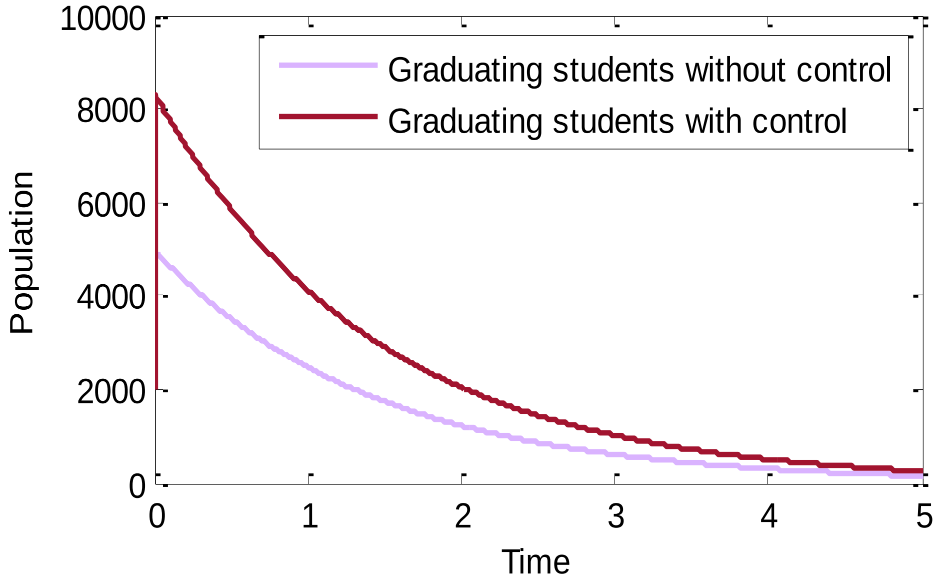


**Figure 9.** Comparison between Dynamics of Graduating Students with and without Control

# CHAPTER VII

## Conclusions and Recommendations

This chapter presents conclusions based on the research findings according to the objective and sub objective(s) of the research and gives recommendations accordingly.

To improve the academic performance of students, we optimize the performance indices to the dynamics describing the academic performance in the form of nonlinear system ODEs. We established the uniform boundedness of the model and the existence and uniqueness result. The independence and interdependence equilibria were found to be locally and globally asymptotically stable. The optimal control analysis was carried out, and lastly, numerical simulation was run to visualize the impact of performance index in optimizing the academic performance.

From the numerical simulation result, it can be observed that Weak students' population dominates other populations. This shows that when there is too much intermingling between Weak students and the other categories of students, it will be to the disadvantage of the other students. The weak students' population can be reduced i.e. the student's level of understanding can be enhanced by incorporating various measures that increase students' abilities. This can go hand in hand with increasing the level of motivation conferred on the Week students.

The significance of the optimal control is also clearly shown. There is drastic increase in the populations of Average, Below – Average, Excellent and Graduating students' population after the application of the control. On the other hand, there is drastic decrease in the population of the Weak students after the application of the control.

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2002 – 2006 **Ajman University,** UAE, AL-Fujairah
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2001 - 2002 **Nafisa Secondary school,** Sultanate of Oman, Saham
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## Skills

- **Communication skills.**
- **Computer Skills:**
  - Programming:
  - MS Office applications: (Word, Excel, and PowerPoint) beside using the Internet skillfully.
- **Report writing skills: (English and Arabic).**
- **Researching skills: (English and Arabic):**
  - Doing researches and thesis.

## Languages:

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## Achievements and Social Activities:

- **Achievement Certificates** from Saham Vocational Center.
- **Achievement Certificate** from L&T Company.
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- **Achievement certificate** as Mathematics Senior Teacher.
- **English at the Work Place Course certificate**.
- **Participation Certificate** from Shinas Technical College.
- **Participation Certificate** in a media workshop.

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## General Interests

- doing multi-works
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- Using the technology in work.
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- Doing Researches and thesis.
- Participating in the activities.
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- Traveling.

## References

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