NEAR EAST UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEM FOR THE SCHRÖDINGER DIFFERENTIAL AND DIFFERENCE EQUATIONS

Ph.D. THESIS

Mesut ÜRÜN

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Supervisor
Prof. Dr. Allaberen ASHYRALYEV

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Approval

We certify that we have read the thesis submitted by **Mesut Ürün** titled **"Time-dependent source identification problem for Schrödinger differential and difference equation"** and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Educational Sciences.

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Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

Mesut Ürün

28/06/2022

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Mesut Ürün

Abstract

Time-dependent source identification problem for Schrödinger differential and difference equation

Ürün, Mesut

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In the present thesis, the time-dependent source identification problem for the Schrödinger equation is investigated.

The stability of the time-dependent source identification problem for the Schrödinger equation in a Hilbert space with the self-adjoint positive definite operator is established. In practice, theorems on stability estimates for the solution of five types of time-dependent source identification problems for one-dimensional and multidimensional Schrödinger equations with local and nonlocal boundary conditions are proved. The absolute stable difference schemes for the approximate solutions of this time-dependent abstract source identification problem are presented. The stability of these difference schemes are established. In applications, stability estimates for the solution of difference schemes for the approximate solutions five types of time-dependent source identification problems for Schrödinger equations are obtained. Numerical results for the first and second-order of accuracy difference schemes of the approximate solution of one-dimensional time-dependent source identification problem for Schrödinger equations with nonlocal, Dirichlet, Neumann, and Robin conditions are provided.

Key Words: Schrödinger equation; Source identification problem; Hilbert spaces; Nonlocal conditions; Neumann conditions; Dirichlet conditions; Robin conditions; Stability; Difference Schemes.

Özet

Schrödinger Diferansiyel ve Fark Denklemi için Zamana Bağlı Kaynak Tanımlama Problemi

Ürün, Mesut

Doktora Tezi, Matematik Anabilim Dalı

Danışman: Prof. Dr. Allaberen Ashyralyev

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Bu tezde, Schrödinger denklemi için zamana bağlı kaynak tanımlama problemi incelenmiştir.

Schrödinger denklemi için zamana bağlı kaynak tanımlama probleminin bir Hilbert uzayında kendine eşlenik pozitif tanımlı operatör ile kararlılığı kurulmuştur. Uygulamada, yerel ve yerel olmayan sınır koşulları ile tek boyutlu ve çok boyutlu Schrödinger denklemi için zamana bağlı beş tür kaynak tanımlama probleminin çözümü için kararlılık tahminleri üzerine teoremler kanıtlanmıştır. Bu zamana bağlı soyut kaynak tanımlama probleminin yaklaşık çözümleri için mutlak kararlı fark şemaları sunulmaktadır. Bu fark şemalarının kararlılığı kurulmuştur.

Uygulamalarda, Schrödinger denklemi için zamana bağlı beş tür kaynak tanımlama probleminin yaklaşık çözümleri için fark şemalarının kararlılık tahminleri elde edilmiştir. Yerel ve yerel olmayan olmayan, Dirichlet, Neumann ve Robin sınır koşullarıyla Schrödinger denklemleri için bir boyutlu zamana bağlı kaynak tanımlama probleminin yaklaşık çözümünün birinci ve ikinci mertebeden doğruluk fark şemaları için sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Schrödinger Denklemi; Kaynak Tanımlama Problemi, Hilbert Uzayı; Lokal Olmayan Sınır Şartları; Neumann sınır Şartları; Dirichlet Sınır Şartları; Robin Sınır Şartları; Kararlılık; Fark Şeması.

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List of Abbreviations and Symbols

SIP	Source Identification Problem
SIPs	Source Identification Problems
PDE	Partial Differential Equation
PDEs	Partial Differential Equations
DS	Difference Scheme
DSs	Difference Schemes
SE	Schrödinger Equation
SEs	Schrödinger Equations
IVP	Initial Value Problem
$\boldsymbol{E_u}$	Error function defined by formula $E_u = \max\limits_{k\in\overline{0,N}} \left(\sum_{n=0}^{M} \lvert u(t,x) - u_n^k\rvert^2 h\right)^{\frac{1}{2}}$,
$\boldsymbol{E_p}$	Error function defined by formula $E_p = \max\limits_{k\in\overline{1,N}} \left\lvert p(t) - p_k\left(\frac{p_k+p_{k-1}}{2}\right)\right\rvert$.

CHAPTER I
Introduction

1.1 Historical Note and Literature Survey

A special case of the Schrödinger equation that admits a statement in those terms are the position-space Schrödinger equation for a single nonrelativistic particle in one dimension:

$$ih\frac{\partial\Psi}{\partial t}=\left[-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2}+V(t,x)\right]\Psi(t,x)$$

Here, $\Psi(x,t)$ is a wave function, a function that assigns a complex number to each point x at each time t. The parameter m is the mass of the particle, and $V(x,t)$ is the potential that represents the environment in which the particle exists. The constant i is the imaginary unit, and h is the reduced Planck constant, which has units of action. It shows that behavior of wave functions and their variation in space and time. It is named after Erwin Schrödinger, who proposed the equation in 1925 and published it in 1926, and formed the basis of the work that later won the Nobel Prize in Physics in 1933 (Schrödinger, 1926a,1926b).

The Schrödinger type equation has many applications such as natural sciences, engineering sciences. The mathematical modeling of many problems in physics, such as quantum mechanics, diffusion equations, heat transfer, quantum physics, and the propagation of sound under water, is based on partial differential equations similar to the Schrödinger equations (Agmon, 1970, 1981; Aguilar & Combes, 1971; Aizenman & Lieb, 1978; Avron & Herbst, 1977; Aizenman & Simon, 1982a, 1982b; J. Avron, Herbst & Simon, 1978; Brdzis & Kato, 1979; Eskin & Ralston, 1995; Burnham et al., 2020; Ita et al., 2020; Biondini, Lottes & Mantzavinos, 2021; Osman et al., 2021; Zhi, 2021). The Schrodinger equation has many technological applications. For example, in modeling quantum devices, electromagnetic wave propagation, underwater acoustics, optics, beam propagation in nonlinear Kerr medium, two-dimensional Schrödinger equation is widely used in modeling problems encountered in relativistic physics or plasmas (Arnold, 1998; Shang et al., 2014; Tappert, 1977; Mayfield, 1989; Manganaro & Parker, 1993; Kopylov, Popov & Vinogradov, 1995a, 1995b). Some of recent studies on the Schrödinger equation are the following. Local and nonlocal problems in the

Schrödinger equation have been extensively studied by many researchers (Antoine, Besse & Mouysset, 2004, 2005; Gordeziani & Avalishvili, 2000; Avalishvili, Avalishvili & Gordeziani, 2005; Avalishvili & Avalishvili, 2014; Avalishvili, Avalishvili & Gordeziani, 2011; Xu, Han & Wu, 2007).

The numerical method for partial differential equations is an effective method in scientific computation. It is not easy to obtain a numerical solution using classical first and second order difference schemes unless many nodes are used. A reasonable way to overcome this disadvantage of classical difference schemes is to design a highly compact finite difference scheme, since the computational overhead will be quite high. The fourth-order compact difference scheme for the linear Schrödinger equation with periodic boundary conditions over a limited region was discussed by Lio and Sun, and by applying the energy method with certain Sobolev embedding inequalities, maximum norm error estimations of the solutions were obtained (Lin Liao, Zhong Sun & Shi, 2010; Liao, Sun, Shi & Wang, 2012; Bratsos, 2010). Proposed two high-order compact finite difference schemes for the one-dimensional nonlinear Schrödinger equation and showed discrete L_2 norm error estimations and convergence speed (Xie, Li Sucheol-Yi, 2009). For the two-dimensional Schrödinger equation, Gao and Xie created fourth-order vari-directional closed compact difference schemes and analyzed the degree of convergence of the schemes (Gao and Xie, 2011).

In the PhD Thesis (Sirma, 2007), the nonlocal boundary value problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} - Au = f(t), 0 \le t \le T, \\ u(0) = \displaystyle\sum_{m=1}^{p} \alpha_m u(\lambda_m) + \varphi, \\ 0 < \lambda_1 < \lambda_1 < \lambda_1 ... < \lambda_1 \le T \end{cases}$$

for the Schrödinger equation in a Hilbert space H with the self-adjoint operator A was considered. Stability estimates for the solution of this problem were established. Two nonlocal boundary value problems were investigated. The first and second order of accuracy difference schemes for the approximate solutions of this nonlocal boundary value problem were presented. The stability of these difference schemes was established. In practice, stability inequalities for the solutions of difference schemes for the Schrödinger equation were obtained. A numerical method was proposed to solve a one-dimensional Schrödinger equation with nonlocal boundary condition. A procedure involving the modified Gauss elimination method was used to solve these difference schemes. The method is illustrated by giving numerical examples. These and other results of this subject were published in papers (Sirma, 2007; Ashyralyev & Sirma 2008, 2009b, 2009a).

In the PhD Thesis (Hicdurmaz, 2015), the initial value problem

$$\begin{cases} iu_t + Au + \int_0^t \gamma(s) D_s^{\alpha} u(s) ds = f(t),\ t \in (0,1), \\ u(0) = 0 \end{cases}$$

for the fractional Schrödinger equation in a Hilbert space H with the self-adjoint operator A was considered.The stability estimates for the solution of the problem and its first order of derivative were established. In practice, one-dimensional fractional Schrödinger differential equation with nonlocal boundary conditions and multidimensional fractional Schrödinger differential equation with the Dirichlet condition were considered. The stability estimates for the solutions of these problems were established. The first order of accuracy difference scheme for approximate solution of this equation was presented. The stability of this difference scheme was established. In applications, the stability estimates for the solutions of difference schemes of the fractional Schrödinger problems were established. These and other results for this subject were published in papers (Ashyralyev, Hicdurmaz, 2011, 2012)

Fractional nonlinear Schrödinger equation was studied by Rida, El-Sherbiny and Arafa, 2007; In these studies, the Adomian decomposition method in applied mathematics was used and analytical and approximate solutions for different kinds of fractional differential equations were investigated (Haydari & Atangana 2019; Sweilam, Hassan & Hassan, 2017; Rida, Sherbiny & Arafa, 2008; Bhrawy, Zaky & Abdelkawy, 2016; Bhrawy & Zaky, 2017; Abdel-Salam, Yousif & El-Aasser, 2016; Asyralyev and Hicdurmaz, 2011, 2012b, 2012a 2016, 2017, 2018b, 2018a; Hicdurmaz 2019, 2020b, 2020a; Asyralyev & Hicdurmaz, 2021; Hicdurmaz, 2021).

The theory and applications of linear and nonlinear time-delayed Schrödinger equations have been widely researched (Agırseven, 2018; Chen, Zhou & Zhao, 2010; Guo & Shao, 2005; Guo & Yang, 2010, 2010a; Wu, 1996; Zhao & Ge, 2011). The existence, uniqueness and regularity properties, Strichartz type estimates for solution of multipoint Cauchy problem for linear and nonlinear Schrödinger equations with general elliptic leading part was obtained in papers (Shakhmurov, 2019, 2020, 2021a, 2021b). Equation involves a involution of integral operators with a general kernel operator functions whose Fourier transform are operator functions defined in a Hilbert space H together with some growth conditions. By assuming enough smoothness on the initial data and the operator functions, the local and global existence and uniqueness of solutions are established. Shakhmurov can obtained a different classes of nonlocal Schrödinger equations by choosing the space H and linear operators, which occur in a wide variety of physical system. The theory and applications of linear time-delay Schrödinger equation

$$\begin{cases} iu_t + Au(t) = bAv([t]), t \in (0, \infty), \\ u(0) = \varphi \end{cases}$$

in a Hilbert space H with the self-adjoint operator A was studied in papers (Erköse, 2021; Agirseven,2018; Ashyralyev & Agirseven, 2019). Theorems on stability estimates for the solution of this problem were established. The applications of theorems for three types of Schrödinger problems were provided. The first and second order of accuracy difference schemes for the approximate solutions of this abstract problem were presented. The theorem on stability estimates for the solutions of these difference schemes was established. The application of theorems on stability of difference schemes for the approximate solutions of the initial boundary value problems for Schrödinger partial differential equation was provided. Additionally, some illustrative numerical results were presented.

Identification problems take an important place in applied sciences and engineering applications and have been studied by many authors (see, Kabanikhin, 2004, 2011; Belov, 2002; Gryazin, Klibanov and Lucas, 1999). The theory and applications of source identification problems for partial differential equations have been given in various papers (Erdogan & Ashyralyev, 2014; Ashyralyev & Prenov, 2014; Ashyralyev & Sazaklioglu, 2014; Kostin, 2013;
Choulli & Yamamoto, 1999; Ashyralyev & Emharab, 2019; Ashyralyev & Sazaklioglu, 2017; Saitoh, Tuan, & Yamamoto, 2002; Ivanchov, 1995; Samarskii & Vabishchevich, 2008; Borukhov & Vabishchevich, 2000; Blasio & Lorenzi, 2007; Ashyralyev, Erdogan & Sazaklioglu, 2019).

The theory and applications of space dependent identification problem for Schrödinger equation

$$\begin{cases} iu_t + Au(t) = f(t) + p, t \in (0,T), \\ u(0) = \varphi, u(T) = \psi, t \in [0,T] \end{cases}$$

in a Hilbert space H with the self-adjoint operator A was studied in papers (Ashyralyev et al., 2019; Ashyralyev and Urun, 2013a, 2013b, 2014; Urun, 2013). The well-posedness of this problem was established. The stability inequalities for the solution of two boundary value problems for the Schrödinger's equation with unknown parameter were obtained. The first and second order of accuracy stable difference schemes for the approximate solution this problem were presented. The well-posedness of these difference schemes was established. Numerical techniques were developed and algorithms were tested on an example.

In the present thesis, we investigate the time-dependent source identification problem for Schrödinger differential equation

$$\begin{cases} i\frac{du}{dt} + Au(t) = p(t)q + f(t), t \in (0,T), \\ u(0) = \varphi, \ B[u(t)] = \psi(t), [0,T] \end{cases} \tag{1.1}$$

in a Hilbert space H with the with self-adjoint positive definite operator A with dense domain $D(A)$ in H. Here $B: H \to R$ is a given linear bounded functional and $\psi(t): [0,T] \to R$ is a given smooth function and $q \in D(A), Bq \neq 0.$

The stability of the differential problem is established. In applications, theorems on stability estimates for the solution of five types of time-dependent source identification problems for Schrödinger equations are obtained. The first of them is the time-dependent source problem for the one dimensional Schrödinger equation with nonlocal conditions. The second of them is the time-dependent source problem for the one dimensional Schrödinger equation with involution and Dirichlet conditions. The third is the time-dependent source problem for the one dimensional Schrödinger equation with Robin conditions. Two of them are the time-dependent source problems for the multidimensional Schrödinger equation with Dirichlet and Neumann conditions. The absolute stable difference schemes for the approximate solutions of this time-dependent abstract source identification problem are investigated. The first and second order of accuracy implicit and second order of accuracy r-modified Crank-Nicolson difference schemes are presented. Stability of these difference schemes are established. In applications, theorems on stability estimates for the solution of difference schemes for the approximate solutions five type of time-dependent source identification problems for Schrödinger equations are obtained. The first of them is the time-dependent source problem for the one dimensional Schrödinger equation with nonlocal conditions. The second of them is the time-dependent source problem for the one dimensional Schrödinger equation with involution and Dirichlet conditions. The third is the time-dependent source problem for the one dimensional Schrödinger equation with Robin conditions. Two of them are the time-dependent source problems for the multidimensional Schrödinger equation with Dirichlet and Neumann conditions.When the analytical methods do not work properly, numerical methods to obtain approximate solutions for partial differential equations play an important role in applied mathematics. We can say that there are many considerable studies in the literature. In present section for the approximate solution of one-dimensional time-dependent source identification problem for Schrödinger equations with nonlocal, Dirichlet, Neumann, and Robin conditions, we use the first and second order of accuracy difference schemes. The error analysis is given. Presently, the time-dependent source identification problem for the fractional Schrödinger type equation was investigated by (Ashurov & Shakarova in 2021, 2022).

1.2 Layout of the Present Thesis

Time-dependent source identification problem for the Schrödinger differential and difference equation has not been investigated before. The main aim the present thesis is a study of the boundedness solution of several time-dependent source identification problem for the schrödinger differential and difference equation. This thesis consists of six chapters.

The first chapter a historical note and literature survey.

The second chapter is to study of the time-dependent source identification problems for several Schrödinger equations. Applying results of Chapter one Fourier series, Laplace and Fourier transform methods, we obtain the exact solution of several time-dependent source identification problems for Schrödinger equations.

In the thirth chapter, the main theorem on stability of the time-dependent source identification problems is established. In applications of the main theorem, stability estimates for the solutions of five type time-dependent source identification problems for the Schrödinger equations with local and nonlocal conditions are obtained. (This chapter was published in TWMS J. Pure Appl. Math. Ashyralyev, Urun, 2022).

In the fourth chapter, single-step absolute stable difference schemes for the approximate solutions of source identication problem are presented. The main theorems on stability of these difference schemes are established. In applications of the main theorems, stability estimates for the solutions of difference schemes for the approximate solutions of the five type of time-dependent source identification problems for Schrödinger equations with local and nonlocal conditions are obtained.

In the fifth chapter, results of numerical experiments are provided with local and nonlocal boundary contions, Dirihlet contions, Neumann conditions, Robin conditions, (This chapter publishe in International Journal of Applied Mathematics and Bulletin of the Karaganda University-Mathematics Ashyralyev,Urun 2021, 2021a).

Finally, in the sixth chapter, the conclusion is given.

1.3 Basic Concept and Definitions

1.3.1 Sturm-Liouville Problem (Arfken, Weber, 2005)

We denote the Sturm Liouville operator as

$$L[v] = -\frac{d}{dx}\left[p(x)\frac{dv}{dx} \right] + q(x)v$$

and consider the Sturm Liouville equation

$$L[v] + \lambda v = 0, \tag{1.2}$$

where $p > 0$ and p and q are continuous functions on the interval $[0, l]$ with local boundary conditions

$$\alpha_1 v(0) + \alpha_2 p(0) v'(0) = 0; \ \ \beta_1 v(l) + \beta_2 p(l) v'(l) = 0, \tag{1.3}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$ or nonlocal boundary conditions

$$v(0) - v(l) = 0, v^{'}(0) - v^{'}(l) = 0, \tag{1.4}$$

The problem of finding a complex number $\lambda = \mu$ such that the boundary value problems (1.2), (1.3) or (1.2), (1.4) have a non trivial solution are called Sturm-Liouville problems.

The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(x, \mu)$ is called an eigenfunction.

We will consider three types of Sturm-Liouville problem.

1. The Sturm-Liouville Problem with Dirichlet Condition

$$-u''(x)+\lambda u(x)=0, 0<x<l, u(0)=u(l)=0$$

has solution

$$u_k(x)=\sin\frac{kx}{l}$$

and

$$\lambda_k=-\left(\frac{k\pi}{l}\right)^2, k=1,2,....$$

In the case when $l=\pi$

$$u_k(x)=\sin kx$$

and

$$\lambda_k=-k^2, k=1,2,....$$

2. The Sturm-Liouville Problem with Neumann Condition

$$-u''(x)+\lambda u(x)=0, 0<x<l, u'(0)=u'(l)=0$$

has solution

$$u_k(x)=\cos\frac{kx}{l}$$

and

$$\lambda_k=(\frac{k\pi}{l}), k=0,1,2,....$$

In the case when $l=\pi$

$$u_k(x)=\cos kx$$

and

$$\lambda_k=-k^2, k=0,1,2,....$$

3. The Sturm-Liouville problem with nonlocal conditions

$$-u''(x)-\lambda u(x)=0, 0<x<l, u(0)=u(l), u'(0)=u'(l)$$

has solution

$$u_k(x)=\cos 2kx, k=0,1,2,...$$

$$u_k(x)=\sin 2kx, k=1,2,...$$

and

$$\lambda_k = 4k^2, k = 0,1,2,....$$

1.3.2 Fourier Series (Brown, Churchyll, 1993)

Let l be a fixed number and $f(x)$ be a periodic function with periodic $2l$, defined on $(-l,l)$. The Fourier series of $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sins and cosines :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}), \tag{1.5}$$

where a_0, a_n and b_n called the Fourier coefficientes of $f(x)$, are given by these formulas

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{p})dx, n = 1,2,...$$

and

$$b_n = \frac{1}{l}\int_{-l}^{i} \sin(\frac{n\pi x}{p})dx, n = 1,2,....$$

1.3.3 Laplace Transform (Franklyn,1 949)

Let $f(t)$ be defined for $t0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or $L\{f(t)\}$, is an integral transform given by the integral

$$F(s) = L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt$$

provided that this (improper) integral exsists i.e that this integral is convergent.

The Laplace transform is operation that transforms a function of t (i.e a function of time domain), defined on $[0,\infty]$ to a function of s (i.e of frequency domain). The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. $F(s)$ is Laplace transform or simply transform of $f(t)$. Together the two functions $f(t)$ and $F(s)$ are called a Laplace transform pair.

1.3.4 Fourier Transform (Bracewell, 1999)

The Fourier transform of a function $f = f(x)$ denoted by $F(s)$ or $F\{f(x)\}$, is an integral transform given by the integral

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-xs}dx.$$

CHAPTER II

Integral Transform Methods Of The Time-Dependent Source Identification Problem For Schrödinger Differential Equations

2.1 Introduction

Time-dependent source identification problem for the Schrödinger equation have the significant role in natural science, applied sciences, engineering, quantum mechanics, diffusion equations, heat equations. Therefore, it is important to study identification problem for the Schrödinger equation. Noted that time-dependent identification problem for Schrödinger equations are not investigated. Therefore, the main aim of chapter two is to study of the time-dependent source identification problems for Schrödinger equations. Applying results of Fourier series, Laplace and Fourier transform methods, we obtain the exact solution of several time-dependent source identification problems for Schrödinger equations.

2.2 Fourier Series Method

We consider Fourier series method for solution of the time-dependent source identification problems for Schrödingerequations with Dirichlet, Neumann and nonlocal boundary conditions.

Problem 2.1. Obtain the Fourier series solution of the following time-dependent source identification problem

$$\begin{cases} i\dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = p(t)sinx - e^{-it}sinx, \\ x \in (0,\pi), t \in (0,1), \\ u(0,x) = sinx, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \int_0^\pi u(t,x)dx = 2e^{it}, t \in [0,1] \end{cases} \tag{2.1}$$

for a one-dimensional Schrödinger equation.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem for Dirichlet condition

$$-u''(x) + \lambda u(x) = 0, u(0) = u(\pi) = 0, x \in (0,\pi)$$

generated by the space operator of problem (2.1). The solution of this Sturm-Liouville problem is

$$\lambda_k = -k^2, u_k(x) = sin(kx), k = 1,2,3,....$$

Then we will obtain the Fourier series solution of problem (2.1) by formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t)\sin(kx). \quad (2.2)$$

Here $A_k(t)$ are unknown functions. Applying this formula to the Schrödinger equation and initial condition, we get

$$\sum_{k=1}^{\infty}[iA_k'(t) + (k^2+1)]\sin(kx) = p(t)sinx - e^{-it}sinx,$$

$$u(0,x) = \sum_{k=1}^{\infty} A_k(0)sin(kx) = sin(x).$$

Moreover, using the integral condition, we get

$$\int_0^{\pi} u(t,x)dx = \int_0^{\pi}\sum_{k=1}^{\infty} A_k(t)\sin(kx)dx = 2e^{it}. \quad (2.3)$$

Equating coefficients of $\sin(kx), k = 1,2,3,..$ to zero, we get

$$iA_k'(t) + k^2A_k(t) = 0, A_k(0) = 0, t \in (0,1)$$

for $k = 2,3,4,...$ and

$$iA_1'(t) + A_1(t) = p(t) - e^{-it}, t \in (0,1), A_1(0) = 1 \quad (2.4)$$

for $k = 1$. It is clear that $A_k(t) = 0, k \neq 1$. From that and formula (2.3), we get

$$-A_1(t)(-2) = 2e^{it},$$

we obtain

$$A_1(t) = e^{it}. \quad (2.5)$$

Putting $A_1(t) = e^{it}$ in the equation (2.4), we get

$$p(t) = e^{-it}.$$

Applying obtaining formulas for $A_k(t), k = 1.2...,$ we can obtain the exact solution of problem (2.2) by formulas

$$\left(u(t,x), p(t)\right) = (e^{it}\sin x, e^{-it}).$$

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x) - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = p(t) q(x) + f(t,x), \\ 0 < t < T, x = (x_1, \dots, x_n) \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, t \in [0,T], \\ \int \dots \int_{\Omega} u(t,x) dx_1 \dots dx_n = \psi(t), t \in [0,T] \end{cases} \tag{2.6}$$

for the multidimensional Schrödinger equation with Dirichlet boundary condition can be investigated. Here and in future $\Omega \subset R^n$ be a bounded open domain with smooth boundary S , $\overline{\Omega} = \Omega \cup S$. Under compatibility conditions problem (2.6) has a unique solutio $(u(t,x), p(t))$ for the smooth functions

$f(t,x), (t,x) \in (o,T) \times \Omega, \alpha_r(x) \ge a > 0,$ $\varphi(x), x \in \overline{\Omega}, \psi(t), t \in [0,T], q(x) = 0, x \in S$

and $\int \dots \int_{\Omega} q(x) dx_1 \dots dx_n \neq 0.$

Problem 2.2. Obtain the Fourier series solution of the following time-dependent source identification problem

$$\begin{cases} i \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)(1 + cosx) - e^{it}, \\ x \in (0,\pi), t \in (0,1), \\ u(0,x) = 1 + cosx, x \in [0,\pi], \\ \frac{\partial u(t,0)}{\partial x} = \frac{\partial u(t,\pi)}{\partial x} = 0, \\ \int_0^{\pi} u(t,x) dx = \pi e^{it}, t \in [0,1] \end{cases} \tag{2.7}$$

for a one-dimensional Schrödinger equation.

Solution. In order to solve the problem, we consider the Sturm-Liouville problem with Neumann condition

$$-u''(x) + \lambda u(x) = 0, u'(0) = u'(\pi) = 0, x \in (0,\pi)$$

generated by the space operator of problem (2.7). It is easy to see that the solution of this Sturm-Liouville problem is

$$\lambda_k = -k^2, u_k(x) = cos(kx), k = 0,1,2,3,\ldots.$$

Therefore, we will seek solution $u(t,x)$ using by the Fourier series

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) cos(kx). \tag{2.8}$$

Here $A_k(t), k = 0,1,2...$ are unknown functions. Putting (2.8) into the equation (2.7), we obtain

$$\sum_{k=0}^{\infty} [iA_k'(t) + (k^2+1)]\cos(kx) = p(t)(1+cosx) - e^{it},$$

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0) cos(kx) = 1 + cos(x),$$

$$\int_0^{\pi} u(t,x)dx = \int_0^{\pi} \sum_{k=0}^{\infty} A_k(t) cos(kx) = \pi e^{it}.$$

Equating the coefficients of $\cos(kx),\ k = 0,1,2,3,4,\ldots$ to zero, we get

$$\begin{cases} iA_k'(t) + (k^2+1)A_k(t) = 0, t \in (0,1), \\ A_k(0) = 0 \end{cases}$$

for $k = 2,3,4,\ldots,$

$$iA_0'(t) + A_0(t) = p(t) - e^{it}, A_0(0) = 1, t \in (0,1)$$

for $k = 0$ and

$$iA_1'(t) + A_1(t) = p(t), A_1(0) = 1, t \in (0,1)$$

for $k = 1$. It is easy that

$$A_0(t) = e^{it}, A_k(t) = 0, k = 2,3,4,\ldots \tag{2.9}$$

Therefore,

$$p(t) = e^{it}. \tag{2.10}$$

Then,we get the following Cauchy problem

$$\begin{cases} iA'_1(t) + A_1(t) = e^{it},\ t \in (0,1) \\ A_1(0) = 1 \end{cases} \tag{2.11}$$

for the first order differential equation. It is clear that

$$A_1(t) = e^{it}. \tag{2.12}$$

Applying the formula(8), we get $u(t,x) = e^{it}(1+cosx)$. Therefore, the exact solution of problem (2.7) $(u(t,x), p(t)) = (e^{it}(1+cosx), e^{it})$.

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x) - \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} + \delta u(t,x) = p(t)q(x) + f(t,x), \\ 0 < t < T, x = (x_1,...,x_n) \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ \dfrac{\partial u(t,x)}{\partial \overline{p}} = 0, x \in S, t \in [0,T], \\ \int\limits_{x \in \overline{\Omega}} \int u(t,x)dx_1...dx_n = \psi(t), t \in [0,T] \end{cases} \tag{2.13}$$

for the multidimensional Schrödinger equation with Neumann condition can be investigated under compatibility conditions problem (2.13) and for the given smooth functions $f(t,x), (t,x) \in (0,T) \times \Omega, a_r(x) \geq a > 0, \delta > 0,$ $\varphi(x), x \in \overline{\Omega}, \psi(t), t \in [0,T],$ $q(x) = 0, x \in S$ and $\int ... \int\limits_{\Omega} q(x)dx_1...dx_n \neq 0.$

Here, $\overline{p}$ is the normal vector to S.

Problem 2.3. Obtain the Fourier series solution of the following time-dependent source identification problem

$$\begin{cases} i\dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)(1+\cos 2x) \\ +(3\cos 2x - 1)e^{it}, x \in (0,\pi), t \in (0,1), \\ u(0,x) = 1 + \cos 2x, x \in [0,\pi], \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \\ \int_0^{\pi} u(t,x)dx = \pi e^{it}, t \in [0,1] \end{cases} \tag{2.14}$$

for a one-dimensional Schrödinger equation with nonlocal conditions.

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) + \lambda u(x) = 0, u(0) = u(\pi), u_x(0) = u_x(\pi), x \in (0, \pi)$$

generated by the space operator of problem (2.14). It is easy to see that the solution of this Sturm-Liouville problem is

$$\begin{cases} \lambda_k = -4k^2, k = 0,1,2,3, \dots \\ u_{k(x)} = \cos(2kx), k = 0,1,2, \dots \\ u_{k(x)} = \sin(2kx), k = 1,2,3, \dots \end{cases}$$

Then, we will obtain the Fourier series of problem (2.14) by formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t) cos(2kx) + \sum_{k=1}^{\infty} B_k(t) \sin(2kx). \tag{2.15}$$

Here $A_k(t), k = 0,1,2..., B_k(t), k = 1,2...,$ are unknown functions. Applying this formula to the Shrödinger's type differential equation and initial condition, we get

$$\begin{cases} \sum_{k=0}^{\infty} [iA_k'(t) + (4k^2+1)A_k(t)] cos(2kx) + \sum_{k=1}^{\infty} [iB_k'(t) + (4k^2+1)B_k(t)] cos(2kx), \\ = p(t)(1 + cos(2x)) + (3cos(2x) - 1)e^{-it}, x \in [0, \pi], t \in [0,1], \\ u(0,x) = \sum_{k=0}^{\infty} A_k(0) cos(2kx) + \sum_{k=1}^{\infty} B_k(0) \sin(2kx) = 1 + cos(2x). \end{cases}$$

Equating coefficients of $\cos(2kx), k = 0,1,2,3, \dots$and $\sin(2kx), k = 1,2,3, \dots$ to zero, we get

$$\begin{cases} iB_k'(t) + (4k^2+1)B_k(t) = 0, t \in (0,1), \\ B_k(0) = 0 \end{cases}$$

for $k = 1,2,,...,$

$$\begin{cases} iA_k'(t) + (4k^2+1)A_k(t) = 0, t \in (0,1), \\ A_k(0) = 0 \end{cases}$$

for $k = 2,3,4,...,$

$$iA_0'(t) + A_0(t) = p(t) - e^{it}, t \in (0,1), A_0(0) = 1$$

for $k = 0$ and

$$iA_1'(t)+5A_1(t)=p(t)+3e^{it},t\in(0,1),A_1(0)=1$$

for $k=1$. It is easy that

$$A_0(t)=e^{it},A_k(t)=0,k=2,3,4,...,B_k(t)=0,k=1,2,3,.... \quad (2.16)$$

Therefore,

$$p(t)=e^{it}. \quad (2.17)$$

Then,we get the following Cauchy problem

$$\begin{cases} iA'_1(t)+5A_1(t)=4e^{it}, \ t\in(0,1) \\ A_1(0)=1 \end{cases} \quad (2.18)$$

for the first order differential equation. It is clear that

$$A_1(t)=e^{it}.$$

Applying formulas (2.15) and (2.16) to (2.17), we get

$$u(t,x)=A_0(t)+A_1(t)\cos 2x+\sum_{k=2}^{\infty}A_k(t)cos(2kx)+\sum_{k=1}^{\infty}B_k(t)\sin(2kx)$$

$$=e^{it}(1+cos2x)$$

We obtain the exact solution of problem (2.14) is

$$(u(t,x),p(t))=(e^{it}(1+\cos 2x),e^{it}).$$

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x)-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(t,x) \\ =p(t)q(x)+f(t,x), \\ 0<t<T,x=(x_1,...,x_n)\in\Omega, \\ u(0,x)=\varphi(x),x\in\overline{\Omega}, \\ u(t,x)|_{S_1}=u(t,x)|_{S_2},\dfrac{\partial u(t,x)}{\partial\overline{p}}|_{S_1}=\dfrac{\partial u(t,x)}{\partial\overline{p}}|_{S_2}, \\ x\in S,S_1\cap S_2=S,t\in[0,T], \\ \int\limits_{\Omega}...\int u(t,x)dx_1...dx_n=\psi(t),t\in[0,T] \end{cases} \quad (2.19)$$

for the multidimensional Schrödinger equation with nonlocal boundary conditions can be investigated. Under compatibility conditions problem (2.19) has a unique solution $\left(u(t,x),p(t)\right)$ for the smooth functions $f(t,x),(t,x)\in$ $\left(0,T\right)\times\Omega, a_r\left(x\right)\geq a>0,\delta>0,\quad \varphi(x),x\in\overline{\Omega},\psi(t),t\in\left[0,T\right],\quad q(x)=0,x\in S$ and $\int\limits_{\overline{\Omega}}\cdots\int q(x)dx_1...dx_n\neq 0.$

2.3 The Laplace transform solution

We consider Laplace transform method for solution of the time dependent source identification problem for the Schrödinger equation.

Problem 2.4. Obtain the Laplace transform solution of the time-dependent source identification problem

$$\begin{cases} i\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2 u(t,x)}{\partial x^2}=p(t)e^{-x}-e^{-it-x},x\in(0,\infty),t\in(0,1), \\ u(0,x)=e^{-x},x\in[0,\infty), \\ u(t,0)=e^{-it},u_x(t,0)=-e^{-it},\int_0^\infty u(t,x)dx=e^{-it},t\in[0,1] \end{cases} \tag{2.20}$$

for a one-dimensional Schrödinger equation.

Solution. Taking the Laplace transform of both sides of the differential equation (2.20), we get

$$\begin{cases} L\{i\dfrac{\partial u(t,x)}{\partial t}\}-s^2L\{u(t,x)\}-su(t,0)-u_x(t,0) \\ =(p(t)-e^{-it})L\{e^{-x}\},t\in(0,1), \\ L\{u(0,x)\}=L\{e^{-x}\}. \end{cases}$$

Putting

$$L\{u(t,x)\}=u(t,s), \tag{2.21}$$

using conditions

$$u(t,0)=e^{-it},u_x(t,0)=-e^{-it}$$

and formula

$$L\{e^{-x}\}=\frac{1}{s+1}, \tag{2.22}$$

we get

$$\begin{cases} iu_t(t,s) - s^2u(t,s) = (1-s)e^{-it} + (p(t) - e^{-it}\dfrac{1}{s+1}, t \in (0,1), \\ u(0,s) = \dfrac{1}{s+1}, u(t,0) = e^{-it}, t \in [0,1]. \end{cases}$$

Now, we taking the Laplace transform with respect to t, we get

$$\begin{cases} (i\mu - s^2)u(\mu,s) = (1-s)\dfrac{1}{\mu+i} + \dfrac{i}{s+1} + (p(\mu) - \dfrac{1}{\mu+i}, \\ u(\mu,0) = \dfrac{1}{\mu+i}. \end{cases}$$

Using condition

$$i\mu\frac{1}{\mu+i} = \frac{1}{\mu+i} + i + p(\mu) - \frac{1}{\mu+i}$$

and

$$p(\mu) = \frac{1}{\mu+i},$$

we get

$$u(\mu,s) = \frac{1}{\mu+i}\frac{1}{s+1}.$$

Taking the inverse Laplace transforms with respect to t and x, we obtain

$$u(t,s) = L^{-1}\{u(\mu,s)\} = e^{-it}\frac{1}{s+1}$$

and

$$u(t,x) = L^{-1}\{u(t,s)\} = e^{-it}e^{-x}$$

Finally, the exact solution of problem (2.20) is

$$(u(t,x), p(t)) = (e^{-it-x}, e^{-it}).$$

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x)-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(t,x)=p(t)q(x)+f(t,x), \\ 0<t<T, x=(x_1,...,x_n)\in\Omega^+, \\ u(0,x)=\varphi(x), x\in\overline{\Omega^+}, \\ u(t,x)\,|_{S+}=\phi(t), \dfrac{\partial u(t,x)}{\partial \overline{x}_k}\,|_{S^+}=\zeta(t), 1\le k\le n, t\in[0,T], \\ \int\limits_{\Omega^+}...\int u(t,x)dx_1...dx_n=\psi(t), t\in[0,T] \end{cases} \tag{2.23}$$

for the multidimensional Schrödinger equation can be investigated. Here and in future $\Omega^+\subset R^n$ be a unit open set in $R^n(x:0<x_k<\infty, 1\le k\le n)$ with boundary S^+, $\overline{\Omega}^+=\Omega^+\cup S^+$. Under compatibility conditions problem (2.23) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x),(t,x)\in(0,T)\times\Omega^+, a_r(x)\ge a>0,$ $\varphi(x), x\in\overline{\Omega^+}, \psi(t), t\in[0,T],$ $q(x)=0, x\in S^+$ and $\int\limits_{\Omega^+}...\int q(x)dx_1...dx_n\ne 0.$

Problem 2.5. Obtain the Laplace transform solution of the time-dependent source identification problem

$$\begin{cases} i\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2 u(t,x)}{\partial x^2}=(p(t)-e^{-it})e^{-x}, x\in(0,\infty), t\in(0,1), \\ u(0,x)=e^{-x}, x\in[0,\infty), \\ u(t,0)=e^{-it}, u(t,\infty)=0, \int_0^\infty u(t,x)dx=e^{-it}, t\in[0,1], \end{cases} \tag{2.24}$$

for a one-dimensional Schrödinger equation.

Solution. Taking Laplace transform of both sides of the Schrödinger equation and using (2.21) and (2.22) and condition $u(t,0)=e^{-it}$, we can write

$$\begin{cases} i\dfrac{\partial u(t,s)}{\partial t}-(s^2u(t,s)-su(t,s)-u_x(t,0))=(p(t)-e^{-it})\dfrac{1}{s+1}, \\ u(0,s)=\dfrac{1}{s+1}. \end{cases}$$

Now, taking the Laplace transform with respect to t, we get

$$(i\mu - s^2)u(\mu, s) = \frac{i}{s+1} - s\frac{1}{\mu+i} - u_x(\mu,0) + p(\mu)\frac{1}{s+1} - \frac{1}{\mu+i}\frac{1}{s+1}. \qquad (25)$$

Using integral condition and definition of Laplace transform, we write

$$u(\mu,0) = \frac{\mu}{\mu+i}.$$

Therefore, putting $s = 0$ in (2.25), we get

$$i\frac{\mu}{\mu+i} = i - u_x(\mu,0) + p(\mu) - \frac{1}{\mu+i}$$

and

$$u_x(\mu,0) = p(\mu) - \frac{2}{\mu+i}.$$

Then

$$u(\mu,s) = \frac{1}{(\mu+i)(s+1)} + \frac{-s}{(s+1)(\mu+i)(s^2-i\mu)} + p(\mu)\frac{s}{(s+1)(s^2-i\mu)}$$

$$= \frac{1}{(\mu+i)(s+1)} + \frac{s}{(s+1)(s^2-i\mu)}(p(\mu) - \frac{1}{\mu+i})$$

$$= \frac{1}{(\mu+i)(s+1)} + \frac{s}{(s+1)(s^2-i\mu)}(p(\mu) - \frac{1}{\mu+i}).$$

Applying formula

$$\frac{s}{(s+1)(s^2-i\mu)} = \frac{1}{2}\frac{s}{s+1}(\frac{1}{s+\sqrt{i\mu}} + \frac{1}{s-\sqrt{i\mu}}),$$

we get

$$u(\mu,s) = \frac{1}{(\mu+i)(s+1)} + (\frac{1}{s+1} - \frac{1}{2}(\frac{1}{s+\sqrt{i\mu}} + \frac{1}{s-\sqrt{i\mu}}))(p(\mu) - \frac{1}{\mu+i}).$$

Taking the inverse Laplace transforms with respect to x, we obtain

$$u(\mu,x) = \frac{1}{(\mu+i)}e^{-x} + [e^{-x} - \frac{1}{2}(e^{-\sqrt{i\mu}x} + e^{\sqrt{i\mu}x})](p(\mu) - \frac{1}{\mu+i}). \qquad (2.26)$$

Using condition

$$\lim_{x\to\infty} u(\mu,x) = 0,$$

we obtain

$$p(\mu) = \frac{1}{\mu+i}$$

and

$$u(\mu,x)=\frac{1}{\mu+i}e^{-x}.$$

Now, taking the inverse Laplace transform with respect to t, we get

$$u(t,x)=L^{-1}\{u(t,s)\}=e^{-it}e^{-x},$$

$$p(t)=L^{-1}\{\frac{1}{\mu+i}\}=e^{-it}.$$

Thus, the exact solution of problem (2.24) is

$$(u(t,x),p(t))=(e^{-it-x},e^{-it}).$$

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x)-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(t,x)=p(t)q(x)+f(t,x), \\ 0<t<T, x=(x_1,...,x_n)\in\Omega^+, \\ u(0,x)=\varphi(x), x\in\overline{\Omega^+}, \\ u(t,x)|_{S+}=\phi(t), 1\le k\le n, t\in[0,T], \\ \int\cdots\int_{\Omega^+}u(t,x)dx_1...dx_n=\psi(t), t\in[0,T] \end{cases} \tag{2.27}$$

for the multidimensional Schrödinger equation can be investigated.

Under compatibility conditions problem (2.27) has a unique solution $(u(t,x),p(t))$ for the smooth functions $f(t,x),(t,x)\in(0,T)\times\Omega^+, a_r(x)\ge a>0, \delta>0, \varphi(x), x\in\overline{\Omega^+}, \psi(t), t\in[0,T], q(x)=0, x\in S^+$ and $\int\cdots\int_{\Omega^+}q(x)dx_1...dx_n\neq 0.$

2.4 The Fourier transform solution

We consider Fourier transform method for solution of the time dependent source identification problem for the Schrödinger equation.

Problem 2.6. Obtain the Fourier transform solution of the following time-dependent source identification problem

$$\begin{cases} i\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2 u(t,x)}{\partial x^2}= p(t)e^{-x^2}-(4x-2)e^{-it-x^2}, \\ t\in(0,1), x\in(-\infty,\infty), \\ u(0,x)=e^{-x^2}, x\in(-\infty,\infty), \\ \int_{-\infty}^{\infty}u(t,x)dx=\sqrt{\pi}e^{-it}, t\in[0,1] \end{cases} \tag{2.28}$$

for a one-dimensional Schrödinger equation.

Solution. Putting

$$F\{u(t,x)\}=u(t,s), F\{e^{-x^2}\}=g(s)$$

and taking the Fourier transform of both sides of the differential equation (2.28) and using definition of Fourier transform and formula

$$F\{u_{xx}(t,x)\}=-s^2u(t,s),$$

we can write

$$\begin{cases} iu_t(t,s)+s^2u(t,s)= p(s)g(s)+s^2g(s)e^{-it}, 0<t<1, \\ u(0,s)=g(s). \end{cases} \tag{2.29}$$

Now, taking the Laplace transform of both sides of the differential equation (2.29) with respect to t, we get

$$(i\mu+s^2)u(t,s)= p(\mu)g(s)+\frac{s^2}{1+i\mu}g(s). \tag{2.30}$$

Applying condition

$$\int_{-\infty}^{\infty}u(t,x)=\sqrt{\pi}e^{-it}, 0<t<1$$

and the definition of Fourier transform, we get

$$u(t,0)=\int_{-\infty}^{\infty}u(t,x)=\sqrt{\pi}e^{-it}, 0<t<1.$$

Then

$$u(\mu,0)=\frac{\sqrt{\pi}}{1+i\mu}. \tag{2.31}$$

From that it follows

$$p(\mu)=\frac{1}{1+\mu}, u(\mu,s)=\frac{g(s)}{1+i\mu}.$$

Now, taking the inverse Laplace transform with respect to t, we get

$$p(t) = e^{-it}, u(t,s) = g(s)e^{-it}.$$

Taking the inverse Fourier transform with respect to x, we get

$$u(t,x) = e^{-x^2}e^{-it}.$$

Thus, the exact solution of problem (2.28) is

$$\big(u(t,x), p(t)\big) = (e^{-it-x^2}, e^{-it}).$$

Note that using similar procedure one can obtain the solution of the following time-dependent source identification problem

$$\begin{cases} iu_t(t,x) - \sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(t,x) = p(t)q(x) + f(t,x), \\ 0 < t < T, x = (x_1,...,x_n) \in R^n, \\ u(0,x) = \varphi(x), x \in R^n, \\ \int\limits_{R^n} \cdots \int u(t,x)dx_1...dx_n = \psi(t), t \in [0,T] \end{cases} \tag{2.32}$$

for the multidimensional Schrödinger equation can be investigated.

Under compatibility conditions problem (2.32) has a unique solution $\big(u(t,x), p(t)\big)$ for the smooth functions $f(t,x), (t,x) \in (0,T) \times R^n, a_r(x) \geq a > 0, \delta > 0,$ $\varphi(x), x \in R^n, \psi(t), t \in [0,T]$ and $\int\limits_{R^n} \cdots \int q(x)dx_1...dx_n \neq 0.$

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the differential equation has constant coefficients. It is well-known that the most general method for solving partial differential equation with dependent in t and in the space variables is operator method.

CHAPTER III

Stability Of The Time-Dependent Source Identification Problems For Schrodinger Equations

3.1 Introduction

In this section, the time-dependent source identification problem for the Schrödinger equation in a Hilbert space with the self-adjoint positive definite operator is studied. The stability of the differential problem is established. In applications, theorems on stability estimates for the solution of five type of time-dependent source identification problems for Schrödinger equations are obtained. The first of them is the time-dependent source problem for the one dimensional Schrödinger equation with nonlocal conditions. The second of them is the time-dependent source identification problem for the one dimensional Schrödinger equation with involution and Dirichlet conditions. The third is the time-dependent source problem for the one dimensional Schrödinger equation with Robin conditions. Two of them are the time-dependent source problems for the multidimensional Schrödinger equation with Dirichlet and Neumann conditions.

3.2 Auxiliary Statements

Necessary definitions, theorems and estimates (Ashyralyev,2014; Kreyszig,1978; Kolmogorov & Fomin, 1957) are given that we will be needed below.

3.2.1 Banach and Hilbert Spaces

Let L be linear space. Then $x, y \in L, \exists x + y \in L$ and $\lambda x \in L,$ λ is a number. $B = (L, \| . \|)$ be normed space

$\forall x \in L, \varphi(x) = \|x\|,$

1. $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = \widetilde{0}$ (zero element),
2. $\|\lambda x\| = |\lambda| \|x\|,$
3. $\|x + y\| \leq \|x\| + \|y\|$for any $x, y \in L.$

Then we say B be Banach spaces if B- normed space and B complete $\Leftrightarrow$ Every Cauchy sequence is convergent $\Leftrightarrow$ From $\|x_n - x_m\| \underset{n,m\to\infty}{\longrightarrow} 0 \Rightarrow \exists x \in B, \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0.$ We denote it by B, the all Banach spaces. $H = (L, \langle\, . \rangle)$ be inner product space

1. $\langle x, y\rangle = \langle y, x\rangle$,
2. $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$,
3. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$,
4. $\langle x, x\rangle = 0 \Leftrightarrow x = \tilde{0}$

$\|x\| = \sqrt{\langle x, x\rangle}$. So all inner product spaces are also normed spaces. We say H be Hilbert space if H - inner product space and H - complete space.

3.2.2 Linear Operators: Boundedness, Norm of Operator

$A: B \to B_1$ is called the linear operator if $D(A)$ is the linear space and

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \text{ for any } \alpha, \beta \text{ numbers}, x, y \in D(A),$$

$$D(A) = \{x \in B, \exists Ax\},$$

$$R(A) = \{z \in B_1, z = Ax \text{ for any } x \in D(A)\}.$$

$B$and B_1 be Banach spaces. In the case when $B_1 = (-\infty, \infty)$, $A: B \to (-\infty, \infty)$ is called the linear functional.

Definition 3.2.2.1. Let B and B_1 are Banach spaces. $A: B \to B_1$ is called the bounded operator if there is a real positive $M > 0$ such that

$$\|Ax\|_{B_1} \leq M\|x\|_B \text{forall} x \in D(A).$$

$\inf M = \|A\|_{B \to B_1}$ is called norm of the operator A. If $B = B_1$,

$$\|A\|_{B \to B_1} = \|A\|_{B \to B} = \|A\|.$$

Theorem 3.2.2.1. We have the following formulas

$$\|A\| = \sup_{\|x\|_B \leq 1} \|Ax\|_B = \sup_{\|x\|_B = 1} \|Ax\|_B = \sup_{\|x\|_B \neq \tilde{0} \in B} \frac{\|Ax\|_B}{\|x\|_B}.$$

3.2.3 Linear Positive Operators in a Hilbert Space

Let $A: H \to H$ be a linearly bounded operator in a Hilbert Space H. Then $A^*: H \to H$ is defined to be the operator satisfying

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for any } x, y \in H.$$

A^* is called the Hilbert adjoint operator A^* to $A. A$ is said to be self adjoint or Hamiltonian, if

$$A = A^* \Rightarrow \langle Ax, y\rangle = \langle x, Ay\rangle \text{ for any } x, y \in.$$

Let $A: H \to H$ is said to be positive and written $A \geq \tilde{0}$ if

$$\langle Ax, x\rangle \geq 0 \text{ for any } x \in H.$$

$A: H \to H$ is said to be positive definite and written $A \geq \delta > \tilde{0}$ if

$$\langle Ax, x\rangle \geq \delta\langle x, x\rangle \text{ for any } x \in H.$$

We consider some examples of positive operators in a Hilbert space

First, let $L_2[0,l]$ be the space of all square integrable functions $\gamma(x)$ difened on $[0,l]$ equipped with the norm

$$||\gamma||_{L_2[0,l]} = \left(\int_0^l |\gamma(x)|^2 dx\right)^2.$$

We introduce the differential operator A defined by the formula

$$Au = -\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x) \tag{3.1}$$

with the domain

$$D(A) = \{u: u, u'' \in L_2[0,l], u(0) = u(l), u'(0) = u'(l)\}.$$

Lemma 3.2.3.1. Let $a(x) \geq 0$ and $a(0) = a(l)$ and A be a differential operator defined by formula (3.1). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0,l]$.

Proof. Assume that $u, v \in D(A)$. Applying the following formula

$$< u, v >= \int_0^l u(x)v(x)dx,$$

we get

$$< Au, v >= \int_0^l Au(x)v(x)dx$$

$$= \int_0^l \left(-\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x)\right) v(x)dx$$

$$= -a(l)u'(l)v(l) + a(0)u'(0)v(0)$$

$$+ \int_0^l a(x)u'(x)v'(x)dx + \int_0^l \delta u(x)v(x)dx \tag{3.2}$$

and

$$< u, Av >= \int_0^l u(x)Av(x)dx$$

$$= \int_0^l u(x)\left(-\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right) + \delta v(x)\right) dx$$

$$= -a(l)v'(l)u(l) + a(0)v'(0)u(0)$$

$$+ \int_0^l a(x)u'(x)v'(x)dx + \int_0^l \delta u(x)v(x)dx. \tag{3.3}$$

From (3.2) and (3.3) it follows

$$< Au, v >=< u, Av >= \int_0^l a(x)u'(x)v'(x)dx + \int_0^l \delta u(x)v(x)dx. \quad (3.4)$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.4), we get

$$< Au, u >= \int_0^l a(x)u'(x)u'(x)dx + \int_0^l \delta u(x)u(x)dx \geq \delta < u, u >.$$

That means A is a positive definite operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Second, we introduce the differential operator A defined by the formula

$$Au = -\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x) \quad (3.5)$$

with the domain

$$D(A) = \{u: u, u'' \in L_2[0, l], u(0) = bu'(0), -u(l) = cu'(l)\}.$$

Lemma 3.2.3.2. Let $a(x) \geq 0, b, c > 0$ and A be a differential operator defined by formula (3.5). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0, l]$.

Proof. Assume that $u, v \in D(A)$. Then, we have formulas (3.2) and (3.3). Applying these formulas, we get

$$< Au, v >=< u, Av >$$

$$= \int_0^l \left(-\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right) + \delta u(x)\right) v(x)dx$$

$$= ca(l)u'(l)v'(l) + a(0)bu'(0)v'(0)$$

$$+ \int_0^l a(x)u'(x)v'(x)dx + \int_0^l \delta u(x)v(x)dx. \quad (3.6)$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.6), we get

$$< Au, u >= ca(l)\left(u'(l)\right)^2 + a(0)b\left(u'(0)\right)^2$$

$$+ \int_0^l a(x)u'(x)u'(x)dx + \int_0^l \delta u(x)u(x)dx \geq \delta < u, u >.$$

That means A is a positive definite operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Third, let $L_2[-l, l]$ be the space of all square integrable functions f defined on $[-l, l]$, equipped with the norm

$$\| f \|_{L_2[-l,l]} = \left\{ \int_{-l}^{l} |f(x)|^2 dx \right\}^{\frac{1}{2}}.$$

The inner product in $L_2[-l, l]$ defined by the following formula

$$\langle u, v\rangle = \int_{-l}^{l} u(x)v(x)dx.$$

We introduce a differential operator A^x defined by the formula

$$Av(x) = -(a(x)v_x(x)_x - \beta\big(a(-x)v_x(-x)\big)_x + \delta v(x) \qquad (3.7)$$

with the domain $D(A^x) = \{u, u_{xx} \in L_2[-l, l]: u(-l) = 0, u(l) = 0\}$.

Lemma 3.2.3.3. Let $a \geq a(x) = a(-x) \geq \sigma > 0$ and $\delta - a|\beta| \geq 0$. Then, the operator A defined by formula (3.7) is the self-adjoint and positive definite operator in $L_2[-l, l]$ space.

Proof. We will prove the following identity and estimate

$$\langle Au, v\rangle = \langle u, Av\rangle, u, v \in D(A), \qquad (3.8)$$

$$\langle Au, u\rangle \geq \delta\langle u, u\rangle, u \in D(A). \qquad (3.9)$$

Applying the definition of the inner product and $u, v \in D(A)$,we get

$$\langle Au, v\rangle = \int_{-l}^{l} \Big(-\big(a(x)u_x(x)\big)_x - \beta\big(a(-x)u_x(-x)\big)_x + \sigma u(x)\Big)v(x)dx \qquad (3.10)$$

$$= -\int_{-l}^{l} (a(x)u_x(x))_x v(x)dx - \beta\int_{-l}^{l} \big(a(-x)u_x(-x)\big)_x v(x)dx + \sigma\int_{-l}^{l} u(x)v(x)dx$$

$$= -a(l)u_x(l)v(l) + a(-l)u_x(-l)v(-l) + \int_{-l}^{l} a(x)u_x(x)v_x(x)dx$$

$$+\beta[-a(-l)u_x(-l)v(-l) + a(l)u_x(l)v(l)] + \beta\int_{-l}^{l} a(-x)u_x(-x)v_x(x)dx$$

$$+\sigma\int_{-l}^{l} u(x)v(x)dx$$

$$= \int_{-l}^{l} a(x)u_x(x)v_x(x)dx + \beta\int_{-l}^{l} a(x)u_x(x)v_x(-x)dx + \sigma\int_{-l}^{l} u(x)v(x)dx,$$

$$\langle u, A^x v\rangle = \int_{-l}^{l} u(x)\Big(-(a(x)v_x(x))_x - \beta\big(a(-x)v_x(-x)\big)_x + \sigma v(x)\Big)dx$$

$$= -\int_{-l}^{l} (a(x)v_x(x))_x u(x)dx - \beta\int_{-l}^{l} \big(a(-x)v_x(-x)\big)_x u(x)dx + \sigma\int_{-l}^{l} u(x)v(x)dx$$

$$= -a(l)v_x(l)u(l) + a(-l)v_x(-l)u(-l) + \int_{-l}^{l} a(x)v_x(x)u_x(x)dx$$

$$+\beta[-a(-l)v_x(-l)u(-l) + a(l)v_x(l)u(l)] + \beta \int_{-l}^{l} a(-x)v_x(-x)u_x(x)dx$$

$$+\sigma \int_{-l}^{l} u(x)v(x)dx$$

$$= \int_{-l}^{l} a(x)u_x(x)v_x(x)dx + \beta \int_{-l}^{l} u_x(x)a(-x)v_x(-x)dx + \sigma \int_{-l}^{l} u(x)v(x)dx.$$

Therefore, from these formulas it follows identity (3.8). Now, we will prove the estimate (3.9). Applying the identity (3.10), we get

$$\langle Au, u\rangle = \int_{-l}^{l} a(x)u_x(x)u_x(x)dx + \beta \int_{-l}^{l} u_x(x)a(-x)u_x(-x)dx$$

$$+\sigma \int_{-l}^{l} u(x)u(x)dx$$

$$\geq \sigma\langle u, u\rangle + \delta \int_{-l}^{l} u_x(x)u_x(x)dx + \beta\delta \int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx.$$

Using the Cauchy inequality, we get

$$\int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx \leq a\left(\int_{-l}^{l} |u_x(x)|^2 dx\right)^{\frac{1}{2}}\left(\int_{-l}^{l} |u_x(-x)|^2 dx\right)^{\frac{1}{2}}$$

$$= a\langle u_x, u_x\rangle.$$

Since $\beta \geq -|\beta|$, we have that

$$\beta \int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx \geq -|\beta| a\langle u_x, u_x\rangle.$$

Then,

$$\langle Au, u\rangle \geq \sigma\langle u, u\rangle + (\delta - |\beta| a)\langle u_x, u_x\rangle \geq \sigma\langle u, u\rangle.$$

Lemma 3.2.3.3 is proved.

3.2.4 Operator-Function Generated by the Positive Operators in a Hilbert Space

Let e^{iAt} is the operator-function generated by the operator A and defined as the solution of the initial value problem

$$i\frac{du(t)}{dt} + Au(t) = 0, t > 0, u(0) = \varphi \tag{3.11}$$

in $H.$ That means

$$u(t) = e^{iAt}\varphi.$$

We have the following formulas

$$\frac{d(e^{iAt})}{dt}\varphi = iAe^{iAt}\varphi \tag{3.12}$$

and estimate

$$\| e^{iAt} \|_{H\to H} \leq 1. \tag{3.13}$$

3.2.5 Banach Fixed-Point Theorem and Its Applications

Definition 3.2.5.1. Let $E = (E, d)$ be a metric space. A fixed point of a mapping $T: E \to E$ of a set E into itself is an element $x \in E$ which is mapped onto itself, that is, $Tx = x,$ the image Tx coincides with $x.$ Note that the Banach fixed-point theorem to be stated below is an existence and uniqueness theorem for fixed points of certain mappings, and it also gives a constructive procedure for obtaining better and better approximations to the solution of the equation

$$x = Tx. \tag{3.14}$$

Actually, we choose an arbitrary $x_0 \in E$ and determine successively a sequence $\{x_j\}_{n=0}^{\infty}$ defined by the relation

$$x_j = Tx_{j-1}, j \in \mathbb{N}_1. \tag{3.15}$$

Here and in this Thesis we will put $_k = \{j \in \mathbb{Z}; j \geq k\}$.

This procedure is called an iteration. Banach's fixed-point theorem gives sufficient conditions for the existence and uniqueness of a fixed point of a class of mappings, called contractions.

Definition 3.2.5.2. A mapping $T: E \to E$ is called a contraction on $E,$ if there is a positive real number $\alpha < 1$ such that for all $x, y \in E$

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{3.16}$$

Theorem 3.2.5.1. Assume that $E \neq \varnothing$ is complete and let T be a contraction mapping on $E.$ Then, T has precisely one fixed point.

Theorem 3.2.5.2. Let T be a mapping of a complete metric space E into itself. Assume that T is a contraction on a closed ball $F = \{x | d(x, x_0) \leq r\}$, that is, T satisfies assumption (3.16) for all $x, y \in F$. Moreover, assume that

$$d(x_0, Tx_0) < (1 - \alpha)r. \tag{3.17}$$

Then, the sequence $\{x_j\}_{j=0}^{\infty}$ defined by recursive formula (3.15) with arbitrary $x_0 \in E$ converges to an $x \in F$. This x is a fixed point of the mapping T and is the only fixed point of T in F. Now, we study the applications of the fixed-point theorem to integral equations.

Definition 3.2.5.3. An integral equation of the form

$$x(t) = \mu \int_a^b k\big(t, s; x(s)\big) ds + f(t) \tag{3.18}$$

is called a Fredholm equation of the second kind. Here, $[a, b]$ is a given interval, μ is a given parameter, f is a given function defined on $[a, b]$, x is an unknown function defined on $[a, b]$. The kernel k of the equation is a given function defined on $[a, b] \times [a, b] \times \mathbb{R}^1$.

Integral equations can be considered on various function spaces. We consider equation (18) on $C[a, b]$, the space of all continuous functions defined on the interval $[a, b]$ with the metric d defined by

$$d(x, y) = \max_{t \in a, b]} |x(t) - y(t)|. \tag{3.19}$$

$C[a, b] = (C[a, b], d)$ is complete. We assume that $f \in C[a, b]$ and k is a continuous function defined on $[a, b] \times [a, b] \times R^1$. Moreover, k satisfies on $[a, b] \times [a, b] \times \mathbb{R}^1$ the Lipschitz condition of the form

$$|k(t, s; u_1) - k(t, s; u_2)| \leq l|u_1 - u_2|. \tag{3.20}$$

Obviously, equation (18) can be written $x = Tx$, where

$$Tx(t) = \mu \int_a^b k\big(t, s; x(s)\big) ds + f(t). \tag{3.21}$$

Since f and k are continuous functions, formula (3.21) defines an operator $T: C[a, b] \to C[a, b]$. We now impose a restriction on μ such that T becomes a contraction. Applying formulas (3.19), (3.21), and condition (3.20), we get

$$d(Tx, Ty) = \max_{t \in a, b]} |Tx(t) - Ty(t)|$$

$$= |\mu| \max_{t \in a, b]} \left| \int_a^b \Big(k\big(t, s; x(s)\big) - k\big(t, s; y(s)\big) \Big) ds \right|$$

$$\leq l|\mu| \max_{t \in a, b]} \int_a^b |x(s) - y(s)| ds \leq l|\mu| \max_{s \in a, b]} |x(s) - y(s)| \int_a^b ds$$

$$= l|\mu|(b-a)d(x,y).$$

So, $d(Tx,Ty) \leq \alpha d(x,y)$, where $\alpha = l|\mu|(b-a)$. We see that T becomes a contraction if

$$|\mu| < \frac{1}{l(b-a)}. \tag{3.22}$$

Banach's fixed-point theorem now gives the following theorem.

Theorem 3.2.5.3. Assume that k and f in equation (3.18) are continuous functions on $[a,b] \times [a,b] \times \mathbb{R}^1$ and $[a,b]$, respectively. Moreover, k satisfies on $[a,b] \times [a,b] \times \mathbb{R}^1$ the Lipschitz condition (3.20). Suppose that μ satisfies condition (3.22). Then, equation (3.18) has a unique solution x defined on $[a,b]$. This function x is the limit of the iterative sequence $\{x_j\}_{j=0}^{\infty}$ defined by the recursive formula

$$x_j(t) = \mu \int_a^b k\left(t,s;x_{j-1}(s)\right)ds + f(t), j \in \mathbb{N}_1, \tag{3.23}$$

$x_0(t)$ is the given continuous function.

Definition 3.2.5.4. An integral equation of the form

$$x(t) = \mu \int_a^t k\big(t,s;x(s)\big)ds + f(t) \tag{3.24}$$

is called a Volterra equation of the second kind. Here, μ is a given parameter, f is a given function defined on $[a,b]$, x is an unknown function defined on $[a,b]$. The kernel k of the equation is a given function defined on $D \times \mathbb{R}^1$, where D is the triangular region in the ts −plane given by $a \leq s \leq t, a \leq t \leq b$.

The difference between (3.18) and (3.24) is that in (3.18) the upper limit of integration b is constant, whereas in (3.24) it is variable. This is essential. In fact, without any restriction on μ we now get the following existence and uniqueness theorem.

Theorem 3.2.5.4. Assume that k and f in equation (3.24) are continuous functions on $[a,b] \times [a,t] \times \mathbb{R}^1$ and $[a,b]$, respectively. Moreover, k satisfies on $[a,b] \times [a,t] \times \mathbb{R}^1$ the Lipschitz condition (3.20). Then, equation (3.18) has a unique solution x defined on $[a,b]$ for every μ. This function x is the limit of the iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recursive formula

$$x_j(t) = \mu \int_a^t k\left(t,s;x_{j-1}(s)\right)ds + f(t), j \in \mathbb{N}_1, \tag{3.25}$$

$x_0(t)$ is a given continuous function.

Proof. We consider equation (3.24) on $C^*[a,b]$, the space of all continuous functions defined on the interval $[a,b]$ with the metric d_* defined by

$$d_*(x,y) = \max_{t\in a,b]} e^{-L(t-a)}|x(t)-y(t)|, L > l|\mu|. \tag{3.26}$$

Since $e^{-L(b-a)} \le e^{-L(t-a)} \le 1,$ we have that

$$e^{-L(b-a)}d(x,y) \le d_*(x,y) \le d(x,y) \text{ for any } x,y \in C[a,b]. \tag{3.27}$$

$C^*[a,b] = (C^*[a,b],d)$ is complete. Obviously, equation (24) can be written as $x = Tx,$ where

$$Tx(t) = \mu \int_a^t k\big(t,s;x(s)\big)ds + f(t). \tag{3.28}$$

Since f and k are continuous functions, formula (3.21) defines an operator $T: C^*[a,b] \to C^*[a,b].$ Applying formulas (3.28), (3.26), and condition (3.20), we get

$$d_*(Tx,Ty) = \max_{t\in a,b]} e^{-L(t-a)}|Tx(t)-Ty(t)|$$

$$= |\mu| \max_{t\in a,b]} e^{-L(t-a)} \left| \int_a^t \Big(k\big(t,s;x(s)\big) - k\big(t,s;y(s)\big)\Big) ds \right|$$

$$\le l|\mu| \max_{t\in a,b]} \int_a^t e^{-L(t-s)}e^{-L(s-a)}|x(s)-y(s)|ds$$

$$\le l|\mu| \max_{s\in a,t]} e^{-L(s-a)}|x(s)-y(s)| \max_{t\in a,b]} \int_a^t e^{-L(t-s)}ds$$

$$= \max_{t\in a,b]} \frac{l|\mu|}{L}(1-e^{-L(t-a)})d_*(x,y) \le \frac{l|\mu|}{L} d_*(x,y).$$

So, $d(Tx,Ty) \le \alpha d(x,y),$ where $\alpha = \frac{l|\mu|}{L}.$ Since $L > l|\mu|,$ we have that $\alpha < 1.$ That means T is a contraction mapping on $C^*[a,b].$ Then, equation (3.18) has a unique solution x defined on $[a,b]$ for every $\mu.$ This function x is the limit of the iterative sequence $\{x_j\}_{j=0}^{\infty}$ defined by recursive formula (3.18). Theorem 3.2.5.4 is proved.

3.3 The Main Theorem On Stability

We consider the time-dependent SIP for the SE

$$\begin{cases} i\frac{du}{dt} + Au(t) = p(t)q + f(t), t \in (0,T), \\ u(0) = \varphi, \ B[u(t)] = \psi(t), [0,T] \end{cases} \tag{3.29}$$

in a Hilbert space H with the self-adjoint positive definite operator A with dense domain $D(A)$ in H. Here $B: H \to \mathrm{R}$ is a given linear bounded functional and $\psi(t): [0,T] \to \mathrm{R}$ is a given smooth function and $q \in D(A), Bq \ne 0.$

By a solution of the time dependent SIP (3.29) we mean a pair $(u(t), p(t))$ satisfying the following conditions:

1. The element $u(t)$ belong to $D(A)$ for all $t \in [0, T]$, and the function $Au(t)$ is continuous on $[0, T]$, $p(t) \in C[0, T]$.
2. $u(t)$ is continuously differentiable on the segment $[0, T]$. The derivative at the end points of the interval are understood as the appropriate unilateral derivative.
3. $(u(t), p(t))$ satisfies the differential equation and conditions.

We denote $u(t)$ and $p(t)$ by formulas

$$u(t) = u(t; \varphi, f(t), \psi(t)), p(t) = p(t; \varphi, f(t), \psi(t)).$$

A solution of problem (3.29) defined in this manner will from now on be referred to as a solution of problem (3.29) in the space $C(H) \times C[0, T]$. Here, $C(H) = C([0, T], H)$ is the space of continuous H-valued functions $u(t)$ defined on $[0, T]$, equipped with the norm

$$||u||_{C(H)} = \max_{0 \le t \le T} ||u(t)||_{H}. \tag{3.30}$$

In this section the main theorem on stability of the SIP (3.29) is established. In applications, stability estimates for the solutions of five type of time-dependent SIPs for SEs with local and nonlocal conditions are obtained.

3.3.1. The Well-Posedness of Differential Problem (3.29)

Theorem 3.1 Let $\varphi \in D(A)$. Suppose that f, $f_t \in C(H)$ and ψ, $\psi' \in C[0, T]$. Then the time dependent SIP (3.29) has a unique solution $(u, p) \in C(H) \times C[0, T]$.

Proof. Assume that $w(t)$ be the solution of the initial value problem (IVP)

$$\begin{cases} i\frac{dw(t)}{dt} + Aw(t) = i\mu(t)Aq + f(t), \ t \in (0, T), \\ w(0) = \varphi \end{cases} \tag{3.31}$$

and $\mu(t)$ be the function determining by

$$\mu(t) = \int_0^t p(s)ds, 0 \le t \le T. \tag{3.32}$$

Then,

$$u(t) = w(t) - i\mu(t)q. \tag{3.33}$$

Using the $B[u(t)] = \psi(t)$ and formula (3.33), we can obtain

$$\mu(t) = \frac{i}{Bq}(\psi(t) - B[w(t)]). \tag{3.34}$$

Since

$$p(t) = \mu_t(t), 0 < t < T, \mu(0) = 0, \tag{3.35}$$

we obtain

$$p(t) = \frac{i}{Bq}(\psi_t(t) - B[w_t(t)]), 0 < t < T. \tag{3.36}$$

Therefore, the following theorem will complete the proof of Theorem 3.1.

Theorem 3.2 Under assumptions of Theorem 3.1, the IVP (3.31) has a unique solution $w(t) \in C(H)$.

Proof. The IVP (3.31) is equivalent to the integral equation

$$w(t) = e^{iAt}\varphi - i\int_0^t e^{iA(t-s)} \left\{\frac{i}{Bq}(\psi(s) - B[w(s)])Aq + f(s)\right\} ds. \tag{3.37}$$

Here, e^{iAt} is operator function generated by the operator A and defined by formula (3.11). Taking the derivative with respect to t, we get

$$\frac{dw(t)}{dt} = iAe^{iAt}\varphi + \int_0^t Ae^{iA(t-s)} \left\{\frac{i}{Bq}(\psi(s) - B[w(s)])Aq + f(s)\right\} ds.$$

From that it follows

$$\frac{dw(t)}{dt} = iAe^{iAt}\varphi + i\left\{\frac{i}{Bq}(\psi(t) - B[w(t)])Aq + f(t)\right\}$$

$$-ie^{iAt}\left\{\frac{i}{Bq}(\psi(0) - B[w(0)])Aq + f(0)\right\}$$

$$+\int_0^t ie^{iA(t-s)}\left\{\frac{i}{Bq}(\psi_s(s) - B[w_s(s)])Aq + f_s(s)\right\} ds. \tag{3.38}$$

Note that (3.38) is a linear Volterra equation of the second kind with respect to t for the $\frac{dw(t)}{dt}$ in $C(H)$. Therefore, the proof of Theorem 3.2 is based on the fixed-point theorem. Actually, the recursive formula for the solution of IVP (3.31) is

$$v_j(t) = iAe^{iAt}\varphi$$

$$+i\left\{\frac{i}{Bq}\left(\psi(t) - B\left[\int_0^t v_{j-1}(s)ds + \varphi\right]\right)Aq + f(t)\right\}$$

$$-ie^{iAt}\left\{\frac{i}{Bq}(\psi(0) - B[\varphi])Aq + f(0)\right\}$$

$$+\int_0^t ie^{iA(t-s)}\left\{\frac{i}{Bq}\big(\psi_s(s) - B[v_{j-1}(s)]\big)Aq + f_s(s)\right\} ds, j \geq 1,$$

$$v_0(t) = iAe^{iAt}\varphi + i\left\{\frac{i}{Bq}(\psi(t) - B[\varphi])Aq + f(t)\right\}$$

$$-ie^{iAt}\left\{\frac{i}{Bq}(\psi(0) - B[\varphi])Aq + f(0)\right\}. \tag{3.39}$$

Therefore,

$$\frac{dw(t)}{dt} = v(t) = v_0(t,x) + \sum_{i=0}^{\infty} \left(v_{i+1}(t,x) - v_i(t,x)\right). \tag{3.40}$$

Applying the triangle inequality and estimate (3.13), we get

$$\|v_0(t)\|_H \leq \| e^{iAt} \|_{H\to H} \|A\varphi\|_H + \left\{\frac{1}{|Bq|}(|\psi(t)| + |B[\varphi]|)\|Aq\|_H + \|f(t)\|_H\right\}$$

$$+\| e^{iAt} \|_{H\to H} \left\{\frac{1}{|Bq|}(|\psi(0)| + |B[\varphi]|)\|Aq\|_H + \|f(0)\|_H\right\}$$

$$\leq \|A\varphi\|_H + \left\{\frac{1}{|Bq|}\left(\max_{t\in[0,T]}|\psi(t)| + |B[\varphi]|\right)\|Aq\|_H + \max_{t\in[0,T]}\|f(t)\|_H\right\}$$

$$+\left\{\frac{1}{|Bq|}(|\psi(0)| + |B[\varphi]|)\|Aq\|_H + \|f(0)\|_H\right\} = M_0,$$

$$\|v_1(t) - v_0(t)\|_H \leq \frac{1}{|Bq|}\int_0^t |B[v_0(s)]|\|Aq\|_H ds$$

$$+\int_0^t \left\|e^{iA(t-s)}\right\|_{H\to H}\frac{1}{|Bq|}|B[v_0(s)]|\|Aq\|_H ds$$

$$\leq \frac{2}{|Bq|}\int_0^t |B[v_0(s)]|\|Aq\|_H ds \leq M_1 t \tag{3.41}$$

for any $t \in [0,T]$. Therefore

$$\|v_1(t)\|_H \leq M_0 + M_1 t$$

for any $t \in [0,T]$. Assume that $j \geq 2$. Then

$$v_{j+1}(t) - v_j(t) = i\left\{\frac{i}{Bq}\left(-B\left[\int_0^t \left[v_j(s) - v_{j-1}(s)\right]ds\right]\right)Aq\right\}$$

$$+\int_0^t ie^{iA(t-s)}\left\{\frac{i}{Bq}\left(-B[v_j(s) - v_{j-1}(s)]\right)Aq\right\}ds.$$

Applying the triangle inequality and estimate (3.13), we get

$$\left\|v_{j+1}(t) - v_j(t)\right\|_H \leq \frac{1}{|Bq|}\int_0^t \left|B\left[v_j(s) - v_{j-1}(s)\right]\right|\|Aq\|_H ds$$

$$+\int_0^t \left\|e^{iA(t-s)}\right\|_{H\to H}\frac{1}{|Bq|}\left|B\left[v_j(s) - v_{j-1}(s)\right]\right|\|Aq\|_H ds$$

$$\leq \frac{2}{|Bq|}\int_0^t \left|B\left[v_j(s) - v_{j-1}(s)\right]\right|\|Aq\|_H ds$$

$$\leq \frac{2}{|Bq|}\int_0^t \|B\|\left\|v_j(s) - v_{j-1}(s)\right\|_H\|Aq\|_H ds$$

$$\leq K\int_0^t \left\|v_j(s) - v_{j-1}(s)\right\|_H ds \tag{3.42}$$

for any $t \in [0,T]$. Using estimates (3.42) and (3.41), we get

$$\|v_2(t)-v_1(t)\|_H \le K\int_0^t \|v_1(s)-v_0(s)\|_H ds \le KM_1\frac{t^2}{2}$$

for any $t \in [0,T]$. Let

$$\left\|v_j(t)-v_{j-1}(t)\right\|_H \le \frac{M_1}{K}\frac{(Kt)^j}{j!}$$

for any $t \in [0,T]$. Then, using estimate (3.42), we get

$$\left\|v_{j+1}(t)-v_j(t)\right\|_H \le K\int_0^t \frac{M_1}{K}\frac{(Ks)^j}{j!}ds = \frac{M_1}{K}\frac{(Kt)^{(j+1)}}{(j+1)!}$$

and

$$\left\|v_{j+1}(t)\right\|_H \le M_0 + M_1 t + \ldots + \frac{M_1}{K}\frac{(Kt)^{(j+1)}}{(j+1)!}$$

for any $t \in [0,T]$ by mathematical induction. From that and formula (3.40) it follows that.

$$\|v(t)\|_H \le \|v_0(t)\|_H + \sum_{j=0}^{\infty}\left(v_{j+1}(t,x)-v_j(t,x)\right)$$

$$\le M_0 + \sum_{j=0}^{\infty}\frac{M_1}{K}\frac{(Kt)^{j+1}}{(j+1)!} \le M_0 + \frac{M_1}{K}e^{Kt}$$

for any $t \in [0,T]$ which proves the existence of a bounded solution of problem (3.31) in $C(H)$.

Now, we will prove uniqueness of this solution of problem (3.31). Assume that there is a bounded solution $z(t,x)$ of problem (3.31) and $z(t,x) \neq w(t,x)$. We denote that $V(t,x) = z(t,x) - v(t,x)$. Therefore, for $V(t,x)$, we have that

$$V((t,x) = \int_0^t e^{iA(t-s)}\ \left\{\frac{i}{Bq}B[V(s,x)]Aq(x)\right\}ds.$$

Applying estimates (3.13), we get

$$\|V(t,\cdot)\|_H \le \frac{1}{|Bq|}\|Aq(.)\|_H\int_0^t |V(s,x)|ds \le K\int_0^t \|V(s,\cdot)\|_H ds$$

for any $t \in [0,T]$. Therefore, using the integral inequality, we get

$$\|V(t,\cdot)\|_H \le 0$$

for any $t \in [0,T]$. From that it follows that $V(t,x) = 0$ which proves the uniqueness of a bounded solution of problem (3.31) in $C(H)$. Theorem 3.2 is proved.

We have the following main theorem on the stability of problem (3.29).

Theorem 3.3 Assume that the assumptions of Theorem 3.1 hold. The solution of SIP (3.29) obeys the stability estimate

$$\|u_t\|_{C(H)} + \|Au\|_{C(H)} + \|p\|_{C[0,T]}$$
$$\leqslant M(\delta,q)\big[\|A\varphi\|_H + \|f(0)\|_H + \|f_t\|_{C(H)}$$
$$+|\psi(0)| + \|\psi_t\|_{C[0,T]}\big]. \tag{3.43}$$

In the present study, $M(\delta,q)$ denotes positive constants, which may different in time and thus it is not a subject of precision.

Proof. Applying formula (3.36), estimates (3.13) and $Bq \neq 0,$ we get the estimate

$$|p(t)| \leq M_1(\delta,q)\big[\|\psi_t\|_{C[0,T]} + \|w_t(t)\|_H\big] \tag{3.44}$$

for all $t \in [0,T]$ and

$$\|p\|_{C[0,T]} \leqslant M_1(\delta,\sigma)\big[\|\psi_t\|_{C[0,T]} + \|w_t\|_{C(H)}\big]. \tag{3.45}$$

Now, applying formulas (3.33) and (3.35), we can write

$$u_t(t) = w_t(t) + p(t)q, 0 < t \leq T.$$

By the triangle inequality, this formula yields us

$$\|u_t\|_{C(H)} \leqslant \|w_t\|_{C(H)} + \|p\|_{C[0,T]}\|q\|_H. \tag{3.46}$$

Then, the proof of estimate (3.43) is based on equation (3.37), estimates (3.45), (3.46) and on the following result of stability estimate.

Theorem 3.4 Assume that the assumptions of Theorem 3.1 hold. The solution of IVP (3.31) obeys the stability estimate

$$\|w_t\|_{C(H)}$$
$$\leqslant M(\delta,q)\big[\|A\varphi\|_H + |\psi(0)| + \|f(0)\|_H + \|f_t\|_{C(H)} + \|\psi_t\|_{C[0,T]}\big]. \tag{3.47}$$

Proof. Applying formula (3.38), we get

$$w_t(t) = ie^{iAt}A\varphi$$
$$-i\int_0^t e^{iA(t-s)}\left\{\frac{i}{Bq}(\psi_s(s) - B[w_s(s)])Aq + f_s(s)\right\}ds$$
$$-ie^{iAt}\left\{\frac{i}{Bq}(\psi(0) - B[w(0)])Aq + f(0)\right\}.$$

Then, applying the triangle inequality and estimate (3.13) and $Bq \neq 0,$ we get the estimate

$$\|w_t(t)\|_H \leq M_3(\delta,q)[|\psi(0)| + \|f(0)\|_H + \|A\varphi\|_{\mathbb{H}}$$
$$+\|\psi_t\|_{C[0,T]} + \|f_t\|_{C(H)}\big] + M_4(\delta,q)\int_0^t \|w_s(s)\|_H ds \tag{3.48}$$

for $0 \leq t \leq T.$ Then, applying the integral inequality, we conclude that the following stability estimate

$$\|w_t(t)\|_H \leq M(\delta,q)[|\psi(0)| + \|f(0)\|_H + \|A\varphi\|_H$$
$$+\|\psi_t\|_{C[0,T]} + \|f_t\|_{C(H)}]e^{M_4(\delta,q)t} \tag{3.49}$$

is satisfied for the solution of IVP (3.31) for every $t \in [0,T]$. From estimate (3.49) it follows estimate (3.47). Theorem 3.4 is established.

3.4. Applications

Now, consider the applications of the main theorem.

Problem 3.4.1. We consider one dimensional time-dependent SIP

$$\begin{cases} iu_t(t,x) - \big(a(x)u_x(t,x)\big)_x + \delta u(t,x) \\ = p(t)q(x) + f(t,x), 0 < t < T, x \in (0,l), \\ u(0,x) = \varphi(x), x \in [0,l], \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), t \in [0,T], \\ \int_0^l u(t,x)dx = \psi(t), t \in [0,T] \end{cases} \tag{3.50}$$

for the SE with nonlocal conditions. Under compatibility conditions problem (3.50) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x)$, $(t,x) \in (0,T) \times (0,l)$, $a(x) \geq a > 0$, $a(l) = a(0)$, $\varphi(x)$, $x \in [0,l]$, $\psi(t)$, $t \in [0,T]$, $q(0) = q(l)$, $q'(0) = q'(l)$ and $\int_0^l q(x)dx \neq 0$.

Problem (3.50) can be written as the time dependent SIP (3.14) in a Hilbert space $H = L_2[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -(a(x)u_x(x)_x + \delta u(x) \tag{3.51}$$

with the domain $D(A^x) = \{u \in W_2^2[0,l]: u(0) = u(l), u_x(0) = u_x(l)\}$. Therefore the main theorem 3.3 permits to get the following result on the stability of problem (3.50).

Theorem 3.5. Assume that $\varphi \in W_2^2[0,l]$ and $f(t,x)$ be a continuously differentiable function in t and square integrable in x, $\psi(t)$ is a continuously differentiable function. Then the SIP (3.50) has a unique solution $u \in C(L_2[0,l]) = C([0,T], L_2[0,l])$, $p \in C[0,T]$ and for the solution of SIP (3.50) the following stability estimates hold

$$\left\|\frac{\partial u}{\partial t}\right\|_{C(L_2[0,l])}+\|u\|_{C(W_2^2[0,l])}+\|p\|_{C[0,T]}\leqslant M_1(q)\left[\|\varphi\|_{W_2^2[0,l]}\right.$$
$$\left.+\|f(0)\|_{L_2[0,l]}+|\psi(0)|+\left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])}+\|\psi'\|_{C[0,T]}\right].$$

Here and in future, the Sobolev space $W_2^2[0,l]$ is defined as the set of all functions $u(x)$ defined on $[0,l]$ such that $u(x)$ and the second order derivative function $u''(x)$ are all locally integrable in $L_2[0,l]$, equipped the norm

$$\|u\|_{W_2^2[0,l]}=\left(\int_0^l |u(x)|^2dx\right)^{\frac{1}{2}}+\left(\int_0^l |u'(x)|^2dx\right)^{\frac{1}{2}}+\left(\int_0^l |u''(x)|^2dx\right)^{\frac{1}{2}}.$$

Proof. The proof of Theorem 3.5 is based on the abstract stability result of the Theorem 3.3, on the self-adjointness and positivity of operator $A=A^x$ defined by the formula (3.51) of Lemma 3.2.3.1 and on boundedness in $L_2[0,l]$ of a linear functional B defined by the formula

$$Bu(t,x)=\int_0^l u(t,x)dx, t\in[0,T]. \tag{3.52}$$

Problem 3.4.2. We consider the time-dependent SIP

$$\begin{cases} iu_t\big(a(x)u_x(t,x)\big)_x-\beta\big(a(-x)u_x(t,-x)\big)_x+\delta u(t,x) \\ p(t)q(x)+f(t,x), 0<t<T, x\in(-l,l) \\ u(0,x)=\varphi(x), x\in[-l,l], \\ u(t,-l)=u(t,l)=0, t\in[0,T], \\ \int_{-l}^l u(t,x)dx=\psi(t), t\in[0,T] \end{cases} \tag{3.53}$$

for the SE with involution and Dirichlet conditions. Under compatibility conditions problem (3.53) has a unique solution $(u(t,x),p(t))$ for the smooth functions $f(t,x)$ $(t,x)\in(0,T)\times(-l,l))$, $a(x)$, $a\geq a(x)=a(-x)\geq\delta>0$, $\delta-a|\beta|\geq 0$, $\varphi(x)$, $x\in 0,l]$, $\psi(t)$, $t\in[0,T]$, $q(-l)=q(l)=0$, and $\int_{-l}^l q(x)dx\neq 0$.

Problem (3.53) can be written as the time dependent identification problem (14) in a Hilbert space $H=L_2[-l,l]$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$Au(x)=-(a(x)u_x(x)_x-\beta\big(a(-x)u_x(-x)\big)_x+\delta u(x) \tag{3.54}$$

with the domain $D(A)=\{u\in W_2^2[-l,l]: u(-l)=u(l)=0\}$.Therefore the main theorem 3.3 permits to get the following result on the stability of problem (3.53).

Theorem 3.6. Suppose that $\varphi \in W_2^2[-l,l]$ and $f(t,x)$ be a continuously differentiable function in t and square integrable in x, $\psi(t)$ is a continuously differentiable function. Then the time source time-dependent SIP (3.53) has a unique solution $u \in C(L_2[-l,l]) = C([0,T], L_2[-l,l])$, $p \in C[0,T]$ and for the solution of the time-dependent SIP (3.53) the following stability estimates hold

$$\left\|\frac{\partial u}{\partial t}\right\|_{C(L_2[-l,l])} + \|u\|_{C(W_2^2[-l,l])} + \|p\|_{C[0,T]} \leqslant M_1(q)\Big[\|\varphi\|_{W_2^2[-l,l]}$$
$$+\|f(0)\|_{L_2[-l,l]} + |\psi(0)| + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[-l,l])} + \|\psi'\|_{C[0,T]}\Big].$$

Here, the Sobolev space $W_2^2[-l,l]$ is defined as the set of all functions $u(x)$ defined on $[-l,l]$ such that $u(x)$ and the second order derivative function $u''(x)$ are all locally integrable in $L_2[-l,l]$, equipped the norm

$$\|u(x)\|_{W_2^2[-l,l]} = \left(\int_{-l}^{l} |u(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_{-l}^{l} |u''(x)|^2 dx\right)^{\frac{1}{2}}.$$

Proof. The proof of Theorem 3.6 is based on the abstract stability result of the Theorem 3.3, on the self-adjointness and positivity of operator $A = A^x$ defined by the formula (3.54) of Lemma 3.2.3.3 and on boundedness in $L_2[-l,l]$ of a linear functional B defined by the formula

$$Bu(t,x) = \int_{-l}^{l} u(t,x)dx, t \in [0,T].$$

Problem 3.4.3. We consider one dimensional time-dependent SIP

$$\begin{cases} iu_t(t,x) - \big(a(x)u_x(t,x)\big)_x + \delta u(t,x) \\ = p(t)q(x) + f(t,x), 0 < t < T, x \in (0,l), \\ u(0,x) = \varphi(x), x \in [0,l], \\ u(t,0) = bu_x(t,0), u(t,l) = -cu_x(t,l), t \in [0,T], \\ \int_0^l u(t,x)dx = \psi(t), t \in [0,T] \end{cases} \quad (3.55)$$

for the SE with Robin boundary conditions. Under compatibility conditions problem (3.55) has a unique solution $\big(u(t,x), p(t)\big)$ for the smooth functions $f(t,x)$, $(t,x) \in (0,T) \times (0,l)$, $a(x) \geq a > 0$, $a(l) = a(0)$, $b,c \geq 0$, $\varphi(x)$, $x \in 0,l]$, $\psi(t)$, $t \in [0,T]$, $q(0) = bq'(0)$, $q(l) = -cq'(l)$ and $\int_0^l q(x)dx \neq 0$.

Problem (3.55) can be written as the time dependent identification problem (3.55) in a Hilbert space $H = L_2[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -(a(x)u_x(x)_x + \delta u(x) \tag{3.56}$$

with the domain $D(A^x) = \{u \in W_2^2[0,l]: u(0) = bu'(0), q(l) = -cu'(l)\}$. Therefore the main theorem 2.3 permits to get the following result on the stability of problem (3.55).

Theorem 3.7. Assume that $\varphi \in W_2^2[0,l]$ and $f(t,x)$ be a continuously differentiable function in t and square integrable in x, $\psi(t)$ is a continuously differentiable function. Then the time-dependent SIP (3.55) has a unique solution $u \in C(L_2[0,l]) = C([0,T], L_2[0,l])$, $p \in C[0,T]$ and for the solution of the time-dependent SIP (3.55) the following stability estimates hold

$$\left\|\frac{\partial u}{\partial t}\right\|_{C(L_2[0,l])} + \|u\|_{C(W_2^2[0,l])} + \|p\|_{C[0,T]} \leqslant M_1(q)\Big[\|\varphi\|_{W_2^2[0,l]}$$
$$+\|f(0)\|_{L_2[0,l]} + |\psi(0)| + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2[0,l])} + \|\psi'\|_{C[0,T]}\Big].$$

Proof. The proof of Theorem 3.7 is based on the abstract stability result of the Theorem 3.3, on the self-adjointness and positivity of operator $A = A^x$ defined by the formula (3.56) of Lemma 3.2.3.2 and on boundedness in $L_2[0,l]$ of a linear functional B defined by the formula (3.52).

Problem 3.4.4. Let $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$ we consider the multidimensional time-dependent SIP

$$\begin{cases} iu_t(t,x) - \sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(t,x) = p(t)q(x) + f(t,x), \\ 0 < t < T, x = (x_1, \dots, x_n) \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, t \in [0,T], \\ \int \underset{\Omega}{\cdots} \int u(t,x)dx_1 \dots dx_n = \psi(t), t \in [0,T] \end{cases} \tag{3.57}$$

for the SE with Dirichlet boundary condition. Under compatibility conditions problem (3.57) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x)$, $(t,x) \in (0,T) \times \Omega$, $a_r(x) \geq a > 0$, $\varphi(x)$, $x \in \overline{\Omega}$, $\psi(t)$, $t \in [0,T]$, $q(x) = 0$, $x \in S$ and

$$\int \underset{\Omega}{\cdots} \int q(x)dx_1 \dots dx_n \neq 0. \tag{3.58}$$

Problem (3.57) can be written as the time-dependent source identification problem (3.57) in a Hilbert space $H = L_2(\overline{\Omega})$ with self-adjoint positive definite operator

$A = A^x$ defined by the formula

$$Au(x) = -\sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(x) \tag{3.59}$$

with domain

$$D(A) = \{u(x): u(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, u(x) = 0, x \in S\}.$$

Therefore the main theorem 3.3 permits to get the following result on the stability of problem (3.57).

Theorem 3.8. Assume that $\varphi \in W_2^2(\overline{\Omega})$ and $f(t,x)$ be a continuously differentiable function in t and square integrable in x, $\psi(t)$ is a continuously differentiable function. Then the time-dependent SIP (3.57) has a unique solution $u \in C(L_2(\overline{\Omega})) = C([0,T], L_2(\overline{\Omega}))$, $p \in C[0,T]$ and for the solution of time-dependent SIP (3.57) the following stability estimates hold

$$\begin{aligned} &\left\|\frac{\partial u}{\partial t}\right\|_{C(L_2(\overline{\Omega}))} + \|u\|_{C(W_2^2(\overline{\Omega}))} + \|p\|_{C[0,T]} \leqslant M_1(q)\Big[\|\varphi\|_{W_2^2(\overline{\Omega})} \\ &+\|f(0)\|_{L_2(\overline{\Omega})} + |\psi(0)| + \left\|\frac{\partial f}{\partial t}\right\|_{C(L_2(\overline{\Omega}))} + \|\psi'\|_{C[0,T]}\Big]. \end{aligned} \tag{3.60}$$

Proof. The proof of Theorem 3.8 is based on the abstract stability result of the Theorem 3.3, on the self-adjointness and positivity of operator $A = A^x$ defined by the formula (3.59) and on boundedness in $L_2(\overline{\Omega})$ of a linear functional B defined by the formula

$$Bu(t,x) = \int \underset{\overline{\Omega}}{\cdots} \int u(t,x)dx_1 \ldots dx_n, t \in [0,T] \tag{3.61}$$

and the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$.

Theorem 3.9. For the solution of the elliptic differential problem (see, Sobolevskii,1975)

$$\begin{cases} Au(x) = w(x), & x \in \Omega, \\ u(x) = 0, & x \in S \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \left\|u_{x_r x_r}\right\|_{L_2(\overline{\Omega})} \leq M\|w\|_{L_2(\overline{\Omega})}.$$

Problem 3.4.5. In $[0,T] \times \Omega$ we consider the multidimensional time-dependent SIP

$$\begin{cases} iu_t(t,x) - \sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(t,x) = p(t)q(x) + f(t,x), \\ 0 < t < T, x = (x_1, \dots, x_n) \in \Omega, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}, \\ \frac{\partial}{\partial \vec{p}} u(t,x) = 0, x \in S, t \in [0,T], \\ \int \underset{\Omega}{\cdots} \int u(t,x) dx_1 \dots dx_n = \psi(t), t \in [0,T] \end{cases} \tag{3.62}$$

for the SE with Neumann boundary condition. Here, $\vec{p}$ is the normal vector to Ω. Under compatibility conditions problem (3.62) has a unique solution $\left(u(t,x), p(t)\right)$ for the smooth functions $f(t,x)$, $(t,x) \in (0,T) \times \Omega$, $a_r(x) \geq a > 0$, $\varphi(x)$, $x \in \overline{\Omega}$, $\psi(t)$, $t \in [0,T]$, $\frac{\partial}{\partial \vec{p}} q(x) = 0$, $x \in S$ and $\int \underset{\Omega}{\cdots} \int q(x) dx_1 \dots dx_n \neq 0$.

Problem (3.62) can be written as the time dependent identification problem (3.14) in a Hilbert space $H = L_2\left(\overline{\Omega}\right)$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$Au(x) = -\sum_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \delta u(x) \tag{3.63}$$

with domain

$$D(A) = \left\{u(x) \colon u(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, \frac{\partial}{\partial \vec{p}} u(x) = 0, x \in S\right\}.$$

Therefore the main theorem 3.3 permits to get the following result on the stability of problem (3.62).

Theorem 3.10. Suppose that assumptions of Theorem 2.8 hold. Then the solutions of the time-dependent SIP (3.62) satisfy the stability estimates (3.60).

Proof. The proof of Theorem 3.10 is based on the abstract theorem 2.3, on boundedness in $L_2(\overline{\Omega})$ of a linear functional B defined by the formula (3.61) and on the self-adjointness and positivity of a differential operator A in $L_2(\overline{\Omega})$ defined by the formula (3.63) with domain

$$D(A) = \left\{u(x) \colon u(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, \frac{\partial}{\partial \vec{p}} u(x) = 0, x \in\right\}$$

and on the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$.

Theorem 3.11. For the solution of the elliptic differential problem (see, Sobolevskii,1975)

$$\begin{cases} A^x u(x) = w(x), & x \in \Omega, \\ \dfrac{\partial}{\partial \overleftarrow{p}} u(x) = 0, & x \in S. \end{cases}$$

CHAPTER IV

Stability Of Difference Schemes

4.1 Introduction

In this section, the absolute stable difference schemes for the approximate solutions of the time-dependent source identification problem for the Schrödinger equation in a Hilbert space with the selfadjoint positive definite operator are investigated. The first and second order of accuracy implicit and second order of accuracy r-modified Crank-Nicolson difference schemes are presented. The stability of these difference schemes are established. In applications, theorems on stability estimates for the solution of difference schemes for the approximate solutions of five type of time-dependent source identification problems for Schrödinger equations are obtained. The first of them is the time-dependent source problem for the one dimensional Schrödinger equation with nonlocal conditions. The second them is the time-dependent source problem for the one dimensional Schrödinger equation with involution and Dirichlet conditions. The third is the time-dependent source problem for the one dimensional Schrödinger equation with Robin conditions. Two of them are the time-dependent source problems for the multidimensional Schrödinger equation with Dirichlet and Neumann conditions.

4.2 Auxiliary Statements

To formulate our results, we introduce normed space $C_\tau(H) = C([0,T]_\tau, H)$ of all abstract grid functions $f^\tau = \{f_k\}_{k=0}^N$ defined on the uniform grid space

$$[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N\tau = T\}$$

with values in H equipped with the norm

$$\|f^\tau\|_{C_\tau(H)} = \max_{0\le k\le N} \|f_k\|_H.$$

4.3 The First Order of Accuracy Difference Scheme

We present the first order of accuracy difference scheme

$$\begin{cases} i\frac{u_k-u_{k-1}}{\tau} + Au_k = p_k q + f_k, f_k = f(t_k), 1 \le k \le N, \\ u_0 = \varphi, \\ Bu_k = \psi_k, \psi_k = \psi(t_k), 0 \le k \le N \end{cases} \tag{4.1}$$

for the approximate solution of the time dependent SIP (3.31).

Now, let us state the stability result for the solution of difference scheme (4.1).

Theorem 4.1 Assume that $\varphi \in D(A)$. Then, the solution of difference scheme (4.1) obeys the stability estimate

$$\left\|\left\{\frac{u_k - u_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)} + \|\{Au_k\}_{k=0}^{N}\|_{C_\tau(H)} + \|\{p_k\}_{k=1}^{N}\|_{C[0,T]_\tau}$$

$$\leq M(\delta, q)[\|A\varphi\|_H + |\psi_0| + \|f_1\|_H$$

$$+ \left\|\left\{\frac{f_k - f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)} + \left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{1}^{N}\right\|_{C[0,T]_\tau}\Bigg]. \tag{4.2}$$

Proof. Assume that grid function $\{w_k\}_{k=0}^{N}$ be the solution of the difference scheme

$$\begin{cases} i\frac{w_k - w_{k-1}}{\tau} + Aw_k = i\mu_k Aq + f_k, 1 \leq k \leq N, \\ w_0 = \varphi \end{cases} \tag{4.3}$$

and $\{\mu_k\}_{k=1}^{N}$ be the grid function determining by formula

$$\mu_k = \sum_{j=1}^{k} p_j \tau, 1 \leq k \leq N, \mu_0 = 0. \tag{4.4}$$

Then,

$$u_k = w_k - i\mu_k q, 0 \leq k \leq N. \tag{4.5}$$

Using the condition $Bu_k = \psi_k$ and formula (4.5), we can obtain

$$\mu_k = \frac{i}{Bq}(\psi_k - B[w_k]), 0 \leq k \leq N. \tag{4.6}$$

Since

$$p_k = \frac{\mu_k - \mu_{k-1}}{\tau}, 1 \leq k \leq N, \tag{4.7}$$

we obtain

$$p_k = \frac{i}{Bq}\left(\frac{\psi_k - \psi_{k-1}}{\tau} - B\left[\frac{w_k - w_{k-1}}{\tau}\right]\right), 1 \leq k \leq N. \tag{4.8}$$

Applying formula (4.8) and $Bq \neq 0$, we get the estimate

$$|p_k| \leq M_1(\delta, q)\left[\left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau} + \left\|\frac{w_k - w_{k-1}}{\tau}\right\|_H\right] \tag{4.9}$$

for any k, $1 \leq k \leq N$ and

$$\|\{p_k\}_{k=1}^{N}\|_{C[0,T]_\tau} \leq M_1(\delta, \sigma)\left[\left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+ \left\|\left\{\frac{w_k - w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C([0,T]_\tau, H)}\right]. \tag{4.10}$$

Now, applying formulas (4.5) and (4.7), we can write

$$\frac{u_k - u_{k-1}}{\tau} = \frac{w_k - w_{k-1}}{\tau} - ip_k q, 1 \leq k \leq N.$$

Then from the triangle inequality and this formula it follows

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_{k=+1}^{N}\right\|_{C_\tau(H)}$$

$$\leq \left\|\left\{\frac{w_k-w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)} + \|\{p_k\}_1^N\|_{C[0,T]_\tau}\|q\|_H, \tag{4.11}$$

Then, the proof of estimate (4.2) is based on equation (4.3), estimates (4.10), (4.11) and on the following result of stability estimate.

Theorem 4.2 Assume that the assumption of Theorem 4.1 holds. The solution of difference scheme (4.3) obeys the stability estimate

$$\left\|\left\{\frac{w_k-w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)}$$

$$\leq M(\delta,q)[\|A\varphi\|_H + |\psi_0| + \|f_1\|_H$$

$$+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)} + \left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\Bigg]. \tag{4.12}$$

Proof. The difference scheme (4.3) is equivalent to the system of difference equations

$$w_k = R^k\varphi - i\sum_{j=1}^{k} R^{k-j+1}\left\{\frac{i}{Bq}\big(\psi_j - B[w_j]\big)Aq + f_j\right\}\tau. \tag{4.13}$$

Here

$$R = (I - \imath\tau A)^{-1}.$$

Applying formula (4.13), we get

$$\frac{w_k-w_{k-1}}{\tau} = iR^kA\varphi - iR^k\left\{\frac{i}{Bq}(\psi_1 - B[w_1])Aq + f_1\right\}$$

$$-i\sum_{j=2}^{k} R^{k-j+1}\left\{\frac{i}{Bq}\big(\psi_j-\psi_{j-1} - B[w_j-w_{j-1}]\big)Aq + f_j - f_{j-1}\right\}\Bigg\} \tag{4.14}$$

for any $k, 1 \leq k \leq N.$ Applying formula (4.14), estimate

$$\| R \|_{H\to H} \leq 1 \tag{4.15}$$

and $Bq \neq 0,$ we get the estimate

$$\left\|\frac{w_k-w_{k-1}}{\tau}\right\|_H \leq M_1(\delta,q)\sum_{j=2}^{k}\|w_j-w_{j-1}\|_H$$

$$+M(\delta,q)\left\{|\psi_0| + \left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\|f_1\|_H + \|A\varphi\|_H + \left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}\right\}$$

for $1 \leq k \leq N.$ Then, applying the discrete analogy of integral inequality, we conclude that the following stability estimate

$$\left\|\frac{w_k - w_{k-1}}{\tau}\right\|_H \leq M(\delta,q)\left\{|\psi_0| + \left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C[0,T]_\tau}\right.$$

$$\left. +\|f_1\|_H + \|A\varphi\|_H + \left\|\left\{\frac{f_k - f_{k-1}}{\tau}\right\}_{k=2}^N\right\|_{C_\tau(H)}\right\} e^{\frac{M_1(\delta,q)(k+N-1)\tau}{1-\tau M_1(\delta,q)}} \qquad (4.16)$$

is satisfied for the solution of difference scheme (4.2) for $1 \leq k \leq N.$ From estimate (4.16) it follows estimate (4.12). Theorem 4.3.2 is established. Now, consider the applications of the main Theorem 4.1. First, we study the absolute stable difference scheme for the approximate solution of the time dependent SIP (3.50). The discretization of time dependent SIP (3.50) is carried out in two stages. In the first stage, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh, \qquad 0 \leq n \leq M, \qquad Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([0,l]_h)$ and $W_{2h}^2 = W_2^2([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_j\}_0^M$ defined on $[0,l]_h,$ equipped with the norms

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x\in 0,l]_h} |\varphi^h(x)|^2 h\right)^{1/2}$$

and

$$\|\varphi^h\|_{W_{2h}^2} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x\in 0,l]_h} \left|(\varphi^h)_{x,j}\right|^2 h\right)^{1/2} + \left(\sum_{x\in 0,l]_h} \left|(\varphi^h)_{x\bar{x},j}\right|^2 h\right)^{1/2},$$

respectively. We denote the self-adjoint positive definite difference operator A_h defined by the formula

$$A_h\varphi^h(x)$$

$$= \left\{-\frac{1}{h^2}\Big(a(x_{n+1})(\varphi_{n+1} - \varphi_n) - a(x_n)(\varphi_{n+1} - \varphi_n)\Big) + \delta\varphi_n\right\}_{n=1}^{M-1} \qquad (4.17)$$

acting in the space of grid functions $\varphi^h(x)$ satisfying the conditions $\varphi_0 = \varphi_M,$ $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}.$

It is well-known that A_h^x is a self-adjoint positive definite operator in $L_{2h}.$ With the help of $A_h^x,$ we reach the time dependent SIP

$$\begin{cases} iu_t^h(t,x) + A_h^x u^h(t,x) = p(t)q^h(x) + f^h(t,x), \\ x \in [0,l]_h, 0 < t < T, \\ u^h(0,x) = \varphi^h(x), x \in [0,l]_h, \\ \sum_{i=1}^{M-1} u^h(t,x_i)h = \psi(t), 0 \le t \le T. \end{cases} \quad (4.18)$$

In the second stage, we replace time dependent SIP (4.18) with a first order of accuracy difference scheme

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p_k q^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), \\ x \in [0,l]_h, 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in [0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h = \psi_k, \psi_k = \psi(t_k), 0 \le k \le N. \end{cases} \quad (4.19)$$

Theorem 4.3 Let τ and h be sufficiently small numbers. For the solution of difference scheme (4.19) the following stability estimates hold

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^{N-1} \right\|_{C([0,T]_\tau, L_{2h})} + \left\| \{u_k^h\}_{k=0}^N \right\|_{C_\tau(W_{2h}^2)} + \|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\le M(\delta,q) \left[\|\varphi^h\|_{W_{2h}^2} + \|f_1^h\|_{L_{2h}} + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_2^N \right\|_{C_\tau(L_{2h})} \right.$$

$$\left. + |\psi_0| + \left\| \left\{ \frac{\psi_k - \psi_{k-1}}{\tau} \right\}_1^N \right\|_{C[0,T]_\tau} \right]. \quad (4.20)$$

Proof. The proof of Theorem 4.3 is based on the abstract Theorem 4.1, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula

$$B^h u^h(t,x) = \sum_{i=1}^{M-1} u^h(t,x_i)h, t \in [0,T]. \quad (4.21)$$

Second, we study the absolute stable difference scheme for the approximate solution of time dependent SIP (3.50). The discretization of time dependent SIP (3.50) is carried out in two stages. In the first stage, we define the grid space

$$[-l,l]_h = \{x = x_n : x_n = nh, -M \le n \le M, Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([-l,l]_h)$ and $W_{2h}^2 = W_2^2([-l,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_j\}_{-M}^M$ defined on $[-l,l]_h$, equipped with the norms

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in -l,l]_h} |\varphi^h(x)|^2 h \right)^{1/2}$$

and

$$\|\varphi^h\|_{W_{2h}^2} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x \in -l,l]_h} \left|(\varphi^h)_{x\bar{x},j}\right|^2 h \right)^{1/2},$$

respectively. To the differential operator A generated by problem (3.50), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}}(x))_{x,r} - \beta(a(-x)\varphi_{\bar{x}}(-x))_{x,r} + \delta\varphi_r\}_{-M+1}^{M-1}, \quad (4.22)$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_{-M}^{M}$ satisfying the conditions $\varphi_{-M} = \varphi_M = 0$.

It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the time dependent SIP

$$\begin{cases} iu_t^h(t,x) + A_h^x u^h(t,x) = p(t)q^h(x) + f^h(t,x), \\ x \in [-l,l]_h, 0 < t < T, \\ u^h(0,x) = \varphi^h(x), x \in [-l,l]_h, \\ \sum_{i=-M+1}^{M-1} u^h(t,x_i)h = \psi(t), 0 \le t \le T. \end{cases} \quad (4.23)$$

In the second stage, we replace time dependent SIP (4.23) with a first order of accuracy difference scheme

$$\begin{cases} i\frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p_k q^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), \\ x \in [-l,l]_h, 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in [-l,l]_h, \\ \sum_{i=-M+1}^{M-1} u_k^h(x_i)h = \psi_k, \psi_k = \psi(t_k), 0 \le k \le N. \end{cases} \quad 4.24)$$

Theorem 4.4 Let τ and h be sufficiently small numbers. For the solution of difference scheme (4.24) the following stability estimates hold

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^{N} \right\|_{C([0,T]_\tau, L_{2h})} + \left\| \{u_k^h\}_{k=0}^{N} \right\|_{\mathbb{C}_\tau(W_{2h}^2)} + \|\{p_k\}_{k=1}^{N}\|_{C[0,T]_\tau}$$

$$\le M(\delta,q) \left[\|\varphi^h\|_{W_{2h}^2} + |\psi_0| + \left\|f_1^h\right\|_{L_{2h}} + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_{1}^{N} \right\|_{C([0,T]_\tau, L_{2h})} \right.$$

$$+|\psi_0| + \left\| \left\{ \frac{\psi_k - \psi_{k-1}}{\tau} \right\}_1^N \right\|_{C[0,T]_\tau} \Bigg].$$

Proof. The proof of Theorem 4.4 is based on the abstract Theorem 4.1, on the self-adjointness and positivity of operator A_h defined by the formula (4.21) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula

$$B^h u^h(t,x) = \sum_{i=-M+1}^{M-1} u^h(t,x_i)h, t \in [0,T]. \tag{4.25}$$

Third, we study the absolute stable difference scheme for the approximate solution of the time dependent SIP (3.53). The discretization of time dependent SIP (3.53) is carried out in two stages. In the first stage, we consider the grid functions $\varphi^h(x)$ on grid space $[0,l]_h$.We denote the self-adjoint positive definite difference operator A_h defined by the formula

$$A_h \varphi^h(x)$$
$$= \left\{ -\frac{1}{h^2}\left(a(x_{n+1})(\varphi_{n+1} - \varphi_n) - a(x_n)(\varphi_{n+1} - \varphi_n)\right) + \delta\varphi_n \right\}_{n=1}^{M-1} \tag{4.26}$$

acting in the space of grid functions $\varphi^h(x)$ satisfying the conditions $\varphi_0 = b\,\frac{\varphi_1 - \varphi_0}{\tau}$,

$$\varphi_M = -c\,\frac{\varphi_M - \varphi_{M-1}}{\tau}.$$

It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the time dependent SIP

$$\begin{cases} iu_t^h(t,x) + A_h^x u^h(t,x) = p(t)q^h(x) + f^h(t,x), \\ x \in [0,l]_h, 0 < t < T, \\ u^h(0,x) = \varphi^h(x), x \in [0,l]_h, \\ \sum_{i=1}^{M-1} u^h(t,x_i)h = \psi(t), 0 \le t \le T. \end{cases} \tag{4.27}$$

In the second stage, we replace time dependent SIP (4.27) with a first order of accuracy difference scheme

$$\begin{cases} i\,\frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p_k q^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), \\ x \in [0,l]_h, 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in [0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h = \psi_k, \psi_k = \psi(t_k), 0 \le k \le N. \end{cases} \tag{4.28}$$

Theorem 4.5 Let τ and h be sufficiently small numbers. For the solution of difference scheme (4.28) the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C([0,T]_\tau,L_{2h})}+\left\|\left\{u_k^h\right\}_{k=0}^{N}\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\leq M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C([0,T]_\tau,L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right]. \tag{4.29}$$

Proof. The proof of Theorem 4.5 is based on the abstract Theorem 3.1, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula

$$B^h u^h(t,x)=\sum_{i=1}^{M-1} u^h(t,x_i)h, t\in[0,T].(30)$$

Fourth, we study the absolute stable difference scheme for the approximate solution of time dependent SIP (3.55). The discretization of time dependent SIP (3.55) is also carried out in two stages. In the first stage, let us define the grid sets

$$\overline{\Omega}_h=\{x=x_r=(h_1r_1,\ldots,h_nr_n),r=(r_1,\ldots,r_n),$$

$$0\leq r_j\leq N_j,h_jN_j=1,j=1,\ldots,n\},$$

$$\Omega_h=\overline{\Omega}_h\cap\Omega, S_h=\overline{\Omega}_h\cap S.$$

We introduce the Banach spaces $L_{2h}=L_2(\overline{\Omega}_h)$ and $W_{2h}^2=W_2^2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1r_1,\ldots,h_nr_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\|\varphi^h\|_{L_{2h}}=\left(\sum_{x\in\overline{\Omega}_h}|\varphi^h(x)|^2h_1\cdots h_n\right)^{1/2}$$

and

$$\|\varphi^h\|_{W_{2h}}=\|\varphi^h\|_{L_{2h}}+\left(\sum_{x\in\overline{\Omega}_h}\sum_{r=1}^{n}\left|(\varphi^h)_{x_r\overline{x}_r,j_r}\right|^2h_1\cdots h_n\right)^{1/2}$$

respectively. To the differential operator A generated by problem (3.55), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x)=-\sum_{r=1}^n\left(a_r(x)u_{\overline{x}_r}^h\right)_{x_r,j_r}+\delta u^h(x) \tag{4.31}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x)=0$ for all $x\in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x, we reach the time dependent SIP

$$\begin{cases} iu_t^h(t,x) + A_h^x u^h(t,x) = p(t)q^h(x) + f^h(t,x), \\ x \in \overline{\Omega}_h, 0 < t < T, \\ u^h(0,x) = \varphi^h(x), x \in \overline{\Omega}_h, \\ \sum_{x\in\overline{\Omega}_h} u^h(t,x)h_1 \cdots h_n = \psi(t), 0 \le t \le T. \end{cases} \tag{4.32}$$

In the second stage, we replace time dependent SIP (3.55) with a first order of accuracy difference scheme

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = p_k q^h(x) + f_k^h(x), f_k^h(x) = f^h(t_k,x), \\ x \in \overline{\Omega}_h, 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in \overline{\Omega}_h, \\ \sum_{x\in\overline{\Omega}_h} u_k^h(x)h_1 \cdots h_n = \psi_k, \psi_k = \psi(t_k), 0 \le k \le N. \end{cases} \tag{4.33}$$

Theorem 4.6 Let τ and h be sufficiently small numbers. For the solution of difference scheme (4.33) the following stability estimates hold

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^N \right\|_{C([0,T]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_{k=0}^N \right\|_{\mathbb{C}_\tau(W_{2h}^2)} + \|\{p_k\}_{k=1}^N\|_{C[0,T]_\tau}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2} + \|f_1^h\|_{L_{2h}} + \left\| \left\{ \frac{f_k^h - f_{k-1}^h}{\tau} \right\}_1^N \right\|_{C([0,T]_\tau, L_{2h})} \right.$$

$$\left. + |\psi_0| + \left\| \left\{ \frac{\psi_k - \psi_{k-1}}{\tau} \right\}_1^N \right\|_{C[0,T]_\tau} \right]. \tag{4.34}$$

Proof. The proof of Theorem 4.6 is based on the abstract Theorem 3.1, on the self-adjointness and positivity of operator A_h defined by the formula (4.31) and on uniformly boundedness a linear functional B^h in $L_2(\overline{\Omega}_h)$ defined by the formula

$$B^h u^h(t,x) = \sum_{x\in\Omega_h} u^h(t,x)h_1 \cdots h_n, t \in [0,T] \tag{4.35}$$

and on the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

Theorem 4.7 For the solutions to the elliptic difference problem

$$\begin{cases} A_h^x u^h(x) = w^h(x), \qquad x \in \Omega_h, \\ u^h(x) = 0, \qquad x \in S_h \end{cases}$$

the following coercivity inequality holds (see, Sobolevskii,1975):

$$\sum_{r=1}^{n} \left\| u^{h}_{x_r \bar{x}_r, j_r} \right\|_{L_2(\overline{\Omega}_h)} \le M \| w^h \|_{L_2(\overline{\Omega}_h)}.$$

Fifth, we study the absolute stable difference scheme for the approximate solution of time dependent SIP (3.62). The discretization of time dependent SIP (3.62) is also carried out in two stages. In the first stage, let us define the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(a_r(x) u^h_{\bar{x}_r}\right)_{x_r, j_r} + \delta u^h(x) \tag{4.36}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $D^h u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x, we also reach the time dependent SIP (4.32). Therefore, in the second stage we get difference scheme (4.33)

Theorem 4.8 Let τ and h be sufficiently small numbers. For the solution of difference scheme (4.33) the stability estimates (4.34) hold.

Proof. The proof of Theorem 4.8 is based on the abstract Theorem 4.1, on the self-adjointness and positivity of operator A_h defined by the formula (4.36) and on uniformly boundedness of a linear functional B^h in $L_2(\overline{\Omega}_h)$ defined by the formula (4.35) and on the following theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

Theorem 4.9 For the solutions to the elliptic difference problem

$$\begin{cases} A_h^x u^h(x) = w^h(x), & x \in \Omega_h, \\ D^h u^h(x) = 0, & x \in S_h \end{cases}$$

the following coercivity inequality holds (see, Sobolevskii,1975):

$$\sum_{r=1}^{n} \left\| u^{h}_{x_r \bar{x}_r, j_r} \right\|_{L_2(\overline{\Omega}_h)} \le M \| w^h \|_{L_2(\overline{\Omega}_h)}.$$

4.4 The Second Order of Accuracy Difference Schemes

We are interested in studying the stability of a high order of accuracy single step absolute stable difference schemes of approximate solutions of the time dependent SIP (3.50). In this section we consider the second order of accuracy r − modified Crank-Nicolson difference schemes generated by

$$\begin{cases} i\frac{u_k-u_{k-1}}{\tau}+Au_k=p_kq+f_k, f_k=f\left(t_k-\frac{\tau}{2}\right), 1\leq k\leq r, \\ i\frac{u_k-u_{k-1}}{\tau}+A\frac{u_k+u_{k-1}}{2}=p_kq+f_k, f_k=f\left(t_k-\frac{\tau}{2}\right), r+1\leq k\leq N, \\ u_0=\varphi, \\ Bu_k=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N \end{cases} \tag{4.37}$$

for the approximate solution of the time dependent SIP (3.50). Note that for $r=0$, we have the Crank-Nicolson difference scheme

$$\begin{cases} i\frac{u_k-u_{k-1}}{\tau}+A\frac{u_k+u_{k-1}}{2}=p_kq+f_k, f_k=f\left(t_k-\frac{\tau}{2}\right), 1\leq k\leq N, \\ u_0=\varphi, \\ Bu_k=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N \end{cases}$$

for the approximate solution of the time dependent SIP (3.50).

Now, let us state the stability result for the solution of difference schemes (4.37).

Theorem 4.10 Assume that $\varphi\in D(A)$. Then, the solution of difference schemes (4.37) satisfy the following stability estimates

$$\begin{aligned} &\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(H)}+\|\{p_k\}_{k=1}^N\|_{C[0,T]_\tau} \\ &+\max_{0\leq k\leq r}\|Au_k\|_H+\max_{r+1\leq k\leq N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_H \\ &\leq M(\delta,q)[\|A\varphi\|_H+|\psi_0|+\|f_1\|_H \\ &+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^N\right\|_{C_\tau(H)}+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\Bigg]. \end{aligned} \tag{4.38}$$

Proof. Suppose that grid function $\{w_k\}_{k=0}^N$ be the solution of the difference scheme

$$\begin{cases} i\frac{w_k-w_{k-1}}{\tau}+Aw_k=i\mu_kAq+f_k, 1\leq k\leq r, \\ i\frac{w_k-w_{k-1}}{\tau}+A\frac{w_k+w_{k-1}}{2}=i\mu_kAq+f_k, r+1\leq k\leq N, \\ w_0=\varphi \end{cases} \tag{4.39}$$

and $\{\mu_k\}_{k=1}^N$ be the grid function determining by formula

$$\mu_k=\sum_{j=1}^k p_j\tau, 1\leq k\leq N, \mu_0=0. \tag{4.40}$$

Then,

$$u_k=w_k-i\mu_kq, 0\leq k\leq N. \tag{4.41}$$

Using the condition $Bu_k = \psi_k$ and formula (4.41), we can get

$$\mu_k = \frac{i}{Bq}(\psi_k - B[w_k]), 0 \leq k \leq N. \tag{4.42}$$

Since

$$p_k = \frac{\mu_k - \mu_{k-1}}{\tau}, 1 \leq k \leq N, \tag{4.43}$$

we get

$$p_k = \frac{i}{Bq}\left(\frac{\psi_k - \psi_{k-1}}{\tau} - B[\frac{w_k - w_{k-1}}{\tau}]\right), 1 \leq k \leq N. \tag{4.44}$$

Applying formula (4.44) and $Bq \neq 0,$ we obtain the estimate

$$|p_k| \leq M_1(\delta, q)\left[\left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C[0,T]_\tau} + \left\|\frac{w_k - w_{k-1}}{\tau}\right\|_H\right] \tag{4.45}$$

for any $k,\ 1 \leq k \leq N$ and

$$\|\{p_k\}_{k=1}^N\|_{C[0,T]_\tau} \leq M_1(\delta, \sigma)\left[\left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C[0,T]_\tau}\right.$$

$$\left. + \left\|\left\{\frac{w_k - w_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(H))}\right]. \tag{4.46}$$

Now, applying formulas (4.41) and (4.43), we can write

$$\frac{u_k - u_{k-1}}{\tau} = \frac{w_k - w_{k-1}}{\tau} - ip_k q, 1 \leq k \leq N.$$

Then from the triangle inequality and this formula it follows

$$\left\|\left\{\frac{u_k - u_{k-1}}{\tau}\right\}_{k=+1}^N\right\|_{C_\tau(H)}$$

$$\leq \left\|\left\{\frac{w_k - w_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(H)} + \|\{p_k\}_1^N\|_{C[0,T]_\tau}\|q\|_H, \tag{4.47}$$

Then, the proof of estimate (4.38) is based on equation (4.39), estimates (4.46), (4.47) and on the following result of stability estimate.

Theorem 4.11 Suppose that the assumption of Theorem 4.1 holds. The solution of difference scheme (4.39) holds the stability estimate

$$\left\|\left\{\frac{w_k - w_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(H)} \leq M(\delta, q)[\|A\varphi\|_H + |\psi_0| + \|f_1\|_H$$

$$+ \left\|\left\{\frac{f_k - f_{k-1}}{\tau}\right\}_{k=2}^N\right\|_{C_\tau(H)} + \left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\Bigg]. \tag{4.48}$$

Proof. The difference scheme (4.39) is equivalent to the system of difference equations

$$w_k = \begin{cases} C^k\varphi - i\sum_{j=1}^{k} C^{k-j+1}\left\{\frac{i}{Bq}\left(\psi_j - B[w_j]\right)Aq + f_j\right\}\tau, 1 \le k \le r, \\ D^{k-r}\left\{C^r\varphi - i\sum_{j=1}^{r} C^{r-j+1}\left\{\frac{i}{Bq}\left(\psi_j - B[w_j]\right)Aq + f_j\right\}\tau\right\} \\ -i\sum_{j=r+1}^{k} D^{k-j+1}\left\{\frac{i}{Bq}\left(\psi_j - B[w_j]\right)Aq + f_j\right\}\tau, r+1 \le k \le N. \end{cases} \quad (4.49)$$

Here $C = \left(I - \frac{\iota\tau A}{2}\right)^{-1}, D = \left(I + \frac{\iota\tau A}{2}\right)\left(I - \frac{\iota\tau A}{2}\right)^{-1}$. Applying formula (4.49), we obtain

$$\frac{w_k - w_{k-1}}{\tau} = \begin{cases} iC^kA\varphi - iC^k\left\{\frac{i}{Bq}(\psi_1 - B[w_1])Aq + f_1\right\} - i\sum_{j=2}^{k} C^{k-j+1} \\ \times\left\{\frac{i}{Bq}\left(\psi_j - \psi_{j-1} - B[w_j - w_{j-1}]\right)Aq + f_j - f_{j-1}\right\}\Big\}, 1 \le k \le r, \end{cases} \quad (4.50)$$

for any $k, 1 \le k \le N$. Applying formula (4.50), estimate

$$\| C \|_{H\to H} \le 1, \| D \|_{H\to H} \le 1 \quad (4.51)$$

and $Bq \neq 0$, we obtain the estimate

$$\left\|\frac{w_k - w_{k-1}}{\tau}\right\|_H \le M_1(\delta,q)\sum_{j=2}^{k}\left\|w_j - w_{j-1}\right\|_H$$

$$+M(\delta,q)\left\{|\psi_0| + \left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\|f_1\|_H + \|A\varphi\|_H + \left\|\left\{\frac{f_k - f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}\right\}$$

for $1 \le k \le N$. Then, applying the discrete analogy of integral inequality, we conclude that the following stability estimate

$$\left\|\frac{w_k - w_{k-1}}{\tau}\right\|_H \le M(\delta,q)\left\{|\psi_0| + \left\|\left\{\frac{\psi_k - \psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\|f_1\|_H + \|A\varphi\|_H + \left\|\left\{\frac{f_k - f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}\right\} e^{\frac{M_1(\delta,q)(k+N-1)\tau}{1-\tau M_1(\delta,q)}} \quad (4.52)$$

is satisfied for the solution of difference scheme (39) for $1 \le k \le N$. From estimate (4.52) it follows estimate (4.48). Theorem 4.11 is established.

Now, we consider the second order of accuracy difference scheme generated by A and A^2

$$\begin{cases} i\frac{u_k-u_{k-1}}{\tau}+A\left(I+\frac{i\tau A}{2}\right)u_k=\left(I+\frac{i\tau A}{2}\right)p_kq+\left(I+\frac{i\tau A}{2}\right)f_k, \\ f_k=f\left(t_k-\frac{\tau}{2}\right), 1\le k\le N, \\ u_0=\varphi, \\ Bu_k=\psi_k, \psi_k=\psi(t_k), 0\le k\le N \end{cases} \tag{4.53}$$

for the approximate solution of the time dependent SIP (4.1).

Let us state the stability result for the solution of difference schemes (4.53).

Theorem 4.12 Assume that $\varphi\in D(A)$. Then, the solution of difference schemes (4.53) satisfies the following stability estimates

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(H)}+\|\{p_k\}_{k=1}^N\|_{C[0,T]_\tau} \tag{4.54}$$

$$+\|\{Au_k\}_{k=1}^N\|_{C_\tau(H)}\le M(\delta,q)[\|A\varphi\|_H+|\psi_0|+\|f_1\|_H$$

$$+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^N\right\|_{C_\tau(H)}+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\Bigg].$$

Proof. Suppose that grid function$\{w_k\}_{k=0}^N$ be the solution of the difference scheme

$$\begin{cases} i\frac{w_k-w_{k-1}}{\tau}+A\left(I+\frac{i\tau A}{2}\right)w_k=i\left(I+\frac{i\tau A}{2}\right)\mu_kAq \\ +\left(I+\frac{i\tau A}{2}\right)f_k, 1\le k\le N, \\ w_0=\varphi \end{cases} \tag{4.55}$$

and $\{\mu_k\}_{k=1}^N$ be the grid function determining by formula

$$\mu_k=\sum_{j=1}^k p_j\tau, 1\le k\le N, \mu_0=0. \tag{4.56}$$

Then,

$$u_k=w_k-i\mu_kq, 0\le k\le N. \tag{4.57}$$

Using the condition $Bu_k=\psi_k$ and formula (4.57), we can get

$$\mu_k=\frac{i}{Bq}(\psi_k-B[w_k]), 0\le k\le N. \tag{4.58}$$

Since

$$p_k=\frac{\mu_k-\mu_{k-1}}{\tau}, 1\le k\le N, \tag{4.59}$$

we get

$$p_k=\frac{i}{Bq}\left(\frac{\psi_k-\psi_{k-1}}{\tau}-B[\frac{w_k-w_{k-1}}{\tau}]\right), 1\le k\le N. \tag{4.60}$$

Applying formula (4.60) and $Bq\neq 0$, we obtain the estimate

$$|p_k| \le M_1(\delta,q)\left[\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}+\left\|\frac{w_k-w_{k-1}}{\tau}\right\|_H\right] \tag{4.61}$$

for any $k,\ 1 \le k \le N$ and

$$\|\{p_k\}_{k=1}^{N}\|_{C[0,T]_\tau} \le M_1(\delta,\sigma)\left[\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\left\|\left\{\frac{w_k-w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)}\right]. \tag{4.62}$$

Now, applying formulas (4.57) and (4.59), we can write

$$\frac{u_k-u_{k-1}}{\tau}=\frac{w_k-w_{k-1}}{\tau}-ip_kq, 1 \le k \le N.$$

Then from the triangle inequality and this formula it follows

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_{k=+1}^{N}\right\|_{C_\tau(H)}$$

$$\le \left\|\left\{\frac{w_k-w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)}+\|\{p_k\}_1^N\|_{C[0,T]_\tau}\|q\|_H, \tag{4.63}$$

Then, the proof of estimate (4.54) is based on equation (4.55), estimates (4.62), (4.63) and on the following result of stability estimate.

Theorem 4.13 Suppose that the assumption of Theorem 4.1 holds. The solution of the difference scheme (4.55) holds the stability estimate

$$\left\|\left\{\frac{w_k-w_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(H)} \le M(\delta,q)[\|A\varphi\|_H+|\psi_0|+\|f_1\|_H$$

$$+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{1}^{N}\right\|_{C[0,T]_\tau}\Bigg]. \tag{4.63}$$

Proof. The difference scheme (4.55) is equivalent to the system of difference equations

$$w_k=\begin{cases} C^k\varphi-i\sum_{j=1}^{k} C^{k-j+1}\left\{\frac{i}{Bq}\left(\psi_j-B[w_j]\right)Aq+f_j\right\}\tau, 1 \le k \le r,\\[2ex] D^{k-r}\left\{C^r\varphi-i\sum_{j=1}^{r} C^{r-j+1}\left\{\frac{i}{Bq}\left(\psi_j-B[w_j]\right)Aq+f_j\right\}\tau\right\}\\[2ex] -i\sum_{j=r+1}^{k} D^{k-j+1}\left\{\frac{i}{Bq}\left(\psi_j-B[w_j]\right)Aq+f_j\right\}\tau,\\[2ex] r+1 \le k \le N. \end{cases} \tag{4.65}$$

Here $C=\left(I-\frac{\iota\tau A}{2}\right)^{-1}, D=\left(I+\frac{\iota\tau A}{2}\right)\left(I-\frac{\iota\tau A}{2}\right)^{-1}$. Applying formula (4.49), we obtain

$$\frac{w_k-w_{k-1}}{\tau}=\begin{cases} iC^kA\varphi-iC^k\left\{\frac{i}{Bq}(\psi_1-B[w_1])Aq+f_1\right\}-i\sum_{j=2}^{k}C^{k-j+1} \\ \times\left\{\frac{i}{Bq}\left(\psi_j-\psi_{j-1}-B[w_j-w_{j-1}]\right)Aq+f_j-f_{j-1}\right\}\Bigg\}, \\ 1\leq k\leq r, \end{cases} \tag{4.66}$$

for any $k, 1 \leq k \leq N.$ Applying formula (4.66), estimate

$$\| C \|_{H\to H}\leq 1, \| B \|_{H\to H}\leq 1 \tag{4.67}$$

and $Bq \neq 0,$ we obtain the estimate

$$\left\|\frac{w_k-w_{k-1}}{\tau}\right\|_H \leq M_1(\delta,q)\sum_{j=2}^{k}\left\|w_j-w_{j-1}\right\|$$

$$+M(\delta,q)\left\{|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\|f_1\|_H+\|A\varphi\|_H+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}\right\}$$

for $1 \leq k \leq N.$ Then, applying the discrete analogy of integral inequality, we conclude that the following stability estimate

$$\left\|\frac{w_k-w_{k-1}}{\tau}\right\|_H \leq M(\delta,q)\left\{|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau}\right.$$

$$\left.+\|f_1\|_H+\|A\varphi\|_{\mathbb{H}}+\left\|\left\{\frac{f_k-f_{k-1}}{\tau}\right\}_{k=2}^{N}\right\|_{C_\tau(H)}\right\}e^{\frac{M_1(\delta,q)(k+N-1)\tau}{1-\tau M_1(\delta,q)}} \tag{4.68}$$

is satisfied for the solution of difference scheme (4.55) for $1 \leq k \leq N.$ From estimate (4.68) it follows estimate (4.64). Theorem 4.13 is established.

Now, consider the applications of the main Theorems 4.10 and 4.12. First, we study the absolute stable difference scheme for the approximate solution of the time dependent SIP (3.50). The discretization of time dependent SIP (3.50) is carried out in two stages. In the first stage, we get the time dependent SIP (4.18). In the second stage, we replace time dependent SIP (4.18) with a second order of accuracy difference schemes

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), 1\le k\le r, x\in[0,l]_h, \\ i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\frac{u_k^h(x)+u_{k-1}^h(x)}{2}=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), r+1\le k\le N, x\in[0,l]_h, \\ u_0^h(x)=\varphi^h(x), x\in[0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\le k\le N. \end{cases} \tag{4.69}$$

and

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\left(I+\frac{i\tau A_h^x}{2}\right)u_k^h(x)=p_k\left(I+\frac{i\tau A_h^x}{2}\right)q^h(x) \\ +\left(I+\frac{i\tau A_h^x}{2}\right)f_k^h(x), f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), x\in[0,l]_h, 1\le k\le N, \\ u_0^h(x)=\varphi^h(x), x\in[0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\le k\le N. \end{cases} \tag{4.70}$$

Theorem 4.14 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.69), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$
$$+\max_{0\le k\le r}\left\|u_k^h\right\|_{W_{2h}^2}+\max_{r+1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$
$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$
$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.14 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.21).

Theorem 4.15 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.70), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(L_{2h})}+\left\|\{u_k^h\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\leq M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.15 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.21).

Second, we study the absolute stable difference scheme for the approximate solution of time dependent SIP (3.53). The discretization of time dependent SIP (3.53) is carried out in two stages. In the first stage, we get the time dependent SIP (4.23). In the second stage, we replace time dependent SIP (4.23) with second order of accuracy difference schemes

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p_k q^h(x)+f_k^h(x),\\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), 1\leq k\leq r, x\in[-l,l]_h,\\ i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x \frac{u_k^h(x)+u_{k-1}^h(x)}{2}=p_k q^h(x)+f_k^h(x),\\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), r+1\leq k\leq N, x\in[-l,l]_h,\\ u_0^h(x)=\varphi^h(x), x\in[-l,l]_h,\\ \sum_{i=1}^{M-1}u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N. \end{cases} \tag{4.71}$$

and

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\left(I+\frac{i\tau A_h^x}{2}\right)u_k^h(x)=p_k\left(I+\frac{i\tau A_h^x}{2}\right)q^h(x)\\ +\left(I+\frac{i\tau A_h^x}{2}\right)f_k^h(x), f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), x\in[-l,l]_h, 1\leq k\leq N,\\ u_0^h(x)=\varphi^h(x), x\in[-l,l]_h,\\ \sum_{i=1}^{M-1}u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N. \end{cases} \tag{4.72}$$

Theorem 4.16 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.71) the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$+\max_{0\le k\le r}\left\|u_k^h\right\|_{W_{2h}^2}+\max_{r+1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.16 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.22).

Theorem 4.17 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.72), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.17 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.25).

Third, we study the absolute stable difference scheme for the approximate solution of the time dependent SIP (3.55). The discretization of time dependent SIP (3.55) is carried out in two stages. In the first stage, we get the time dependent SIP (4.27). In the second stage, we replace time dependent SIP (4.27) with second order of accuracy difference schemes

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), 1\le k\le r, x\in[0,l]_h, \\ i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\frac{u_k^h(x)+u_{k-1}^h(x)}{2}=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), r+1\le k\le N, x\in[0,l]_h, \\ u_0^h(x)=\varphi^h(x), x\in[0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\le k\le N. \end{cases}, \tag{4.73}$$

and

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\left(I+\frac{i\tau A_h^x}{2}\right)u_k^h(x)=p_k\left(I+\frac{i\tau A_h^x}{2}\right)q^h(x) \\ +\left(I+\frac{i\tau A_h^x}{2}\right)f_k^h(x), f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), x\in[0,l]_h, 1\le k\le N, \\ u_0^h(x)=\varphi^h(x), x\in[0,l]_h, \\ \sum_{i=1}^{M-1} u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\le k\le N. \end{cases} \tag{4.74}$$

Theorem 4.18 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.73), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$
$$+\max_{0\le k\le r}\|u_k^h\|_{W_{2h}^2}+\max_{r+1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$
$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\|f_1^h\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$
$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.18 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.30).

Theorem 4.19 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.74), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(L_{2h})}+\left\|\{u_k^h\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\leq M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.19 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.30).

Fourth, we study the absolute stable difference schemes for the approximate solution of the time dependent SIP (3.57). The discretization of the time dependent SIP (3.57) is carried out in two stages. In the first stage, we get the time dependent SIP (4.32). In the second stage, we replace time dependent SIP (4.32) with the second order of accuracy difference schemes

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), 1\leq k\leq r, x\in\overline{\Omega}_h, \\ i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\frac{u_k^h(x)+u_{k-1}^h(x)}{2}=p_k q^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), r+1\leq k\leq N, x\in\overline{\Omega}_h, \\ u_0^h(x)=\varphi^h(x), x\in\overline{\Omega}_h, \\ \sum_{i=1}^{M-1}u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N. \end{cases} \tag{4.75}$$

and

$$\begin{cases} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x\left(I+\frac{i\tau A_h^x}{2}\right)u_k^h(x)=p_k\left(I+\frac{i\tau A_h^x}{2}\right)q^h(x) \\ +\left(I+\frac{i\tau A_h^x}{2}\right)f_k^h(x), f_k^h(x)=f^h(t_k-\frac{\tau}{2},x), x\in[0,l]_h, 1\leq k\leq N, \\ u_0^h(x)=\varphi^h(x), x\in[0,l]_h, \\ \sum_{i=1}^{M-1}u_k^h(x_i)h=\psi_k, \psi_k=\psi(t_k), 0\leq k\leq N. \end{cases} \tag{76}$$

Theorem 4.20 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.75), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$+\max_{0\le k\le r}\left\|u_k^h\right\|_{W_{2h}^2}+\max_{r+1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.20 is based on the abstract Theorem 4.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.35) and on the following Theorem 4.7 on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

Theorem 4.21 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.76), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.21 is based on the abstract Theorem 4.12, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.35) and on the following Theorem 4.7 on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

Fifth, we study the absolute stable difference schemes for the approximate solution of the time dependent SIP (3.62). The discretization of time dependent SIP (3.62) is carried out in two stages. In the first stage, we also get the time dependent SIP

(4.32). In the second stage, we replace time dependent SIP (4.32) with second order of accuracy difference schemes (4.75) and (4.76).

Theorem 4.22 Let τ and h be sufficiently small numbers. For the solution of difference schemes (4.75), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N-1}\right\|_{C_\tau(L_{2h})}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$+\max_{0\le k\le r}\left\|u_k^h\right\|_{W_{2h}^2}+\max_{r+1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.22 is based on the abstract Theorem 3.10, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.35) and on the following Theorem 4.8 on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

Theorem 4.23 Let τ and h be sufficiently small numbers. For the solution of the difference scheme (4.76), the following stability estimates hold

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_k^h\right\}_{k=1}^N\right\|_{C_\tau(W_{2h}^2)}+\|\{p_k\}_{k=0}^N\|_{C[0,T]_\tau}$$

$$\le M(\delta,q)\left[\|\varphi^h\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\left\|\left\{\frac{f_k^h-f_{k-1}^h}{\tau}\right\}_2^N\right\|_{C_\tau(L_{2h})}\right.$$

$$\left.+|\psi_0|+\left\|\left\{\frac{\psi_k-\psi_{k-1}}{\tau}\right\}_1^N\right\|_{C[0,T]_\tau}\right].$$

Proof. The proof of Theorem 4.23 is based on the abstract Theorem 3.12, on the self-adjointness and positivity of operator A_h defined by the formula (4.17) and on uniformly boundedness a linear functional B^h in L_{2h} defined by the formula (4.35) and on the following Theorem 4.8 on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega}_h)$.

CHAPTER V

Numerical Experiments

1 Introduction

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature. In present section for the approximate solution of one-dimensional time-dependent source identification problem for Schrödinger equations with nonlocal, Dirichlet, Neumann, and Robin conditions, we use the first and second order of accuracy difference schemes. The error analysis is given.

2 Numerical Results

5.2.1 Time-Dependent SIP with Nonlocal Conditions

We study the first and second order of accuracy difference schemes for the the numerical solution of the following SIP

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}+u(t,x)=p(t)(1+\sin 2x) \\ +(3\sin(2x)-1)e^{it}, x\in(0,\pi), t\in(0,1), \\ u(0,x)=1+\sin 2x, x\in[0,\pi], \\ u(t,0)=u(t,\pi), u_x(t,0)=u_x(t,\pi), \\ \int_0^\pi u(t,x)dx=\pi e^{it}, t\in[0,1] \end{cases} \tag{5.1}$$

for a one dimensional time-dependent SE with nonlocal conditions. The exact solution of this problem is $\big(u(t,x),p(t)\big)=\big((1+\sin 2x)e^{it},e^{it}\big)$.

First, we consider the first order of accuracy Rothe DS

$$\begin{cases} i\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k \\ =p_k(1+\sin 2x_n)+(3\sin 2x_n-1)e^{it_k}, \\ t_k=k\tau, x_n=nh, 1\le k\le N, 1\le n\le M-1, \\ u_n^0=1+\sin 2x_n, 0\le n\le M, Mh=\pi, N\tau=1, \\ u_M^k=u_0^k, u_M^k-u_{M-1}^k=u_1^k-u_0^k, \\ \sum_{m=1}^M u_m^k h=\pi e^{it_k}, 0\le k\le N. \end{cases} \tag{5.2}$$

Algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of first order DS (5.2) contains three steps. We introduce η_k by the formula

$$\eta_k = \sum_{m=1}^k p_m \tau, k \in \overline{1,N}, \eta_0 = 0. \tag{5.3}$$

Then,

$$p_k = \frac{\eta_k - \eta_{k-1}}{\tau}, k \in \overline{1,N}. \tag{5.4}$$

We have that

$$u_n^k = w_n^k - i\eta_k(1 + sin2x_n), k \in \overline{0,N}, \quad n \in \overline{0,M}, \tag{5.5}$$

where w_n^k is the solution of the following DS

$$\begin{cases} i\frac{w_n^k - w_n^{k-1}}{\tau} - \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + w_n^k - z_n h \sum_{k=1}^M w_m^k = \varphi^k, \\ \varphi^k = z_n \pi e^{it_k} + (3\sin 2x_n - 1)e^{it_k}, k \in \overline{1,N}, n \in \overline{1,M-1}, \\ w_n^0 = 1 + \sin 2x_n, n \in \overline{1,M-1}, \\ w_M^k = w_0^k, w_M^k - w_{M-1}^k = w_1^k - w_0^k, \end{cases} \tag{5.6}$$

where z_n is defined by formula

$$z_n = \frac{1}{\pi + dh}\left[sin2x_n\left(\frac{1 - cos2h}{h^2} - \frac{1}{2}\right) - \frac{1}{2}\right], n \in \overline{1,M-1}.$$

Using the integral condition

$$\sum_{m=1}^M u_m^k h = \pi e^{it_k}, k \in \overline{0,N},$$

we get

$$\eta_k = \frac{\sum_{m=1}^M w_m^k h - \pi e^{it_k}}{i(\pi + dh)}, k \in \overline{1,N}, \tag{5.7}$$

$$d = \sum_{m=1}^M sin2x_m.$$

Second, we present the second order of accuracy Crank-Nicolson DS

$$\begin{cases} i\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2}-\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}+\frac{u_n^k+u_n^{k-1}}{2} \\ =\frac{p_k+p_{k-1}}{2}(1+\sin 2x_n)+(3\sin 2x_n-1)e^{i\left(t_k-\frac{\tau}{2}\right)}, \\ t_k=k\tau, x_n=nh, 1\le k\le N, 1\le n\le M-1, \\ u_n^0=1+\sin 2x_n, 0\le n\le M, Mh=\pi, N\tau=1, \\ u_M^k=u_0^k, u_M^k-u_{M-1}^k=u_1^k-u_0^k, \\ \sum_{m=1}^{M} u_m^k h=\pi e^{it_k}, 0\le k\le N. \end{cases} \tag{5.8}$$

Algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of second order DS (5.8) contains also three steps. We introduce η_k by the formula

$$\eta_k=\frac{p_0+p_k}{2}\tau+\sum_{m=1}^{k-1} p_m\tau, k\in\overline{1,N}, \eta_0=0. \tag{5.9}$$

Then,

$$\frac{p_k+p_{k-1}}{2}=\frac{\eta_k-\eta_{k-1}}{\tau}, k\in\overline{1,N}. \tag{5.10}$$

We will use formula (5.5), where w_n^k is the solution of the following DS

$$\begin{cases} i\frac{w_n^k-w_n^{k-1}}{\tau}-\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{2h^2}-\frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{2h^2}+\frac{w_n^k+w_n^{k-1}}{2} \\ -z_nh\sum_{k=1}^{M} w_m^k-z_nh\sum_{k=1}^{M} w_m^{k-1}=z_n\pi(e^{it_k}+e^{it_{k-1}}) \\ +(3\sin 2x_n-1)e^{i\left(t_k-\frac{\tau}{2}\right)}, k\in\overline{1,N}, n\in\overline{1,M-1} \\ w_n^0=1+\sin 2x_n, n\in\overline{1,M-1}, \\ w_M^k=w_0^k, w_M^k-w_{M-1}^k=w_1^k-w_0^k. \end{cases} \tag{5.11}$$

Algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of DS (5.8) contains also three steps. For the fist step,will obtain $\{\{w_n^k\}_{k=0}^N\}_{n=0}^M$. by (5.11).

It is clear that we can written (5.6) and (5.11) as the initial value problem the first-order difference equation concerning k and matrix coefficients

$$\begin{cases} Aw^k+Bw^{k-1}=\varphi^k, 1\le k\le N-1, \\ w^0=\{1+sin(2x_n)\}_{n=1}^M. \end{cases} \tag{5.12}$$

Then,

$$w^k=inv(A)(\varphi_1^k-Bw^{k-1}),$$

where A,B are $(M+1)\times(M+1)$ square matrices and φ^k is $(M+1)\times 1$ column matrix and

$$A=\begin{bmatrix}1 & 0 & 0 & \cdot & 0 & -1 \\ a & b-hz_1 & a-hz_1 & \cdot & -hz_1 & -hz_1 \\ 0 & a-hz_2 & b-hz_2 & \cdot & -hz_2 & -hz_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & -hz_{M-2} & -hz_{M-2} & \cdot & a-hz_{M-2} & -hz_{M-2} \\ 0 & -hz_{M-1} & -hz_{M-1} & \cdot & b-hz_{M-1} & a-hz_{M-1} \\ 1 & -1 & 0 & \cdot & -1 & 1\end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$B=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&c&0&0&\cdot&0&0&0\\0&0&c&0&\cdot&0&0&0\\0&0&0&c&\cdot&0&0&0\\0&0&0&0&\cdot&0&0&0\\0&0&0&0&\cdot&0&0&0\\0&0&0&0&\cdot&c&0&0\\0&0&0&0&\cdot&0&c&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix}_{(M+1)\times(M+1)},$$

$$a=-\frac{1}{h^2}, b=\frac{i}{\tau}+\frac{2}{h^2}+1, c=-\frac{i}{\tau},$$

$$f_n^k=\begin{bmatrix}0\\ \varphi_1^k\\ \vdots\\ \varphi_{M-1}^k\\ 0\end{bmatrix}_{(M+1)\times 1},$$

$$w^k=\begin{bmatrix}w_0^k\\ w_1^k\\ \vdots\\ w_{M-1}^k\\ w_M^k\end{bmatrix}_{(M+1)\times 1}$$

for the first order of accuracy Rothe DS and

$$A=\begin{bmatrix}1 & 0 & 0 & \cdot & 0 & -1 & 0 \\ a & b-hz_1 & a-hz_1 & \cdot & -hz_1 & -hz_1 & 0 \\ 0 & a-hz_2 & b-hz_2 & \cdot & -hz_2 & -hz_2 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & -hz_{M-2} & -hz_{M-2} & \cdot & a-hz_{M-2} & -hz_{M-2} & 0 \\ 0 & -hz_{M-1} & -hz_{M-1} & \cdot & b-hz_{M-1} & a-hz_{M-1} & 0 \\ 1 & -1 & 0 & \cdot & -1 & 1 & 0\end{bmatrix}_{(M+1)\times(M+1)},$$

and

$$B=\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ a & a-hz_1 & c-hz_1 & \cdot & -hz_1 & -hz_1 & 0 \\ 0 & a-hz_2 & c-hz_2 & \cdot & -hz_2 & -hz_2 & 0 \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ 0 & -hz_{M-2} & -hz_{M-2} & \cdot & c-hz_{M-2} & a-hz_{M-2} & 0 \\ 0 & -hz_{M-1} & -hz_{M-1} & \cdot & a-hz_{M-1} & c-hz_{M-1} & a \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$a=-\frac{1}{2h^2}, b=\frac{i}{\tau}+\frac{1}{h^2}+\frac{1}{2}, c=\frac{-i}{\tau}+\frac{1}{h^2}+\frac{1}{2},$$

$$\varphi_n^k=\begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$w^k=\begin{bmatrix} w_0^k \\ w_1^k \\ \vdots \\ w_{M-1}^k \\ w_M^k \end{bmatrix}_{(M+1)\times 1},$$

$$\varphi_n^k=z_n\pi(e^{it_k}+e^{it_{k-1}})+(3sin2x_n-1)e^{i(t_k-\tau/2)}$$

for the second order of accuracy Crank-Nicolson DS (5.11). Second, we will find η_k and p_k and $\frac{p_k+p_{k-1}}{2}$ by formulas (5.4),(5.10), (5.9) and (5.5).

Third, we will find $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ by formulas (5.5). The errors are computed by formulas

$$E_u=\max_{k\in\overline{0,N}}\left(\sum_{n=0}^M |u(t,x)-u_n^k|^2 h\right)^{\frac{1}{2}}, \tag{5.13}$$

$$E_p=\max_{k\in\overline{1,N}}\left|p(t)-p_k\left(\frac{p_k+p_{k-1}}{2}\right)\right|. \tag{5.14}$$

Numerical solutions of $u(t,x)$ at (t_k,x_n) is u_n^k and of $p(t)$ at t_k is p_k. The numerical results of SIP (5.2) are provided.

Table 5.2.1

The errors between the exact and the numerical solutions of (5.2) for different values of N and M

$Error$	$M = N = 20$	$M = N = 40$	$M = N = 80$
E_p	0.0022	0.0011	0.0005
E_u	0.034	0.017	0.008

As it is seen in Table 5.2.1, if M and N are multiplied by 2, the value of errors decreases approximately $1/2$ for the DS. This shows that it has the first order of accuracy and numerical solutions for second order of accuracy Crank-Nicolson DS of $u(t,x)$ at (t_k, x_n) is u_n^k and of $p(t)$ at t_k is $\frac{p_k+p_{k-1}}{2}$. The numerical results of SIP (5.2) are provided.

Table 5.2.2

The errors between the exact and the numerical solutions of (5.2) for different values of N and M

$Error$	$M = N = 20$	$M = N = 40$	$M = N = 80$
E_p	0.0002	0.00005	0.00001
E_u	0.017	0.0043	0.0011

As it is seen in Table 5.2.2, if M and N are multiplied by 2, the value of errors decreases approximately $1/4$ for the DS. This shows that it has the second order of accuracy.

5.2.2 Time-Dependent SIP with Dirichlet Condition

We study the numerical solution of the following SIP

$$\begin{cases} iu_t - u_{xx} = p(t)sin(x) - e^{-it}sinx, x \in (0,\pi), t \in (0,1), \\ u(0,x) = sinx, x \in 0,\pi], \\ u(t,0) = u(t,\pi) = 0, \\ \int_0^\pi u(t,x)dx = 2e^{it}, t \in 0,1] \end{cases} \tag{5.15}$$

for a one dimensional SE with Dirichlet condition. The exact solution of this problem is $(u(t,x), p(t)) = \left(e^{it}sinx, e^{-it}\right)$. We study the following first order of accuracy difference scheme

$$\begin{cases}\frac{i}{\tau}(u_n^k-u_n^{k-1})-\frac{1}{h^2}\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right)\\ =p_k\sin x_n+\sin x_n e^{-it_k},\\ t_k=k\tau, x_n=nh, k=\overline{1,N}, n=\overline{1,M-1},\\ u_n^0=\sin x_n, n=\overline{0,M}, Mh=\pi, N\tau=1,\\ u_M^k=u_0^k=0,\\ \sum_{m=1}^M u_m^k h=2e^{it_k}, k=\overline{0,N}.\end{cases} \tag{5.16}$$

The algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_0^M$ and $\{p_k\}_1^N$ of DS (5.16) contains three steps. We introduce η_k by the formula

$$\eta_k=\sum_{m=1}^k p_m\tau, k=\overline{1,N}, \eta_0=0. \tag{5.17}$$

We have that

$$u_n^k=v_n^k-i\eta_k\sin x_n, k=\overline{0,N}, n=\overline{0,M}, \tag{5.18}$$

where v_n^k is the solution of the DS

$$\begin{cases}i\frac{v_n^k-v_n^{k-1}}{\tau}-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}+r_n\sum_{m=1}^M v_m^k=f_n^k,\\ f_n^k=\frac{2e^{it_k}}{h}r_n-e^{-it_k}\sin x_n, k=\overline{1,N}, n=\overline{1,M-1},\\ r_n=\frac{2\sin x_n(cosh-1}{dh^2}, d=\sum_{m=1}^M sinx_m, n\in\overline{1,M-1},\\ v_n^0=\sin x_n, n=\overline{0,M},\\ v_0^k=v_M^k=0, k=\overline{0,N}.\end{cases} \tag{5.19}$$

Using the integral condition

$$\sum_{m=1}^M u_m^k h=2e^{it_k}, k\in\overline{0,N},$$

we get

$$\eta_k=\frac{\sum_{m=1}^M v_m^k h-2e^{it_k}}{idh}, k\in\overline{1,N}, \tag{5.20}$$

$$p_k=\frac{\eta_k-\eta_{k-1}}{\tau}, k=\overline{1,N}. \tag{5.21}$$

In the first step, we find the solution $\{\{v_n^k\}_{k=0}^N\}_{n=0}^M$ of the corresponding first order of accuracy difference scheme (5.19). For obtaining it, we will write difference scheme

(5.19) in matrix form as

$$\begin{cases} Av^k + Bv^{k-1} = \varphi^k, 1 \le k \le N-1, \\ \\ v^0 = \{\sin x_n\}_{n=1}^{M}, \end{cases} \tag{5.22}$$

where A, B are $(M+1)\times(M+1)$ square matrices and φ^k is $(M+1)\times 1$ column matrix and

$$A = \begin{bmatrix} 1 & & 0 & \cdot & 0 & 0 \\ a & b-hr_1 & a-hr_1 & \cdot & -hr_1 & -hr_1 \\ 0 & a-hr_2 & b-hr_2 & \cdot & -hr_2 & -hr_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -hr_{M-2} & -hr_{M-2} & \cdot & a-hr_{M-2} & -hr_{M-2} \\ 0 & -hr_{M-1} & -hr_{M-1} & \cdot & b-hr_{M-1} & a-hr_{M-1} \\ 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & c & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & c & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$a = -\frac{1}{h^2}, b = \frac{i}{\tau} + \frac{2}{h^2}, c = -\frac{i}{\tau},$$

$$\varphi_n^k = \begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$v^k = \begin{bmatrix} v_0^k \\ v_1^k \\ \vdots \\ v_{M-1}^k \\ v_M^k \end{bmatrix}_{(M+1)\times 1}.$$

In the second step we will find $\{\eta_k\}_{k=0}^{N}$, $\{p_k\}_{k=1}^{N}$ by formulas (5.20) and (5.21). In the third step we will find $\{\{u_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (5.17) and (5.18).

The errors are computed by formulas (5.13) and (5.14).

Numerical solutions of $u(t,x)$ at (t_k, x_n) is u_n^k and of $p(t)$ at t_k is p_k.

The results of numerical experiments for problem (5.15) are provided in Table 4.2.1.

As it is seen in Table 5.2.3, if M and N are multiplied by 2, the value of errors decreases approximately $1/2$ for the DS. This shows that it has the first order of accuracy.

Table 5.2.3

The errors between the exact and the numerical solutions of (5.15) for different values of N and M

$Error$	$M = N = 20$	$M = N = 40$	$M = N = 80$
E_p	0.0710	0.036	0.017
E_u	0.0292	0.0142	0.0710

5.2.3 Time-Dependent SIP with Neumann Condition

We study the numerical solution of the following SIP

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(t)(1+\cos 2x) \\ +(3\cos(2x)-1)e^{it}, x \in (0,\pi), t \in (0,1), \\ u(0,x) = 1+\cos 2x, x \in [0,\pi], \\ u_x(t,0) = u_x(t,\pi) = 0, \\ \int_0^\pi u(t,x)dx = \pi e^{it}, t \in [0,1] \end{cases} \tag{5.23}$$

for a one dimensional SE with Neumann condition. The exact solution of this problem is $\left(u(t,x), p(t)\right) = \left((1+\cos 2x)e^{it}, e^{it}\right)$. We study the following first order of accuracy difference scheme

$$\begin{cases} i\frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} + u_n^k \\ = p_k(1+\cos 2x_n) + (3\cos 2x_n - 1)e^{it_k}, \\ t_k = k\tau, x_n = nh, 1 \le k \le N, 1 \le n \le M-1, \\ u_n^0 = 1+\cos 2x_n, 0 \le n \le M, Mh = \pi, N\tau = 1, \\ u_M^k = u_0^k, u_M^k - u_{M-1}^k = u_1^k - u_0^k, \\ \sum_{m=1}^{M} u_m^k h = \pi e^{it_k}, 0 \le k \le N. \end{cases} \tag{5.24}$$

The algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of DS (5.24) contains three steps. We introduce η_k by the formula

$$\eta_k = \sum_{m=1}^{k} p_m \tau, k \in \overline{1,N}, \eta_0 = 0. \tag{5.25}$$

We have that

$$u_n^k = w_n^k - i\eta_k(1 + \cos 2x), k \in \overline{0,N}, n \in 0, M], \tag{5.26}$$

where w_n^k is the solution of the DS

$$\begin{cases} i\frac{w_n^k - w_n^{k-1}}{\tau} - \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + w_n^k + r_n h \sum_{m=1}^{M} w_m^k \\ = \varphi_n^k, k \in \overline{1,N}, n \in \overline{1, M-1}, \\ \varphi_n^k = e^{it_k}(\pi r_n + 3\cos 2x_n - 1), \\ r_n = \frac{1}{\pi + dh}\left[2\cos x_n\left(\frac{\cos h - 1}{h^2} - \frac{1}{2}\right) - 1\right], \\ w_n^0 = 1 + \cos 2x_n, n \in \overline{0,M}, \\ w_1^k - w_0^k = w_M^k - w_{M-1} = 0, k \in \overline{0,N}. \end{cases} \tag{5.27}$$

Using the integral condition

$$\sum_{m=1}^{M} u_m^k h = \pi e^{it_k}, 0 \le k \le N,$$

we get

$$\eta_k = \frac{\sum_{m=1}^{M} w_m^k h - \pi e^{it_k}}{i(\pi + dh)}, k \in \overline{1,N}. \tag{5.28}$$

$$p_k = \frac{\eta_k - \eta_{k-1}}{\tau}, \overline{1,N}. \tag{5.29}$$

In the first step, we find the solution $\{\{w_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ of the corresponding first order of accuracy difference scheme (5.27). For obtaining it, we will write difference scheme (5.27) in matrix form as

$$\begin{cases} Aw^k + Bw^{k-1} = \varphi^k, 1 \le k \le N-1, \\ w^0 = \{1 + \cos(2x_n)\}_{n=1}^{M}, \end{cases} \tag{30}$$

where A, B are $(M+1) \times (M+1)$ square matrices and φ^k is $(M+1) \times 1$ column matrix and

$$A=\begin{bmatrix} -1 & 1 & 0 & \cdot & 0 & 0 \\ a & b-hr_1 & a-hr_1 & \cdot & -hr_1 & -hr_1 \\ 0 & a-hr_2 & b-hr_2 & \cdot & -hr_2 & -hr_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -hr_{M-2} & -hr_{M-2} & \cdot & a-hr_{M-2} & -hr_{M-2} \\ 0 & -hr_{M-1} & -hr_{M-1} & \cdot & b-hr_{M-1} & a-hr_{M-1} \\ 0 & 0 & 0 & \cdot & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & c & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & c & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$a=-\frac{1}{h^2}, b=\frac{i}{\tau}+\frac{2}{h^2}+1, c=-\frac{i}{\tau},$$

$$\varphi_n^k=\begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$w^k=\begin{bmatrix} w_0^k \\ w_1^k \\ \vdots \\ w_{M-1}^k \\ w_M^k \end{bmatrix}_{(M+1)\times 1}.$$

In the second step, we will find $\{\eta_k\}_{k=0}^N$, $\{p_k\}_{k=1}^N$ by formulas (5.28) and (5.29). In the third step, will find $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ by formulas (5.25) and (5.26).

The errors are computed by formulas (5.13) and (5.14). Numerical solutions of $u(t,x)$ at (t_k,x_n) is u_n^k and of $p(t)$ at t_k is p_k. The results of numerical experiments for problem (5.23) are provided in Table 5.2.4 As, it is seen in Table Table 5.2.4 if M and N are multiplied by 2, the value of errors decreases approximately $1/2$ for the DS. This shows that it has the first order of accuracy.

Table 5.2.3

The errors between the exact and the numerical solutions of (5.23) for different values of N and M

$Error$	$M = N = 20$	$M = N = 40$	$M = N = 80$
E_p	0.0980	0.0493	0.0247
E_u	0.0072	0.0036	0.0018

5.2.4 Time-Dependent SIP with Robin Condition

We study the numerical solution of the following SIP

$$\begin{cases} iu_t - u_{xx} + u = \frac{5}{4}\cos\frac{x}{2}e^{-it} + p(t)\cos\frac{x}{2}, \\ x \in (0,\pi), t \in (0,1), \\ u(0,x) = \cos\frac{x}{2}, x \in 0,\pi], \\ u(t,0) - e^{-it} = u_x(t,0), \\ -u(t,\pi) - \frac{1}{2}e^{-it} = u_x(t,\pi), \\ \int_0^\pi u(t,x)dx = 2e^{-it_k}, t \in 0,1] \end{cases} \tag{5.31}$$

for a one dimensional SE with Robin condition. The exact solution of this problem is $(u,p) = \left(e^{-it}\cos\frac{x}{2}, e^{-it}\right)$. We study the following first order accuracy difference scheme

$$\begin{cases} \frac{i}{\tau}(u_n^k - u_n^{k-1}) - \frac{1}{h^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) + u_n^k \\ = \frac{5}{4}\cos\frac{x_n}{2}e^{-it_k} + p_k\cos\frac{x_n}{2}, \\ t_k = k\tau, x_n = nh, k \in \overline{1,N}, n \in \overline{1,M-1}, \\ u_n^0 - e^{-it_k} = \frac{u_1 - u_0}{h}, n \in \overline{0,M}, Mh = \pi, N\tau = 1, \\ -u_n^0 - \frac{1}{2}e^{-it_k} = \frac{u_M - u_{M-1}}{h}, \sum_{m=1}^M u_m^k h = 2e^{-it_k}, k \in \overline{0,N}, \end{cases} \tag{5.32}$$

The algorithm for obtaining the solution $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of DS (5.32) contains three steps. We introduce η_k by the formula

$$\eta_k = \sum_{m=1}^k p_m\tau, k \in \overline{1,N}, \eta_0 = 0. \tag{5.33}$$

We have that

$$u_n^k = w_n^k - i\eta_k \cos(\frac{x_n}{2}), k \in \overline{0,N}, n \in 0, M], \tag{5.34}$$

where w_n^k is the solution of the DS

$$\begin{cases} i\,\frac{w_n^k - w_n^{k-1}}{\tau} - \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + w_n^k + r_m \sum_{m=1}^{M} {}_m^k = \varphi_n^k \\ \varphi_n^k = \left(\frac{5}{4}\cos\frac{x_n}{2} + \frac{2r_m}{hd}\right) e^{-it_k}, k \in \overline{1,N}, n \in \overline{1,M-1}, \\ r_m = \frac{1}{d} 2\cos\frac{x_n}{2}\left(\frac{cos\frac{h}{2} - 1}{h^2} + 0.5\right), d = \sum_{m=1}^{M} \cos\frac{x_n}{2} \\ w_n^0 = \cos\frac{x_n}{2}, n \in \overline{0,M}, \\ w_0^k - e^{-it_k} = \frac{1}{h}(w_1^k - w_0^k), k \in \overline{0,N}, \\ -w_M^k - \frac{1}{2}e^{-it_k} = \frac{1}{h}(w_M^k - w_{M-1}), k \in \overline{0,N}. \end{cases} \tag{5.35}$$

Using the integral condition

$$\sum_{m=1}^{M} u_m^k h = 2e^{-it_k}, k \in \overline{0,N},$$

we get

$$\eta_k = \frac{\sum_{m=1}^{M} w_m^k h - 2e^{-it_k}}{idh}, k \in \overline{1,N}, \tag{5.36}$$

$$p_k = \frac{\eta_k - \eta_{k-1}}{\tau}, \overline{1,N}. \tag{5.37}$$

In the first step, we find the solution $\{\{w_n^k\}_{k=0}^N\}_{n=0}^M$ of the corresponding first order of accuracy difference scheme (5.35). For obtaining it, we will write difference scheme (5.35) in matrix form as

$$\begin{cases} Aw^k + Bw^{k-1} = \varphi_1^k, 1 \le k \le N-1, \\ w^0 = \{\cos(\frac{x_n}{2})\}_{n=1}^M, \end{cases}$$

where A, B are $(M+1)\times(M+1)$ square matrices and φ^k is $(M+1)\times 1$ column matrix and

$$A=\begin{bmatrix} \frac{h+1}{h} & \frac{-1}{h} & 0 & \cdot & 0 & 0 \\ a & b-hr_1 & a-hr_1 & \cdot & -hr_1 & -hr_1 \\ 0 & a-hr_2 & b-hr_2 & \cdot & -hr_2 & -hr_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -hr_{M-2} & -hr_{M-2} & \cdot & a-hr_{M-2} & -hr_{M-2} \\ 0 & -hr_{M-1} & -hr_{M-1} & \cdot & b-hr_{M-1} & a-hr_{M-1} \\ 0 & 0 & 0 & \cdot & \frac{1}{h} & -\frac{h+1}{h} \end{bmatrix}_{(M+1)\times(M+1)}$$

and

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & c & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & c & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$a=-\frac{1}{h^2}, b=\frac{i}{\tau}+\frac{2}{h^2}+1, c=-\frac{i}{\tau},$$

$$\varphi_n^k=\begin{bmatrix} e^{-it_k} \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ \frac{1}{2}e^{-it_k} \end{bmatrix}_{(M+1)\times 1},$$

$$w^k=\begin{bmatrix} w_0^k \\ w_1^k \\ \vdots \\ w_{M-1}^k \\ w_M^k \end{bmatrix}_{(M+1)\times 1}.$$

In the second step, we will find $\{\eta_k\}_{k=0}^N$, $\{p_k\}_{k=1}^N$ by formulas (5.36) and (5.37). In the third step, we will find $\{\{u_n^k\}_{k=0}^N\}_{n=0}^M$ by formulas (5.33) and (5.34). The errors are computed by formulas (5.13) and (5.14). Numerical solutions of $u(t,x)$ at (t_k,x_n) is u_n^k and of $p(t)$ at t_k is p_k. The results of numerical experiments for problem (5.31) are provided in Table 5.2.5. As it is seen in Table 5.2.5, if M and N are multiplied by 2, the value of errors decreases approximately $1/2$ for the DS. This shows that it has the first order of accuracy.

Table 5.2.3

The errors between the exact and the numerical solutions of (5.31) for different values of N and M

$Error$	$M = N = 20$	$M = N = 40$	$M = N = 80$
E_p	0.0341	0.0171	0.0085
E_u	0.055	0.0028	0.0014

CHAPTER VI

Conclusion

1. The history of direct and inverse boundary value problems for SIPs are studied.
2. Fourier series, Laplace transform and Fourier transform methods are applied for the solution of six identification problems for SIPs.
3. The stability of the time-dependent SIP for the SE in a Hilbert space with the self-adjoint positive definite operator is established.
4. First and second order of accuracy single step difference schemes for the numerical solution of this time-dependent SIP are presented. The absolute stability of these difference schemes is established.
5. Applications, five time-dependent SIPs for SEs are studied.

Stability estimates are created for the solution of these SIPs and their difference schemes for the numerical solution of the time-dependent SIPs for SEs are obtained.

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Appendices

Appendix A

Matlab Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (5.2.1)

```
function(Nonlocal condition)
clear all; close all;
N=40;
M=40;
i=sqrt(-1);
h=pi/M;
tau=1/N;
a=-1/(2*h^2);
b=(i/tau)+(1/h^2)+0.5;
c=-i/tau+1/(h^2)+0.5;
d=0;
for m=1:M
d=d+sin(2*m*h);
end
for m=1:M     z(m)=(sin(2*m*h)*((1-cos(2*h))/(h^2)+0.5)+0.5)/(pi+d*h);
end
A=zeros(M+1,M+1);
B=zeros(M+1,M+1);
A(1,1)=1;
A(1,M+1)=-1;
A(M+1,1)=1;
A(M+1,2)=-1;
A(M+1,M)=-1;
A(M+1,M+1)=1;
for m=2:M
A(m,m)=b;
A(m,m-1)=a;
A(m,m+1)=a;
B(m,m)=c;
B(m,m-1)=a;
```

```
B(m,m+1)=a;
end
for m=2:M
temp=h*z(m-1);
for j=2:M+1
A(m,j)=A(m,j)-temp;
B(m,j)=B(m,j)-temp;
end
end
W=zeros(M+1,N+1);
fii=zeros(M+1,N+1);
for j=1:M+1
x=(j-1)*h;
W(j,1)=1+sin(2*x);
end
for k=2:N+1
for j=2:M
x=(j-1)*h;
t=(k-1)*tau;
fii(j,k)=-(pi*z(j-1))*(exp(i*t)+exp(i*(t-tau)))
+(3*sin(2*x)-1)*exp(i*(t-tau/2));
end
W(:,k)=A\(-B*W(:,k-1)+fii(:,k));
end
eta(1)=0;
for k=2:N+1
S=0;
for j=2:M+1
S=S+W(j,k);
end
S=S*h;
t=(k-1)*tau;
eta(k)=(S-pi*exp(i*t))/(i*(pi+d*h));
end
```

```
for k=1:N
r(k)=(eta(k+1)-eta(k))/tau;
end
u=zeros(M+1,N+1);
for k=1:N+1
for j=1:M+1
x=(j-1)*h;
u(j,k)=W(j,k)-i*eta(k)*(1+sin(2*x));
end
end
```

Exact Solution of this PDE

```
for k=1:N
t=(k-1/2)*tau;
ep(k)=exp(i*t);
end;
for j=1:M+1
for k=1:N+1
t=(k-1)*tau;
x=(j-1)*h;
eu(j,k)=exp(i*t)*(1+sin(2*x));
end;
end;
for k=1:N+1
for j=1:M+1
t=(k-1)*tau;
x=(j-1)*h;
ew(j,k)=(2*exp(i*t)-1)*(1+sin(2*x));
end;
end;
```

Absolute Differences

```
absdiffw=max(max(abs(ew-W)))
absdiffp=max(abs(ep-r))
absdiffu=max(max(abs(eu-u))
```

Appendix B

Matlab Implementation of One Dimension Second Order (Crank-Nicolson) of Accuracy Difference Schemes of Problem (5.2.1)

```
function(Nonlocal condition)
clear all; close all;
N=40;
M=40;
i=sqrt(-1);
h=pi/M;
tau=1/N;
a=-1/(2*h^2);
b=(i/tau)+(1/h^2)+0.5;
c=-i/tau+1/(h^2)+0.5;
d=0;
for m=1:M
d=d+sin(2*m*h);
end
for m=1:M
z(m)=(sin(2*m*h)*((1-cos(2*h))/(h^2)+0.5)+0.5)/(pi+d*h);
end
A=zeros(M+1,M+1);
B=zeros(M+1,M+1);
A(1,1)=1;
A(1,M+1)=-1;
A(M+1,1)=1;
A(M+1,2)=-1;
A(M+1,M)=-1;
A(M+1,M+1)=1;
for m=2:M
A(m,m)=b;
A(m,m-1)=a;
A(m,m+1)=a;
B(m,m)=c;
B(m,m-1)=a;
```

```
B(m,m+1)=a;
end
for m=2:M
temp=h*z(m-1);
for j=2:M+1
A(m,j)=A(m,j)-temp;
B(m,j)=B(m,j)-temp;
end
end
W=zeros(M+1,N+1);
fii=zeros(M+1,N+1);
for j=1:M+1
x=(j-1)*h;
W(j,1)=1+sin(2*x);
end
for k=2:N+1
for j=2:M
x=(j-1)*h;
t=(k-1)*tau;
fii(j,k)=-(pi*z(j-1))*(exp(i*t)+exp(i*(t-tau)))+(3*sin(2*x)-1)*exp(i*(t-tau/2));
end
W(:,k)=A\(-B*W(:,k-1)+fii(:,k));
end
eta(1)=0;
for k=2:N+1
S=0;
for j=2:M+1
S=S+W(j,k);
end
S=S*h;
t=(k-1)*tau;
eta(k)=(S-pi*exp(i*t))/(i*(pi+d*h));
end
for k=1:N
```

```
r(k)=(eta(k+1)-eta(k))/tau;
end
u=zeros(M+1,N+1);
for k=1:N+1
for j=1:M+1
x=(j-1)*h;
u(j,k)=W(j,k)-i*eta(k)*(1+sin(2*x));
end
end
```

Exact Solution Of This Pde

```
for k=1:N
t=(k-1/2)*tau;
ep(k)=exp(i*t);
end;
for j=1:M+1
for k=1:N+1
t=(k-1)*tau;
x=(j-1)*h;
eu(j,k)=exp(i*t)*(1+sin(2*x));
end;
end;
for k=1:N+1
for j=1:M+1
t=(k-1)*tau;
x=(j-1)*h;
ew(j,k)=(2*exp(i*t)-1)*(1+sin(2*x));
end;
end;
```

Absolute Differences

```
absdiffw=max(max(abs(ew-W)))
absdiffp=max(abs(ep-r))
absdiffu=max(max(abs(eu-u)))
```

Appendix C

Matlab Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (5.2.2)

```
function(Drichlet condition)
clear all;
close all;
N=80;
M=80;
i=sqrt(-1);
h=pi/M;
tau=1/N;
a=(i/tau)+(2/h^2)+1;
b=-i/tau;
c=-1/(h^2);
A=zeros(M+1,M+1);
for m=2:M
for j=2:M
A(m,j)=-h/pi;
end
end
for m=2:M
A(m,m)=a-(h/pi);
end
for m=2:M-1
A(m,m+1)=c-(h/pi);
end
for m=3:M
A(m,m-1)=c-(h/pi);
end
A(1,1)=1;
A(1,M+1)=-1;
A(2,1)=c;
A(M,M+1)=c;
A(M+1,1)=1;
```

```
A(M+1,2)=-1;
A(M+1,M+1)=1;
A(M+1,M)=-1;
B=zeros(M+1,M+1);
for n=2:M
B(n,n)=b;
end
B;
W=zeros(M+1,N+1);
fii=zeros(M+1,N+1);
for j=1:M+1
x=(j-1)*h;
W(j,1)=sin(2*x)+1;
end
for k=2:N+1
for j=2:M
x=(j-1)*h;
t=(k-1)*tau;
fii(j,k)=exp(i*t)*(-2+4*sin(2*x));
end
W(:,k)=A\(-(B*W(:,k-1))+fii(:,k));
end
eta(1)=0;
for k=2:N+1
S=0;
for j=2:M+1
S=S+W(j,k);
end
S=S*h;
t=(k-1)*tau;
eta(k)=(S-pi*exp(i*t))/(i*pi);
end
for k=2:N
p(k)=(eta(k)-eta(k-1))/tau;
```

```
end
u=zeros(M+1,N+1);
for k=1:N+1
for   j=1:M+1
u(j,k)=W(j,k)-i*eta(k);
end
end
Exact Solution Of This Pde
for k=2:N
t=(k-1)*tau;
ep(k)=exp(i*t);
end;
for j=1:M+1
for k=1:N+1
t=(k-1)*tau;
x=(j-1)*h;
eu(j,k)=exp(i*t)*(1+sin(2*x));
end;
end;
for k=1:N+1
for j=1:M+1
t=(k-1)*tau;
x=(j-1)*h;
es(j,k)=-1+exp(i*t)*(2+sin(2*x));
end;
end;
Absolute Differences
absdiffW=max(max(abs(es-W)))
absdiffp=max(abs(ep-p))
absdiffu=max(max(abs(eu-u)))
```

Appendix D

Matlab Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (5.2.3)

```
function (Neumann condition)
clear all;
close all;
N=20;
M=20;
i=sqrt(-1);
h=pi/M;
tau=1/N;
a=-1/(h^2);
b=(i/tau)+(2/h^2)+1;
c=-i/tau;
d=0;
for m=1:M
d=d+cos(2*m*h);
end
for m=1:M
r(m)=(2*cos(2*m*h)*((cos(2*h)-1)/h^2-0.5)-1)/(pi+h*d);
end
A=zeros(M+1,M+1);
B=zeros(M+1,M+1);
A(1,1)=-1;
A(1,2)=1;
A(M+1,M)=-1;
A(M+1,M+1)=1;
for m=2:M
A(m,m)=b;
A(m,m-1)=a;
A(m,m+1)=a;
B(m,m)=c;
end
for m=2:M
```

```
temp=h*r(m);
for j=2:M+1
A(m,j)=A(m,j)+temp;
end
end
v=zeros(M+1,N+1);
fii=zeros(M+1,N+1)
for j=1:M+1
x=(j-1)*h;
v(j,1)=1+cos(2*x);
end
for k=2:N+1
for j=2:M
x=(j-1)*h;
t=(k-1)*tau;
fii(j,k)=(pi*r(j-1)+3*cos(2*x)-1)*exp(i*t); % r(j-1)
end
v(:,k)=A\(-B*v(:,k-1)+fii(:,k));
end
eta(1)=0;
for k=2:N+1
S=0;
for j=2:M+1
S=S+v(j,k);
end
S=S*h;
t=(k-1)*tau;
eta(k)=(S-pi*exp(i*t))/(i*(pi+d*h));
end
eta(1)=0;
for k=2:N
p(k)=(eta(k)-eta(k-1))/tau;
end
u=zeros(M+1,N+1);
```

```
for k=1:N+1
for j=1:M+1
x=(j-1)*h;
u(j,k)=v(j,k)-i*eta(k)*(1+cos(2*x));
end
end
Exact Solution Of This Pde
for k=2:N
t=(k-1)*tau;
ep(k)=exp(i*t);
end;
for j=1:M+1
for k=1:N+1
t=(k-1)*tau;
x=(j-1)*h;
eu(j,k)=exp(i*t)*(1+cos(2*x));
end;
end;
for k=1:N+1
for j=1:M+1
t=(k-1)*tau;
x=(j-1)*h;
ev(j,k)=(exp(i*t)+exp(i)-1)*(1+cos(2*x));
end;
end;
Absolute Differences
absdiffw=max(max(abs(ev-v)))
absdiffp=max(abs(ep-p))
absdiffu=max(max(abs(eu-u)))
```

Appendix E

Matlab Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (5.2.4)

```
function (Robin condition)
clear all;
close all;
N=20;
M=20;
i=sqrt(-1);
h=pi/M;
tau=1/N;
b=-1/(h^2);
a=-i/tau;
c=(i/tau)+(2/h^2);
d=0;
for m=1:M
d=d+cos(m*h/2);
end
for m=2:M+1
r(m)=(2/(d*h^2))*(cos(m*h/2)*(cos(h/2)-1));
end
r;
A=zeros(M+1,M+1);
for m=2:M+1
for j=2:M
A(m,j)=r(m);
end
end
for m=2:M
A(m,m)=c+h*r(m);
end
for m=2:M-1
A(m,m+1)=b+h*r(m);
end
```

```
for m=3:M-1
A(m,m-1)=b+h*r(m);
end
A;
A(1,1)=1+1/h;
A(1,2)=-1/h;
A(2,1)=b;
A(M,M+1)=b;
A(M+1,M-1)=-1/h;
A(M+1,M+1)=1+1/h;
B=zeros(M+1,M+1);
for m=2:M
B(m,m)=a;
end
W=zeros(M+1,N+1);
fii=zeros(M+1,N+1);
for j=2:M+1
x=(j-1)*h;
W(j,1)=cos(x/2);
end
for k=1:N+1
t=(k-1)*tau;
fii(1,k)=exp(-i*t);
fii(M+1,k)=-0.5*exp(-i*t);
end
for k=2:N+1
for j=2:M+1
x=(j-1)*h;
t=(k-1)*tau;
fii(j,k)=(r(m)/(h)+1.25*cos(x/2))*exp(-i*t);
end
W(:,k)=A\(-B*W(:,k-1)+fii(:,k));
end
eta(1)=0;
```

```
for k=2:N+1
S=0;
for j=2:M+1
S=S+W(j,k);
end
S=S*h;
t=(k-1)*tau;
eta(k)=(S-2*exp(-i*t))/(i*d*h);
end
eta(1)=0;
for k=2:N+1
p(k)=(eta(k)-eta(k-1))/tau;
end
u=zeros(M+1,N+1);
for k=1:N+1
for   j=1:M+1
x=(j-1)*h;
u(j,k)=W(j,k)-i*eta(k)*(cos(x/2));
end
end
Exact Solution Of This Pde
for k=2:N+1
t=(k-1)*tau;
ep(k)=exp(-i*t);
end;
for j=2:M+1
for k=2:N+1
t=(k-1)*tau;
x=(j-1)*h;
eu(j,k)=exp(-i*t)*cos(x/2);
end;
end;
for k=1:N+1
for j=1:M+1
```

```
t=(k-1)*tau;
x=(j-1)*h;
es(j,k)=(exp(-i*t)+exp(-i*t)-1)*(cos(x/2));
end;
end;
```

Absolute Differences

```
absdiffu=max(max(abs(es-W)))
absdiffp=max(abs(ep-p(k)))
absdiffu=max(max(abs(eu-u)))
```

Appendix F

NEAR EAST UNIVERSITY

ETHICAL APROVAL DOCUMENT

Date: 28/06/2022

To Graduate School of Applied Sciences

The research project title "**Time-dependent source identification problem for Schrödinger differential and difference equation**" has been evaluated. Since the researcher(s) will not collect primary data from humans, animals, plants or earth, this project does not need to go through the ethics committee.

Title: Prof. Dr.

Name Surname: Allaberen Ashyralyev

Signature:

Role in the Research Project: Supervisor

Appendix G

Turnitin Similarity Report

1.06.2022 21:58 Turnitin

Skip to Main Content
Ödevler
Öğrenciler
Not Defteri
Kütüphaneler
Takvim
Tartışma
Tercihler
Bu sayfa hakkında

Bu sizin ödev kutunuzdur. Bir yazılı ödevi görüntülemek için yazılı ödevin başlığını seçin. Bir Benzerlik Raporunu görüntülemek için yazılı ödevin benzerlik sütunundaki Benzerlik Raporu ikonunu seçin. Tıklanabilir durumda olmayan bir ikon Benzerlik Raporunun henüz oluşturulmadığını gösterir.

Tez

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

Sil | İndir | Şuraya taşı...

Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
Mesut Ürün	Abstract	%0 %0	0%	0%	0%	--	--	ödev indir	1848623313	01-Haz-2022
Mesut Ürün	Conclusion	%0 %0	0%	0%	0%	--	--	ödev indir	1848626290	01-Haz-2022
Mesut Ürün	Chapter 4	%4 %4	12%	4%	9%	--	--	ödev indir	1848625310	01-Haz-2022
Mesut Ürün	All Thesis	%5 %5	6%	12%	0%	--	--	ödev indir	1848627024	01-Haz-2022
Mesut Ürün	Chapter 1	%5 %5	9%	5%	7%	--	--	ödev indir	1848623669	01-Haz-2022
Mesut Ürün	Chapter 2	%7 %7	3%	3%	4%	--	--	ödev indir	1848624331	01-Haz-2022
Mesut Ürün	Chapter 3	%7 %7	13%	0%	12%	--	--	ödev indir	1848624833	01-Haz-2022
Mesut Ürün	Chapter 5	%7 %7	11%	7%	0%	--	--	ödev indir	1848625992	01-Haz-2022

Prof. Dr. A. Ashyraliyev

Appendix H

Curriculum Vita (CV)

Personal Information:

Full Name: Mesut Ürün

Nationality: Turkey

Data of birth: (01-November-1984).

Marital Status: Married and have a boy

E-mail address: mesuturun@gmail.com

Education:

- 2019-2022 **PhD.**, *Near East University*,Mersin
  Applied Mathematics
- 2015-2019 **BSc.**, *İstanbul Technical University*, İstanbul
  Mechanical Engineering
- 2013-2015 **PhD.**, *Istanbul University*, Istanbul
  Applied Mathematics
- 2009-2012 **MSc.**, *İstanbul University*, İstanbul
  Applied Mathematics
- 2003-2008 **BSc.**, *Marmara University*, İstanbul
  Faculty of Technical Education

PhD thesis:

Title: *Time-Dependet Source Identification Problem for Schrödinger* Differantial and Difference Equation

Supervisors: Prof. Dr. Allaberen Ashyralyev

Master thesis:

Title: *Determination of Control Parameter for The Schrödinger Equation*

Supervisors: Prof. Dr. Allaberen Ashyralyev

Publications:

- Ashyralyev, A., & Urun, M., (2022). Stability of Time-dependent source identification problem for Schrödinger differential equations, TWMS J. Pure Appl. Math. (2022)
- Ashyralyev, A., & Urun, M., (2022). I.C.Parmaksizoglu, Mathematical modeling of the energy consumption problem, Karaganda.vol(1).

- Ashyralyev, A., & Urun, M. (2021). On the Crank-Nicolson difference scheme for the time-dependent source identification problem. *Karaganda*, (2), 35-44.
- Ashyralyev, A., & Urun, M., (2021). Time-dependent source identification Schrödinger type problem. *International Journal of Applied Mathematics*, *34*(2), 297.
- Ashyralyev, A., & Urun, M. (2014). A second order of accuracy difference scheme for Schrödinger equations with an unknown parameter. *Filomat*, *28*(5), 981-993.
- Ashyralyev, A., & Urun, M. (2013). Abstract and Applied Analysis 2013. *Article ID, 548201*, 1-8.
- Ashyralyev, A., & Urun, M., (2013). A Second Order of Accuracy Difference Scheme for Schrödinger Equations With an Unknown Parameter. *International Conference of Numerical Analysis and Aplplied Mathematics June*,2013 (ICAAMM2013), Istanbul.

Conference Proceeding:

- A. Ashyralyev, M. Urun, and I.C. Parmaksizoglu, Mathematical modeling of the energy consumption problem, (ICAAM2022),Istanbul.
- A. Ashyralyev and M. Urun, Time-Dependent Source Identification Schrödinger Type Problem, (ICAAM2022),Istanbul.
- A. Ashyralyev and M. Urun, Time-Dependent Source Identification Problem for the Schrödinger Equation with Nonlocal Boundary Conditions, *3rd International Conference of Mathematical Sciences* (ICMS2019),İstanbul.
- A. Ashyralyev and M. Urun, Source Identification Problems for Schrödinger Differential and Difference Equations, *AIP Conference Proceedings*, 020014, (ICAAM2018), Cyprus.
- A. Ashyralyev and M. Urun, Determination of a Control Param- eter of the r-modified Crank-Nicholson Difference Scheme for the Schrödinger Equation, *AIP Conf. Proc.* 1759, 020086, (ICAAM 2016), Istanbul.
- International Conference of Numerical Analysis and Applied Mathematics (ICAAM 2011), Gümüşhane.

Work Experience:

- Galatasaray University, Maritime Vocational School, Department of Marine Engines Operation

2020-Present **Lecturer**, Beşiktaş-İstanbul

- Final VIP

2021-2022 **Mathematics Teacher**, Altunizade-İstanbul

- Murat Education Institution

2008-2019 **Mathematics Teacher**, Kadiköy-Beşiktaş-İstanbul

Calculus, ordinary differential equations, partial differential equations, statistics, probability courses are given to teachers who want to become mathematics teachers in public schools. In addition, the ALES exam is equivalent to the GRE-GMAT. I lecture mathematics course required for these exams.

- Özköseoğlu Stainless Chemical

2007-2008 **Production Manager**, İkitelli-İstanbul

The projects in the production department of the company and the personnel working in the company are guided, the manufacturing process is planned, the coordination and control in the production area is ensured.

- Sınav Education Institution

2003-2007 **Mathematics Teacher**, Gaziosmanpaşa-İstanbul

Trainings:

- Udemy Python and Programming, Data analysis

Languages:

- Turkish
- Kurdish
- English

Computer Skills:

- MATLAB, LaTeX, AUTOCAD, MICROSOFT OFFICE, PASCAL, PYTHON

Lectures:

- Mathematics (2013-present)
- Statistics (2013-present)
- Calculus (2013-present)
- Analysis-1 (2013-present), for Mathematics Teachers
- Analysis-2 (2013-present), for Mathematics Teachers

- Differential Equation (2013-present), for Mathematics Teachers and Engineers
- Probability (2013-present), for Mathematics Teachers and Engineers
- Linear Algebra (2013-present), for Mathematics Teachers (ÖABT)
- Mathematics and Geometry (2003-present),
- Fluid Mechanics
- Heat Transfer
- Mechanical Vibrations
- Thermodynamics

Research Areas:

- Schrödinger Equation
- Ordinary Differential Equation
- Partial Differential Equations
- Numerical Analysis
- Heat Transfer
- Delay Differential Equation
- Mathematical Modeling

References:

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- Prof. Dr. Müjgan TEZ, Department of Statistics, Marmara University, +90(532) 727 61 13, mtez@marmara.edu.tr