BISHAR CHATO HASO
THE TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEM FOR THE DELAY HYPERBOLIC EQUATIONS
M.Sc. THESIS
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Bishar Chato HASO

Supervisor
Prof. Dr. Allaberen ASHYRALYEV

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Approval

We certify that we have read the thesis submitted by Bishar Chato Haso titled " **The time-dependent source identification problem for the delay hyperbolic equations** " and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Educational Sciences.

Examining Committee	Name-Surname	Signature
Head of the Committee:	Prof. Dr. Deniz Agirseven	....
Committee Member*:	Assoc. Prof. Dr. Nuriye Sancar	....
Supervisor:	Prof. Dr. Allaberen Ashyralyev	....

Approved by the Head of the Department

28../06./ 2022

Prof. Dr. Evren Hinçal

Head of Department

Approved by the Institute of Graduate Studies

...../...../ 2022

Prof. Dr. Kemal Hüsnü Can Başer

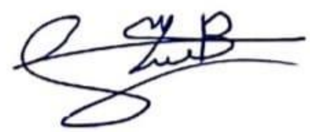
Head of the Institute

Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

Bishar Chato Haso

28 / 06 / 2022



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Bishar Chato Haso

Abstract

The Time-Dependent Source Identification Problem for the Delay Hyperbolic Equations

Haso, Bishar Chato

Master Thesis, Department of Mathematics

Supervisor: Prof. Dr. Allaberen Ashyralyev

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Our project is aimed to investigate the time-dependent source identification problem for delay hyperbolic partial differential equations. This thesis deals with analytical and approximate solutions of several problems for delay hyperbolic partial differential equations. In the present study, a time-dependent source identification problem with local and nonlocal conditions for a one-dimensional delay hyperbolic equation is investigated. Stability estimates for the solutions of the time-dependent source identification problems are established. Furthermore, a first order of accuracy difference scheme for the numerical solutions of the time-dependent source identification problems for delay hyperbolic equations with local and nonlocal conditions are presented. New absolute stable difference scheme for the approximate solution of the one dimensional delay hyperbolic equation is constructed and a numerical algorithm is presented. Additionally, illustrative numerical results are provided.

Key Words: Hyperbolic differential equation, Time delay, Source identification problem, Stability, Difference Schemes.

Özet

Zaman Gecikmeli Hiperbolik Denklemler İçin Kaynak Tanımlama Problemi

Haso, Bishar Chato

Yüksek Lisans Tezi, Matematik Anabilim Dalı

Danışman: Prof. Dr. Allaberen Ashyralyev

Haziran, 2022, (113) sayfa

Projemiz, gecikmeli hiperbolik kısmi diferansiyel denklemler için zamana bağlı kaynak tanımlama problemini araştırmayı amaçlamaktadır. Bu tez, gecikmeli hiperbolik kısmi diferansiyel denklemler için çeşitli problemlerin analitik ve yaklaşık çözümlerini ele almaktadır. Bu çalışmada, tek boyutlu bir gecikme hiperbolik denklemi için yerel ve yerel olmayan koşullarla zamana bağlı bir kaynak belirleme problemi incelenmiştir. Zamana bağlı kaynak tanımlama problemlerinin çözümleri için kararlılık tahminleri oluşturulmuştur. Ayrıca, yerel ve yerel olmayan koşullara sahip gecikmeli hiperbolik denklemler için zamana bağlı kaynak tanımlama problemlerinin sayısal çözümleri için bir doğruluk farkı şeması sunulmaktadır. Tek boyutlu gecikmeli hiperbolik denklemin yaklaşık çözümü için yeni mutlak kararlı fark şeması oluşturulmuş ve sayısal algoritma sunulmuştur. Ek olarak, açıklayıcı sayısal sonuçlar sağlanmaktadır.

Anahtar Kelimeler: hiperbolik diferansiyel denklem, zaman gecikmesi, kaynak tanımlama sorunu, istikrar, fark şemaları.

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List of Abbreviations and Symbols

DHE	Delay Hyperbolic Equation
DHPDE	Delay Hyperbolic Partial Differential Equation
DHPDEs	Delay Hyperbolic Partial Differential Equations
IVP	Initial Value Problem
IVPs	Initial Value Problems
BVP	Boundary Value Problem
BVPs	Boundary Value Problems
DS	Difference Scheme
DSs	Difference Schemes
SIP	Source Identification Problem
SIPs	Source Identification Problems
$\boldsymbol{R^1 = I}$	Real Line $(-\infty, \infty)$
$\boldsymbol{mE_u}$	Error function defined by formula $\max\limits_{(m-1)N \le k \le mN} \left(\sum\limits_{n=1}^{M-1} \lvert u(t_k, x_n) - mu_n^k \rvert^2 h \right)^{\frac{1}{2}}$, $m = 1,2,3 \ldots$
$\boldsymbol{mE_p}$	Error function defined by formula $\max\limits_{(m-1)N+1 \le k \le mN-1} \lvert p(t_k) - mp_k \rvert$, $m = 1,2,3 \ldots$

CHAPTER I
Introduction

1.1 Historical Note and Literature Survey

Delay differential equations, differential integral equations and functional differential equations have been studied for at least 200 years. During the last 50 years, the theory of functional differential equations has been developed extensively and has become part of the vocabulary of researchers dealing with specific applications such as viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flow, neural networks, combustion, interaction of species, microbiology, learning models, epidemiology, physiology, as well as many others. Stochastic effects are also being considered but the theory is not as well developed by Hale, J. K. (2006). Delay hyperbolic differential equation have been studied in several papers, for example: Ashyralyev, Agirseven, 2019; Son, Thao, 2019; Monteghetti, Haine, Matignon, 2017; Zhang, Zhang, Deng, 2014; Prakash, Harikrishnan, 2012; Vyazmin, Sorokin, 2017; Farkas, 2003. However, Shah, Wiener, 1985, studied the existence and uniqueness of the bounded solution of nonlinear one dimensional delay hyperbolic differential equation with constant coefficients. Ashyralyev and Agirseven in 2019 studied the existence and uniqueness of a bounded solution a semilinear time delay hyperbolic equation in a Hilbert space. In applications, theorems on the existence and uniqueness of bounded solutions of four problems for semilinear time delay differential equations of hyperbolic type were obtained. The two-steps of a first order of accuracy difference scheme was presented, the main theorem on the existence and uniqueness of uniformly bounded solution of the difference scheme with respect to time step size was proved. Numerical results were presented. In the paper of Prakash and Harikrishnan, 2012, a class of impulsive vector hyperbolic differential equation with delays was investigated. They studied different sufficient conditions for H-oscillation of solutions systems subject to the Neumann boundary condition by employing certain second-order impulsive differential inequality, where H is a unite vector in R^M. Allaberen Ashyralyev and Deniz Agirseven in 2014 studied the source identification problem for a delay parabolic equation with nonlocal conditions. The stability estimates in Hölder norms for the solution of the problem was established. In 2020 the absolute stable difference schemes for third order delay partial differential equations have been studied. The absolute stable of a first order of accuracy difference scheme for the approximate solution of the delay partial differential equation in a Hilbert space was presented. However, the theorem on the stability of the difference scheme was proved. In practice, stability estimates for the solutions of three-step difference schemes for different types of delay partial differential equations were obtained. Numerical results were given by Ashyralyev, A., Hınçal, E., Ibrahim, S. Numerical solutions of source identification problem for hyperbolic-parabolic equations have been studied, partial differential equations with unknown source terms were widely used in mathematical modeling of real-life systems in many different fields of science and engineering. Various local and nonlocal boundary value problems for hyperbolic-parabolic equations with unknown sources have been reduced to the boundary value problem for the differential equation with parameter p. In applications, the stability inequalities for the solution of three source identification problems for hyperbolic-parabolic equations were obtained. The first and second order of accuracy difference scheme for the approximate solution were constructed and investigated by Maral Ashyralyyeva and Maksat Ashyraliyev, 2016. There is always a major interest for the theory of source identification problems for partial differential equations since they have widespread applications in modern physics and technology. For this effort, the stability of various source identification problems for partial differential and difference equations has also been studied extensively by many researchers (see, for examle, Ashyralyev, A., Agirseven, D., 2014; Blasio, G. Di., Lorenzi,A.2007; Kabanikhin, S.I. 2004; Orazov, I., Sadybekov, M.A., 2012; Ashyralyev, A., Emharab, F., 2019; Ashyralyev, A., Ashyralyyev, C., 2014; Ashyralyev, A., Al-Hammouri,A., 2020; Ashyralyev, A., Al-Hammouri, A., Ashyralyyev, C., 2021; Ashyralyev, A., Erdogan, A.S., 2014; Ashyralyev, A., Urun, M., 2021; Sadybekov, M.A., Dildabek,G., Ivanova,M.B., 2018; Saitoh, S., Tuan, V.K., Yamamoto, M., 2002; Sakamoto, K.,Yamamoto, M., 2011; Samarskii, A.A., Vabishchevich, P.N., 2007; Ashyralyev, A., Agirseven, D., Agarwal, R.P., 2020; Emharab, F., 2019; Ahmad Mohammad Salem Al-Hammauri, 2020; Erdogan, A.S., 2010; Ashyraliyev,M., Ashyralyyeva,M.A., Ashyralyev,A., 2020; Ashurov, R.R., Shakarova M.D., 2022). In many fields of the contemporary science and technology, systems with delaying terms appear. The dynamical processes are described by systems of delay ordinary and partial differential and difference equations. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The stability of the delay differential and difference equations has been studied in many papers (see, for example, Al-Mutib, A.N., 1984; Ashyralyev, A., Akca, H., 2001; Ashyralyev, A., Akca, H., Yenicerioglu, A. F., 2003; Ashyralyev, A., Sobolevskii, P.E., 2001; Bellen, A., Jackiewicz, Z., Zennaro, M., 1988; Torelli, L., 1989; Yeniçerioğlu, A. F., Yalçinbaş, S., 2004; Yeniçerioğlu, A. F., 2008; Ashyralyev, A., Agirseven, D., 2020; Agirseven, D., 2018). Delay partial differential equations arise in many applications such as control theory, climate models, medicine, biology, and much more (for example, see Wu, J., 1996 and the references therein).

1.2 Layout of the Present Thesis

The time-dependent source identification problem for delay hyperbolic partial differential equations has not been investigated before. The main aim of the present Thesis is to study the boundedness solution of several time-dependent identification problems for delay hyperbolic equations. This thesis consists of five Chapters. First chapter is the introduction. Second chapter, six examples of the second order differential equation with time-dependent identification problems for delay hyperbolic equations are investigated. We obtained the exact solution of the initial boundary value problem for a one dimensional delay hyperbolic equation. Third chapter, Theorems on stability estimates for the solution of the initial boundary value problem for the second order of hyperbolic differential equations with time delay are proved. In Chapter Four, we obtain the algorithms of numerical solution for the IVP for the one dimensional delay hyperbolic partial differential equation with Dirichlet, Neumann and nonlocal boundary conditions. We will present the first order of accuracy difference schemes for the numerical solutions of delay hyperbolic equations. Numerical analysis is provided. Based on the main results of the thesis, reports were made at the Satellite Conference "Numerical Functional Analysis - 2021" of ICAAM November 22 - 24 , 2021 ISTANBUL, TURKEY. Chapter Five presents some conclusions which are obtained from Chapters Two, Three and Four. Two expanded abstracts are published in AIP Conference Proceedings 2022. One paper is submitted in the journal "Bulletin of the Karaganda University" and one paper is submitted in the international journal of Applied Mathematics. Besides, some ideas are given for working in the future.

1.3.Basic Concepts and Definitions :

This section highlights basic concepts and definitions in the theory of ordinary and partial differential equations with Delay Hyperbolic equation leading us to conduct and understand the works in this thesis.

1.3.1 *Sturm-Liouville problem (Arfken, Weber, 2005)*

We denote the Sturm-Liouville operator as

$$L[y] = -\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y$$

and consider the Sturm-Liouville equation

$$L[y] + \lambda y = 0, \tag{1.1}$$

where $p > 0$ and p and q are continuous functions on the interval $[0, l]$ with local boundary conditions

$$\alpha_1 y(0) + \alpha_2 p(0)y'(0) = 0; \beta_1 y(l) + \beta_2 p(l)y'(l) = 0, \tag{1.2}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$ or nonlocal boundary conditions

$$y(0) - y(l) = 0; y'(0) - y'(l) = 0. \tag{1.3}$$

The problem of finding a complex number $\lambda = \mu$ such that the BVPs $(1.1), (1.2)$ or $(1.1), (1.3)$ have a non trivial solution are called Sturm-Liouville problems. The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(x, \mu)$ is called an eigenfunction. We will consider three types of Sturm-Liouville problem.

1.3.1.1. The Sturm-Liouville Problem with Dirichlet Condition.

$$-u''(x) + \lambda u(x) = 0, 0 < x < l, u(0) = u(l) = 0 \tag{1.4}$$

has solution

$$u_k(x) = \sin\frac{k\pi x}{l} \quad and \quad \lambda_k = -\left(\frac{k\pi}{l}\right)^2, k = 1,2,3, \ldots.$$

In the case when $l = \pi,$ we have that

$$u_k(x) = \sin kx \quad and \quad \lambda_k = -k^2, k = 1,2,3, \ldots.$$

1.3.1.2. The Sturm-Liouville Problem with Neumann Condition.

$$-u''(x) + \lambda u(x) = 0, 0 < x < l, u'(0) = u'(l) = 0 \tag{1.5}$$

has solution

$$u_k(x) = \cos\frac{k\pi x}{l} \quad and \quad \lambda_k = -\left(\frac{k\pi}{l}\right)^2, k = 0,1,2, \ldots.$$

In the case when $l = \pi,$ we have that

$$u_k(x) = \cos kx \quad and \quad \lambda_k = -k^2, k = 0,1,2,\ldots.$$

1.3.1.3. The Sturm-Liouville Problem with Nonlocal Conditions.

$$-u''(x) + \lambda u(x) = 0, 0 < x < l, u(0) = u(l), u'(0) = u'(l) \tag{1.6}$$

has solution

$$u_k(x) = \cos\frac{2k\pi x}{l} \;, k = 0,1,2,\ldots$$

$$u_k(x) = \sin\frac{2k\pi x}{l} \;, k = 1,2,\ldots$$

and

$$\lambda_k = -4\left(\frac{k\pi}{l}\right)^2, k = 0,1,2,\ldots.$$

In the case when $l = \pi,$ we have that

$$u_k(x) = \cos 2kx \quad , k = 0,1,2,\ldots$$

$$u_k(x) = \sin 2kx \quad , k = 1,2,\ldots$$

and

$$\lambda_k = -4k^2, k = 0,1,2,\ldots.$$

1.3.2 *Fourier Series (Serov, V. (2017))*

Let L be a fixed number and $f(x)$ be a periodic function with periodic $2L$, defined on $(-L, L)$. The Fourier series of $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right) \tag{1.7}$$

where the Fourier coefficients a_0,a_m and b_m are defined by the integrals

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{1.8}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx \,, m = 1,2,3,\ldots \tag{1.9}$$

And

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \,, m = 1,2,3,\ldots. \tag{1.10}$$

1.3.3 *The Laplace Transform (Finan, M. B. 2010)*

The Laplace transform can be helpful in solving ordinary and partial differential equations because it can replace an ODE with an algebraic equation or replace a PDE with an ODE. Another reason that the Laplace transform is useful is that it can help deal with the boundary conditions of a PDE on an infinite domain.

Definition 1. Let f be a real valued function of the real variable t, defined for $t \geq 0$. Let s be a variable that we will assume to be real, and consider the function F defined by

$$L\{f(t)\} = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \qquad (1.11)$$

for all values of s for which this integral exists. The function F defined by the integral (1.11) is called the Laplace transform of the function f. We will denote the Laplace transform F of f by $L\{f\}$ and will denote $F(s)$ by $L\{f(t)\}$ .Note that for those $s \in C$ for which the integral makes sense $F(s)$ is a complex-valued function of complex number.

1.3.4 *The Fourier transform (Bracewell, 1999)*

There are several ways to define the Fourier transform of a function $f: R \to C$.

Definition. Let f be a real valued function of the real variable x, defined for $x \in (-\infty,\infty)$. Let s be a variable and consider the function F defined by

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-isx}dx, \qquad (1.12)$$

for all values of s for which this integral exists. The function F defined by the integral (1.12) is called the Fourier transform of the function f. We will denote the Fourier transform F of f by $F\{f\}$ and will denote $F(s)$ by $F\{f(x)\}$. Note that for those $s \in C$ for which the integral makes sense $F(s)$ is a complex-valued function of complex number.

CHAPTER II

Integral Transform Methods of Time-Dependent Identification Problem for Delay Hyperbolic Equations

2.1 Introduction

Delay hyperbolic equations appear in mathematical models of applied mathematics, physics, biology, and population dynamics. Therefore, it is important to study hyperbolic type differential equations with time delay terms. Note that time-dependent identification problems for delay hyperbolic equations are not investigated. Therefore, the main aim of Chapter Two is to study the time-dependent identification problems for several hyperbolic equations. Applying results of Chapter One and Fourier series, Laplace and Fourier transform methods, we obtain the exact solution of several time-dependent identification problems for delay hyperbolic equations.

2.2 Fourier Series Method

We consider the Fourier series method for the solution of the time-dependent identification problems for delay hyperbolic differential equations with Dirichlet, Neumann and non-local boundary conditions.

Problem 1. we consider the time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = b\dfrac{\partial^2 u(t-\omega,x)}{\partial x^2} + p(t)q(x) + f(t,x), \\ 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in [0,l], \\ u(t,0) = u(t,l) = 0, \displaystyle\int_0^l u(t,x)dx = \zeta(t), t \ge 0 \end{cases} \tag{2.1}$$

for a one dimensional delay hyperbolic equation with Dirichlet condition. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (2.1) has a unique solution $\big(u(t,x), p(t)\big)$ for the smooth functions $f(t,x)\big(t \in (0,\infty), x \in (0,l)\big), g(t,x), \zeta(t), q(x)$. Here b is a constant. Assume that $\int_0^l q(x)dx \neq 0$, and $q(0) = q(l) = 0$, and $g(t,0) = g(t,l) = 0, t \in [-\omega, 0]$, $f(t,0) = f(t,l) = 0, t \in [0,\infty)$.

For example, we consider the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)\sin x + bu_{xx}(t-\pi, x) \\ -\sin t \sin x - b\sin t \sin x, t > 0, 0 < x < \pi, \\ u(t,x) = \sin t \sin x, -\pi \le t \le 0, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \int_0^\pi u(t,x)dx = 2\sin t, t \ge 0 \end{cases} \tag{2.2}$$

for a one dimensional delay hyperbolic equation with Dirichlet condition.

Solution. For this case $\omega = \pi, l = \pi, g(t,x) = \sin t \sin x, -\pi \le t \le 0, 0 \le x \le \pi; f(t,x) = -\sin t \sin x - b\sin t \sin x, t > 0, 0 < x < \pi, \zeta(t) = 2\sin t, t \ge 0$. In order to solve the problem (2.2), we consider the Sturm-Liouville problem

$$u''(x) - \lambda u(x) = 0, 0 < x < \pi, u(0) = u(\pi) = 0$$

generated by the space operator of problem (2.2). Note that the solution of this Sturm–Liouville problem is

$$u_k(x) = \sin kx, \lambda_k = -k^2, k = 1,2,3, \ldots.$$

Therefore, we will seek the Fourier series solution $u(t,x)$ by the formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t)\sin kx. \tag{2.3}$$

Here $A_k(t), k = 1,2,3, \ldots$ are unknown functions. Putting (2.3) into the equation (2.2) and using given initial and boundary conditions, we obtain

$$\sum_{k=1}^{\infty} [A_k''(t) + k^2[A_k(t) + bA_k(t-\pi)]\sin kx$$
$$= p(t)\sin x - \sin t \sin x - b\sin t \sin x, 0 < t < \infty$$

and

$$\sum_{k=1}^{\infty} A_k(t)\sin kx = \sin t \sin x, -\pi \le t \le 0.$$

Equating coefficients of $\sin kx, k = 1,2,3, \ldots$ to zero,we get

$$\begin{cases} A_1''(t) + A_1(t) + bA_1(t-\pi) = p(t) - \sin t - b\sin t, k = 1, \\ A_k''(t) + k^2[A_k(t) + bA_k(t-\pi)] = 0, k \ne 1, 0 < t < \infty \end{cases}$$

and

$$\begin{cases} A_1(t) = \sin t, k = 1, \\ A_k(t) = 0, k \neq 1, -\pi \leq t \leq 0. \end{cases}$$

First, we obtain $A_k(t), k \neq 1$. It is clear that $A_k(t)$ be solution of the following IVP

$$\begin{cases} A_k''(t) + k^2 A_k(t) + bk^2 A_k(t-\pi) = 0, 0 < t < \infty, \\ A_k(t) = 0, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$A_k(t) = \{A_{k,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,3, \dots\},$$

where $A_{k,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A_{k,1}''(t) + k^2 A_{k,1}(t) = 0, 0 < t < \pi, \\ A_{k,1}(0) = 0, A_{k,1}'(0) = 0, \\ A_{k,m}''(t) + k^2 A_{k,m}(t) + bk^2 A_{k,m-1}(t-\pi) = 0, (m-1)\pi < t < m\pi, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. For obtaining $A_{k,1}(t)$, we will consider the auxilliary equation

$$q^2 + k^2 = 0.$$

We have that $q = \pm ki$. Therefore,

$$A_{k,1}(t) = c_1 \cos(kt) + c_2 \sin(kt).$$

Taking the derivative, we get

$$A_{k,1}'(t) = -kc_1 \sin(kt) + kc_2 \cos(kt).$$

Using initial conditions $A_{k,1}(0) = 0, A_{k,1}'(0) = 0$, we get

$$c_1 = 0, c_2 = 0.$$

Therefore,

$$A_{k,1}(t) = 0, 0 \leq t \leq \pi.$$

Now, suppose that

$$A_{k,m}(t) = 0, (m-1)\pi \leq t \leq \mathrm{m}\pi.$$

Then, $A_{k,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A_{k,m}''(t) + k^2 A_{k,m}(t) = 0, (m-1)\pi < t < m\pi, \\ A_{k,m}((m-1)\pi) = 0, A_{k,m}'((m-1)\pi) = 0, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. In the same manner, we can write

$$A_{k,m}(t) = c_1 \cos(k(t-(m-1)\pi)) + c_2 \sin(k(t-(m-1)\pi)).$$

Using initial conditions $A_{k,m}((m-1)\pi)=0, A'_{k,m}((m-1)\pi)=0$, we get

$$c_1=0, c_2=0.$$

Therefore,

$$A_{k,m}(t)=0, (m-1)\pi\le t\le m\pi.$$

Applying mathematical induction,

$$A_{k,m}(t)=0, m\pi\le t\le (m+1)\pi$$

is true for any $m\ge 1$. Thus,

$$A_k(t)=\{A_{k,m}(t), (m-1)\pi\le t\le m\pi, m=1,2,\dots\}=0 \tag{2.4}$$

for all $k\neq 1$. Applying formula (2.3) and condition $\int_0^\pi u(t,x)dx=2\sin t$, we get

$$\int_0^\pi u(t,x)dx=\sum_{k=1}^{\infty}\frac{2A_{2k-1}(t)}{2k-1}=2\sin t, 0\le t<\infty. \tag{2.5}$$

Second, we obtain $A_1(t)$. Applying formulas (2.4) and (2.5), we get

$$2A_1(t)=2\sin t.$$

Then, $A_1(t)=\sin t$.Thus,

$$u(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx=A_1(t)\sin x=\sin t\sin x.$$

Third, we obtain $p(t)$. It is clear that $A_1(t)$ be the solution of the following BVP

$$\begin{cases}A''_1(t)+A_1(t)+bA_1(t-\pi)=p(t)-\sin t-b\sin t, 0<t<\infty,\\ A_1(t)=\sin t, -\pi\le t\le 0\end{cases}$$

for the second order ordinary differential equation with time delay. Since $A_1(t)=\sin t$, we have that

$$p(t)=\sin t.$$

Therefore,

$$(u(t,x),p(t))=(\sin t\sin x,\sin t)$$

is the exact solution of the problem (2.2).

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} - \mathrm{b} \sum_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t-\pi,x)}{\partial x_r^2} \\ = p(t)q(x) + f(t,x), \\ 0 < t < \infty, x = (x_1, \cdots, x_n) \in \Omega, \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in \overline{\Omega}, \\ u(t,x) = 0, 1 \le r \le n, 0 \le t < \infty, x \in S, \\ \int \ldots \int_{x \in \overline{\Omega}} u(t,x) dx_1 \ldots dx_n = \zeta(t), t \ge 0 \end{cases} \tag{2.6}$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x), (t \in (0,\infty), x \in \Omega), g(t,x)(t \in [-\omega, 0], x \in \overline{\Omega})$ are given smooth functions. Here and in the future Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n (0 < x_k < 1, 1 \le k \le n)$ with the boundary S and $\overline{\Omega} = \Omega \cup S$.

Unfortunately, The Fourier series method described in solving (2.6) can be used only in the case when (2.6) has constant coefficients.

Problem 2. we consider the time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = b \dfrac{\partial^2 u(t-\omega,x)}{\partial x^2} + p(t)q(x) + f(t,x), \\ 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in [0,l] \\ u_x(t,0) = u_x(t,l) = 0, \int_0^l u(t,x) dx = \zeta(t), t \ge 0 \end{cases} \tag{2.7}$$

for a one dimensional delay hyperbolic equation with Neumann condition. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (2.7) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x)(t \in (0,\infty), x \in (0,l)), g(t,x), \zeta(t), q(x)$. Here b is a constant. Assume that $\int_0^l q(x)dx \ne 0$, and $q'(0) = q'(l) = 0$, and $g_x(t,0) = g_x(t,l) = 0, t \in [-\omega, 0], f_x(t,0) = f_x(t,l) = 0, t \in [0,\infty)$.

For example, we consider the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)(1 + \cos x) + bu_{xx}(t - \pi, x) \\ -\sin t(2 + \cos x) - b\sin t\cos x, t > 0, 0 < x < \pi, \\ u(t, x) = \sin t(1 + \cos x), -\pi \le t \le 0, 0 \le x \le \pi, \\ u_x(t, 0) = u_x(t, \pi) = 0, \int_0^\pi u(t, x)dx = \pi\sin t, t \ge 0 \end{cases} \quad (2.8)$$

for a one dimensional delay hyperbolic equation with Neumann condition.

Solution. For this case $\omega = \pi, l = \pi, g(t, x) = \sin t(1 + \cos x), -\pi \le t \le 0, 0 \le x \le \pi; f(t, x) = -\sin t(2 + \cos x) - b\sin t\cos x, t > 0, 0 < x < \pi, \zeta(t) = \pi\sin t, t \ge 0$. In order to solve problem (2.8), we consider the Sturm-Liouville problem

$$u''(x) - \lambda u(x) = 0, 0 < x < \pi, u'(0) = u'(\pi) = 0$$

generated by the space operator of problem (2.8). Note that the solution of this Sturm–Liouville problem is

$$u_k(x) = \cos kx, \lambda_k = -k^2, k = 0,1,2,3, \ldots.$$

Therefore, we will seek the Fourier series solution $u(t, x)$ by the formula

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t)\cos kx. \quad (2.9)$$

Here $A_k(t), k = 0,1,2, \ldots$ are unknown functions. Putting (2.9) into the equation (2.8) and using given initial and boundary conditions, we obtain

$$\sum_{k=0}^{\infty}[A_k''(t) + k^2[A_k(t) + bA_k(t - \pi)]]\cos kx$$
$$= p(t)(1 + \cos x) - \sin t(2 + \cos x) - b\sin t\cos x, t > 0$$

and

$$\sum_{k=0}^{\infty} A_k(t)\cos kx = \sin t(1 + \cos x), -\pi \le t \le 0.$$

Equating coefficients of $\cos kx, k = 0,1,2, \ldots$ to zero,we get

$$\begin{cases} A_1''(t) + A_1(t) + bA_1(t - \pi) = p(t) - \sin t - b\sin t, k = 1, \\ A_0''(t) = p(t) - 2\sin t, k = 0, \\ A_k''(t) + k^2[A_k(t) + bA_k(t - \pi)] = 0, k \ne 0,1, t > 0 \end{cases}$$

and

$$\begin{cases} A_1(t) = \sin t, k = 1, \\ A_0(t) = \sin t, k = 0, \\ A_k(t) = 0, k \neq 0,1, -\pi \leq t \leq 0. \end{cases}$$

First, we obtain $A_k(t), k \neq 0,1$. It is clear that $A_k(t)$ be solution of the following IVP

$$\begin{cases} A_k''(t) + k^2 A_k(t) + bk^2 A_k(t-\pi) = 0, t > 0, \\ A_k(t) = 0, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$A_k(t) = \{A_{k,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,3, \dots\},$$

where $A_{k,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following initial value problems

$$\begin{cases} A_{k,1}''(t) + k^2 A_{k,1}(t) = 0, 0 < t < \pi, \\ A_{k,1}(0) = 0, A_{k,1}'(0) = 0, \\ A_{k,m}''(t) + k^2 A_{k,m}(t) + bk^2 A_{k,m-1}(t-\pi) = 0, (m-1)\pi < t < m\pi, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. For obtaining $A_{k,1}(t)$, we will consider the auxilliary equation

$$q^2 + k^2 = 0.$$

We have that $q = \pm ki$. Therefore,

$$A_{k,1}(t) = c_1 \cos(kt) + c_2 \sin(kt).$$

Taking the derivative, we get

$$A_{k,1}'(t) = -kc_1 \sin(kt) + kc_2 \cos(kt).$$

Using the initial conditions $A_{k,1}(0) = 0, A_{k,1}'(0) = 0$, we get

$$c_1 = 0, c_2 = 0.$$

Therefore,

$$A_{k,1}(t) = 0, 0 \leq t \leq \pi.$$

Now, suppose that

$$A_{k,m}(t) = 0, (m-1)\pi \leq t \leq \mathrm{m}\pi.$$

Then, $A_{k,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A_{k,m}''(t) + k^2 A_{k,m}(t) = 0, (m-1)\pi < t < m\pi, \\ A_{k,m}((m-1)\pi) = 0, A_{k,m}'((m-1)\pi) = 0, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. In the same manner, we can write

$$A_{k,m}(t) = c_1 \cos(k(t-(m-1)\pi)) + c_2 \sin(k(t-(m-1)\pi)).$$

Using initial conditions $A_{k,m}((m-1)\pi) = 0, A'_{k,m}((m-1)\pi) = 0$, we get

$$c_1 = 0, c_2 = 0.$$

Therefore,

$$A_{k,m}(t) = 0, (m-1)\pi \leq t \leq m\pi.$$

Applying mathematical induction,

$$A_{k,m}(t) = 0, m\pi \leq t \leq (m+1)\pi$$

is true for any $m \geq 1$. Thus,

$$A_k(t) = \left\{A_{k,m}(t), (m-1)\pi \leq t \leq m\pi, m = 1,2, \ldots\right\} = 0 \tag{2.10}$$

for all $k \neq 0,1$.

Second, we obtain $A_0(t)$. Applying formula (2.9) and condition $\int_0^\pi u(t,x)dx = \pi \sin t$, we get

$$\int_0^\pi u(t,x)dx = \int_0^\pi \sum_{k=0}^{\infty} A_k(t) \cos kx = A_0(t)\pi = \pi \sin t, t \geq 0.$$

From that it follows that

$$A_0(t) = \sin t.$$

Third, we obtain $p(t)$. It is clear that $A_0(t)$ be the solution of the following BVP

$$\begin{cases} A_0''(t) = p(t) - 2\sin t, t > 0, \\ A_0(t) = \sin t, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. Since $A_0(t) = \sin t$, we have that

$$p(t) = \sin t.$$

Fourth, we obtain $A_1(t), k = 1$. It is clear that $A_1(t)$ be solution of the following IVP

$$\begin{cases} A_1''(t) + A_1(t) + bA_1(t-\pi) = -b\sin t, t > 0, \\ A_1(t) = \sin t, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$A_1(t) = \{A_{1,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,3, \ldots\},$$

where $A_{1,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A''_{1,1}(t) + A_{1,1}(t) = 0, 0 < t < \pi, \\ A_{1,1}(0) = 0, A'_{1,1}(0) = 1, \\ A''_{1,m}(t) + A_{1,m}(t) + bA_{1,m-1}(t-\pi) = -b\sin t, (m-1)\pi < t < m\pi, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. For obtaining $A_{1,1}(t)$, we will consider the auxilliary equation

$$q^2 + 1 = 0.$$

We have that $q = \pm i$. Therefore,

$$A_{1,1}(t) = c_1 \cos(t) + c_2 \sin(t).$$

Taking the derivative, we get

$$A'_{1,1}(t) = -c_1 \sin(t) + c_2 \cos(t).$$

Using the initial conditions $A_{1,1}(0) = 0, A'_{1,1}(0) = 1$, we get

$$c_1 = 0, c_2 = 1.$$

Therefore,

$$A_{1,1}(t) = \sin t, 0 \leq t \leq \pi.$$

Now, suppose that

$$A_{1,m}(t) = \sin t, (m-1)\pi \leq t \leq \mathrm{m}\pi.$$

Then, $A_{1,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A''_{1,m}(t) + A_{1,m}(t) = 0, (m-1)\pi < t < m\pi, \\ A_{1,m}((m-1)\pi) = 0, A'_{1,m}((m-1)\pi) = 1, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. In the same manner, we can write

$$A_{1,m}(t) = c_1 \cos(k(t-(m-1)\pi)) + c_2 \sin(k(t-(m-1)\pi)).$$

Using initial conditions $A_{1,m}((m-1)\pi) = 0, A'_{1,m}((m-1)\pi) = 1$, we get

$$c_1 = 0, c_2 = 1.$$

Therefore,

$$A_{1,m}(t) = \sin t, (m-1)\pi \leq t \leq m\pi.$$

Applying mathematical induction,

$$A_{1,m}(t) = \sin t, m\pi \leq t \leq (m+1)\pi$$

is true for any $m \geq 1$. Thus,

$$A_1(t) = \{A_{1,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,\dots\} = \sin t.$$

Therefore,

$$u(t,x)=\sum_{k=0}^{\infty}A_k(t)\cos kx=A_0(t)+A_1(t)\cos x=\sin t(1+\cos x).$$

Hence,

$$(u(t,x),p(t))=(\sin t(1+\cos x),\sin t)$$

is the exact solution of the problem (2.8).

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases}\dfrac{\partial^2 u(t,x)}{\partial t^2}-\sum_{r=1}^{n}\alpha_r\dfrac{\partial^2 u(t,x)}{\partial x_r^2}-\mathrm{b}\sum_{r=1}^{n}\alpha_r\dfrac{\partial^2 u(t-\pi,x)}{\partial x_r^2}\\ =p(t)q(x)+f(t,x),\\ 0<t<\infty, x=(x_1,\cdots,x_n)\in\Omega,\\ u(t,x)=g(t,x),-\omega\le t\le 0, x\in\overline{\Omega},\\ \dfrac{\partial u(t,x)}{\partial\overline{m}}=0, 1\le r\le n, 0\le t<\infty, x\in S,\\ \int\cdots\int_{x\in\overline{\Omega}}u(t,x)dx_1\ldots dx_n=\zeta(t), t\ge 0\end{cases}\tag{2.11}$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r>\alpha>0$ and $f(t,x),q(x),(t\in(0,\infty),x\in\Omega),g(t,x)(t\in[-\omega,0],x\in\overline{\Omega})$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to S. However, The Fourier series method described in solving (2.11) can be used only in the case when (2.11) has constant coefficients.

Problem 3. we consider the time-dependent identification problem

$$\begin{cases}\dfrac{\partial^2 u(t,x)}{\partial t^2}-\dfrac{\partial^2 u(t,x)}{\partial x^2}=b\dfrac{\partial^2 u(t-\omega,x)}{\partial x^2}+p(t)q(x)+f(t,x),\\ 0<t<\infty, x\in(0,l),\\ u(t,x)=g(t,x),-\omega\le t\le 0, x\in[0,l],\\ u(t,0)=u(t,l), u_x(t,0)=u_x(t,l),\\ \int_0^l u(t,x)dx=\zeta(t), t\ge 0\end{cases}\tag{2.12}$$

for a one dimensional delay hyperbolic equation with non-local condition. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (2.12) has a unique solution $(u(t,x),p(t))$ for the smooth functions $f(t,x)(t \in (0,\infty), x \in (0,l)), g(t,x), \zeta(t), q(x)$. Here b is a constant. Assume that $\int_0^l q(x)dx \neq 0$, and $q(0)=q(l), q'(0)=q'(l)$ and $g(t,0)=g(t,l), g_x(t,0)=g_x(t,l), t \in [-\omega,0], f(t,0)=f(t,l), f_x(t,0)=f_x(t,l), t \in [0,\infty)$.

For example, we consider the time-dependent identification problem

$$\begin{cases} u_{tt}-u_{xx}=p(t)(1+\cos 2x)+bu_{xx}(t-\pi,x) \\ -\sin 2t(5+\cos 2x)+4b\sin 2t\cos 2x, t>0, 0<x<\pi, \\ u(t,x)=\sin 2t(1+\cos 2x), -\pi \leq t \leq 0, 0 \leq x \leq \pi, \\ u(t,0)=u(t,\pi), u_x(t,0)=u_x(t,\pi), \\ \int\limits_0^\pi u(t,x)dx=\pi \sin 2t, t \geq 0 \end{cases} \tag{2.13}$$

for a one dimensional delay hyperbolic equation with non-local condition.

Solution. For this case $\omega=\pi, l=\pi, g(t,x)=\sin 2t(1+\cos 2x), -\pi \leq t \leq 0, 0 \leq x \leq \pi; f(t,x)=-\sin 2t(5+\cos 2x)+4b\sin 2t\cos 2x, t>0, 0<x<\pi, \zeta(t)=\pi\sin 2t, t \geq 0$. In order to solve problem (2.13), we consider the Sturm-Liouville problem

$$u''(x)-\lambda u(x)=0, 0<x<\pi, u(0)=u(\pi), u'(0)=u'(\pi)$$

generated by the space operator of problem (2.13). Note that the solution of this Sturm–Liouville problem is

$$u_k(x)=\cos 2kx, \lambda_k=-4k^2, k=0,1,2,3,\ldots,$$

and

$$u_k(x)=\sin 2kx, \lambda_k=-4k^2, k=1,2,3,\ldots.$$

Therefore, we will seek the Fourier series solution $u(t,x)$ by the formula

$$u(t,x)=\sum_{k=0}^{\infty} A_k(t)\cos 2kx+\sum_{k=1}^{\infty} B_k(t)\sin 2kx. \tag{2.14}$$

Here $A_k(t), k=0,1,2,\ldots$ and $B_k(t), k=1,2,..$ are unknown functions. Putting (2.14) into the equation (2.13) and using given initial and boundary conditions, we obtain

$$\sum_{k=0}^{\infty}[A_k''(t)+4k^2[A_k(t)+bA_k(t-\pi)]\cos 2kx$$
$$+\sum_{k=1}^{\infty}[B_k''(t)+4k^2[B_k(t)+bB_k(t-\pi)]]\sin 2kx$$
$$=p(t)(1+\cos 2x)-\sin 2t(5+\cos 2x)+4b\sin 2t\cos 2x, t>0$$

and

$$\sum_{k=0}^{\infty}A_k(t)\cos 2kx+\sum_{k=1}^{\infty}B_k(t)\sin 2kx=\sin 2t(1+\cos 2x), -\pi\le t\le 0.$$

Equating coefficients of $\cos 2kx, k=0,1,2,\ldots$ to zero,we get

$$\begin{cases} A_1''(t)+4A_1(t)+4bA_1(t-\pi)=p(t)-\sin 2t+4b\sin 2t, k=1, \\ A_0''(t)=p(t)-5\sin 2t, k=0, \\ A_k''(t)+4k^2[A_k(t)+bA_k(t-\pi)]=0, k\neq 0,1, t>0 \end{cases}$$

and

$$\begin{cases} A_1(t)=\sin 2t, k=1, \\ A_0(t)=\sin 2t, k=0, \\ A_k(t)=0, k\neq 0,1, -\pi\le t\le 0. \end{cases}$$

Also we have that

$$B_k''(t)+4k^2B_k(t)+4bk^2B_k(t-\pi)=0, t\ge 1$$

it is clear that $B_k(t)=0$ for $k\ge 1$.

First, we obtain $A_k(t), k\neq 0,1$. It is clear that $A_k(t)$ be solution of the following IVP

$$\begin{cases} A_k''(t)+4k^2A_k(t)+4bk^2A_k(t-\pi)=0, t>0, \\ A_k(t)=0, -\pi\le t\le 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$A_k(t)=\{A_{k,m}(t), (m-1)\pi\le t\le m\pi, m=0,1,2,3,\ldots\},$$

where $A_{k,m}(t), (m-1)\pi\le t\le m\pi$ be solutions of the following IVPs

$$\begin{cases} A_{k,1}''(t)+4k^2A_{k,1}(t)=0, 0<t<\pi, \\ A_{k,1}(0)=0, A_{k,1}'(0)=0, \\ A_{k,m}''(t)+4k^2A_{k,m}(t)+4bk^2A_{k,m-1}(t-\pi)=0, (m-1)\pi<t<m\pi, m\ge 2 \end{cases}$$

for the second order ordinary differential equation with time delay. For obtaining

$A_{k,1}(t)$, we will consider the auxilliary equation

$$q^2 + 4k^2 = 0.$$

We have that $q = \pm 2ki$. Therefore,

$$A_{k,1}(t) = c_1\cos(2kt) + c_2\sin(2kt).$$

Taking the derivative, we get

$$A'_{k,1}(t) = -2kc_1\sin(2kt) + 2kc_2\cos(2kt).$$

Using the initial conditions $A_{k,1}(0) = 0, A'_{k,1}(0) = 0$, we get

$$c_1 = 0, c_2 = 0.$$

Therefore,

$$A_{k,1}(t) = 0, 0 \leq t \leq \pi.$$

Now, suppose that

$$A_{k,m}(t) = 0, (m-1)\pi \leq t \leq \mathrm{m}\pi.$$

Then, $A_{k,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A''_{k,m}(t) + 4k^2 A_{k,m}(t) = 0, (m-1)\pi < t < m\pi, \\ A_{k,m}((m-1)\pi) = 0, A'_{k,m}((m-1)\pi) = 0, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. In the same manner, we can write

$$A_{k,m}(t) = c_1\cos(2k(t-(m-1)\pi)) + c_2\sin(2k(t-(m-1)\pi)).$$

Using initial conditions $A_{k,m}((m-1)\pi) = 0, A'_{k,m}((m-1)\pi) = 0$, we get

$$c_1 = 0, c_2 = 0.$$

Therefore,

$$A_{k,m}(t) = 0, (m-1)\pi \leq t \leq m\pi.$$

Applying mathematical induction,

$$A_{k,m}(t) = 0, m\pi \leq t \leq (m+1)\pi$$

is true for any $m \geq 1$. Thus,

$$A_k(t) = \{A_{k,m}(t), (m-1)\pi \leq t \leq m\pi, m = 1,2,\dots\} = 0 \qquad (2.15)$$

for all $k \neq 0,1$.

Second, we obtain $A_0(t)$. Applying formula (2.14) and condition $\int_0^\pi u(t,x)dx = \pi\sin 2t$, we get

$$\int_0^\pi u(t,x)dx = \int_0^\pi \left[\sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx\right] dx$$

$$= A_0(t)\pi + \sum_{k=1}^{\infty} \frac{A_k(t)\sin 2kx}{2k}\Bigg]_0^\pi - \sum_{k=1}^{\infty} \frac{\mathrm{B_k(t)cos2kx}}{\mathrm{2k}}\Bigg]_0^\pi$$

$$= A_0(t)\pi = \pi\sin 2t, t \geq 0.$$

From that it follows that

$$A_0(t) = \sin 2t.$$

Third, we obtain $p(t)$. It is clear that $A_0(t)$ be the solution of the following BVP

$$\begin{cases} A_0''(t) = p(t) - 5\sin 2t \ \ , t > 0, \\ A_0(t) = \sin 2t, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. Since $A_0(t) = \sin 2t,$ we have that

$$p(t) = \sin 2t.$$

Fourth, we obtain $A_1(t), k = 1$. It is clear that $A_1(t)$ be solution of the following IVP

$$\begin{cases} A_1''(t) + 4A_1(t) + 4bA_1(t-\pi) = 4b\sin 2t, t > 0, \\ A_1(t) = \sin 2t, -\pi \leq t \leq 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$A_1(t) = \{A_{1,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,3, \ldots\},$$

where $A_{1,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A_{1,1}''(t) + 4A_{1,1}(t) = 0, 0 < t < \pi, \\ A_{1,1}(0) = 0, A_{1,1}'(0) = 2, \\ A_{1,m}''(t) + 4A_{1,m}(t) + 4bA_{1,m-1}(t-\pi) = 4b\sin 2t, (m-1)\pi < t < m\pi, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. For obtaining $A_{1,1}(t)$, we will consider the auxilliary equation

$$q^2 + 4 = 0.$$

We have that $q = \pm 2i$. Therefore,

$$A_{1,1}(t) = c_1\cos(2t) + c_2\sin(2t).$$

Taking the derivative, we get

$$A_{1,1}'(t) = -2c_1\sin(2t) + 2c_2\cos(2t).$$

Using the initial conditions $A_{1,1}(0) = 0, A'_{1,1}(0) = 2$, we get

$$c_1 = 0, c_2 = 1.$$

Therefore,

$$A_{1,1}(t) = \sin 2t, 0 \leq t \leq \pi.$$

Now, suppose that

$$A_{1,m}(t) = \sin 2t, (m-1)\pi \leq t \leq \mathrm{m}\pi.$$

Then, $A_{1,m}(t), (m-1)\pi \leq t \leq m\pi$ be solutions of the following IVPs

$$\begin{cases} A''_{1,m}(t) + 4A_{1,m}(t) = 0, (m-1)\pi < t < m\pi, \\ A_{1,m}((m-1)\pi) = 0, A'_{1,m}((m-1)\pi) = 2, m \geq 2 \end{cases}$$

for the second order ordinary differential equation with time delay. In the same manner, we can write

$$A_{1,m}(t) = c_1 \cos(2k(t-(m-1)\pi)) + c_2 \sin(2k(t-(m-1)\pi)).$$

Using initial conditions $A_{1,m}((m-1)\pi) = 0, A'_{1,m}((m-1)\pi) = 2$, we get

$$c_1 = 0, c_2 = 1.$$

Therefore,

$$A_{1,m}(t) = \sin 2t, (m-1)\pi \leq t \leq m\pi.$$

Applying mathematical induction,

$$A_{1,m}(t) = \sin 2t, m\pi \leq t \leq (m+1)\pi$$

is true for any $m \geq 1$. Thus,

$$A_1(t) = \{A_{1,m}(t), (m-1)\pi \leq t \leq m\pi, m = 0,1,2,\ldots\} = \sin 2t.$$

Therefore,

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx$$

$$= A_0(t) + A_1(t)\cos 2x = \sin 2t(1+\cos 2x).$$

Hence,

$$(u(t,x), p(t)) = (\sin 2t(1+\cos 2x), \sin 2t)$$

is the exact solution of the problem (2.13).

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t,x)}{\partial x_r^2} - \mathrm{b} \sum_{r=1}^{n} \alpha_r \dfrac{\partial^2 u(t-\pi,x)}{\partial x_r^2} \\ = p(t)q(x) + f(t,x), \\ 0 < t < \infty, x = (x_1, \cdots, x_n) \in \Omega, \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \left.\dfrac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_1} = \left.\dfrac{\partial u(t,x)}{\partial \overline{m}}\right|_{S_2}, \\ \int \cdots \int_{x \in \overline{\Omega}} u(t,x) dx_1 \ldots dx_n = \zeta(t), t \ge 0 \end{cases} \tag{2.16}$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x), (t \in (0,\infty), x \in \Omega), g(t,x)(t \in [-\omega, 0], x \in \overline{\Omega})$ are given smooth functions. Here and in the future $S = S_1 \cup S_2, S_1 \cap S_2 = \varnothing, x \in S$. However, The Fourier series method described in solving (2.16) can be used only in the case when (2.16) has constant coefficients.

2.3 Laplace Transform Method

We consider the Laplace transform method for the solution of the time-dependent identification problem for delay hyperbolic equations.

Problem 4. Obtain the Laplace transform solution of the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)e^{-x} + bu_{xx}(t-\pi,x) \\ -3\sin(t)\, e^{-x} + b\sin(t)e^{-x}, t > 0, x > 0, \\ u(t,x) = \sin(t)\, e^{-x}, -\pi \le t \le 0, x \ge 0, \\ u(t,0) = \sin t, u_x(t,0) = -\sin t, t \ge 0, \\ \int_0^\infty u(t,x) dx = \sin t, t \ge 0 \end{cases} \tag{2.17}$$

for a one dimensional delay hyperbolic differential equation.

Solution. Here and in future, we will denote

$$L\{u(t,x)\} = u(t,s).$$

Using the formula

$$L\{e^{-x}\} = \frac{1}{s+1}$$

and taking the Laplace transform of both sides of the problem (2.17), we can write

$$\begin{cases} L\{u_{tt}(t,x)\} - L\{u_{xx}(t,x)\} - bL\{u_{xx}(t-\pi,x)\} \\ = p(t)L(e^{-x}\} - 3\text{sin}t \;\; L(e^{-x}\} + b\text{sin}t \;\; L(e^{-x}\}, \\ \;0 < t < \infty, \\ L\{u(t,x)\} = \text{sin}t \;\; L(e^{-x}\}, -\pi \le t \le 0. \end{cases}$$

Applying the definition of Laplace transform and initial conditions, $u(t,0) = \text{sin}t, u_x(t,0) = -\text{sin}t$, we can write

$$\begin{cases} u_{tt}(t,s) - s^2u(t,s) - bs^2u(t-\pi,s) = \text{sin}t - s\text{sin}t + bs\text{sin}t - b\text{sin}t \\ +p(t)\dfrac{1}{s+1} - 3\dfrac{1}{s+1}\text{sin}t \;\; + \dfrac{1}{s+1}b\text{sin}t, \\ u(t,s) = \dfrac{1}{s+1}\text{sin}t \;\; . \end{cases}$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following IVP

$$\begin{cases} u_{tt}(t,s) - s^2u(t,s) - bs^2u(t-\pi,s) = \text{sin}t - s\text{sin}t + bs\text{sin}t - b\text{sin}t \\ +p(t)\dfrac{1}{s+1} - 3\dfrac{1}{s+1}\text{sin}t \;\; + \dfrac{1}{s+1}b\text{sin}t, \\ u(t,s) = \dfrac{1}{s+1}\text{sin}t, -\pi \le t \le 0 \end{cases}$$

for the second order delay ordinary differential equation with time delay. We denote that

$$u(t,s) = \{u_m(t,s), (m-1)\pi \le t \le m\pi, m = 1,2,3, \dots\}.$$

Since

$$u_1(t-\pi,s) = -\frac{1}{s+1}\text{sin}t, -\pi \le t \le 0$$

we have that

$$\begin{cases} u_{1,tt}(t,s) - s^2u_1(t,s) - bs^2u_1(t-\pi,s) = \text{sin}t - s\text{sin}t + bs\text{sin}t - b\text{sin}t \\ +p(t)\dfrac{1}{s+1} - \dfrac{3}{s+1}\text{sin}t \;\; + \dfrac{1}{s+1}b\text{sin}t, 0 < t < \pi, \\ u_1(0,s) = 0, u_{1,t}(0,s) = \dfrac{1}{s+1}, -\pi \le t \le 0 \end{cases}$$

or

$$\begin{cases} u_{1,tt}(t,s) - s^2 u_1(t,s) = -bs^2 \text{sin}t \ \ \dfrac{1}{s+1} + \text{sin}t - s\text{sin}t + bs\text{sin}t - b\text{sin}t \\ +p(t)\dfrac{1}{s+1} - \dfrac{3}{s+1}\text{sin}t \ \ + \dfrac{1}{s+1} b\text{sin}t, 0 < t < \pi, \\ u_1(0,s) = 0, u_{1,t}(0,s) = \dfrac{1}{s+1}. \end{cases}$$

Taking the Laplace transform of both sides with respect to t, we get

$$\mu^2 u_1(\mu,s) - \mu u_1(0,s) - u_{1,t}(0,s) - s^2 u_1(\mu,s)$$
$$= -bs^2 \frac{1}{(\mu^2+1)(s+1)} + \frac{1}{\mu^2+1} - \frac{s}{\mu^2+1} + \frac{bs}{\mu^2+1} - \frac{b}{\mu^2+1}$$
$$+p(\mu)\frac{1}{s+1} - \frac{3}{(\mu^2+1)(s+1)} + \frac{b}{(\mu^2+1)(s+1)}$$

or

$$(\mu^2 - s^2)u_1(\mu,s) = \left(1 - \tfrac{bs^2}{\mu^2+1} + p(\mu) - \tfrac{3}{\mu^2+1} + \tfrac{b}{\mu^2+1}\right)\tfrac{1}{s+1} + \tfrac{1-s+bs-b}{\mu^2+1}. \quad (2.18)$$

Since $\int_0^\infty u(t,x)dx = \text{sin}t$ and by definition of Laplace transform, we get

$$L\{u(t,x)\} = \int_0^\infty e^{-sx} u(t,x)dx$$
$$u(t,s) = \int_0^\infty e^{-sx} u(t,x)dx$$

putting $s = 0$, we get

$$u(t,0) = \int_0^\infty u(t,x)dx = \text{sin}t.$$

Taking the Laplace transform of both sides with respect to t, we get

$$u(\mu,0) = \frac{1}{\mu^2+1}. \quad (2.19)$$

Putting $s = 0$ into equation (2.18), we get

$$\mu^2 u_1(\mu,0) = 1 - \frac{2}{\mu^2+1} + p(\mu)$$
$$u_1(\mu,0) = \frac{1}{\mu^2}\left[1 - \frac{2}{\mu^2+1} + p(\mu)\right] \quad (2.20)$$

From (2.19) and (2.20), we get

$$\frac{1}{\mu^2+1}=\frac{1}{\mu^2}\left[1-\frac{2}{\mu^2+1}+p(\mu)\right]$$

From that it follows that

$$p(\mu)=\frac{1}{\mu^2+1}.$$

Putting $p(\mu)=\frac{1}{\mu^2+1}$ into equation (2.18), we obtain $u_1(\mu,s)$, then

$$(\mu^2-s^2)u_1(\mu,s)=\frac{\mu^2-s^2-bs^2+bs^2}{(\mu^2+1)(s+1)}$$

or

$$u_1(\mu,s)=\frac{1}{(\mu^2+1)(s+1)}.$$

Therefore, we have that

$$u_1(\mu,s)=\frac{1}{(\mu^2+1)(s+1)},p(\mu)=\frac{1}{\mu^2+1}.$$

Now, taking the inverse Laplace transform with respect to t,we get

$$\begin{cases} u_1(t,s)=\frac{1}{s+1}\sin(t), 0\le t\le\pi, \\ p(t)=\sin(t). \end{cases}$$

Suppose that

$$u_{m-1}(t,s)=\frac{1}{s+1}\sin(t),(m-1)\pi\le t\le \mathrm{m}\pi.$$

Now, we obtain $u_m(t,s)$ as the solution of the following problem

$$\begin{cases} u_{m,tt}(t,s)-s^2u_m(t,s)-bs^2u_m(t-\pi,s) \\ =\sin t-s\sin t+bs\sin t-b\sin t \\ -\frac{2}{s+1}\sin t \ +\frac{1}{s+1}b\sin t,(m-1)\pi\le t\le m\pi, \\ u_m(t,s)=\frac{1}{s+1}\sin(t),(m-2)\pi\le t\le(m-1)\pi. \end{cases}$$

Since

$$u_m(t-\pi,s)=u_{m-1}(t-\pi,s)=-\frac{1}{s+1}\sin(t).$$

We have that

$$\begin{cases} u_{m,tt}(t,s) - s^2 u_m(t,s) = -\dfrac{1}{s+1} bs^2 \sin(t) \; + \sin(t) - s\sin(t) \\ + bs\sin(t) - b\sin(t) - \dfrac{2}{s+1}\sin(t) \; + \dfrac{1}{s+1} b\sin t, \\ (m-1)\pi \le t \le m\pi, \\ u_m(m\pi, s) = 0, u_{m,t}(m\pi, s) = \dfrac{1}{s+1}\cos(m\pi). \end{cases}$$

Therefore,

$$u_m(t,s) = \frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi.$$

Applying mathematical induction,

$$u_m(t,s) = \frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi$$

is true for any $m \ge 1$. Thus,

$$u(t,s) = \{\frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi, m = 1,2,3, \dots\} = \frac{1}{s+1}\sin(t).$$

Now, taking the inverse Laplace transform with respect to x,we get

$$u(t,x) = \sin(t)e^{-x}.$$

Therefore,

$$(u(t,x), p(t)) = (\sin(t)e^{-x}, \sin(t))$$

is the exact solution of the problem (2.17).

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - \mathrm{b} \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t-\pi,x)}{\partial x_r^2} \\ = p(t)q(x) + f(t,x), \\ 0 < t < \infty, x = (x_1, \cdots, x_n) \in \Omega^+ \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in \overline{\Omega}^+ \\ u(t,x) = \alpha(t,x), u_{x_r}(t,x) = \beta(t,x), \\ 1 \le r \le n, 0 \le t < \infty, x \in S^+, \\ \int \cdots \int_{x \in \overline{\Omega}} u(t,x) dx_1 \ldots dx_n = \zeta(t), t \ge 0 \end{cases} \quad (2.21)$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r > \alpha > 0$ and $f(t,x), q(x), (t \in (0,\infty), x \in \Omega^+), g(t,x)(t \in [-\omega, 0], x \in \overline{\Omega}^+), \alpha(t,x),$ $\beta(t,x), (t \in (0,\infty), x \in S^+,$ are given smooth functions. Here and in the future Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n (0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and $\overline{\Omega}^+ = \Omega^+ \cup S^+$.
Unfortunately, The Laplace transform method described in solving (2.21) can be used only in the case when (2.21) has constant coefficients.

Problem 5. Obtain the Laplace transform solution of the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)e^{-x} + b u_{xx}(t-\pi,x) \\ -3\sin(t)\, e^{-x} + b\sin(t) e^{-x}, t > 0, x > 0, \\ u(t,x) = \sin(t)\, e^{-x}, -\pi \le t \le 0, x \ge 0, \\ u_x(t,0) = -\sin t, u(t,\infty) = 0, t \ge 0, \\ \int_0^{\infty} u(t,x) dx = \sin t, t \ge 0 \end{cases} \quad (2.22)$$

for a one dimensional delay hyperbolic differential equation.

Solution. Here and in future, we will denote

$$L\{u(t,x)\} = u(t,s).$$

Using the formula

$$L\{e^{-x}\} = \frac{1}{s+1}$$

and taking the Laplace transform of both sides of the differential equation (2.22), we can write

$$\begin{cases} L\{u_{tt}(t,x)\} - L\{u_{xx}(t,x)\} - bL\{u_{xx}(t-\pi,x)\} \\ = p(t)L(e^{-x}\} - 3\text{sin}t \;\; L(e^{-x}\} + b\text{sin}t \;\; L(e^{-x}\}, \\ \;\; 0 < t < \infty, \\ L\{u(t,x)\} = \text{sin}t \;\; L(e^{-x}\}, -\pi \le t \le 0. \end{cases}$$

Applying the definition of Laplace transform and initial condition, $u_x(t,0) = -\text{sin}t$, and denoting $u(t,0) = \xi_1(t)$, we can write

$$\begin{cases} u_{tt}(t,s) - s^2u(t,s) - bs^2u(t-\pi,s) = \text{sin}t - b\text{sin}t - s\xi_1(t) - bs\xi_1(t-\pi) \\ +p(t)\frac{1}{s+1} - 3\frac{1}{s+1}\text{sin}t \;\; + \frac{1}{s+1}b\text{sin}t, \\ u(t,s) = \frac{1}{s+1}\text{sin}t, u(t,\infty) = 0. \end{cases}$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following IVP

$$\begin{cases} u_{tt}(t,s) - s^2u(t,s) - bs^2u(t-\pi,s) = \text{sin}t - b\text{sin}t - s\xi_1(t) \\ -bs\xi_1(t-\pi) + p(t)\frac{1}{s+1} - 3\frac{1}{s+1}\text{sin}t \;\; + \frac{1}{s+1}b\text{sin}t, \\ u(t,s) = \frac{1}{s+1}\text{sin}t, -\pi \le t \le 0, \\ u(t,\infty) = 0 \end{cases}$$

for the second order ordinary differential equation with time delay. We denote that

$$u(t,s) = \{u_m(t,s), (m-1)\pi \le t \le m\pi, m = 1,2,3,\ldots\}.$$

Since

$$u_1(t-\pi,s) = -\frac{1}{s+1}\text{sin}t, -\pi \le t \le 0.$$

We have that

$$\begin{cases} u_{1,tt}(t,s) - s^2 u_1(t,s) - bs^2 u_1(t-\pi,s) = \sin t - b\sin t - s\xi_1(t) - bs\xi_1(t-\pi) \\ +p(t)\dfrac{1}{s+1} - \dfrac{3}{s+1}\sin t \ \ + \dfrac{1}{s+1} b\sin t, 0 < t < \pi, \\ u_1(0,s) = 0, u_{1,t}(0,s) = \dfrac{1}{s+1}, -\pi \le t \le 0 \end{cases}$$

or

$$\begin{cases} u_{1,tt}(t,s) - s^2 u_1(t,s) = -bs^2 \sin t \ \ \dfrac{1}{s+1} + \sin t - b\sin t - s\xi_1(t) + bs\sin t \\ +p(t)\dfrac{1}{s+1} - \dfrac{3}{s+1}\sin t \ \ + \dfrac{1}{s+1} b\sin t, 0 < t < \pi, \\ u_1(0,s) = 0, u_{1,t}(0,s) = \dfrac{1}{s+1}, u_1(t,\infty) = 0. \end{cases}$$

Taking the Laplace transform of both sides with respect to t, we get

$$\mu^2 u_1(\mu,s) - \mu u_1(0,s) - u_{1,t}(0,s) - s^2 u_1(\mu,s)$$
$$= -bs^2 \frac{1}{(\mu^2+1)(s+1)} + \frac{1}{\mu^2+1} - \frac{b}{\mu^2+1} - s\xi_1(\mu) + \frac{bs}{\mu^2+1}$$
$$+p(\mu)\frac{1}{s+1} - \frac{3}{(\mu^2+1)(s+1)} + \frac{b}{(\mu^2+1)(s+1)}$$

or

$$(\mu^2 - s^2)u_1(\mu,s) = \left(1 - \frac{bs^2}{\mu^2+1} + p(\mu) - \frac{3}{\mu^2+1} + \frac{b}{\mu^2+1}\right)\frac{1}{s+1} \qquad (2.23)$$
$$+ \frac{1-b+bs}{\mu^2+1} - s\xi_1(\mu).$$

We know that $u(t,x) = \sin(t)e^{-x}$, then

$$u(t,0) = \sin t. \qquad (2.24)$$

Since

$$u(t,0) = \xi_1(t). \qquad (2.25)$$

From (2.24) and (2.25) we have that

$$\xi_1(t) = \sin t.$$

Taking the Laplace transform with respect to t, we obtain

$$\xi_1(\mu) = \frac{1}{\mu^2+1}. \qquad (2.26)$$

Now, putting (2.26) into (2.23), we get

$$(\mu^2 - s^2)u_1(\mu, s) = \left(1 - \frac{bs^2}{\mu^2+1} + p(\mu) - \frac{3}{\mu^2+1} + \frac{b}{\mu^2+1}\right)\frac{1}{s+1} + \frac{1-b+bs-s}{\mu^2+1}. \quad (2.27)$$

Since $\int_0^\infty u(t,x)dx = \sin t$ and by definition of Laplace transform, we get

$$L\{u(t,x)\} = \int_0^\infty e^{-sx}u(t,x)dx$$

$$u(t,s) = \int_0^\infty e^{-sx}u(t,x)dx$$

putting $s = 0$, we get

$$u(t,0) = \int_0^\infty u(t,x)dx = \sin t.$$

Taking the Laplace transform of both sides with respect to t, we get

$$u(\mu,0) = \frac{1}{\mu^2+1}. \quad (2.28)$$

Putting $s = 0$ into equation (2.27), we get

$$\mu^2 u_1(\mu,0) = 1 - \frac{2}{\mu^2+1} + p(\mu)$$

$$u_1(\mu,0) = \frac{1}{\mu^2}\left[1 - \frac{2}{\mu^2+1} + p(\mu)\right]. \quad (2.29)$$

From (2.28) and (2.29), we get

$$\frac{1}{\mu^2+1} = \frac{1}{\mu^2}\left[1 - \frac{2}{\mu^2+1} + p(\mu)\right]$$

From that it follows that

$$p(\mu) = \frac{1}{\mu^2+1}.$$

Putting $p(\mu) = \frac{1}{\mu^2+1}$ into equation (2.27), we obtain $u_1(\mu,s)$, then

$$(\mu^2 - s^2)u_1(\mu,s) = \frac{\mu^2 - s^2 - bs^2 + bs^2}{(\mu^2+1)(s+1)}$$

or

$$u_1(\mu,s) = \frac{1}{(\mu^2+1)(s+1)}.$$

Therefore, we have that

$$u_1(\mu, s) = \frac{1}{(\mu^2+1)(s+1)}, p(\mu) = \frac{1}{\mu^2+1}.$$

Now, taking the inverse Laplace transform with respect to t,we get

$$\begin{cases} u_1(t,s) = \frac{1}{s+1}\sin(t), 0 \le t \le \pi, \\ p(t) = \sin(t). \end{cases}$$

Suppose that

$$u_{m-1}(t,s) = \frac{1}{s+1}\sin(t), (m-2)\pi \le t \le (m-1)\pi.$$

Now, we obtain $u_m(t,s)$ as the solution of the following problem

$$\begin{cases} u_{m,tt}(t,s) - s^2 u_m(t,s) - bs^2 u_m(t-\pi,s) = \sin t - s\sin t + bs\sin t - b\sin t \\ -\frac{2}{s+1}\sin t + \frac{1}{s+1} b\sin t, (m-1)\pi \le t \le m\pi, \\ u_m(t,s) = \frac{1}{s+1}\sin(t), (m-2)\pi \le t \le (m-1)\pi. \end{cases}$$

Since

$$u_m(t-\pi,s) = u_{m-1}(t-\pi,s) = -\frac{1}{s+1}\sin(t).$$

We have that

$$\begin{cases} u_{m,tt}(t,s) - s^2 u_m(t,s) = -\frac{1}{s+1} bs^2 \sin(t) \\ +\sin(t) - s\sin(t) + bs\sin(t) - b\sin(t) \\ -\frac{2}{s+1}\sin(t) + \frac{1}{s+1} b\sin t, (m-1)\pi \le t \le m\pi, \\ u_m(m\pi,s) = 0, u_{m,t}(m\pi,s) = \frac{1}{s+1}\cos(m\pi). \end{cases}$$

Therefore,

$$u_m(t,s) = \frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi.$$

Applying mathematical induction,

$$u_m(t,s) = \frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi$$

is true for any $m \ge 1$. Thus,

$$u(t,s) = \{\frac{1}{s+1}\sin(t), (m-1)\pi \le t \le m\pi, m = 1,2,3,\dots\} = \frac{1}{s+1}\sin(t).$$

Now, taking the inverse Laplace transform with respect to x,we get

$$u(t,x)=\sin(t)e^{-x}$$

Therefore,

$$(u(t,x),p(t))=(\sin(t)e^{-x},\sin(t))$$

is the exact solution of the problem (2.22).

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases}\frac{\partial^2 u(t,x)}{\partial t^2}-\sum_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}-\mathrm{b}\sum_{r=1}^{n}\alpha_r\frac{\partial^2 u(t-\pi,x)}{\partial x_r^2}\\ =p(t)q(x)+f(t,x),\\ 0<t<\infty, x=(x_1,\cdots,x_n)\in\Omega^+,\\ u(t,x)=g(t,x),-\omega\le t\le 0, x\in\overline{\Omega}^+,\\ u(t,x)=\alpha(t,x),u_{x_r}(t,x)=\beta(t,x),\\ 1\le r\le n, 0\le t<\infty, x\in S^+,\\ \int\cdots\int_{x\in\overline{\Omega}} u(t,x)dx_1\ldots dx_n=\zeta(t), t\ge 0\end{cases} \tag{2.30}$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r>\alpha>0$ and $f(t,x),q(x),(t\in(0,\infty),x\in\Omega^+),g(t,x)(t\in[-\omega,0],x\in\overline{\Omega}^+),\alpha(t,x),\beta(t,x),(t\in(0,\infty),x\in S^+,$ are given smooth functions. Here and in the future Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n(0<x_k<\infty,1\le k\le n)$ with the boundary S^+ and $\overline{\Omega}^+=\Omega^+\cup S^+$.

Unfortunately, The Laplace transform method described in solving (2.30) can be used only in the case when (2.30) has constant coefficients.

3.4 Fourier Transform Method

We consider the Fourier transform method for the solution of the time-dependent identification problem for delay hyperbolic equations.

Problem 6. Obtain the Fourier transform solution of the time-dependent identification problem

$$\begin{cases} u_{tt}-u_{xx}=p(t)e^{-x^2}+bu_{xx}(t-\pi,x)-\sin(t)\,e^{-x^2} \\ -\sin(t)\,(4x^2-1)e^{-x^2}+b\sin(t)(4x^2-2)e^{-x^2}, t>0, x\in R^1, \\ u(t,x)=\sin(t)\,e^{-x^2}, -\pi\le t\le 0, x\in R^1, \\ \int_{-\infty}^{\infty} u(t,x)dx=\sqrt{\pi}\sin t, t\ge 0 \end{cases} \tag{2.31}$$

for a one dimensional delay hyperbolic differential equation.

Solution. Here and in the future, we will denote

$$F\{u(t,x)\}=u(t,s).$$

Taking the Fourier transform of both sides of the problem (2.31), we can write

$$F\{u_{tt}(t,x)\}-F\{u_{xx}(t,x)\}-bF\{u_{xx}(t-\pi,x)\}$$
$$=p(t)F\{e^{-x^2}\}-2\sin(t)\ F\{e^{-x^2}\}-\sin(t)F\{4x^2e^{-x^2}-e^{x^2}\}$$
$$+b\sin(t)\ F\{4x^2e^{-x^2}-2e^{x^2}\},0<t<\infty$$

and

$$F\{u(t,x)\}=\sin(t)\ F\{e^{-x^2}\}, -\pi\le t\le 0, x\in R^1.$$

Applying definition of Fourier transform. Therefore,

$$\begin{cases} u_{tt}(t,s)+s^2u(t,s)+bs^2u(t-\pi,s)=p(t)F\{e^{-x^2}\} \\ -2\sin(t)F\{e^{-x^2}\}+s^2\sin(t)F\{e^{-x^2}\}-bs^2\sin(t)F\{e^{-x^2}\}, t>0, \\ u(t,s)=\sin(t)F\{e^{-x^2}\}, -\pi\le t\le 0, x\in R^1. \end{cases}$$

Now, we obtain $u(t,s)$. It is clear that $u(t,s)$ is solution of the following IVP

$$\begin{cases} u_{tt}(t,s)+s^2u(t,s)+bs^2u(t-\pi,s)=p(t)F\{e^{-x^2}\} \\ -2\sin(t)F\{e^{-x^2}\}+s^2\sin(t)F\{e^{-x^2}\}-bs^2\sin(t)F\{e^{-x^2}\}, t>0, \\ u(t,s)=\sin(t)F\{e^{-x^2}\}, -\pi\le t\le 0 \end{cases}$$

for the second order ordinary differential equation with time delay, we denote that

$$u(t,s)=\{u_m(t,s),(m-1)\pi\le t\le m\pi, m=0,1,2,3,\dots\}.$$

Since, $u_1(t-\pi,s)=-\sin(t)F\{e^{-x^2}\}, -\pi\le t\le 0,$ therefore,

$$\begin{cases} u_{1,tt}(t,s) + s^2 u(t,s) = p(t)F\{e^{-x^2}\} - 2\sin(t)F\{e^{-x^2}\} \\ +s^2 \sin(t)F\{e^{-x^2}\},0 < t < \infty, \\ u_1(0,s) = 0, u_{1,t}(0,s) = F\{e^{-x^2}\}. \end{cases} \tag{2.32}$$

Now, taking the Laplace transform of both sides of the differential equation (2.32) with respect to t, we get

$$(\mu^2 + s^2)u_1(\mu,s) = F\{e^{-x^2}\} + (p(\mu) + \frac{s^2 - 2}{\mu^2 + 1})F\{e^{-x^2}\}.$$

Using formula

$$F\{e^{-x^2}\} = \sqrt{\pi}e^{-\frac{s^2}{4}}.$$

Then,

$$(\mu^2 + s^2)u_1(\mu,s) = \left(1 + p(\mu) + \frac{s^2 - 2}{\mu^2 + 1}\right)\sqrt{\pi}e^{-\frac{s^2}{4}} \tag{2.33}$$

putting $s = 0$ into equation (2.33), we get

$$\mu^2 u_1(\mu,0) = \left(1 + p(\mu) - \frac{2}{\mu^2 + 1}\right)\sqrt{\pi},$$

$$u_1(\mu,0) = \frac{\sqrt{\pi}}{\mu^2}\left(1 + p(\mu) - \frac{2}{\mu^2 + 1}\right). \tag{2.34}$$

Applying condition

$$\int_{-\infty}^{\infty} u(t,x)dx = \sqrt{\pi}\sin(t), t \geq 0$$

and the definition of Fourier transform, we get

$$u(t,0) = \int_{-\infty}^{\infty} u(t,x)dx = \sqrt{\pi}\sin(t), t \geq 0.$$

Taking the Laplace transform of both sides with respect to t, we get

$$u(\mu,0) = \frac{\sqrt{\pi}}{\mu^2 + 1}. \tag{2.35}$$

Therefore, using (2.34) and (2.35),we get

$$\frac{\sqrt{\pi}}{\mu^2 + 1} = \frac{\sqrt{\pi}}{\mu^2}\left(1 + p(\mu) - \frac{2}{\mu^2 + 1}\right).$$

From that it follows that

$$p(\mu) = \frac{1}{\mu^2 + 1}.$$

Putting $p(\mu) = \frac{1}{\mu^2+1}$ into equation (2.33), we get

$$(\mu^2 + s^2)u_1(\mu, s) = \left(1 + \frac{1}{\mu^2 + 1} + \frac{s^2 - 2}{\mu^2 + 1}\right)\sqrt{\pi}e^{-\frac{s^2}{4}}.$$

From that it follows that

$$u_1(\mu, s) = \frac{1}{\mu^2 + 1}\sqrt{\pi}e^{-\frac{s^2}{4}}.$$

Since,

$$\sqrt{\pi}e^{-\frac{s^2}{4}} = F\{e^{-x^2}\}.$$

Then,

$$u_1(\mu, s) = \frac{1}{\mu^2 + 1}F\{e^{-x^2}\}.$$

Now, taking the invers Laplace transform with respect to t, we obtain

$$u_1(t, s) = \sin(t)F\{e^{-x^2}\}, 0 \leq t \leq \pi.$$

Suppose that

$$u_{m-1}(t, s) = \sin(t)F\{e^{-x^2}\}, (m - 2)\pi \leq t \leq (m - 1)\pi.$$

Now, we obtain $u_m(t, s)$ as the solution of the following problem

$$\begin{cases} u_{m,tt}(t, s) + s^2u_m(t, s) + bs^2u_m(t - \pi, s) = -\sin(t)F\{e^{-x^2}\} \\ +s^2\sin(t)F\{e^{-x^2}\} - bs^2\sin(t)F\{e^{-x^2}\} + b\sin(t)F\{e^{-x^2}\}, t > 0, \\ u_m(t, s) = \sin(t)F\{e^{-x^2}\}, (m - 1)\pi \leq t \leq \mathrm{m}\pi. \end{cases}$$

Since, $u_m(t - \pi, s) = u_{m-1}(t - \pi, s) = -\sin(t)F\{e^{-x^2}\}$, we have that

$$\begin{cases} u_{m,tt}(t, s) + s^2u_m(t, s) = -\sin(t)F\{e^{-x^2}\} + s^2\sin(t)F\{e^{-x^2}\} \\ +b\sin(t)F\{e^{-x^2}\}, (m - 1)\pi \leq t \leq m\pi, \\ u_m(m\pi, s) = 0, u_{m,t}(m\pi, s) = \cos(m\pi)F\{e^{-x^2}\}. \end{cases}$$

Therefore,

$$u_m(t, s) = \sin(t)F\{e^{-x^2}\}, (m - 1)\pi \leq t \leq m\pi.$$

Applying mathematical induction,

$$u_m(t, s) = \sin(t)F\{e^{-x^2}\}, (m - 1)\pi \leq t \leq m\pi.$$

is true for any $m \geq 1$. Thus,

$$u(t, s) = \{\sin(t)F\{e^{-x^2}\}, (m - 1)\pi \leq t \leq m\pi, m = 1,2,3, \ldots\} = \sin(t)F\{e^{-x^2}\}.$$

Therefore,

$$u(t,s) = \sin(t)F\{e^{-x^2}\}.$$

Now, taking the inverse Fourier transform with respect to x, we obtain

$$u(t,x) = \sin(t)e^{-x^2}.$$

Therefore, the exact solution of the problem (2.31) is

$$(u(t,x), p(t)) = (\sin(t)e^{-x^2}, \sin(t)).$$

Note that using similar procedure one can obtain the solution of the following time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{|r|=2m} \alpha_r \dfrac{\partial^{|r|+1} u(t,x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} - \mathrm{b} \sum\limits_{|r|=2m} \alpha_r \dfrac{\partial^{|r|+1} u(t-\omega,x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \\ = p(t)q(x) + f(t,x), \\ 0 < t < \infty, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(t,x) = g(t,x), -\omega \leq t \leq 0, x \in \mathbb{R}^n, \\ \int \cdots \int\limits_{x \in \overline{\Omega}} u(t,x) dx_1 \dots dx_n = \zeta(t), t \geq 0 \end{cases} \tag{2.36}$$

for the multidimensional hyperbolic partial differential equation with a delay term. Assume that $\alpha_r \geq \alpha \geq 0$ and $f(t,x), q(x), (t \in (0,\infty), x \in \mathbb{R}^n), g(t,x)(t \in [-\omega, 0], x \in \mathbb{R}^n)$, are given smooth functions. However, The Fourier transform method described in solving (2.36) can be used only in the case when (2.36) has constant coefficients.

CHAPTER III

Stability of the Time-Dependent Identification Problem for Delay Hyperbolic Equations

3.1 Introduction

In the present section, two time-dependent identification problems for one dimensional delay hyperbolic equations are considered. The theorems on the stability estimates for the solution of these problems are established.

3.2 Basic Formulas

Two basic formulas are given.

3.2.1 *Dalambert's Formula (Wyley, Sons, 1993)*

$$u(t) = \cos(ct)\varphi + \frac{1}{c}\sin(ct)\psi + \int_0^t \frac{1}{c}\sin(c(t-y))f(y)dy$$

is the exact solution of the initial value problem

$$u_{tt}(t) + c^2u(t) = f(t), t > 0, u(0) = \varphi, u'(0) = \psi$$

for second order ordinary linear differential equation with constant coefficients

3.2.2 *Dalambert's Formula for Hyperbolic Equations (Dalambert, 1749)*

$$u(x,t) = \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi + \int_0^t \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,\xi)d\xi d\tau \quad (3.1)$$

is the exact solution of the initial value problem

$$\frac{\partial^2 u(t,x)}{\partial t^2} - c^2\mathrm{u}_{xx}(t,x) = f(t,x), t > 0,$$

$$u(0,x) = \varphi(x), u'(0,x) = \psi(x), x \in (-\infty,\infty)$$

for the one-dimensional wave equation with constant coefficients and initial conditions at $t = 0.$ It is named after the mathematician Jean le Rond d'Alembert, who derived it in 1747 as a solution to the problem of a vibrating string.

3.2.3 *Operator-Functions Generated by the Positive Operator.*

Let $c(t)$ is operator-function generated by the operator A and defined as the solution of the initial value problem for a second order differential equation

$$u_{tt}(t) + Au(t) = 0, 0 < t < \infty, u(0) = \varphi, u_t(0) = 0 \quad (3.2)$$

in a Hilbert space $H,$ that is

$$u(t) = c(t)\varphi.$$

Similarly, $s(t)$ is operator-function generated by the operator A and defined as the solution of the initial value problem for a second order differential equation

$$v_{tt}(t) + Av(t) = 0, 0 < t < \infty, v(0) = 0, v_t(0) = \psi \tag{3.3}$$

in a Hilbert space H, namely

$$v(t) = s(t)\psi.$$

By definitions of $c(t)$ and $s(t)$, we have that

$$s'(t) = c(t), c'(t) = -As(t). \tag{3.4}$$

We cosider the second order differential operator A determined by

$$Av = -\big(a(x)v_x(x)\big)_x \tag{3.5}$$

In $\mathbb{L}_2[0,l]$ with domain $\mathbb{D}(A) = \{v: v, v'' \in \mathbb{L}_2[0,l], v(0) = v(l) = 0\}$ dense in $\mathbb{L}_2[0,l]$. It is well-known that A is the positive-definite and self-adjoint operator in $\mathbb{L}_2[0,l]$. Let us give estimates (formula (3.6)) that will be needed below

$$\begin{cases} \| A^{-\frac{1}{2}} \|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le l^{-\frac{1}{2}}, \quad \| s(t) \|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le t, \\ \| c(t) \|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le 1, \quad \| A^{\frac{1}{2}}s(t) \|_{\mathbb{L}_2[0,l]\to\mathbb{L}_2[0,l]} \le 1. \end{cases} \tag{3.6}$$

3.3 Stability of the Time-Dependent Identification Problems.

First, the time-dependent identification problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = b\dfrac{\partial^2 u(t-\omega,x)}{\partial x^2} + p(t)q(x) + f(t,x), \\ 0 < t < \infty, x \in (-\infty,\infty), \\ u(t,x) = g(t,x), -\omega \le t \le 0, x \in (-\infty,\infty), \\ \displaystyle\int_{-\infty}^{\infty} \alpha(x)u(t,x)dx = \zeta(t), t \ge 0 \end{cases} \tag{3.7}$$

for one dimensional delay hyperbolic equation is considered. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (3.7) has a unique solution $(u(t,x), p(t))$ for the smooth functions $f(t,x)(t \in (0,\infty), x \in (-\infty,\infty)), g(t,x)(t \in [-\omega,0], x \in (-\infty,\infty)), \zeta(t)(t \ge 0), q(x), \alpha(x), x \in (-\infty,\infty)$. Here b is a constant.

We have the following theorems on the stability of problem (3.7).

Theorem 3.1. Assume that $\int_{-\infty}^{\infty}\alpha(x)q(x)dx \neq 0$ and $\int_{-\infty}^{\infty}|\alpha(x)|dx \leq \alpha < \infty$. Then for the solution of problem (3.7) the following stability estimate holds:

$$\max_{0\leq t\leq\omega}|p(t)|, \max_{0\leq t\leq\omega}\|u_{tt}\|_{C(-\infty,\infty)}, \max_{0\leq t\leq\omega}\|u_t\|_{C^{(1)}(-\infty,\infty)}, \max_{0\leq t\leq\omega}\|u\|_{C^{(2)}(-\infty,\infty)} \tag{3.8}$$

$$\leq M(q,\alpha)\left[a_0 + \max_{0\leq t\leq\omega}\|f'(t)\|_{C(-\infty,\infty)} + \|f(0)\|_{C(-\infty,\infty)} + \max_{0\leq t\leq\omega}|\zeta''|\right],$$

$$a_0 = \max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_{C(-\infty,\infty)}, \max_{-\omega\leq t\leq 0}\|g_t(t)\|_{C^{(1)}(-\infty,\infty)},\right.$$

$$\left.\max_{-\omega\leq t\leq 0}\|g(t)\|_{C^{(2)}(-\infty,\infty)}\right\},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}|p(t)|, \quad \max_{n\omega\leq t\leq(n+1)\omega}\|u_{tt}\|_{C(-\infty,\infty)}, \max_{n\omega\leq t\leq(n+1)\omega}\|u_t\|_{C^{(1)}(-\infty,\infty)}, \tag{3.9}$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|u\|_{C^{(2)}(-\infty,\infty)} \leq M(q,\alpha)\left[a_n + \max_{(n-1)\omega\leq t\leq n\omega}|p(t)|\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\|f'(t)\|_{C(-\infty,\infty)} + \|f(n\omega)\|_{C(-\infty,\infty)} + \max_{n\omega\leq t\leq(n+1)\omega}|\zeta''|\right],$$

$$a_n = \max\left\{\max_{(n-1)\omega\leq t\leq n\omega}\|u_{tt}(t)\|_{C(-\infty,\infty)}, \max_{(n-1)\omega\leq t\leq n\omega}\|u_t(t)\|_{C^{(1)}(-\infty,\infty)},\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega}\|u(t)\|_{C^{(2)}(-\infty,\infty)}\right\}, n = 1,2,\cdots.$$

Here $C(-\infty,\infty)$ refers to the vector space of continuous functions $w(x)$ from the entire real line to $R = (-\infty,\infty)$ with norm

$$\|w\|_{C(-\infty,\infty)} = \sup_{x\in(-\infty,\infty)}|w(x)|.$$

Proof. We will seek $u(t,x)$, using the substitution

$$u(t,x) = w(t,x) + \eta(t)q(x), \tag{3.10}$$

where $\eta(t)$ is the function defined by the formula

$$\eta(t) = \int_{(n-1)\omega}^{t}(t-s)p(s)ds, \eta\big((n-1)\omega\big) = \eta'\big((n-1)\omega\big) = 0, n = 1,\ldots. \tag{3.11}$$

It is easy to see that $w(t,x)$ is the solution of the problems

$$\begin{cases}\dfrac{\partial^2 w(t,x)}{\partial t^2} - \dfrac{\partial^2 w(t,x)}{\partial x^2} = \eta(t)q''(x) + bg_{xx}(t-\omega,x) + f(t,x), \\ 0 < t < \omega, x \in (-\infty,\infty), \\ w(0,x) = g(0,x), w_t(0,x) = g_t(0,x), x \in (-\infty,\infty),\end{cases} \tag{3.12}$$

and

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} - \dfrac{\partial^2 w(t,x)}{\partial x^2} = b\dfrac{\partial^2 w(t-\omega,x)}{\partial x^2} \\ +\big(\eta(t) + b\eta(t-\omega)\big)q''(x) + f(t,x), \\ (n-1)\omega < t < n\omega, x \in (-\infty,\infty), n = 2,3,\cdots, \\ w\big((n-1)\omega+,x\big) = w\big((n-1)\omega-,x\big), \\ w_t\big((n-1)\omega+,x\big) = w_t\big((n-1)\omega-,x\big), \\ x \in (-\infty,\infty), n = 2,3,\cdots. \end{cases} \tag{3.13}$$

Now we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition

$$\int_{-\infty}^{\infty} \alpha(x)u(t,x)dx = \zeta(t)$$

and substitution (3.10), we get

$$\eta(t) = \frac{\zeta(t) - \int_{-\infty}^{\infty} \alpha(x)w(t,x)dx}{\int_{-\infty}^{\infty} \alpha(x)q(x)dx}$$

From that and $p(t) = \eta''(t),$ it follows that

$$p(t) = \frac{\zeta''(t) - \int_{-\infty}^{\infty} \alpha(x)\frac{\partial^2}{\partial t^2}w(t,x)dx}{\int_{-\infty}^{\infty} \alpha(x)q(x)dx}.$$

Then, using the triangle inequality, we obtain

$$|p(t)| \le \frac{|\zeta''(t)| + \int_{-\infty}^{\infty} \left|\alpha(x)\frac{\partial^2}{\partial t^2}w(t,x)\right| dx}{\left|\int_{-\infty}^{\infty} \alpha(x)q(x)dx\right|} \tag{3.14}$$

$$\le k(q,\alpha)\left[|\zeta''(t)| + \left\|\frac{\partial^2}{\partial t^2}w(t,.)\right\|_{C(-\infty,\infty)}\right]$$

for all $t \in (0,\infty)$. Now, using substitution (3.10), we get

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 w(t,x)}{\partial t^2} + p(t)q(x).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u(t,\cdot)}{\partial t^2}\right\|_{C(-\infty,\infty)} \le \left\|\frac{\partial^2 w(t,\cdot)}{\partial t^2}\right\|_{C(-\infty,\infty)} + |p(t)|\|q\|_{C(-\infty,\infty)} \tag{3.15}$$

for all $t \in (0,\infty)$. Therefore, the proof of Theorem 3.1 is based on the following theorem.

Theorem 3.2. Under assumptions of Theorem 3.1, for the solution of problems (3.12) and (3.13) the following stability estimate holds:

$$\max_{0\le t\le\omega}\|w_{tt}\|_{C(-\infty,\infty)},\quad \max_{0\le t\le\omega}\|w_t\|_{C^{(1)}(-\infty,\infty)},\quad \max_{0\le t\le\omega}\|w\|_{C^{(2)}(-\infty,\infty)} \tag{3.16}$$

$$\le M(q,\alpha)\left[a_0+\max_{0\le t\le\omega}\|f'(t)\|_{C(-\infty,\infty)}+\|f(0)\|_{C(-\infty,\infty)}+\max_{0\le t\le\omega}|\zeta''|\right],$$

$$a_0=\max\left\{\max_{-\omega\le t\le 0}\|g_{tt}(t)\|_{C(-\infty,\infty)},\max_{-\omega\le t\le 0}\|g_t(t)\|_{C^{(1)}(-\infty,\infty)},\right.$$

$$\left.\max_{-\omega\le t\le 0}\|g(t)\|_{C^{(2)}(-\infty,\infty)}\right\},$$

$$\max_{n\omega\le t\le(n+1)\omega}\|w_{tt}\|_{C(-\infty,\infty)},\max_{n\omega\le t\le(n+1)\omega}\|w_t\|_{C^{(1)}(-\infty,\infty)},\max_{n\omega\le t\le(n+1)\omega}\|w\|_{C^{(2)}(-\infty,\infty)} \tag{3.17}$$

$$\le M(q,\alpha)\left[a_n+\max_{n\omega\le t\le(n+1)\omega}\|f'(t)\|_{C(-\infty,\infty)}+\|f(n\omega)\|_{C(-\infty,\infty)}\right.$$

$$\left.+\max_{n\omega\le t\le(n+1)\omega}|\zeta''|\right]$$

$$a_n=\max\left\{\max_{(n-1)\omega\le t\le n\omega}\|w_{tt}(t)\|_{C(-\infty,\infty)},\max_{(n-1)\omega\le t\le n\omega}\|w_t(t)\|_{C^{(1)}(-\infty,\infty)},\right.$$

$$\left.\max_{(n-1)\omega\le t\le n\omega}\|w(t)\|_{C^{(2)}(-\infty,\infty)}\right\},n=1,2,\cdots.$$

Proof. First, we will prove that

$$\max_{0\le t\le\omega}\|w_{tt}\|_{C(-\infty,\infty)} \tag{3.18}$$

$$\le M(q,\alpha)\left[a_0+\max_{0\le t\le\omega}\|f'(t)\|_{C(-\infty,\infty)}+\|f(0)\|_{C(-\infty,\infty)}+\max_{0\le t\le\omega}|\zeta''|\right].$$

Applying the Dalambert's formula(3.1), we get the following formula

$$w(t,x)=\frac{g(0,x+t)+g(0,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g_t(0,\xi)d\xi$$

$$+\int_0^t\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}\left[\eta(\tau)q''(\xi)+bg_{\xi\xi}(\tau-\omega,\xi)+f(\tau,\xi)\right]d\xi d\tau$$

for any $t\in[0,\omega],x\in(-\infty,\infty)$. From that it follows that

$$w(t,x)=\frac{g(0,x+t)+g(0,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g_t(0,\xi)d\xi$$

$$+\int_0^t\frac{\eta(\tau)}{2}\left[q_{x+(t-\tau)}(x+(t-\tau))-q_{x-(t-\tau)}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{b}{2}\left[g_{x+(t-\tau)}(\tau-\omega,x+(t-\tau))-g_{x-(t-\tau)}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)d\xi d\tau.$$

Taking the derivatives, we get

$$w_t(t,x)=\frac{g_t(0,x+t)+g_t(0,x-t)}{2}+\frac{1}{2}[g_t(0,x+t)-g_t(0,x-t)]$$

$$+\int_0^t \frac{\eta(\tau)}{2}\left[q_{x+(t-\tau),t}(x+(t-\tau))-q_{x-(t-\tau),t}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{b}{2}\left[g_{x+(t-\tau),t}(\tau-\omega,x+(t-\tau))-g_{x-(t-\tau),t}(\tau-\omega,x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{1}{2}[f(\tau,x+(t-\tau))-f(\tau,x-(t-\tau))]d\tau,$$

$$w_{tt}(t,x)=\frac{g_{tt}(0,x+t)+g_{tt}(0,x-t)}{2}+\frac{1}{2}[g_{tt}(0,x+t)-g_{tt}(0,x-t)]$$

$$+\int_0^t \frac{\eta(\tau)}{2}\left[q_{x+(t-\tau),tt}(x+(t-\tau))-q_{x-(t-\tau),tt}(x-(t-\tau))\right]d\tau$$

$$+\int_0^t \frac{b}{2}[g_{tt}(-\omega,x+t)-g_{tt}(-\omega,x-t)]d\tau$$

$$+\int_0^t \frac{1}{2}[f_t(\tau,x+(t-\tau))-f_t(\tau,x-(t-\tau))]d\tau.$$

Applying this formula and the triangle inequality and estimate (3.14), we get

$$\|w_{tt}(t,\cdot)\|\le M(q,\alpha)\left[a_0+\max_{0\le t\le\omega}\|f'(t)\|_{C(-\infty,\infty)}+\|f(0)\|_{C(-\infty,\infty)}+|\zeta''(t)|\right]$$

$$+M(\mathrm{q})\int_0^t \|w_{\tau\tau}(\tau,\cdot)\|d\tau$$

for any $t\in[0,\omega]$. By the integral inequality, we get the estimate (3.18). Applying equation (3.12) and triangle inequality and estimate (3.18), we get estimate (3.16).

Second, we will prove that

$$\max_{n\omega\le t\le(n+1)\omega}\left\|\frac{\partial^2 w(t,\cdot)}{\partial t^2}\right\|_{C(-\infty,\infty)} \tag{3.19}$$

$$\le M(q,\alpha)\Big[a_n+\max_{(n-1)\omega\le t\le n\omega}|p(t)|+\max_{n\omega\le t\le(n+1)\omega}\|f'(t)\|_{C(-\infty,\infty)}$$

$$+\|f(\mathrm{n}\omega)\|_{C(-\infty,\infty)}+\max_{n\omega\le t\le(n+1)\omega}|\zeta''|\Big],n=1,2,\cdots$$

Applying the Dalambert's formula(3.1), we get the following formula

$$w(t,x)=\frac{w(\mathrm{n}\omega,x+t)+w(\mathrm{n}\omega,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}w_t(\mathrm{n}\omega,\xi)d\xi$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}\big[(\eta(\tau)+b\eta(\tau-\omega))q''(\xi)+bw_{\xi\xi}(\tau-\omega,\xi)+f(\tau,\xi)\big]d\xi d\tau.$$

for any $t\in[\mathrm{n}\omega,(n+1)\omega],x\in(-\infty,\infty)$. From that it follows that

$$w(t,x)=\frac{w(\mathrm{n}\omega,x+t)+w(\mathrm{n}\omega,x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}w_t(\mathrm{n}\omega,\xi)d\xi$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{(\eta(\tau)+b\eta(\tau-\omega))}{2}\big[q_{x+(t-\tau)}(x+(t-\tau))-q_{x-(t-\tau)}(x-(t-\tau))\big]d\tau$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{b}{2}\big[w_{x+(t-\tau)}(\tau-\omega,x+(t-\tau))-w_{x-(t-\tau)}(\tau-\omega,x-(t-\tau))\big]d\tau$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}f(\tau,\xi)d\xi d\tau.$$

Taking the derivatives, we get

$$w_t(t,x)=\frac{w_t(\mathrm{n}\omega,x+t)+w_t(\mathrm{n}\omega,x-t)}{2}$$

$$+\frac{1}{2}[w_t(\mathrm{n}\omega,x+t)-w_t(\mathrm{n}\omega,x-t)]$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{(\eta(\tau)+b\eta(\tau-\omega))}{2}\big[q_{x+(t-\tau),t}(x+(t-\tau))-q_{x-(t-\tau),t}(x-(t-\tau))\big]d\tau$$

$$+\int_{\mathrm{n}\omega}^{t}\frac{b}{2}\big[w_{x+(t-\tau),t}(\tau-\omega,x+(t-\tau))-w_{x-(t-\tau),t}(\tau-\omega,x-(t-\tau))\big]d\tau$$

$$+\int_{n\omega}^{t}\frac{1}{2}[f(\tau,x+(t-\tau))-f(\tau,x-(t-\tau))]d\tau,$$

$$w_{tt}(t,x)=\frac{w_{tt}(n\omega,x+t)+w_{tt}(n\omega,x-t)}{2}$$

$$+\frac{1}{2}[w_{tt}(n\omega,x+t)-w_{tt}(n\omega,x-t)]$$

$$+\int_{n\omega}^{t}\frac{(\eta(\tau)+b\eta(\tau-\omega))}{2}\left[q_{x+(t-\tau),tt}(x+(t-\tau))-q_{x-(t-\tau),tt}(x-(t-\tau))\right]d\tau$$

$$+\int_{n\omega}^{t}\frac{b}{2}[w_{tt}(-\omega,x+t)-w_{tt}(-\omega,x-t)]d\tau$$

$$+\int_{n\omega}^{t}\frac{1}{2}[f_t(\tau,x+(t-\tau))-f_t(\tau,x-(t-\tau))]d\tau.$$

Applying this formula and the triangle inequality and estimate (3.14), we get

$$\|w_{tt}(t,\cdot)\|\leq M(q,\alpha)\left[a_n+\max_{(n-1)\omega\leq t\leq n\omega}|p(t)|\right.$$

$$\left.+\max_{n\omega\leq t\leq(n+1)\omega}\|f'(t)\|_{C(-\infty,\infty)}+\|f(n\omega)\|_{C(-\infty,\infty)}+\max_{n\omega\leq t\leq(n+1)\omega}|\zeta''|\right]$$

$$+M(q)\int_{n\omega}^{t}\|w_{\tau\tau}(\tau,\cdot)\|\,d\tau$$

for any $t\in[n\omega,(n+1)\omega]$. By the integral inequality, we get the estimate (3.16). Applying equation (3.13) and triangle inequality and estimate (3.16), we get estimate (3.17). This completes the proof of Theorem 3.2.

Moreover, we have that

Theorem 3.3. Assume that $\int_{-\infty}^{\infty}\alpha(x)q(x)dx\neq 0$ and $\int_{-\infty}^{\infty}|\alpha(x)|^q dx\leq\alpha<\infty, 1\leq q<\infty, \frac{1}{q}+\frac{1}{p}=1$. Then for the solution of problem (3.7) the following stability estimate holds:

$$\max_{0\leq t\leq\omega}|p(t)|, \max_{0\leq t\leq\omega}\|u_{tt}\|_{L_p(-\infty,\infty)}, \max_{0\leq t\leq\omega}\|u_t\|_{W_p^1(-\infty,\infty)}, \max_{0\leq t\leq\omega}\|u\|_{W_p^2(-\infty,\infty)}$$

$$\leq M(q,\alpha)\left[a_0+\max_{0\leq t\leq\omega}\|f'(t)\|_{L_p(-\infty,\infty)}+\|f(0)\|_{L_p(-\infty,\infty)}+\max_{0\leq t\leq\omega}|\zeta''|\right],$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_{L_p(-\infty,\infty)}, \max_{-\omega\leq t\leq 0}\|g_t(t)\|_{W_p^1(-\infty,\infty)},\right.$$

$$\left.\max_{-\omega\leq t\leq 0}\|g(t)\|_{W_p^2(-\infty,\infty)}\right\},$$

$$\max_{n\omega\le t\le(n+1)\omega}|p(t)|,\quad \max_{n\omega\le t\le(n+1)\omega}\|u_{tt}\|_{L_p(-\infty,\infty)}, \max_{n\omega\le t\le(n+1)\omega}\|u_t\|_{W_p^1(-\infty,\infty)},$$

$$\max_{n\omega\le t\le(n+1)\omega}\|u\|_{W_p^2(-\infty,\infty)} \le M(q,\alpha)\Big[a_n + \max_{(n-1)\omega\le t\le n\omega}|p(t)|$$

$$+\max_{n\omega\le t\le(n+1)\omega}\|f'(t)\|_{L_p(-\infty,\infty)} + \|f(\mathrm{n}\omega)\|_{L_p(-\infty,\infty)} + \max_{n\omega\le t\le(n+1)\omega}|\zeta''|\Big],$$

$$a_n = \max\Big\{\max_{(n-1)\omega\le t\le n\omega}\|u_{tt}(t)\|_{L_p(-\infty,\infty)}, \max_{(n-1)\omega\le t\le n\omega}\|u_t(t)\|_{W_p^1(-\infty,\infty)},$$

$$\max_{(n-1)\omega\le t\le n\omega}\|u(t)\|_{W_p^2(-\infty,\infty)}\Big\}, n = 1,2,\cdots.$$

Here $L_p(-\infty,\infty)$ refers to the vector space of functions $w(x)$ from the entire real line to $R=(-\infty,\infty)$ satisfy the condition

$$\int_{-\infty}^{\infty}|w(x)|^p dx < \infty.$$

Second, the time-dependent identification problem

$$\begin{cases}\dfrac{\partial^2 u(t,x)}{\partial t^2} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = b\dfrac{\partial^2 u(t-\omega,x)}{\partial x^2} + p(t)q(x) + f(t,x),\\ 0<t<\infty, x\in(0,l),\\ u(t,x) = g(t,x), -\omega\le t\le 0, x\in[0,l],\\ u(t,0)=u(t,l)=0, t\ge 0,\\ \displaystyle\int_0^l u(t,x)dx = \zeta(t), t\ge 0\end{cases} \tag{3.20}$$

for one dimensional delay hyperbolic equation is considered. Here $u(t,x)$ and $p(t)$ are unknown functions. Under compatibility conditions, problem (3.20) has a unique solution $(u(t,x),p(t))$ for the smooth functions $f(t,x)(t\in(0,\infty),x\in(0,l)), g(t,x)(t\in[-\omega,0],x\in[0,l]), \zeta(t)(t\ge 0), q(x), x\in(0,l)$. Here b is a constant.

We have the following theorem on the stability of problem (3.20).

Theorem 3.4. Assume that $\int_0^l q(x)dx \neq 0$. Then for the solution of problem (3.20) the following stability estimate holds:

$$\max_{0\le t\le\omega}|p(t)|,\quad \max_{0\le t\le\omega}\|u_{tt}\|_{\mathbb{L}_2[0,l]}, \max_{0\le t\le\omega}\|u_t\|_{\mathbb{W}_2^1[0,l]},\quad \max_{0\le t\le\omega}\|u\|_{\mathbb{W}_2^2[0,l]} \tag{3.21}$$

$$\le M(q,\alpha)\Big[a_0 + \max_{0\le t\le\omega}\|f'(t)\|_{\mathbb{L}_2[0,l]} + \|f(0)\|_{\mathbb{L}_2[0,l]} + \max_{0\le t\le\omega}|\zeta''|\Big]$$

$$a_0 = \max\left\{\max_{-\omega\le t\le 0}\|g_{tt}(t)\|_{\mathbb{L}_2[0,l]}, \max_{-\omega\le t\le 0}\|g_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$
$$\left.\max_{-\omega\le t\le 0}\|g(t)\|_{\mathbb{W}_2^2[0,l]}\right\},$$

$$\max_{n\omega\le t\le(n+1)\omega}|p(t)|,\quad \max_{n\omega\le t\le(n+1)\omega}\|u_{tt}\|_{\mathbb{L}_2[0,l]}, \max_{n\omega\le t\le(n+1)\omega}\|u_t\|_{\mathbb{W}_2^1[0,l]}, \tag{3.22}$$
$$\max_{n\omega\le t\le(n+1)\omega}\|u\|_{\mathbb{W}_2^2[0,l]} \le M(q,\alpha)\left[a_n + \max_{(n-1)\omega\le t\le n\omega}|p(t)|\right.$$
$$\left.+\max_{n\omega\le t\le(n+1)\omega}\|f'(t)\|_{\mathbb{L}_2[0,l]} + \|f(n\omega)\|_{\mathbb{L}_2[0,l]} + \max_{n\omega\le t\le(n+1)\omega}|\zeta''|\right],$$
$$a_n = \max\left\{\max_{(n-1)\omega\le t\le n\omega}\|u_{tt}(t)\|_{\mathbb{L}_2[0,l]}, \max_{(n-1)\omega\le t\le n\omega}\|u_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$
$$\left.\max_{(n-1)\omega\le t\le n\omega}\|u(t)\|_{\mathbb{W}_2^2[0,l]}\right\}, n = 1,2,\cdots.$$

Here $\mathbb{L}_2[0,l]$ be the space of all square integrable functions $w(x)$ defined on $[0,l]$ and $\mathbb{W}_2^k[0,l], k = 1,2$ be Sobolev spaces equipped with norms

$$\|w\|_{\mathbb{W}_2^1[0,l]} = \left(\int_0^l [w^2(z) + w_z^2(z)]dz\right)^{\frac{1}{2}},$$

$$\|w\|_{\mathbb{W}_2^2[0,l]} = \left(\int_0^l [w^2(z) + w_{zz}^2(z)]dz\right)^{\frac{1}{2}},$$

respectively.

Proof. We will seek $u(t,x)$, using the substitution

$$u(t,x) = w(t,x) + \eta(t)q(x), \tag{3.23}$$

where $\eta(t)$ is the function defined by the formula

$$\begin{cases}\eta(t) = \displaystyle\int_{(n-1)\omega}^{t}(t-s)p(s)ds,\\ \eta\big((n-1)\omega\big) = \eta'\big((n-1)\omega\big) = 0, n = 1,2,\ldots.\end{cases} \tag{3.24}$$

It is easy to see that $w(t,\ x)$ is the solution of the problems

$$\begin{cases}\dfrac{\partial^2 w(t,x)}{\partial t^2} - \dfrac{\partial^2 w(t,x)}{\partial x^2} = \eta(t)q''(x) + bg_{xx}(t-\omega,x) + f(t,x),\\ 0 < t < \omega, x \in (0,l),\\ w(0,x) = g(0,x), w_t(0,x) = g_t(0,x), x \in (0,l),\\ w(t,0) = w(t,l) = 0, t \ge 0\end{cases} \tag{3.25}$$

and

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} - \dfrac{\partial^2 w(t,x)}{\partial x^2} = b\dfrac{\partial^2 w(t-\omega,x)}{\partial x^2} \\ +\big(\eta(t) + b\eta(t-\omega)\big)q''(x) + f(t,x), \\ (n-1)\omega < t < n\omega, x \in (0,l), n = 2,3,\cdots, \\ w\big((n-1)\omega+,x\big) = w\big((n-1)\omega-,x\big), \\ w_t\big((n-1)\omega+,x\big) = w_t\big((n-1)\omega-,x\big), \\ x \in (0,l), n = 2,3,\cdots, \\ w(t,0) = w(t,l) = 0, t \geq 0. \end{cases} \tag{3.26}$$

Now we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition

$$\int_0^l u(t,x)dx = \zeta(t)$$

and substitution (3.23), we get

$$\eta(t) = \frac{\zeta(t) - \int_0^l w(t,x)dx}{\int_0^l q(x)dx}.$$

From that and $p(t) = \eta''(t)$, it follows that

$$p(t) = \frac{\zeta''(t) - \int_0^l \frac{\partial^2}{\partial t^2} w(t,x)dx}{\int_0^l q(x)dx}.$$

Then, using the triangle inequality, we obtain

$$|p(t)| \leq \frac{|\zeta''(t)| + \int_0^l \left|\frac{\partial^2}{\partial t^2} w(t,x)\right| dx}{\left|\int_0^l q(x)dx\right|} \tag{3.27}$$

$$\leq k(q,l)\left[|\zeta''(t)| + \left\|\frac{\partial^2}{\partial t^2} w(t,.)\right\|_{\mathbb{L}_2[0,l]}\right]$$

for all $t \in (0,\infty)$. Now, using substitution (3.23), we get

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 w(t,x)}{\partial t^2} + p(t)q(x).$$

Applying the triangle inequality, we obtain

$$\left\|\frac{\partial^2 u(t,\cdot)}{\partial t^2}\right\|_{\mathbb{L}_2[0,l]} \leq \left\|\frac{\partial^2 w(t,\cdot)}{\partial t^2}\right\|_{\mathbb{L}_2[0,l]} + |p(t)|\|q\|_{\mathbb{L}_2[0,l]} \tag{3.28}$$

for all $t \in (0,\infty)$. Therefore, the proof of Theorem 3.4 is based on the following theorem.

Theorem 3.5. Under assumptions of Theorem 3.4, for the solution of problems (3.25) and (3.26) the following stability estimate holds:

$$\max_{0\le t\le \omega}\|w_{tt}\|_{\mathbb{L}_2[0,l]},\quad \max_{0\le t\le \omega}\|w_t\|_{\mathbb{W}_2^1[0,l]},\quad \max_{0\le t\le \omega}\|w\|_{\mathbb{W}_2^2[0,l]} \tag{3.29}$$
$$\le M(q,l)\left[a_0+\max_{0\le t\le \omega}\|f'(t)\|_{\mathbb{L}_2[0,l]}+\|f(0)\|_{\mathbb{L}_2[0,l]}+\max_{0\le t\le \omega}|\zeta''|\right],$$
$$a_0=\max\left\{\max_{-\omega\le t\le 0}\|g_{tt}(t)\|_{\mathbb{L}_2[0,l]},\ \max_{-\omega\le t\le 0}\|g_t(t)\|_{\mathbb{W}_2^1[0,l]},\ \max_{-\omega\le t\le 0}\|g(t)\|_{\mathbb{W}_2^2[0,l]}\right\},$$

$$\max_{n\omega\le t\le (n+1)\omega}\|w_{tt}\|_{\mathbb{L}_2[0,l]},\quad \max_{n\omega\le t\le (n+1)\omega}\|w_t\|_{\mathbb{W}_2^1[0,l]},\ \max_{n\omega\le t\le (n+1)\omega}\|w\|_{\mathbb{W}_2^2[0,l]} \tag{3.30}$$
$$\le M(q,l)\left[a_n+\max_{n\omega\le t\le (n+1)\omega}\|f'(t)\|_{\mathbb{L}_2[0,l]}+\|f(\mathrm{n\omega})\|_{\mathbb{L}_2[0,l]}+\max_{n\omega\le t\le (n+1)\omega}|\zeta''|\right],$$
$$a_n=\max\left\{\max_{(n-1)\omega\le t\le n\omega}\|w_{tt}(t)\|_{\mathbb{L}_2[0,l]},\ \max_{(n-1)\omega\le t\le n\omega}\|w_t(t)\|_{\mathbb{W}_2^1[0,l]},\right.$$
$$\left.\max_{(n-1)\omega\le t\le n\omega}\|w(t)\|_{\mathbb{W}_2^2[0,l]}\right\},n=1,2,\cdots.$$

Proof. It is clear that the mixed problems (3.25) and (3.26) can be written as the IVPs

$$\begin{cases} w''(t)+Aw(t)+\mu(t)Aq=bAg(t-\omega)+f(t), t\in(0,\omega),\\ w(0)=g(0), w'(0)=g_t(0)\end{cases} \tag{3.31}$$

and

$$\begin{cases} w''(t)+Aw(t)+\mu(t)Aq=bAw(t-\omega)+f(t),\\ (n-1)\omega<t<n\omega,\quad n=2,3,\cdots,\\ w\big((n-1)\omega+\big)=w\big((n-1)\omega-\big), w'\big((n-1)\omega+\big)=w'\big((n-1)\omega-\big),\\ n=2,3,\cdots\end{cases} \tag{3.32}$$

in a Hilbert space $\mathbb{H}=\mathbb{L}_2[0,l]$ with A determining by (3.5). From (3.24) and (3.27) it follows that

$$|p(t)|,|\mu(t)|\le k(q,l)[|\zeta''(t)|+\|w_{tt}(t)\|_H] \tag{3.33}$$

for all $t\in(0,\infty)$. Therefore, the proof of Theorem 3.5 is based on the following abstract theorem.

Theorem 3.6. Under assumptions of Theorem 3.5, for the solution of problems (3.31) and (3.32) the following stability estimate holds:

$$\max_{0\le t\le \omega}\|w_{tt}\|_H,\ \max_{0\le t\le \omega}\left\|A^{\frac{1}{2}}w_t\right\|_H,\ \max_{0\le t\le \omega}\|Aw\|_H \tag{3.34}$$

$$\leq M(q,l)\left[a_0+\max_{0\leq t\leq\omega}\|f'(t)\|_H+\|f(0)\|_H+\max_{0\leq t\leq\omega}|\zeta''|\right]$$

$$a_0=\max\left\{\max_{-\omega\leq t\leq 0}\|g_{tt}(t)\|_H,\max_{-\omega\leq t\leq 0}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H,\max_{-\omega\leq t\leq 0}\|Ag(t)\|_H\right\},$$

$$\max_{n\omega\leq t\leq(n+1)\omega}\|w_{tt}\|_H,\max_{n\omega\leq t\leq(n+1)\omega}\|w_t\|_H,\max_{n\omega\leq t\leq(n+1)\omega}\|Aw\|_H \tag{3.35}$$

$$\leq M(q,l)\left[a_n+\max_{n\omega\leq t\leq(n+1)\omega}\|f'(t)\|_H+\|f(\mathrm{n\omega})\|_H+\max_{n\omega\leq t\leq(n+1)\omega}|\zeta''|\right],$$

$$a_n=\max\left\{\max_{(n-1)\omega\leq t\leq n\omega}\|w_{tt}(t)\|_H,\max_{(n-1)\omega\leq t\leq n\omega}\left\|A^{\frac{1}{2}}w_t(t)\right\|_H,\right.$$

$$\left.\max_{(n-1)\omega\leq t\leq n\omega}\|Aw(t)\|_H\right\},n=1,2,\cdots.$$

Proof. The initial value problems (3.31) and (3.32) are equivalent to the integral equations

$$w(t)=c(t)g(0)+s(t)g_t(0) \tag{3.36}$$

$$+\int_0^t s(t-z)[-\mu(z)Aq+bAg(z-\omega)+f(z)]dz,0\leq t\leq\omega$$

$$w(t)=c(t-(n-1)\omega)w((n-1)\omega)+s(t-(n-1)\omega)w_t((n-1)\omega) \tag{3.37}$$

$$+\int_{(n-1)\omega}^t s(t-z)[-\mu(z)Aq+bAg(z-\omega)+f(z)]dz,$$

$$(n-1)\omega\leq t\leq n\omega,n=2,\cdots$$

in H, respectively. Let $t\in[0,\omega]$. Applying equation (3.31) and formula (3.36), we get

$$Aw(t)=c(t)Ag(0)+s(t)Ag_t(0)$$

$$+\int_0^t As(t-z)[-\mu(z)Aq+bAg(z-\omega)+f(z)]dz$$

$$=c(t)Ag(0)+s(t)Ag_t(0)$$

$$-\mu(t)Aq+bAg(t-\omega)+f(t)-c(t)[bAg(-\omega)+f(0)]$$

$$-\int_0^t c(t-z)[-\mu'(z)Aq+bAg'(z-\omega)+f'(z)]dz.$$

Therefore, applying this formula, the triangle inequality and estimates (3.6) and (3.33), we get

$$\|w_{tt}(t)\|_{\mathbb{H}}\leqslant\|Ag(0)\|_{\mathbb{H}}+\left\|A^{\frac{1}{2}}g_t(0)\right\|_{\mathbb{H}}+\|f(0)\|_{\mathbb{H}}+\omega\max_{t\in[0,\omega]}\|f_t\|_{\mathbb{H}}$$

$$+\max_{-\omega\leq t\leq 0}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H+M_3(q,l)\max_{0\leq t\leq\omega}|\zeta''|+M_3(q,l)\int_0^t\|w_{zz}(z)\|_H dz.$$

Using the integral inequality, we get

$$\max_{0\le t\le\omega}\|w_{tt}\|_H \le \ M(q,l)\left[a_0+\max_{0\le t\le\omega}\|f'(t)\|_H+\|f(0)\|_H+\max_{0\le t\le\omega}|\zeta''|\right].$$

In the same manner, we can obtain

$$\max_{0\le t\le\omega}\left\|A^{\frac{1}{2}}w_t\right\|_H \le \ M(q,l)\left[a_0+\max_{0\le t\le\omega}\|f'(t)\|_H+\|f(0)\|_H+\ \max_{0\le t\le\omega}|\zeta''|\right].$$

From that and equation (3.31) it follows estimate for $\max_{0\le t\le\omega}\|Aw\|_H$.

Let $t\in[(n-1)\omega,n\omega],n=2,\cdots$. Applying equation (3.32) and formula (3.37),

$$Aw(t)=c(t-(n-1)\omega)Aw((n-1)\omega)+s(t-(n-1)\omega)Aw_t((n-1)\omega)$$
$$+\int_{(n-1)\omega}^{t}As(t-z)[-\mu(z)Aq+bAg(z-\omega)+f(z)]dz$$
$$=c(t)Aw\big((n-1)\omega\big)+s(t)Aw_t\big((n-1)\omega\big)\ -\mu(t-(n-1)\omega)Aq$$
$$+bAw(t-n\omega)+f(t)-c(t-(n-1)\omega)[bAg(-n\omega)+f((n-1)\omega)]$$
$$-\int_{(n-1)\omega}^{t}c(t-z)[-\mu'(z)Aq+bAg'(z-\omega)+f'(z)]dz.$$

Applying this formula, the triangle inequality and estimates (3.6) and (3.33), we get

$$\|w_{tt}(t)\|_{\mathbb{H}}\leqslant\|Aw((n-1)\omega)\|_{\mathbb{H}}+\left\|A^{\frac{1}{2}}w_t((n-1)\omega)\ \right\|_{\mathbb{H}}+\left\|f\big((n-1)\omega\big)\right\|_{\mathbb{H}}$$
$$+\omega\max_{t\in[(n-1)\omega,n\omega]}\|f_t\|_{\mathbb{H}}+\max_{(n-1)\omega\le t\le n\omega}\left\|A^{\frac{1}{2}}g_t(t)\right\|_H$$
$$+M_3(q,l)\ \max_{(n-1)\omega\le t\le n\omega}|\zeta''|+M_3(q,l)\int_{(n-1)\omega}^{t}\|w_{zz}(z)\|_H dz.$$

Using the integral inequality, we get

$$\max_{(n-1)\omega\le t\le n\omega}\|w_{tt}\|_H$$
$$\le\ M(q,l)\left[a_n+\max_{(n-1)\omega\le t\le n\omega}\|f'(t)\|_H+\left\|f\big((n-1)\omega\big)\right\|_H+\ \max_{(n-1)\omega\le t\le n\omega}|\zeta''|\right].$$

In the same manner, we can obtain

$$\max_{(n-1)\omega\le t\le n\omega}\left\|A^{\frac{1}{2}}w_t\right\|_H$$
$$\le\ M(q,l)\left[a_n+\max_{(n-1)\omega\le t\le n\omega}\|f'(t)\|_H+\left\|f\big((n-1)\omega\big)\right\|_H+\ \max_{(n-1)\omega\le t\le n\omega}|\zeta''|\right].$$

From that and equation (3.32) it follows estimate for $\max_{(n-1)\omega\le t\le n\omega}\|Aw\|_H$. Theorem 3.6 is established.

CHAPTER IV

Difference Schemes for the Solution of Time-Dependent Identification Problem for Delay Hyperbolic Equations

4.1 Introduction

It is important to know that when the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We need numerical applications when one cannot know concrete values of constants in stability estimates. Therefore, we can use the numerical methods to get approximate solutions of local and nonlocal problems for the time-dependent identification problem for delay hyperbolic partial differential equations. In this chapter we obtain the algorithms of numerical solution for the initial-boundary-value problem for the one dimensional delay hyperbolic partial differential equations with Dirichlet, Neumann and nonlocal boundary conditions. Therefore, the first order of accuracy DSs for the solution of one-dimensional DHPDEs are presented.

4.2 Absolute Stable Difference Schemes for the Solution of Time-Dependent Identification Problems for Delay Hyperbolic Equations with Dirichlet Boundary Condition.

We consider the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)\sin x + 0.01u_{xx}(t-\pi, x) - 1.01\sin t\sin x, \\ t > 0, 0 < x < \pi, \\ u(t,x) = \sin t\sin x, -\pi \le t \le 0, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \int\limits_0^\pi u(t,x)dx = 2\sin t, t \ge 0, \end{cases} \tag{4.1}$$

for a one dimentional delay hyperbolic differential equation with Dirichlet condition. Recall that

$$(u(t,x), p(t)) = ((mu(t,x), mp(t)))_{m=1}^{\infty},$$

where $(mu(t,x), mp(t))$ is exact solution pair of the problem (4.1) on $t \in [(m-1)\pi, m\pi], m \ge 1$. The exact solution pair of the problem (4.1) is $(u(t,x), p(t)) = (\sin(t)\sin(x), \sin(t))$. For the numerical solution of problem (4.1), we present the following first order of accuracy difference scheme for the approximate solution for the problem (4.1)

$$\begin{cases} \dfrac{mu_n^{k+1} - 2(mu)_n^k + mu_n^{k-1}}{\tau^2} - \dfrac{mu_{n+1}^{k+1} - 2(mu)_n^{k+1} + mu_{n-1}^{k+1}}{h^2} \\ = mp_k \sin(x_n) - \sin(t_{k+1}) \sin(x_n), m = 1, \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \dfrac{mu_n^{k+1} - 2(mu)_n^k + mu_n^{k-1}}{\tau^2} - \dfrac{mu_{n+1}^{k+1} - 2(mu)_n^{k+1} + mu_{n-1}^{k+1}}{h^2} \\ = mp_k \sin(x_n) - 1.01 \sin(t_{k+1}) \sin(x_n), \\ +0.01 \dfrac{(m-1)u_{n+1}^{k-N} - 2\left((m-1)u\right)_n^{k-N} + (m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k = k\tau, x_n = nh, \\ (m-1)N + 1 \le k \le mN - 1, \\ 1 \le n \le M-1, N\tau = \pi, Mh = \pi, m = 2,3, \ldots, \\ mu_n^{(m-1)N} = 0, \dfrac{mu_n^{(m-1)N+1} - mu_n^{(m-1)N}}{\tau} = \sin(x_n), 0 \le n \le M, m = 1, \\ mu_n^{(m-1)N} = (m-1)u_n^{(m-1)N}, \\ \dfrac{mu_n^{(m-1)N+1} - mu_n^{(m-1)N}}{\tau} = \dfrac{(m-1)u_n^{(m-1)N} - (m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, m \ge 2, \\ mu_0^{k+1} = mu_M^{k+1} = 0, \sum_{i=1}^{M-1} mu_i^{k+1} h = 2\sin(t_{k+1}), \\ (m-1)N \le k \le mN, m = 1,2, \ldots. \end{cases} \tag{4.2}$$

We consider two cases: $m = 1$ and $m \ge 2$. First, let $m = 1$, then $0 \le k \le N$. From problem (4.2) it follows that

$$\begin{cases} \dfrac{1u_n^{k+1} - 2(1u)_n^k + 1u_n^{k-1}}{\tau^2} - \dfrac{1u_{n+1}^{k+1} - 2(1u)_n^{k+1} + 1u_{n-1}^{k+1}}{h^2} \\ = 1p_k \sin(x_n) - \sin(t_{k+1}) \sin(x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, N\tau = \pi, Mh = \pi, \\ 1u_n^0 = 0, \dfrac{1u_n^1 - 1u_n^0}{\tau} = \sin(x_n), 0 \le n \le M, \\ 1u_0^{k+1} = 1u_M^{k+1} = 0, \displaystyle\sum_{i=1}^{M-1} 1u_i^{k+1} h = 2\sin(t_{k+1}), 0 \le k \le N. \end{cases} \tag{4.3}$$

Algorithm for obtaining the solution of the time-dependent identification problem (4.3) $\{1u_k\}_{k=0}^N = \{\{1u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{1p_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$1u_n^k = 1\omega_n^k + 1\eta_k \sin(x_n), 0 \le k \le N, 0 \le n \le M, \tag{4.4}$$

Applying difference scheme (4.3) and formula (4.4), we will obtain formula

$$1\eta_{k+1} = \frac{2\sin(t_{k+1}) - \sum_{i=1}^{M-1} 1\omega_i^{k+1} h}{\sum_{i=1}^{M-1} \sin(x_i) h}, -1 \le k \le N-1, \tag{4.5}$$

and the difference scheme

$$\begin{cases} \dfrac{1\omega_n^{k+1} - 2(1\omega)_n^k + 1\omega_n^{k-1}}{\tau^2} - \dfrac{1\omega_{n+1}^{k+1} - 2(1\omega)_n^{k+1} + 1\omega_{n-1}^{k+1}}{h^2} \\ + \dfrac{\sum_{i=1}^{M-1} 1\omega_i^{k+1} h}{\sum_{i=1}^{M-1} \sin(x_i) h} \sin(x_n) \dfrac{2(\cos(h) - 1)}{h^2} \\ = \left[\dfrac{2}{\sum_{i=1}^{M-1} \sin(x_i) h} \dfrac{2(\cos(h) - 1)}{h^2} - 1 \right] \sin(t_{k+1}) \sin(x_n), \\ t_k = k\tau, x_n = nh, 1 \le k \le N-1, 1 \le n \le M-1, \\ 1\omega_n^0 = 0, \dfrac{1\omega_n^1 - 1\omega_n^0}{\tau} = \sin(x_n), 0 \le n \le M, \\ 1\omega_0^{k+1} = 1\omega_M^{k+1} = 0, -1 \le k \le N-1. \end{cases} \tag{4.6}$$

In the first stage, we find numerical solution $\{\{1\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.6). For obtaining the solution of difference scheme (4.6), we will write it in the matrix form as

$$\begin{cases} A(1\omega)^{k+1} + B(1\omega)^{k} + C(1\omega)^{k-1} = (1f)^{k}, 1 \le k \le N-1, \\ 1\omega^{0} = 0, 1\omega^{1} = \tau \sin(x_n), \end{cases} \tag{4.7}$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $1\omega^{s}, s = k, k \pm 1, 1f^{k}$ are $(M+1) \times 1$ column matrices and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & a + \frac{c_1}{d} & b + \frac{c_1}{d} & \cdots & \frac{c_1}{d} & \frac{c_1}{d} & 0 \\ 0 & b + \frac{c_2}{d} & a + \frac{c_2}{d} & \cdots & \frac{c_2}{d} & \frac{c_2}{d} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{c_{M-2}}{d} & \frac{c_{M-2}}{d} & \cdots & a + \frac{c_{M-2}}{d} & b + \frac{c_{M-2}}{d} & 0 \\ 0 & \frac{c_{M-1}}{d} & \frac{c_{M-1}}{d} & \cdots & b + \frac{c_{M-1}}{d} & a + \frac{c_{M-1}}{d} & b \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & g & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & g & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$1f^{k} = \begin{bmatrix} 0 \\ 1f(t_k, x_1) \\ . \\ 1f(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$1\omega^{s} = \begin{bmatrix} 1\omega_0^{s} \\ 1\omega_1^{s} \\ . \\ 1\omega_{M-1}^{s} \\ 1\omega_M^{s} \end{bmatrix}_{(M+1)\times 1}, \text{for} \quad s = k, k \pm 1.$$

Here, $a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, c_n = \sin(x_n) \frac{2(\cos(h)-1)}{h}, d = \sum_{i=1}^{M-1} \sin(x_i)h, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2}, 1f(t_k, x_n) = \left[\frac{2}{\sum_{i=1}^{M-1} \sin(x_i)h} \frac{2(\cos(h)-1)}{h^2} - 1\right] \sin(t_{k+1}) \sin(x_n), 1 \le k \le N-1, 1 \le n \le M-1.$

So, we have the IVP for the second order difference equation (4.7) with respect to k with matrix coefficients A, B and C: Since ω^0 and ω^1 are given, we can obtain the solution of (4.7) by direct formula

$$1\omega^{k+1} = A^{-1}(1f^k - B(1\omega)^k - C(1\omega)^{k-1}), k = 1, \dots, N-1. \quad (4.8)$$

Applying formula $1\eta_{k+1} = \sum_{i=1}^{k} (k+1-i)(1p)_i \tau^2, 1 \le k \le N-1, \eta_0 = \eta_1 = 0$, we can obtain

$$1p_k = \frac{1\eta_{k+1} - 2(1\eta)_k + 1\eta_{k-1}}{\tau^2}, 1 \le k \le N-1. \quad (4.9)$$

In the second stage, we will obtain $\{1p_k\}_{k=1}^{N-1}$ by formulas (4.5) and (4.9). Finally, in the third stage, we will obtain $\{\{1u_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.4) and (4.5). The errors are computed by

$$1E_u = \max_{0 \le k \le N} \left(\sum_{n=1}^{M-1} |u(t_k, x_n) - 1u_n^k|^2 h \right)^{\frac{1}{2}}, \quad (4.10)$$

$$1E_p = \max_{1 \le k \le N-1} |p(t_k) - 1p_k|,$$

where $u(t,x), p(t)$ represent the exact solution, $1u_n^k$ represent the numerical solutions at (t_k, x_n) and $1p_k$ represent the numerical solutions at t_k.

Second, let $m \ge 2,$ then $(m-1)N \le k \le mN.$ From problem (4.2) it follows that

$$\begin{cases} \dfrac{mu_n^{k+1} - 2(mu)_n^k + mu_n^{k-1}}{\tau^2} - \dfrac{mu_{n+1}^{k+1} - 2(mu)_n^{k+1} + mu_{n-1}^{k+1}}{h^2} \\ = mp_k \sin(x_n) - 1.01 \sin(t_{k+1}) \sin(x_n), \\ +0.01 \dfrac{(m-1)u_{n+1}^{k-N} - 2\left((m-1)u\right)_n^{k-N} + (m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k = k\tau, x_n = nh, \\ (m-1)N + 1 \le k \le mN - 1, \\ 1 \le n \le M-1, N\tau = \pi, Mh = \pi, \\ mu_n^{(m-1)N} = (m-1)u_n^{(m-1)N}, \\ \dfrac{mu_n^{(m-1)N+1} - mu_n^{(m-1)N}}{\tau} = \dfrac{(m-1)u_n^{(m-1)N} - (m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ mu_0^{k+1} = mu_M^{k+1} = 0, \sum_{i=1}^{M-1} mu_i^{k+1} h = 2\sin(t_{k+1}), \\ (m-1)N \le k \le mN, m \ge 2. \end{cases} \tag{4.11}$$

In the same manner, algorithm for obtaining the solution of the time-dependent identification problem (4.11) $\{mu_k\}_{k=0}^{N} = \{\{mu_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ and $\{mp_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$mu_n^k = m\omega_n^k + m\eta_k \sin(x_n), (m-1)N \le k \le mN, 0 \le n \le M, \tag{4.12}$$

Applying difference scheme (4.11) and formula (4.12), we will obtain formula

$$m\eta_{k+1} = \frac{2\sin(t_{k+1}) - \sum_{i=1}^{M-1} m\omega_i^{k+1} h}{\sum_{i=1}^{M-1} \sin(x_i) h}, (m-1)N - 1 \le k \le mN - 1, \tag{4.13}$$

and the difference scheme

$$\begin{cases} \frac{m\omega_n^{k+1} - 2(m\omega)_n^k + m\omega_n^{k-1}}{\tau^2} - \frac{m\omega_{n+1}^{k+1} - 2(m\omega)_n^{k+1} + m\omega_{n-1}^{k+1}}{h^2} \\ + \frac{\sum_{i=1}^{M-1} m\omega_i^{k+1} h}{\sum_{i=1}^{M-1} \sin(x_i)\, h} \sin(x_n) \frac{2(\cos(h) - 1)}{h^2} \\ = 0.01 \frac{((m-1)w)_{n+1}^{k-N} - 2\big((m-1)w\big)_n^{k-N} + ((m-1)w)_{n-1}^{k-N}}{h^2} \\ + \left[\frac{2}{\sum_{i=1}^{M-1} \sin(x_i)\, h} \frac{2(\cos(h) - 1)}{h^2} - 1.01\right] \sin(t_{k+1}) \sin(x_n), \\ (m-1)N + 1 \le k \le mN - 1, \\ m\omega_n^{(m-1)N} = (m-1)\omega_n^{(m-1)N}, \\ \frac{m\omega_n^{(m-1)N+1} - m\omega_n^{(m-1)N}}{\tau} = \frac{(m-1)\omega_n^{(m-1)N} - (m-1)\omega_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ m\omega_0^{k+1} = m\omega_M^{k+1} = 0, (m-1)N \le k \le mN, m \ge 2. \end{cases} \tag{4.14}$$

In the first stage, we find numerical solution $\{\{m\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.14). For obtaining the solution of difference scheme (4.14), we will write it in the matrix form as

$$\begin{cases} A(m\omega)^{k+1} + B(m\omega)^k + C(m\omega)^{k-1} = (mf)^k, \\ (m-1)N + 1 \le k \le mN - 1, \\ (m\omega)_n^{(m-1)N} = ((m-1)\omega)_n^{(m-1)N}, \\ (m\omega)_n^{(m-1)N+1} = 2((m-1)\omega)_n^{(m-1)N} - ((m-1)\omega)_n^{(m-1)N-1} \end{cases} \tag{4.15}$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $m\omega^s, s = k, k \pm 1, mf^k$ are $(M+1) \times 1$ column matrices and

$$mf^k = \begin{bmatrix} 0 \\ mf(t_k, x_1) \\ . \\ mf(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$m\omega^s = \begin{bmatrix} m\omega_0^s \\ m\omega_1^s \\ . \\ m\omega_{M-1}^s \\ m\omega_M^s \end{bmatrix}_{(M+1)\times 1}, \text{for} \quad s = k, k \pm 1.$$

So, we have the initial value problem for the second order difference equation (4.15) with respect to k with matrix coefficients A, B and C: Since $m\omega_n^N$ and $m\omega_n^{N+1}$ are given, we can obtain the solution of (4.15) by direct formula

$$\begin{cases} (m\omega)^{k+1} = A^{-1}((mf)^k - B(m\omega)^k - C(m\omega)^{k-1}), \\ N+1 \le k \le mN-1. \end{cases} \quad (4.16)$$

Applying formula $m\eta_{k+1} = \sum_{i=1}^{k} (k+1-i)(mp)_i \tau^2, (m-1)N+1 \le k \le mN-1, m\eta_{(m-1)N} = m\eta_{(m-1)N+1} = 0$, we can obtain

$$mp_k = \frac{m\eta_{k+1} - 2(m\eta)_k + m\eta_{k-1}}{\tau^2}, (m-1)N+1 \le k \le mN-1. \quad (4.17)$$

In the second stage, we will obtain $\{mp_k\}_{k=1}^{N-1}$ by formulas (4.13) and (4.17). Finally, in the third stage, we will obtain $\{\{mu_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.12) and (4.13). The errors are computed by

$$mE_u = \max_{(m-1)N \le k \le mN} \left(\sum_{n=1}^{M-1} |u(t_k, x_n) - mu_n^k|^2 h \right)^{\frac{1}{2}}, \quad (4.18)$$

$$mE_p = \max_{(m-1)N+1 \le k \le mN-1} |p(t_k) - mp_k|,$$

where $u(t,x), p(t)$ represent the exact solution, mu_n^k represent the numerical solutions at (t_k, x_n) and mp_k represent the numerical solutions at t_k. The numerical results are given in the following table.

Table 4.1.

Error Analysis for Difference Schemes (4.6) and (4.14)

$Error$	$N=M=20$	$N=M=40$	$N=M=80$	$N=M=160$
$1E_u$	0.1267	0.0669	0.0345	0.0176
$1E_p$	0.1564	0.0785	0.0393	0.0196
$2E_u$	0.2942	0.1655	0.0883	0.0456
$2E_p$	0.1404	0.0747	0.0379	0.0190
$3E_u$	0.4341	0.2567	0.1408	0.0739
$3E_p$	0.2185	0.1418	0.1027	0.0830

As it is seen in Table 1, if M and N are multiplied by 2, the value of errors decreases approximately $1/2$ for the DS. This shows that it has the first order of accuracy.

4.3 Absolute Stable Difference Schemes for the Solution of Time-Dependent Identification Problems for Delay Hyperbolic Equations with Neumann Boundary Condition.

We consider the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)(1 + \cos x) + 0.01u_{xx}(t - \pi, x) \\ -\sin t(2 + \cos x) - 0.01\sin t\cos x, t > 0, 0 < x < \pi, \\ u(t, x) = \sin t(1 + \cos x), -\pi \leq t \leq 0, 0 \leq x \leq \pi, \\ u_x(t, 0) = u_x(t, \pi) = 0, \int\limits_0^\pi u(t, x)dx = \pi\sin t, t \geq 0 \end{cases} \quad (4.19)$$

for a one dimentional delay hyperbolic differential equation with Neumann condition. Recall that

$$(u(t, x), p(t)) = ((mu(t, x), mp(t)))_{m=1}^{\infty},$$

where $(mu(t, x), mp(t))$ is exact solution pair of the problem (4.19) on $t \in [(m - 1)\pi, m\pi], m \geq 1$. The exact solution pair of the problem (4.19) is $(u(t, x), p(t)) = (\sin t(1 + \cos x), \sin t)$. For the numerical solution of problem (4.19), we present the following first order of accuracy difference scheme for the approximate solution for the problem (4.19)

$$\begin{cases} \frac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\frac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ = mp_k(1+\cos(x_n))-\sin(t_{k+1})\,(2+\cos(x_n)), m=1, \\ 1\le k\le N-1, 1\le n\le M-1, \\ \frac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\frac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ = mp_k(1+\cos(x_n))-2\sin(t_{k+1})-1.01\sin(t_{k+1})\cos(x_n), \\ +0.01\frac{(m-1)u_{n+1}^{k-N}-2\big((m-1)u\big)_n^{k-N}+(m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k=k\tau, x_n=nh, \\ (m-1)N+1\le k\le mN-1, \\ 1\le n\le M-1, N\tau=\pi, Mh=\pi, m=2,3,\ldots, \\ mu_n^{(m-1)N}=0, \frac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=1+\cos(x_n), \\ 0\le n\le M, m=1, \\ mu_n^{(m-1)N}=(m-1)u_n^{(m-1)N}, \\ \frac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=\frac{(m-1)u_n^{(m-1)N}-(m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0\le n\le M, m\ge 2, \\ mu_1^{k+1}-mu_0^{k+1}=mu_M^{k+1}-mu_{M-1}^{k+1}=0, \\ \sum_{i=0}^{M-1}\mathrm{m}u_i^{k+1}h=\pi\sin(tk+1), (m-1)N\le k\le mN, m=1,2,\ldots. \end{cases} \tag{4.20}$$

We consider two cases: $m=1$ and $m\ge 2$. First, let $m=1$, then $0\le k\le N$. From problem (4.20) it follows that

$$\begin{cases}\dfrac{1u_n^{k+1}-2(1u)_n^k+1u_n^{k-1}}{\tau^2}-\dfrac{1u_{n+1}^{k+1}-2(1u)_n^{k+1}+1u_{n-1}^{k+1}}{h^2}\\ =1p_k(1+\cos(x_n))-\sin(t_{k+1})\,(2+\cos(x_n)),\\ 1\le k\le N-1,1\le n\le M-1,N\tau=\pi,Mh=\pi,\\ 1u_n^0=0,\dfrac{1u_n^1-1u_n^0}{\tau}=1+\cos(x_n)\,,0\le n\le M,\\ 1u_1^{k+1}-1u_0^{k+1}=1u_M^{k+1}-1u_{M-1}^{k+1}=0,\\ \sum_{i=0}^{M-1}\mathrm{m}u_i^{k+1}h=\pi\sin(t_{k+1})\,,0\le k\le N.\end{cases} \quad (4.21)$$

Algorithm for obtaining the solution of the time-dependent identification problem (4.21) $\{1u_k\}_{k=0}^N=\{\{1u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{1p_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$1u_n^k=1\omega_n^k+1\eta_k\big(1+\cos(x_n)\big),0\le k\le N,0\le n\le M, \quad (4.22)$$

Applying difference scheme (4.21) and formula (4.22), we will obtain formula

$$1\eta_{k+1}=\frac{\pi\sin(t_{k+1})-\sum_{i=0}^{M-1}1\omega_i^{k+1}h}{\pi},-1\le k\le N-1, \quad (4.23)$$

and the difference scheme

$$\begin{cases}\dfrac{1\omega_n^{k+1}-2(1\omega)_n^k+1\omega_n^{k-1}}{\tau^2}-\dfrac{1\omega_{n+1}^{k+1}-2(1\omega)_n^{k+1}+1\omega_{n-1}^{k+1}}{h^2}\\ +\sum_{i=0}^{M-1}1\omega_i^{k+1}\cos(x_n)\dfrac{2(\cos(h)-1)}{\pi h}\\ =\left[\dfrac{2(\cos(h)-1)}{h^2}-1\right]\sin(t_{k+1})\cos(x_n)-2\sin(t_{k+1}),\\ t_k=k\tau,x_n=nh,1\le k\le N-1,1\le n\le M-1,\\ 1\omega_n^0=0,\dfrac{1\omega_n^1-1\omega_n^0}{\tau}=1+\cos(x_n)\,,0\le n\le M,\\ 1\omega_1^{k+1}-1\omega_0^{k+1}=1\omega_M^{k+1}-1\omega_{M-1}^{k+1}=0,-1\le k\le N-1.\end{cases} \quad (4.24)$$

In the first stage, we find numerical solution $\{\{1\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.24). For obtaining the solution of difference scheme (4.24), we will write it in the matrix form as

$$\begin{cases} A(1\omega)^{k+1} + B(1\omega)^k + C(1\omega)^{k-1} = (1f)^k, 1 \le k \le N-1, \\ 1\omega^0 = 0, 1\omega^1 = \tau(1+\cos(x_n)), \end{cases} \tag{4.25}$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $1\omega^s, s = k, k \pm 1, 1f^k$ are $(M+1)\times 1$ column matrices and

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ b & a+c_1 & b+c_1 & \cdots & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdots & c_2 & c_2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c_{M-2} & c_{M-2} & \cdots & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdots & b+c_{M-1} & a+c_{M-1} & b \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & e & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & g & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & g & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$1f^k = \begin{bmatrix} 0 \\ 1f(t_k, x_1) \\ . \\ 1f(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$1\omega^s = \begin{bmatrix} 1\omega_0^s \\ 1\omega_1^s \\ . \\ 1\omega_{M-1}^s \\ 1\omega_M^s \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1.$$

Here, $a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, c_n = \cos(x_n)\frac{2(\cos(h)-1)}{\pi h}, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2},$

$$1f(t_k, x_n) = \left[\frac{2(\cos(h)-1)}{h^2} - 1\right]\sin(t_{k+1})\cos(x_n) - 2\sin(t_{k+1}),$$

$$1 \le k \le N-1, 1 \le n \le M-1.$$

So, we have the IVP for the second order difference equation (4.25) with respect to k

with matrix coefficients A, B and C: Since $1\omega^0$ and $1\omega^1$ are given, we can obtain the solution of (4.25) by direct formula

$$1\omega^{k+1} = A^{-1}(1f^k - B(1\omega)^k - C(1\omega)^{k-1}), k = 1, \dots, N-1. \qquad (4.26)$$

Applying formula $1\eta_{k+1} = \sum_{i=1}^{k} (k+1-i)(1p)_i \tau^2, 1 \le k \le N-1, \eta_0 = \eta_1 = 0$, we can obtain

$$1p_k = \frac{1\eta_{k+1} - 2(1\eta)_k + 1\eta_{k-1}}{\tau^2}, 1 \le k \le N-1. \qquad (4.27)$$

In the second stage, we will obtain $\{1p_k\}_{k=1}^{N-1}$ by formulas (4.23) and (4.27). Finally, in the third stage, we will obtain $\{\{1u_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.22) and (4.23). The errors are computed by

$$1E_u = \max_{0 \le k \le N} \left(\sum_{n=0}^{M-1} |u(t_k, x_n) - 1u_n^k|^2 h \right)^{\frac{1}{2}}, \qquad (4.28)$$

$$1E_p = \max_{1 \le k \le N-1} |p(t_k) - 1p_k|,$$

where $u(t,x), p(t)$ represent the exact solution, $1u_n^k$ represent the numerical solutions at (t_k, x_n) and $1p_k$ represent the numerical solutions at t_k.

Second, let $m \ge 2,$ then $(m-1)N \le k \le mN.$ From problem (4.20) it follows that

$$\begin{cases}\dfrac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\dfrac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ =mp_k(1+\cos(x_n))-2\sin(t_{k+1})-1.01\sin(t_{k+1})\cos(x_n), \\ +0.01\dfrac{(m-1)u_{n+1}^{k-N}-2\big((m-1)u\big)_n^{k-N}+(m-1)u_{n-1}^{k-N}}{h^2}, t_k=k\tau, x_n=nh, \\ (m-1)N+1\le k\le mN-1, \\ 1\le n\le M-1, N\tau=\pi, Mh=\pi, \\ mu_n^{(m-1)N}=(m-1)u_n^{(m-1)N}, \\ \dfrac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=\dfrac{(m-1)u_n^{(m-1)N}-(m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0\le n\le M, m\ge 2, \\ mu_1^{k+1}-mu_0^{k+1}=mu_M^{k+1}-mu_{M-1}^{k+1}=0, \\ \displaystyle\sum_{i=0}^{M-1}1u_i^{k+1}h=\pi\sin(t_{k+1}), (m-1)N\le k\le mN, m\ge 2.\end{cases} \tag{4.29}$$

In the same manner, algorithm for obtaining the solution of the time-dependent identification problem (4.29) $\{mu_k\}_{k=0}^N=\{\{mu_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{mp_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$mu_n^k=m\omega_n^k+m\eta_k\big(1+\cos(x_n)\big), (m-1)N\le k\le mN, 0\le n\le M, \tag{4.30}$$

Applying difference scheme (4.29) and formula (4.30), we will obtain formula

$$m\eta_{k+1}=\frac{\pi\sin(t_{k+1})-\sum_{i=0}^{M-1}m\omega_i^{k+1}h}{\pi}, (m-1)N-1\le k\le mN-1, \tag{4.31}$$

and the difference scheme

$$\begin{cases} \dfrac{m\omega_n^{k+1} - 2(m\omega)_n^k + m\omega_n^{k-1}}{\tau^2} - \dfrac{m\omega_{n+1}^{k+1} - 2(m\omega)_n^{k+1} + m\omega_{n-1}^{k+1}}{h^2} \\ + \displaystyle\sum_{i=0}^{M-1} m\omega_i^{k+1} \cos(x_n) \frac{2(\cos(h) - 1)}{\pi h} \\ = 0.01 \dfrac{((m-1)w)_{n+1}^{k-N} - 2\big((m-1)w\big)_n^{k-N} + ((m-1)w)_{n-1}^{k-N}}{h^2} \\ + \left[\dfrac{2(\cos(h) - 1)}{h^2} - 1.01\right] \sin(t_{k+1}) \cos(x_n) - 2\sin(t_{k+1}), \\ (m-1)N + 1 \le k \le mN - 1, \\ m\omega_n^{(m-1)N} = (m-1)\omega_n^{(m-1)N}, \\ \dfrac{m\omega_n^{(m-1)N+1} - m\omega_n^{(m-1)N}}{\tau} = \dfrac{(m-1)\omega_n^{(m-1)N} - (m-1)\omega_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ mu_1^{k+1} - mu_0^{k+1} = mu_M^{k+1} - mu_{M-1}^{k+1} = 0, \\ (m-1)N \le k \le mN, m \ge 2. \end{cases} \tag{4.32}$$

In the first stage, we find numerical solution $\{\{m\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.32). For obtaining the solution of difference scheme (4.32), we will write it in the matrix form as

$$\begin{cases} A(m\omega)^{k+1} + B(m\omega)^k + C(m\omega)^{k-1} = (mf)^k, \\ (m-1)N + 1 \le k \le mN - 1, \\ (m\omega)_n^{(m-1)N} = ((m-1)\omega)_n^{(m-1)N}, \\ (m\omega)_n^{(m-1)N+1} = 2((m-1)\omega)_n^{(m-1)N} - ((m-1)\omega)_n^{(m-1)N-1}, \end{cases} \tag{4.33}$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $m\omega^s, s = k, k \pm 1, mf^k$ are $(M+1) \times 1$ column matrices and

$$mf^k = \begin{bmatrix} 0 \\ mf(t_k, x_1) \\ . \\ mf(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$m\omega^s = \begin{bmatrix} m\omega_0^s \\ m\omega_1^s \\ . \\ m\omega_{M-1}^s \\ m\omega_M^s \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1.$$

So, we have the initial value problem for the second order difference equation (4.33) with respect to k with matrix coefficients A, B and C: Since $m\omega_n^N$ and $m\omega_n^{N+1}$ are given, we can obtain the solution of (4.33) by direct formula

$$(m\omega)^{k+1} = A^{-1}((mf)^k - B(m\omega)^k - C(m\omega)^{k-1}), \tag{4.34}$$

$$(m-1), N+1 \leq k \leq mN-1.$$

Applying formula $m\eta_{k+1} = \sum_{i=1}^{k} (k+1-i)(mp)_i\tau^2, (m-1)N+1 \leq k \leq mN-1, m\eta_{(m-1)N} = m\eta_{(m-1)N+1} = 0$, we can obtain

$$mp_k = \frac{m\eta_{k+1} - 2(m\eta)_k + m\eta_{k-1}}{\tau^2}, (m-1)N+1 \leq k \leq mN-1. \tag{4.35}$$

In the second stage, we will obtain $\{mp_k\}_{k=1}^{N-1}$ by formulas (4.31) and (4.35). Finally, in the third stage, we will obtain $\{\{mu_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.30) and (4.31). The errors are computed by

$$mE_u = \max_{(m-1)N \leq k \leq mN} \left(\sum_{n=0}^{M-1} |u(t_k, x_n) - mu_n^k|^2 h \right)^{\frac{1}{2}}, \tag{4.36}$$

$$mE_p = \max_{(m-1)N+1 \leq k \leq mN-1} |p(t_k) - mp_k|,$$

where $u(t,x), p(t)$ represent the exact solution, mu_n^k represent the numerical solutions at (t_k, x_n) and mp_k represent the numerical solutions at t_k. The numerical results are given in the following table.

Table 4.2.

Error Analysis for Difference Schemes (4.24) and (4.32)

$Error$	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
$1E_u$	0.1754	0.1112	0.0625	0.0331
$1E_p$	0.2018	0.0967	0.0475	0.0235
$2E_u$	0.6868	0.3775	0.1947	0.0937
$2E_p$	0.2270	0.1052	0.0499	0.0242
$3E_u$	0.8276	0.4675	0.2869	0.2023
$3E_p$	0.2490	0.1119	0.0516	0.0245

4.4 Absolute Stable Difference Schemes for the Solution of Time-Dependent Identification Problems for Delay Hyperbolic Equations with Nonlocal Boundary Condition.

we consider the time-dependent identification problem

$$\begin{cases} u_{tt} - u_{xx} = p(t)(1 + \cos 2x) + 0.01u_{xx}(t - \pi, x) \\ -\sin 2t(5 + \cos 2x) + 0.04\sin 2t\cos 2x, t > 0, 0 < x < \pi, \\ u(t,x) = \sin 2t(1 + \cos 2x), -\pi \leq t \leq 0, 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \\ \int_0^\pi u(t,x)dx = \pi\sin 2t, t \geq 0 \end{cases} \tag{4.37}$$

for a one dimentional delay hyperbolic differential equation with nonlocal condition. Recall that

$$(u(t,x), p(t)) = ((mu(t,x), mp(t)))_{m=1}^{\infty},$$

where $(mu(t,x), mp(t))$ is exact solution pair of the problem (4.37) on $t \in [(m-1)\pi, m\pi], m \geq 1$. The exact solution pair of the problem (4.37) is $(u(t,x), p(t)) = (\sin 2t(1 + \cos 2x), \sin 2t)$. For the numerical solution of problem (4.37), we present the following first order of accuracy difference scheme for the approximate solution for the problem (4.37)

$$\begin{cases} \dfrac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\dfrac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ = mp_k(1+\cos(2x_n))-\sin(2t_{k+1})(5+\cos(2x_n)), m=1, \\ 1\le k\le N-1, 1\le n\le M-1, \\ \dfrac{mu_n^{k+1}-2(mu)_n^k+mu_n^{k-1}}{\tau^2}-\dfrac{mu_{n+1}^{k+1}-2(mu)_n^{k+1}+mu_{n-1}^{k+1}}{h^2} \\ = mp_k(1+\cos(2x_n))-5\sin(2t_{k+1})-0.96\sin(2t_{k+1})\cos(2x_n) \\ +0.01\dfrac{(m-1)u_{n+1}^{k-N}-2\big((m-1)u\big)_n^{k-N}+(m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k=k\tau, x_n=nh, \\ (m-1)N+1\le k\le mN-1, \\ 1\le n\le M-1, N\tau=\pi, Mh=\pi, m=2,3,\ldots, \\ mu_n^{(m-1)N}=0, \dfrac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=2(1+\cos(2x_n)), \\ 0\le n\le M, m=1, \\ mu_n^{(m-1)N}=(m-1)u_n^{(m-1)N}, \\ \dfrac{mu_n^{(m-1)N+1}-mu_n^{(m-1)N}}{\tau}=\dfrac{(m-1)u_n^{(m-1)N}-(m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0\le n\le M, m\ge 2, \\ mu_0^{k+1}=mu_M^{k+1}, mu_1^{k+1}-mu_0^{k+1}=mu_M^{k+1}-mu_{M-1}^{k+1}, \\ \sum_{i=0}^{M-1}\mathrm{m}u_i^{k+1}h=\pi\sin(2tk+1), (m-1)N\le k\le mN, m=1,2,\ldots. \end{cases} \tag{4.38}$$

We consider two cases: $m=1$ and $m\ge 2$. First, let $m=1$, then $0\le k\le N$. From problem (4.38) it follows that

$$\begin{cases} \frac{1u_n^{k+1}-2(1u)_n^k+1u_n^{k-1}}{\tau^2}-\frac{1u_{n+1}^{k+1}-2(1u)_n^{k+1}+1u_{n-1}^{k+1}}{h^2} \\ =1p_k\big(1+\cos(2x_n)\big)-\sin(2t_{k+1})\big(5+\cos(2x_n)\big), \\ 1\le k\le N-1, 1\le n\le M-1, N\tau=\pi, Mh=\pi, \\ 1u_n^0=0, \frac{1u_n^1-1u_n^0}{\tau}=2(1+\cos(2x_n)), 0\le n\le M, \\ 1u_0^{k+1}=1u_M^{k+1}, 1u_1^{k+1}-1u_0^{k+1}=1u_M^{k+1}-1u_{M-1}^{k+1}, \\ \sum_{i=0}^{M-1} 1u_i^{k+1}h=\pi\sin(2t_{k+1}), 0\le k\le N. \end{cases} \tag{4.39}$$

Algorithm for obtaining the solution of the time-dependent identification problem (4.39) $\{1u_k\}_{k=0}^N=\{\{1u_n^k\}_{k=0}^N\}_{n=0}^M$ and $\{1p_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$1u_n^k=1\omega_n^k+1\eta_k\big(1+\cos(2x_n)\big), 0\le k\le N, 0\le n\le M, \tag{4.40}$$

Applying difference scheme (4.39) and formula (4.40), we will obtain formula

$$1\eta_{k+1}=\frac{\pi\sin(2t_{k+1})-\sum_{i=0}^{M-1} 1\omega_i^{k+1}h}{\pi}, -1\le k\le N-1, \tag{4.41}$$

and the difference scheme

$$\begin{cases} \frac{1\omega_n^{k+1}-2(1\omega)_n^k+1\omega_n^{k-1}}{\tau^2}-\frac{1\omega_{n+1}^{k+1}-2(1\omega)_n^{k+1}+1\omega_{n-1}^{k+1}}{h^2} \\ +\sum_{i=0}^{M-1} 1\omega_i^{k+1}\cos(2x_n)\frac{2(\cos(2h)-1)}{\pi h} \\ =\left[\frac{2(\cos(2h)-1)}{h^2}-1\right]\sin(2t_{k+1})\cos(2x_n)-5\sin(2t_{k+1}), \\ t_k=k\tau, x_n=nh, 1\le k\le N-1, 1\le n\le M-1, \\ 1\omega_n^0=0, \frac{1\omega_n^1-1\omega_n^0}{\tau}=2(1+\cos(2x_n)), 0\le n\le M, \\ 1\omega_0^{k+1}=1\omega_M^{k+1}, 1\omega_1^{k+1}-1\omega_0^{k+1}=1\omega_M^{k+1}-1\omega_{M-1}^{k+1}, \\ -1\le k\le N-1 \end{cases} \tag{4.42}$$

In the first stage, we find numerical solution $\{\{1\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.42). For obtaining the solution of difference scheme (4.42), we will write it in the matrix form as

$$\begin{cases} A(1\omega)^{k+1} + B(1\omega)^{k} + C(1\omega)^{k-1} = (1f)^{k}, 1 \le k \le N-1, \\ 1\omega^{0} = 0, 1\omega^{1} = 2\tau(1+\cos(2x_n)), \end{cases} \tag{4.43}$$

where A, B, C are $(M+1)\times(M+1)$ square matrices, $1\omega^{s}, s = k, k\pm 1, 1f^{k}$ are $(M+1)\times 1$ column matrices and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ b & a+c_1 & b+c_1 & \cdots & c_1 & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdots & c_2 & c_2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & c_{M-2} & c_{M-2} & \cdots & a+c_{M-2} & b+c_{M-2} & 0 \\ 0 & c_{M-1} & c_{M-1} & \cdots & b+c_{M-1} & a+c_{M-1} & b \\ -1 & 1 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & e & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & g & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & g & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$1f^{k} = \begin{bmatrix} 0 \\ 1f(t_k, x_1) \\ . \\ 1f(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$1\omega^{s} = \begin{bmatrix} 1\omega_0^{s} \\ 1\omega_1^{s} \\ . \\ 1\omega_{M-1}^{s} \\ 1\omega_M^{s} \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k\pm 1.$$

Here,

$$a = \frac{1}{\tau^2} + \frac{2}{h^2}, b = -\frac{1}{h^2}, c_n = \cos(2x_n)\frac{2(\cos(2h)-1)}{\pi h}, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2},$$

$$1f(t_k, x_n) = \left[\frac{2(\cos(2h) - 1)}{h^2} - 1\right] \sin(2t_{k+1}) \cos(2x_n) - 5 \sin(2t_{k+1}),$$

$$1 \leq k \leq N - 1, 1 \leq n \leq M - 1.$$

So, we have the IVP for the second order difference equation (4.43) with respect to k with matrix coefficients A, B and C: Since $1\omega^0$ and $1\omega^1$ are given, we can obtain the solution of (4.43) by direct formula

$$1\omega^{k+1} = A^{-1}(1f^k - B(1\omega)^k - C(1\omega)^{k-1}), k = 1, \ldots, N - 1. \quad (4.44)$$

Applying formula $1\eta_{k+1} = \sum_{i=1}^{k} (k + 1 - i)(1p)_i \tau^2, 1 \leq k \leq N - 1, \eta_0 = \eta_1 = 0$, we can obtain

$$1p_k = \frac{1\eta_{k+1} - 2(1\eta)_k + 1\eta_{k-1}}{\tau^2}, 1 \leq k \leq N - 1. \quad (4.45)$$

In the second stage, we will obtain $\{1p_k\}_{k=1}^{N-1}$ by formulas (4.41) and (4.45). Finally, in the third stage, we will obtain $\{\{1u_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.40) and (4.41). The errors are computed by

$$1E_u = \max_{0 \leq k \leq N} \left(\sum_{n=0}^{M-1} |u(t_k, x_n) - 1u_n^k|^2 h \right)^{\frac{1}{2}}, \quad (4.46)$$

$$1E_p = \max_{1 \leq k \leq N-1} |p(t_k) - 1p_k|,$$

where $u(t, x), p(t)$ represent the exact solution, $1u_n^k$ represent the numerical solutions at (t_k, x_n) and $1p_k$ represent the numerical solutions at t_k.

Second, let $m \geq 2,$ then $(m - 1)N \leq k \leq mN.$ From problem (4.38) it follows that

$$\begin{cases} \dfrac{mu_n^{k+1} - 2(mu)_n^k + mu_n^{k-1}}{\tau^2} - \dfrac{mu_{n+1}^{k+1} - 2(mu)_n^{k+1} + mu_{n-1}^{k+1}}{h^2} \\ = mp_k\big(1 + \cos(2x_n)\big) - 5\sin(2t_{k+1}) - 0.96\sin(2t_{k+1})\cos(x_n), \\ +0.01\dfrac{(m-1)u_{n+1}^{k-N} - 2\big((m-1)u\big)_n^{k-N} + (m-1)u_{n-1}^{k-N}}{h^2}, \\ t_k = k\tau, x_n = nh, \\ (m-1)N + 1 \le k \le mN - 1, \\ 1 \le n \le M - 1, N\tau = \pi, Mh = \pi, \\ mu_n^{(m-1)N} = (m-1)u_n^{(m-1)N}, \\ \dfrac{mu_n^{(m-1)N+1} - mu_n^{(m-1)N}}{\tau} = \dfrac{(m-1)u_n^{(m-1)N} - (m-1)u_n^{(m-1)N-1}}{\tau}, \\ 0 \le n \le M, \\ mu_0^{k+1} = mu_M^{k+1}, mu_1^{k+1} - mu_0^{k+1} = mu_M^{k+1} - mu_{M-1}^{k+1}, \\ \sum_{i=0}^{M-1} \mathrm{m}u_i^{k+1}h = \pi\sin(2t_{k+1}), (m-1)N \le k \le mN, m \ge 2. \end{cases} \tag{4.47}$$

In the same manner, algorithm for obtaining the solution of the time-dependent identification problem (4.47) $\{mu_k\}_{k=0}^{N} = \{\{mu_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ and $\{mp_k\}_{k=1}^{N-1}$ contains three stages. Actually, let us define

$$mu_n^k = m\omega_n^k + m\eta_k\big(1 + \cos(x_n)\big), (m-1)N \le k \le mN, 0 \le n \le M, \tag{4.48}$$

Applying difference scheme (4.47) and formula (4.48), we will obtain formula

$$m\eta_{k+1} = \frac{\pi\sin(2t_{k+1}) - \sum_{i=0}^{M-1} m\omega_i^{k+1}h}{\pi}, (m-1)N - 1 \le k \le mN - 1, \tag{4.49}$$

and the difference scheme

$$
\begin{cases}
\dfrac{m\omega_n^{k+1} - 2(m\omega)_n^k + m\omega_n^{k-1}}{\tau^2} - \dfrac{m\omega_{n+1}^{k+1} - 2(m\omega)_n^{k+1} + m\omega_{n-1}^{k+1}}{h^2} \\
+ \displaystyle\sum_{i=0}^{M-1} m\omega_i^{k+1} \cos(2x_n) \dfrac{2(\cos(2h) - 1)}{h^2} \\
= 0.01 \dfrac{((m-1)w)_{n+1}^{k-N} - 2\big((m-1)w\big)_n^{k-N} + ((m-1)w)_{n-1}^{k-N}}{h^2} \\
+ \left[\dfrac{2(\cos(2h) - 1)}{h^2} - 0.96\right] \sin(2t_{k+1}) \cos(2x_n) - 5\sin(2t_{k+1}), \\
(m-1)N + 1 \le k \le mN - 1, \\
m\omega_n^{(m-1)N} = (m-1)\omega_n^{(m-1)N}, \\
\dfrac{m\omega_n^{(m-1)N+1} - m\omega_n^{(m-1)N}}{\tau} = \dfrac{(m-1)\omega_n^{(m-1)N} - (m-1)\omega_n^{(m-1)N-1}}{\tau}, \\
0 \le n \le M, \\
mu_0^{k+1} = mu_M^{k+1}, mu_1^{k+1} - mu_0^{k+1} = mu_M^{k+1} - mu_{M-1}^{k+1}, \\
(m-1)N \le k \le mN, m \ge 2.
\end{cases}
\tag{4.50}
$$

In the first stage, we find numerical solution $\{\{m\omega_n^k\}_{k=0}^N\}_{n=0}^M$ of corresponding first order of accuracy auxiliary difference scheme (4.50). For obtaining the solution of difference scheme (4.50), we will write it in the matrix form as

$$
\begin{cases}
A(m\omega)^{k+1} + B(m\omega)^k + C(m\omega)^{k-1} = (mf)^k, \\
(m-1)N + 1 \le k \le mN - 1 \\
(m\omega)_n^{(m-1)N} = ((m-1)\omega)_n^{(m-1)N}, \\
(m\omega)_n^{(m-1)N+1} = 2((m-1)\omega)_n^{(m-1)N} - ((m-1)\omega)_n^{(m-1)N-1},
\end{cases}
\tag{4.51}
$$

where A, B, C are $(M+1) \times (M+1)$ square matrices, $m\omega^s, s = k, k \pm 1, mf^k$ are $(M+1) \times 1$ column matrices and

$$
mf^k = \begin{bmatrix} 0 \\ mf(t_k, x_1) \\ . \\ mf(t_k, x_{M-1}) \\ 0 \end{bmatrix}_{(M+1)\times 1},
$$

$$m\omega^s = \begin{bmatrix} m\omega_0^s \\ m\omega_1^s \\ . \\ m\omega_{M-1}^s \\ m\omega_M^s \end{bmatrix}_{(M+1)\times 1}, \text{for } s = k, k \pm 1.$$

So, we have the initial value problem for the second order difference equation (4.51) with respect to k with matrix coefficients A, B and C: Since $m\omega_n^N$ and $m\omega_n^{N+1}$ are given, we can obtain the solution of (4.51) by direct formula

$$\begin{cases} (m\omega)^{k+1} = A^{-1}((mf)^k - B(m\omega)^k - C(m\omega)^{k-1}), \\ (m-1)N + 1 \leq k \leq mN - 1. \end{cases} \quad (4.52)$$

Applying formula $m\eta_{k+1} = \sum_{i=1}^{k} (k+1-i)(mp)_i\tau^2, (m-1)N + 1 \leq k \leq mN - 1, m\eta_{(m-1)N} = m\eta_{(m-1)N+1} = 0$, we can obtain

$$mp_k = \frac{m\eta_{k+1} - 2(m\eta)_k + m\eta_{k-1}}{\tau^2}, (m-1)N + 1 \leq k \leq mN - 1. \quad (4.53)$$

In the second stage, we will obtain $\{mp_k\}_{k=1}^{N-1}$ by formulas (4.49) and (4.53). Finally, in the third stage, we will obtain $\{\{mu_n^k\}_{k=0}^{N}\}_{n=0}^{M}$ by formulas (4.48) and (4.49). The errors are computed by

$$mE_u = \max_{(m-1)N \leq k \leq mN} \left(\sum_{n=0}^{M-1} |u(t_k, x_n) - mu_n^k|^2 h \right)^{\frac{1}{2}}, \quad (4.54)$$

$$mE_p = \max_{(m-1)N+1 \leq k \leq mN-1} |p(t_k) - mp_k|,$$

where $u(t,x), p(t)$ represent the exact solution, mu_n^k represent the numerical solutions at (t_k, x_n) and mp_k represent the numerical solutions at t_k. The numerical results are given in the following table.

Table 4.3.

Error Analysis for Difference Schemes (4.42) and (4.50)

$Error$	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
$1E_u$	1.1609	0.7991	0.4562	0.2424
$1E_p$	0.5665	0.2845	0.1384	0.0678
$2E_u$	2.3541	2.3087	1.4187	0.7631
$2E_p$	0.5251	0.3449	0.1957	0.1020
$3E_u$	2.5509	4.6543	3.3288	1.8975
$3E_p$	0.7032	0.2679	0.2075	0.1240

CHAPTER V

Conclusion

This thesis is devoted to the time-dependent source identification problems for delay hyperbolic differential equations with unknown parameter p(t). The following results are established:

• The history of direct and inverse boundary value problems for delay hyperbolic differential equations is considered.

• Fourier series, Laplace transform and Fourier transform methods are applied for the solution of six identification problems for delay hyperbolic differential equations.

• The main theorems on the stability estimates for the solution of the time-dependent source identification problems for delay hyperbolic differential equations are established.

• The first order of accuracy difference schemes for the approximate solution of the one dimensional time-dependent source identification problems for delay hyperbolic differential equations with local and non-local conditions are given.

• The Matlab implementation of these difference schemes is presented.

• The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

Our Future Plan is

• Investigate a high order of accuracy absolute stable difference schemes for the numerical solution of the time-dependent SIP for the DHE.

• Study the numerical realization for the numerical solution of two and three dimensional time-dependent SIP for the DHE.

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Appendices

Appendix A

Matlab Implementation of one Dimension First Order of Accuracy Difference Schemes of Problem (4.1)

```
function pb1(N,M)
h=pi/M;tau=pi/N;
a=(1/(tau^2))+(2/(h^2));
e=-2/(tau^2);
b=-1/(h^2);
g=1/(tau^2);
d=0;
for i=1:M-1;
d=d+h*sin(i*h);
end;
z=2*(cos(h)-1)/(d*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=z*sin((i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+(z*sin((i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(z*sin((i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(z*sin((i-1)*h));
end;
A(1,1)=1;A(M+1,M+1)=1;A(2,1)=b;A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1,1);
for j=1:M+1;
for k=2:N;
fii1(j,k)=((4*(cos(h)-1)/(d*(h^2)))-1)*sin(k*tau)*sin((j-1)*h);
end;
```

```
end;
fii1;
G=inv(A);
W1=zeros(M+1,1);
for j=1:M+1;
W1(j,1)=0;
W1(j,2)=(tau)*sin((j-1)*h);
for k=3:N+1;
W1(:,k)=G*(-(B*W1(:,k-1))-(C*W1(:,k-2))+fii1(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s1(j)=D+(W1(j,k+1)-2*(W1(j,k))+W1(j,k-1));
D=s1(j);
end;
p1(k)=(2*sin((k+1)*tau)-4*sin(k*tau)+2*sin((k-1)*tau)-(h*D))/(d*(tau^2));
end;
p1(k);
L=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
L(i,j)=0;
end;
end;
for i=2:M;
L(i,i)=a;
end;
for i=2:M-1;
L(i,i+1)=b;
end;
for i=3:M;
L(i,i-1)=b;
end;
L(1,1)=1;
L(M+1,M+1)=1;
L;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1,1);
for j=1:M+1;
```

```
for k=2:N;
x=(j-1)*h;
fii1(j,k)=(p1(k)*sin(x))-sin(k*tau)*sin(x);
end;
end;
fii1;
G=inv(L);
u1=zeros(M+1,1);
for j=1:M+1;
x=(j-1)*h;
u1(j,1)=0;
u1(j,2)=(tau)*sin(x);
end;
for k=3:N+1;
u1(:,k)=G*(-(B*u1(:,k-1))-(C*u1(:,k-2))+fii1(:,k-1));
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE'\%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es1(j,k)=(2*sin(t)-t)*sin(x);
eu1(j,k)=sin(t)*sin(x);
end;
end;
for k=2:N;
t=(k-1)*tau;
ep1(k)=sin(t);
end;
%ABSOLUTE DIFFERENCES;
absdifW1=max(max(abs(es1-W1)));
absdifu1=max(max(abs(eu1-u1)));
absdifp1=max(max(abs(ep1-p1)));
display([absdifW1,absdifu1,absdifp1])

%SECOND STEP;
fii2=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii2(j,k)=((0.01)/h^2)*(W1(j+1,k)-2*W1(j,k)+W1(j-1,k))+(((4*(cos(h)-1)/(d*(h^2)))-
1)*sin((k+N)*tau)-((0.01)*sin(k*tau)))*sin((j-1)*h);
end;
end;
fii2;
G=inv(A);
W2=zeros(M+1,1);
for j=1:M+1;
W2(j,1)=W1(j,N+1);
W2(j,2)=2*W1(j,N+1)-W1(j,N);
for k=3:N+1;
```

```
W2(:,k)=G*(-(B*W2(:,k-1))-(C*W2(:,k-2))+fii2(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s2(j)=D+(W2(j,k+1)-2*(W2(j,k))+W2(j,k-1));
D=s2(j);
end;
p2(k)=(2*sin((k+N+1)*tau)-4*sin((k+N)*tau)+2*sin((k+N-1)*tau)-(h*D))/(d*(tau^2
));
end;
p2(k);
fii2=zeros(M+1,1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii2(j,k)=(p2(k)-sin((k+N)*tau)-(0.01)*sin(k*tau))*sin(x)+((0.01)/h^2)*(W1(j+1,k)-2
*W1(j,k)+W1(j-1,k));
end;
end;
fii2;
G=inv(L);
u2=zeros(M+1,1);
for j=1:M+1;
x=(j-1)*h;
u2(j,1)=u1(j,N+1);
u2(j,2)=2*u1(j,N+1)-u1(j,N);
end;
for k=3:N+1;
u2(:,k)=G*(-(B*u2(:,k-1))-(C*u2(:,k-2))+fii2(:,k-1));
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE'\%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
t=(k+N-1)*tau;
x=(j-1)*h;
es2(j,k)=(2*sin(t)-t)*sin(x);
eu2(j,k)=sin(t)*sin(x);
end;
end;
for k=2:N;
t=(k+N-1)*tau;
ep2(k)=sin(t);
end;
%ABSOLUTE DIFFERENCES;
absdifW2=max(max(abs(es2-W2)));
absdifu2=max(max(abs(eu2-u2)));
absdifp2=max(max(abs(ep2-p2)));
display([absdifW2,absdifu2,absdifp2])
```

```
%THIRD STEP;
fii3=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii3(j,k)=((0.01)/h^2)*(W2(j+1,k)-2*W2(j,k)+W2(j-1,k))+(((4*(cos(h)-1)/(d*(h^2)))-
1)*sin((k+(2*N))*tau)-((0.01)*sin(k*tau)))*sin((j-1)*h);
end;
end;
fii3;
G=inv(A);
W3=zeros(M+1,1);
for j=1:M+1;
W3(j,1)=W2(j,N+1);
W3(j,2)=2*W2(j,N+1)-W2(j,N);
for k=3:N+1;
W3(:,k)=G*(-(B*W3(:,k-1))-(C*W3(:,k-2))+fii3(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s3(j)=D+(W3(j,k+1)-2*(W3(j,k))+W3(j,k-1));
D=s3(j);
end;
p3(k)=(2*sin((k+2*N+1)*tau)-4*sin((k+2*N)*tau)+2*sin((k+2*N-1)*tau)-(h*D))/(d
*(tau^2));
end;
p3(k);
fii3=zeros(M+1,1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii3(j,k)=(p3(k)-sin((k+2*N)*tau)-(0.01)*sin(k*tau))*sin(x)+((0.01)/h^2)*(W2(j+1,k
)-2*W2(j,k)+W2(j-1,k));
end;
end;
fii3;
G=inv(L);
u3=zeros(M+1,1);
for j=1:M+1;
x=(j-1)*h;
u3(j,1)=u2(j,N+1);
u3(j,2)=2*u2(j,N+1)-u2(j,N);
end;
for k=3:N+1;
u3(:,k)=G*(-(B*u3(:,k-1))-(C*u3(:,k-2))+fii3(:,k-1));
end;
%\%\%\%\%\%'EXACT SOLUTION OF THIS PDE'\%\%\%\%\%\%\%\%
for j=1:M+1;
for k=1:N+1;
```

```
t=(k+2*N-1)*tau;
x=(j-1)*h;
es3(j,k)=(2*sin(t)-t)*sin(x);
eu3(j,k)=sin(t)*sin(x);
end;
end;
for k=2:N;
t=(k+2*N-1)*tau;
ep3(k)=sin(t);
end;
%ABSOLUTE DIFFERENCES;
absdifW3=max(max(abs(es3-W3)));
absdifu3=max(max(abs(eu3-u3)));
absdifp3=max(max(abs(ep3-p3)));
display([absdifW3,absdifu3,absdifp3])
```

Appendix B

Matlab Implementation of one Dimension First Order of Accuracy Difference Schemes of Problem (4.19)

```
function pb2(N,M)
h=pi/M;tau=pi/N;
a=(1/(tau^2))+(2/(h^2));
e=-2/(tau^2);
b=-1/(h^2);
g=1/(tau^2);
z=2*(cos(h)-1)/(pi*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=z*cos((i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+(z*cos((i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(z*cos((i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(z*cos((i-1)*h));
end;
A(1,1)=-1;A(1,2)=1;A(M+1,M+1)=1;A(M+1,M)=-1;
A(2,1)=b;A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1,1);
for j=2:M;
for k=2:N; fii1(j,k)=((2*(cos(h)-1)/(h^2))-1)*sin(k*tau)*cos((j-1)*h)-2*sin(k*tau);
end;
end;
fii1;
G=inv(A);
W1=zeros(M+1);
for j=1:M+1;
```

```
W1(j,1)=0;
W1(j,2)=(tau)*(1+cos((j-1)*h));
for k=3:N+1;
W1(:,k)=G*(-(B*W1(:,k-1))-(C*W1(:,k-2))+fii1(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s1(j)=D+(W1(j,k+1)-2*(W1(j,k))+W1(j,k-1));
D=s1(j);
end;
p1(k)=((sin((k+1)*tau)-2*sin(k*tau)+sin((k-1)*tau))/((tau)^2))-((h*D)/(pi*(tau)^2));
end;
L=zeros(M+1);
for i=2:M;
L(i,i)=a;
end;
for i=2:M-1;
L(i,i+1)=b;
end;
for i=3:M;
L(i,i-1)=b;
end;
L(1,1)=-1;L(1,2)=1;
L(M+1,M+1)=1;L(M+1,M)=-1;
L(2,1)=b;L(M,M+1)=b;
L;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii1(j,k)=(p1(k)*(1+cos(x)))-sin(k*tau)*(2+cos(x));
end;
end;
fii1;
G=inv(L);
u1=zeros(M+1);
for j=1:M+1;
x=(j-1)*h;
```

```
u1(j,1)=0;
u1(j,2)=(tau)*(1+cos(x));
end;
for k=3:N+1;
u1(:,k)=G*(-(B*u1(:,k-1))-(C*u1(:,k-2))+fii1(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es1(j,k)=(2*sin(t)-t)*(1+cos(x));
eu1(j,k)=sin(t)*(1+cos(x));
end;
end;
for k=2:N;
t=(k-1)*tau;
ep1(k)=sin(t);
end;
%ABSOLUTE DIFFERENCES;
absdifW1=max(max(abs(es1-W1)));
absdifu1=max(max(abs(eu1-u1)));
absdifp1=max(max(abs(ep1-p1)));
display([absdifW1,absdifu1,absdifp1])
%SECOND STEP;
fii2=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii2(j,k)=((0.01)/h^2)*(W1(j+1,k)-2*W1(j,k)+W1(j-1,k))+((2*(cos(h)-1)/(h^2)-1)*(si
n((k+N)*tau)-((0.01)*sin(k*tau)))*cos((j-1)*h))-2*sin((k+N)*tau);
end;
end;
fii2;
G=inv(A);
W2=zeros(M+1);
for j=1:M+1;
W2(j,1)=W1(j,N+1);
W2(j,2)=2*W1(j,N+1)-W1(j,N);
for k=3:N+1;
W2(:,k)=G*(-(B*W2(:,k-1))-(C*W2(:,k-2))+fii2(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s2(j)=D+(W2(j,k+1)-2*(W2(j,k))+W2(j,k-1));
D=s2(j);
end;
```

```
p2(k)=((sin((k+N+1)*tau)-2*sin((k+N)*tau)+sin((k+N-1)*tau))/((tau)^2))-((h*D)/(pi
*(tau)^2));
end;
fii2=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii2(j,k)=p2(k)*(1+cos(x))-2*sin((k+N)*tau)-sin((k+N)*tau)*cos(x)-(0.01)*sin(k*tau
)*cos(x)+(((0.01)/h^2)*(W1(j+1,k)-2*W1(j,k)+W1(j-1,k)));
end;
end;
fii2;
G=inv(L);
u2=zeros(M+1);
for j=1:M+1;
x=(j-1)*h;
u2(j,1)=u1(j,N+1);
u2(j,2)=2*u1(j,N+1)-u1(j,N);
end;
for k=3:N+1;
u2(:,k)=G*(-(B*u2(:,k-1))-(C*u2(:,k-2))+fii2(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k+N-1)*tau;
x=(j-1)*h;
es2(j,k)=(2*sin(t)-t)*(1+cos(x));
eu2(j,k)=sin(t)*(1+cos(x));
end;
end;
for k=2:N;
t=(k+N-1)*tau;
ep2(k)=sin(t);
end;
%ABSOLUTE DIFFERENCES;
absdifW2=max(max(abs(es2-W2)));
absdifu2=max(max(abs(eu2-u2)));
absdifp2=max(max(abs(ep2-p2)));
display([absdifW2,absdifu2,absdifp2])
%THIRD STEP;
fii3=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii3(j,k)=((0.01)/h^2)*(W2(j+1,k)-2*W2(j,k)+W2(j-1,k))+((2*(cos(h)-1)/(h^2)-1)*(si
n((k+(2*N))*tau)-((0.01)*sin(k*tau)))*cos((j-1)*h))-2*sin((k+(2*N))*tau);
end;
end;
fii3;
```

```
G=inv(A);
W3=zeros(M+1);
for j=1:M+1;
W3(j,1)=W2(j,N+1);
W3(j,2)=2*W2(j,N+1)-W2(j,N);
for k=3:N+1;
W3(:,k)=G*(-(B*W3(:,k-1))-(C*W3(:,k-2))+fii3(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s3(j)=D+(W3(j,k+1)-2*(W3(j,k))+W3(j,k-1));
D=s3(j);
end;
p3(k)=((sin((k+2*N+1)*tau)-2*sin((k+2*N)*tau)+sin((k+2*N-1)*tau))/((tau)^2))-((h
*D)/(pi*(tau)^2));
end;
fii3=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii3(j,k)=p3(k)*(1+cos(x))-2*sin((k+2*N)*tau)-sin((k+2*N)*tau)*cos(x)-(0.01)*sin(
k*tau)*cos(x)+(((0.01)/h^2)*(W2(j+1,k)-2*W2(j,k)+W2(j-1,k)));
end;
end;
fii3;
G=inv(L);
u3=zeros(M+1);
for j=1:M+1;
x=(j-1)*h;
u3(j,1)=u2(j,N+1);
u3(j,2)=2*u2(j,N+1)-u2(j,N);
end;
for k=3:N+1;
u3(:,k)=G*(-(B*u3(:,k-1))-(C*u3(:,k-2))+fii3(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k+2*N-1)*tau;
x=(j-1)*h;
es3(j,k)=(2*sin(t)-t)*(1+cos(x));
eu3(j,k)=sin(t)*(1+cos(x));
end;
end;
for k=2:N;
t=(k+2*N-1)*tau;
ep3(k)=sin(t);
end;
```

```
%ABSOLUTE DIFFERENCES;
absdifW3=max(max(abs(es3-W3)));
absdifu3=max(max(abs(eu3-u3)));
absdifp3=max(max(abs(ep3-p3)));
display([absdifW3,absdifu3,absdifp3])
```

Appendix C

Matlab Implementation of one Dimension First Order of Accuracy Difference Schemes of Problem (4.37)

```
function pb3(N,M)

h=pi/M;tau=pi/N;
a=(1/(tau^2))+(2/(h^2));
e=-2/(tau^2);
b=-1/(h^2);
g=1/(tau^2);
z=2*(cos(2*h)-1)/(pi*h);
A=zeros(M+1,M+1);
for i=2:M;
for j=2:M;
A(i,j)=z*cos(2*(i-1)*h);
end;
end;
for i=2:M
A(i,i)=a+(z*cos(2*(i-1)*h));
end;
for i=2:M-1;
A(i,i+1)=b+(z*cos(2*(i-1)*h));
end;
for i=3:M;
A(i,i-1)=b+(z*cos(2*(i-1)*h));
end;
A(1,1)=1;A(1,M+1)=-1;A(M+1,1)=-1;A(M+1,2)=1;A(M+1,M)=1;A(M+1,M+1)=-1;
A(2,1)=b;A(M,M+1)=b;
A;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end;
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii1(j,k)=((2*(cos(2*h)-1)/(h^2))-1)*sin(2*k*tau)*cos(2*(j-1)*h)-5*sin(2*k*tau);
end;
end;
fii1;
G=inv(A);
W1=zeros(M+1);
```

```
for j=1:M+1;
W1(j,1)=0;
W1(j,2)=2*(tau)*(1+cos(2*(j-1)*h));
for k=3:N+1;
W1(:,k)=G*(-(B*W1(:,k-1))-(C*W1(:,k-2))+fii1(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s1(j)=D+(W1(j,k+1)-2*(W1(j,k))+W1(j,k-1));
D=s1(j);
end;
p1(k)=((sin(2*(k+1)*tau)-2*sin(2*k*tau)+sin(2*(k-1)*tau))/((tau)^2))-((h*D)/(pi*(ta
u)^2));
end;
L=zeros(M+1);
for i=2:M;
L(i,i)=a;
end;
for i=2:M-1;
L(i,i+1)=b;
end;
for i=3:M;
L(i,i-1)=b;
end;
L(1,1)=1;L(1,M+1)=-1;
L(M+1,1)=-1;L(M+1,2)=1;L(M+1,M)=1;L(M+1,M+1)=-1;
L(2,1)=b;L(M,M+1)=b;
L;
B=zeros(M+1,M+1);
for n=2:M;
B(n,n)=e;
end
B;
C=zeros(M+1,M+1);
for n=2:M;
C(n,n)=g;
end;
C;
fii1=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii1(j,k)=(p1(k)*(1+cos(2*x)))-sin(2*k*tau)*(5+cos(2*x));
end;
end;
fii1;
G=inv(L);
u1=zeros(M+1);
```

```
for j=1:M+1;
x=(j-1)*h;
u1(j,1)=0;
u1(j,2)=2*(tau)*(1+cos(2*x));
end;
for k=3:N+1;
u1(:,k)=G*(-(B*u1(:,k-1))-(C*u1(:,k-2))+fii1(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es1(j,k)=((5/4)*sin(2*t)-(1/2)*t)*(1+cos(2*x));
eu1(j,k)=sin(2*t)*(1+cos(2*x));
end;
end;
for k=2:N;
t=(k-1)*tau;
ep1(k)=sin(2*t);
end;
%ABSOLUTE DIFFERENCES;
absdifW1=max(max(abs(es1-W1)));
absdifu1=max(max(abs(eu1-u1)));
absdifp1=max(max(abs(ep1-p1)));
display([absdifW1,absdifu1,absdifp1])
%SECOND STEP;
fii2=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii2(j,k)=((-0.01)/h^2)*(W1(j+1,k)-2*W1(j,k)+W1(j-1,k))+((2*(cos(2*h)-1)/(h^2)-1)
*(sin(2*(k+N)*tau)-(4*(0.01)*sin(2*k*tau)))*cos(2*(j-1)*h))-5*sin(2*(k+N)*tau);
end;
end;
fii2;
G=inv(A);
W2=zeros(M+1);
for j=1:M+1;
W2(j,1)=W1(j,N+1);
W2(j,2)=2*W1(j,N+1)-W1(j,N);
for k=3:N+1;
W2(:,k)=G*(-(B*W2(:,k-1))-(C*W2(:,k-2))+fii2(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s2(j)=D+(W2(j,k+1)-2*(W2(j,k))+W2(j,k-1));
D=s2(j);
end;
```

```
p2(k)=((sin(2*(k+N+1)*tau)-2*sin(2*(k+N)*tau)+sin(2*(k+N-1)*tau))/((tau)^2))-((h
*D)/(pi*(tau)^2));
end;
fii2=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii2(j,k)=p2(k)*(1+cos(2*x))-5*sin(2*(k+N)*tau)-sin(2*(k+N)*tau)*cos(2*x)-4*(0.0
1)*sin(2*k*tau)*cos(2*x)-(((0.01)/h^2)*(W1(j+1,k)-2*W1(j,k)+W1(j-1,k)));
end;
end;
fii2;
G=inv(L);
u2=zeros(M+1);
for j=1:M+1;
x=(j-1)*h;
u2(j,1)=u1(j,N+1);
u2(j,2)=2*u1(j,N+1)-u1(j,N);
end;
for k=3:N+1;
u2(:,k)=G*(-(B*u2(:,k-1))-(C*u2(:,k-2))+fii2(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k+N-1)*tau;
x=(j-1)*h;
es2(j,k)=((5/4)*sin(2*t)-(1/2)*t)*(1+cos(2*x));
eu2(j,k)=sin(2*t)*(1+cos(2*x));
end;
end;
for k=2:N;
t=(k+N-1)*tau;
ep2(k)=sin(2*t);
end;
%ABSOLUTE DIFFERENCES;
absdifW2=max(max(abs(es2-W2)));
absdifu2=max(max(abs(eu2-u2)));
absdifp2=max(max(abs(ep2-p2)));
display([absdifW2,absdifu2,absdifp2])
%THIRD STEP;
fii3=zeros(M+1,1);
for j=2:M;
for k=2:N;
fii3(j,k)=((-0.01)/h^2)*(W2(j+1,k)-2*W2(j,k)+W2(j-1,k))+((2*(cos(2*h)-1)/(h^2)-1)
*(sin(2*(k+2*N)*tau)-(4*(0.01)*sin(2*k*tau)))*cos(2*(j-1)*h))-5*sin(2*(k+2*N)*t
au);
end;
end;
fii3;
```

```
G=inv(A);
W3=zeros(M+1);
for j=1:M+1;
W3(j,1)=W2(j,N+1);
W3(j,2)=2*W2(j,N+1)-W2(j,N);
for k=3:N+1;
W3(:,k)=G*(-(B*W3(:,k-1))-(C*W3(:,k-2))+fii3(:,k-1));
end;
end;
for k=2:N;
D=0;
for j=1:M-1;
s3(j)=D+(W3(j,k+1)-2*(W3(j,k))+W3(j,k-1));
D=s3(j);
end;
p3(k)=((sin(2*(k+2*N+1)*tau)-2*sin(2*(k+2*N)*tau)+sin(2*(k+2*N-1)*tau))/((tau)
^2))-((h*D)/(pi*(tau)^2));
end;
fii3=zeros(M+1);
for j=2:M;
for k=2:N;
x=(j-1)*h;
fii3(j,k)=p3(k)*(1+cos(2*x))-5*sin(2*(k+2*N)*tau)-sin(2*(k+2*N)*tau)*cos(2*x)-4
*(0.01)*sin(2*k*tau)*cos(2*x)-(((0.01)/h^2)*(W2(j+1,k)-2*W2(j,k)+W2(j-1,k)));
end;
end;
fii3;
G=inv(L);
u3=zeros(M+1);
for j=1:M+1;
x=(j-1)*h;
u3(j,1)=u2(j,N+1);
u3(j,2)=2*u2(j,N+1)-u2(j,N);
end;
for k=3:N+1;
u3(:,k)=G*(-(B*u3(:,k-1))-(C*u3(:,k-2))+fii3(:,k-1));
end;
%n%n%n%n%n%?EXACT SOLUTION OF THIS PDE?n%n%n%n%n%n%n%
for j=1:M+1;
for k=1:N+1;
t=(k+2*N-1)*tau;
x=(j-1)*h;
es3(j,k)=((5/4)*sin(2*t)-(1/2)*t)*(1+cos(2*x));
eu3(j,k)=sin(2*t)*(1+cos(2*x));
end;
end;
for k=2:N;
t=(k+2*N-1)*tau;
ep3(k)=sin(2*t);
end;
```

```
%ABSOLUTE DIFFERENCES;
absdifW3=max(max(abs(es3-W3)));
absdifu3=max(max(abs(eu3-u3)));
absdifp3=max(max(abs(ep3-p3)));
display([absdifW3,absdifu3,absdifp3])
```

Appendix D

NEAR EAST UNIVERSITY

Ethical Approval Document

Date: 28 / 06 /2022

To Graduate School of Applied Sciences

The research project title **"The time-dependent source identification problem for the delay hyperbolic equations"** has been evaluated. Since the researcher(s) will not collect primary data from humans, animals, plants or earth, this project does not need to go through the ethics committee.

Title: Prof. Dr.

Name Surname: Allaberen Ashyralyev

Signature:

Role in the Research Project: Supervisor

Appendix E

Turnitin Similarity Report

Tez

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

Sil | İndir | Şuraya taşı...

Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
Bishar Chato Haso	Abstract	%0 %0	0%	0%	0%	--	--	ödev indir	1806152762	09-Nis-2022
Bishar Chato Haso	Conclusion	%0 %0	0%	0%	0%	--	--	ödev indir	1806153906	09-Nis-2022
Bishar Chato Haso	Chapter 3	%9 %9	10%	4%	0%	--	--	ödev indir	1806153508	09-Nis-2022
Bishar Chato Haso	All Thesis	%10 %10	5%	12%	6%	--	--	ödev indir	1806154261	09-Nis-2022
Bishar Chato Haso	Chapter 1	%10 %10	12%	11%	6%	--	--	ödev indir	1806152955	09-Nis-2022
Bishar Chato Haso	Chapter 2	%10 %10	9%	12%	4%	--	--	ödev indir	1806153216	09-Nis-2022
Bishar Chato Haso	Chapter 4	%11 %11	8%	10%	0%	--	--	ödev indir	1806153726	09-Nis-2022

Appendix F

Curriculum Vita (CV)

Personal information

Full Name: Bishar Chato Haeo

Nationality: Iraq

Data and Place of birth: Nineveh – Iraq, (01-Jan-1988).

Marital Status: Married

E-mail address: bashar.chato1988@gmail.com

Education

Degree	**Institute**	**Year of Graduation**
B.Sc. In Mathematics	University of Zakho, Department of Mathematics	2013

Professional Experience:

- Worked as a teacher at Sinuny Preparatory Mixed School in Sinuny, Sinjar, Nineveh for one year (2013-2014)
- Worked as a teacher teaching math at Shingal Institute for Teacher Training for one year (2014-2015)
- Worked as a lecturer at Ronahy Preparatory Mixed School in Sharya-Duhok for four years (2015-2019).

Languages:

Kurdish	Listening	Writing	Speaking	Reading
Arabic	Listening	Writing	Speaking	Reading
English	Listening	Writing	Speaking	Reading

IT Skills:

- Computer literate with good working knowledge of Microsoft Office programs including Word, Excel, Power Point, etc.
- Good at surfing internet and well managing with social networking websites such as: Skype, Gmail, Yahoo, Facebook, etc.

Motivation:

- Enjoy a challenge and work hard to achieve objectives.

Technical Expertise:

- Have a good ability for solving problems.
- Commitment to best practice.
- Attention to details.
- Able to motivate others.
- Have experience in coaching and training

INTERNATIONAL PUBLICATIONS:

Allaberen Ashyralyev and Bishar Haso, " On the Stability of the Time-Dependent Identification Problem for the Delay Hyperbolic Equation." AIP Conference Proceedings (2022).

Allaberen Ashyralyev and Bishar Haso, "Numerical Solution of the Time-Dependent Source Identification Problem for the Delay Hyperbolic Equation." AIP Conference Proceedings (2022).

Allaberen Ashyralyev and Bishar Haso, " Stability of the time-dependent identification problem for delay hyperbolic equations." Bulletin of the Karaganda University (2022).

Allaberen Ashyralyev and Bishar Haso, " Stability of the time-dependent identification problem for delay hyperbolic partial differential equation with Dirichlet boundary conditions." International Journal of Applied Mathematics (2022).