NEAR EAST UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

TELEGRAPH TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

M.Sc. THESIS

Ogulbabek BATYROVA

Nicosia

February, 2023

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Approval

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To my parents...

ABSTRACT

The aim of this thesis is to investigate the initial boundary value problem for the telegraph type involutory partial differential equations. This thesis deals with analytical and approximate solutions of several problems for involutory telegraph equations. Moreover, the stability of the initial value problem for the second order differential equations with damping term and involution is established. New absolute stable difference schemes for the approximate solution of the one dimensional involutory telegraph differential equation are constructed and a numerical algorithm is presented. Numerical analysis is provided. In Chapter Two, we obtain an equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order differential equations with damping term and involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with damping term and involution is proved. Theorem on existence and uniqueness of bounded solution of initial value problem for the second order ordinary nonlinear differential equation with damping term and ınvolution is established. In Chapter Three, we get exact solutions of the several problems for involutory telegraph equations by using the result of Chapter Two and Fourier series method, Laplace and Fourier transform methods. Involutory telegraph equations have not been investigated before. Therefore, this approach is applied for the first time in this thesis. Note that these methods can be used for multidimensional telegraph type involutory partial differential equations. In Chapter Four, new absolute stable difference schemes for the numerical solution of the one dimensional involutory telegraph equations are presented. For the first and second order of accuracy difference schemes have been built, a program is written, examples are solved, and numerical results have been tabulated. Comparisons of errors are made between the exact and numerical solutions in the maximum norm. All the computer programs are written in Matlab.

Keyword: Involutory telegraph differential equations; Fourier series method; Laplace transform solution; Fourier transform solution; Difference scheme; Numerical algorithm; Computer programs.

ÖZET

Bu tezin amacı, telgraf tipi kısmi diferansiyel denklemler içeren başlangıç sınır değer problemini incelemektir. Bu tez, telgraf denklemleri için çeşitli problemlerin analitik ve yaklaşık çözümlerini ele almaktadır. Ayrıca, sönüm terimli ve involüsyonlu ikinci mertebeden diferansiyel denklemler için başlangıç değer probleminin kararlılığı kurulmuştur. Tek boyutlu telgraf diferansiyel denkleminin yaklaşık çözümü için yeni mutlak kararlı fark şemaları oluşturulmuş ve sayısal bir algoritma sunulmuştur. Sayısal analiz sağlanmıştır. İkinci Bölümde, dördüncü mertebe adi diferansiyel denklemler için sönüm terimli ve involüsyonlu ikinci mertebe diferansiyel denklemler için başlangıç değer problemine eşdeğer bir başlangıç değer problemi elde ediyoruz. Sönüm terimli ve involüsyonlu ikinci mertebeden adi lineer diferansiyel denklemin başlangıç değer probleminin çözümü için kararlılık tahminlerine ilişkin teorem kanıtlanmıştır. Sönüm terimli ve involüsyonlu ikinci mertebeden adi lineer olmayan diferansiyel denklem için başlangıç değer probleminin sınırlı çözümünün varlığı ve tekliği üzerine teorem kurulmustur. Üçüncü Bölüm'de, İkinci Bölüm ve Fourier seri yöntemi, Laplace ve Fourier dönüşüm yöntemlerinin sonuçlarını kullanarak, telgraf denklemleri için çeşitli problemlerin kesin çözümlerini elde edilmiştir. İnvolüsyon telgraf denklemleri daha önce araştırılmamış. Bu nedenle bu yaklaşım ilk kez bu tezde uygulanmıştır. Bu yöntemlerin çok boyutlu telgraf tipi kısmi diferansiyel denklemler için kullanılabilir. Dördüncü Bölüm'de, tek boyutlu telgraf denklemlerinin sayısal çözümü için yeni mutlak kararlı fark şemaları sunulmaktadır. Birinci ve ikinci doğruluk mertebesi için fark şemaları oluşturulmuş, program yazılmış, örnekler çözülmüş ve sayısal sonuçlar tablolaştırılmıştır. Maksimum normdaki kesin ve sayısal çözümler arasında hata karşılaştırmaları yapılır. Tüm bilgisayar programları Matlab ile yazılmıştır.

Anahtar Kelimeler: İnvolüsyon telgraf diferansiyel denklemleri; Fourier serisi yöntemi; Laplace dönüşümü çözümü; Fourier dönüşümü çözümü; Fark şeması; Sayısal algoritma; Bilgisayar programları

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LIST OF ABBREVIATIONS AND SYMBOLS

DEI	Differential Equation with Involution
IDE	Involutory Differential Equation
DS	Difference Scheme
DSs	Difference Schemes

CHAPTER 1

INTRODUCTION

1.1 Historical Note and Literature Survey

Delay differential equations are universal phenomena applied to their models in engineering systems to behave like a real process (Vlasov & Rautian, 2016; Bhalekar & Patade, 2016; Srividhya & Gopinathan, 2006; Sriram & Gopinathan, 2004). Initial conditions in one point are not enough to get the solution of delay differential equations. For the first time (Falbo, 2013), in an experiment measuring the population growth of a species of water fleas, Nisbet tried to use a delay differential model in his study. He clarified the form of the population equation in

$$N'(t) = aN(t-d) + bN(t).$$

The obstacle in his investigation was that he did not have enough information about reasonable history function $N(t)$ on $[-d, 0]$ to get a solution to this problem. He reversed time to get the solution of functional differential equations. He used time reversal in order to seek the population before the initial time $t = 0$ (Nisbet, 1997). An involutory differential equation is a type of equation and a time reversal problem is its special case. It is called an involutory differential equation, if it is involving an unknown function y at t and $d - t$.

Definition 1.1.1. Assume that $u(t)$ is the involutory function, that is, $u(u(t)) = t$ and t_0 is the fixed point of u. Then

$$y'(t) = f(t; y(t); y(u(t))$$

is called the involutory differential equation or differential equation with involution. Really, Falbo in 2006 has been surprised to learn differential equations with delay and involution terms have deeply different properties of solutions. Moreover, the theory of differential equations with involution and without involution is deeply different. Therefore, it is important to study the properties of differential equations with involution. Involutions have been an interesting subject of research at least since Rothe first computed the number of different involutions on finite sets in 1800. After that, Babbage published in 1816 the foundational paper in which functional equations are first considered, in particular, those of the form $u(u(t)) = t$.

The theory and applications of ordinary differential equations with involution have been studied by many authors. The algebraic and analytic aspects of the theory of ordinary differential equations with involution were discussed in the monographs of Przeworska-Rolewicz and Wiener 1993(Przeworska-Rolewicz, 1973; Wiener, 1993). Spectral problems arising in connection with differential operators with involution were considered in papers of Kurdyumov, Khromov, Cabada, Tojo, (Kurdyumov & Khromov, 2008; Cabada & Tojo, 2014) for first-order operators and in papers of Sadybekov, Sarsenbi; Kopzhassarova, Sarsenbi(Sadybekov & Sarsenbi, 2012; Kopzhassarova & Sarsenbi, 2012) for second-order operators with involution. Despite the progress of the theory of functional equations, we have waited for Silberstein, who in 1940 solved the first functional differential equation with an involution (Silberstein, 1940). The interest in differential equations with involutions is retaken by Wiener in 1968. Wiener, together with Watkins, will lead the discoveries in this direction in the following decades (Aftabizadeh, Huang, & Wiener, 1988) Quite a lot of work has been done ever since by several authors. In 2013 the first Green's function for a differential equation with an involution was computed by Cabada, and Tojo, 2013 and the field rapidly expanded (Cabada & Tojo, 2013, 2017) Cabada and Tojo in the monograph cover the existing results regarding Green's functions for differential equations with involutions (DEI). The first part of the book is devoted to the study of the most useful aspects of involutions from an analytical point of view and the associated algebras of differential operators. The work combines the state of the art regarding the existence and uniqueness results for DEI and new theorems describing how to obtain Green's functions, proving that the theory can be extended to operators (not necessarily involutions) of a similar nature, such as the Hilbert transform or projections, due to their analogous algebraic properties. Obtaining a Green's function for these operators leads to new results on the qualitative properties of the solutions, in particular maximum and antimaximum principles. The existence and uniqueness of the bounded solution of a nonlinear one-dimensional delay hyperbolic differential equation with constant coefficients were proved in by Shah, Poorkarimi, and Wiener, 1986 (Shah, Poorkarimi, & Wiener, 1986). Note that for investigating a wide class of multidimensional delay hyperbolic nonlinear differential equations the approach of this paper is not applicable. Recently, delay hyperbolic differential equations have been studied in several papers by Ashyralyev, and Agirseven, 2014, 2019; Son, Thao, 2019; Monteghetti, Haine, Matignon, 2017; Zhang, Zhang, Deng, 2014; Prakash, Harikrishnan, 2012; Vyazmin, Sorokin, 2017, 2002; Farkas, 2003.(Son & Thao, 2019; Monteghetti, Haine, & Matignon, 2017; Prakash & Harikrishnan, 2012; Vyazmin & Sorokin, 2017; Farkas, 2003)

Ashyralyev and Agirseven 2019 studied the existence and uniqueness of a bounded solution in a semi linear time delay hyperbolic equation in a Hilbert space (Ashyralyev & Agirseven, 2019). In applications, theorems on the existence and uniqueness of bounded solutions of four problems for semi linear time delay differential equations of hyperbolic type are obtained. The two-step difference scheme of a first order of accuracy is presented the mean theorem on the existence and uniqueness of a uniformly bounded solution of this difference scheme with respect to time step size is proved. In applications, theorems on the existence and uniqueness of uniformly bounded solutions with respect to time and space step sizes of difference schemes for four semi linear time delay partial differential equations are established. Numerical results are presented.

In the paper of Prakash and Harikrishnan, 2012, a class of impulsive vector hyperbolic differential equations with delays were investigated (Prakash & Harikrishnan, 2012). They studied different sufficient conditions for H-oscillation of solutions systems subject to the Neumann boundary condition by employing certain second-order impulsive differential inequality, where H is a unite vector in R^M. The main results are illustrated by two examples.

Differential equations with involution appear in mathematical models of ecology, biology, and population dynamics (Przeworska-Rolewicz, 1973; Wiener, 1993). In recent decades, one-dimensional partial differential equations with involution in x have been investigted by many scientists in papers (Ashyralyev & Sarsenbi, 2017b, 2015; Ashyralyev, Karabaeva, &

Sarsenbi, 2016; Ashyralyev & Sarsenbi, 2017a) In the study of Ashyralyev, Sarsenbi, 2017, the mixed problem of one dimensional parabolic equation with involution in x was investigated. Applying operator tools, the stability and coercive stability estimates in Holder norms for the solution to this problem were established. In the paper of Ashyralyev, Sarsenbi, 2015, a mixed problem for two dimensional elliptic equation with involution was studied. This problem was reduced to a boundary value problem for the abstract elliptic equation in Hilbert space with a self-adjoint positive definite operator. Operator tools permit us to obtain stability and coercive stability estimates in Holder norms, in one variable, for the solution. In the paper of Ashyralyev, Kakabaeva, and Sarsenbi, 2016, a stable difference scheme for the approximate solution of elliptic equations with involution was constructed. Theorem on stability and almost coercive stability and coercive stability of this difference scheme was established. The theoretical statements for statements for the solution of this difference scheme were supported by the results of the numerical experiment. In the paper of Ashyralyev and Sarsenbi, 2017, a mixed problem of one dimensional hyperbolic equation with the involution in x was investigated. The stability estimates in the maximum norm in t for the solution of this problem are established. In the Ph.D. thesis of Sarsenbi 2019 under Turmetov and Ashyralyev's supervise, the theory of the basic property of eigenfunctions of second order differential operators with involution was investigated, on this basis, the Fourier method was justified for solving direct and inverse problems for one dimensional parabolic equations with involution in x. The applied value of these results in their importance in the study of several mathematical models containing partial differential equations with involution in space variables. The existence and uniqueness of the solution of a mixed problem for a parabolic equation with an involution in x in the form of a Fourier series were established. The classes of solvability of ill-posed problems for a parabolic equation with involution in x were considered. The questions of solvability of inverse problems for the heat equation and their fractional analogs were investigated. The solvability of inverse problems for a parabolic equation with an involution in x was proved.

As mentioned before we need the values of unknown functions at the previous time for solving delay differential equations. Therefore, it is important to study hyperbolic type differential equations with time involution. Noted that partial differential equations with time involution are not investigated before.

Abbas, 2019, in his master thesis investigated a Schrödinger type involutory partial differential equations (Ashyralyev & Ahmed, 2019). He obtained the solutions to several Schrödinger type involutory ordinary and partial differential problems. The numerical solution to the first order of the accuracy difference scheme for involutory one dimensional Schrödinger type partial differential equations was investigated. Moreover, this difference scheme was tested by an example and numerical results were given.

Mohammed, 2019, in his master's thesis, investigated a parabolic type involuntary partial differential equation. He obtained the solutions of the several parabolic type involuntary partial differential problems. The numerical solutions of the initial boundary value problem to the first and second order accuracy difference schemes were investigated. Moreover, these difference schemes were tested by an example and some numerical results were given.

Abdalmohammed, 2020, in his master thesis, investigated hyperbolic type involuntary partial differential equations. He obtained the solutions to several hyperbolic type involuntary partial differential problems. The numerical solutions of the initial boundary value problem to the first and second order of accuracy difference schemes were investigated. Moreover, these difference schemes were tested by an example and some numerical results were given. (Ashyralyev & Abdalmohammed, 2020b, 2020a)

1.2 Layout of the Present Thesis

Involutory telegraph type partial differential equation with damping term is not investigated before. The main aim of the present Thesis is to study of the boundedness solution of several involutory telegraph type partial differential equation with damping term. This thesis is classified in five Chapter. Chapter one is introduction. Chapter two, the second order differential equation with damping term is investigated. We obtain equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear equations with damping term and involution. Theorem on stability estimates for the solution of the initial value problem for the second order linear involutory differential equation with damping term and involution is proved. Finally, Theorem on existence and uniqueness of bounded solution of initial value problem for second order ordinary nonlinear differential equation with damping term and involution is established.

In Chapter Three, the initial value problem for the telegraph type involutory in t second order linear partial differential equation with damping term is investigated. The equivalent initial value problem for the fourth order partial differential equations to the initial value problem for this second order linear partial differential equations with involution and damping term is obtained. Applying the operator tools, the stability estimates for the solution and its first and second order derivatives of this problem are established.

In Chapter Four, we obtain the algorithms of numerical solution for the initial-boundary-value problem for the one dimensional telegraph type involutory partial differential equation with a damping term for Dirichlet and Neumann boundary conditions. We will present the first and second order of accuracy difference schemes for the numerical solutions of involutory problems. We used the procedure of the modified Gauss elimination method for solving these difference schemes. Numerical analysis is provided. Chapter Five is the conclusion.

1.3 Basic concepts and definitions

This section highlights basic concepts and definitions in the theory of ordinary differential equations with involution leading us to conduct and understand the works in this thesis.

1.3.1 Sturm-Liouville problem

(Arfken & WEBER, 1970)

We denote the Sturm-Liouville operator as

$$L[y] = -\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y$$

and consider the Sturm-Liouville equation

$$L[y] + \lambda y = 0, \tag{1.1}$$

where $p > 0$ and p and q are continuous functions on the interval $[0, l]$ with local boundary conditions

$$\alpha_1 y(0) + \alpha_2 p(0) y'(0) = 0; \beta_1 y(l) + \beta_2 p(l) y'(l) = 0, \tag{1.2}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$ or nonlocal boundary conditions

$$y(0) - y(l) = 0, \; y'(0) - y'(l) = 0, \tag{1.3}$$

The problem of finding a complex number $\lambda = \mu$ such that the boundary value problems (1.1), (1.2) or (1.1), (1.3) have a non trivial solution are called Sturm-Liouville problems.

The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(x, \mu)$ is called an eigenfunction.

We will consider three type of Sturm-Liouville problem.

1. The Sturm-Liouville problem with Dirichlet condition

$$-u''(x) + \lambda u(x) = 0, \; 0 < x < l, \; u(0) = u(l) = 0$$

has solution

$$u_k(x) = \sin \frac{kx}{l} \text{ and } \lambda_k = -\left(\frac{k\pi}{l}\right)^2, \; k = 1, 2, ...$$

In the case when $l = \pi$

$$u_k(x) = \sin kx \text{ and } \lambda_k = -k^2, \; k = 1, 2, ...$$

2. The Sturm-Liouville problem with Neumann condition

$$-u^{''}(x) + \lambda u(x) = 0,\ 0 < x < l,\ u'(0) = u'(l) = 0$$

has solution

$$u_k(x) = \cos\frac{kx}{l} \text{ and } \lambda_k = (\frac{k\pi}{l}),\ k = 0, 1, 2, ...$$

In the case when $l = \pi$

$$u_k(x) = \cos kx \text{ and } \lambda_k = -k^2,\ k = 0, 1, 2, ...$$

3.The Sturm-Liouville problem with nonlocal conditions

$$-u^{''}(x) - \lambda u(x) = 0,\ 0 < x < l,\ u(0) = u(l),\ u^{'}(0) = u^{'}(l)$$

has solution

$$u_k(x) = \cos 2kx,\ k = 0, 1, 2, ...$$

$$u_k(x) = \sin 2kx,\ k = 1, 2, ...$$

and

$$\lambda_k = 4k^2,\ k = 0, 1, 2, ...$$

1.3.2Fourier series

(Arfken & WEBER, 1970)

Let l be a fixed number and $f(x)$ be a periodic function with periodic $2l$, defined on $(-l, l)$. The Fourier series of $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sins and cosines :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) \tag{1.4}$$

where a_0, a_n and b_n called the Fourier coefficients of $f(x)$, are given by these formulas

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx,\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{p})dx,\ n = 1, 2, ...$$

and

$$b_n = \frac{1}{l}\int_{-l}^{i} \sin(\frac{n\pi x}{p})dx,\ n = 1, 2, ...$$

1.3.3 Laplace transform

(Franklin & Trent, 1959)

Let $f(t)$ be defined for $t \geqslant 0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or $\{f(t)\}$, is an integral transform given by the integral

$$F(s) = \{f(t)\} = \int\limits_{0}^{\infty} f(t)e^{-st}dt$$

Provided that this (improper) integral exsists i.e that this integral convergent.

The Laplace transform is operation that transforms a function of t (i.e a function of time domain), defined on $[0, \infty]$ to a function of s (i.e of frequency domain). The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. $F(s)$ is Laplace transform or simply transform of $f(t)$. Together the two functions $f(t)$ and $F(s)$ are called a Laplace transform pair.

1.3.4 Fourier transform

(Bracewell, 1999)

The Fourier transform of a function $f = f(x)$ denoted by $F(s)$ or $F\{f(x)\}$, is an integral transform given by the integral

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-xs}dx.$$

1.3.5 Dalamberts fourmula

(Wyley, Sons, 1993)

$$u(t) = \cos(ct)\varphi + \frac{1}{c}\sin(ct)\psi + \int_{0}^{t} \frac{1}{c}\sin(c\,(t-y))f(y)dy$$

is the general solution of the initial value problem

$$u_{tt}(t) + c^2u(t) = f(t),\ t > 0,\ u(0) = \varphi,\ u'(0) = \psi$$

for second order ordinary linear differential equations with constant coefficients

1.3.6 Dalamberts formula for hyperbolic equations

(Dalambert, 1749)

In mathematics, and specifically partial differential equations (PDEs), d'Alembert's formula is the general solution to the one-dimensional wave equation

$$\frac{\partial^2 u\,(t,x)}{\partial t^2} - c^2u_{xx}\,(t,x) = f(t,x).$$

The solution depends on the boundary conditions at $t = 0 : u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$:

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi + \int_{0}^{t} \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,\xi)d\xi d\tau.$$

It is named after the mathematician Jean le Rond d'Alembert, who derived it in 1747 as a solution to the problem of a vibrating string.

1.3.7Banach fixed-point theorem

(Kreyszig, 1993)(Ashyralyev, 2014)

Definition 1.3.1. Let $E = (E,d)$ be a metric space. A fixed point of a mapping $T : E \to E$

of a set E into itself is an element $x \in E$ which is mapped onto itself, that is, $Tx = x$, the image Tx coincides with x.

Note that the Banach fixed-point theorem to be stated below is existence and uniqueness theorem for fixed points of certain mappings and it also gives a constructive procedure for obtaining better and better approximations to the solution of the equation

$$x = Tx.$$

Actually, we choose an arbitrary $x_0 \in E$ and determine successively a sequence $\{x_n\}_{n=0}^{\infty}$ defined by the relation

$$x_n = Tx_{n-1},\ n \in_1 . \tag{1.5}$$

Here and in this Thesis, we will put $\mathbb{N}_k = \{n \in \mathbb{Z};\ n \geq k\}$.

This procedure is called an iteration. Iteration procedures are used in many fields of applied mathematics. Banach's fixed-point theorem gives sufficient conditions for the existence and uniqueness of a fixed point of a class of mappings, called contractions.

Definition 1.3.2. A mapping $T : E \to E$ is called a contraction on E, if there is a positive real number $\alpha < 1$ such that for all $x, y \in E$

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{1.6}$$

Theorem 1.3.1. Assume that $E \neq \varnothing$ is complete and let T be a contraction mapping on E. Then, T has precisely one fixed point.

CHAPTER 2

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH DAMPING TERM AND INVOLUTION

2.1 Introduction

We consider the initial value problem for the second order ordinary differential equation with damping term and involution

$$y''(t) = f(t, y(t), y^{'}(t), y(u(t))),\ t \in I = (-\infty, \infty),\ y(t_0) = y_0,\ y'(t_0) = y_0'. \quad (2.1)$$

Here $u(t)$ is involutory, that is $u(u(t)) = t$, and t_0 is fixed point of u. Problem (2.1) does not seem to yield directly to any techniques that ordinary differential equations without a damping term can be used in (2.1). Therefore, in Chapter Two, we consider the second order linear involutory differential equations with damping terms. We obtain an equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with involution and dumping terms. This result permits us to obtain bounded solutions of initial boundary value problems for involutory telegraph equations in Chapter Three. Moreover, the Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution and damping term is proved. Finally, the Theorem on the existence and uniqueness of the bounded solution of the initial value problem for the second order ordinary nonlinear differential equation with involution and damping term is established.

2.2 Linear ordinary differential equation with damping term and involution

Let $C^{\infty}[I]$ be a set of all differentiable functions for all degrees.

Theorem 2.1. Let $a(t)$, $b(t), \alpha(t)$ be a functions of class C^{∞} on I, such that $b(t)$ does not vanish on the interval I, then the problem

$$y^{''}(t) + \alpha(t)y^{'}(t) = a(t)y(t) + b(t)y(-t) + f(t),\ t \in I,\ y(0) = \varphi,\ y^{'}(0) = \psi \quad (2.2)$$

is equivalent to the following problem for the fourth order ordinary differential equation

$$\begin{cases} y^{iv}(t) = p(t)y(t) + q(t)y^{'}(t) + r(t)y^{''}(t) + s(t)y^{'''}(t) + F(t),\ t \in I, \\ y(0) = \varphi,\ y^{'}(0) = \psi, \\ y^{''}(0) = a(0)\varphi + b(0)\varphi - \alpha(0)\psi + f(0), \\ y^{'''}(0) = \left[-\alpha(0)\left[a(0) + b(0)\right] + a^{'}(0) + b^{'}(0)\right]\varphi \\ + \left[-\alpha^{'}(0) + \alpha^{2}(0) + a(0) - b(0)\right]\psi + f^{'}(0) - \alpha(0) f(0), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} p(t) &= a^{''}(t) + b(-t)b(t) - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{1}{b(t)}a^{'}(t) \\ &\quad - \left[b^{''}(t) + b(t)\,a(-t) - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{b^{'}(t)}{b(t)}\right]\frac{a(t)}{b(t)}, \end{aligned}$$

$$\begin{aligned} q(t) &= -\alpha^{''}(t) + 2a^{'}(t) + \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{1}{b(t)}\left[\alpha^{'}(t) - a(t)\right] \\ &\quad + \left[b^{''}(t) + b(t)\,a(-t) - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{b^{'}(t)}{b(t)}\right]\frac{\alpha(t)}{b(t)}, \end{aligned}$$

$$\begin{aligned} r(t) &= -2\alpha^{'}(t) + a(t) + \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{\alpha(t)}{b(t)} \\ &\quad + \left[b^{''}(t) + b(t)\,a(-t) - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{b^{'}(t)}{b(t)}\right]\frac{1}{b(t)}, \end{aligned}$$

$$s(t) = \left[2b^{'}(t) + b(t)\left(\alpha(-t) - \alpha(t)\right)\right]\frac{1}{b(t)}$$

and

$$\begin{aligned} F(t) &= -\left[b^{''}(t) + b(t)\,a(-t) - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{1}{b(t)}b^{'}(t)\right]\frac{f(t)}{b(t)} \\ &\quad - \left[2b^{'}(t) + b(t)\,\alpha(-t)\right]\frac{f'(t)}{b(t)} + b(t)f(-t) + f^{''}(t). \end{aligned}$$

Proof. Differentiating the equation (3.28) with respect to t, we get

$$y^{'''}(t) = -\alpha^{'}(t)y^{'}(t) - \alpha(t)y^{''}(t) + a^{'}(t)y(t) + a(t)y^{'}(t) + b^{'}(t)y(-t) - b(t)y^{'}(-t) + f^{'}(t). \quad (2.4)$$

From that it follows

$$y^{'}(-t) = \frac{-y^{'''}(t) - \alpha^{'}(t)y^{'}(t) - \alpha(t)y^{''}(t) + a^{'}(t)y(t) + a(t)y^{'}(t)}{b(t)} \quad (2.5)$$

$$+\frac{b^{'}(t)y(-t) + f^{'}(t)}{b(t)}.$$

Replacing t in equation (3.28) with $-t$, we get

$$y^{''}(-t) = -\alpha(-t)y^{'}(-t) + a(-t)y(-t) + b(-t)y(t) + f(-t). \quad (2.6)$$

Applying equation (3.28), we can write

$$y(-t) = \frac{-y^{''}(t) - \alpha(t)y^{'}(t) + a(t)y(t) + f(t)}{-b(t)}. \quad (2.7)$$

Differentiating the equation (3.28) two times, we get

$$y^{iv}(t) = -\alpha^{''}(t)y^{'}(t) - 2\alpha^{'}(t)y^{''}(t) - \alpha(t)y^{'''}(t) + a^{''}(t)y(t) + 2a^{'}(t)y^{'}(t) \quad (2.8)$$

$$+a(t)y^{''}(t) + b^{''}(t)y(-t) - 2b^{'}(t)y^{'}(-t) + b(t)y^{''}(-t) + f^{''}(t).$$

Substituting $y^{''}(-t)$ and $y^{'}(-t)$ from equations (2.5) and (2.6) in equation (2.8), we get

$$y^{iv}(t) = -\alpha(t)y^{'''}(t) + \Big(a(t) - 2\alpha^{'}(t)\Big)y^{''}(t) + \Big(2a^{'}(t) - \alpha^{''}(t)\Big)y^{'}(t)$$

$$+\Big(a^{''}(t) + b(t)b(-t)\Big)y(t) + f^{''}(t) + b(t)f(-t)$$

$$+\Big(b^{''}(t) + b(t)a(-t)\Big)y(-t) - \Big(2b^{'}(t) + b(t)\alpha(-t)\Big)y^{'}(-t)$$

$$= -\alpha(t)y^{'''}(t) + \Big(a(t) - 2\alpha^{'}(t)\Big)y^{''}(t) + \Big(2a^{'}(t) - \alpha^{''}(t)\Big)y^{'}(t)$$

$$+\Big(a^{''}(t)+b(t)b(-t)\Big)\,y(t)+f^{''}(t)+b(t)f(-t)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{f^{'}(t)}{b(t)}$$

$$-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{-y^{'''}(t)-\alpha(t)y^{''}(t)+a^{'}(t)y(t)+\big(a(t)-\alpha^{'}(t)\big)\,y^{'}(t)}{b(t)}$$

$$+\left[\Big(b^{''}(t)+b(t)a(-t)\Big)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{b^{'}(t)}{b(t)}\right]y(-t).$$

Using this equation and formula for $y(-t)$ from equation (2.7) in equation (2.8), we get

$$y^{iv}(t)=\Big(2b^{'}(t)+b(t)\left(\alpha(-t)\right)-\alpha(t)\Big)\,\frac{y^{'''}(t)}{b(t)}$$

$$+\Big(a(t)-2\alpha^{'}(t)\Big)\,y^{''}(t)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{-\alpha(t)y^{''}(t)}{b(t)}$$

$$+\left[\Big(b^{''}(t)+b(t)a(-t)\Big)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{b^{'}(t)}{b(t)}\right]\frac{-y^{''}(t)}{-b(t)}$$

$$+\Big(2a^{'}(t)-\alpha^{''}(t)\Big)\,y^{'}(t)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{\big(a(t)-\alpha^{'}(t)\big)\,y^{'}(t)}{b(t)}$$

$$+\left[\Big(b^{''}(t)+b(t)a(-t)\Big)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{b^{'}(t)}{b(t)}\right]\frac{-\alpha(t)y^{'}(t)}{-b(t)}$$

$$+\Big(a^{''}(t)+b(t)b(-t)\Big)\,y(t)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{a^{'}(t)y(t)}{b(t)}$$

$$+\left[\Big(b^{''}(t)+b(t)a(-t)\Big)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{b^{'}(t)}{b(t)}\right]\frac{a(t)y(t)}{-b(t)}$$

$$+f^{''}(t)+b(t)f(-t)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{f^{'}(t)}{b(t)}$$

$$-\left[\Big(b^{''}(t)+b(t)a(-t)\Big)-\Big(2b^{'}(t)+b(t)\alpha(-t)\Big)\,\frac{b^{'}(t)}{b(t)}\right]\frac{f(t)}{b(t)}$$

$$=p(t)y(t)+q(t)y^{'}(t)+r(t)y^{''}(t)+s(t)y^{'''}(t)+F(t).$$

Using the equation (3.28) and formula (2.4) and initial conditions $y(0)=\varphi$ and $y^{'}(0)=\psi$, we get

$$y^{'''}(0)=\Big[-\alpha\left(0\right)\left[a(0)+b(0)\right]+a^{'}(0)+b^{'}(0)\Big]\,\varphi$$

$$+(-\alpha^{'}(0)+\alpha^{2}(0)+a(0)-b(0))\psi+f^{'}(0)-\alpha(0)f(0).$$

and

$$y^{''}(0)=(a(0)+b(0)\varphi-\alpha(0)\psi+f(0).$$

So, the problem (3.29) is presented. Now, we will get (3.28) from (3.29). Denote that

$$L(t)=y^{''}(t)+\alpha(t)y^{'}(t)-a(t)y(t)-b(t)y(-t)-f(t),\ t\in I.$$

It is easy to see that $L(t)$ is the solution of the following problem

$$L^{''}(t)+\alpha(t)L^{'}(t)=a(t)L(t)+b(t)L(-t),\ t\in I,\ L(0)=0,\ L^{'}(0)=0.$$

From that it follows $L(t)\equiv 0$. Theorem 2.1 is proved. Now, we consider the applications of theorem 2.1. First, we consider the initial value problem

$$\begin{cases} y^{''}(t)+y'(t)=-5y(-t)+4y(t)-10\sin t+\cos t,\ t\in I=(-\infty,\infty), \\ y(\frac{\pi}{2})=1,\ y'(\frac{\pi}{2})=0 \end{cases} \tag{2.9}$$

for the second order differential equation with damping term and involution. Noted that it is an easy problem without involution term. But, we can not use classical methods for solving problem without involution term directly for the problem with damping term and involution. In this simple example, we will show how we study such kind of problem. In the same manner as Theorem 2.1 to problem (2.9), we can obtain the following equivalent initial value problem for the fourth order differential equation

$$y^{(4)}(t)-8y^{''}(t)-9y(t)=0,\ t\in I,\ y(\frac{\pi}{2})=1,\ y'(\frac{\pi}{2})=0,\ y''(\frac{\pi}{2})=-1,\ y'''(\frac{\pi}{2})=0.$$

The auxiliary equation is

$$m^4-8m^2-9=0$$

We have four roots $\pm i$ and ± 3. Therefore, the general solution of

$$y^{(4)}(t) - 8y^{''}(t) - 9y(t) = 0$$

is

$$y(t) = c_1 \sin t + c_2 \cos t + c_3 e^{3t} + c_4 e^{-3t}.$$

Therefore, the exact solution is

$$y(t) = \sin t.$$

Second, we consider the initial value problem

$$\begin{cases} y^{''}(t) + \alpha y^{'}(t) = by(-t) + ay(t) + f(t), t \in I, \\ \\ y(0) = \varphi, \ y'(0) = \psi \end{cases} \tag{2.10}$$

for the second order involutory ordinary differential equation with damping term. We are intersted in studying the stability of problem (2.10) on I. It is important to study several problems in applicatons. In general cases of α, a and b the solution of (2.10) is not bounded on I. Applying Theorem 2.1, we get the equivalent initial value problem

$$
\begin{cases}
y^{iv}(t)+(a^2-b^2)y(t)-(2a+\alpha^2)\,y^{''}(t)=F(t), \\
\\
F(t)=-af(t)+bf(-t)-\alpha f^{'}(t)+f^{''}(t),\ t\in I, \\
\\
y(0)=\varphi,\ y^{'}(0)=\psi, \\
\\
y^{''}(0)=(b+a)\varphi+f(0)-\alpha\psi, \\
\\
y^{'''}(0)=-\alpha\,(b+a)\,\varphi+(-b+a+\alpha^2)\psi+f'(0)-\alpha f(0)
\end{cases}
\tag{2.11}
$$

for the fourth order ordinary differential equation. We will obtain the solution of the problem (2.11). Assume that $|b|\,\langle|a|\,, a\in\left(-\left(\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}\right), -\frac{\alpha^2}{2}\right)$. Then it is easy to see that

$$
\begin{aligned}
&\frac{d^4y(t)}{dt^4}-(2a+\alpha^2)\frac{d^2y(t)}{dt^2}+(a^2-b^2)y(t) \\
= \;&\left(\frac{d^2}{dt^2}-\left(a+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\right)\left(\frac{d^2}{dt^2}-\left(a+\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\right)y(t).
\end{aligned}
$$

Therefore, problem (2.11) can be written as initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right) y(t) = v(t) \\ y(0) = \varphi, y^{'}(0) = \psi, \\ \left(\frac{d^2}{dt^2} - \left(a + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\right) v(t) = F(t) \\ F(t) = -af(t) + bf(-t) - \alpha f^{'}(t) + f^{''}(t),\ t \in I, \\ v(0) = \left(b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right) \varphi + f(0) - \alpha\psi, \\ v^{'}(0) = -\alpha(b+a)\varphi \\ -\left(b + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right) \psi + f^{'}(0) - \alpha f(0) \end{cases} \tag{2.12}$$

for the system of second order differential equations. Applying the d' Alembert's formula, we get

$$y(t) = \cos(mt)\varphi + \frac{\sin(mt)}{m}\psi + \int_0^t \frac{\sin(m\,(t-s))}{m} v(s)ds, \tag{2.13}$$

$$\begin{aligned} v(t) \;&=\; \cos(nt)\left[\left(b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\varphi + f(0) - \alpha\psi\right] \\ &+ \frac{\sin(nt)}{n}\left[-\alpha(b+a)\varphi - \left(b + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right)\psi + f^{'}(0) - \alpha f(0)\right] \\ &+ \int_0^t \frac{\sin(n\,(t-s))}{n} F(s)ds, \end{aligned} \tag{2.14}$$

where

$$m=\sqrt{a+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}},$$

$$n=\sqrt{a+\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}}.$$

Since $F(t)=-af(t)+bf(-t)-\alpha f^{'}(t)+f^{''}(t)$and

$$\int_0^t \frac{\sin(n\,(t-s))}{n} f^{'}(t)ds=-\frac{\sin nt}{n}f(0)+\int_0^t \cos(n\,(t-s))f(s)ds,$$

$$\int_0^t \frac{\sin(n\,(t-s))}{n} f^{''}(s)ds$$
$$=-\frac{\sin nt}{n}f^{'}(0)-\cos(nt)f(0)+f(t)-\int_0^t n\sin(n\,(t-s))f(s)ds,$$

we can write

$$\begin{aligned} v(t) \;=\; & \cos(nt)\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi\right] \\ & +\frac{\sin(nt)}{n}\left[-\alpha(b+a)\varphi-\left(b+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi\right] \\ & -a\int_0^t \frac{\sin(n\,(t-s))}{n}f(s)ds+b\int_{-t}^0 \frac{\sin(n\,(t+s))}{n}f(s)ds \\ & -\alpha\int_0^t \cos(n\,(t-s))f(s)ds+f(t)-\int_0^t n\sin(n\,(t-s))f(s)ds. \end{aligned} \tag{2.15}$$

Applying formulas (2.13) and (2.15) we get

$$y(t)=\cos(mt)\varphi+\frac{\sin(mt)}{m}\psi \tag{2.16}$$

$$+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi\right]$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left[-\alpha(b+a)\varphi-\left(b+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi\right]$$

$$+\frac{1}{m^2-n^2}\int_0^t[-n\sin(n(t-s))+m\sin(m(t-s))]f(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t[\cos(n(t-s))-\cos(m(t-s))]f(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t[-n\sin(n(t-s))+m\sin(m(t-s))]f(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0[-\frac{1}{n}\sin(n(t+s))+\frac{1}{m}\sin(m(t+s))]f(s)ds.$$

Theorem 2.2. Assume that $|b|<|a|$, $a\in(-\left(\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}\right),-\frac{\alpha^2}{2})$. Then the problem (2.10) is stable and the following stability estimate holds

$$\sup_{t\in I}|y(t)|\le M(a,b,\alpha)\left[|\varphi|+|\psi|+|f(0)|+\int_{-\infty}^{\infty}|f(s)|\,ds\right].$$

The proof is based on the formula (2.16) and the triangle inequality.

2.3 Nonlinear ordinary differential equation with involution

We consider the initial value problem

$$\begin{cases} y''(t)+\alpha y'(t)=by(-t)+ay(t)+f(t,y(t),y'(t)),\ t\in I, \\ y(0)=\varphi,\ y'(0)=\psi \end{cases} \quad (2.17)$$

for the second order nonliner involutory ordinary differential equation with damping term. We are intersted in studying the existence and uniqueness of bounded solution of problem (2.17) on I. In general cases of α, a and b the solution of (2.17) is not bounded on I.We will applied a fixed point theorem.

Let $C^{(1)}(I)$ be metric space of all continuously differentiable functions defined on the interval I with the metric d defined by

$$d(x,y) = \sup_{t\in I}|x(t)-y(t)| + \sup_{t\in I}\left|\frac{dx\,(t)}{dt} - \frac{dy\,(t)}{dt}\right|.$$

Note that $C^{(1)}(I)$ is the complete space. This is first condition of a fixed point theorem in metric space(Ashyralyev & Sarsenbi, 2015)

Theorem 2.3. Assume that $|b| < |a|$, $a \in \left(-\left(\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}\right), -\frac{\alpha^2}{2}\right)$and f is continuous and bounded function on the region

$$P = \{(t,x,y) : -\infty < t < \infty,\ \ |x-\varphi| < M,\ \ |y-\psi| < M\}$$

and $f(0,\varphi,\psi) = 0$. Suppose that f satisfies a Lipschitz condition on P with respect to its second and third arguments, that is, there is a constant l such that for $(t,x,u)\,,\ (t,y,v) \in P$

$$|f(t,x,u) - f(t,y,v)| \leq l\left(|x-y| + |u-v|\right). \tag{2.18}$$

Then, initial value problem (2.17) has a unique solution $y \in C^{(1)}(I)$.

Proof. The procedure of proving theorem on the existence and uniqueness of a bounded solution of problem (2.17) is based on reducing this problem to an integral equation

$$y(t) = Ty(t), \tag{2.19}$$

where

$$Ty(t) = \cos(mt)\varphi + \frac{\sin\,(mt)}{m}\psi$$

$$+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi\right]$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left[-\alpha(b+a)\varphi-\left(b+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi+\alpha f(0)\right]$$

$$+\frac{1}{m^2-n^2}\int_0^t[-n\sin(n(t-s))+m\sin(m(t-s))]f(s,y(s),y^{'}(s))ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t[\cos(n(t-s))-\cos(m(t-s))]f(s,y(s),y^{'}(s))ds.$$

$$-\frac{a}{m^2-n^2}\int_0^t[-n\sin(n(t-s))+m\sin(m(t-s))]f(s,y(s),y^{'}(s))ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0[-\frac{1}{n}\sin(n(t+s))+\frac{1}{m}\sin(m(t+s))]f(s,y(s),y^{'}(s))ds.$$

The proof of equation (2.17) is based on the formula (2.19). Note that intergal form is a Volterra type integro-differential equation of the second kind. Therefore, the recursive formula for the solution of problem (2.19) is

$$y_0(t)=\cos(mt)\varphi+\frac{\sin(mt)}{m}\psi$$

$$+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\left[\left(b-\frac{\alpha^2}{2}-\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\varphi\right]$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left[-\alpha(b+a)\varphi-\left(b+\frac{\alpha^2}{2}+\sqrt{a\alpha^2+\frac{\alpha^4}{4}+b^2}\right)\psi+\alpha f(0)\right],$$

$$y_j(t)=y_0(t) \tag{2.20}$$

$$+\frac{1}{m^2-n^2}\int_0^t[-n\sin(n\,(t-s))+m\sin(m\,(t-s))]f(s,y_{j-1}(s),y_{j-1}^{'}(s))ds$$
$$-\frac{\alpha}{m^2-n^2}\int_0^t[\cos(n\,(t-s))-\cos(m\,(t-s))]f(s,y_{j-1}(s),y_{j-1}^{''}(s))ds$$
$$-\frac{a}{m^2-n^2}\int_0^t[-n\sin(n\,(t-s))+m\sin(m\,(t-s))]f(s,y_{j-1}(s),y_{j-1}^{'}(s))ds$$
$$-\frac{b}{m^2-n^2}\int_{-t}^0[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))]f(s,y_{j-1}(s),y_{j-1}^{'}(s))ds, j\geq 1.$$

According to the method of recursive approximation (2.20), we get

$$y(t)=y_0(t)+\sum_{j=0}^{\infty}\left[y_{j+1}(t)-y_j(t)\right]. \tag{2.21}$$

We have that

$$\begin{aligned} y_{j+1}(t)-y_j(t) &= \frac{1}{m^2-n^2}\int_0^t[-n\sin(n\,(t-s))+m\sin(m\,(t-s))] \qquad (2.22)\\ &\times\left[f(s,y_j(s),y_j^{'}(s))-f(s,y_{j-1}(s),y_{j-1}^{'}(s))\right]ds\\ &-\frac{\alpha}{m^2-n^2}\int_0^t[\cos(n\,(t-s))-\cos(m\,(t-s))]\\ &\times\left[f(s,y_j(s),y_j^{'}(s)-f(s,y_{j-1}(s),y_{j-1}^{'}(s))\right]ds\\ &-\frac{a}{m^2-n^2}\int_0^t[-n\sin(n\,(t-s))+m\sin(m\,(t-s))]\\ &\times\left[f(s,y_j(s),y_j^{'}(s)-f(s,y_{j-1}(s),y_{j-1}^{'}(s))\right]ds\\ &-\frac{b}{m^2-n^2}\int_{-t}^0[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))]\\ &\times\left[f(s,y_j(s),y_j^{'}(s)-f(s,y_{j-1}(s),y_{j-1}^{'}(s))\right]ds,\\ j &\geq 1. \end{aligned}$$

Therefore, applying the triangle inequality, formula (2.22) and Lipschitz condition (2.18),

we get

$$\begin{aligned} & \left|y_{j+1}(t)-y_j(t)\right|,\left|y_{j+1}'(t)-y_j'(t)\right| \\ \leq M(a,b,\alpha)l \quad & \int_{-|t|}^{|t|}\left[\left|y_j(s)-y_{j-1}(s)\right|+\left|y_j'(s)-y_{j-1}'(s)\right|\right]ds \end{aligned} \tag{2.23}$$

for any $t \in I$ and $j \geq 1$.Moreover, applying the triangle inequality, we get

$$\left|y_0(t)\right|,\left|y_0'(t)\right| \leq M_1(a,b,\alpha,\varphi,\psi),$$

$$\left|y_1(t)-y_0(t)\right|,\left|y_1'(t)-y_0'(t)\right| \leq M(a,b,\alpha)\left|t\right|, \tag{2.24}$$

for any $t \in I$. Applying estimates (2.23) and (2.24) , we can prove that

$$\begin{aligned} & \left|y_{j+1}(t)-y_j(t)\right|,\left|y_{j+1}'(t)-y_j'(t)\right| \\ \leq \quad & [4M(a,b,\alpha)lM_2(a,b,\alpha)]^j \frac{|t|^{j+1}}{(j+1)!} \end{aligned} \tag{2.25}$$

for any $t \in I$ and $j \geq 1$.Therefore, applying the triangle inequality, formula (2.21) and and estimates (2.23) and (2.25), we get

$$\begin{aligned} & \left|y(t)-y_n(t)\right|,\left|y'(t)-y_n'(t)\right| \\ \leq \quad & \sum_{j=n+1}^{\infty}[4M(a,b,\alpha)lM_2(a,b,\alpha)]^j \frac{|t|^{j+1}}{(j+1)!} \to 0, n \to \infty, \end{aligned}$$

$$\begin{aligned} & \left|y(t)\right|,\left|y'(t)\right| \leq M_1(a,b,\alpha,\varphi,\psi)+M_2(a,b,\alpha)\left|t\right| \\ & + \sum_{j=n+1}^{\infty}[4M(a,b,\alpha)lM_2(a,b,\alpha)]^j \frac{|t|^{j+1}}{(j+1)!} \end{aligned}$$

for any $t \in I$.Theorem 2.3 is proved.

CHAPTER 3

METHODS OF SOLUTION OF TELEGRAPH TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

3.1 Introduction

Differential equations with involution appear in mathematical models of ecology, biology, and population dynamics. As it noted in Introduction, in recent decades, one-dimensional elliptic and parabolic type partial differential and difference equations with involution in x have been investigated by many authors. A mixed problem of one dimensional telegraph equation with the involution in x was investigated in the paper of Ashyralyev, and Sarsenbi, 2017(Ashyralyev & Sarsenbi, 2017a). The stability estimates in the maximum norm in t for the solution to this problem to be established. As mentioned before we need the values of unknown functions at the previous times for solving delay differential equations. Therefore, it is important to study telegraph type differential equations with time involution. Noted that telegraph type differential equations with time involution are not investigated. Therefore, the main aim of Chapter Three is to study the boundedness solution of several involutory telegraph equations. Applying the results of Chapter Two and the Fourier series, Laplace and Fourier transform methods, we obtain the exact solutions of several problems for involutory telegraph equations.

3.2 The Fourier series method

First, we consider the initial boundary value problem

$$
\begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2}+\alpha\frac{\partial u(t,x)}{\partial t}-au_{xx}\left(t,x\right)-bu_{xx}\left(-t,x\right)=g(t,x),\\
\\
x\in\left(0,\pi\right),\ -\infty<t<\infty,\\
\\
u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ x\in[0,\pi],\\
\\
u(t,0)=u(t,\pi)=0,\ t\in(-\infty,\infty)
\end{cases}
\tag{3.1}
$$

for one dimensional telegraph type involutory equation. Here $g(t,x)\,(t\in I,\ x\in(0,\pi))$ and $\varphi(x),\ \psi(x)(x\in[0,\pi])$ are given smooth functions and $0\leq\alpha$. Assume that $g(t,0)=g(t,\pi)=0$,

$t\in I$ and $\varphi(0)=\varphi(\pi)=\psi(0)=\psi(\pi)=0$. For solving this problem, we consider the Sturm-Liouville problem

$$-u''(x)-\lambda u(x)=0,\ 0<x<\pi,u(\pi)=u(0)=0$$

generated by the space operator of the problem (3.1). As noted in Chapter 1 the solution to this Sturm-Liouville problem is

$$\lambda_k=k^2,\ u_k(x)=\sin kx,\ k=1,2,....$$

Then, applying formulas

$$g(t,x)=\sum_{k=1}^{\infty}g_k(t)\sin kx,\ g_k(t)=\frac{2}{\pi}\int_0^{\pi}g(t,y)\sin kydy,$$

$$\varphi(x)=\sum_{k=1}^{\infty}\varphi_k\sin kx,\ \varphi_k=\frac{2}{\pi}\int_0^{\pi}\varphi(y)\sin kydy,$$

$$\psi(x)=\sum_{k=1}^{\infty}\psi_k\sin kx,\ \psi_k=\frac{2}{\pi}\int_0^{\pi}\psi(y)\sin kydy,$$

we obtain the Fourier series solution of mixed problem (3.1) by the formula

$$u(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx, \tag{3.2}$$

where $A_k(t)$ are unknown functions. Applying this equation and initial conditions, we get

$$\sum_{k=1}^{\infty}A_k''(t)\sin kx+\alpha\sum_{k=1}^{\infty}A_k'(t)\sin kx$$

$$+a\sum_{k=1}^{\infty}k^2A_k(t)\sin kx+b\sum_{k=1}^{\infty}k^2A_k(-t)\sin kx$$

$$=\sum_{k=1}^{\infty}g_k(t)\sin kx,\ x\in(0,\pi),\ -\infty<t<\infty,$$

$$\sum_{k=1}^{\infty}A_k\,(0)\sin kx=\sum_{k=1}^{\infty}\varphi_k\sin kx,\ x\in[0,\pi],$$

$$\sum_{k=1}^{\infty}A_k'\,(0)\sin kx=\sum_{k=1}^{\infty}\psi_k\sin kx,\ x\in[0,\pi].$$

Equating coefficients $\sin kx,\ k=1,2,...$to zero, we get the initial value problems

$$\begin{cases}A_k''(t)+\alpha A_k'(t)+ak^2A_k(t)+bk^2A_k(-t)=g_k(t),\ -\infty<t<\infty,\\ \\ A_k(0)=\varphi_k,\ A_k'(0)=\psi_k\end{cases} \tag{3.3}$$

for the second order involutory ordinary differential equations. Applying results of Chapter Two, we get

$$A_k(t)=\cos(mt)\varphi_k+\frac{\sin(mt)}{m}\psi_k+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}$$

$$\times\left[bk^2-\frac{\alpha^2}{2}-\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}\right]\varphi_k \tag{3.4}$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$-\alpha(bk^2+ak^2)\varphi_k-\left[bk^2+\frac{\alpha^2}{2}+\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}\right]\psi_k$$

$$+\frac{1}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t\left[\cos(n\,(t-s))-\cos(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{ak^2}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{bk^2}{m^2-n^2}\int_{-t}^0\left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right]g_k(s)ds,$$

where

$$m=\sqrt{ak^2+\frac{\alpha^2}{2}+\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}}$$

$$n=\sqrt{ak^2+\frac{\alpha^2}{2}-\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}}$$

Then, applying formula (3.2), we can obtain Fourier series solution of mixed problem (3.1) by the following formula

$$u(t,x)=\sum_{k=1}^{\infty}\sin kx\left\{\cos(mt)\varphi_k+\frac{\sin(mt)}{m}\psi_k\right.$$

$$+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\left[bk^2-\frac{\alpha^2}{2}-\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}\right]\varphi_k$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left(-\alpha(bk^2+ak^2)\varphi_k-\left[bk^2+\frac{\alpha^2}{2}+\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}\right]\psi_k\right)$$

$$+\frac{1}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t\left[\cos(n\,(t-s))-\cos(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{ak^2}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{bk^2}{m^2-n^2}\int_{-t}^0\left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right]g_k(s)ds\}.$$

For example, for the involutory telegraph problem

$$\begin{cases} \frac{\partial^2u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-au_{xx}\,(t,x)-bu_{xx}\,(-t,x) \\ =ae^{-t}\sin\,(x)+be^{t}\sin\,(x)\,, \\ x\in(0,\pi)\,,\ -\infty<t<\infty, \\ u(0,x)=\sin(x),\ u_t(0,x)=-\sin(x),\ x\in[0,\pi], \\ u(t,0)=u(t,\pi)=0,\ t\in(-\infty,\infty) \end{cases}$$

$g(t,x) = ae^{-t}\sin(x) + be^{t}\sin(x)$, $\varphi(x) = \sin(x)$, $\psi(x) = -\sin(x)$. Therefore,

$$g_k(t) = \begin{cases} ae^{-t} + be^{t},\ k = 1, \\ \\ 0, k \neq 1, \end{cases}$$

$$\varphi_k = \begin{cases} 1,\ k = 1, \\ \\ 0,\ k \neq 1, \end{cases}$$

$$\psi_k = \begin{cases} -1,\ k = 1, \\ \\ 0,\ k \neq 1. \end{cases}$$

Applying formula (3.4), we get $A_k(t) = 0$ for $k \neq 1$ and

$$A_1(t) = \cos(mt)\varphi_1 + \frac{\sin(mt)}{m}\psi_1$$

$$+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\left[b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right]\varphi_1$$

$$+\frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2-n^2}\left(-\alpha(b+a)\varphi_1 - \left[b + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right]\psi_1\right)$$

$$+\frac{1}{m^2-n^2}\int_0^t \left[-n\sin(n(t-s)) + m\sin(m(t-s))\right] g_1(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos(n(t-s)) - \cos(m(t-s))\right] g_1(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t \left[-n\sin(n(t-s)) + m\sin(m(t-s))\right] g_1(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin(n(t+s)) + \frac{1}{m}\sin(m(t+s))\right] g_1(s)ds$$

$$
= \cos(mt) - \frac{\sin(mt)}{m} + \frac{\cos(nt) - \cos(mt)}{m^2 - n^2}\left[b - \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right]
$$

$$
+\frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2 - n^2}\left(-\alpha(b+a) + \left[b + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}\right]\right)
$$

$$
+\frac{1}{m^2 - n^2}\int_0^t \left[-n\sin(n\,(t-s)) + m\sin(m\,(t-s))\right]\left(ae^{-s} + be^{s}\right)ds
$$

$$
-\frac{1}{m^2 - n^2}\int_0^t \left[\cos(n\,(t-s)) - \cos(m\,(t-s))\right]\left(ae^{-s} + be^{s}\right)ds
$$

$$
-\frac{a}{m^2 - n^2}\int_0^t \left[-n\sin(n\,(t-s)) + m\sin(m\,(t-s))\right]\left(ae^{-s} + be^{s}\right)ds
$$

$$
-\frac{b}{m^2 - n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin(n\,(t+s)) + \frac{1}{m}\sin(m\,(t+s))\right]\left(ae^{-s} + be^{s}\right)ds.
$$

Since

$$
m = \sqrt{a + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}},
$$

$$
n = \sqrt{a + \frac{\alpha^2}{2} - \sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2}},
$$

$$
m^2 - n^2 = 2\sqrt{a\alpha^2 + \frac{\alpha^4}{4} + b^2},
$$

we have that

$$
A_1(t) = e^{-t}.
$$

Then,

$$u(t,x)=e^{-t}\sin x.$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+\alpha\frac{\partial u(t,x)}{\partial t}-a\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}-b\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2}=g(t,x), \\ \\ x=(x_1,...,x_n)\in\Omega,\ -\infty<t<\infty, \\ \\ u(\frac{d}{2},x)=\psi(x),\ u_t((\frac{d}{2},x)=\varphi(x),\ x\in\overline{\Omega},\ d\geq 0, \\ \\ u(t,x)=0,\ x\in S,\ t\in(-\infty,\infty) \end{cases} \tag{3.5}$$

for the multidimensional involutory telegraph type differential equation. Suppose that $\alpha_r > \alpha > 0$ and $g\,(t,x)\,(t\in(-\infty,\infty),x\in\Omega)\,,\ \psi(x),\ \varphi(x)\left(t\in(-\infty,\infty),x\in\overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n\;(0<x_k<1, 1\leq k\leq n)$ with the boundary

$$S,\overline{\Omega}=\Omega\cup S.$$

However Fourier series method described in solving (3.5) can be used only in the case when (3.5) has constant coefficients.

Second, we consider initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - a u_{xx}(t,x) - b u_{xx}(-t,x) = g(t,x), \\ x \in (0,\pi), \ -\infty < t < \infty, \\ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x), \ x \in [0,\pi], \\ u_x(t,0) = u_x(t,\pi) = 0, \ t \in (-\infty,\infty) \end{cases} \tag{3.6}$$

for one dimensional telegraph type involutory equation. Here $g(t,x)\,(t \in I, x \in (0,\pi))$ and $\varphi(x),\ \psi(x)(x \in [0,\pi])$ are given smooth functions and $\alpha \geq 0$. Assume that $g_x(t,0) = g_x(t,\pi) = 0$,

$t \in I$ and $\varphi'(0) = \varphi'(\pi) = \psi'(0) = \psi'(\pi) = 0$. For solving this problem, we consider the Sturm Liouville problem

$$-u''(x) - \lambda u(x) = 0,\ 0 < x < \pi, u'(\pi) = u'(0) = 0$$

generated by the space operator of problem (3.6). As noted in Chapter 1 the solution of this Sturm-Liouville problem is

$$\lambda_k = k^2,\ u_k(x) = \cos kx,\ k = 0, 1, 2, ...$$

Then, applying formulas

$$g(t,x) = \sum_{k=0}^{\infty} g_k(t) \cos kx,$$

$$g_k(t) = \frac{2}{\pi} \int_0^{\pi} g(t,y) \cos ky dy,\ k \neq 0,$$

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k \cos kx,\ \varphi_k = \frac{2}{\pi} \int_0^{\pi} \varphi(y) \cos kydy,\ k \neq 0,$$

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k \cos kx,\ \psi_k = \frac{2}{\pi} \int_0^{\pi} \psi(y) \cos kydy,\ k \neq 0,$$

$$g_0(t) = \frac{1}{\pi} \int_0^{\pi} g(t, y) dy,$$

$$\varphi_0(t) = \frac{1}{\pi} \int_0^{\pi} \varphi(y) dy,\ \psi_0(t) = \frac{1}{\pi} \int_0^{\pi} \psi(y) dy,$$

we obtain Fourier series solution of mixed problem (3.6) by the formula

$$u(t, x) = \sum_{k=0}^{\infty} A_k(t) \cos kx, \tag{3.7}$$

Here $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$\sum_{k=0}^{\infty} A_k''(t) \cos kx + \alpha \sum_{k=0}^{\infty} A_k'(t) \cos kx$$

$$+a \sum_{k=0}^{\infty} k^2 A_k(t) \cos kx - b \sum_{k=0}^{\infty} k^2 A_k(-t) \cos kx$$

$$= \sum_{k=0}^{\infty} g_k(t) \cos kx, x \in (0, \pi), -\infty < t < \infty,$$

$$\sum_{k=0}^{\infty} A_k(0) \cos kx = \sum_{k=0}^{\infty} \varphi_k \cos kx,\ x \in [0, \pi],$$

$$\sum_{k=0}^{\infty} A_k'(0) \cos kx = \sum_{k=0}^{\infty} \psi_k \cos kx,\ x \in [0, \pi].$$

Equating coefficients $\cos kx, k = 0, 1, 2, ...$to zero, we get

$$\begin{cases} A_0''(t) + \alpha A_0'(t) = g_0(t), \ -\infty < t < \infty, \\ \\ A_0(0) = \varphi_0, \ A_0'(0) = \psi_0 \end{cases} \tag{3.8}$$

and

$$\begin{cases} A_k''(t) + \alpha A_k'(t) + ak^2 A_k(t) + bk^2 A_k(-t) = g_k(t), \\ \\ -\infty < t < \infty, \\ \\ A_k(0) = \varphi_k, \ A_k'(0) = \psi_k, k \neq 0 \end{cases} \tag{3.9}$$

for the second order involutory ordinary differential equations. Applying results of Chapter Two, we get

$$A_k(t) = \cos(mt)\varphi_k + \frac{\sin(mt)}{m}\psi_k + \frac{\cos(nt) - \cos(mt)}{m^2 - n^2}$$

$$\times \left[bk^2 - \frac{\alpha^2}{2} - \sqrt{\left(a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4 \right)} \right] \varphi_k \tag{3.10}$$

$$+ \frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2 - n^2} \left(-\alpha(bk^2 + ak^2)\varphi_k \right.$$

$$\left. - \left[bk^2 + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4} \right] \psi_k \right)$$

$$+ \frac{1}{m^2 - n^2} \int_0^t \left[-n \sin(n(t-s)) + m \sin(m(t-s)) \right] g_k(s) ds$$

$$- \frac{\alpha}{m^2 - n^2} \int_0^t \left[\cos(n(t-s)) - \cos(m(t-s)) \right] g_k(s) ds$$

$$- \frac{a}{m^2 - n^2} \int_0^t \left[-n \sin(n(t-s)) + m \sin(m(t-s)) \right] g_k(s) ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^{0}\left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right]g_k(s)ds, k\neq 0,$$

The problem (3.8) is not involutory problem. It is easy to see that

$$(A_0'(t)+\alpha A_0(t))'=g_0(t).$$

Then

$$A_0'(t)+\alpha A_0(t)=A_0'(0)+\alpha A_0(0)+_0^t g_0(s)ds, -\infty<t<\infty$$

or

$$A_0'(t)+\alpha A_0(t)=\psi_0+\alpha\varphi_0+_0^t g_0(s)ds, -\infty<t<\infty, A_0(0)=\varphi_0.$$

Solving this problem, we can obtain

$$A_0(t)=e^{-\alpha t}\varphi_0+(\psi_0+\alpha\varphi_0)_0^t e^{-\alpha(t-y)}dy+_0^t e_0^{-\alpha(t-y)y}g_0(s)dsdy$$

$$=e^{-\alpha t}\varphi_0+(\psi_0+\alpha\varphi_0)\frac{1}{\alpha}\left(1-e^{-\alpha t}\right)+_0^t\frac{1}{\alpha}\left(1-e^{-\alpha(t-s)}\right)g_0(s)ds$$

$$=\frac{1}{\alpha}\varphi_0+\frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\psi_0+_0^t\frac{1}{\alpha}\left(1-e^{-\alpha(t-s)}\right)g_0(s)ds. \tag{3.11}$$

Then, applying formulas (3.10) and (3.11), we can obtain Fourier series solution of mixed problem (3.1) by the following formula

$$u(t,x)=\frac{1}{\alpha}\varphi_0+\frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\psi_0+_0^t\frac{1}{\alpha}\left(1-e^{-\alpha(t-s)}\right)g_0(s)ds$$

$$+\sum_{k=1}^{\infty}\cos kx\left\{\cos(mt)\varphi_k+\frac{\sin(mt)}{m}\psi_k+\frac{\cos(nt)-\cos(mt)}{m^2-n^2}\right.$$

$$\times\left[bk^2-\frac{\alpha^2}{2}-\sqrt{\left(a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4\right)}\right]\varphi_k$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left(-\alpha(bk^2+ak^2)\varphi_k-\left[bk^2+\frac{\alpha^2}{2}+\sqrt{a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4}\right]\psi_k\right)$$

$$+\frac{1}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t\left[\cos(n\,(t-s))-\cos(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t\left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right]g_k(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0\left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right]g_k(s)ds\}.$$

For example, for the involutory telegraph problem

$$\begin{cases}\frac{\partial^2u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-au_{xx}\,(t,x)-bu_{xx}\,(-t,x)\\[2ex] =\left(ae^{-t}-be^{t}\right)\cos\left(x\right),\ x\in\left(0,\pi\right),\ -\infty<t<\infty,\\[2ex] u(0,x)=\cos(x),\ u_t(0,x)=-\cos(x),\ x\in[0,\pi],\\[2ex] u_x(t,0)=u_x(t,\pi)=0,\ t\in(-\infty,\infty)\end{cases}$$

$g(t,x)=\left(ae^{-t}-be^{t}\right)\cos\left(x\right)$, $\varphi(x)=\cos(x)$, $\psi(x)=-\cos(x)$. Therefore,

$$g_k(t)=\begin{cases}\left(ae^{-t}+be^{t}\right),\ k=1,\\[2ex] 0,\ k\neq 1,\end{cases}$$

$$\varphi_k = \begin{cases} 1,\ k=1, \\ \\ 0,\ k\neq 1, \end{cases}$$

$$\psi_k = \begin{cases} -1,\ k=1, \\ \\ 0,\ k\neq 1. \end{cases}$$

Applying formula (3.4), we get $A_1(t) = e^{-t}$ and

$$u(t,x) = e^{-t}\cos x.$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2} \\ \\ -b\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(d-t,x)}{\partial x_r^2} = g(t,x), \\ \\ x=(x_1,...,x_n)\in\overline{\Omega},\ -\infty<t<\infty, \\ \\ u(\frac{d}{2},x)=\psi(x),\ u_t(\frac{d}{2},x)=\varphi(x),\ x\in\overline{\Omega}, d\geq 0, \\ \\ \frac{\partial u}{\partial \overline{m}}(t,x)=0,\ x\in S,\ t\in(-\infty,\infty) \end{cases} \tag{3.12}$$

for the multidimensional involutory hyperbolic type differential equation. Assume that $\alpha_r > \alpha > 0, 0\leq\alpha$ and $g(t,x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$, $\psi(x)$, $\varphi(x)\left(t\in(-\infty,\infty), x\in\overline{\Omega}\right)$ are the smooth functions. Here and in future $\overline{m}$ is the normal to S. However Fourier series method described in solving (3.12) can be used only in the case when (3.12) has constant coefficients.

Third, we consider initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - a u_{xx}(t,x) \\ -b u_{xx}(-t,x) = g(t,x), \\ x \in (0,\pi), \ -\infty < t < \infty, \\ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x), \ \ x \in [0,\pi], \\ u(t,0) = u(t,\pi), \ u_x(t,0) = u_x(t,\pi), \ t \in (-\infty,\infty). \end{cases} \tag{3.13}$$

for one dimensional hyperbolic type involutory equation. Here $g(t,x)$ $(t \in I, x \in (0,\pi))$ and $\varphi(x), \psi(x) (x \in [0,\pi])$ are given smooth functions and $0 \leq \alpha$. Assume that $g(t,0) = g(t,\pi)$, $g_x(t,0) = g_x(t,\pi)$, $t \in I$ and

$\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$, $\psi(0) = \psi(\pi)$, $\psi'(0) = \psi'(\pi)$. For solving this problem, we consider the Sturm Liouville problem

$$-u''(x) - \lambda u(x) = 0, \ 0 < x < \pi$$

$u(\pi) = u(0)$, $u'(\pi) = u'(0)$ generated by the space operator of problem (3.13). As noted in Chapter 1 the solution of this Sturm-Liouville problem is

$$\lambda_k = 4k^2, \ u_k(x) = \cos(2kx), \ k = 0, 1, 2, ..., \ u_k(x) = \sin(2kx), \ k = 1, 2, ...$$

Then, applying formulas

$$g(t,x) = \sum_{k=0}^{\infty} g_k(t)\cos 2kx + \sum_{k=1}^{\infty} f_k(t)\sin 2kx,$$

$$g_k(t) = \frac{2}{\pi}\int_0^{\pi} g(t,y)\cos 2ky dy,\ k \neq 0,$$

$$g_0(t) = \frac{1}{\pi}\int_0^{\pi} g(t,y) dy,$$

$$f_k(t) = \frac{2}{\pi}\int_0^{\pi} g(t,y)\sin 2ky dy,$$

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k \cos 2kx + \sum_{k=1}^{\infty} \xi_k \sin 2kx,$$

$$\varphi_k = \frac{2}{\pi}\int_0^{\pi} \varphi(y)\cos 2ky dy,\ k \neq 0,$$

$$\xi_k = \frac{2}{\pi}\int_0^{\pi} \varphi(y)\sin 2ky dy,$$

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k \cos 2kx + \sum_{k=1}^{\infty} \omega_k \sin 2kx,$$

$$\psi_k = \frac{2}{\pi}\int_0^{\pi} \psi(y)\cos 2ky dy,\ k \neq 0,$$

$$\varphi_0 = \frac{1}{\pi}\int_0^{\pi} \varphi(y) dy,\ \psi_0 = \frac{1}{\pi}\int_0^{\pi} \psi(y) dy,$$

$$\omega_k = \frac{2}{\pi}\int_0^{\pi} \psi(y)\sin 2ky dy,$$

we obtain Fourier series solution of mixed problem (3.13) by the formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos 2kx + \sum_{k=1}^{\infty} B_k(t)\sin 2kx, \tag{3.14}$$

where $A_k(t)$, $k = 0, 1, 2, ...$, and $B_k(t)$, $k = 1, 2, ...$ are unknown functions.

Equating the coefficients of $\cos 2kx$, $k = 0, 1, 2, ...$, and $\sin 2kx$, $k = 1, 2, ...$ to zero, we get initial value problems

$$\begin{cases} B_k''(t) + \alpha B_k'(t) + 4ak^2 B_k(t) + 4bk^2 B_k(-t) = f_k(t), \\ \\ -\infty < t < \infty, \\ \\ B_k(0) = \xi_k,\ B_k'(0) = \omega_k,\ k = 1, 2, ..., \end{cases}$$

$$\begin{cases} A_k''(t) + \alpha A_k'(t) + 4ak^2 A_k(t) + 4bk^2 A_k(-t) = g_k(t), \\ \\ -\infty < t < \infty, \\ \\ A_k(0) = \varphi_k,\ A_k'(0) = \psi_k,\ k = 0, 1, 2, ... \end{cases}$$

$$\begin{cases} A_0''(t) + \alpha A_0'(t) = g_0(t),\ -\infty < t < \infty, \\ \\ A_0(0) = \varphi_0,\ A_0'(0) = \psi_0 \end{cases}$$

for the second order involutory ordinary differential equations. Applying results of Chapter Two and , we get

$$A_k(t) = \cos(mt)\varphi_k + \frac{\sin(mt)}{m}\psi_k + \frac{\cos(nt) - \cos(mt)}{m^2 - n^2}$$

$$\times \left[bk^2 - \frac{\alpha^2}{2} - \sqrt{\left(a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4 \right)} \right] \varphi_k$$

$$+ \frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2 - n^2}$$

$$\times \left(-\alpha(bk^2 + ak^2)\varphi_k - \left[bk^2 + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4} \right] \psi_k \right)$$

$$+ \frac{1}{m^2 - n^2} \int_0^t \left[-n \sin(n(t-s)) + m \sin(m(t-s)) \right] g_k(s) ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos(n\,(t-s))-\cos(m\,(t-s))\right] g_k(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t \left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right] g_k(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right] g_k(s)ds, k\neq 0,$$

$$B_k(t)=\cos(mt)\xi_k+\frac{\sin(mt)}{m}\omega_k+\frac{\cos(nt)-\cos(mt)}{m^2-n^2} \tag{3.15}$$

$$\times\left[bk^2-\frac{\alpha^2}{2}-\sqrt{\left(a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4\right)}\right]\xi_k$$

$$+\frac{\frac{1}{n}\sin(nt)-\frac{1}{m}\sin(mt)}{m^2-n^2}$$

$$\times\left(-\alpha(bk^2+ak^2)\xi_k-\left[bk^2+\frac{\alpha^2}{2}+\sqrt{\left(a\alpha^2k^2+\frac{\alpha^4}{4}+b^2k^4\right)}\right]\omega_k\right)$$

$$+\frac{1}{m^2-n^2}\int_0^t \left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos(n\,(t-s))-\cos(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t \left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right] f_k(s)ds, k\neq 0,$$

Then, applying formula (3.14) and (3.11)we can obtain Fourier series solution of mixed problem (3.13) by the following formula

$$u(t,x) = \frac{1}{\alpha}\varphi_0 + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\psi_0 +_0^t \frac{1}{\alpha}\left(1 - e^{-\alpha(t-s)}\right) g_0(s)ds$$

$$+\sum_{k=1}^{\infty} \cos 2kx \left\{ \cos(mt)\varphi_k + \frac{\sin(mt)}{m}\psi_k + \frac{\cos(nt) - \cos(mt)}{m^2 - n^2} \right.$$

$$\times \left[bk^2 - \frac{\alpha^2}{2} - \sqrt{\left(a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4 \right)} \right] \varphi_k$$

$$+\frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2 - n^2}$$

$$\times \left(-\alpha(bk^2 + ak^2)\varphi_k - \left[bk^2 + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4} \right] \psi_k \right)$$

$$+\frac{1}{m^2 - n^2} \int_0^t \left[-n\sin(n(t-s)) + m\sin(m(t-s)) \right] g_k(s)ds$$

$$-\frac{\alpha}{m^2 - n^2} \int_0^t \left[\cos(n(t-s)) - \cos(m(t-s)) \right] g_k(s)ds$$

$$-\frac{a}{m^2 - n^2} \int_0^t \left[-n\sin(n(t-s)) + m\sin(m(t-s)) \right] g_k(s)ds$$

$$-\frac{b}{m^2 - n^2} \int_{-t}^0 \left[-\frac{1}{n}\sin(n(t+s)) + \frac{1}{m}\sin(m(t+s)) \right] g_k(s)ds\}$$

$$+\sum_{k=1}^{\infty} \sin 2kx \left\{ \cos(mt)\xi_k + \frac{\sin(mt)}{m}\omega_k + \frac{\cos(nt) - \cos(mt)}{m^2 - n^2} \right.$$

$$\times \left[bk^2 - \frac{\alpha^2}{2} - \sqrt{\left(a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4 \right)} \right] \xi_k$$

$$+\frac{\frac{1}{n}\sin(nt) - \frac{1}{m}\sin(mt)}{m^2 - n^2}$$

$$\times \left(-\alpha(bk^2 + ak^2)\xi_k - \left[bk^2 + \frac{\alpha^2}{2} + \sqrt{a\alpha^2 k^2 + \frac{\alpha^4}{4} + b^2 k^4} \right] \omega_k \right)$$

$$+\frac{1}{m^2-n^2}\int_0^t \left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{\alpha}{m^2-n^2}\int_0^t \left[\cos(n\,(t-s))-\cos(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{a}{m^2-n^2}\int_0^t \left[-n\sin(n\,(t-s))+m\sin(m\,(t-s))\right] f_k(s)ds$$

$$-\frac{b}{m^2-n^2}\int_{-t}^0 \left[-\frac{1}{n}\sin(n\,(t+s))+\frac{1}{m}\sin(m\,(t+s))\right] f_k(s)ds\}$$

For example, for the involutory telegraph problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-au_{xx}\,(t,x)-bu_{xx}\,(-t,x) \\ =\left(4ae^{-t}+4be^{t}\right)\sin\left(2x\right), x\in\left(0,\pi\right),\ -\infty<t<\infty, \\ u(0,x)=1+\sin(2x),\ u_t(0,x)=-\left(1+\sin(2x)\right),\ x\in[0,\pi], \\ u(t,0)=u(t,\pi),\ u_x(t,0)=u_x(t,\pi),\ t\in(-\infty,\infty) \end{cases} \tag{3.16}$$

$g(t,x)=\left(4ae^{-t}+4be^{t}\right)\sin\left(2x\right)$, $\varphi(x)=1+\sin(2x)$, $\psi(x)=-\left(1+\sin(2x)\right)$. Therefore,

$$g_k(t)=\begin{cases} e^{-t},\ k=0, \\ 0,\ k\neq 0, \end{cases}$$

$$f_k(t)=\begin{cases} 4ae^{-t}+4be^{t},\ k=1, \\ 0,\ k\neq 1, \end{cases}$$

$$\varphi_k = \begin{cases} 1, \ k = 0, \\ \\ 0, \ k \neq 0, \end{cases}$$

$$\xi_k = \begin{cases} 1, \ k = 1, \\ \\ 0, \ k \neq 1, \end{cases}$$

$$\psi_k = \begin{cases} -1, \ k = 0, \\ \\ 0, \ k \neq 0, \end{cases}$$

$$\omega_k = \begin{cases} -1, \ k = 1, \\ \\ 0, \ k \neq 1. \end{cases}$$

we obtain Fourier series solution of mixed problem (3.14)equating the coefficients of $\cos 2kx$, $k = 0, 1, 2, ...$, and $\sin 2kx$, $k = 1, 2, ...$ to zero, we get initial value problems

$$\begin{cases} B_1''(t) + B_1'(t) + 4aB_1(t) + 4bB_1(-t) \\ \\ = 4ae^{-t} + 4be^{t}, \\ \\ -\infty < t < \infty, \\ \\ B_1(0) = 1, \ B_1'(0) = -1, \end{cases}$$

$$\begin{cases} A_0''(t) + A_0'(t) = 0, \ -\infty < t < \infty, \\ \\ A_0(0) = 1, \ A_0'(0) = -1. \end{cases}$$

Applying formulas (3.11) and (3.15), we get $B_1(t) = e^{-t}$, $A_0(t) = e^{-t}$ and

$$u(t,x) = A_0(t) + B_1(t)\sin 2kx = e^{-t} + e^{-t}\sin(2x).$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1,...,x_n) \in \overline{\Omega},\ -\infty < t < \infty, \\ u(\frac{d}{2},x) = \psi(x),\ u_t(\frac{d}{2},x) = \varphi(x),\ x \in \overline{\Omega},\ d \geq 0, \\ u(t,x)|_{S_1} = u(t,x)|_{S_2},\ \frac{\partial u(t,x)}{\partial \overline{p}}\Big|_{S_1} = \frac{\partial u(t,x)}{\partial \overline{p}}\Big|_{S_2},\ t \in (-\infty,\infty) \end{cases} \tag{3.17}$$

for the multidimensional involutory telegraph type equation. Assume that $a_r > a_0 > 0, 0 \leq \alpha$ and $g(t,x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$, $\psi(x)\left(t \in (-\infty,\infty), x \in \overline{\Omega}\right)$ are smooth functions. Here $S = S_1 \cup S_2$, $\varnothing = S_1 \cap S_2$. However Fourier series method described in solving (3.17) can be used only in the case when (3.17) has constant coefficients.

3.3 The Laplace transform solution

First, we consider the initial boundary value problem

$$
\begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-au_{xx}\left(t,x\right)-bu_{xx}\left(-t,x\right) \\
=(-4ae^{-t}-4be^{t})e^{-2x}, \\
x\in\left(0,\infty\right),\ t\in\left(-\infty,\infty\right) \\
u(0,x)=e^{-2x},\ u_t(0,x)=-e^{-2x},\ x\in[0,\infty), \\
u(t,0)=e^{-t},\ u_x(t,0)=-2e^{-t},\ t\in(-\infty,\infty)
\end{cases}
\tag{3.18}
$$

for a one dimensional involutory telegraph equation. For solving the problem (3.18) we denote that

$$
u\left(t,s\right)=L\left\{u\left(t,x\right)\right\}.
$$

Appling Laplace transform of both side with respect to x, we get

$$
u_{tt}\left(t,s\right)+u_t\left(t,s\right)-a\left\{s^2u\left(t,s\right)-se^{-t}-\left(-2e^{-t}\right)\right\}
$$

$$
-b\left\{s^2u\left(-t,s\right)-se^{t}-\left(-2e^{t}\right)\right\}=(-4ae^{-t}-4be^{t})\frac{1}{s+2},
$$

$$
u\left(0,s\right)=\frac{1}{s+2},\ u_t\left(0,s\right)=-\frac{1}{s+2}.
$$

From that it follows initial value problem

$$
u_{tt}\left(t,s\right)+u_t\left(t,s\right)-as^2u\left(t,s\right)-bs^2u\left(-t,s\right)=a\left(s\right)e^{-t}+b\left(s\right)e^{t},
$$

$$u(0,s)=\frac{1}{s+2},\ u_t(0,s)=-\frac{1}{s+2}$$

for the second order involutory ordinary differential equation. Here

$$a(s)=-\frac{as^2}{s+2},\ b(s)=-\frac{bs^2}{s+2}.$$

Applying Theorem 2:1, we get the equivalent initial value problem.

$$\begin{cases} u^{(4)}(t,s)-(2as^2+1)\,u_{tt}(t,s)+(a^2-b^2)\,s^4u(t,s) \\ =\frac{e^{-t}}{s+2}\left[a^2s^4-b^2s^4-2as^2\right],\ t\in I, \\ u(0,s)=\frac{1}{s+2},\ u_t(0,s)=-\frac{1}{s+2},\ u_{tt}(0,s)=\frac{2bs^2+1}{s+2}, \\ u^{(3)}(0,s)=\frac{1}{s+2}\left(2bs+2as^2-1\right) \end{cases} \tag{3.19}$$

Then it is easy to see that

$$u^{(4)}(t,s)-\left(2as^2+1\right)u_{tt}(t,s)+\left(a^2-b^2\right)s^4u(t,s)$$

$$=\left(\frac{d^2}{dt^2}-\left(as^2+\frac{1}{2}+\sqrt{as^2+\frac{1}{4}+b^2s^4}\right)\right)$$

$$\times \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4} \right) \right) u\,(t,s)$$

Therefore, problem (3.19) can be written as initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} + \sqrt{as^2 + \frac{1}{4} + b^2 s^4} \right) \right) u\,(t,s) = v(t,s) \\ u\,(0,s) = \frac{1}{s+2},\ u_t\,(0,s) = -\frac{1}{s+2}, \\ \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4} \right) \right) v(0,s) \\ = \frac{e^{-t}}{s+2} \left[a^2 s^4 - b^2 s^4 - 2as^2 \right],\ t \in I, \\ v(0,s) = \left(bs^2 - \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4} \right) \frac{1}{s+2} + \frac{1-(a+b)s^2}{s+2} \\ v'(0,s) = \frac{(a-b)s^2}{s+2} + \frac{1}{s+2} \left(bs^2 - \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4} \right) \end{cases}$$

Applying results of Chapter Two, we get

$$u\,(t,s) = \frac{e^{-t}}{s+2}.$$

Therefore,

$$u\,(t,x) = e^{-t} e^{-2x}.$$

is the exact solution of problem (3.18).

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - a \sum_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \psi(x), \ u_t(\frac{d}{2}, x) = \varphi(x), \ x \in \overline{\Omega}^+, \\ u(t,x) = \alpha\left(t,x\right), \ u_{x_r}(t,x) = \beta_r\left(t,x\right), \ 1 \le r \le n, \ t \in I, \ x \in S^+ \end{cases} \tag{3.20}$$

for the multidimensional telegraph type involutory partial differential equations. Assume that $1 \le \alpha, a_r > a_0 > 0$ and $g\left(t,x\right) \left(t \in I, x \in \overline{\Omega}^+\right)$, $\psi(x)$, $\varphi(x) \left(x \in \overline{\Omega}^+\right)$, $\alpha\left(t,x\right)$, $\beta_r\left(t,x\right) \left(t \in I, x \in S^+\right)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (3.20) can be used only in the case when (3.20) has constant or polynomial coefficients.

Second, we consider the initial-value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) \\ = -4ae^{-2x} - 4be^{-2x}, \\ x \in (0,\infty),\ t \in (-\infty,\infty) \\ u(0,x) = e^{-2x},\ u_t(0,x) = 0, x \in [0,\infty), \\ u(t,0) = 1,\ u(t,\infty) = 0,\ t \in (-\infty,\infty) \end{cases} \tag{3.21}$$

for a one dimensional involutory telegraph equation with dumping term. For solving the problem (3.21) applying Laplace transform of both sides with respect to x, we get

$$u_{tt}(t,s) + u_t(t,s) - a\left\{s^2 u(t,s) - s - \beta(t)\right\}$$
$$-b\left\{s^2 u(-t,s) - s - \beta(-t)\right\}$$
$$= -(a+b)s - a\beta(t) - b\beta(-t) - \frac{4(a+b)}{s+2},$$
$$u(0,s) = \frac{1}{s+2}, u_t(0,s) = 0.$$

where $\beta(t)$ is unknown function and

$$\beta(t) = u_x(t,0).$$

From that it follows the following problem

$$u_{tt}(t,s) + u_t(t,s) - as^2 u(t,s) - bs^2 u(-t,s)$$

$$= -(a+b)s - a\beta(t) - b\beta(-t) - \frac{4(a+b)}{s+2}, \tag{3.22}$$

$$u(0,s) = \frac{1}{s+2},\ u_t(0,s) = 0$$

for second order involutory ordinary differential equations.We will obtain $u(t,s)$. In the same manner in Chapter Two, we get equivalent to (3.21) the following problem

$$\begin{cases} u^{(4)}(t,s) - (2as^2+1)u_{tt}(t,s) + (a^2-b^2)s^4 u(t,s) \\ = -a\beta''(t) - b\beta''(-t) + \frac{a}{bs^2}\beta'(t) - \frac{1}{s^2}\beta'(-t) \\ +(a^2-b^2)s^2\beta(t) + (a^2-b^2)s^3 + \frac{4(a^2-b^2)s^2}{s+2},\ t \in I, \\ u(0,s) = \frac{1}{s+2},\ u_t(0,s) = 0,\ u_{tt}(0,s) = -(a+b)[\beta(0)+2], \\ u^{(3)}(0,s) = (b-a)\beta'(0) \end{cases} \tag{3.23}$$

Then it is easy to see that

$$u^{(4)}(t,s) - \left(2as^2+1\right)u_{tt}(t,s) + \left(a^2-b^2\right)s^4 u(t,s)$$

$$= \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} + \sqrt{as^2 + \frac{1}{4} + b^2s^4}\right)\right)$$

$$\times \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2s^4}\right)\right)u(t,s)$$

Therefore, problem (3.23) can be written as initial value problem

$$\begin{cases} \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} + \sqrt{as^2 + \frac{1}{4} + b^2 s^4}\right)\right) u\left(t, s\right) = v(t, s) \\ \\ u\left(0, s\right) = \frac{1}{s+2},\ u_t\left(0, s\right) = 0, \\ \\ \left(\frac{d^2}{dt^2} - \left(as^2 + \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4}\right)\right) v(t, s) \\ \\ = -a\beta''\left(t\right) - b\beta''\left(-t\right) + \frac{a}{bs^2}\beta'\left(t\right) - \frac{1}{s^2}\beta'\left(-t\right) \\ \\ v(0, s) = \left(bs^2 - \frac{1}{2} - \sqrt{as^2 + \frac{1}{4} + b^2 s^4}\right)\frac{1}{s+2} - 4a - 4b \\ \\ v'(0, s) = \frac{-bs^2 - as^2}{s+2} + 4a + 4b \end{cases}$$

It is easy to see that $\beta\left(s\right) = -\frac{2}{s}$, $u\left(t, s\right) = \frac{1}{s+2}$. Then,

$$u(t, x) = e^{-2x}$$

is the exact solution of problem (3.21).

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - a\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} \\ \\ -b\sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = g(t,x), \\ \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ -\infty < t < \infty, \\ \\ u(\frac{d}{2}, x) = \psi, \ u_t(\frac{d}{2}, x) = 0, \ x \in \overline{\Omega}^+, \\ \\ u(t,x) = \alpha\left(t,x\right), \ t \in I, \ x \in S^+ \end{cases} \tag{3.24}$$

for the multidimensional telegraph type involutory partial differential equations. Assume that $1 \le \alpha, a_r > a_0 > 0$ and $g\left(t,x\right)\left(t \in I, x \in \overline{\Omega}^+\right), \psi(x)\left(x \in \overline{\Omega}^+\right), \alpha\left(t,x\right), \beta_r\left(t,x\right)\left(t \in I, x \in S^+\right)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (3.24) can be used only in the case when (3.24) has constant or polynomial coefficients.

3.4 The Fourier transform solution

First, we consider the initial-value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+\frac{\partial u(t,x)}{\partial t}-au_{xx}\left(t,x\right)-bu_{xx}\left(-t,x\right) \\ =\left(2-a(4x^2-2)\right)e^{-2t}e^{-x^2}-b(4x^2-2)e^{2t}e^{-x^2}, \\ x\in\left(-\infty,\infty\right),\ -\infty<t<\infty, \\ u(0,x)=e^{-x^2},\ u_t(0,x)=-2e^{-x^2},\ x\in(-\infty,\infty). \end{cases} \tag{3.25}$$

for telegraph type involutory partial differential equation. We will obtain Fourier transform for solving problem (3.25). Taking the Fourier transform, we get initial value problem

$$\begin{cases} u_{tt}\left(t,s\right)+u_t\left(t,s\right)+as^2u\left(t,s\right)+bs^2u\left(-t,s\right) \\ =2e^{-2t}q(s)+as^2e^{-2t}q(s)+bs^2e^{2t}q(s), \\ u\left(0,s\right)=q(s),\ u_t\left(0,s\right)=-2q(s) \end{cases} \tag{3.26}$$

for second order involutory ordinary differential equation. Here

$$u\left(t,s\right)=F\left\{u\left(t,x\right)\right\},\ q(s)=F\left\{e^{-x^2}\right\}.$$

In the similar manner we get equivalent to problem (3.26) the following initial value problem

$$
\begin{cases}
u^{(4)}(t,s) + (2as^2-1)\,u_{tt}(t,s) + (a^2-b^2)\,s^4 u(t,s) \\
= 12e^{-2t}q(s) + 8as^2e^{-2t}q(s) + (a^2-b^2)s^4e^{-2t}q(s), \\
u(0,s) = q(s),\ u_t(0,s) = -2q(s),\ u_{tt}(0,s) = 4q(s), \\
u^{(3)}(t,s) = -8q(s)
\end{cases}
$$

for the fourth order ordinary differential equation. It is easy to see that

$$
u(t,s) = q(s)e^{-2t} = e^{-2t}F\left\{e^{-x^2}\right\}.
$$

Therefore, the exact solution of the problem (3.25) is

$$
u(t,x) = e^{-2t}e^{-x^2}.
$$

Finally, we study the stability of the solution of the initial value problem for telegraph type involutory partial differential equation

$$
\begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2} + \alpha\frac{\partial u(t,x)}{\partial t} - au_{xx}(t,x) - bu_{xx}(-t,x) = g(t,x),\ t,\ x\in I, \\
u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x\in I.
\end{cases}
\tag{3.27}
$$

Here $g(t,x)\,(t,x\in I)$ and $\varphi(x)$, $\psi(x)\,(x\in I)$ are given smooth and bounded functions and $|b| < a, \frac{\alpha^2}{2} < a \leq \frac{\alpha^2}{4} + \frac{b^2}{\alpha^2}, \alpha \geq 0.$

Problem (3.27) can be written as abstract initial value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + aAu\left(t\right) + bAu\left(-t\right) = g(t),\ t \in I \\ \\ u(0) = \varphi,\ u'(0) = \psi \end{cases} \tag{3.28}$$

in a Banach space $C(I)$ of all continuous bounded functions $f(x)$ defined on I with norm

$$\|f\|_{C(I)} = \sup_{x \in I} \left|f(x)\right|.$$

Here, positive operator A defined by the formula

$$Au = -u''(x)$$

with domain $D(A) = \{u : u(x),\ u''(x) \in C(I)\}$, $g(t) = g(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions with values in $C(I)$ and $\varphi = \varphi(x),\ \psi = \psi(x)$ are unknown elements of $C(I)$. The normed space $C_1(I)$ is the all continuous real-valud functions $f(x)$ on I and norm defined by

$$\|f\|_{C_1(I)} = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

Theorem 3.1. Assume that $|b| < a, 0 \leq \alpha,\ a \in (\frac{\alpha^2}{2}, \frac{\alpha^2}{4} + \frac{b^2}{\alpha^2}]$.Let $g(t)$ be a smooth and bounded abstract functions on I and $g(t), g_t(t), g_{tt}(t) \in C_1(I)$ and $g(t), \varphi, \psi \in D(A)$, then the problem (3.28) is equivalent to the following initial value problem

$$\begin{cases} \frac{d^4u(t)}{dt^4} + (2a-\alpha^2)\, A\frac{d^2u(t)}{dt^2} + (a^2-b^2)\, A^2u\,(t) = F(t), \\ \\ F(t) = aAg(t) - bAg(-t) - \alpha g_t(t) + g_{tt}(t),\ t \in I, \\ \\ u(0) = \varphi,\ u'(0) = \psi,\ u''\,(0) = -\,(a+b)\, A\varphi - \alpha\psi + g(0), \\ \\ u'''\,(0) = (-a+b)\, A\psi + \alpha\,(a+b)\, A\varphi + \alpha^2\psi + g_t(0) - \alpha g(0) \end{cases} \tag{3.29}$$

for the fourth order ordinary differential equation in a Banach space $C(I)$.

Proof. Differentiating the equation (3.28) with respect to t, we get

$$\frac{d^3u\,(t)}{dt^3} + \alpha\frac{d^2u\,(t)}{dt^2} + aAu'\,(t) - bAu'\,(-t) = g_t(t), \tag{3.30}$$

$$\frac{d^4u\,(t)}{dt^4} + \alpha\frac{d^3u\,(t)}{dt^3} + aAu''\,(t) + bAu''\,(-t) = g_{tt}(t). \tag{3.31}$$

Using these equations and initial condition and equation in problem (3.28), we get

$$\begin{cases} u(0) = \varphi,\ u'(0) = \psi, \\ \\ u''\,(0) = -\,(a+b)\, A\varphi - \alpha\psi + g(0), \\ \\ u'''\,(0) = -\,(a-b)\, A\psi + \alpha\,(a+b)\, A\varphi + \alpha^2\psi + g_t(0) - \alpha g(0). \end{cases} \tag{3.32}$$

Putting $-t$ instead of t equation (3.28), we get

$$u_{tt}(-t) + \alpha u_t(-t) + aAu\,(-t) + bAu\,(t) = g(-t). \tag{3.33}$$

Applying equations (3.28), (3.31) and (3.33), we get

$$\frac{d^4u\,(t)}{dt^4}+\alpha\frac{d^3u\,(t)}{dt^3}+aA\frac{d^2u\,(t)}{dt^2}$$

$$+bA\left[-\alpha u_t\,(-t)-aAu\,(-t)-bAu\,(t)+g(-t)\right]=g_{tt}(t),$$

$$bAu\,(-t)=-\frac{d^2u\,(t)}{dt^2}-\alpha\frac{du\,(t)}{dt}-aAu\,(t)+g(t).$$

From these equations it follows equation

$$\frac{d^4u\,(t)}{dt^4}+\alpha\frac{d^3u\,(t)}{dt^3}+aA\frac{d^2u\,(t)}{dt^2}$$

$$+\alpha\left[-\frac{d^3u\,(t)}{dt^3}-\alpha\frac{d^2u\,(t)}{dt^2}-aA\frac{du\,(t)}{dt}+g_t(t)\right]$$

$$+aA\left[\frac{d^2u\,(t)}{dt^2}+\alpha\frac{du\,(t)}{dt}+aAu\,(t)-g(t)\right]-b^2A^2u\,(t)$$

$$=-bAg(-t)+g_{tt}(t)$$

or

$$\frac{d^4u\,(t)}{dt^4}+\left(2a-\alpha^2\right)A\frac{d^2u\,(t)}{dt^2}+\left(a^2-b^2\right)A^2u\,(t)$$

$$=aAg(t)-bAg(-t)-\alpha g_t(t)+g_{tt}(t).$$

So, the problem (3.29) is presented. Now, we will get (3.28) from (3.29). Denote that

$$L(t)=\frac{d^2u\,(t)}{dt^2}+\alpha\frac{du\,(t)}{dt}+aAu\,(t)+bAu\,(-t)-g(t),\ t\in I.$$

It is easy to see that $L(t)$ is the solution of the following problem

$$L^{''}(t)+\alpha L^{'}(t)+aAL(t)+bAL(-t)=0,\ t\in I,\ L(0)=0,\ L^{'}(0)=0.$$

From that it follows $L(t)\equiv 0$. Theorem 3.1 is proved.

Now we will obtain solution of the initial value problem (3.29). It is easy to see that

$$\frac{d^4u(t)}{dt^4}+\left(2a-\alpha^2\right)A\frac{d^2u(t)}{dt^2}+\left(a^2-b^2\right)A^2u(t)$$

$$=\left(\frac{d^2}{dt^2}+q^2A\right)\left(\frac{d^2}{dt^2}+p^2A\right)u(t),$$

where

$$p^2=\left(a-\frac{\alpha^2}{2}+\sqrt{-a\alpha^2+\frac{\alpha^4}{4}+b^2}\right),q^2=\left(a-\frac{\alpha^2}{2}-\sqrt{-a\alpha^2+\frac{\alpha^4}{4}+b^2}\right).$$

Therefore, problem (3.29) can be written as abstract initial value problem

$$\begin{cases}\left(\frac{d^2}{dt^2}+p^2A\right)u(t)=v(t),\ u(0)=\varphi,\ u'(0)=\psi,\\ \left(\frac{d^2}{dt^2}+q^2A\right)v(t)=F(t),\\ F(t)=aAg(t)-bAg(-t)-\alpha g_t(t)+g_{tt}(t),\ t\in I,\\ v(0)=\left(-b-a+p^2\right)A\varphi-\alpha\psi+g(0),\\ v'(0)=\alpha\left(a+b\right)A\varphi+\left(b-a+p^2\right)A\psi+\alpha^2\psi-\alpha g(0)+g'(0)\end{cases}\tag{3.34}$$

for the system of second order abstract differential equations in a Banach space $C(I)$. Problem (3.34) can be written as initial value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - p^2 u_{xx}(t,x) = v(t,x), \ t, \ x \in I, \\ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x), \ \ x \in I, \\ \frac{\partial^2 v(t,x)}{\partial t^2} - q^2 v_{xx}(t,x) = F(t,x), F(t,x) = -a g_{xx}(t,x) \\ +b g_{xx}(-t,x) + g_{tt}(t,x) - \alpha g_t(t,x), \ t, \ x \in I, \\ v(0,x) = (b+a-p^2)\,\varphi_{xx}(x) - \alpha\psi(x) + g(0,x), \\ v_t(0,x) = -\alpha\,(a+b)\,\varphi_{xx}(x) \\ -\,(b-a+p^2)\,\psi_{xx}(x) + \alpha^2\psi(x) - \alpha g(0,x) + g'(0,x), \ x \in I \end{cases} \tag{3.35}$$

for the system of telegraph equations. Applying the d'Alembert's formula, we get

$$u(t,x) = \frac{1}{2}\left(\varphi(x+pt) + \varphi(x-pt)\right) \tag{3.36}$$

$$+\frac{1}{2p}\int_{x-pt}^{x+pt}\psi(\xi)d\xi + \frac{1}{2p}\int_0^t\int_{x-p(t-\tau)}^{x+p(t-\tau)} v(\tau,\xi)d\xi d\tau,$$

$$v(t,x) = \frac{1}{2}\left[\left(b+a-p^2\right)\varphi_{xx}(x+qt) - \alpha\psi(x+qt) + g(0,x+qt)\right. \tag{3.37}$$

$$\left. + \left(b+a-p^2\right)\varphi_{xx}(x-qt) - \alpha\psi(x-qt)\right] + g(0,x-qt)$$

$$+\frac{1}{2q}\int_{x-qt}^{x+qt}\left[-\alpha\,(a+b)\,\varphi_{\lambda\lambda}(\lambda) - \left(b-a+p^2\right)\psi_{\lambda\lambda}(\lambda) + \alpha^2\psi(\lambda) - \alpha g(0,\lambda) + g'(0,\lambda)\right]d\lambda$$

$$+\frac{1}{2q}\int_0^t\int_{x-q(t-r)}^{x+q(t-r)} F(r,\lambda)d\lambda dr.$$

Applying formulas (3.36) and (3.37), we get

$$u(t,x)=\frac{1}{2}\left(\varphi(x+pt)+\varphi(x-pt)\right)+\frac{1}{2p}\int\limits_{x-pt}^{x+pt}\psi(\xi)d\xi \tag{3.38}$$

$$+\frac{1}{2p}\int\limits_{0}^{t}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[\left(b+a-p^2\right)\varphi_{\xi\xi}(\xi+q\tau)-\alpha\psi(\xi+q\tau)\right.$$

$$\left.+\left(b+a-p^2\right)\varphi_{\xi\xi}\left(\xi-q\tau\right)-\alpha\psi\left(\xi-q\tau\right)\right]d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^2-b^2}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha\left(a+b\right)\varphi_{\lambda\lambda}(\lambda)\right.$$

$$\left.-\left(b-a+p^2\right)\psi_{\lambda\lambda}(\lambda)+\alpha^2\psi(\lambda)\right]d\lambda d\xi d\tau$$

$$+\frac{1}{2p}\int\limits_{0}^{t}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right]d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^2-b^2}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left[-\alpha g(0,\lambda)+g'(0,\lambda)\right]d\lambda d\xi d\tau$$

$$+\int\limits_{0}^{t}\frac{1}{4\sqrt{a^2-b^2}}\int\limits_{x-p(t-\tau)}^{x+p(t-\tau)}\int\limits_{0}^{\tau}\int\limits_{\xi-q(\tau-r)}^{\xi+q(\tau-r)}F(r,\lambda)d\lambda dr d\xi d\tau.$$

Theorem 3.2. Assume that $|b|<a, 0\leq\alpha$, $a\in(\frac{\alpha^2}{2},\frac{\alpha^2}{4}+\frac{b^2}{\alpha^2}]$. Let $g(t,x)\in C(I\times I), g(t,x)\in C_1(I\times I)$ and $\varphi(x),\varphi_x(x),\varphi_{xx}(x),\psi(x)\in C_1(I),\psi(x),\psi_x(x)\in C(I)$ and

$$\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}\left|g(0,z)\right|dzd\tau,\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left|\psi(\lambda)\right|d\lambda d\xi d\tau,$$

$$\int\limits_{0}^{|t|}\int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int\limits_{\xi-q\tau}^{\xi+q\tau}\left|g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)\right|d\lambda d\xi d\tau,$$

$$\int_{0}^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(\lambda)|\,d\lambda d\tau,\ \int_{-\infty}^{\infty}|g(t,x)|\,dydx<\infty$$

for any $t,x\in I$.Then, for solutions of problem (3.27) we have following stability estimates

$$\sup_{t,x\in I}|u(t,x)|\leq M_1(a,b)\left[\sup_{x\in I}|\varphi(x)|+\int_{-\infty}^{\infty}|\psi(y)|\,dy\right.$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(y,x)|\,dydx+\sup_{t,x\in I}\int_{0}^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|g(0,z)|\,dzd\tau$$

$$+\alpha\int_{-\infty}^{\infty}|\varphi(x)|\,dx+\alpha^2\sup_{t,x\in I}\int_{0}^{|t|}\int_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}|\psi(\lambda)|\,d\lambda d\xi d\tau$$

$$+\alpha\sup_{t,x\in I}\int_{0}^{|t|}\int_{x-p(|t|-\tau)}^{x+p(|t|-\tau)}\int_{\xi-q\tau}^{\xi+q\tau}\left|g(\tau-\frac{1}{q}\left|\xi-\lambda\right|,\lambda)\right|d\lambda d\xi d\tau$$

$$\left.+\alpha\sup_{t,x\in I}\int_{0}^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(z)|\,dzd\tau\right], \tag{3.39}$$

$$\sup_{t,x\in I}|u_t(t,x)|+\sup_{t,x\in I}|u_x(t,x)|\leq M_2(a,b)\left[\sup_{x\in I}|\varphi_x(x)|+\sup_{x\in I}|\psi(x)|\right.$$

$$+\sup_{y\in I}\int_{-\infty}^{\infty}|g(y,x)|\,dx+\alpha^2\sup_{t,x\in I}\int_{0}^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(z)|\,dzd\tau$$

$$\left.+\alpha\int_{-\infty}^{\infty}|\psi(y)|\,dy+\alpha\sup_{x\in I}|\varphi(x)|+\alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(y,x)|\,dydx\right], \tag{3.40}$$

$$\sup_{t,x\in I}|u_{tt}(t,x)|+\sup_{t,x\in I}|u_{xx}(t,x)|+\sup_{t,x\in I}|u_{tx}(t,x)|$$

$$\leq M_3(a,b)\left[\sup_{x\in I}|\varphi_{xx}(x)|+\sup_{x\in I}|\psi_x(x)|+\sup_{t,x\in I}|g(t,x)|\right.$$

$$+\alpha \sup_{x\in I}|\psi(x)| + \alpha \sup_{x\in I}|\varphi_x(x)| + \alpha^2 \int_{-\infty}^{\infty} |\psi(y)|\, dy + \alpha \int_{-\infty}^{\infty} \sup_{y\in I}|g(y,x)|\, dx\Bigg]. \quad (3.41)$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha,\beta,...)$ to stress the fact that the constant depends only on $\alpha,\beta,....$

Proof. We have that

$$u(t) = J_1(t,x) + J_2(t,x) + J_3(t,x) + J_4(t,x),$$

where

$$J_1(t,x) = \frac{1}{2}\left(\varphi(x+pt) + \varphi(x-pt)\right) + \frac{1}{2p}\int_{x-pt}^{x+pt} \psi(\xi)d\xi,$$

$$J_2(t,x) = \frac{1}{2p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left[\left(b+a-p^2\right)\varphi_{\xi\xi}(\xi+q\tau) - \alpha\psi(\xi+q\tau)\right.$$

$$\left. + \left(b+a-p^2\right)\varphi_{\xi\xi}\left(\xi-q\tau\right) - \alpha\psi\left(\xi-q\tau\right)\right] d\xi d\tau,$$

$$J_3(t,x) = \int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_{\xi-q\tau}^{\xi+q\tau} \left[-\alpha\left(a+b\right)\varphi_{\lambda\lambda}(\lambda)\right.$$

$$\left. - \left(b-a+p^2\right)\psi_{\lambda\lambda}(\lambda) + \alpha^2\psi(\lambda)\right] d\lambda d\xi d\tau,$$

$$J_4(t,x) = \int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_0^{\tau} \int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)} F(r,\lambda) d\lambda dr d\xi d\tau$$

$$+\frac{1}{2p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left[g(0,\xi+q\tau) + g(0,\xi-q\tau)\right] d\xi d\tau$$

$$+\int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_{\xi-q\tau}^{\xi+q\tau} \left[-\alpha g(0,\lambda)+g'(0,\lambda)\right] d\lambda d\xi d\tau.$$

We prove estimate (3.39). We will estimate $J_k(t,x)$, $k=1,2,3,4$, separately. Applying the triangle inequality, we get

$$|J_1(t,x)| \le M_{11}(a,b)\left[\sup_{x\in I}|\varphi(x)| + \int_{-\infty}^{\infty} |\psi(y)|\, dy\right] \tag{3.42}$$

for any t, $x \in I$. Now, let us estimate $J_2(t,x)$. We have that

$$\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left(\varphi_{\xi\xi}(\xi+q\tau)+\varphi_{\xi\xi}\left(\xi-q\tau\right)\right) d\xi d\tau$$

$$= \frac{1}{2}\int_0^t \big[\big(\varphi_{x+p(t-\tau)}(x+p\,(t-\tau)+q\tau)+\varphi_{x+p(t-\tau)}\left(x+p\,(t-\tau)-q\tau\right)\big)$$

$$-\big(\varphi_{x-p(t-\tau)}(x-p\,(t-\tau)+q\tau)+\varphi_{x-p(t-\tau)}\left(x-p\,(t-\tau)-q\tau\right)\big)\big]\, d\tau$$

$$= \frac{1}{p}\left[\varphi(x+pt)+\varphi\left(x-pt\right)-\varphi(x+qt)-\varphi\left(x-qt\right)\right].$$

Then

$$J_2(t,x) = \frac{(b+a-p^2)}{2\,p}\left[\varphi(x+pt)+\varphi\left(x-pt\right)-\varphi(x+qt)-\varphi\left(x-qt\right)\right] \tag{3.43}$$

$$-\frac{\alpha}{4p}\int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \left(\psi(\xi+q\tau)+\psi\left(\xi-q\tau\right)\right) d\xi d\tau.$$

Applying the triangle inequality, we get

$$|J_2(t,x)| \le M_{12}(a,b)\left[\sup_{x\in I}|\varphi(x)| + \alpha \int_0^{|t|} \int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau} |\psi(z)|\, dz d\tau\right] \tag{3.44}$$

for any $t, x \in I$. Third, let us estimate $J_3(t,x)$. It is easy to see that

$$J_3(t,x) = -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \left[\varphi_{\xi+q\tau}(\xi+q\tau) - \varphi_{\xi+q\tau}(\xi-q\tau)\right] d\xi d\tau \tag{3.45}$$

$$-\frac{b-a-p^2}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \left[\psi_{\xi+q\tau}(\xi+q\tau) - \psi_{\xi+q\tau}(\xi-q\tau)\right] d\xi d\tau$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \int\limits_{\xi-q\tau}^{\xi+q\tau} \psi(\lambda) d\lambda d\xi d\tau$$

$$= -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}} \int\limits_0^t \left[\varphi(x+p\,(t-\tau)+q\tau) - \varphi(x-p\,(t-\tau)+q\tau)\right.$$

$$\left. -\varphi(x+p\,(t-\tau)-q\tau) + \varphi(x-p\,(t-\tau)-q\tau)\right] d\tau$$

$$-\frac{(b-a-p^2)}{4\sqrt{a^2-b^2}} \int\limits_0^t \left[\psi(x+p\,(t-\tau)+q\tau) - \psi(x-p\,(t-\tau)+q\tau)\right.$$

$$\left. -\psi(x+p\,(t-\tau)-q\tau) + \psi(x-p\,(t-\tau)-q\tau)\right] d\tau$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \int\limits_{\xi-q\tau}^{\xi+q\tau} \psi(\lambda) d\lambda d\xi d\tau.$$

Applying the triangle inequality, we get

$$|J_3(t,x)| \le M_{13}(a,b) \left[\int\limits_{-\infty}^{\infty} |\psi(x)|\, dx \right.$$

$$\left. +\alpha \int\limits_{-\infty}^{\infty} |\varphi(x)|\, dx + \alpha^2 \int\limits_0^{|t|} \int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)} \int\limits_{\xi-q\tau}^{\xi+q\tau} |\psi(\lambda)|\, d\lambda d\xi d\tau \right] \tag{3.46}$$

for any $t, x \in I$. We have that

$$J_4(t,x) = \frac{1}{4\sqrt{a^2-b^2}} \int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_0^\tau \int_{\xi-p(\tau-r)}^{\xi+q(\tau-r)} \left[-ag_{\lambda\lambda}(r,\lambda) + bg_{\lambda\lambda}(-r,\lambda)\right] d\lambda dr d\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}} \int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_0^\tau \int_{\xi-p(\tau-r)}^{\xi+q(\tau-r)} \left[g_{rr}(r,\lambda) - \alpha g_r(r,\lambda)\right] d\lambda dr d\xi d\tau$$

$$+\frac{1}{2p} \int_0^t \int_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2} \left[g(0,\xi+q\tau) + g(0,\xi-q\tau)\right] d\xi d\tau$$

$$+\int_0^t \frac{1}{4\sqrt{a^2-b^2}} \int_{x-p(t-\tau)}^{x+p(t-\tau)} \int_{\xi-q\tau}^{\xi+q\tau} \left[-\alpha g(0,\lambda) + g'(0,\lambda)\right] d\lambda d\xi d\tau.$$

Applying formulas

$$\int_0^\tau \int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)} \left[-ag_{\lambda\lambda}(r,\lambda) + bg_{\lambda\lambda}(-r,\lambda)\right] d\lambda dr = \frac{2a}{q} g(\tau,\xi)$$

$$-\frac{a-b}{q}\left(g(0,\xi+p\tau) + g(0,\xi-p\tau)\right) - \frac{2b}{q} g(-\tau,\xi),$$

$$\int_0^\tau \int_{\xi-q(\tau-r)}^{\xi+q(\tau-r)} \left[g_{rr}(r,\lambda) - \alpha g_r(r,\lambda)\right] d\lambda dr$$

$$= \int_{\xi-q\tau}^{\xi} \int_0^{\tau-\frac{1}{q}(\xi-\lambda)} \left[g_{rr}(r,\lambda) - \alpha g_r(r,\lambda)\right] d\lambda dr$$

$$+ \int_{\xi}^{\xi+q\tau} \int_0^{\tau+\frac{1}{q}(\xi-\lambda)} \left[g_{rr}(r,\lambda) - \alpha g_r(r,\lambda)\right] d\lambda dr$$

$$= 2qg(\tau,\xi) - qg(0,\xi-q\tau) - qg(0,\xi+q\tau)$$

$$-\int_{\xi-q\tau}^{\xi+q\tau} g'(0,\lambda)\, d\lambda + \alpha \int_{\xi-q\tau}^{\xi+q\tau} g(0,\lambda)\, d\lambda - \alpha \int_{\xi-q\tau}^{\xi+q\tau} g(\tau - \frac{1}{q}\left|\xi-\lambda\right|, \lambda) d\lambda,$$

we get

$$J_4(t,x) = \frac{1}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \left[\frac{2a}{q} g(\tau,\xi)\right.$$

$$\left. -\frac{a-b}{q}\left(g(0,\xi+p\tau)+g(0,\xi-p\tau)\right) - \frac{2b}{q} g(-\tau,\xi)\right] d\xi d\tau$$

$$+\frac{1}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \left[2qg(\tau,\xi) - qg(0,\xi-q\tau) - qg(0,\xi+q\tau)\right] d\xi d\tau$$

$$+\frac{1}{2p} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \frac{1}{2}\left[g(0,\xi+q\tau)+g(0,\xi-q\tau)\right] d\xi d\tau$$

$$-\frac{\alpha}{4\sqrt{a^2-b^2}} \int\limits_0^t \int\limits_{x-p(t-\tau)}^{x+p(t-\tau)} \int\limits_{\xi-q\tau}^{\xi+q\tau} g(\tau - \frac{1}{q}\left|\xi-\lambda\right|, \lambda) d\lambda d\xi d\tau. \tag{3.47}$$

Applying the triangle inequality, we get

$$|J_4(t,x)| \le M_{14}(a,b) \left[\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |g(y,x)|\, dydx\right.$$

$$+\int\limits_0^{|t|} \int\limits_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau} |g(0,z)|\, dzd\tau$$

$$\left.+\alpha \int\limits_0^{|t|} \int\limits_{x-p(|t|-\tau)}^{x+p(|t|-\tau)} \int\limits_{\xi-q\tau}^{\xi+q\tau} \left|g(\tau - \frac{1}{q}\left|\xi-\lambda\right|, \lambda)\right| d\lambda d\xi d\tau\right] \tag{3.48}$$

for any $t,\ x \in I$. Combining the estimates for $J_k(t,x), k = 1,2,3,4,$ we obtain estimate (3.39). Now, we prove estimate (3.40). We will estimate $J_{k,t}(t,x)$ and $J_{k,x}(t,x)$, $k = 1,2,3,4,$ separately. First, we start with estimates for $J_{1,t}(t,x)$ and $J_{1,x}(t,x)$. We have that

$$J_{1,t}(t,x)$$

$$= \frac{p}{2} \left(\varphi_{x+pt}(x+pt) - \varphi_{x-pt}(x-pt)\right) + \frac{1}{2} \left[\psi(x+pt) + \psi(x-pt)\right], \tag{3.49}$$

$$J_{1,x}(t,x)$$

$$= \frac{1}{2} \left(\varphi_{x+pt}(x+pt) + \varphi_{x-pt}(x-pt)\right) + \frac{1}{2p} \left[\psi(x+pt) - \psi(x-pt)\right]. \tag{3.50}$$

Applying the triangle inequality, we get

$$|J_{1,t}(t,x)|, \ |J_{1,x}(t,x)| \le M_{21}(a,b) \left[\sup_{x\in I} |\varphi_x(x)| + \sup_{x\in I} |\psi(x)|\right] \tag{3.51}$$

for any $t, x \in I$. Second, let us estimate $J_{2,t}(t,x)$ and $J_{2,x}(t,x)$. Applying the formula(3.43), we get

$$J_{2,t}(t,x) = \frac{(b+a-p^2)}{2p} [p\varphi_{x+pt}(x+pt) \tag{3.52}$$

$$-p\varphi_{x-pt}(x-pt) - q\varphi_{x+qt}(x+qt) + q\varphi_{x-qt}(x-qt)]$$

$$-\frac{\alpha}{4p} \int_0^t [p\psi(x+p(t-\tau)+q\tau) + p(x+p(t-\tau)-q\tau)$$

$$+p\psi(x-p(t-\tau)+q\tau) + p\psi(x-p(t-\tau)-q\tau)]\, d\tau$$

$$J_{2,x}(t,x) = \frac{(b+a-p^2)}{2p} [\varphi_{x+pt}(x+pt) \tag{3.53}$$

$$+\varphi_{x-pt}(x-pt) - \varphi_{x+qt}(x+qt) + \varphi_{x-qt}(x-qt)]$$

$$-\frac{\alpha}{4p} \int_0^t [\psi(x+p(t-\tau)+q\tau) + \psi(x+p(t-\tau)-q\tau)$$

$$-\psi(x-p(t-\tau)+q\tau) + \psi(x-p(t-\tau)-q\tau)]\, d\tau.$$

Applying the triangle inequality, we get

$$|J_{2,t}(t,x)|, \ |J_{2,x}(t,x)| \le M_{22}(a,b) \left[\sup_{x\in I} |\varphi_x(x)| + \alpha \int_{-\infty}^{\infty} |\psi(y)|\, dy\right] \tag{3.54}$$

for any $t, x \in I$. Third, let us estimate $J_{3,t}(t,x)$ and $J_{3,x}(t,x)$.Applying the formula(3.45),

we get

$$J_{3,t}(t,x) = -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}\left[\frac{p}{p+q}\left(\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)-\varphi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)-\varphi(x-pt)\right)\right] \tag{3.55}$$

$$-\frac{(b-a-p^2)}{4\sqrt{a^2-b^2}}\left[\frac{p}{p+q}\left(\psi(x+qt)+\psi(x-qt)-\psi(x+pt)-\psi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\psi(x+qt)+\psi(x-qt)-\psi(x+pt)-\psi(x-pt)\right)\right]$$

$$+\frac{p\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau}\psi(\lambda)d\lambda+\int_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau}\psi(\lambda)d\lambda\right]d\tau,$$

$$J_{3,x}(t,x) = -\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}\left[\frac{1}{p+q}\left(-\varphi(x+qt)+\varphi(x-qt)-\varphi(x+pt)+\varphi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(-\varphi(x+qt)+\varphi(x-qt)+\varphi(x+pt)-\varphi(x-pt)\right)\right]$$

$$-\frac{(b-a-p^2)}{4\sqrt{a^2-b^2}}\left[\frac{1}{p+q}\left(-\psi(x+qt)+\psi(x-qt)-\psi(x+pt)+\psi(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(-\psi(x+qt)+\psi(x-qt)+\psi(x+pt)-\psi(x-pt)\right)\right]$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int_0^t\left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau}\psi(\lambda)d\lambda-\int_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau}\psi(\lambda)d\lambda\right]d\tau. \tag{3.56}$$

Applying the triangle inequality, we get

$$|J_{3,t}(t,x)|\,,\;\; |J_{3,x}(t,x)|$$

$$\leq M_{23}(a,b)\left[\alpha\sup_{x\in I}|\varphi(x)|+\sup_{x\in I}|\psi(x)|+\alpha^2\int_0^{|t|}\int_{x-p(|t|-\tau)-q\tau}^{x+p(|t|-\tau)+q\tau}|\psi(\xi)|\,d\xi d\tau\right] \tag{3.57}$$

for any $t,\, x\in I$. Fourth, let us estimate $J_{4,t}(t,x)$ and $J_{4,x}(t,x)$. Applying formula (3.47),we get

$$J_{4,t}(t,x) = \frac{p}{4\sqrt{a^2-b^2}}$$

$$\times \int_0^t \left[\left(\frac{2a}{q} + 2q \right) \left[g(\tau, x + p\,(t-\tau)) + g(\tau, x - p\,(t-\tau)) \right] \right.$$

$$- \frac{2b}{q} \left[g(-\tau, x + p\,(t-\tau)) + g(-\tau, x - p\,(t-\tau)) \right]$$

$$- \left(\frac{a-b}{q} + q \right) \left[g(0, x + p\,(t-\tau) + q\tau) + g(0, x - p\,(t-\tau) + q\tau) \right]$$

$$\left. - \left(\frac{a-b}{q} + q \right) \left[g(0, x + p\,(t-\tau) - q\tau) + g(0, x - p\,(t-\tau) - q\tau) \right] \right] d\tau$$

$$+ \frac{1}{4} \int_0^t \left[g(0, x + p\,(t-\tau) + q\tau) + g(0, x - p\,(t-\tau) + q\tau) \right.$$

$$\left. + g(0, x + p\,(t-\tau) - q\tau) + g(0, x - p\,(t-\tau) - q\tau) \right] d\tau$$

$$- \frac{\alpha p}{4\sqrt{a^2-b^2}} \int_0^t \left[\int_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau} g(\tau - \frac{1}{q} \left| x + p\,(t-\tau) - \lambda \right|, \lambda) d\lambda \right.$$

$$\left. + \int_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau} g(\tau - \frac{1}{q} \left| x - p\,(t-\tau) - \lambda \right|, \lambda) d\tau \right] d\tau, \qquad (3.58)$$

$$J_{4,x}(t,x) = \frac{1}{4\sqrt{a^2-b^2}}$$

$$\times \int_0^t \left[\left(\frac{2a}{q} + 2q \right) \left[g(\tau, x + p\,(t-\tau)) - g(\tau, x - p\,(t-\tau)) \right] \right.$$

$$- \frac{2b}{q} \left[g(-\tau, x + p\,(t-\tau)) - g(-\tau, x - p\,(t-\tau)) \right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\,(t-\tau)+q\tau)-g(0,x-p\,(t-\tau)+q\tau)\right]$$

$$\left.-\left(\frac{a-b}{q}+q\right)\left[g(0,x+p\,(t-\tau)-q\tau)-g(0,x-p\,(t-\tau)-q\tau)\right]\right] d\tau$$

$$+\frac{1}{4}\int\limits_{0}^{t}\left[g(0,x+p\,(t-\tau)+q\tau)-g(0,x-p\,(t-\tau)+q\tau)\right.$$

$$\left.+g(0,x+p\,(t-\tau)-q\tau)-g(0,x-p\,(t-\tau)-q\tau)\right] d\tau$$

$$-\frac{\alpha}{4\sqrt{a^2-b^2}}\int\limits_{0}^{t}\left[\int\limits_{x+p(t-\tau)-q\tau}^{x+p(t-\tau)+q\tau} g(\tau-\frac{1}{q}\left|x+p\,(t-\tau)-\lambda\right|,\lambda)d\lambda\right.$$

$$\left.-\int\limits_{x-p(t-\tau)-q\tau}^{x-p(t-\tau)+q\tau} g(\tau-\frac{1}{q}\left|x-p\,(t-\tau)-\lambda\right|,\lambda)d\lambda\right] d\tau. \tag{3.59}$$

Applying the triangle inequality, we get

$$\left|J_{4,t}(t,x)\right|,\left|J_{4,x}(t,x)\right|$$

$$\leq M_{24}(a,b)\left[\sup_{y\in I}\int\limits_{-\infty}^{\infty}\left|g(y,x)\right|dx+\alpha\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left|g(y,x)\right|dydx\right] \tag{3.60}$$

for any $t,\ x\in I$. Combining the estimates for $J_{k,t}(t,x)$ and $J_{k,x}(t,x), k=1,2,3,4,$ we obtain estimate (3.40).

Now, we will prove estimate (3.41). We will estimate $J_{k,tt}(t,x)$, $J_{k,tx}(t,x)$ and $J_{k,xx}(t,x)$, $k=1,2,3,4,$ separately. First, we will estimate $J_{1,tt}(t,x)$, $J_{1,tx}(t,x)$ and $J_{1,xx}(t,x)$. Using formulas (3.49), (3.50) and taking the derivative, we get

$$J_{1,tt}(t,x)=\frac{p^2}{2}\left(\varphi_{x+pt,x+pt}(x+pt)+\varphi_{x-pt,x-pt}(x-pt)\right)$$

$$+\frac{p}{2}\left[\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right],$$

$$J_{1,tx}(t,x)=\frac{p}{2}\left(\varphi_{x+pt,x+pt}(x+pt)-\varphi_{x-pt,x-pt}(x-pt)\right)$$

$$+\frac{1}{2}\left[\psi_{x+pt}(x+pt)+\psi_{x-pt}(x-pt)\right],$$

$$J_{1,xx}(t,x)=\frac{1}{2}\left(\varphi_{x+pt,x+pt}(x+pt)+\varphi_{x-pt,x+pt}(x-pt)\right)$$

$$+\frac{1}{2p}\left[\psi_{x+pt}(x+pt)-\psi_{x+pt}(x-pt)\right].$$

Applying the triangle inequality, we get

$$\left|J_{1,tt}(t,x)\right|,\left|J_{1,tx}(t,x)\right|,\ \ \left|J_{1,xx}(t,x)\right|$$

$$\leq M_{31}(a,b)\left[\sup_{x\in I}\left|\varphi_{xx}(x)\right|+\sup_{x\in I}\left|\psi_{x}(x)\right|\right] \tag{3.61}$$

for any $t,\,x\in I$. Second, we estimate $J_{2,tt}(t,x)$, $J_{2,tx}(t,x)$ and $J_{2,xx}(t,x)$. Using formulas (3.49), (3.50) and taking the derivative, we get

$$J_{2,tt}(t,x)=\frac{(b+a-p^{2})}{2p}\left[p^{2}\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.+p^{2}\varphi_{x-pt,x-pt}(x-pt)-q^{2}\varphi_{x+qt,x+qt}(x+qt)-q^{2}\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\left[p\psi(x+pt)+p\psi(x-qt)+p\psi(x+qt)+p\psi(x-qt)\right]$$

$$-\frac{\alpha}{4p}\int_{0}^{t}\left[p^{2}\psi_{x+p(t-\tau)+q\tau}(x+p\,(t-\tau)+q\tau)+p^{2}\psi(x+p\,(t-\tau)-q\tau)\right.$$

$$\left.-p^{2}\psi_{x-p(t-\tau)+q\tau}(x-p\,(t-\tau)+q\tau)+p^{2}\psi_{x-p(t-\tau)-q\tau}(x-p\,(t-\tau)-q\tau)\right]d\tau$$

$$=\frac{(b+a-p^{2})}{2p}\left[p^{2}\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$\left.+p^{2}\varphi_{x-pt,x-pt}(x-pt)-q^{2}\varphi_{x+qt,x+qt}(x+qt)-q^{2}\varphi_{x-qt,x-qt}(x-qt)\right]$$

$$-\frac{\alpha}{4p}\left[p\psi(x+pt)+p\psi(x-qt)+p\psi(x+qt)+p\psi(x-qt)\right]$$

$$-\frac{\alpha}{4}\left[\frac{p}{-p+q}\left(\psi(x+qt)-\psi(x+pt)\right)-\frac{p}{p+q}\left(\psi(x-qt)-\psi(x+pt)\right)\right.$$

$$\left.-\frac{p}{p+q}\left(\psi(x+qt)-\psi(x-pt)\right)+\frac{p}{p-q}\left(\psi(x-qt)-\psi(x-pt)\right)\right],$$

$$J_{2,xt}(t,x) = \frac{(b+a-p^2)}{2p}\left[p\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$-p\varphi_{x-pt,x-pt}(x-pt) - q\varphi_{x+qt,x+qt}(x+qt) - q\varphi_{x-qt,x-qt}(x-qt)]$$

$$-\frac{\alpha}{4}\int_0^t \left[\psi_{x+p(t-\tau)+q\tau}(x+p(t-\tau)+q\tau) + \psi_{x+p(t-\tau)-q\tau}(x+p(t-\tau)-q\tau)\right.$$

$$\left.+\psi_{x-p(t-\tau)+q\tau}(x-p(t-\tau)+q\tau) - \psi_{x-p(t-\tau)-q\tau}(x-p(t-\tau)-q\tau)\right] d\tau$$

$$= \frac{(b+a-p^2)}{2p}\left[p\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$-p\varphi_{x-pt,x-pt}(x-pt) - q\varphi_{x+qt,x+qt}(x+qt) - q\varphi_{x-qt,x-qt}(x-qt)]$$

$$-\frac{\alpha}{4}\left[\frac{1}{-p+q}(\psi(x+qt)-\psi(x+pt)) - \frac{1}{p+q}(\psi(x-qt)-\psi(x+pt))\right.$$

$$\left.+\frac{1}{p+q}(\psi(x+qt)-\psi(x-pt)) - \frac{1}{p-q}(\psi(x-qt)-\psi(x-pt))\right],$$

$$J_{2,xx}(t,x) = \frac{(b+a-p^2)}{2p}\left[\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$-\varphi_{x-pt,x-pt}(x-pt) - \varphi_{x+qt,x+qt}(x+qt) + \varphi_{x-qt,x-qt}(x-qt)]$$

$$-\frac{\alpha}{4p}\int_0^t \left[\psi_{x+p(t-\tau)+q\tau}(x+p(t-\tau)+q\tau) + \psi_{x+p(t-\tau)-q\tau}(x+p(t-\tau)-q\tau)\right.$$

$$\left.-\psi_{x-p(t-\tau)+q\tau}(x-p(t-\tau)+q\tau) + \psi_{x-p(t-\tau)-q\tau}(x-p(t-\tau)-q\tau)\right] d\tau$$

$$= \frac{(b+a-p^2)}{2p}\left[\varphi_{x+pt,x+pt}(x+pt)\right.$$

$$-\varphi_{x-pt,x-pt}(x-pt) - \varphi_{x+qt,x+qt}(x+qt) + \varphi_{x-qt,x-qt}(x-qt)]$$

$$-\frac{\alpha}{4p}\left[\frac{1}{-p+q}(\psi(x+qt)-\psi(x+pt)) - \frac{1}{p+q}(\psi(x-qt)-\psi(x+pt))\right.$$

$$\left.-\frac{1}{p+q}(\psi(x+qt)-\psi(x-pt)) + \frac{1}{p-q}(\psi(x-qt)-\psi(x-pt))\right].$$

Applying the triangle inequality, we get

$$|J_{2,tt}(t,x)|, |J_{2,tx}(t,x)|, \ |J_{2,xx}(t,x)|$$

$$\leq M_{32}(a,b)\left[\sup_{x\in I}|\varphi_{xx}(x)|+\alpha\sup_{x\in I}|\psi(x)|\right]$$

for any $t,\ x\in I$. Third, we estimate $J_{3,tt}(t,x), J_{3,tx}(t,x)$ and $J_{3,xx}(t,x)$. Using formulas (3.55), (3.56) and taking the derivative, we get

$$J_{3,tt}(t,x)=-\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(q\varphi_{x+qt}(x+qt)-q\varphi_{x-qt}(x-qt)-p\varphi_{x+pt}(x+pt)+p\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(q\varphi_{x+qt}(x+qt)-q\varphi_{x-qt}(x-qt)-p\varphi_{x+pt}(x+pt)+p\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{(b-a+p^2)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(q\psi_{x+qt}(x+qt)-q\psi_{x-qt}(x-qt)-p\psi_{x+pt}(x+pt)+p\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(q\psi_{x+qt}(x+qt)-q\psi_{x-qt}(x-qt)-p\psi_{x+pt}(x+pt)+p\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{p\alpha^2}{2\sqrt{a^2-b^2}}\int\limits_{x-qt}^{x+qt}\psi(\lambda)d\lambda+\frac{p^2\alpha^2}{4\sqrt{a^2-b^2}}\int\limits_{0}^{t}\left[\psi(x+p\,(t-\tau)+q\tau)\right.$$

$$\left.-\psi(x+p\,(t-\tau)-q\tau)-\psi(x-p\,(t-\tau)+q\tau)+\psi(x-p\,(t-\tau)-q\tau)\right]d\tau,$$

$$J_{3,tx}(t,x)=-\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{(b-a+p^2)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{p}{p+q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{p\alpha^2}{4\sqrt{a^2-b^2}}\int\limits_0^t [\psi(x+p\,(t-\tau)+q\tau)$$

$$-\psi(x+p\,(t-\tau)-q\tau)-\psi(x-p\,(t-\tau)+q\tau)+\psi(x-p\,(t-\tau)-q\tau)]\,d\tau,$$

$$J_{3,xx}(t,x)=-\frac{\alpha\,(a+b)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{1}{p+q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\varphi_{x+qt}(x+qt)+\varphi_{x-qt}(x-qt)-\varphi_{x+pt}(x+pt)-\varphi_{x-pt}(x-pt)\right)\right]$$

$$-\frac{(b-a+p^2)}{4\sqrt{a^2-b^2}}$$

$$\times\left[\frac{1}{p+q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right.$$

$$\left.+\frac{p}{p-q}\left(\psi_{x+qt}(x+qt)+\psi_{x-qt}(x-qt)-\psi_{x+pt}(x+pt)-\psi_{x-pt}(x-pt)\right)\right]$$

$$+\frac{\alpha^2}{4\sqrt{a^2-b^2}}\int\limits_0^t [\psi(x+p\,(t-\tau)+q\tau)$$

$$-\psi(x+p\,(t-\tau)-q\tau)-\psi(x-p\,(t-\tau)+q\tau)+\psi(x-p\,(t-\tau)-q\tau)]\,d\tau.$$

Applying the triangle inequality, we get

$$|J_{3,tt}(t,x)|\,,\;\;|J_{3,tx}(t,x)|\,,\;\;|J_{3,xx}(t,x)|\leq M_{33}(a,b)\left[\sup_{x\in I}|\psi_x(x)|\right.$$

$$\left.+\alpha\sup_{x\in I}|\varphi_x(x)|+\alpha^2\int\limits_{-\infty}^{\infty}|\psi(x)|\,dx\right] \tag{3.62}$$

for any t, $x\in I$. Fourth, we estimate $J_{4,tt}(t,x)$, $J_{4,tx}(t,x)$ and $J_{4,xx}(t,x)$. Using formulas (3.58), (3.59) and taking the derivative, we get

$$J_{4,tt}(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)p\,[g(0,x+pt)+g(0,x-pt)]\right.$$

$$-\frac{2b}{q}p\,[g(0,x+pt)+g(0,x-pt)]-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)[2g(0,x+qt)$$

$$+\frac{p}{-p+q}\left[g(0,x+qt)-g(0,x+pt)\right]-\frac{p}{p+q}\left[g(0,x+qt)-g(0,x-pt)\right]\Big]$$

$$-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\Big[2g(0,x-qt)$$

$$+\frac{p}{p-q}\left[g(0,x-qt)-g(0,x+pt)\right]-\frac{p}{p+q}\left[g(0,x-qt)-g(0,x-pt)\right]\Big]$$

$$-\frac{\alpha p}{2\sqrt{a^2-b^2}}\int\limits_{x-qt}^{x+qt} g(t-\frac{1}{q}\left|x-\lambda\right|,\lambda)d\lambda$$

$$-\frac{\alpha p^2}{4\sqrt{a^2-b^2}}\int\limits_0^t \left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right.$$

$$\left.-g(0,x-p\left(t-\tau\right)+q\tau)+g(0,x+p\left(t-\tau\right)-q\tau)\right]d\tau,$$

$$J_{4,tx}(t,x)=\frac{1}{4\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)\frac{1}{p}\left[g(0,x+pt)-g(0,x-pt)\right]\right.$$

$$-\frac{2b}{qp}\left[g(0,x+pt)-g(0,x-pt)\right]-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\Big[\frac{1}{-p+q}$$

$$\times\left[g(0,x+qt)-g(0,x+pt)\right]+\frac{1}{p+q}\left[g(0,x+qt)-g(0,x-pt)\right]\Big]$$

$$-\left(\frac{a-b}{q}+q+\frac{1}{4}\right)\Big[-\frac{1}{p+q}$$

$$\times\left[g(0,x-qt)-g(0,x+pt)\right]+\frac{1}{p-q}\left[g(0,x-qt)-g(0,x-pt)\right]\Big]$$

$$-\frac{\alpha p}{4\sqrt{a^2-b^2}}\int\limits_0^t \left[g(0,x+p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right.$$

$$\left.+g(0,x-p\left(t-\tau\right)+q\tau)-g(0,x+p\left(t-\tau\right)-q\tau)\right]d\tau,$$

$$J_{4,xx}(t,x)=\frac{1}{4p\sqrt{a^2-b^2}}\left[\left(\frac{2a}{q}+2q\right)[-2g(t,x)+g(0,x+pt)+g(0,x-pt)]\right.$$

$$-\frac{2b}{q}[-2g(-t,x)+g(0,x+pt)+g(0,x-pt)]$$

$$-\left(\frac{a-b}{q}+q\right)\left[\frac{1}{-p+q}(g(0,x+qt)-g(0,x+pt))-\frac{1}{p+q}(g(0,x+qt)-g(0,x-pt))\right]$$

$$-\left(\frac{a-b}{q}+q\right)\left[-\frac{1}{p+q}(g(0,x-qt)-g(0,x+pt))-\frac{1}{p-q}(g(0,x-qt)-g(0,x-pt))\right]$$

$$+\frac{1}{4}\left[\frac{1}{-p+q}(g(0,x+qt)-g(0,x+pt))-\frac{1}{p+q}(g(0,x+qt)-g(0,x-pt))\right.$$

$$\left.-\frac{1}{p+q}(g(0,x-qt)-g(0,x+pt))-\frac{1}{p-q}(g(0,x-qt)-g(0,x-pt))\right]$$

$$-\frac{\alpha p}{4\sqrt{a^2-b^2}}\int\limits_0^t[g(0,x+p(t-\tau)+q\tau)-g(0,x+p(t-\tau)-q\tau)$$

$$+g(0,x-p(t-\tau)+q\tau)-g(0,x+p(t-\tau)-q\tau)]\,d\tau.$$

Applying the triangle inequality, we get

$$|J_{4,tt}(t,x)|\,,\ |J_{4,tx}(t,x)|\,,\ |J_{4,xx}(t,x)|\leq M_{43}(a,b)\left[\sup_{t,x\in I}|g(t,x)|\right.$$

$$\left.+\alpha\int\limits_{-\infty}^{\infty}\sup_{y\in I}|g(y,x)|\,dx\right] \tag{3.63}$$

for any $t,\ x\in I$. Combining the estimates for $J_{k,tt}(t,x)$, $J_{k,tx}(t,x)$ and $J_{k,xx}(t,x)$, $k=1,2,3,4$, we obtain estimate (3.41). Theorem 3.2 is proved. Note that using similar procedure we can get the solution of following

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial^2 u(t,x)}{\partial t^2} - a \sum_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} - b \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(d-t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1, ..., x_n) \in \mathbb{R}^n, \ -\infty < t < \infty, \\ u(\frac{d}{2}, x) = \psi(x), \ u_t(\frac{d}{2}, x) = \varphi(x), \ x \in \mathbb{R}^n \end{cases} \tag{3.64}$$

for a multidimensional telegraph involutory partial differential equations. Assume that $a_r > a_0 > 0$ and $g\left(t,x\right) \left(t \in I, x \in \mathbb{R}^n\right), \psi(x), \varphi(x) \left(x \in \mathbb{R}^n\right)$ are smooth functions. However Fourier transform method described in solving (3.64) can be used only in the case when (3.64) has constant coefficients.

CHAPTER 4

DIFFERENCE METHOD FOR THE SOLUTION OF TELEGRAPH TYPE INVOLUTORY PARTIAL DIFFERENTIAL EQUATIONS

4.1 Introduction

If the analytical methods do not work correctly, we can use the numerical methods to get approximate solutions of local and nonlocal problems for the telegraph type involutory partial differential equations. In this chapter, we obtain the algorithms of numerical solution for the initial-boundary-value the problem for the one dimensional telegraph type involutory partial differential equation with Dirichlet and Neumann boundary conditions. We will present the first and second order accuracy difference schemes for the numerical solutions of involutory problems. We use the procedure of modified Gauss elimination method for solving these difference schemes.

For the construction of the approximate solutions, we define sets of grid points

$$[-T,T]_\tau = \left\{t_k : t_k = k\tau, -N \le k \le N, N\tau = T\right\},$$

$$[0,l]_h = \{x_n : x_n = nh,\ 0 \le n \le M, Mh = l\},$$

$$[-\pi,\pi]_\tau \times [0,\pi]_h$$

$$= \left\{(t_k, x_n) : t_k = k\tau, -N \le k \le N, N\tau = \pi, x_n = nh,\ 0 \le n \le M, Mh = \pi\right\}.$$

Definition 4.1.1. (Sobolevskii, 1975)

$$v(t,\tau) = o(\tau^p) \text{ as } \tau \to 0+$$

means that there exists a constant $M \ge 0$ such that , we have $|v(t,\tau)| \le M|\tau|^p$.

The construction difference schemes are based on the Taylors decomposition of three points.

Theorem 4.1. (Ashyralyev & Sobolevskii, 2004) Let the function $v(t)$ have a fourth order continuous derivative and $t_k, t_{k\pm1} \in [-T,T]_\tau$. Then the following relation holds

$$v(t_{k+1}) - 2v(t_k) + v(t_{k-1}) = \tau^2 v^{''}(t_{k+1}) + o(\tau^3), \tag{4.1}$$

$$v(t_{k+1}) - 2v(t_k) + v(t_{k-1}) = \frac{\tau^2}{2} v^{''}(t_k) + \frac{\tau^2}{4}\left[v^{''}(t_{k+1}) + v^{''}(t_{k-1})\right] + o(\tau^4). \tag{4.2}$$

Now, we will give well-known (Sobolevskii, 1975) approximation formulas for first and second order derivatives for smooth functions

$$u^{''}(x_n) = \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} + o(h^2), \tag{4.3}$$

$$\begin{cases} u^{'}(0) = \frac{-u(2h)+4u(h)-3u(0)}{2h} + o(h^2), \\ \\ u^{'}(\pi) = \frac{u(\pi-2h)-4u(\pi-h)+3u(\pi)}{2h} + o(h^2), \end{cases} \tag{4.4}$$

$$v^{'}(0) = \begin{cases} \frac{v(\tau)-v(0)}{\tau} + o(\tau), \\ \\ \frac{-v(2\tau)+4v(\tau)-3v(0)}{2\tau} + o(\tau^2). \end{cases} \tag{4.5}$$

4.2 Involutory differential equation with Dirichlet boundary condition

We consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - u_{xx}(t,x) - p u_{xx}(-t,x) \\ \\ = (\cos(t) - p\sin(t))\sin(x), \\ \\ x \in (0,\pi), \ -\pi < t < \pi, \\ \\ u(0,x) = 0, \ u_t(0,x) = \sin(x), \ x \in [0,\pi], \\ \\ u(t,0) = u(t,\pi) = 0, \ t \in [-\pi,\pi] \end{cases} \tag{4.6}$$

for the one dimensional telegraph type involutory partial differential equation with Dirichlet boundary condition. The exact solution of the equation (4.6) is $u(t,x) = \sin t \sin(x)$, $0 \leq x \leq \pi$, $-\pi \leq t \leq \pi$. Applying formulas (4.1), (4.2), (4.3), (4.4) and (4.5), we present the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + \frac{u_n^{k+1}-u_n^k}{\tau} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -p\frac{u_{n+1}^{-k-1}-2u_n^{-k-1}+u_{n-1}^{-k-1}}{h^2} = (\cos(t_{k+1}) - p\sin(t_{k+1}))\sin(x_n), \\ t_k = k\tau,\ x_n = nh, N\tau = \pi,\ Mh = \pi, \\ -N+1 \leq k \leq N-1,\ 1 \leq n \leq M-1, \\ u_n^0 = 0,\ \frac{u_n^1-u_n^0}{\tau} = \sin(x_n),\ 0 \leq n \leq M, \\ u_0^k = u_M^k = 0,\ -N \leq k \leq N \end{cases} \tag{4.7}$$

and the second order of accuracy difference scheme

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} \\ -\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2}-\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2}-p\frac{u_{n+1}^{-k}-2u_n^{-k}+u_{n-1}^{-k}}{2h^2} \\ -p\frac{u_{n+1}^{-k+1}-2u_n^{-k+1}+u_{n-1}^{-k+1}}{4h^2}-p\frac{u_{n+1}^{-k-1}-2u_n^{-k-1}+u_{n-1}^{-k-1}}{4h^2} \\ =(\cos(t_k)-p\sin(t_k))\sin(x_n), \\ t_k=k\tau, x_n=nh, N\tau=\pi, Mh=\pi, \\ -N+1\le k\le N-1,\ 1\le n\le M-1, \\ u_n^0=0,\ \frac{-u_n^2+4u_n^1-3u_n^0}{2\tau}=\sin(x_n),\ 0\le n\le M, \\ u_0^k=u_M^k=0,\ -N\le k\le N \end{cases} \tag{4.8}$$

They are systems of algebraic equations and they can be written in the matrix form

$$Au_{n-1}+Bu_n+Cu_{n+1}=D\varphi_n,\ 1\le n\le M-1,\ u_0=\overrightarrow{0},\ u_M=\overrightarrow{0}. \tag{4.9}$$

Here an in future A, B, C are $(2N+1)\times(2N+1)$ matrices and $D=I_{2N+1}$ is the identity matrix, φ_n and u_s are $(2N+1)\times 1$ column vectors

$$D\varphi_n = \begin{bmatrix} 0 \\ \tau \sin(x_n) \\ (\cos(t_k) - b\sin(t_k))\sin(x_n) \\ \cdot \\ (\cos(t_{N-1}) - b\sin(t_{N-1}))\sin(x_n) \end{bmatrix}_{(2N+1)\times 1}, \quad u_s = \begin{bmatrix} u_s^{-N} \\ u_s^{-N+1} \\ \cdot \\ u_s^{N-1} \\ u_s^{N} \end{bmatrix}_{(2N+1)\times 1}$$

and

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & d & 0 & 0 \\ 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 & \cdot & d & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & a & 0 & d & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & a+d & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & d & 0 & a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & d & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & a & 0 & 0 \\ 0 & d & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & a & 0 \\ d & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & a \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & -1 & 1 & \cdot & 0 & 0 & 0 & 0 \\ b & c & e & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & f & 0 & 0 \\ 0 & b & c & e & \cdot & 0 & 0 & 0 & \cdot & f & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & e & 0 & f & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & c & e+f & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & b+f & c & e & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & f & 0 & \cdot & 0 & 0 & 0 & \cdot & c & e & 0 & 0 \\ 0 & f & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & b & c & e & 0 \\ f & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & b & c & e \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$a = -\frac{1}{h^2}, b = \frac{1}{\tau^2}, c = -\frac{2}{\tau^2} - \frac{1}{\tau}, d = -\frac{p}{h^2}, e = \frac{1}{\tau^2} + \frac{1}{\tau} + \frac{2}{h^2}$ and $f = \frac{2p}{h^2}$ for the difference scheme (4.7) and

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ b & a & b & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & d & c & d \\ 0 & b & a & b & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & d & c & d & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & b & a & b+d & c & d & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & b+d & a+c & b+d & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & d & c & b+d & a & b & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & d & c & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & b & a & 0 & 0 \\ 0 & d & c & d & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & b & a & b & 0 \\ d & c & d & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & b & a & b \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & -3 & 4 & -1 & 0 & \cdot & 0 & 0 & 0 & 0 \\ q & e & f & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & t & g & t \\ 0 & q & e & f & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & t & g & t & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & q & e & f+t & g & t & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & q+t & e+g & f+t & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & t & g & q+t & e & f & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & t & g & & 0 & 0 & 0 & 0 & 0 & \cdot & e & f & 0 & 0 \\ 0 & t & g & t & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & q & e & f & 0 \\ t & g & t & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & q & e & f \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$a = -\frac{1}{2h^2}$, $b = -\frac{1}{4h^2}$, $c = -\frac{p}{2h^2}$, $d = -\frac{p}{4h^2}$, $e = -\frac{2}{\tau^2} + \frac{1}{h^2}$, $f = \frac{1}{\tau^2} + \frac{1}{2\tau} + \frac{1}{2h^2}$, $q = \frac{1}{\tau^2} - \frac{1}{2\tau} + \frac{1}{2h^2}$, $g = \frac{p}{h^2}$, and $t = \frac{p}{2h^2}$ for the difference scheme (4.8).

For getting the solution of the matrix (4.9), we will apply the modified Gauss elimination method. We are using the following form for getting the solution of the matrix equation

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1},\ n = M-1,\ \ldots,\ 1, \tag{4.10}$$

where $u_M = \overrightarrow{0}$, $\alpha_j\ (j = 1, ..., M-1)$ are $(2N+1) \times (2N+1)$ square matrices, $\beta_j\ (j = 1\ , \ldots, M-1)$ are $(2N+1) \times 1$ column matrices, $\alpha_1,\ \beta_1$ are zero matrices and

$$\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n + C\beta_n),\ n = 1,\ \ldots,\ M-1. \end{cases}$$

NUMERICAL ANALYSIS

The different values of N and M are recorded to the numerical solutions, and u_n^k represents the numerical solution of these difference schemes at $u(t_k, x_n)$. Table 1 is established for $N = M = 40$, 80, 160 respectively and the errors are found by

$$E_M^N = \max_{-N \le k \le N,\ 1 \le n \le M-1} |u(t_k, x_n) - u_n^k|. \tag{4.11}$$

If N and M are doubled, the values of the errors between the exact and approximate solution are decreases by a factor of approximately $1/2$ for the first order difference scheme (4.7) and $1/4$ for the second order of accuracy scheme (4.8). We presented the errors in this table and it shows the accuracy of difference shemes. The accuracy increases with the second order approximation.

TABLE 4.1. Error Analysis E_M^N

Difference schemes$/N = M$	40	80	160
(4.7)	0.1077	0.0457	0.0290
(4.8)	0.0081	0.0020	$5.0462e-04$

Applying this method, we can obtain approximate solutions of several problems for one and two dimensional the telegraph type involutory partial differential equations with dependent coefficients.

4.3 Involutory Telegraph type differential equation with Neumann boundary condition

We consider the initial-boundary-value problem

$$
\begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - u_{xx}(t,x) - p u_{xx}(-t,x) \\
= (\cos(t) - p\sin(t))\cos(x), \\
x \in (0,\pi), \ -\pi < t < \pi, \\
u(0,x) = 0, \ u_t(0,x) = \cos(x), \ x \in [0,\pi], \\
u_x(t,0) = u_x(t,\pi) = 0, \ t \in [-\pi,\pi]
\end{cases}
\tag{4.12}
$$

for the one dimensional telegraph type involutory partial differential equation with Neumann condition. The exact solution problem (4.6)is $u(t,x) = \sin(t)\cos(x)$, $0 \le x \le \pi$, $-\pi \le t \le \pi$. Applying formulas (4.1), (4.2), (4.3), (4.4) and (4.5), we present the following first order of accuracy difference scheme

$$
\begin{cases}
\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{u_n^{k+1} - u_n^k}{\tau} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} \\
-b\frac{u_{n+1}^{-k-1} - 2u_n^{-k-1} + u_{n-1}^{-k-1}}{h^2} = (\cos(t_{k+1}) - p\sin(t_{k+1}))\cos(x_n), \\
-N+1 \le k \le N-1, \ 1 \le n \le M-1, \\
u_n^0 = 0, \ \frac{u_n^1 - u_n^0}{\tau} = \cos(x_n), \ 0 \le n \le M, \\
u_1^k = u_0^k = 0, \ u_M^k = u_{M-1}^k = 0, \ -N \le k \le N
\end{cases}
\tag{4.13}
$$

and second of accuracy in t difference scheme

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\frac{u_n^{k+1}-u_n^k}{2\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} \\ -\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2}-\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2}-b\frac{u_{n+1}^{-k}-2u_n^{-k}+u_{n-1}^{-k}}{2h^2} \\ -p\frac{u_{n+1}^{-k+1}-2u_n^{-k+1}+u_{n-1}^{-k+1}}{4h^2}-p\frac{u_{n+1}^{-k-1}-2u_n^{-k-1}+u_{n-1}^{-k-1}}{4h^2} \\ =(\cos(t_k)-p\sin(t_k))\cos(x_n), \\ -N+1\le k\le N-1,\ 1\le n\le M-1, \\ u_n^0=0, \frac{-u_n^2+4u_n^1-3u_n^0}{2\tau}=\cos(x_n),\ 0\le n\le M, \\ -u_2^k+4u_1^k-3u_0^k=0,\ -3u_M^k+4u_{M-1}^k-u_{M-2}^k=0, \\ -N\le k\le N. \end{cases} \tag{4.14}$$

They are systems of algebraic equations and they can be written in the matrix form

$$Au_{n-1}+Bu_n+Cu_{n+1}=D\varphi_n,\ 1\le n\le M-1,\ u_0=u_1,\ u_M=u_{M-1} \tag{4.15}$$

for difference scheme (4.13) and

$$Au_{n-1}+Bu_n+Cu_{n+1}=D\varphi_n,\ 1\le n\le M-1, \tag{4.16}$$

$$3u_0=4u_1-u_2,\ 3u_M=4u_{M-1}-u_{M-2}$$

for difference scheme (4.14). For the solutions of (4.15), we will apply modified Gauss elimination method by the following form

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1},\ n = M-1,\ ...,\ 1,$$

where $u_M = (I - \alpha_M)^{-1}\beta_M$, α_j $(j = 1, ..., M-1)$ are $(2N+1)\times(2N+1)$ square matrices, β_j $(j = 1, ..., M-1)$ are $(2N+1)\times 1$ column matrices, $\alpha_1 = I$, $\beta_1$is zero matrices and

$$\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n + C\beta_n),\ n = 1,\ ...,\ M-1. \end{cases}$$

For the solutions of (4.16), we will apply same modified Gauss elimination method by formula

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1},\ n = M-1,\ ...,\ 1,$$

$$u_M = ((B + 4A)\alpha_M + C - 3A)^{-1}\{-(B + 4A)\beta_M + D\varphi_{M-1}\},$$

where α_j $(j = 1, ..., M-1)$ are $(2N+1)\times(2N+1)$ square matrices, β_j $(j = 1, ..., M-1)$ are $(2N+1)\times 1$ column matrices defined by formula

, $\alpha_1 = -(A - 3C)^{-1}(B + 4C)$, $\beta_1 = (A - 3C)^{-1}D\varphi_1$ and

$$\begin{cases} \alpha_{n+1} = -(B + C\alpha_n)^{-1}A,\ \alpha_1 = -(A - 3C)^{-1}(B + 4C), \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n + C\beta_n),\ \beta_1 = (A - 3C)^{-1}D\varphi_1 \\ n = 1,\ ...,\ M-1. \end{cases}$$

NUMERICAL ANALYSIS

As we consider before the numerical solutions are recorded for different values of N and

M, and u_n^k represents the numerical solution of this difference scheme at $u(t_k, x_n)$. Table 2 is constructed for $N = M = 40$, 80, 160 respectively and the errors are computed by formula (4.11). If N and M are doubled, the values of the errors are decreases by a factor of approximately $1/2$ for the first order difference scheme (4.13) and $1/4$ for the second order of accuracy scheme (4.14). The errors presented in this table indicates the accuracy of difference scheme. We conclude that, the accuracy increases with the second order approximation.

TABLE 4.2. Error Analysis E_M^N

Difference schemes/$N = M$	40	80	160
(4.13)	0.1013	0.0496	0.0300
(4.14)	0.0080	0.0020	$5.0452e-04$

CHAPTER 5

CONCLUSION

This thesis is devoted to initial boundary value problem for telegraph type involutory partial differential equations. The following results are established:

The history of telegraph type involutory differential equations is studied.

Following original results are obtained: Fourier series, Laplace transform and Fourier transform are applied for the solution of several telegraph type involutory partial differential equations. The main theorem on stability estimates telegraph differential equations is proved.

The first and second order of accuracy difference schemes for the approximate solution of the one dimensional telegraph partial differential equations with Dirichlet and Neuman conditions are given.The Matlab implementation of these difference schemes are presented.

As noted these methods can be used for multidimensional telegraph type involutory partial differential equations. These formulas for the solutions are important for the solving applied problems involving involution term. Finally, stability of initial boundary value problem for telegraph type involutory partial differential can be investigated. Stable difference schemes a higher order of accuracy for the approximate solutions of these differential problems can be presented and stability can be studied.

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APPENDIX A

APPENDIX

A.1 Matlab Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (3.1)

function drihlet1st(N,M);

if nargin ¡ 1; end;

close; close;

tau=pi/N;

h=pi/M;

p=1;a=-1/h^2;$b = 1/tau^2; c = (-2/tau^2) - (1/tau); d = -p/h^2;$

$e = (1/tau^2) + (1/tau) + (2/h^2); f = (2 * p/h^2);$

$A = zeros(2 * N + 1, 2 * N + 1);$

$A(N + 1, N + 1) = a + d;$

$fork = 3 : N;$

$A(k, k) = a; A(k, 2 * N + 2 - k) = d;$

$end;$

$fork = N + 2 : 2 * N + 1;$

$A(k, k) = a; A(k, 2 * N + 2 - k) = d;$

$end;$

$C = A;$

$B = zeros(2 * N + 1, 2 * N + 1);$

$B(1, N + 1) = 1;$

$B(2, N + 1) = -1;$

$B(2, N + 2) = 1;$

$fork = 3 : N;$

$B(k, k - 2) = b; B(k, k - 1) = c; B(k, k) = e; B(k, 2 * N + 2 - k) = f;$

$end;$

$B(N + 1, N - 1) = b; B(N + 1, N) = c; B(N + 1, N + 1) = e + f;$

$B(N+2,N)=b+f; B(N+2,N+1)=c; B(N+2,N+2)=e;$

$fork=N+3:2*N+1;$

$B(k,2*N+2-k)=f;$

$B(k,k-2)=b;$

$B(k,k-1)=c;$

$B(k,k)=e;$

$end;$

$D=eye(2*N+1,2*N+1);$

$forj=2:M;$

$fii(1,j)=0;$

$fii(2,j)=tau*sin((j-1)*h);$

$fork=3:2*N+1;$

$fii(k,j)=(cos((k-1-N)*tau)-p*sin((k-1-N)*tau))*sin((j-1)*h);$

$end;$

$end;$

$alpha1=zeros(2*N+1,2*N+1);$

$betha1=zeros(2*N+1,1);$

$forj=2:M;$

$Q=inv(B+C*alphaj-1);$

$alphaj=-Q*A;$

$bethaj=Q*(D*(fii(:,j))-C*bethaj-1);$

$end;$

$U=zeros(2*N+1,M+1);$

$forj=M:-1:1;$

$U(:,j)=alphaj*U(:,j+1)+bethaj;$

end

$'EXACTSOLUTIONOFTHISPROBLEM';$

$forj=1:M+1;$

$fork=1:2*N+1;$

$es(k,j)=sin((k-1-N)*tau)*sin((j-1)*h);$

$end;$

$end;$

$maxes = max(max(abs(es)));$

$maxerror = max(max(abs(es - U)));$

$relativeerror = maxerror/maxes;$

$cevap1 = [N, M, maxerror, relativeerror]$

A.2 Matlab Implementation of the second Order of Accuracy Difference Scheme of Problem (3.1)

function drihlet1st(N,M);

if nargin ¡ 1; end;

close;close;

tau=pi/N;

h=pi/M;

p=1;

a=-1/(2*h^2);

$b = -1/(4 * h^2);$

$c = -p/(2 * h^2);$

$d = -p/(4 * h^2);$

$e = (-2/tau^2) + (1/(h^2));$

$f = (1/(tau^2)) + 1/(2 * tau) + (1/(2 * h^2));$

$g = p/h^2;$

$q = (1/(tau^2)) - (1/(2 * tau)) + (1/(2 * h^2));$

$t = p/(2 * h^2);$

$A = zeros(2 * N + 1, 2 * N + 1);$

for k=3:N;

A(k,k-2)=b;

A(k,k-1)=a;

A(k,k)=b;

```
A(k,2*N+2-k)=d;
A(k,2*N+3-k)=c;
A(k,2*N+4-k)=d;
end;
A(N+1,N-1)=b;A(N+1,N)=a;A(N+1,N+1)=b+d;A(N+1,N+2)=c;A(N+1,N+3)=d;
A(N+2,N)=b+d; A(N+2,N+1)=a+c; A(N+2,N+2)=b+d;
A(N+3,N-1)=d; A(N+3,N)=c; A(N+3,N+1)=b+d;
A(N+3,N+2)=a;A(N+3,N+3)=b;
for k=N+4:2*N+1;
A(k,k-2)=b;
A(k,k-1)=a;
A(k,k)=b;
A(k,2*N+2-k)=d;
A(k,2*N+3-k)=c;
A(k,2*N+4-k)=d;
end;
A;
C=A;
B=zeros(2*N+1,2*N+1);
B(1,N)=1;
B(2,N)=-3;
B(2,N+1)=4;
B(2,N+2)=-1;

for k=3:N;
B(k,k-2)=q;
B(k,k-1)=e;
B(k,k)=f;
B(k,2*N+2-k)=t;
B(k,2*N+3-k)=g;
```

```
B(k,2*N+4-k)=t;
end;
B(N+1,N-1)=q;
B(N+1,N)=e;B(N+1,N+1)=f+t;B(N+1,N+2)=g;B(N+1,N+3)=t;
B(N+2,N)=q+t; B(N+2,N+1)=e+g; B(N+2,N+2)=f+t;
B(N+3,N-1)=t;B(N+3,N)=g;
B(N+3,N+1)=q+t;B(N+3,N+2)=e;B(N+3,N+3)=f;
for k=N+4:2*N+1;
B(k,k-2)=q;
B(k,k-1)=e;
B(k,k)=f;
B(k,2*N+2-k)=t;
B(k,2*N+3-k)=g;
B(k,2*N+4-k)=t;
end;
B;
D=eye(2*N+1,2*N+1);
for j=2:M;
fii(1,j)=0;
fii(2,j)=2*tau*sin((j-1)*h);
for k=3:2*N+1;
fii(k,j)=(cos((k-2-N)*tau)-p*sin((k-2-N)*tau))*sin((j-1)*h);
end;
end;
alpha1=zeros(2*N+1,2*N+1);
betha1=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alphaj-1);
alphaj=-Q*A;
bethaj=Q*(D*(fii(:,j))-C*bethaj-1); end;
```

```
U=zeros(2*N+1,1);
U(:,M+1)=zeros(2*N+1,1);
for j=M:-1:1;
U(:,j)=alphaj*U(:,j+1)+bethaj;
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
es(k,j)=sin((k-1-N)*tau)*sin((j-1)*h);
end;
end;
maxes=max(max(abs(es)));
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap1=[N,M, maxerror,relativeerror]
```

A.3 Matlab Implementation of the first Order of Accuracy Difference Scheme of Problem (3.2)

```
function neuman1st(N,M)
if nargin ¡ 1;
end;
close;close;
tau=pi/N;
h=pi/M;
```

p=1;a=-1/h^2; $b = 1/tau^2; c = (-2/tau^2) - (1/tau); d = -p/h^2;$

$e = (1/tau^2) + (1/tau) + (2/h^2); f = (2 * p/h^2);$

$A = zeros(2 * N + 1, 2 * N + 1);$

$A(N + 1, N + 1) = a + d;$

$fork = 3 : N;$

$A(k, k) = a;$

$A(k, 2 * N + 2 - k) = d;$

$end;$

$for k = N + 2 : 2 * N + 1;$

$A(k, k) = a;$

$A(k, 2 * N + 2 - k) = d;$

$end;$

$C = A;$

$B = zeros(2 * N + 1, 2 * N + 1);$

$B(1, N + 1) = 1;$

$B(2, N + 1) = -1;$

$B(2, N + 2) = 1;$

$for k = 3 : N;$

$B(k, k - 2) = b; B(k, k - 1) = c; B(k, k) = e; B(k, 2 * N + 2 - k) = f;$

$end;$

$B(N + 1, N - 1) = b; B(N + 1, N) = c; B(N + 1, N + 1) = e + f;$

$B(N + 2, N) = b + f; B(N + 2, N + 1) = c; B(N + 2, N + 2) = e;$

$for k = N + 3 : 2 * N + 1;$

$B(k, 2 * N + 2 - k) = f;$

$B(k, k - 2) = b;$

$B(k, k - 1) = c;$

$B(k, k) = e;$

$end;$

$D = eye(2 * N + 1, 2 * N + 1);$

$for j = 2 : M;$

$fii(1, j) = 0;$

$fii(2, j) = tau * cos((j - 1) * h);$

$for k = 3 : 2 * N + 1;$

$fii(k, j) = (cos((k - 1 - N) * tau) - p * sin((k - 1 - N) * tau)) * cos((j - 1) * h);$

$end;$

$end;$

$alpha1 = eye(2 * N + 1, 2 * N + 1);$

$betha1 = zeros(2 * N + 1, 1);$

$forj = 2 : M;$

$Q = inv(B + C * alphaj - 1);$

$alphaj = -Q * A;$

$bethaj = Q * (D * (fii(:, j)) - C * bethaj - 1);$

$end;$

$U = zeros(2 * N + 1, 1);$

$U(:, M + 1) = inv(D - alphaM) * bethaM;$

$forj = M : -1 : 1;$

$U(:, j) = alphaj * U(:, j + 1) + bethaj;$

end

$'EXACTSOLUTIONOFTHISPROBLEM';$

$forj = 1 : M + 1;$

$fork = 1 : 2 * N + 1;$

$es(k, j) = sin((k - 1 - N) * tau) * cos((j - 1) * h);$

$end;$

$end;$

$maxes = max(max(abs(es)));$

$maxerror = max(max(abs(es - U)));$

$relativeerror = maxerror/maxes;$

$cevap1 = [N, M, maxerror, relativeerror]$

end

A.4 Matlab Implementation of the second Order of Accuracy Difference Scheme of Problem (3.2)

function neuman2ndog(N,M)

if nargin ¡ 1; end;

close;close;

tau=pi/N;

h=pi/M;

p=1;

a=-1/(2*h^2);

$b = -1/(4 * h^2);$

$c = -p/(2 * h^2);$

$d = -p/(4 * h^2);$

$e = (-2/tau^2) + (1/(h^2));$

$f = (1/(tau^2)) + 1/(2 * tau) + (1/(2 * h^2));$

$g = p/h^2;$

$q = (1/(tau^2)) - (1/(2 * tau)) + (1/(2 * h^2));$

$t = p/(2 * h^2);$

$A = zeros(2 * N + 1, 2 * N + 1);$

$fork = 3 : N;$

$A(k, k-2) = b;$

$A(k, k-1) = a;$

$A(k, k) = b;$

$A(k, 2 * N + 2 - k) = d;$

$A(k, 2 * N + 3 - k) = c;$

$A(k, 2 * N + 4 - k) = d;$

$end;$

$A(N+1, N-1) = b;$

$A(N+1, N) = a;$

$A(N+1, N+1) = b + d;$

$A(N+1, N+2) = c;$

$A(N+1, N+3) = d;$

$A(N+2, N) = b + d;$

$A(N+2, N+1) = a + c;$

$A(N+2, N+2) = b + d;$

$A(N+3, N-1) = d;$

$A(N+3, N) = c;$

$A(N+3, N+1) = b + d;$

$A(N+3,N+2)=a;$

$A(N+3,N+3)=b;$

$fork=N+4:2*N+1;$

$A(k,k-2)=b;$

$A(k,k-1)=a;$

$A(k,k)=b;$

$A(k,2*N+2-k)=d;$

$A(k,2*N+3-k)=c;$

$A(k,2*N+4-k)=d;$

$end;$

$C=A;$

$B=zeros(2*N+1,2*N+1);$

$B(1,N)=1;$

$B(2,N)=-3;$

$B(2,N+1)=4;$

$B(2,N+2)=-1;$

```
for k=3:N;
B(k,k-2)=q;
B(k,k-1)=e;
B(k,k)=f;
B(k,2*N+2-k)=t;
B(k,2*N+3-k)=g;
B(k,2*N+4-k)=t;
end;
B(N+1,N-1)=q;
B(N+1,N)=e;
B(N+1,N+1)=f+t;
B(N+1,N+2)=g;
B(N+1,N+3)=t;
```

```
B(N+2,N)=q+t;
B(N+2,N+1)=e+g;
B(N+2,N+2)=f+t;
B(N+3,N-1)=t;
B(N+3,N)=g;
B(N+3,N+1)=q+t;
B(N+3,N+2)=e;
B(N+3,N+3)=f;
for k=N+4:2*N+1;
B(k,k-2)=q;
B(k,k-1)=e;
B(k,k)=f;
B(k,2*N+2-k)=t;
B(k,2*N+3-k)=g;
B(k,2*N+4-k)=t;
end;
D=eye(2*N+1,2*N+1);
for j=2:M;
fii(1,j)=0;
fii(2,j)=2*tau*cos((j-1)*h);
for k=3:2*N+1;
fii(k,j)=-p*sin((k-2-N)*tau)*cos((j-1)*h);
end;
end;
alpha1=zeros(2*N+1,2*N+1);—
betha1=zeros(2*N+1,1);
for j=2:M;
Q=inv(B+C*alphaj-1);
alphaj=-Q*A;
bethaj=Q*(D*(fii(:,j))-C*bethaj-1);
```

```
end;
U=zeros(2*N+1,1);
U(:,M+1)=inv(D-alphaM)*bethaj;
for j=M:-1:1;
U(:,j)=alphaj*U(:,j+1)+bethaj;
end
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:2*N+1;
es(k,j)=sin((k-2-N)*tau)*cos((j-1)*h);
end;
end;
maxes=max(max(abs(es)));
maxerror=max(max(abs(es-U)));
relativeerror=maxerror/maxes;
cevap1=[N,M, maxerror,relativeerror]
end
```

OGULBABEK BATYROVA

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

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Ogulbabek Batyrova	Conclusion	%0 %0	0%	0%	0%	--	--	ödev indir	2007560953	06-Şub-2023
Ogulbabek Batyrova	Chapter 3	%5 %5	5%	0%	0%	--	--	ödev indir	2007560615	06-Şub-2023
Ogulbabek Batyrova	Chapter 4	%6 %6	11%	13%	10%	--	--	ödev indir	2007560786	06-Şub-2023
Ogulbabek Batyrova	Chapter 2	%8 %8	0%	13%	8%	--	--	ödev indir	2007560367	06-Şub-2023
Ogulbabek Batyrova	Chapter 1	%10 %10	5%	5%	0%	--	--	ödev indir	2007560162	06-Şub-2023
Ogulbabek Batyrova	All Thesis	%11 %11	0%	0%	11%	--	--	ödev indir	2007559965	06-Şub-2023