MOHAMMED SUBHI HADI

Fractional Modeling and Analysis on Optimal

PhD THESIS

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NEAR EAST UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

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Nicosia
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Approval

We certify that we have read the thesis submitted by Mohammed Subhi Hadi titled **"Fractional Modeling and Analysis on Optimal Control Policies"** and that in our combined opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctoral of Science.

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Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

Mohammed Subhi HADI

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To my parents…

Abstract

Fractional Modeling and Analysis on Optimal Control Policies

HADI, Mohammed Subhi Hadi
PhD, Department of Electrical and Electronic Engineering

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In this thesis, fractional-order modeling was implemented to create a COVID-19 epidemic model which takes awareness and vaccination into consideration. The model was formulated based on Caputo – Fabrizio's form of fractional differential equations. The main goal of the thesis is to investigate the effect of applying different control policies using optimal control on the created model. The epidemiological model used in this study consisted of five classes where awareness about the epidemic and vaccination against the epidemic has been introduced; susceptible unaware, susceptible aware, susceptible vaccinated, infected, and recovered. Equilibrium points of the model were established and the basic reproduction rate was calculated afterward. The existence and uniqueness property of the fractional model was determined.

The fractional optimal control problem was formulated and investigated based on the fractional model of COVID-19. The control policies used to control the COVID-19 model were three time-dependent functions; awareness campaign for the unaware population, vaccination for the aware population, and optimal vaccination. Using numerical simulation, the effects of a different combination of these policies were performed. Results showed drastic decay in the infected population when a combination of all three control polices was applied.

The study shows the importance of implementing successive policies on controlling the spread of epidemic, especially public awareness and vaccination.

Keywords: Optimal control; mathematical model; fractional order model; fractional optimal control; COVID-19

Özet

Fractional Modeling and Analysis on Optimal Control Policies

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PhD, Elektrik-Elektronik Mühendisliği Bölümü
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Bu tezde, farkındalık ve aşılamayı dikkate alan bir COVID-19 salgın modeli oluşturmak için kesirli sıralı modelleme uygulanmıştır. Model, kesirli diferansiyel denklemlerin Caputo – Fabrizio formuna dayalı olarak formüle edilmiştir. Tezin temel amacı, optimal kontrol kullanılarak farklı kontrol politikalarının uygulanmasının oluşturulan model üzerindeki etkisini araştırmaktır. Bu çalışmada kullanılan epidemiyolojik model, salgın konusunda farkındalığın ve salgına karşı aşılamanın tanıtıldığı beş sınıftan oluşmaktadır; duyarlı farkında değil, duyarlı farkında, duyarlı aşılanmış, enfekte olmuş ve iyileşmiş. Modelin denge noktaları belirlenmiş ve daha sonra temel yeniden üretim hızı hesaplanmıştır. Kesirli modelin varlık ve teklik özelliği belirlendi.

Kesirli optimal kontrol problemi, COVID-19'un kesirli modeline dayalı olarak formüle edildi ve analiz edildi. COVID-19 modelini kontrol etmek için kullanılan kontrol politikaları, zamana bağlı üç işlevdi; bilinçsiz nüfus için bilinçlendirme kampanyası, bilinçli nüfus için aşılama, optimal aşılama. Sayısal simülasyon kullanılarak, bu politikaların farklı kombinasyonlarının etkileri gerçekleştirilmiştir. Sonuçlar, üç kontrol politikasının tümünün kombinasyonu uygulandığında, enfekte olmuş popülasyonda ciddi bir bozulma olduğunu gösterdi.

Çalışma, başta halkın bilinçlendirilmesi ve aşılama olmak üzere, salgının yayılmasını kontrol altına almak için birbirini takip eden politikaların uygulanmasının önemini göstermektedir.

Anahtar kelimeler: Optimal kontrol; matematiksel model; kesirli sıra modeli; kesirli optimal kontrol; COVID-19

Table of Contents

CHAPTER 3

CHAPTER 4

CHAPTER 5

List of Tables

List of Figures

List of Abbreviations

BVP:	Boundary Value Problem
COVID-19	Coronavirus Disease of 2019
FDE:	Fractional Differential Equation
FO:	Fractional Order
FOC:	Fractional Optimal Control
FOCP:	Fractional Optimal Control Problem
FOMPC:	Fractional Order Model Predictive Controller
FOPID:	Fractional Order Proportional Integral Differential
FOSMC:	Fractional Order Sliding Mode Controller
FOTF:	Fractional Order Transfer Function
IO:	Integer Order
IVP:	Initial Value Problem
NLP:	Non-linear Optimization Problem
OC:	Optimal Control
OCP:	Optimal Control Problem
ODE:	Ordinary Differential Equation
PMP:	Pontryangin’s Maximum Principle
RK-4:	Runge-Kutta Fourth Order

CHAPTER 1

Introduction

1.1 Introduction

A mathematical model is a way of describing a system through the use of mathematical equations and notations, with the goal of making it simpler to analyze the effects of different modules of a system or to provide a proper explanation of a system (Abramowitz and Stegun, 1968). Mathematical modeling describes the procedure for creating such a model (Press et al., 1987). Engineering, biological sciences, health, economics, and the social sciences are only a few of the fields where mathematical modeling has become increasingly significant in the research and design of solutions to modern challenges. Mathematical modeling has been a mainstay in public health field research for the past several decades, and it has been used as a crucial tool in the field of studying and controlling of infectious diseases.

Since December 2019, COVID-19 infection started and spread worldwide. Still, there is continued transmission in many countries in the world. Epidemiologists' experience in dealing with Ebola, TB, cholera, HIV, etc. pandemics extremely aided the government in conferring measures that include isolating the infected patients, border closures, lockdown, and disinfecting of contaminated surfaces on a regular basis, consecutively, to face the menace caused by the disease. Many state-of-the-art technologies were used to accelerate the mitigation process, these include; next-generation gene sequencing for pathogen identification, artificial intelligence-based algorithms for the classification of infected cases, mathematical model-based analysis for characterization of the spread dynamics of the disease, and big-data methods to trail the mobility of the population (He et al., 2020; Jiang et al., 2021; Mohamadou et al., 2020; Tang et al., 2020; Vaishya et al., 2020). Particularly, mathematical models are used in characterizing stages of disease spread in a specified populace and also help to optimize disbursement related to involvement and management of hospital facilities. On this matter, this thesis examined the applications of successive optimal control policies on the COVID-19 epidemic to create disease control strategies. To aid public health authorities, this research can be used to suggest initiatives to stem the spread of an epidemic.

1.2 Background of Study

By December 31, 2019, World Health Organization (WHO) was informed by Chine's health authorities in Wuhan City of the occurrence of several cases of unknown pneumonia etiology. By January 7, 2019-nCoV as originally abbreviated by WHO was screened by a sample of throat swabs from certain patients (RT, 2020). After that, some study groups renamed severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), and afterward WHO renamed the disease coronavirus disease 2019 (COVID-19). By January 30, statistics showed that 7736 and 12167 confirmed and supposed cases had been respectively counted in China. Similarly, 18 other countries reported confirmed cases reached 82. WHO on the same day gave the status of Public Health Emergency of International Concern (PHEIC) to the SARS-CoV-2 epidemic (Transmission, 2020).

Considering the epidemiological aspects that cause to the pandemic, evidence from Li et al. report where the SEIR model was used for analysis shows that about 86% of the infections that occurred before Jan 23, 2020, in China, were not documented (Li et al., 2020). Although executing random and rapid testing policies reduced the percentage of unreported cases but still, a substantial proportion of the cases have gone undetected. Hence it is concluded that the core cause of the recurrence of the disease when restrictions are lifted is the hidden asymptomatic patients (Fang et al., 2020; Kassa et al., 2020). To substantiate the occurrence of both asymptomatic and pre-symptomatic disease transmission, the Cluster busting approach has been used (Fang et al., 2020). Fang et al. also show that super is responsible for almost 80% of the spread of the disease (Fang et al., 2020). To facilitate the effective implementation of containment strategies it is important to recognize hidden asymptomatic transmission nodes or unreported infections. Values of R_0 and R_e show that involvement responses such as closures of borders, lockdowns, social distances, and verdicts to lift or reduce restrictions have a major bearing on how the epidemic develops (University JH).

A comprehensive understanding of the dynamics of the disease spread between populations is vital for predicting infectious diseases and their future characteristics and planning for reliable programs to implement control interventions for preventing disease breakout (Lin et al., 2020; Yang et al., 2020). The theory of mathematical modeling and optimal control play a central role in studying the dynamics of infectious diseases and their management (Lin et al., 2020; Sofia et al., 2015). Yet, because of the parameter uncertainties, complexity and inherent non-linearity related to epidemical models modeling and control analysis of such systems is a tough and complex process. Moreover to the financial constraints of minimizing the cost of achieving the control goal and maximizing the efficiency with which they are achieved, this also has to be taken into account (Djouima et al., 2017; Gambhire et al., 2020).

Optimal control theory is another useful tool for studying mathematical models and learning more about the dynamics of a disease, especially contagious ones. The theory was first introduced in 1986, after the development of the well-known Pontryagin Maximum Principle (PMP) (Pontryagin and Boltyanskii, 1986). These days, optimal control theory is practically required for all models of infectious diseases (Okyere et al.; Mojaver and Kheiri, 2016; Karrakchou et al., 2006; Adams et al., 2004; Gul et al., 2008; Mukandavire et al., 2009b; Makinde and Okosun, 2011; Yusuf and Benyah, 2012; Mwanga et al., 2014; Choi et al., 2015; Rihan et al., 2014).

Because the majority of physical phenomena, such as biological systems, have an after-effect or a persistent memory property, it is possible that fractional differential equations could more accurately describe these phenomena. This is because fractional differential equations also have an after-effect memory built into their structure. Thus, fractional model is considered in this thesis.

Fractional optimal control problem (FOCP) is the general form of the classic optimal control problem (OCP) and arise when optimal control theory is applied to fractional order models (models formed using fractional calculus). In FOCP, integer order differential equation is replaced by fractional differential equation (FDE), and the performance index is expressed as a fractional integration operator (Ali et al., 2016). Various articles were published concerning the theory of FOCPs and their formulations using analytical and numerical methods (Agrawal, 2004; Agrawal and Baleanu, 2007; Agrawal, 2008; Jelicic and Petrovacki, 2009; Agrawal et al., 2010; Odzijewicz et al., 2012; Kamocki, 2014; Chinnathambi et al., 2019; Al-Mdallal and

Abu Omer, 2018; Al-Mdallal and Hajji, 2015; Hajji and Al-Mdallal, 2018). The FOCPs are often used in modern infectious disease models for their nature of memory dependence due to the fractional-order model, which allows for faster and more accurate disease control. As a result, we can conclude that the FOCPs has all the makings of the most suitable instrument that can be used to model infectious diseases and other biologically linked systems that possess memory in their very nature.

1.3 Research Problem and Statement

Discrimination, anxiety, and poverty are only some of the negative outcomes of COVID-19. Collapsed public health programs, inadequate funding in underdeveloped nations, widespread public ignorance, and exogenous re-infection, where new strains are constantly evolving, have all been blamed for the uptick in reported cases.

Significant strides have been completed in creating a framework for theory involving the dynamics and control strategies of COVID - 19, but many difficult and detailed open questions remained unanswered and uninvestigated. For example, integrating public awareness on COVID – 19 model for controlling its spread is rare. Many researchers have suggested that COVID – 19 occurrences could be evaded by considering awareness and isolation in the control method. Optimal control applications also require more research that compares and contrasts various control strategies to demonstrate the relative impact of each.

There has been significant development of a theoretical framework for analyzing the dynamics and control approaches of COVID - 19, but many difficult and crucial open questions remain. For instance, public awareness models for preventing COVID - 19 are uncommon. Isolation and awareness have been suggested as potential prevention strategies for COVID - 19 by numerous researchers. More research is required on optimal control applications to deliver cost-effectiveness analysis and comparison control techniques for demonstrating the relative importance of one control method over another.

This study will consider the implementation of fractional optimal control problem theory to analyze the COVID-19 epidemic by incorporating awareness and vaccination into the mathematical model of the disease.

1.4 Research Aim and Objectives

This thesis aims to study the use of successive optimal control policies applied to the fractional order model. COVID – 19 epidemics will be modeled using fractional order differential equations and subsequently analyzed and optimal control theory will be applied to it using different control policies. To demonstrate the performance of the applied policies numerical simulations will be carried on.

The following objectives of research would finally accomplish the main aim of this thesis:

- To explore the nature of the COVID – 19 epidemics
- To formulate a novel fractional order model of the COVID – 19 epidemics incorporating awareness.
- To find the existence and uniqueness of the Solutions for the fractional order model.
- To formulate the successive optimal control problem for the fractional order model.
- Investigate the effects of applying different control policies using optimal control for different susceptible epidemiological model classes.

1.5 Scope of the Study

The core scope of this study is on carrying out an examination of the dynamics of COVID – 19 epidemics with fractional optimal control policies. The research considered a five-compartmental COVID – 19 model consisting of (susceptible unaware - susceptible aware - susceptible vaccinated - infected - recovered). With this model the population has been divided into five classes namely; susceptible unaware (healthy people but can contract the disease who are unaware of the disease), susceptible aware (healthy people but can contract the disease who are aware of the disease), susceptible vaccinated (healthy people but can contract the disease who are vaccinated against the disease), infected (afflicted people who have reached the latter stages of the disease), recovered (recovered individuals from the disease). Three time-dependent control policies were applied to the fractional-order model; a susceptible class of unaware population with awareness control is detected, a susceptible class of aware population with vaccine control is found, and a susceptible class with a vaccinated population with optimal vaccination control is found. MATLAB/SIMULINK 2019b will be used to perform all the numerical simulations.

1.6 Thesis Organization

This thesis was arranged into five chapters. All chapters have been organized as follows:

Chapter 1: This chapter contains the introduction and background of the study. Also, discusses the problem statement, aim and objective finally, the scope of the study.

Chapter 2: This chapter provides a literature review on the basic theories of optimal control and fractional calculus. Also, reviews other related studies.

Chapter 3: This chapter demonstrates the development of a new model of COVID – 19 dynamics, and analysis of the proposed model.

Chapter 4: This chapter studies optimal control development and its analysis as well as numerical simulations of the solutions of the fractional-order model.

Chapter 5: This chapter presents the summary and the conclusion of the thesis.

CHAPTER 2

Literature Review

A method used in trading off among stability, fuel or time optimality, transient and steady state performance, etc., is termed as control. When there are mutual constrains, then the way of finding a balance among various performance indices under some factors is termed as Control. It is one of the most essential fields in modern technology. The research space about Control is still advancing greatly over the period of time. The area has been receiving serious attention in many research fields. For about 2000 years ago, Arabs, Greeks, and Ancient Rome carried out research based on the principle of feedback. This leads to many significant projects of control systems, which were built. They include shower systems in the imperial palace, the float valve level regulator for water clocks, and the automatic gates in the temples (Lewis, 1990). Zhang Heng, a Chinese polymath use the principle of a "suspended pendulum" to invent the seismograph in 132 A.D. (Han Dynasty), which is one of the most profound applications of control. The official adoption of the automatic control system in the modern sense was first about the speed regulator of the steam engine that was invented by James in the year 1788 (Nof, 2009).

Going by history, the area of control has a significant contribution in almost all aspects of our day-to-day activities, including healthcare, military services, Agriculture, industries, etc. The application of controls can be as simple or difficult as launching a rocket, depending on the situation.

2.1 Optimal Control

Optimal Control which is usually abbreviated as OC can be defined as a method or policy for obtaining an optimal outcome in a given system. OC can also be termed as the way in which some of the parameters in a model are controlled in order to obtain optimized output through obtaining control and the state of trajectory in a dynamic system within a given time frame after minimizing a performance index (Bryson Jr., 1996).

The history of Optimal Control is traced back to the augmentation of the calculus of variations. This started in the 17th century when the first result for the calculus of variations was publicized. This is due to the challenge made by Bernoulli to the entire world of renown mathematicians, which says that “in case of a very small body which is traveling under the effect of gravity, what will happen? Which of the sides between the two fixed sides of the body will allow it to travel in the smallest possible time?’’

Lagrange and Euler found and formulated solutions to some specific problems in the calculus of variations. This idea of the calculus of variations was applied to the area of Hamiltonian’s Principle or the Least Action Principle in the field of Theoretical physics. In the late 1920s and early 1930s, the idea was extended to applications to economics, and thereafter, more and more applications were published occasionally by people like Hotelling, Evans, Ross, and Ramsey, (Sussmann and Willems, 1997).

Since the year 1950 Optimal control theory emerged from the calculus of variations as a generalization form of it. This was inspired by its tremendous applications in the military. Russian mathematician by the name Lev S. Pontryagin and his team came up with the first and most notable result between the years 1908 to 1988 by coming with the famous Pontryagin Maximum Principle (Pontryagin and Boltyanskii, 1986). This important result gave rise to the most important results needed in optimizing problems related to differential equations (Leitmann, 1997).

Bellman discovered that Dynamic Programming uses the optimal control principle and it is the most proper tool that can be used to solve discrete problems because it allows a momentous reduction in the time and complexity involved in finding the optimal controls. After this discovery, the theory provides a new technique of Dynamic Programming in mathematics (Kirk, 1998). The idea of the Hamilton-Jacobi-Bellman equation which was used sometimes to solve partial differential equations came as a result of the optimal control principle which allows the possibility of obtaining different techniques for continuous problems. This gives rise to the relationship between the OCP and Lyapunov’s stability theory.

The applications of OCP were limited to simple problems before the generation of computers, but with the arrival of the computer, it became possible to apply the OC theory to more sophisticated problems.

It is now possible to apply the idea of OC in order to get the desired result in different types of equations, be it ODE, PDE, Stochastic DE, discrete DE, integral – differential equations, the merger of discrete and continuous systems, or even fractional order differential equations (FODE).

2.1.1 Optimal Control Problem

A traditional optimal control problem must have a performance index or cost function ($J[x(t), u(t)]$), a set of state variables ($x(t) \in X$), a set of control variables ($u(t) \in U$) within a time t, while $t_0 \leq t \leq t_f$. Its main aim is to find a continuous piecewise control $u(t)$ together with the associated state variable $x(t)$ that maximize the given objective functional. Below is an example of a typical OCP in Lagrange formulation.

Definition 2.1 **(**Lagrange formulation)**:** The optimal control problem in Lagrange form is given in the form:

$$\max_u \; J[x(t), u(t)] = \int_{t_0}^{t_f} f(t, x(t), u(t))\, dt,$$

$$s.t. \qquad \dot{x}(t) = g\,(t\,, x(t)\,, u(t)), \qquad (2.1)$$

$$x(t_0) = x_0.$$

The value $x(t_f)$ is not restricted, this means, it can take any value, and it can also be fixed as $x(t_f) = x_f$.

Here, the functions f and g are continuous and differentiable, and the control set U is a Lebesgue measurable function which will make the control(s) and related states variables to be piecewise continuous.

It is always possible to swap either back or forth between either maximizing or minimizing a given function by directly reversing the cost function, hence in most of the OCP notes emphasis is been given on maximizing a function. See below:

$$min\{J\} = -max\{-J\}. \qquad (2.2)$$

2.1.2 Optimal Control Formulation

The formulation of the OCP given above is termed as Lagrange. The other two methods that can be used to formulate OCP problems are; the Bolza and Mayer methods (Chachuat, 2007).

Definition 2.2 **(**Bolza method)**:** The formulation of optimal control problem in Bolza form is given as:

$$\max_{u} J\,[\,x(t)\,,u(t)] = \emptyset\big(t_0\,,x(t_0),t_f,x(t_f)\big) + \int_{t_0}^{t_f} f(t,x(t),u(t))\,dt,$$

$$s.t. \qquad \dot{x}(t) = g\,(t\,,x(t),u(t)), \qquad (2.3)$$

$$x(t_0) = x_0.$$

$$where\ \emptyset\ is\ a\ continuous\ differentiable\ function.$$

Definition 2.3 **(**Mayer method)**:** The formulation of optimal control problem in Mayer form is given as:

$$\max_{u} J\,[x(t)\,,u(t)] = \emptyset\big(t_0,x(t_0\,,t_f\,,x(t_f)\,)\big)$$

$$s.t. \qquad \dot{x}(t) = g\,\big(t\,,x(t)\,,u(t)\big), \qquad (2.4)$$

$$x(t_0) = x_0.$$

2.1.3 Pontryagin's Maximum Principle (PMP)

Pontryagin and his team in the 20th century constructed the most useful result in OCP that is paramount in finding the optimal control. The result, which was termed the greatest achievement in the era, gives an idea on how to use an adjoint function of a DE to fix to the objective function. These Adjoint functions attached constraints to the function that will either be optimized.

Definition 2.4 **(**Hamiltonian Equation)**:** Given an optimal control problem in Lagrange form then the function:

$$H\big(t,x(t)\,,u(t)\,,\lambda\,(\mathrm{t})\big) = f\big(t\,,x(t)\,,u(t)\big) + \lambda(\mathrm{t})g\big(t\,,x(t)\,,u(t)\big), \qquad (2.5)$$

is termed as Hamiltonian function while $\lambda(\mathrm{t})$ is the adjoint variable.

Theorem 2.1 **(**Pontryagin's Maximum Principle (PMP))**:** If $u^*(t)$ and $x^*(t)$ are optimal values for a given OCP then, there must be piecewise and differentiable adjoint variable $\lambda(\mathrm{t})$ with:

$$H\big(t, x^*(t), u(t), \lambda(t)\big) \leq H\big(t, x^*(t), u^*(t), \lambda(t)\big). \qquad (2.6)$$

all control functions u at each time t.

where H is the Hamiltonian which was described previously and $\lambda'(t)$ given as follows:

$$\lambda'(t) = \frac{\partial H\big(t, x^*(t), u^*(t), \lambda(t)\big)}{\partial x}, \qquad (2.7)$$

$$\lambda\big(t_f\big) = 0.$$

(Pontryagin and Boltyanskii, 1986).

Remark: The above condition, $\lambda\big(t_f\big) = 0$ is known as the transversality condition, which is used when the OCP does not have terminal values in its state variables.

The most significant contribution of PMP is its ability to change the problem of finding a control value which optimizes the objective function in the state ODE and specified initial conditions to become a problem of optimization of Hamiltonian. Hence considering the Hamiltonian and the adjoint equation, we have:

$$\frac{\partial H}{\partial u} = 0. \qquad (2.8)$$

Hence, using the Hamiltonian alone, we can get our necessary conditions by calculating the integral in the objective function.

2.1.4 Optimal Control along Payoff Terms

In some instances, we need to optimize the terms in the entire time interval, in some other instances we only need to optimize our function in a specified time interval. Certain cases warrant that the state values must be taken into consideration by the objective function (Lenhart and Workman, 2007).

Definition 2.5 **(**OCP for payoff term)**:** An OCP alongside a payoff term can be given in the following form:

$$\max_{u} \; J\,[x(t), u(t)] = \emptyset\,\big(x\,(t_f)\big) + \int_{t_0}^{t_f} f(t\,, x(t)\,, u(t))\,dt,$$

$$s.t. \qquad \dot{x}(t) = g(t\,, x(t)\,, u(t)), \qquad (2.9)$$

$$x(t_0) = x_0.$$

$where\ \emptyset\big(x(t_f)\big)$is the desired result with respect to the level of the population $x(t_f)$ and it is known to be the payoff or salvage.

The necessary conditions of the OCP with payoff can be derived by applying PMP to be:

Proposition 2.1(The conditions needed): If $u^*(t)$ and $x^*(t)$ are the optimal values of a given OCP (like the one in definition 5) then, there must be a differentiable and piecewise adjoint variable λ(t) with:

$$H\big(t, x^*(t)\,, u(t)\,, \lambda\,(\mathrm{t})\big) \leq H\big(t, x^*(t), u^*(t)\,, \lambda(\mathrm{t})\big). \qquad (2.10)$$

when all controls u at every time t,

where H is the Hamiltonian parameter which was well-defined previously and

$$\lambda'(t) = \frac{\partial H\big(t, x^*(t), u^*(t), \lambda(\mathrm{t})\big)}{\partial x}\,(adjoint\ condition),$$

$$\frac{\partial H}{\partial u} = 0 \qquad (Optimality\ condition), \qquad (2.11)$$

$$\lambda\big(t_f\big) = \emptyset'\big(x(t_f)\big)(Transversality\ condition).$$

2.1.5 Optimal Control for Bounded Control

Definition 2.6 (OCP for bounded control). An OCP alongside bounded control can be summarized by the following:

$$\max_u J[x(t), u(t)] = \int_{t_0}^{t_f} f(t, x(t), u(t))\, dt,$$

$$s.t. \qquad \dot{x}(t) = g(t, x(t), u(t)), \qquad (2.12)$$

$$x(t_0) = x_0,$$

$$a \leq u(t) \leq b.$$

$$where\ a, b\ are\ fixed\ real\ constants\ and\ a < b.$$

For problems with bounds on their controls, there is a need for different required conditions.

Proposition 2.2 (The conditions needed). If $u^*(t)$ and $x^*(t)$ are the optimal values of a given OC problem (like the one in definition 5) then, there must be a piecewise and differentiable adjoint variable λ(t) with:

$$H\big(t, x^*(t), u(t), \lambda(\mathrm{t})\big) \leq H\big(t, x^*(t), u^*(t), \lambda(\mathrm{t})\big). \qquad (2.13)$$

when all controls u at every time t,

where H is Hamiltonian which was previously defined and

$$\lambda'(t) = \frac{\partial H\big(t, x^*(t), u^*(t), \lambda(\mathrm{t})\big)}{\partial x} (adjoint\ condition), \qquad (2.14)$$

$$\lambda(t_f) = 0 \qquad (transversality\ condition),$$

By using PMP, the optimal control must satisfy the following optimality condition:

$$u^* = \begin{cases} a, & if \quad \frac{\partial H}{\partial u} < 0, \\ a < \breve{u} < b & if \quad \frac{\partial H}{\partial u} = 0, \\ b, & if \quad \frac{\partial H}{\partial u} > 0. \end{cases} \qquad (optimality\ condition)$$

It means that the maximization may occur at all the allowed controls, and $\breve{u}$ it can be found by using the following:

$$\frac{\partial H}{\partial u} = 0 \qquad (2.15)$$

Precisely, the control u^* maximizes H pointwise optimally with regards to $a \leq u \leq b$.

(Kamien and Schwartz, 1991).

2.2 Solving Optimal Control Problems

Recently there is development in the area of computational mathematics especially related to the methods of obtaining numerical solutions of both integral and differential equations. This gave rise to methods of solving highly complex real-world problems.

Similarly, optimal control problems and the numerical methods for obtaining their solutions as well as their algorithms have been enriched significantly.

2.2.1 Numerical Solutions for Dynamical Systems

Dynamical systems are described by a set of ODEs. It is usually presented as a system of n-ODEs for $t_0 \leq t \leq t_f$, as:

$$\dot{y} = \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \\ . \\ . \\ . \\ \dot{y_n} \end{bmatrix} = \begin{bmatrix} f_1(\, y_1(t)\,, \ldots y_n(t)\,, t) \\ f_2(\, y_1(t)\,, \ldots y_n(t)\,, t) \\ . \\ . \\ . \\ f_n(y_1(t)\,, \ldots y_2(t)\,, t) \end{bmatrix} \qquad (2.16)$$

ODE problems can be subdivided into two, namely: Initial Value Problems (IVP) and Boundary Value Problems (BVP). This is to do with the specification of conditions related to the margin in the domain. While in IVP the specification of the conditions is made at the initial state, in BVP the specifications of the conditions are made at both the initial and final point.

Some of the numerical techniques used in solving IVP in literature are, the Euler method and the Runge-Kutta method, for BVP one of the techniques, is the method of shooting.

a. Euler method

This is the most popular technique that is used for the numerical solution of dynamic systems. Given a differential equation in the following form:

$\dot{x} = f(x(t), t)$, then is possible to make an appropriate approximation as this:

$$x_{n+1} \cong x_n + hf(x(t_n), t_n). \qquad (2.17)$$

Approximation of x_{n+1} as $x(t)$ for time t_{n+1} possesses an error with the order of h^2. This indicates that the accuracy of the calculation relies on the selected value of h. Generally, a decrease in the value of h leads to an increase in the accuracy of calculation but leads to elongation in time intervals.

For systems with much higher orders, the Euler approximation method is not generally effective. Hence there is a need for bit more exact and complex methods. The Runge-Kutta method is one of those techniques.

b. Runge-Kutta technique

The Runge-Kutta technique is a multi-step technique, where at any time t_{k+1} we find the solution from a set of preceding values $t_{j-k}, \dots, t_k$ where j denotes the step number. Here, if the DE is in given as $\dot{x} = f(x(t), t)$, we can make the following approximations using Runge-Kutta second order as follows:

$$x_{n+1} \cong x_n + \frac{h}{2}[f(x_n(t), t_n) + f(x_{n+1}, t_{n+1})], \qquad (2.18)$$

Alternatively, using Runge-Kutta fourth order, we have

$$x_{n+1} \cong x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \qquad (2.19)$$

Where;

$$k_1 = f(x(t), t),$$

$$k_2 = f\left(x(t) + \frac{h}{2}k_1, t + \frac{h}{2}\right),$$

$$k_3 = f\left(x(t) + \frac{h}{2}k_2, t + \frac{h}{2}\right),$$

$$k_4 = f(x(t) + hk_3, t + h).$$

The above approximation x_{n+1} of $x(t)$ at the point t_{n+1} has an error that depends on h^3 and h^5 for Runge-Kutta second-order and Runge-Kutta fourth-order techniques respectively.

2.2.2 Solution of Optimal Control Problems by Numerical Means

Bellman in the 1950s provided a means of solving OCP by numerical methods. From that period up to now, a lot of sophisticated techniques and many applications for the complexities exist in Literature (Rao, 2009).

Indirect and direct techniques are the two main methods of solving OCP. OCP is indirectly solved by converting it to a BVP, with the help of the PMP in the indirect method. On the other hand, the optimal problem’s solution is found by directly duplicating the optimization problem with an infinite dimension to a problem with a finite dimension for the direct method.

a. Indirect techniques

For an indirect technique, the optimal conditions in the first-order original OCP can be found by using PMP. This technique directs to a BVP with multiple point which can be solved to find a prospect of the optimal trajectory known as extremals.

It is necessary to have control equations notably stated as well as the transversality conditions and all the adjoint equations provided they exist in case of an indirect method. Note that the formation of the problem and the technique employed in solving the problem do not have any direct relationship between the two. Is possible to consider any method of solving OCP to solve a problem formulated directly or indirectly. Example of a numerical approach using an indirect method of solving OC problems is the Forward-Backward sweep method.

b. Direct methods

Another class of numerical technique for optimization of dynamic systems has evolved, and it was called direct methods.

This development was as a result of the demand of solving complex problems in optimization, the technique becomes popular by the help of the rapid increase in the computational world.

In this technique an array of points $x_1, x_2, \dots, x^*$ is constructed in such a way that the objective function is minimized, and typically, $F(x_1) > F(x_2) > \cdots > F(x^*)$. In this method approximation of the state variables and/or control, variables are done by a suitable function of approximation (like piecewise constant parameterization or polynomial approximation). At the same time, the approximation cost function is done by the function of approximation. Then, the problem will be formulated again in a normal nonlinear optimization problem (NLP) form by treating the coefficients of the approximating function as variables of optimization as follows:

$$\min_{x,u} F(x)$$

$$s.t. \quad C_i(x) = 0, \qquad i \in E \tag{2.20}$$

$$C_j(x) \geq 0, \qquad j \in I$$

where $C_i, i \in E$ *is* the set of equality constraint *and* $C_j, j \in I$ is the set of inequality constraint.

The NLP is much simpler to solve compared to the BVP, because of its sparsity and the availability of too many notable software programs designed to deals with its features. Hence due to this, the number of different kind of problems that may be solved by the use of direct methods is far more than those that may be solved by the use of indirect methods. Therefore the direct methods becomes more famous these days and many research also written a highly developed software programs for the usage of these methods.

2.2.3 Optimal Control Software

Here we give examples of some software programs that are developed specifically for this purpose.

a. OC-ODE

In 2009, Optimal Control of Ordinary-Differential Equations (OC-ODE) was presented (Gerdts, 2009). This combines the routines of OCP in FORTRAN 77 with ODEs. It uses an automatic direct discretization method to change OCP to NLP. It contains some procedures that can be used to analyze the estimation for numerical adjoint and sensitivity analysis.

b. DOTcvp

This is a MATLAB toolbox for dynamic optimization (Hirmajer et al., 2009). It provides a space for the FORTRAN to build its files of the ordinary differental equation, sensitivities, and Jacobian. In calculating the profiles of the optimal control, it uses a method of vector parameterization, especially when a solution to the control is provided in the sense of piecewise.

DOTcvp is modified by the SUNDIALS tool (Hindmarsh et al., 2005). This can be used to solve IVP and Jacobian automatic generation and can also be used for the gradients. Furthermore, the Adams or BDF linear method can also be used in solving the initial value problem with Newton, Functional iteration module.

c. Muscod-II

Muscod-II an acronym for Multiple Shooting CODe is one of the recent Optimal Control solvers (Kuhl et al., 2007). It is an advanced version of AMPL that can be used in solving a combination of integer nonlinear ODE and DAE constrained OCP.

Fourier, Gay, and Kernighan introduced AMPL in 2002 (Fourer et al., 2002). It is a mathematical programming language for modeling. It possesses the ability of handling large amount of data. It can also be used as in machine solvers and independent solvers. TACO Toolkit was introduced to simplify the use of AMPL.

2.3 Fractional Calculus

The fractional calculus was originated nearly the same period as ordinary calculus was introduced. After the formulation of ordinary differentiation and integration by Newton and Leibniz in the 17th century, Leibniz received a letter from L'Hopital demanding his idea if the derivative has an order of $\frac{1}{2}$ (Podlubny, 1999a). This letter is what led to the eventual start of fractional integrals and derivatives theories.

The advance of fractional calculus completes the theory of operation in Mathematics. Machado et al. give the recent history of fractional calculus (Machado et al., 2011). This is a remarkable tool that is used in explaining different phenomena of physics that couldn't be explained by conventional mathematics. Phenomena that possess an extended memory, dependence on long ranges, etc are best explained by the theory of Fractional calculus.

2.3.1 Preliminaries

a. Definitions

Definitions for fractional order integrals and differentiations exist in literature in more than 10 forms (Miller and Ross, 1993). Here we give some of the most commonly used definitions for convenience (Magin, 2006).

Definition 2.7 (Qian and Wong, 2010): The fractional derivative of order $\alpha \in [n-1, n)$ of $f(x)$ for Rieman-Liouville can be defined as:

$$ {}_{a}^{RL}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt\,, \quad n = [\alpha] + 1. \quad (2.21)$$

Definition 2.8 (Qian and Wong, 2010): The fractional derivative of order $\alpha \in (n-1, n]$ of $f(x)$ for Caputo is defined as:

$$ {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t) dt, \quad n = [\alpha] + 1. \qquad (2.22)$$

Definition 2.9 (Ortiz et al., 2013): **(Linearity)**

If f, g are continuous and l, d are scalars, then

$$ {}_a^{RL} D_x^\alpha [lf(x) + dg(x)] = l\, {}_a^{RL} D_x^\alpha f(x) + d\, {}_a^{RL} D_x^\alpha g(x), \qquad (2.23)$$

$$ {}_a^C D_x^\alpha [lf(x) + dg(x)] = l\, {}_a^C D_x^\alpha f(x) + d\, {}_a^C D_x^\alpha g(x).$$

Definition 2.10 (Baba, 2019): **(Contraction)**

For an operator $f: X \to X$ which mapped a metric space onto itself, it is contractive for $0 < q < 1$

$$ d\left(f(x), f(y)\right) = q d(x, y), \qquad \forall\, x, y \in X. \qquad (2.24)$$

b. Functions

Here we give some of the most commonly used functions for convenience (Magin, 2006).

- ***Gamma function***

Gamma function s is the essential factor in practically all fractional integral definitions. It is defined as the factorial of non-integer numbers as follows;

The integral formula of the gamma function is written as:

$$ \Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, p > 0. \qquad (2.25)$$

Some gamma function properties to remember include:

$$ \Gamma(1) = 1; \qquad \Gamma(n+1) = n! \quad (n = 0,1,2 \ldots)$$

$$ \Gamma\left(1/2\right) = \sqrt{\pi}; \qquad \Gamma(x+1) = x\,\Gamma(x). \qquad (2.26)$$

The fractional order derivative of a variable that has the same fractional order power as the variable in question is a constant, much like the integer derivative,

$$\frac{d^{\alpha}}{dx^{\alpha}}x^{\alpha} = \frac{\Gamma\,(\,\alpha+1)}{\Gamma\,(\,\alpha-\alpha+1)}x^{\alpha-\alpha} = \Gamma\,(\,\alpha\,+1). \qquad (2.27)$$

- ***Mittag-Leffer function***

The exponential function acts a vital part in the solution of ordinary differential equations. This is the case for Mittag-Leffer function (M-L) as it is the generalized form of an exponential function. The two most common forms of Mittag-Leffer function (M-L) are given below (Prajapati and Shukla, 2012; Chaurasia and Pandey, 2010).

$$E_{\alpha}(x) = \sum\nolimits_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}(\alpha > 0) \qquad (2.28)$$

$$E_{\alpha,\beta}(x) = \sum\nolimits_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}(\alpha > 0, \beta > 0). \qquad (2.29)$$

Properties of the M-L function that need to be mentioned as follows;

$$E_{1,1}(x) = e^{x};$$

$$E_{1,2}(x) = \frac{e^{x}-1}{x} \qquad (2.30)$$

- ***Error function***

Another special function that needs to be mentioned is the error function which can be defined as follow:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du, \qquad -\infty < x < \infty. \qquad (2.31)$$

It has the following properties;

$$erf(0) = 0$$

$$erf(\infty) = 1 \qquad (2.32)$$

$$erf(x) + erfc(x) = 1$$

Where $erfc(x)$ $is\ called\ the\ complimentary\ error\ function$

- ***Confluent hypergeometric function***

This function is used to get the solution for the equations of confluent hypergeometric, and is represented as follows;

$$ {}_1F_1(a\,;c\,;x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!}, \quad -\infty < x < \infty \qquad (2.33)$$

Where the Pochhammer symbols are defined $(a)_n$ and $(c)_n$.

$$(a)_n = \frac{\Gamma\,(\,a+1)}{\Gamma\,(a)}, \qquad and \qquad (2.34)$$

$$(c)_n = \frac{\Gamma\,(\,c+1)}{\Gamma\,(c)}.n = 0,1,2\,...$$

Below are some of the hypergeometric function's most popular characteristics:

$$ {}_1F_1(1;1;x) = e^x, \qquad (2.35)$$

$$\frac{1}{\Gamma(\alpha+1)}\ {}_1F_1(a;\alpha+1;at) = E_{1,2}(at).$$

c. Important theorems

Theorem 2.2 (Baba, 2019)**:** ***(Principle of Banach contraction mapping)***

Any operator of contractive that mapped a metric space against the operator itself and will create a unique fixed point. Moreover, when $f{:}X \to X$ is an operator of contractive that mapped a metric space onto itself with its fixed point a: $f(\boldsymbol{a}) = \boldsymbol{a}$; then for any continual sequence:

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1), \dots, x_{n+1} = f(x_n), \dots, \qquad (2.36)$$

that converges to $\boldsymbol{a}$.

Then we said that $\boldsymbol{a}$ is a solution or equilibrium for the continuous dynamical system and the discrete dynamical system is a fixed point.

Theorem 2.3 (Matignon, 1996): For the equilibrium solutions x^* of a given system say $(*)$ to be asymptotically stable locally, then all its eigenvalues λ_i in its Jacobian matrix $\frac{\partial f}{\partial x_i}$ which is evaluated for the equilibrium points must satisfy the condition below:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \qquad 0 < \alpha < 1. \tag{2.37}$$

Theorem 2.4 (Delvari et al., 2012): If $x = 0$ is an equilibrium solution of system $(*)$, and $\Omega \subseteq \mathbb{R}^n$ is a domain comprising $x = 0$.

If $V(t,\ x)$: $[t_0,\ \infty] \times \Omega \to \mathbb{R}$ is a continuously differentiable function given as:

$$W_1(x) \leq V(t,x) \leq W_2(x) \quad \text{and} \tag{2.38}$$

$${}_0^C D_t^\alpha V(t\,,x) \leq -W_3(x), \qquad for\ \ t \geq 0, \qquad x \in \Omega.$$

where, $W_1(x), W_2(x)\ and\ W_3(x)$ are definite functions that are continuous and positive on Ω and V is a contender function of Lyapunov, then $x = 0$ is globaly asymptoticaly stable.

Theorem 2.5 (Vergas-De-Leon, 2015): Let $x(t) \in \mathbb{R}^+$be a derivable and continuous function. Then, for any time instant $t \geq t_0$ and $\alpha \in (0, 1)$

$${}_0^C D_t^\alpha [x(t) - x^* - x^* \ln\left(\frac{x(t)}{x^*}\right)] \leq \left(1 - \frac{x(t)}{x^*}\right) {}_0^C D_t^\alpha x(t), \quad x^* \in \mathbb{R}^+. \tag{2.39}$$

2.3.2 Fractional Order Differential Equations (FODEs)

These are some of the essential tools that are used to describe fractional-order dynamic systems. Hence, they are very important and need to be studied. Here we give the definitions of linear and nonlinear Fractional order differential equations.

a. Linear Fractional order differential equations

These are the most used as per as fractional order controls are concerned. This is due to their simplicity and regularity. For their general expression, see below:

$$a_1\,{}_0^CD_t^{\alpha_1}y(t) + a_2\,{}_0^CD_t^{\alpha_2}y(t) + \cdots + a_n\,{}_0^CD_t^{\alpha_n}y(t) = b_1\,{}_0^CD_t^{\beta_1}u(t) + b_2\,{}_0^CD_t^{\beta_2}u(t) + \cdots + b_n\,{}_0^CD_t^{\beta_n}u(t), \quad (2.40)$$

$wheretheorders, \alpha_i, \beta_j \;\; (i, j = 1,2, \dots)$ can be arbitrary real numbers, i.e., $\alpha_i, \beta_j \in \mathbb{R}$. If α_i and β_j are integer multiples of a common factor, the equation is considered to have a commensurate order; and if there is no common factor occurs it is supposed to be of non-commensurate order (Vinagre and Feliu, 2000).

b. Nonlinear Fractional order differential equations

This is defined as Fractional order differential equations that are not linear.

2.4 Fractional Order Controllers

Since the solutions of fractional order models can be found now by either analytic or numerical means, people frequently use them to model real-world problems. This leads to the many usages of fractional calculus in different areas of applications like control theory and electrical circuits theory. Chen explains about four situations for a fractional control of closed-loop control systems (Chen, 2006). These are; Integer order (IO) model with Integer order (IO) controller, the Integer order (IO) model with the Fractional order (FO) controller, the Fractional order (FO) model with Integer order (IO) controller, and Fractional order (FO) model with Fractional order (FO) controller.

Many shreds of evidence showed that a controller designed from the best fractional order model performs better than the one designed from the corresponding integer order model. A lot of researchers gave reasons as to why is better to use fractional order control than integer order control (Monje, 2006; Monje et al., 2008). It was also discovered that using fractional order controllers gives a higher chance of adjustability in changing both gain characteristics as well as the phase characteristics of the controller. This flexibility makes fractional order (FO) controllers one of the greatest tools used to design a robust control system. This indicates that a fractional order controller designed with a few tuning knobs can have almost the same robustness as that of an integer order (IO) design with very high tuning knobs.

Different types of controllers designed with fractional order exist in the literature, here we give examples of some of these controllers.

2.4.1 Proportional Integral and Derivative (PID) Controllers

Examples of these controllers can be found in many industrial process control applications. It was estimated that 95% of controllers used in this regard are PID controllers. PID controllers consist of a combination of three different controllers logically that give rise to a single controlled output.

Elmer Sperry was the pioneer of the PID controller in 1911. He invented the basic Proportional controller. In 1933 the Taylor Instrumental Company (TIC) developed the first Pneumatic controller which is fully tunable. After some years, control engineers put their heads together to find a way of removing the error that was in the steady state of the Proportional controllers. They achieved that through bringing back some of the false values until the mistake is not zero. The method of eliminating the error gives what is called Proportional Integral (PI) controller. Using derivative action to decrease the overshooting problems, the Pneumatic PID controller was invented in 1940.

Fractional order PID controllers are of the form $PI^{\lambda}D^{\mu}$. These controllers were studied by (Podlubny, 1999b) in the time domain and by (Petras, 1999) in the frequency domain. Its general form is given as:

$$C(S) = \frac{U(S)}{E(S)} = K_p + T_i S^{-\lambda} + T_d S^{\sigma}. \qquad (2.41)$$

with λ and $\boldsymbol{\sigma}$ being real numbers (positive), $K_p$is the gain of proportional, T_i *is* the constant of integration while T_d *is* the constant of differentiation. It can be seen that by taking $\lambda = 1$ and $\boldsymbol{\sigma}= 1$, we have the traditional (IO) PID controller when $\lambda = 0$ $(T_i = 0)$we have the PD^{σ} controller, and when $\sigma = 0$ $(T_d = 0)$ we have the PI^{λ} controller etc.

All the above classes of controllers are distinct classes of the $PI^{\lambda}D^{\sigma} the$ controller that has output formula given as:

$$U(t) = K_p e(t) + T_i {D_t}^{-\lambda} e(t) + T_d {D_t}^{\sigma} e(t). \qquad (2.42)$$

Some of the works on fractional order PID controllers found in (Axtell and Bise, 1990; Blas et al., 2002; Manabe, 1961; Mehaut et al., 2004; Monje et al., 2008; Monje et al., 2008; Oustaloup, 2006; Xue and Chen, 2002).

2.4.2 Fractional Order Model Predictive Controllers

Model predictive control (MPC) is defined as optimal control theory using the numerical optimization method. Plan responses and future control efforts can be predicted by the use of a system model predictive control. This can be done by optimizing the system at given intervals of time with regard to a given performance function. Recently, predictive control is among the popular progressive control techniques that are used in many industries (Rawlings, 2000; Muske and Rawlings, 1993; Bemporad, 2006; Morari and Lee, 1999; Garcia et al., 1989).

For MPC, models are mainly used to predict the possible output and also the control efforts needed to obtain the earmarked trajectory. Hence, in the case of MPC, the model's accuracy always gives the control as well as the exact prospect trajectory of the input. This provides the basic principle of the operation of MPC. Therefore, MPC is not a single technique but more of a methodology. Hence, it possesses many names; Model Predictive Control (MPC), Receding Horizon Control (RHC), Model Based Predictive Control (MBPC), Internal Model Control (IMC), Moving Horizon Control (MHC), etc.

Fractional order MPC refers to those systems that are fractional in nature and possess fractional MPC controllers. To design MPC there is a need for a state space model. The general form of the FOTF is given as:

$$G(s) = \frac{a}{S^{\alpha} + b}, \quad a, b \in \mathbb{R}. \tag{2.43}$$

with, $\alpha < 1$.

2.4.3 Fractional Order Sliding Mode Controllers

The sliding mode control popularly known as SMC is non–linear in nature and changes the dynamics of the system by using a non – continuous control signal. This induces the system to slide through a transition of the initial behavior of the systems. SMC has a peculiarity of activating control law that forces the states of the systems to change from their original states to a new sliding surface. Sliding Mode Controllers are one of the most successful control methods, hence their application to many complex systems in engineering and sciences. Many problems concerning fractional order models use

SMC methodology in literature (Hosseinnia et al., 2010; Tavazoei and Haeri, 2008). Some of the applications of FO SMC have been given by (Yin et al., 2013; Yin et al., 2014).

2.4.4 Fractional Order Optimal Controllers

Fractional Optimal Control Problems (FOCPs) are those problems of optimal control that contains fractional order models, they can be termed as the universal form of traditional optimal control problems (OCPs). The differential equations in FOCP are of fractional order that is FDEs, and its performance index is represented with the fractional operator of integration (Choi et al., 2015). Several works in the literature give basic theories and essential foundation for FOCPs, many of them studied in detail the procedure of designing FOCPs and found the conditions of the optimal control for different states variables by the use of both numerical technique and analytical techniques (Agarwal, 2004; Agarwal and Baleanu, 2007; Agarwal, 2008; Jelicic and Petrovacki, 2009; Agarwal et al., 2010; Odzijewicz et al., 2012; Kamocki, 2014; Chinnathambi et al., 2019; Al-Mdallal and Abu Omer, 2018; Al-Mdallal and Hajji, 2015; Hajji and Al-Mdallal, 2018).

The general method of forming and solving the problem of fractional optimal control (FOCP) is given by (Agrawal et al., 2004). In their formulation, they use the left and right R-L definitions of FO derivatives as in the form expressed below:

$$J(u) = \int_0^T F(x, u, t)\, dt. \tag{2.44}$$

Based on the following constraints of the dynamic system.

$${}_a^C D_t^\alpha x = G(x, u, t\ )(0 < \alpha < 1), \tag{2.45}$$

and the initial conditions:

$$x(0) = x_0$$

where $x(t)$is the state variable.

The cost criteria for the integral in the quadratic form are given below:

$$J(u) = \frac{1}{2}\int_0^1 [\ q\ (\ t)x^2(t) + r(\ t)u^2\ ]\, dt. \tag{2.46}$$

Subject to the following:

$$ {}_{a}^{C}D_{t}^{\alpha}x = a(t)x + b(t)u. \qquad (2.47) $$

Using the derivation given in (Agrawal et al., 2004), then the Euler-Lagrange equations for the above FOCP can be obtained as:

$$ {}_{a}^{C}D_{t}^{\alpha}x = G(x\,, u\,, t\,), $$

$$ {}_{t}^{C}D_{1}^{\alpha}\lambda \;\; = \frac{\partial F}{\partial x} + \lambda \frac{G}{\partial x} \qquad (2.48) $$

$$ 0 = \frac{\partial F}{\partial x} + \lambda \frac{G}{\partial u} $$

with $x(0) = x_0$ and $\lambda(1) = 0$.

The solution of the fractional Euler-Lagrange equation and many more methods of FOCPs with their results were all available in the literature.

2.5 COVID – 19 Mathematical Models

Since the beginning of COVID –19 pandemic many researchers collaborated to provide mathematical models of the disease dynamics to analyze the effect of the pandemic on susceptible populations some of the models are presented as follows;

- ***SIR Model***

The SIR model is one of the most common structures to model diseases the basic idea of the model is to create 3 classes or compartments of the population the first one is a susceptible population which the letter ‘S’ stand for, and then the infected population which ‘I’ stand for, lastly ‘R’ is the recovered population. The model dynamic of disease transmission will be added through these parameters and equations would be derived subsequently (Gul et al., 2008).

- ***SEIR Model***

This model is similar to the SIR model but one extra compartment is added which is the exposed population which ‘E’ stand for. This model was used frequently to model COVID -19 since the exposed population was important to be analyzed especially in lockdown phases (Guihua, and Zhen, 2005; Alqahtani and Yusuf, 2022).

- ***Modified Compartment Model***

These models' compartments were modified by adding new classes to the model and changing the methods of disease transmission rates. For example, adding quarantine population in SUQC and SEIQR models. In this thesis, a unique modified model was used for modeling COVID-19 which more specified classes were added regarding awareness and vaccination.

CHAPTER 3

Fractional COVID – 19 Modeling and Analysis on Successive Optimal Control Policies

3.1 Introduction

Many sectors of life were heavily impacted by the damaging effects of COVID-19 which started by the end of December 2019. The world economy still recovering from the effect of that pandemic. Many people die from the pandemic, while many have been infected and are battling for their lives. The COVID -19 outbreak lifts many unanswered questions for researchers to answer. Some critical biological information about COVID-19 is still unknown. Many research works were dedicated to finding new and adequate vaccines for the disease. Many items such as ventilators were used to help infected individuals in many countries. The main target is to reduce the number of infected individuals and subsequently deaths due to the pandemic which is why many countries adopt non–pharmaceutical measures such as lockdowns, Airport closures, use of sanitizers, and social distancing. Many studies from theoretical to practical points of view about the pandemic are carried out (Al-sheikh et al., 2011; Owolabi and Atanga, 2019; Do and Lee, 2016; Chowell et al., 2015; Liu et al., 2020; Chen et al., 2020; Khan and Atanga, 2020; Chen et al., 2020; Coccia, 2021a; Coccia, 2021b).

While 75% of the infected individuals recover without falling seriously sick, most of the infected individuals recover naturally (Ivorra et al., 2020). Throat infection, chest pain, runny nose or nasal congestion, losing smell and taste, vomiting, diarrhea, and nausea are some of the symptoms of COVID – 19. In most cases, these symptoms appear slowly. Older age suffers major complications compared to younger age. In general, an infected person takes two days to two weeks to show symptoms of the disease (Zamir et al., 2021). Mostly mild cases take two weeks to recover, whereas critical cases take three to six weeks to recover (Gomes, 2020). Now that COVID –19 vaccine is available and the non–pharmaceutical interventions to avert the transmission of the outbreak such as; quarantine, self-isolation, social distancing, and use of (PPE) personal protective equipment (face masks, hand gloves, etc.) Also, using sanitizer and washing hands regularly, avoiding contact with people showing the symptoms, and reporting any suspected case. There is a need to increase awareness levels among people. This will help in total compliance and subsequent eradication of the disease.

Since the inception of the pandemic in 2019, it caused millions of infections and thousands of deaths. It also caused a predicament in the socio-economic growth of the entire world. Hence, there is an urgent need to clearly understand the transmission dynamics of the disease. This leads to the need of developing mathematical models that study the dynamics of the disease and the impact of the control measures in curtailing the spread of the disease.

Because the majority of physical phenomena, such as biological systems, have an after-effect or a persistent memory property, it is possible that fractional differential equations could more accurately describe these phenomena. This is because fractional differential equations also have an after-effect memory built into their structure. This is why many researchers about real-life phenomena use fractional order differential equations (Escalante et al., 2018a; Escalante et al., 2018b; Ullah et al., 2018; Gomez, 2018). Caputo-Fabrizio (CF) fractional-order derivative is one of the recent senses of fractional-order differential equations that was introduced in 2015. The CF derivative is based on a kernel of 1 exponential more details of the equation are found in (Caputo, 2015). Caputo-Fabrizio sense was implemented in modeling many systems in various fields (Saad and Gomez, 2018; Abdeljawad, 2017; Abdeljawad and Baleanu, 2017), also used in modeling COVID -19 pandemic (Thabet et al., 2021; Bonyah et al., 2022; Pandey et al., 2022; Kumar et al., 2022a). The Caputo–Fabrizio fractional derivative introduces fewer noises than the Riemann–Liouville fractional derivative (Atanga, 2018). Therefore, Caputo–Fabrizio fractional derivative was selected to be used in this research.

Most mathematical models of COVID – 19 that studied control in literature did not consider time-dependent control strategies which are the most realistic approach (Baba et al., 2022; Baba et al., 2021; Baba et al., 2020; Baba and Nasidi, 2021; Baba et al., 2020; Baba and Nasidi, 2020; Baba and Baleanu, 2020; Ahmed et al., 2020).

However, very little research in this direction does exist, such as (Jajarmi et al., 2019; Baleanu et al., 2019; Sweilam et al., 2019; Yildiz et al., 2018a; Yildiz et al., 2018b; Baleanu et al., 2016) and this sort of policy could be used to propose or design programs of epidemic controls (Baba et al., 2020; Treesatayapum, 2022). Many researchers consider different parameters such as geolocation in different countries as (Pandey et al., 2022) for India, (Kumar et al., 2022b) for Japan, and (Batiha et al., 2022) for Saudi Arabia. The global and local dynamics of COVID - 19 may be completely characterized by mathematical models operating under fractional order derivatives. In addition, models of this type that make use of fractional calculus are superior in terms of their ability to precisely and accurately represent the observed occurrences (Nunez et al., 2021; Saha et al., 2020; Batiha et al., 2022; Nana-kyere et al., 2022; Ghosh et al., 2021; Khan et al., 2022; Dhar et al., 2022; Mohammadi and Rezapour, 2022; Nadim et al., 2021). The researchers utilize models to track the evolution epidemic over a period such as SEIR in (Alqahtani and Yusuf, 2022) which consider four compartments as follows; Susceptible, Exposed, Infected, and Recovered.

In (Zeb et al., 2022; Benati and Coccia, 2022; Coccia, 2022a) researchers developed models and applied optimal control for vaccination or restriction methods. In (Coccia, 2022b) conclude that regardless of control measures and the vaccination process COVID – 19 is pretentious by environmental and seasonal factors.

3.2 Formation of the Model

Consists of a system of fractional order differential equations the model was formed by Caputo – Fabrizio sense with five compartments. The compartments are; Us (t), As (t), Vs (t), I(t), and R(t) stand for susceptible unaware compartment, Susceptible aware compartment, Susceptible vaccinated compartment, Infected compartment, and recovered compartment respectively. The model is given below;

$${}^{CF}_{0}D^{\alpha}_{t}U_s(t) = \pi^{\alpha} - \beta_1^{\alpha}U_s(t)I(t) - \mu^{\alpha}U_s(t),$$

$${}^{CF}_{0}D^{\alpha}_{t}A_s(t) = -\beta_2^{\alpha}A_s(t)I(t) - \mu^{\alpha}A_s(t),$$

$${}^{CF}_{0}D^{\alpha}_{t}V_s(t) = -\beta_3^{\alpha}V_s(t)I(t) - \mu^{\alpha}V_s(t),$$

$${}^{CF}_{0}D^{\alpha}_{t}I(t) = \beta_1^{\alpha}U_s(t)I(t) + \beta_2^{\alpha}A_s(t)I(t) + \beta_3^{\alpha}V_s(t)I(t) - (\mu^{\alpha} + \gamma^{\alpha} + \delta^{\alpha})I(t),$$

$${}^{CF}_{0}D^{\alpha}_{t}R(t) = \delta^{\alpha}I(t) - \mu^{\alpha}R(t),$$

The initial conditions used for this model are presented below;

$U_s(0) = a_1, A_s(0) = a_2, V_s(0) = a_3, I(0) = a_4 and R(0) = a_5$.

Table 1 below shows the parameters used in the model and their respective meanings.

Table 3.1. Meaning of Parameters

Parameter	Meaning
π	Recruitment rate
β_1	The transmission rate of COVID–19 in a susceptible unaware compartment
$\beta_2 < \beta_1$	The transmission rate of COVID–19 in a susceptible aware compartment
$\beta_3 < \beta_2 < \beta_1$	The transmission rate of COVID–19 in a susceptible vaccinated compartment
μ	Natural death rate
γ	Recovery rate
δ	Disease-induced death rate
$0 < \alpha < 1$	Fraction order

3.3 Analysis of the Model

Here Equilibria, basic reproduction number, existence, and uniqueness analysis of the solution of the model are carried out.

3.3.1 Equilibria Analysis and Deriving the Basic reproduction number

The method of finding equilibrium solutions is straightforward, whereby equating the equations in the model to zero afterward the system should simultaneously be solved. five equilibrium solutions were attained;

i. Disease-free equilibrium (E_0)

$$E_0 = \{U_s^0, A_s^0, V_s^0, I^0, R^0\} = \left\{\frac{\pi^\alpha}{\mu^\alpha}, 0,0,0,0\right\}.$$

ii. Endemic with respect to U_s only (E_1)

$$E_1 = \{U_s^1, I^1, R^1\}$$
$$= \left\{\frac{\mu^\alpha + \gamma^\alpha + \delta^\alpha}{\beta_1^\alpha}, \frac{\pi^\alpha \beta_1^\alpha - \mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)}{\beta_1^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)}, \frac{\delta^\alpha[\pi^\alpha \beta_1^\alpha - \mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)]}{\mu^\alpha \beta_1^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)}\right\}.$$

iii. Endemic with respect to A_s only (E_2)

This equilibrium point doesn't exist as we have;

$$I^2 = \frac{-\mu^\alpha}{\beta_1^\alpha},$$

which is not biologically meaningful as we don't have a negative population.

iv. Endemic with respect to V_s only (E_3)

This equilibrium point doesn't exist as we have;

$$I^3 = \frac{-\mu^\alpha}{\beta_2^\alpha},$$

which is not biologically meaningful as we don't have a negative population.

v. Endemic with respect to $U_s, A_s and V_s$ (E_4)

This equilibrium point doesn't exist as we have;

$$I^4 = \frac{-\mu^\alpha}{\beta_1^\alpha} or I^3 = \frac{-\mu^\alpha}{\beta_2^\alpha},$$

which is not biologically meaningful as we don't have a negative population.

Hence the only feasible endemic equilibrium solution is E_1.

Now E_1 only exists if

$$\frac{\pi^\alpha \beta_1^\alpha - \mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)}{\beta_1^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)} > 0$$

This implies;

$$\frac{\pi^{\alpha}\beta_1^{\alpha}}{\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})} > 1.$$

Let,

$$\frac{\pi^{\alpha}\beta_1^{\alpha}}{\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})} = R_0,$$

whereR_0 is the basic reproduction ratio.

3.3.2 Existence and Uniqueness of a Solution of The Model

In this section, a fixed-point result is applied to check the existence and uniqueness of the solution of the model. Let the system be rewritten as;

$${}_{0}^{CF}D_t^{\alpha}U_s(t) = F_1(t, U_s),$$

$${}_{0}^{CF}D_t^{\alpha}A_s(t) = F_2(t, A_s),$$

$${}_{0}^{CF}D_t^{\alpha}V_s(t) = F_3(t, U_s),$$

$${}_{0}^{CF}D_t^{\alpha}I(t) = F_4(t, I),$$

$${}_{0}^{CF}D_t^{\alpha}R(t) = F_5(t, R).$$

Applying the Caputo – Fabrizio operator, the system becomes;

$$U_s(t) - U_s(0) = \frac{2\,(1-\alpha)}{(2-\alpha)M(\alpha)}F_1(t, U_s) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_1(\eta, U_s)d\eta,$$

$$A_s(t) - A_s(0) = \frac{2\,(1-\alpha)}{(2-\alpha)M(\alpha)}F_2(t, A_s) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_2(\eta, A_s)d\eta,$$

$$V_s(t) - V_s(0) = \frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)}F_3(t, V_s) + \frac{2\alpha}{(2-\alpha)\,M(\alpha)}\int_0^t F_3(\eta, V_s)d\eta,$$

$$I(t) - I(0) = \frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)}F_4(t, I) + \frac{2\alpha}{(2-\alpha)\,M(\alpha)}\int_0^t F_4(\eta, I)d\eta,$$

$$R(t) - R(0) = \frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)} F_5(t,R) + \frac{2\alpha}{(2-\alpha)\,M(\alpha)} \int_0^t F_5(\eta,R)d\eta.$$

Now, we need to prove $F_1, \dots, F_5$ satisfy Lipschitz continuity and contraction. See the theorem below;

Theorem 3.1:F_1 *is Lipschitz and if*

$$0 \le \beta_1^\alpha h_1 + \mu^\alpha < 1,$$

it is a contraction.

Proof of Theorem 3.1:

$$\begin{aligned}\|F_1(t,U_s) - F_1(t,U_{s1})\| &= \|\pi^\alpha - \beta_1^\alpha U_s(t)I(t) - \mu^\alpha U_s(t) - \pi^\alpha - \beta_1^\alpha U_{s1}(t)I(t) - \mu^\alpha U_{s1}(t)\| \\ &= \left\|-\beta_1^\alpha I(t)\big(U_s(t) - U_{s1}(t)\big) - \mu^\alpha\big(U_s(t) - U_{s1}(t)\big)\right\| \\ &\le \beta_1^\alpha \|I(t)\|\|U_s(t) - U_{s1}(t)\| + \mu^\alpha\|U_s(t) - U_{s1}(t)\| \\ &\le (\beta_1^\alpha h_1 + \mu^\alpha)\|U_s(t) - U_{s1}(t)\| \\ &\le L_1\|U_s(t) - U_{s1}(t)\|,\end{aligned}$$

Where,

$$L_1 = \beta_1^\alpha h_1 + \mu^\alpha and h_1 \ge \|I(t)\|.$$

In the same way, the Lipschitz continuity and contraction were shown for $F_2, \dots, F_5$, where we obtain $L_2, \dots, L_5$ respectively as their Lipschitz constants.

In recursive form, let

$$q_{1n}(t) = U_{s_n}(t) - U_{s_{n-1}}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right)\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta,$$

$$q_{2n}(t) = A_{s_n}(t) - A_{s_{n-1}}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_2\left(t, A_{s_{n-1}}\right) - F_2\left(t, A_{s_{n-2}}\right)\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_2\left(\eta, A_{s_{n-1}}\right) - F_2\left(\eta, A_{s_{n-2}}\right)\right) d\eta,$$

$$q_{3n}(t) = V_{s_n}(t) - V_{s_{n-1}}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_3\left(t, V_{s_{n-1}}\right) - F_3\left(t, V_{s_{n-2}}\right)\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_3\left(\eta, V_{s_{n-1}}\right) - F_3\left(\eta, V_{s_{n-2}}\right)\right) d\eta,$$

$$q_{4n}(t) = I_n(t) - I_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_4(t, I_{n-1}) - F_4(t, I_{n-2})\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_4(\eta, I_{n-1}) - F_4(\eta, I_{n-2})\right) d\eta,$$

$$q_{5n}(t) = R_n(t) - R_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_5(t, R_{n-1}) - F_5(t, R_{n-2})\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_5(\eta, R_{n-1}) - F_5(\eta, R_{n-2})\right) d\eta,$$

with initial conditions;

$$U_s{}^0(t) = U_s(0), A_s{}^0(t) = A_s(0), V_s{}^0(t) = V_s(0), I_0(0) = I(0) and R_0(0) = R(0).$$

Taking norm of q_{1n}, we have;

$$\|q_{1n}(t)\| = \left\|U_{s_n}(t) - U_{s_{n-1}}(t)\right\|$$
$$= \left\|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right)\right)\right.$$
$$\left. + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta\right\|.$$

Applying triangular inequality, we have;

$$\|q_{1n}(t)\| = \left\|U_{s_n}(t) - U_{s_{n-1}}(t)\right\|$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left\|F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right)\right\|$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta\right\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \left\|U_{s_{n-1}} - U_{s_{n-2}}\right\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \left\|U_{s_{n-1}} - U_{s_{n-2}}\right\| d\eta.$$

This implies;

$$\|q_{1n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \|q_{1n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \|q_{1n-1}(t)\| d\eta.$$

Similarly,

$$\|q_{2n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_2 \|q_{2n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_2 \int_0^t \|q_{2n-1}(t)\| d\eta,$$

$$\|q_{3n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_3 \|q_{3n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_3 \int_0^t \|q_{3n-1}(t)\| d\eta,$$

$$\|q_{4n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_4 \|q_{4n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_4 \int_0^t \|q_{4n-1}(t)\| d\eta,$$

$$\|q_{5n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_5 \|q_{5n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_5 \int_0^t \|q_{5n-1}(t)\| d\eta.$$

Subsequently, we have;

$$U_{s_n}(t) = \sum_{i=1}^{n} q_{1i}(t)\,, A_{s_n}(t) = \sum_{i=1}^{n} q_{2i}(t), V_{s_n}(t) = \sum_{i=1}^{n} q_{3i}(t), I_n(t)$$
$$= \sum_{i=1}^{n} q_{4i}(t), R_n(t) = \sum_{i=1}^{n} q_{5i}(t).$$

To show the existence of the solution, the following theorem was proven;

Theorem 3.2: *The solution exists if there exist* t_1 *such that the following inequality is true,*

$$\frac{2\,(1-\alpha)}{(2-\alpha)\,M\,(\alpha)} L_i + \frac{2\alpha t_1}{(2-\alpha)\,M\,(\alpha)} L_i < 1, \quad i = 1, \dots, 5.$$

Proof of Theorem 3.2: Recursively, we have

$$\|q_{1n}(t)\| \leq \|U_{s_n}(0)\| \left[\frac{2\,(1-\alpha)}{(2-\alpha)\,M\,(\alpha)}\; L_1 + \frac{2\alpha}{(2-\alpha)\,M\,(\alpha)}\; L_1\right]^n,$$

$$\|q_{2n}(t)\| \leq \|A_{s_n}(0)\| \left[\frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)} L_2 + \frac{2\alpha}{(2-\alpha)\,M(\alpha)} L_2\right]^n,$$

$$\|q_{3n}(t)\| \leq \|V_{s_n}(0)\| \left[\frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)} L_3 + \frac{2\alpha}{(2-\alpha)\,M(\alpha)} L_3\right]^n,$$

$$\|q_{4n}(t)\| \leq \|I_n(0)\| \left[\frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)} L_4 + \frac{2\alpha}{(2-\alpha)\,M(\alpha)} L_4\right]^n,$$

$$\|q_{5n}(t)\| \leq \|R_n(0)\| \left[\frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)} L_5 + \frac{2\alpha}{(2-\alpha)\,M(\alpha)} L_5\right]^n.$$

Hence solutions are existed and continuous. To demonstrate that the above functions construct the solutions, consider;

$$U_s(t) - U_s(0) = U_{s_n}(t) - K_{1_n}(t),$$

$$A_s(t) - A_s(0) = A_{s_n}(t) - K_{2_n}(t),$$

$$V_s(t) - V_s(0) = V_{s_n}(t) - K_{3_n}(t),$$

$$I(t) - I(0) = I_n(t) - K_{4_n}(t),$$

$$R(t) - R(0) = R_n(t) - K_{5_n}(t).$$

Hence,

$$\|K_{1_n}(t)\| = \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left(F_1(t, U_{s_{n-1}}) - F_1(t, U_{s_{n-2}})\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(F_1(\eta, U_{s_{n-1}}) - F_1(\eta, U_{s_{n-2}})\right) d\eta \right\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|F_1(t, U_{s_{n-1}}) - F_1(t, U_{s_{n-2}})\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left\| \int_0^t \left(F_1(\eta, U_{s_{n-1}}) - F_1(\eta, U_{s_{n-2}})\right) d\eta \right\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \|U_s - U_{s_{n-1}}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \|U_s - U_{s_{n-1}}\| t.$$

Carrying out the procedure, we get

$$\|K_{1_n}(t)\| \leq \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha t}{(2-\alpha)M(\alpha)}\right]^{n+1} {L_1}^{n+1} k.$$

At $t = t_1$, we get

$$\|K_{1_n}(t)\| \leq \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha t_1}{(2-\alpha)M(\alpha)}\right]^{n+1} {L_1}^{n+1} k.$$

Taking the limit as $n \to \infty$, we get

$$\|K_{1_n}(t)\| \to 0.$$

Similarly, we get

$$\| K_{2_n}(t) \|, \| K_{3_n}(t) \|, \| K_{4_n}(t) \|, \| K_{5_n}(t) \| \to 0.$$

Finally, to show uniqueness, assume there exists some solutions say, ${U_s}^1(t), {A_s}^1(t), {V_s}^1(t), I^1(t)$ and $R^1(t),$ then

$$\|U_s(t) - {U_s}^1(t)\| \left(1 - \frac{2\,(1-\alpha)}{(2-\alpha)\,M\,(\alpha)} L_1 - \frac{2\alpha t}{(2-\alpha)\,M\,(\alpha)} L_1\right) \leq 0.$$

The following theorem completes the result.

Theorem 3.3: *If*

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1-\frac{2\alpha t}{(2-\alpha)M(\alpha)}L_1\right)>0,$$

then the solution is unique.

Proof of Theorem 3.3: Consider

$$\left\|U_s(t)-U_s{}^1(t)\right\|\left(1-\frac{2\,(1-\alpha)}{(2-\alpha)\,M(\alpha)}\;L_1-\frac{2\alpha t}{(2-\alpha)\,M(\alpha)}\;L_1\right)\leq 0.$$

Since,

$$\left(1-\frac{2\,(1-\alpha)}{(2-\alpha)\,M\,(\alpha)}\;L_1-\frac{2\;\alpha t}{(2-\alpha)\,M\,(\alpha)}\;L_{\,1}\right)>0,$$

then

$$\left\|U_s(t)-U_s{}^1(t)\right\|=0.$$

This implies,

$$U_s(t)=U_s{}^1(t).$$

This applies to the remaining functions.

CHAPTER 4

Optimal Control and Numerical Simulation

4.1 Optimal Control Analysis

This chapter follows the next phase of research which includes formulating the optimal control problem and finding the existence of a solution for the problem. Furthermore, the numerical simulation will be presented with a detailed discussion and interpretation.

4.1.1. Formation of Optimal Control Problem

The dynamics of the control system can be described by the following system of Fractional-order differential equations in the Caputo–Fabrizio sense;

$$
\begin{aligned}
{}^{CF}_{0}D_t^{\alpha} U_S(t) &= \pi^{\alpha} - \beta_1^{\alpha} U_S I - \mu^{\alpha} U_S - \theta u_1 U_S + \in A_S,\\
{}^{CF}_{0}D_t^{\alpha} A_S(t) &= \beta_2^{\alpha} A_S I - \mu^{\alpha} A_S - \in A_S - \emptyset u_2 A_S + \rho u_3 V_S, \qquad (4.1)\\
{}^{CF}_{0}D_t^{\alpha} V_S(t) &= \emptyset u_2 A_S - \beta_3^{\alpha} V_S I - \mu^{\alpha} V_S - \rho u_3 V_S,\\
{}^{CF}_{0}D_t^{\alpha} I(t) &= \beta_1^{\alpha} U_S I + \beta_2^{\alpha} A_S I + \beta_3^{\alpha} V_S I - (\mu^{\alpha} + \gamma^{\alpha} + \delta^{\alpha}) I,\\
{}^{CF}_{0}D_t^{\alpha} R(t) &= \delta^{\alpha} I - \mu^{\alpha} R
\end{aligned}
$$

Where,

$u_1 = Awareness\ campaign\ about\ COVID - 19$

$u_2 = vaccination\ for\ the\ aware\ class$

$u_3 = taking\ optimal\ vaccine$

The following objective function will be minimized:

$$J(u_1, u_2, u_3) = \int_0^{t_f} (aU_S + bA_S + cV_S + du_1^2 + eu_2^2 + fu_3^2)\, dt, \qquad (4.2)$$

The objective here is minimizing U_S, A_S and V_S at the same time to minimize the cost of the three controls $u_1, u_2\ and\ u_3$. Hence, we need to get the optimal control $u_1^*, u_2^*\ and\ u_3^*$ such that:

$$J(u_1^*, u_2^*, u_3^*) = \min_{u_1, u_2} \{J(u_1, u_2, u_3) | u_1, u_2, u_3 \in \Omega\}. \qquad (4.3)$$

The set of control as:

$\Omega = \{(u_1, u_2, u_3)|u_i: [0, t_f] \to [0, \infty) Lebesguemeasurable,\ i = 1,2,3\}.$

The expenses of minimizing U_S is represented by the term aU_S, and that of minimizing $A_S$is represented by bA_S, while the one for minimizing $V_S$is represented by cV_S. Likewise, all the expenses associated with the control u_1 is represented by du_1^2, all the expenses associated with the control u_2 are represented by eu_2^2 and also all the expenses associated with the control u_3 is represented by fu_3^2 The sufficient conditions required for the optimal control to be fulfilled can be found by using the most popular PMP. The said principle can be used to turn (4.1) and (4.3) equations to a point-wise minimizing problem of the Hamiltonian H for($u_1, u_2 u_3$) stated as follows:

$$H = aU_S + bA_S + cV_S + du_1^2 + eu_2^2 + fu_3^2 + \lambda_{U_S}\{\pi^\alpha - \beta_1^\alpha U_S I - \mu^\alpha U_S - \theta u_1 U_S + \in A_S\} + \lambda_{A_S}\{\beta_2^\alpha A_S I - \mu^\alpha A_S - \in A_S - \emptyset u_2 A_S + \rho u_3 V_S\} + \lambda_{V_S}\{\emptyset u_2 A_S - \beta_3^\alpha V_S I - \mu^\alpha V_S - \rho u_3 V_S\} + \lambda_I\{\beta_1^\alpha U_S I + \beta_2^\alpha A_S I + \beta_3^\alpha V_S I - (\mu^\alpha + \gamma^\alpha + \delta^\alpha) I\} + \lambda_R\{\delta^\alpha I - \mu^\alpha R\} \quad (4.4)$$

where,$\lambda_{U_S}, \lambda_{A_S}, \lambda_{V_S}, \lambda_I,\ and \lambda_R$ are the adjoint variables or co-state variables.

$$-\frac{d\lambda_{U_S}}{dt} = \frac{\partial H}{\partial U_S} = a + \lambda_{U_S}\{-\beta_1^\alpha I - \mu^\alpha - \theta u_1\} + \lambda_I \beta_1^\alpha I,$$

$$-\frac{d\lambda_{A_S}}{dt} = \frac{\partial H}{\partial A_S} = b + \lambda_{U_S} \in + \lambda_{A_S}\{\beta_2^\alpha I - \mu^\alpha - \in - \emptyset u_2\} + \lambda_I \beta_2^\alpha I, \quad (4.5)$$

$$-\frac{d\lambda_{V_S}}{dt} = \frac{\partial H}{\partial V_S} = c + \lambda_{A_S}\rho u_3 + \lambda_{V_S}\{-\beta_3^\alpha I - \mu^\alpha - \rho u_3\} + \lambda_I \beta_3^\alpha I,$$

$$-\frac{d\lambda_I}{dt} = \frac{\partial H}{\partial I} = -\lambda_{U_S}\beta_1^\alpha U_S + \lambda_{A_S}\beta_2^\alpha A_S + \lambda_{V_S}\beta_3^\alpha V_S + \lambda_I\{\beta_1^\alpha U_S + \beta_2^\alpha A_S + \beta_3^\alpha V_S - (\mu^\alpha + \gamma^\alpha + \delta^\alpha)\},$$

$$-\frac{d\lambda_R}{dt} = \frac{\partial H}{\partial R} = -\lambda_R \mu^\alpha R$$

The transversality conditions are $\lambda_{U_S}(t_f) = \lambda_{A_S}(t_f) = \lambda_{V_S}(t_f) = \lambda_I(t_f) = \lambda_R(t_f) = 0$, for 0 <u_i<1, for i= 1, 2,3,

From the interior of the controls, we have:

$$\frac{\partial H}{\partial u_1} = 2du_1 - \lambda_{U_S}\theta U_S = 0,$$

$$\frac{\partial H}{\partial u_2} = 2eu_2 - \lambda_{A_S}\emptyset A_S + \lambda_{V_S}\emptyset A_S = 0. \quad (4.6)$$

$$\frac{\partial H}{\partial u_3} = 2fu_3 + \lambda_{A_S}\rho V_S - \lambda_{V_S}\rho V_S = 0$$

From where;

$$u_1 = \frac{1}{2d}\lambda_{U_S}\theta U_S,$$

$$u_2 = \frac{1}{2e}\emptyset A_S[\lambda_{A_S} - \lambda_{V_S}], \quad (4.7)$$

$$u_3 = \frac{1}{2f}\rho V_S[\lambda_{V_S} - \lambda_{A_S}].$$

4.1.2. Existence of Optimal Solutions

We give the following theorem for the existence of the optimal controls;

Theorem 4.1: *The control values (u_1^*, u_2^*, u_3^*) which can minimize(u_1, u_2, u_3) over U are given by,*

$$u_1^* = max\left\{0, min\left[1, \frac{1}{2d}\lambda_{U_S}\theta U_S\right]\right\},$$

$$u_2^* = max\left\{0, min\left[1, \frac{1}{2e}\emptyset A_S[\lambda_{A_S} - \lambda_{V_S}]\right]\right\}, \quad (4.8)$$

$$u_3^* = max\left\{0, min\left[1, \frac{1}{2f}\rho V_S[\lambda_{V_S} - \lambda_{A_S}]\right]\right\},$$

Where,$\lambda_{U_S}, \lambda_{A_S}, \lambda_{V_S}, \lambda_I$, $and \lambda_R$are, co-state variables that satisfy(4.1- 4.8)as well as the transversality conditions that follow$\lambda_{U_S}(t_f) = \lambda_{A_S}(t_f) = \lambda_{V_S}(t_f) = \lambda_I(t_f) = \lambda_R(t_f) = 0$ and

$$u_1^* = \begin{cases} 0, & if u_1 \leq 0, \\ u_1, & if \quad 0 < u_1 < 1, \\ 1, & if u_1 \geq 0, \end{cases}$$

$$u_2^* = \begin{cases} 0, & if u_2 \leq 0, \\ u_2, & if \quad 0 < u_2 < 1, \\ 1, & if u_2 \geq 0. \end{cases} \quad (4.9)$$

$$u_3^* = \begin{cases} 0, & if u_3 \leq 0, \\ u_3, & if \quad 0 < u_3 < 1, \\ 1, & if u_3 \geq 0. \end{cases}$$

Proof of Theorem 4.1: To prove the existence of the optimal control solution we use the convexity of the integrand of J to controlsu_1, u_2 and u_3 for the boundedness of the solutions of the state and the Lipschitz properties of the state concerning the variables of the state. Hence, we apply PMP and get the following:

$$ {}^{CF}_{0}D^{\alpha}_{t}\lambda_{U_S}(t) = \frac{\partial H}{\partial U_S}; \quad {}^{CF}_{0}D^{\alpha}_{t}\lambda_{A_S}(t) = \frac{\partial H}{\partial A_S}; \qquad (4.10)$$

$$ {}^{CF}_{0}D^{\alpha}_{t}\lambda_{V_S}(t) = \frac{\partial H}{\partial V_S}; \; {}^{CF}_{0}D^{\alpha}_{t}\lambda_{I}(t) = \frac{\partial H}{\partial I}; \; {}^{CF}_{0}D^{\alpha}_{t}\lambda_{R}(t) = \frac{\partial H}{\partial R}; $$

with, $\lambda_{U_S}(t_f) = \lambda_{A_S}(t_f) = \lambda_{V_S}(t_f) = \lambda_I(t_f) = \lambda_R(t_f) = 0$

The conditions for the optimality can be gotten after differentiating the Hamiltonian $H$with respect to u_1, u_2 andu_3:

$$ \frac{\partial H}{\partial u_1} = 0 ; \quad \frac{\partial H}{\partial u_2} = 0 \; ; \quad \frac{\partial H}{\partial u_3} = 0 \qquad (4.11)$$

The adjoint system (4.4) and (4.5) come from the solution of (3.1), and the optimal controls (4.7) can be gotten from (4.8). The optimal system is comprised of the controlled system (4.1) and its initial conditions, the system of adjoint (4.4), and conditions for transversality.

4.2. Numerical Simulation

The numerical simulations were done by Matlab 2021b software. Variable and parameter values are given as, $\pi = 1, \beta_1 = 0.0007, \beta_2 = 0.00007, \beta_3 = 0.000007, \mu = 0.02, \gamma = 0.2, \delta = 0.01, \theta = 0.002, \emptyset = 0.0012, \mathrm{p} = 0.001.$

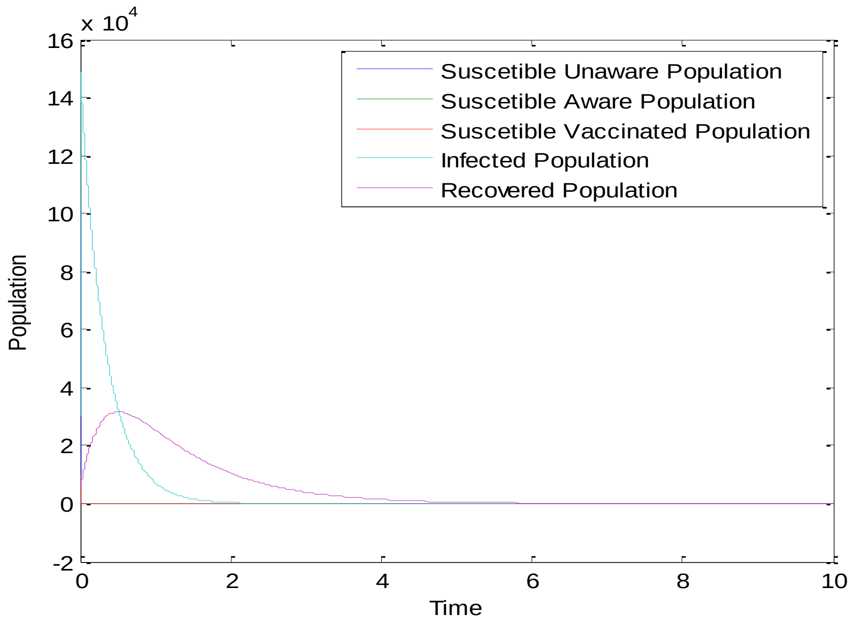

Figure 1. Dynamics of the model

Figure 1 depicts the dynamics of the model. It can be seen that without any control, the susceptible unaware population, susceptible aware population, and susceptible vaccinated populations all go to extinction, whereas infected and recovered populations proliferate. This clearly shows the need for the application of various control measures to control the pandemic.

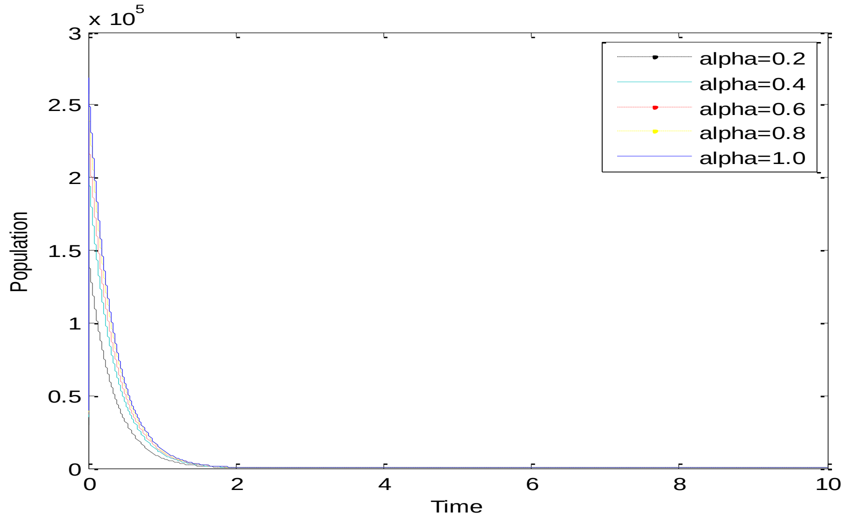

Figure 2. Dynamics of the infected population for various values of α

Figure 2 shows the variation of biological behavior of the infected population when fractional-order α is varied. It can be noticed that the population of the infected class was reduced when α the fraction order is decreased from 1 to 0.2. Hence, the memory effect can be seen clearly.

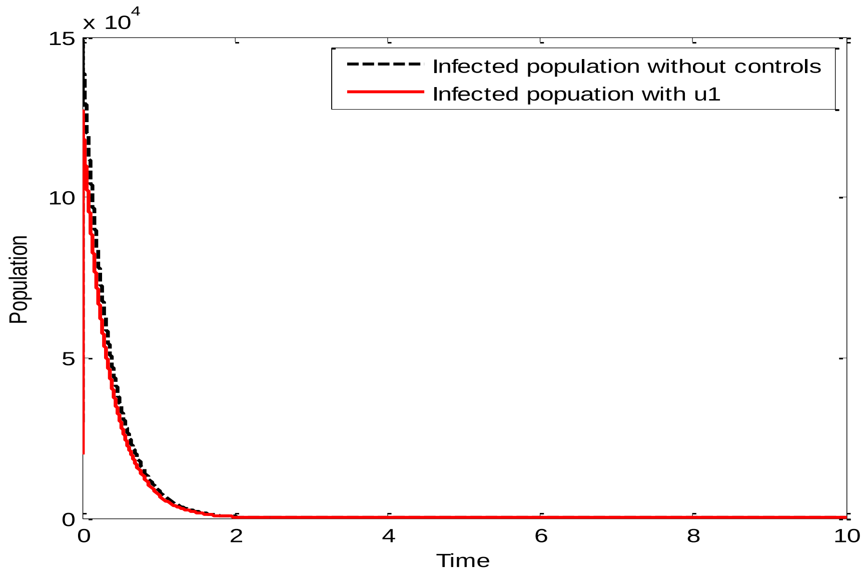

Figure 3. Comparing the dynamics of the infected population without control and with control u_1

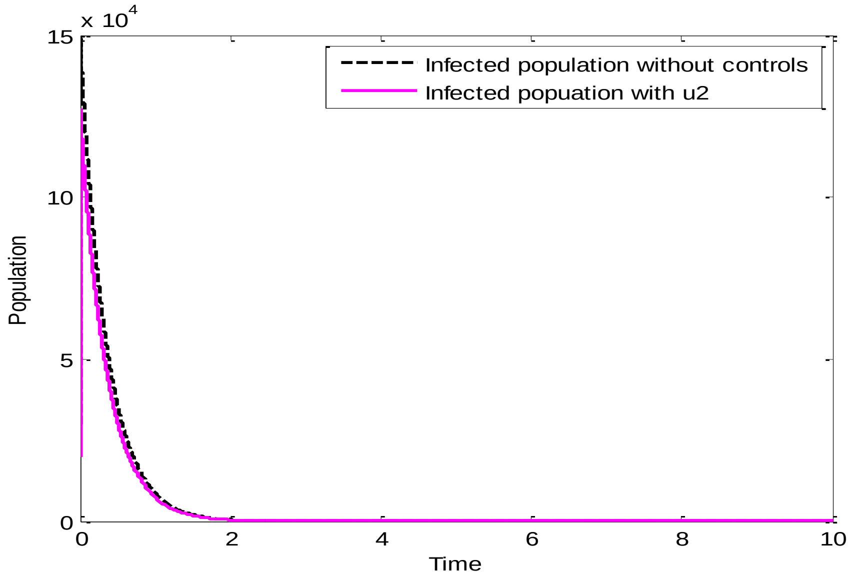

Figure 4. Comparing the dynamics of the infected population without control and with control u_2

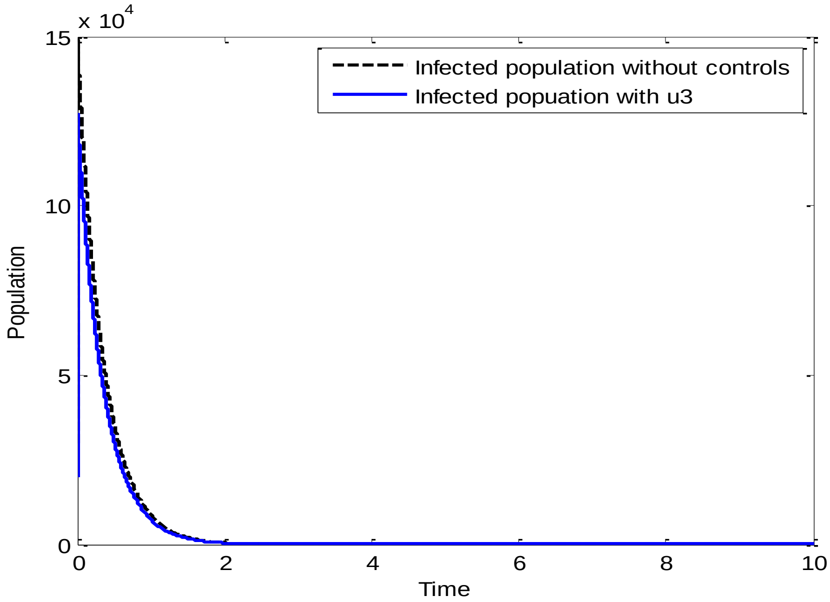

Figure 5. Comparing the dynamics of the infected population without control and with control u_3

Figures 3,4 and 5 compare the effect of controls $u_1, u_2 \& u_3$ respectively on the dynamics of the infected population. It is clear that when any control is observed, the population of the infected individuals is reduced. This is a positive effect and hence there is a need for compliance with the control measures.

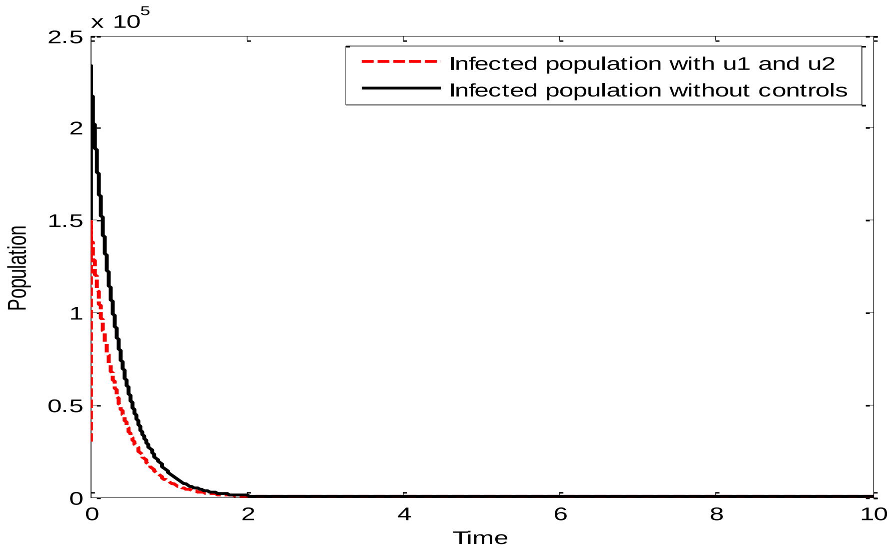

Figure 6. Comparing the dynamics of the infected population without control and with control u_1 &u_2

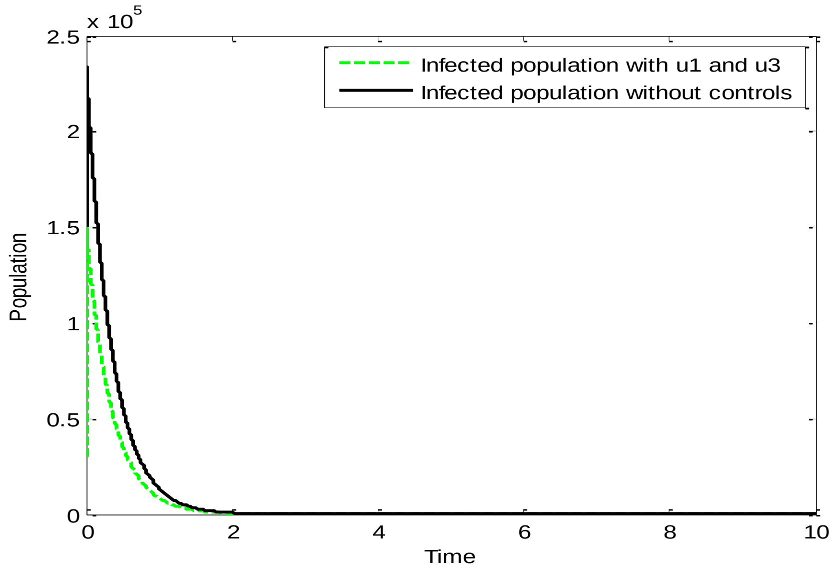

Figure 7. Comparing the dynamics of the infected population without control and with control u_1&u_3

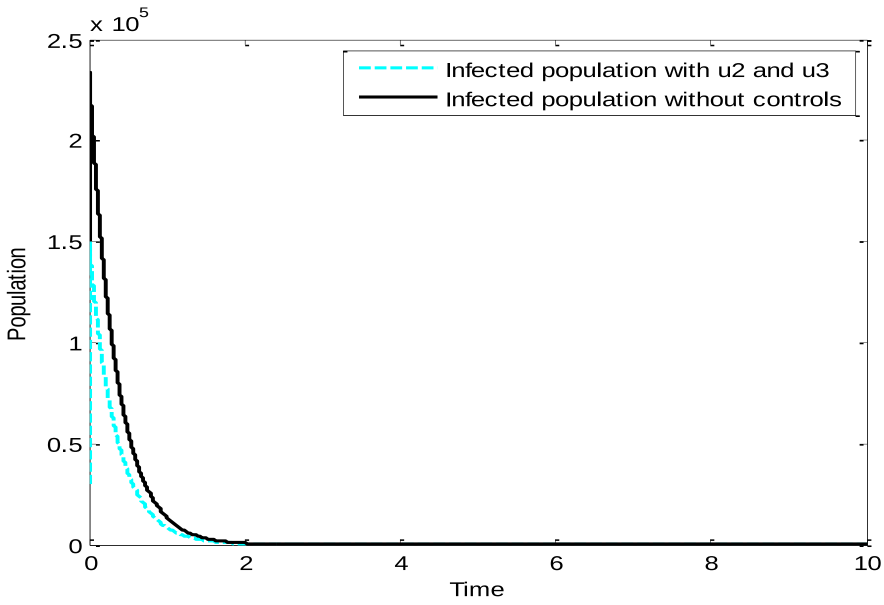

Figure 8. Comparing the dynamics of the infected population without control and with control $u_2 \& u_3$

Figures 6, 7, and 8 compare the effect of two controls, i.e. $u_1 \& u_2, u_1 \& u_3$, $and u_2 \& u_3$respectively on the dynamics of the infected population. It is clear that when two controls are applied the drastic change in the population of infected individuals is seen more than in the application of a single control. Hence to control the pandemic there is a need for the application of more than one control measure. However, the economic implication of combining and applying more than one control measure must be taken into consideration.

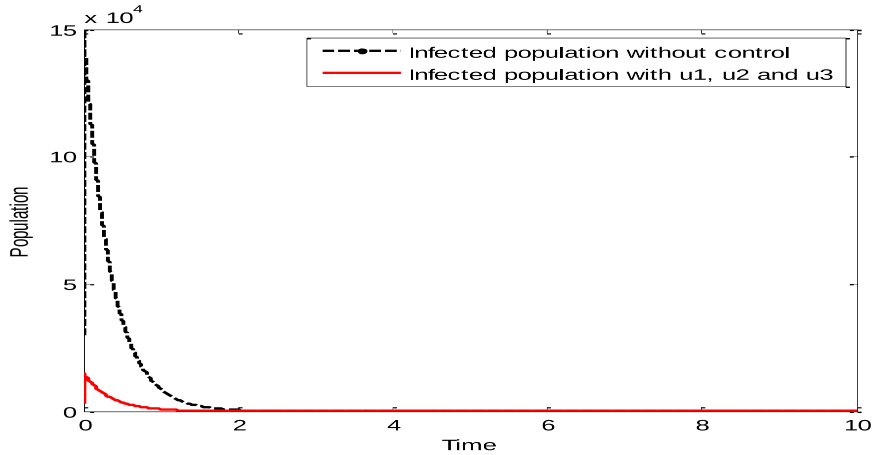

Figure 9. Comparing the dynamics of the infected population without control and with control $u_1, u_2 \& u_3$.

Figure 9 compares the effect of the three controls, i.e. u_1, $u_2 \& u_3$on the dynamics of the infected population. The application of all the control measures in the partitioned susceptible population leads to the desired outcome. The effect is seen clearly. Hence to obtain the desired result, there is a need for awareness, and not only vaccinating the susceptible population but making sure that full dosage is given.

These results show the significant impact of awareness about COVID – 19 and the vaccination process, other models investigate the optimal control of vaccination or the restriction measures applied to susceptible classes which do not reflect the social awareness about infections.

CHAPTER 5

Conclusion

In this thesis, Caputo – Fabrizio's sense is used to develop the fractional-order COVID -19 model, which consists of five compartments: susceptible unaware compartment, susceptible aware compartment, susceptible vaccinated compartment, infected compartment, and recovered compartment. Three types of susceptible classes are studied in this paper: a susceptible class of an unaware population with awareness control is detected, a susceptible class of an aware population with vaccine control is found, and a susceptible class of the vaccinated population with optimal vaccination control is found. The calculation of equilibrium points leads to the determination of the basic reproduction ratio. The model's properties of existence and uniqueness are confirmed. Also, the optimal control formula was developed and consequently analyzed the presence of an optimal solution was achieved. The biological significance of fractional order modeling is established by the use of numerical simulations, which are conducted. By utilizing a variety of control functions, it is evident that combining the three control methods has a significant impact on decreasing the number of infected individuals. This study approach incorporates both vaccination and awareness into consideration about the COVID-19 epidemic. For further studies, it's suggested to utilize the environmental conditions with the awareness of the susceptible class to see the impact of optimal control on such a model.

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APPENDIX 1

Ethical Approval Letter

APPENDIX 2

Similarity Report

APPENDIX 3

Curriculum Vitae

Personal Information

Surname, Name : HADI, Mohammed Subhi Hadi Hadi

Nationality : Iraqi

Date and Place of Birth : 31 October 1988, Mosul

Marital Status : Married

Education

Degree	Institution	Year of Graduation
M.Sc.	Newcastle University, Newcastle Upon Tyne, UK. Automation and control Engineering	2012
B.Sc.	University of Duhok, Duhok, Iraq, Electrical and Computer Engineering	2010

Work Experience

Year Enrollment	Place	
2011 – Present	Department of Electrical and Computer Engineering, UoD	Assistant Lecturer
2017 – 2018	Department of Electrical and Electronics Engineering, NEU	Assistant

Foreign Languages

Kurdish, Mother language.

English, spoken and written very good.

Arabic, spoken and written fluently.

Honors and Awards

- HCDP Scholarship, 2010

Publication In International Refereed Journals (In Coverage Of Ssci And Sci-Expanded):

- Hadi, M. S., & Bilgehan, B. (2022). Fractional COVID-19 Modeling and Analysis on Successive Optimal Control Policies. *Fractal and Fractional*, 6(10), 533. https://doi.org/10.3390/fractalfract6100533

Publication In International Refereed Journals:

- SAMANN, F. E., & SUBHI HADI, M. (2018). Human To Television Interface For Disabled People Based On EOG. *Journal of Duhok University*, *21*(1), 54-64. https://doi.org/10.26682/sjuod.2018.21.1.5

Bulletin Presented In International Academic Meetings And Published In Proceeding Books:

- Hadi, M. S., & Esmaili, P. (2019, October). Brain Computer Interface (BCI) For Controlling Path Planning Mobile Robots: A Review. In 2019 3rd International Symposium on Multidisciplinary Studies and Innovative Technologies (ISMSIT) (pp. 1-4). https://doi.org/10.1109/ISMSIT.2019.8932902

Courses Given (*from 2011 to 2022*)

Undergraduate:

- Mechatronic Components and Instruments.
- Control engineering.
- Computer Architecture.
- Operating Systems.
- Power Electronic.
- DC Machines and Transformer.
- Data Structure.
- Control Lab.
- Mechatronic Lab.
- Power Electronic Lab

HOBBIES

- Coin collecting, Swimming.