NEAR EAST UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

APPLICATION OF FLOQUET THEORY ON HILL'S EQUATION

M.Sc. THESIS

John Olugbenga EDUNJOBI

Nicosia
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February, 2023

APPROVAL

We certify that we have read the thesis submitted by John Olugbenga EDUNJOBI titled **"APPLICATION OF FLOQUET THEORY ON HILL'S EQUATION"** and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Mathematical Sciences.

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DECLARATION

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

John Olugbenga EDUNJOBI

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John Olugbenga EDUNJOBI

… to my mentor; Rev (Dr) Chibueze Amadi.

… to my lovely Mama; Pastor (Mrs.) IB Amadi.

ABSTRACT

Application OF Floquet Theory on Hill's Equation

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M Sc., Department of Mathematics

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As an effective method of solving linear systems with periodic coefficients and obtaining stability properties of periodic solutions of linear and nonlinear systems with periodic coefficients, the Floquet theory was presented, which relies on the computation of monodromy matrices for solving linear and nonlinear systems with periodic coefficients. Periodic coefficient type of differential equations were studied for their linear stability in floquet theory. A central concept in this theory are the Floquet exponents, which are similar to eigenvalues of variational matrices (Jacobians) at a state of equilibrium. Non-autonomous periodic differential equations can be analyzed using Floquet theory. The stability or non-stability of the Hill's equation under some cases were considered.

Key Words*: *floquet exponents, fundamental matrix, stability, eigen value, differential equation

Özet

Application OF Floquet Theory on Hill's Equation

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Periyodik katsayılı lineer sistemleri çözmenin ve periyodik katsayılı lineer ve lineer olmayan sistemlerin periyodik çözümlerinin stabilite özelliklerini elde etmenin etkili bir yöntemi olarak, periyodik katsayılı lineer ve lineer olmayan sistemleri çözmek için monodromi matrislerinin hesaplanmasına dayanan Floquet teorisi sunuldu. .

Periyodik katsayılı türdeki diferansiyel denklemler, floket teorisinde doğrusal kararlılıkları için incelenmiştir. Bu teorideki merkezi bir kavram, bir denge durumunda varyasyonel matrislerin (Jacobians) özdeğerlerine benzeyen Floquet üstelleridir.

Otonom olmayan periyodik diferansiyel denklemler, Floquet teorisi kullanılarak analiz edilebilir.

Bazı durumlarda Hill denkleminin kararlı olup olmadığı dikkate alınmıştır.

Anahtar Kelimeler: floquet üsleri, temel matris, kararlılık, özdeğer, diferansiyel denklem

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CHAPTER I

1.1 Introduction

A study of differential systems with periodic data is commonly used in physical and natural sciences (such as elasticity, astronomy, ecology) for studying multiple phenomena dynamics interactions (or interactions of multi-species). Thus, fundamental questions such as existence, uniqueness and stability arise when dealing with systems of periodic (ordinary) differential equations.

An important tool for the study and management of time-varying systems, the Floquet Theory is specifically examined in this thesis. Based on Floquet's (1883) pioneering work and Lyapunov's (1892) contribution, this theory is a systematic way of studying linear systems including periodic coefficients. Further, it provides a formula for resolving the resolution of a linear differential system with constant coefficients as well as a modification of the variables that result in a linear differential system with periodic coefficients.

The purpose of this thesis is to demonstrate the Floquet theory for assessing the stability of periodic results whether linear or nonlinear to differential systems.

We will now look at some definitions and generalities

Definition 1.1. An ***ordinary differential equation*** **(ODE)** is simply a differential equation that involves the derivatives of one or more functions of one variable as its unknown.

Definition 1.2. A ***linear differential equation*** is an equation defined by a linear polynomial in the unknown function and its derivatives, or one that has the following form:

$$a_0(p)q + a_1(p)q' + a_2(p)q'' + \cdots + a_n(p)q^n = b(p)$$

where $a_0(p)$,..., $a_n(p)$ and $b(p)$ are arbitrary differentiable functions that do not need to be linear, and $q, \dots, q^{(n)}$ are successive derivatives of the unknown function q of the variable p.

Definition 1.3. A ***periodic function*** is a function whose values repeat every so often. The trigonometric functions, which repeat every two radians, provide a suitable illustration. Oscillations, waves and other periodic phenomena are described by these functions across science. Any function that is not periodic is called ***aperiodic***.

Definition 1.4. In matrix algebra, a ***fundamental matrix*** of a system of *n* homogeneous linear ODE:

$$\dot{x} = A(t)x(t)$$

is a matrix-valued function $\psi(t)$ whose columns are systems of linearly independent solutions. We can write every solution to this system as $x(t) = \psi(t)c$, for some constant vector *c* (written as a column vector of height *n*).

Definition 1.5. A ***linear space*** (or vector space) consists a set of vectors V, a scalar field $\mathcal{F}$ and two operations $+$ and $*$ called vector addition and multiplication respectively and complies with the following conditions:.

i. $(x + y) + z = x + (y + z)\ \forall\ x, y, z \in V$.
ii. $x + 0 = x\ \forall\ 0 \in V$ and $x \in X$.
iii. For every $x \in X$, there is a vector in X written $-x$ and called the negative of x such that $x + (-x) = 0$.
iv. $(xy)z = x(yz) \forall\ x, y \in \mathcal{F}$ and $z \in X$.
v. $(a + b)p = ap + bp$ and $a(p + q) = ap + aq\ \forall\ a, b \in \mathcal{F}$ and $p, q \in X$.
vi. $1p = p\ \forall\ p \in X$.

Definition 1.6. A mapping $p(.) = \|.\|: X \to \mathbb{R}$ where X is a linear space over a field $\mathcal{F}$ and $\mathcal{F}$ holds either for $\mathbb{R}^n$ or $\mathbb{C}$ is said to be a ***norm*** if the following hold:

i) $\|p\| \geq 0\ \forall\ p \in X$ and $\|p\| = 0 \Leftrightarrow p = 0$

ii) $\|cp\| = |c|\|p\|\ \forall\ c \in \mathcal{F}, p \in X$

iii) $\|p + q\| \leq \|p\| + \|q\|\ \forall\ p, q \in X$

Definition 1.7. A linear space on a mapping $p(.) = \|.\|: X \to \mathbb{R}$ over a field $\mathcal{F}$ is a ***normed space*** provided the following conditions hold:

i) $\|ap\| = |a|\|p\|\ \forall\ a \in \mathcal{F}, p \in X$

ii) $\|p + q\| \leq \|p\| + \|q\|\ \forall\ p, q \in X$

A pair of a linear space X and a norm $\|.\|$ on X written as $(X, \|.\|)$ is referred to as ***normed linear space*** over field $\mathcal{F}$.

Example 1.8. The examples below define a norm on the vector space $\mathbb{R}^n$:

i) **Absolute norm**:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \forall\, y = (x, \ldots, x_n) \in \mathbb{R}^n$$

ii) **Euclidean norm**:

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, \forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

iii) **Maximum norm**:

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|,\ \forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Example 1.9. We will consider $X = C([0,1])$ as a vector space for all continuous real-valued functions on $[0,1]$. On the vector space $C[0,1]$, each of the following expressions defines a norm which is commonly used:

1) $\|f\|_1 = \int_0^1 |f(t)| dt$ for every $f \in C([0,1])$.
2) $\|f\|_2 = (\int_0^1 (|f(t)|)^{\frac{1}{2}} dt)^{\frac{1}{2}}$ for every $C([0,1])$.
3) $\|f\|_\infty = max\,\{|\, f(t)|: t \in [0,1]\}$.

Definition 1.10 (Equivalent norms). Let $\|.\|_1$and $\|.\|_2$ be two norms defined on the normed linear space X and α, β some constants. Then, $\|.\|_1$and $\|.\|_2$ are said to be equivalent if $\exists\ \alpha > 0$ and $\beta > 0$:

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1\ \forall p \in X$$

Definition 1.11. A metric d defined on $E \times E$ is canonically endowed with every normed linear space E by:

$$d(x, y) = \|x - y\|\ \forall\, x, y \in X$$

Definition 1.12 (Cauchy sequence). Let $(x_n)_{n\ge1}$ be a sequence of elements of a normed vector space X. $(x_n)_{n\ge1}$ is Cauchy if:

$$\lim_{n,m\to\infty} \|p_n - p_m\| = 0,$$

that is, for any $\varepsilon > 0$, $\exists$ an integer $N = N(\varepsilon)$ such that $\|p_n - p_m\| < \varepsilon$ whenever $n \ge N$ and m$\ge N$.

Remark 1.13. Every Cauchy sequence $(x_n)_{n\ge1}$ in a normed linear space is enclosed; i.e. $\exists\ M \ge 0$ (a constant) in such a way that $\|x_n\| \le M, \forall n \ge 1$.

Definition 1.14 (Convergent sequence). The sequence $(x_n)_{n\ge1}$ is said to be convergent if its elements in the normed vector space X converges to an element $x \in X$ where:

$$\lim_{n\to\infty} \|x_n - x_m\| = 0$$

Remark 1.15. Any convergent sequence is Cauchy in a normed linear space.

Definition 1.16. If every Cauchy sequence in a normed linear space $(X, \|.\|)$ converges, such a space is complete (i.e. has its limit in X). A complete normed linear space is called a ***Banach space***.

Example 1.17. The normed linear space $(C([0,1]); \|.\|_\infty)$ is a good example of a Banach space.

Definition 1.18. Let X be a normed linear space. The sets:

$$B_r(p) = \{p \in X: \|p\| < r\} \text{ and } \bar{B}_r(p) = \{p \in X: \|p\| \leq r\}$$

are referred to as open balls and closed balls respectively, with radius $r > 0$ and centered at a point $x \in X$. If $\forall\ x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$, a nonempty subset A of normed linear space X is said to be an ***open set***. Furthermore, if $X\backslash A$ is open, then A is a ***closed set***.

Proposition 1.19. Let X be a normed linear space. A subset A of X is closed if every convergent sequence $(a_n)_{n\geq 1}$ of elements of A has its limit in A.

Definition 1.20. For a subset A of a normed linear space X, the ***interior*** of A (written as $int(A)$) is the union of all open sets in A. The intersection of all closed sets in A is called the ***closure*** of A (written as $cl(A)$ or $\bar{A}$).

Theorem 1.21. Let A a normed linear space X and A a subset of X. Then

a) $p \in int(A)$ iff $\forall\ r > 0, B(p,r) \subseteq A$.

b) $p \in cl(A)$ iff $\forall\ r > 0, B(p,r) \cap A \neq \emptyset$.

Remark 1.22. In a given subset A and normed linear space X,

$$x \in \bar{A} \Leftrightarrow \exists (a_n)_n \subset A |\lim_{n\to\infty} a_n = x$$

Definition 1.23 (Neighbourhood). In a given normed linear space X, with $x \in X$ and V a subset of X containing x. V is called a neighbourhood of x if $\exists$ open set U of X containing x and enclosed in V. The collection of all neighbourhoods of x is denoted by $N(x)$.

Theorem 1.24: A normed linear space is finite dimensional iff its closed unit ball is compact, that is, every bounded sequence of the closed unit ball has a convergent subsequence. This is called Riesz/Heine-Borel theorem**.**

1.2 Linear operators

Given two normed linear spaces X and Y, over a field $\mathcal{F}$,

Definition 1.25 (Linear operators). A $\mathcal{F}$-linear operator T from X into Y is a map $T: X \to Y$:

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

$\forall\ \alpha, \beta \in \mathcal{F}$ and $x, y \in X$.

When $Y = \mathcal{F}$, the map is known as a ***linear functional***.

Proposition 1.26. A set of $\mathcal{F}$-linear operators from X into Y has a natural structure of linear space over $\mathcal{F}$ and can be written as $L(X, Y)$. $L(X, X)$ can simply be denoted as $L(X)$.

Proposition 1.27. If Z is a linear space, then

$$f \in L(X, Y) \text{ and } g \in L(Y, Z) \implies gof \in L(X, Z).$$

Theorem 1.28. Let $T \in L(X, Y)$. Then the following are equivalent:

i) At the origin, T is continuous if $\{x_n\}_n$is a sequence in X such tha $x_n \to 0$ as $n \to \infty$, then $T(x_n) \to 0$ in Y as $n \to \infty$.

ii) T is Lipschitz, that is, $\exists$ a constant $K \geq 0$ and for every $x \in X$,

$$\|T(x)\| \leq K\|x\|$$

iii) Image of $T(\bar{B}_1(0))$ is enclosed, $T(\bar{B}_1(0))$ being a closed unit ball.

Definition 1.29: If $\exists$ some $k \geq 0$ such that

$$\|T(x)\| \leq k\|x\| \forall\ x \in X,$$

Then the linear operator $T: X \to Y$ is **bounded**

If T is bounded, the norm of T is defined as:

$$\|T\| = inf\{k: \|T(x)\| \leq \|x\|, x \in X\}$$

We denote the set of linear bounded operators from X to Y as $B(X, Y)$. We simply write $B(X)$ if $X = Y$.

Proposition 1.30. Suppose $Y \neq \{0\}$ and $T \in B(Y)$, then:

$$\|T\| = \sup_{\|y\| \leq 1} \|T(y)\| = \sup_{\|y\| = 1} \|T(y)\| = \sup_{\|y\| \neq 0} \frac{\|T(y)\|}{y}$$

1.3 Matrix Concept and Basic Operator Theory

Definition 1.31. An $x \times y$ matrix A is a rectangular array of numbers that has m rows and n columns and $x, y \in \mathbb{R}$ or $\mathbb{C}$. Let a_{ij} denote numbers that appear in the row i and column j of A. A can be written in its extended form as:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x1} & a_{x2} & \cdots & a_{xy} \end{pmatrix}$$

or in more compact form as:

$$\left(a_{ij}\right)_{x,y}$$

Let's denote $A = \left(a_{ij}\right)_{x,y}$ and $1 \leq j \leq y, 1 \leq i \leq x$.

Definition 1.32. If the rows and columns of A are interchanged, a matrix A^t that is known as the transpose of A is created. Thus, if $A = \left(a_{ij}\right)_{x,y}$, then $A^t = \left(a_{ji}\right)_{y,x}$.

Definition 1.33. The trace of B denoted by $tr(B)$ is the sum of diagonal elements of matrix B i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Definition 1.34.

Two $m \times n$ matrices A and B are said to be **equal** if $a_{ij} = b_{ij}$ for each i and j

Let $D = \left(d_{ij}\right)_{m,n}$, $A = \left(a_{ij}\right)_{m,n}$ and $B = \left(b_{ij}\right)_{n.p}$ be matrices.

i) $$D + A = \left(d_{ij} + a_{ij}\right)_{m,n}$$

Similarly,

$$D - A = \left(d_{ij} - a_{ij}\right)_{m,n}$$

ii) Matrix A multiplied by a scalar α gives: $\alpha A = \left(\alpha a_{ij}\right)_{m,n}$

iii) An $m \times n$ matrix A multiplied by an $x \times y$ matrix B gives:

$$AB = \left(\sum_{k=1}^{x} a_{ik} b_{kj}\right)_{mp}$$

Some special matrices

i) A matrix with one (1) row is called a **row-vector** B or $1 \times m$ matrix i.e.

$$B = (b_1, b_2, \dots, b_n),$$

where the $a_i^1 s$ are scalars.

ii) A matrix with only one (1) column is called a **column-vector** B or $n \times 1$ matrix i.e.

$$B = \begin{pmatrix} b_1 \\ \vdots \\ B_n \end{pmatrix},$$

where the $a_i^1 s$ are scalars.

iii) A matrix whose entries are all zero (0) is called a ***zero matrix*** is. It also serves as the additive identity of n matrices. The zero $m \times n$ matrix is denoted by $0_{m,n}$ or simply as 0.

iv) A matrix whose number of rows and columns are equal is called a **square matrix** i.e. $(n \times n)$.

v) An **identity matrix** with order *n* has zeros everywhere else and one along its principal diagonal, which runs from top left to bottom right. It is denoted by $I_n = (\delta_{ij})$.

For a square matrix B, we have that

$$BI = IB = B$$

vi) A ***non-singular*** matrix is a square matrix whose column vectors are independent along linear axes or the determinant is not equal to zero, i.e. $det(B) \neq 0$.

If $det(B) = 0$, then B is said to be singular or degenerated.

vii) A ***diagonal matrix*** is a square matrix whose non-zero elements lie on the principal diagonal.

viii) A square matrix M in which $\exists\ k \in \mathbb{Z}^+$ such that $M^K = 0$ is referred to as ***nilpotent***.

Definition 1.35. Two $n \times n$ matrices *C* and *D* are said to be similar (denoted by $C \sim D$) if ∃ a nonsingular matrix *T* such that $T^{-1}CT = D$.

Definition 1.36. A scalar λ of a square matrix B is an eigenvalue of I if $\exists\ v \in \mathbb{R}^n$: $Bv = \lambda v$ $(v \neq 0)$, where v is a vector. The roots of the characteristic polynomial $p(\lambda) = det(A - \lambda I)$ with $p(\lambda)$ which has degree n are the eigenvalues λ of B.

Remark 1.38. When the matrix A contains repeated eigenvalues, diagonalization cannot be achieved. As a result, it is important to generalize the eigenvectors.

Definition 1.39. An eigenvector ranked k of B, is a nonzero vector *v* associated with an eigenvalue λ iff:

$$(B - \lambda I)^k v = 0 \; and \; (B - \lambda I)^{k-1} v \neq 0$$

Lemma 1.40. The vectors $v, (B-\lambda I)v, \dots, (B-\lambda I)^{k-1}v$ are linearly independent if v is a generalized eigenvector of rank k.

The vectors $v_1, v_2, \dots, v_k$ are linearly dependent if ∃ some scalars $c_1, c_2, \dots, c_k$ (not necessarily zero) and

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$

If the set of vectors $v_1, v_2, \dots, v_k$is not linearly dependent, then it is linearly independent.

Definition 1.41 (Jordan form). We build a new basis for $\mathbb{C}^n$ as a result of the aforementioned lemma such that the matrix representation of B with regard to that new basis is what we call the Jordan canonical form (J).

Theorem 1.42. For each $n \times n$ complex matrix B with eigenvalues $\lambda_1, \dots, \lambda_s$ of multiplicities $n_1, \dots, n_s$ respectively, ∃ a non-singular $n \times n$ matrix P such that

$$P^{-1}BP = J = diag(J_1, \dots, J_s),$$

where each of block matrices $J_1, \dots, J_s$ has the form:

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \cdots & \dots & \ddots & \dots & \vdots \\ 0 & \cdots & \dots & \lambda_k & 1 \\ 0 & \cdots & \dots & 0 & \lambda_k \end{pmatrix},$$

$k = 1, \dots s$ and $\sum_{k=1}^{s} n_k = n$.

The Jordan blocks, where J is the Jordan canonical form of A, are represented by the block matrices $J_1, \dots, J_s$. It is noteworthy that any Jordan block $J_k(\lambda)$ may be expressed as $J_k = \lambda_k I + N_k$, where N_k is a nilpotent of order k and A is similar to J i.e. $A \sim J$.

1.4 Limits on operator of sequences

Limits in the norm and strong limits of operator sequences are introduced in this section. As part of our discussion of series of operators, we shall introduce derivatives and integrals of operators that depend on a criterion.

Definition 1.43 (Convergence). In a Banach space X and sequence (A_n) of operators in $\mathcal{L}(X)$, A_n is said to converge in norm to an operator $A \in \mathcal{L}(X)$ if

$$\lim_{n\to\infty} \|A_n - A\| = 0 \qquad (1.1)$$

If, for each element $x \in X$,

$$\lim_{n\to\infty}\|A_n x - Ax\| = 0, \quad (1.2)$$

then A_n strongly converges to $A \in \mathcal{L}(X)$.

Hence, (1.1) ⇒ (1.2), but in general, the converse is not true.

However, it is true for finite dimensional Banach spaces, if we put $x = e^{(i)}$.

From (1.2), we have that:

$$\lim_{n\to\infty}\left\|a_{i1}^{(1)}, \dots, a_{ik}^{(k)}\right\| = 0 \ \ for\ i = 1, \dots, k.$$

The limit is uniform since there are only a finite number of components, and as a result, all components tend to zero.

Now, we may define the series $\sum_{s=1}^{\infty} A_s$.

The series $\sum_{s=1}^{\infty} A_s$ is said to be convergent if the partial sums $\sum_{s=1}^{N} A_s$ form a sequence that converges in $\mathcal{L}(X)$.

Matrix $\sum_{s=1}^{N} A_s$ in this case corresponds to a matrix whose elements are $\sum_{s=1}^{N} a_{ij}^{(s)} = a_{ij}$.

If the series $\sum_{s=1}^{\infty} A_s$ converges, then we say that $\sum_{s=1}^{\infty} A_s$ converges absolutely.

This occurs in the case of matrices iff $\sum_{s=1}^{\infty}\left|a_{ij}{}^{(s)}\right|$converges with $A_s = \left(a_{ij}^{(s)}\right)$.

Proposition 1.44. In a normed linear space X, the series $\sum_0^{\infty}\frac{A^n}{n!}$ is absolutely convergent if $A \in \mathcal{L}(X)$,.

Proof. It is sufficient if we can prove that the partial sums $\{S_N\}_{N=1}^{\infty}$ for $\sum_0^{\infty}\frac{\|A^n\|}{n!}$ is a Cauchy sequence.

We will define $\sum_0^{\infty}\frac{A^n}{n!}$ and note that a Cauchy sequence is formed by the partial sums of the convergent series of real number $\sum_0^{\infty}\frac{\|A^n\|}{n!} = e^{\|A\|}$.

This fact implies that S_N is a Cauchy sequence in $\mathcal{L}(X)$, hence converges absolutely.

Definition 1.45. An exponential map is the map defined as:

$$exp\colon g \to G,$$

G being a normed linear space and $g \in G$.

In series form, we can define the exponential map as:

$$e^A = \sum_0^{\infty}\frac{A^n}{n!}$$

The next proposition gives the main properties of the exponential map:

Proposition 1.46. Suppose $A, B \in \mathcal{L}(\mathbb{R}^n)$.

i) $e^{-1} = (e^A)^{-1}$.

ii) $\|e^A\| \leq e^{\|A\|}$.

iii) $e^A \in \mathcal{L}(\mathbb{R}^n)$ if $A \in \mathcal{L}(\mathbb{R}^n)$.

iv) $B^{-1}e^A B = e^{B^{-1}AB}$ if b is nonsingular.

Definition 1.47. Let A be an operator that depends on a real parameter t with $t_0 \in [a, b]$ and $a \leq t \leq b$. The operator

$$\frac{A(t_0 + h) - A(t_0)}{h}$$

is defined if h is sufficiently small. A is ***differentiable*** at t with respect to t if its limit as $h \to 0$ exists.

Definition 1.48. The symbol for the limiting operator is $\frac{d}{dt}A(t)$. Thus, we have:

$$\lim_{h \to 0} \left\| \frac{A(t_0 + h) - A(t_0)}{h} - \frac{dA}{dt} \right\| = 0$$

If each component a_{ij} is differentiable, the limit exists for all matrices, so we have

$$\frac{d}{dt}A(t) = \left(\frac{d}{dt}a_{ij}(t)\right)$$

If A and B are two differentiable operators in $\mathcal{L}(X)$, then

$$\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}$$

Definition 1.49. The series of operators converges uniformly if for every $\mathcal{E} > 0$, $\exists$ a $v_{\mathcal{E}} > 0$ such that $\forall\ t \in [a, b]$ and every $v > v_\varepsilon$,

$$\left\| \sum_{s=v}^{\infty} A_s(t) \right\| < \varepsilon$$

Definition 1.50. If the operator $A_s(t)$ is differentiable and $\sum_{s=0}^{\infty} \frac{d}{dt} A_s(t)$ is uniformly convergent, then the operator $A = \sum_{s=v}^{\infty} A_s(t)$ is differentiable, and

$$\frac{dA}{dt} = \sum_{s=0}^{\infty} \frac{d}{dt} A_s$$

The following result serves as an illustration of this notion:

Definition 1.51. If $A \in \mathcal{L}(X)$, then $tA \in \mathcal{L}(X)$ if $\forall\ t \in \mathbb{R}$, $t \longmapsto e^{tA}$ is differentiable and

$$\frac{d}{dt}(e^{tA}) = Ae^{tA}$$

If A is a nonsingular matrix, then the logarithm of A, denoted by $ln(A)$ is a well-defined matrix.

Definition 1.52. Let A be an $n \times n$ matrix that is nonsingular. Then $\exists$ an $n \times n$ matrix B (called the ***logarithm*** of A) such that $A = e^B$.

Definition 1.53. Suppose that the component a_{ij} of the matrix $A(t)$, which depends on the parameter t are all integrable functions over the interval $[t_0, t]$. The matrix

$$\int_{t_0}^{t} A(T)dT$$

is called the ***integral*** of $A(t)$ between t_0 and t.

Definition 1.54. If the functions a_{ij} are continuous in the definition,

$$\frac{d}{dt}\left(a_{ij}\right)_{m,n} = \left(\frac{da_{ij}}{dt}\right)_{m,n},$$

then

$$\frac{d}{dt}\int A(T)dT = A(t).$$

Definition 1.55. If A is an $n \times n$ matrix and $L \geq \left|a_{ij}\right|$, then

$$\left\|\int_{t_0}^{t} A(T)dT\right\| \leq \sqrt{m.n}.L|t - t_0|$$

1.5 Calculus in review

Here, we want to define differentiability in its broadest sense.

Let U be a nonempty open subset of Banach spaces X and Y and $\|.\|$ be the norm in both Banach spaces.

Definition 1.56. A function $f: U \to Y$ is said to be differentiable at $x \in U$ if there exists a linear map $A \in \mathcal{L}(X, Y)$ such that:

$$\lim_{a \to 0} \frac{\|f(x + a) + f(x) - Aa\|}{\|a\|} = 0$$

Remark 1.57. This kind of map is unique if it exists and can be written as $A = f^{I}(x)$, which is referred to as the derivative of f at x. Df and f_x are other common notations for the derivative.

Here are some general derivatives facts.

We will look at some standard facts about derivatives and the symbols X, Y, X_i and Y_i will denote Banach spaces.

i) If $f: X \to Y$ is differentiable at $a \in X$, then f is also continuous at a.

ii) If $f: X \to Y_1 \times \ldots \times Y_n$ is given by $f(x) = (f_1(x), \ldots, f_n(x))$, and if f_i is differentiable for each i, then so is f and $Df(x) = (Df_1(x), \ldots, Df_n(x))$.

iii) If the function $f: X_1 \times X_2 \times \ldots \times X_n \to Y$ is given by $x_1, x_2, \ldots, xz_n) \longmapsto f(x_1, x_2, \ldots, x_n)$, the ith partial derivative of f at $(a_1, a_2, \ldots, a_n) \in X_1 \times X_2 \times \ldots \times X_n$ is the derivative of the function $g: X_j \to Y$ defined by $g(x_j) = f(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$. This derivative is denoted by $D_j f(a)$. All partial derivatives of f exist if f is differentiable.. If $h = (h_1, h_2, \ldots, h_n)$, then:

$$Df(x)h = \sum_{j=1}^{n} D_i f(x)\, h_j$$

f is said to be continuously differentiable in U if all of its partial derivatives exist and are continuous in the open set $U \subset X_1 \times X_2 \times \ldots \times X_n$, but the converse is not true.

Theorem 1.50. Assume $[a, b]$ is a closed interval, and $f: [a, b] \to Y$ is a continuous function. If f is a differentiable function on the open interval (a, b) and $\exists$ some number $M > 0$ such that $\|f'(t)\| \leq M \ \forall\ t \in (a, b)$, then

$$\|f(b) - f(a)\| \leq M(b - a)$$

Theorem 1.51 (Mean value Theorem). Suppose that $f: X \to Y$ is differentiable on an open set $U \subseteq X$ with $a, b \in U$ and $a + t(b - a) \in U$ for $0 \leq t \leq 1$. If $\exists$ some $M > 0$ such that:

$$\sup_{0 \leq t \leq 1} \|Df(a + t(b - a))\| \leq M,$$

then

$$\|f(b) - f(a)\| \leq M\|b - a\|$$

1.6 Integration in Banach Space

Definition 1.52. Sequel to definition 1.16, a Banach space $(X, \|.\|)$ is a normed linear space (over $\mathbb{R}$ or $\mathbb{C}$) that is complete wrt $d(x,y) = \|x - y\|$.

Let X be a Banach space, $I = [a,b] \subset \mathbb{R}$ and $a < b$.

Definition 1.53. The following formula defines the integral in the sense of Riemann if $f: I \to X$ is continuous on I:

$$\int_a^b f(t)dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k\frac{b-a}{n}\right)$$

and we know that:

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\| dt.$$

Theorem 1.54. Suppose U is an open, nonempty subset of X. If $x + ty \in U$ for $0 \leq t \leq 1$ and $f: X \to Y$ is a differentiable function, then:

$$f(x+y) - f(x) = \int_0^1 Df(x+ty)y dt$$

CHAPTER II

Basic notions of ordinary differential equations

What is an ordinary differential equation solution, for example? and other fundamental questions are covered at the beginning of this chapter. Do differential equations have solutions all the time? Are differential equation solutions unique? The second strategy involves studying ordinary differential equations (ODEs) qualitatively and considering the notion of stability.

2.1 General concept of ODE

Let $f: I \times \mu \to \mathbb{R}^n$ be a vector-valued function defined by $(t, x) \longmapsto f(t, x)$ and let $I \subseteq R$ and $\mu \subseteq \mathbb{R}^n$ be nonempty open sets.

We will consider an ODE system of the following first order:

$$x' = f(t, x) \tag{2.1}$$

where $f = (f_1, \dots, f_n)$ is a vector-valued function as defined above, the unknown function $x = (x_1, \dots, x_n)$ is a vector, and the prime implies differentiation wrt t (an independent variable which is usually a measure of time). It should be noted that (2.1) can be rewritten as:

$$x' = f(x), \text{ where } x \in \mu$$

This differential equation is known as an autonomous differential equation.

If not, (2.1) is referred to as a non-autonomous differential equation.

Let $I \subseteq R$ and $\mu \subseteq \mathbb{R}^n$ be nonempty open sets and let $f: I \times \mu \to \mathbb{R}^n$ be a vector-valued function defined by $(t, x) \longmapsto f(t, x)$.

Note that if a vector-valued function f does not depend explicitly on the independent variable t, then the corresponding differential equation is called autonomous differential equation and (2.1) can be rewritten as:

$$x' = f(x), x \in \mu$$

If not, (2.1) is referred to as a non-autonomous differential equation.

An IVP associated to (2.1) for (t_0, x_0) is given by the differential equation together with an IVP as follows:

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

Example 2.1. In the IVP

$$\begin{cases} x^i = f(t,x) \\ x(t_0) = x_0 \end{cases}, \tag{2.3}$$

the vector valued function f is defined by $f(t,x) = -x + e^{-t}$ in $\mathbb{R} \times \mathbb{R}$, the point $(0,1)$ corresponds to the IVP $x(0) = 1$.

Example 2.2. The IVP

$$x' = Ax \quad x(0) = (1,0)^t \tag{2.4}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $x = (x_1, x_2)^t \in \mathbb{R}^2$

Definition 2.3. We refer to the vector-valued function $x: I_0 \to \mu$ defined on some non-empty open subinterval I_0 of I as the ***local solution*** of the differential equation (2.1) and such that:

(i) $(t, f(t, x(t)) \in I \; \forall \; t \in I_0$,

(ii) $x'(t) = f(t, x(t)) \; \forall \; t \in I_0$.

Definition 2.4. Let x be a solution function of $x' = f(t,x)$ on an interval (α, β) where $\alpha < \beta$. (α, β) is a ***maximal domain*** of x if $\nexists$ any extension of x over any of the intervals $(\alpha - \epsilon, \beta)$ or $(\alpha, \beta + \epsilon)$ with $\epsilon > 0$, such that x is still a solution of $x' = f(t,x)$.

Remark 2.5. If $t_0 \in I_0$ satisfies the IVP $x(t_0) = x_0$ and x is a solution of the differential equation above, then x is a solution of the initial condition (2.2).

Example 2.6. The initial condition (2.3) above ha $x(t) = (1 + t)e^{-t}$ as solution, and valid $\forall \; t \in \mathbb{R}$. $\mathbb{R}$ is thus the maximum interval of existence.

Example 2.7. The scalar differential equation $x' = x^2$ has the solution

$$x(t) = \frac{1}{1-t}$$

defined on $I = (-1,1)$. That solution is considered continuous to the left to $-\infty$, but not continuous to the right. In this case, the interval I is not the maximal interval of existence of the solution.

Proposition 2.8. An nth order system:

$$x^{(n)} = F(t, x, x', \dots, x^{(n-1)}) \tag{2.5}$$

where $x: \mathbb{R} \to \mathbb{R}$ is an unknown function, $x^{(n)} = \frac{d^n x}{dt^n}$ and F is defined on $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ may also be treated as a type of the system (2.1).

Proof. If we set

$$\frac{d^k x}{dt^k} = y_{k+1}$$

for $0 \leq k \leq n-1$, in which case

$$y'_k = y_{k+1}, 0 \leq k \leq n-1$$
$$y^I_n = F(t, y_1, \dots, . y_n)$$

i.e. if $y = (y_1, y_2, \dots, y_n)$ and

$$f(t,y) = (y_2, \dots, y_n, F(t, y_1, \dots, y_n)),$$

we have

$$y' = f(t,y)$$

where f is considered as a vector-valued function defined on $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$, that is a system of first order and has the form (2.3) as required.

Example 2.9. If we set $x_1 = x$ and $x_2 = x_1$, the equation $x'' + x = b(t)$ can be written as the system $x'_1 = x_2$, $x'_2 = -x_1 + b(t)$ or equivalent in the form (2.1) with $f(t,x) = Ax + B(t)$ where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ b(t) \end{pmatrix}$$

2.2 General existence and uniqueness theorem of system solutions

Let $f: G \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function.

Definition 2.10. We say f is ***continuous*** if every component of f is continuous in G.

Definition 2.11. f is considered to be ***uniformly Lipschitz*** on G wrt x if $\forall\ t \in I$, $\exists$ a constant $K > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| \tag{2.6}$$

$\forall\ (t,x), (t,y) \in G$.

The constant K is called a Lipschitz constant for f on G.

For example, the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(t,x) = t - x$ satisfies the Lipschitz condition in $\mathbb{R}^2$ with $K = 1$.

Definition 2.12. A vector-valued function f defined on $G \subseteq \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ is ***locally Lipschitz*** wrt x if $\forall\ (t, x) \in G$, $\exists$ a neighborhood of (t, x) for which f is Lipschitz.

Remark 2.13. If $\partial f_i / \partial x_j\ (i, j: 1, \dots, n)$ exist and these partial derivatives are continuous on some region $G \subseteq \mathbb{R} \times \mathbb{R}^n$, then f satisfies a condition that is Lipschitz in G.

Theorem 2.14. Let the function f be continuous vector-valued and defined thus:

$$U = \{(t, x) : |t - t_0| \leq a \text{ and } \|x - x_0\| \leq r\} \text{ where } a > 0,\ r > 0;$$

Suppose f satisfies the Lipschitz condition on U and there is a constant $M > 0$ such that:

$$\|f(t, x)\| \leq M\ \forall\ (t, x) \in U.$$

then there is a unique solution on (2.4) on the interval

$$I_\alpha = \{t: |t - t_0| \leq \alpha\} \text{ where } \alpha = min\left\{\alpha, \frac{r}{M}, \frac{1}{K+1}\right\}.$$

This is called the **Local Existence Theorem**.

A good knowledge of the following lemma and theorems will be needed for the proof of the local existence theorem:

Lemma 2.15. A function $\emptyset$ is an IVP solution on the interval I iff it is an integral equation solution:

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)\text{ds},\ \forall\ t \in I \qquad (2.7)$$

Proof: Assume that $\emptyset$ is an IVP solution on I, then $\emptyset$ is differentiable on I and

$$\emptyset^I(t) = f(t, \emptyset(t)$$

If we take the integral from t_0 to any *t* in *I*, we get:

$$\emptyset(t) - \emptyset(t_0) = \int_{t_0}^{t} f\big(s, \emptyset(s)\big)\text{ds}$$

Using the initial condition $\emptyset(t_0) = x_0$, we see that $\emptyset$ satisfies (2.7)

By applying the fundamental theorem of calculus to (2.7), we get that $\emptyset$ satisfies

$$\emptyset^I(t) = f(t, \emptyset(t)) \qquad (t \in I)$$

and putting $t = t_0$ in (2.7), we have $\emptyset(t_0) = x_0$. This completes the proof.

Theorem 2.16 (Contraction mapping principle). If (X, d) is a complete metric space and $T: X \to X$ a contraction mapping with $k \in [0,1)$ a contraction constant, then $\bar{x} \in$

X is a unique fixed point on T. More so, for arbitrary $a \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by:

$$\begin{cases} x_0 = a \\ x_n = T(x_{n-1}), for\ n \geq 1 \end{cases}$$

converges to $\bar{x}$.

The contraction mapping concept is generalized by the Picard's iteration theorem. The iteration method is extremely helpful for solving ordinary differential equations even though the appropriate integral operator is not a contraction.

Theorem 2.17. Let (X, d) be a complete metric space and $T: X \to X$ a continuous map. Let $x_0 \in X$ and define $x_{n+1} = T(x_n)$ by induction.

If $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$, the sequence $(x_n)_{n\geq 0}$ converges to a fixed point $\bar{x}$ of T and

$$d(\bar{x}, x_n) \leq \sum_{k=n}^{\infty} d(x_k, x_{k+1})$$

This is called the ***Picard iteration***.

With the above lemma and theorems, we can now prove the local existence theorem.

Local existence theorem proof.

It is sufficient to prove from lemma (2.15) the existence and uniqueness of solution of (2.7) in I_0.

Let $X = C(I_0)$, $S = \{x: x \in X\ and\ \|x - x_0\| \leq r\}$.

S is a closed subset of X and $(X, \|.\|)$ is complete. Hence, S is complete ($\|.\|$ in this case is the supermom norm on X).

We will consider the map:

$$T: S \to S$$

$$x \longmapsto Tx$$

defined by

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)ds \qquad \forall\ t \in I_\alpha$$

We need to show that T is well defined.

Indeed, if $x \in S$, then $\|Tx - x_0\| = \sup_{t\in I_\alpha} \left\| \int_{t_0}^{t} f\big(s, x(s)\big)\mathrm{ds} \right\|$

$\forall\ t \in I_\alpha$, it implies that:

$$\left\| \int_{t_0}^{t} f(s, x(s)) \mathrm{ds} \right\| \leq \left| \int_{t_0}^{t} \| f(s, x(s)) \| \mathrm{ds} \right|$$

$$\leq M|t_0 - t|$$

$$\leq M\alpha$$

$$\leq r$$

Therefore, $\|Tx - x_0\| \leq r$.

Claim. *T* is a strict contraction on *S*.

For every $x \in X, y \in X$ and $t \in I_\alpha$, one has

$$\|Tx - Ty\| = \left\| \int_{t_0}^{t} f(s, x(s)) \mathrm{ds} - \int_{t_0}^{t} f(s, y(s)) \mathrm{ds} \right\|$$

$$\leq \int_{t_0}^{t} \| f(s, x(s)) - f(s, y(s)) \| \mathrm{ds}$$

$$\leq K\|x - y\||t_0 - t|$$

$$\leq K\alpha\|x - y\|$$

Since $\alpha \leq \frac{1}{K+1}$, therefore $\lambda = K\alpha < 1$.

T is thus a strict contraction on *S* and has a unique fixed point (assume x) by the contraction mapping theorem. Furthermore, this point is the unique solution of equation (2.6).

Lemma 2.18. Let $f: D \to \mathbb{R}^n$ be continuous in some region *D* of $\mathbb{R}^{n+1}$ and assume $\exists\ M > 0$ such that $\|f(t, x)\| \leq M\ \forall\ (t, x) \in D$. Let *x* be a solution of IVP (2.4) that exists on a finite interval $J = (\alpha, \beta)$. Then $\lim_{t \to \alpha^+} x(t)$ and $\lim_{t \to \beta^-} x(t)$ exist and are finite.

Simply put, *x* can by extension be continuous to α and β.

Proof. Let's define t_1 and t_2 as two points on the interval J such that $t_1 < t_2$. Then, since *x* satisfies the integral equation (2.7),

$$x(t_1) = x_0 + \int_{t_0}^{t_1} f(s, x(s)) ds \qquad (2.8)$$

$$x(t_2) = x_0 + \int_{t_0}^{t_2} f(s, x(s)) ds \qquad (2.9)$$

Subtraction equation (2.9) from (2.8) gives:

$$x(t_1) - x(t_2) = \int_{t_1}^{t_2} f(s, x(s)) ds$$

and the assumption $\|f(t, x(t))\| \leq M$ for $(t, x) \in D$ now gives

$$\|x(t_1) - x(t_2)\| \leq M|t_1 - t_2|$$

The Cauchy convergence criteria demonstrates that $x(t)$ tends to a limit as t tends to β from below since the right side tends to zero as t_1 and t_2 both tend to β from below. By allowing t_1 and t_2 tend to α from above, we may analogously demonstrate that $x(t)$ tends to a limit as t tends to α from above.

Let us now define $x_\alpha = \lim_{t\to\alpha^+} x(t)$ and $x_\beta = \lim_{t\to\beta^-} x(t)$ and we have a solution x defined on the closed interval $[\alpha, \beta]$.

Theorem 2.19. Let $\alpha < \beta \in \mathbb{R}$, and $f: D \to \mathbb{R}^n$ be continuous and suppose $\exists\ M > 0$ such that $\|f(t,x)\| \leq M$ for all $(t,x) \in D$ and $\big(\alpha, x(\alpha)\big), (\beta, x(\beta) \in D$.

Then the solution x of (2.4) in the interval (α, β) can be extended to

$(\alpha, \beta + \varepsilon)$ or $(\alpha - \varepsilon, \beta)$ with $\varepsilon > 0$.

Proof. From lemma (2.18) above, $x_\alpha = \lim_{t\to\alpha^+} x(t)$ and $x_\beta = \lim_{t\to\beta^-} x(t)$ exist.

We can define function u as follows:

$$u(t) = \begin{cases} x(t) \text{ if } t \in (\alpha, \beta) \\ x_\beta \text{ if } t = \beta \end{cases}$$

Then $\forall\ t \in (\alpha, \beta]$, we have

$$u(t) = u(\beta) + \int_\beta^t f\big(s, u(s)\big) ds$$

$$= x_0 + \int_{t_0}^\beta f\big(s, u(s)\big) ds + \int_\beta^t f\big(s, u(s)\big) ds$$

$$= x_0 + \int_{t_0}^t f\big(s, x(s)\big) ds,$$

the left-hand derivative $u_l'(\beta)$ exists and $u_l'(\beta) = f(\beta, u(\beta))$. u is thus an extension of x in $(\alpha, \beta]$.

Let $v = v(t)$ be a solution of the IVP:

$$\begin{cases} x' = f(t,x) \\ x(\beta) = u(\beta), \end{cases}$$

Then for some $\varepsilon > 0$, v exists in the interval $[\beta, \beta + \varepsilon)$.

If we define the function $w(t)$ thus,

$$w(t) = \begin{cases} u(t) \text{ if } t \in (\alpha, \beta) \\ v(t) \text{ if } \mathrm{t} \in [\beta, \beta + \varepsilon) \end{cases}$$

then *w* is continuous and differentiable in $(\alpha, \beta) \cup (\beta, \beta+\varepsilon)$. Also, w'_β and $w'_r(\beta)$ both exist and are equal.

So, $w'_l(\beta) = u'(\beta) = f(t, u(\beta))$ and $w'_r(\beta) = v'(\beta) = f\big(t, v(\beta)\big) = f\big(t, u(\beta)\big)$.

Hence, *w* is a solution of IVP (2.4) in the interval $(\alpha, \beta - \varepsilon)$, and an extension of *x* in the interval $(\alpha, \beta - \varepsilon)$.

It can be shown that x can also be extended to $(\alpha - \varepsilon, \beta)$ if we follow the same argument, which completes the proof.

We next give the local existence theorem without Lipschitz condition in a generalized

Theorem 2.20. Let *f* be a continuous vector-valued function defined on

$$\triangle = \{(t, x) \colon |t - t_0| \leq a \text{ and } \|x\| < \infty\}\ a > 0$$

and assume that f satisfies a Lipschitz condition on the operator $\triangle$ and there exists a constant $M > 0$ such that

$$\|f(t, x)\| \leq M\ \forall (t, x) \in \triangle$$

Then (2.4) has a unique solution in the entire interval $[t_0 - a, t_0 + a]$.

Proof. From the local existence theorem, $\exists$ a unique solution *x* of (2.4) and $\alpha > 0$ that exist on $I = [t_0 - \alpha, t_0 + \alpha]$.

Assuming I to be the maximum interval of existence of solution, we shall set $t_1 = t_0 - \alpha$ and $t_2 = t_0 + \alpha$.

By lemma (2.18), $x(t_1) = \lim_{t \to t_1^+} x(t)$ and $x(t_2) = \lim_{t \to t_2^-} x(t)$ exist.

We claim that $t_2 = t_0 + a, t_1 = t_0 + a$.

By contradiction, suppose $t_2 < t_0 + a$, since $\big(t_1, x(t_1)\big) \in \triangle$, by theorem (2.19), we can extend the IVP solution of x over $(t_1, t_2 + \varepsilon)$ for some $\varepsilon > 0$ such that $t_2 < t_2 + \varepsilon < t_0 + a$. Since I_α is the maximal interval of existence of the solution, this is a contradiction.

So, $t_2 = t_0 + a$.

Similarly, we show that $t_1 = t_0 + a$ so that $I = [t_0 - a, t_0 + a]$.

Hence, the proof.

Theorem 2.21. Let the region $D \in \mathbb{R}^{n+1}$ be a continuous function $f \colon D \to \mathbb{R}^n$. Suppose $M > 0$ exists such that $\|f(t, x)\| \leq M\ \forall\ (t, x) \in D$ and *f* is uniformly Lipschitz on D, then the unique solution x of (2.4) can be extended up to the point where the graph of D meets the boundary.

Proof. Let's assume that x cannot be extended up to the boundary of D, but can be extended to the right only to $[t_0, s)$.

$x(s) = \lim_{t \to s^-} x(t)$ exists by lemma (2.18).

The proof is complete if $(s, x(s))$ is a boundary point of D.

If $(s, x(s))$ is not a boundary point of D, then there exists a box centered at $(s, x(s))$ that lies in D. However, by the method in theorem (2.19), we may now extend the solution x to the right of s, which is a contradiction. It implies that x can be extended to the boundary of D. This completes the proof.

2.3 Linear systems of Ordinary Differential Equations

We say that (2.3) is a linear system if f is linear, i.e. each component f_i of f is of the form:

$$f_i(t, x) = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t) \qquad (i = 1, \dots, n)$$

This can be written as:

$$x' = A(t)x + b(t) \tag{2.10}$$

where $A(t)$ is an $n \times n$ matrix whose elements are $a_{ij}(t), b(t) \in n \times 1$ vector with $b_i(x)$ components and $x(t) \in n \times 1$ unknown vector with $x_i(t)$ components.

Definition 2.22. If $b(t) = 0$, (2.11) below is called a ***homogeneous linear system***

$$x' = A(t)x \tag{2.11}$$

If otherwise, we have a ***nonhomogeneous*** system.

The ODE in (2.8) is a good example of a linear system.

We will make $\|x\|$ any convenient norm on $\mathbb{R}^n$ and $\sup_{\|x\|=1} \|A(x)\|$ in this section.

Proposition 2.23. The linear system (2.10) where $t \longmapsto A(t)$ and $t \longmapsto b(t)$ are continuous functions on an interval $I \in t_0$. If $x_0 \in \mu$, then $\exists$ one and only one solution x of (2.12) on I.

$$\begin{cases} x' = A(t)x + b(t) \\ \quad x(t_0) = x_0 \end{cases} \tag{2.12}$$

Proof. Each function f_i may be shown to satisfy the Lipchitz condition (2.6) on I, which is required to prove this contradiction, using the global existence theorem. We assume that I is both closed and finite. Otherwise, we choose a closed, finite subinterval I_0 of I. The absolute value of any function a_{ij} is therefore less than some positive number $\forall\ t \in I$ since $t \longmapsto A(t)$ is continuous on I. If N is the biggest of these numbers, then $\forall\ t \in I$, we have:

$$\left|a_{ij}(t)\right| \leq N \qquad i = 1, \dots, n \text{ and } j = 1, \dots, n$$

Let $t \in I$ be fixed and $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}
\|f_i(t,x) - f_i(t,y)\| &= \left\|\sum_{j=1}^{n} a_{ij}(t)x_j - \sum_{j=1}^{n} a_{ij}(t)y_j\right\| \\
&\leq \sum_{j=1}^{n} \left|a_{ij}(t)(x_j - y_j)\right| \\
&\leq \sum_{j=1}^{n} \left|a_{ij}\right|\left|x_j - y_j\right| \\
&\leq N \sum_{j=1}^{n} \left|x_j - y_j\right| \\
&\leq N\|x - y\| \quad \forall\ x, y \in \mathbb{R}^n.
\end{aligned}$$

From the inequality above, since each f_i $(i = 1, \dots, n)$ is Lipschitz on J with a Lipschitz constant N, the vector-valued function f is Lipschitz on I. Hence, the initial condition (2.12) has a unique solution on the interval I from the global existence theorem.

Definition 2.24. If all of the n solutions to the ODE in (2.12) defined on the same interval I are linearly independent functions on I, then the set of n solutions is referred to as a ***fundamental set of solutions*** on I.

Definition 2.25. Let $X = (x_1 \dots x_n)$ and $x_1 \dots x_n$ be n solutions of (2.11). X is called a ***fundamental matrix*** of (2.11) if the n solutions are linearly independent.

Furthermore, if $X(t_0) = I$, X is called a ***principal fundamental matrix***.

Theorem 2.26. If X is a fundamental matrix on I for $x' = A(t)x$ and C is a nonsingular constant matrix, then XC is likewise a fundamental matrix of $x' = A(t)x$ on I.

Corollary 2.27. A solution u of $x' = A(t)x$ is of the form:

$$u(t) = X(t).c$$

where c is a constant vector and X = *X(t)* is a fundamental matrix solution.

Corollary 2.28. The matrix $X(t) = e^{tA}$ is the fundamental matrix of $x' = Ax$ if A is a constant matrix. The IVP $x' = Ax, x(0) = x_0$ therefore has a unique solution

$x(t) = e^{tA}x_0$.

- The solutions are linear combinations of $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$, i.e. $x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t}$ if the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are all distinct
- If the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are not all distinct and n_j are their respective multiplicities with $n_1 + \cdots + n_k = n,$ then the solutions are linear combinations of $p_{n_1}e^{\lambda_1 t}, \dots, p_{n_k}e^{\lambda_k t}$, where the degree of polynomial $p_j \le (n_j - 1)$, $j = 1,2, \dots, k$.

Theorem 2.29. If ϕ and ψ are two solutions of system (2.10) and are defined on some interval (a, b) with $a < b$. Then $t \longmapsto c_1\phi(t) + c_2\psi(t)$ is also a solution if c_1 and c_2 are constants and defined on the same interval.

Theorem 2.30. Assume that X is a fundamental matrix of the homogeneous system (2.11) on I, an open interval. If $t_0 \in I$ and the Wronskian $W(t)$ of $X(t)$ is $det(X(t))$, then,

$$W(t) = W(t_0)exp\left(\int_{t_0}^{t} tr(a(s)\right)ds$$

Proof. $t \longmapsto X(t)$ is a differentiable function that is a matrix solution. Expanding around t_0 in a Taylor series of order 1 gives:

$$X(t) = X(t_0) + (t - t_0)X'(t_0) + o((t - t_0)$$
$$= X(t_0) + (t - t_0)A(t_0)X(t_0) + o((t - t_0)$$
$$= [I + (t - t_0)A(t_0)]X(t_0) + o((t - t_0)$$

so that

$$\det\big(X(t)\big) = \det[I + (t - t_0)A(t_0)]\det(X(t_0))$$
$$\mathrm{W(t)} = \det[I + (t - t_0)A(t_0)]\,W(t_0)$$

It is clear that the function $\varepsilon \longmapsto \phi(\varepsilon) = det(I + \varepsilon C)$ is differentiable.

If C is any nonsingular square matrix and we expand around 0 in a Taylor series of order 1, then:

$$det(I + \varepsilon C) = 1 + \varepsilon tr(C)o(\varepsilon),$$

From this last equation, we have that:

$$W(t) = W(t_0)[1 + (t - t_0)tr(A(t_0))]$$

By expanding $t \longmapsto W(t)$ in a Taylor series, we now obtain:

$$W(t) = W(t_0) + (t - t_0)W'(t_0) + o\big((t - t_0)\big),$$

so that

$$W'(t_0) = W(t_0)tr(A((t_0)).$$

Since no assumption about t_0 has been made, we can then write

$$W'(t) = W(t)tr(A(t)$$

Hence, the solution to this is given by:

$$W(t) = W(t_0)exp\left(\int_{t_0}^{t} tr(A(s))\right)ds$$

Theorem 2.31. If $\bar{x}$ is any particular solution of (2.10), then any solution of such can be expressed in the form:

$$x(t) = \bar{x} + X(t)c \tag{2.13}$$

with c a constant vector and X a fundamental matrix solution of (2.11).

Proof. Setting $x(t) = \bar{x} + X(t)c$, we have:

$$\begin{aligned} x'(t) &= \bar{x}'(t) + X'(t)c \\ &= A(t)\bar{x} + b(t) + A(t)Xc \\ &= A(t)(\bar{x} + X(t)c) + b(t) \\ &= A(t)x(t) + b(t) \end{aligned}$$

Hence, $x(t)$ is a solution of (2.10), a nonhomogeneous equation.

Setting $u(t)$ as any solution of (2.10), it is required to prove that $u(t)$ is of the form (2.13).

Whereas,

$$u'(t) = A(t)u + b(t) \tag{2.14}$$

As $\bar{x}$ is a particular solution of (2.10),

$$\bar{x}'(t) = A(t)\bar{x} + b(t); \tag{2.15}$$

Subtracting (2.15) from (2.14), we obtain

$$u'(t) - \bar{x}^{I}(t) = A(t)(u - \bar{x}) \tag{2.16}$$

(2.16) means that $(u - \bar{x})$ is a solution of (2.11), a homogeneous equation.

The solution of (2.11) has a fundamental matrix X and $\exists$ a constant vector c so that

$$u(t) - \bar{x}(t) = X(t)c$$

Hence,

$$u(t) = \bar{x}(t) + X(t)c$$

and this completes the proof.

Theorem 2.32 (Variation of constant). If X is a fundamental matrix solution of (2.11), then,

$$\bar{x}(t) = X(t).\int_{t_0}^{t} X^{-\prime}(s)b(s)ds$$

is a particular solution to nonhomogeneous equation (2.10).

CHAPTER III

Stability Theory

Approximations are commonly used in various applications to enter (numerical) data into differential systems (with a certain degree of error). Thus, the subject of stability, often referred to as robustness, sensitivity, or continuity, concerns how solutions, which may potentially take the form of physical systems, respond to small perturbations of the data.

Nonetheless, it should be noted that a physical system is stable if only a very small deviation from its current state results in a very small change in the state. If no such exists, such system is unstable.

3.1 Phase space

The autonomous systems below will be taken into consideration:

$$x' = f(x); \qquad x \in M \subset \mathbb{R}^n \tag{3.1}$$

where the vector-valued function f is continuous on M to $\mathbb{R}^n$.

We can plot the solution x of (3.1) above in the space $x_1 - x_n$. This space is called ***phase space***.

A solution of an ODE can be referred to by several different geometric terms namely: ***orbit, trajectory and integral curve***.

Definition 3.1. The ***integral curve*** of an ODE solution, shown by the graph $\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x = x(t), t \in I\}$ represents the time interval throughout which the solution x exists.

Definition 3.2. The projection of an integral curve along the axis t in phase space is referred to as a phase curve or ***trajectories of solutions***.

Definition 3.3. In the phase space of, let x_0 be a point (3.1). For $x_0 \in M \subset \mathbb{R}^n$, the ***orbit*** through x_0 denoted $\varphi(x_0)$ is defined as:

$$\varphi(x_0) = \{x \in \mathbb{R}^n : x = x(t, t_0, x_0), t \in I\}$$

We will look at the example below to illustrate the concept of the phase space:

Example 3.4. Consider the differential equation:

$$\frac{d^2u}{dt^2} + u = 0 \qquad (t, u) \in \mathbb{R} \times \mathbb{R}$$

By setting $x = u$ and $y = x'$, this system of equation can be transformed to the form:

$$X' = AX, \quad X \in \mathbb{R}^2$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda_1 = i$ and $\lambda_2 = -i$ are the eigenvalues of A. An eigenvector corresponding to $\lambda = i$ is $v = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $k \neq 0$ is constant. Hence, the general solution is $X(t) = e^{it}v$.

The phase space is at plane $\mathbb{R}^2$.

The solution through $(x, y) = (1,0)$ at $t = 0$ is:

$$(x(t), y(t)) = (cos\, t, -sin\, t)$$

The integral curve through $(1,0)$ at $t = 0$ is:

$$\{(t, (cos\, t, sin\, t)) : t \in \mathcal{R}\}$$

The orbit through $(0,1)$ is given by: $x^2 + y^2 = 1$ (which corresponds to the equation of a circle).

In addition to the different types of orbits that can be obtained by trying to solve ordinary differential equations, two special types will be defined: the ***rest point*** and the ***periodic orbit***.

Definition 3.4. A ***rest point***, also called critical point, singular point, stationary point, steady state or an equilibrium point is a point $x_e \in \mathbb{R}^n$such that $f(x_e) = 0$. This type of solution remains constant over time.

Definition 3.5. In the ODE (3.1), the solution x is known as a ***periodic solution*** if there exists a constant T, such that $(t + T) = x(t)\ \forall\ t \in I$. Periodic solutions have closed phase curves, known as ***cycles*** or periodic orbits.

3.2 General definition of stability

Definition 3.6. The equilibrium point x_e of (3.1) is ***stable*** if for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that for any solution *x* of (3.1), if $\|x_e - x(t_0\| < \delta$, then the solution *x* exists $\forall\ t \geq t_0$ and $\|x_e - x(t)\| < \varepsilon\ \forall\ t > t_0$.

Example 3.7. The equation $x' = 1 - x$ is stable at the equilibrium point $x_e = 1$.

Definition 3.8. If x_e of (3.1) is stable and $\exists\ \delta_0 >\ 0$ such that

$$\lim_{t \to \infty} x(t) = x_e \text{ if } \|x(t_0) - x_e\| < \delta_0$$

Then the equilibrium solution x_e of (3.1) is said to be ***asymptotically stable.***

Example 3.9. The equilibrium solution $x_e = 0$ of $x' = -ax, a > 0$ is not only stable, but asymptotically stable.

If *x* is any solution of $x' = -ax$, the solution *x* of $x' = -ax$ with the IVP $x(0) = 1$ is $x(t) = e^{-at}$, so $\lim_{t\to\infty} x(t) = 0 = x_\varepsilon$.

If it is not stable, then the equilibrium solution x_ε is ***unstable***

Example 3.10. The solution $x_\varepsilon = 0$ of $x' = x^2$ is unstable, since for $t_0, x_0 > 0$, the solution $x(t) = \frac{x_0}{1+x_0(t_0-t)}$ fails to exist at $t = x_0^{-1} + t_0$.

To understand stability of solution of non-autonomous systems in (2.3), we will generalize to arbitrary solutions, where the real valued-vector function f is defined and continuous in $D = \{\,(t,x) \colon 0\ \le t < \infty, \|x\| < a\}$, where $a \ge 0$ is a constant. Furthermore, we will set $x := x(t, t_0, x_0)$ to be any solution of (2.3) with IVP $x(t_0) = x_{0,}$ where $t_0 \ge 0$.

Definition 3.11. If $\forall\ \varepsilon > 0, \exists\ \delta = \delta(\varepsilon, t_0)$ such that for any other solution $y := y(t, t_0, x_0)$ of (2.3) existing for $t > t_0$ and satisfying $|x(t_0) - y(t_0)| < \delta$, one has $|x(t) - y(t) < \varepsilon|$ for $t > t_0$, then the solution *x* of (2.3) is stable.

Definition 3.12. The solution *x* of (2.3) is asymptotically stable if it is stable and there is a constant $\delta_0 = \delta_0(t_0) > 0$ such that, if $|x(t_0) - y(t_0)| < \delta_0$, then $\lim_{t\to\infty} y(t) = x(t)$.

Example 3.13. Every solution of $x' = -tx$ is asymptotically stable, and hence stable.

Next, we will find a method of examining the stability of a solution of an ODE.

To understand how stable x is, we need to know the nature of the solution around x. This is accomplished via a term called ***linearization***.

Definition 3.14 (Linearization). Linearization involves approximating a complicated nonlinear system to a linear one. The concept of linearization is to approximate a nonlinear map with one that is linear.

We will now describe the method of linearization for any solution *x* of (.3.1).

Suppose that $u = x + y$ is solution of (3.1), then u satisfies (3.1) and the Taylor's expansion of f around *x* gives:

$$u' = x' + y' = f(x(t)) + Df(x(t))y + o(\|y\|) \tag{3.2}$$

where Df is the derivative of f, $o(\|y\|) = \|y\| \in (y)$ with $\lim_{y \to 0} \varepsilon(y) = 0$.

Since *x* is a solution of (3.1), then (3.2) above becomes:

$$y' = Df(x(t))y + \ o(\|y\|) \tag{3.3}$$

and this explains the evolution of orbits around *x*.

The question of stability concerns solutions arbitrarily close to x, so we can answer this question by investigating the related linear system:

$$y' = Df(x(t))y \tag{3.4}$$

In the case of constant coefficients, stability of x therefore entails stability of an eigenvalue question, which is an eigenvalue problem.

3.3 Linear systems' stability

Considering the linear system below:

$$x' = A(t)x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \tag{3.5}$$

$t \to A(t)$ is a continuous matrix-valued function and *X* a fundamental matrix of (3.5) that satisfies $X(t_0) = I$.

Theorem 3.15. All solutions of (3.5) are stable iff each solution is bounded.

Proof. We will make *X* a fundamental matrix of (3.5). There exists a constant *M* such that $\|X(t)\| \leq M \ \forall \ t \in \mathbb{R}$ if (3.5) has all solutions bounded.

Given any $\varepsilon > 0$, then

$\|x_0 - y_0\| < \frac{\varepsilon}{M+1} \Rightarrow \|x(t, t_0, x_0) - y(t, t_0, y_0)\| = \|X(t)(x_0 - y_0)\| \leq M\|x_0 - y_0\| < \varepsilon,$

y being any solution with IVP $y(t_0) = y_0$. All solutions are therefore stable.

In converse, $x(t, t_0, 0) = 0$ is stable if all solutions are stable.

Given $\varepsilon > 0$, $\exists \ \delta = \delta(\varepsilon)$ such that $\|y_0\| < \delta$ implies

$$\|0 - y(t, t_0, y_0)\| = \|X(t)y_0\| < \varepsilon.$$

If we let y_0 be a vector and $\frac{\delta}{2}$ in the *i*th place with zero everywhere else, then

$$\|X(t)y_0\| = \|X_i(t)\| \frac{\delta}{2} < \varepsilon$$

where X_i is the *ith* column of *X*. Hence,

$$\|X(t)\| < 2n\varepsilon\delta^{-1} = k$$

Therefore for any solution,

$$\|y(t,t_0,y_0)\| = \|X(t)x_0\| < k\|x_0\|$$

and hence, all solutions are bounded.

We next consider the ODE (3.5) when the matrix $A = A(t)$ is constant.

Definition 3.16. A polynomial $P(\lambda)$ is stable when all solutions (roots) of the characteristic polynomial $P(\lambda)$ have negative real parts.

Theorem 3.17. If $\lambda_1, \lambda_2, \dots, \lambda_k$ eigenvalues of A that are distinct, with λ_j having multiplicity n_j and $n_1 + n_2 + \cdots + n_k = n$. If $p > \max_{j=1,\dots,k} Re(\lambda_j)$, then $\exists$ a constant $M > 0$ such that:

$$\left\|e^{(tA)}\right\| \leq Me^{pt} (0 \leq t \leq \infty)$$

***Proof*.** From (2.28), e^{tA} is a fundamental matrix of $x^I = Ax$ (a linear ODE) and of the form:

$$e^{tA} = \sum_j^k p_j(t)e^{\lambda_j t},$$

Where the polynomial $p_j(t) \leq (n_j - 1)$.

Recall: For $\lambda_j \in \mathbb{C}$, $\left\|e^{\lambda_j t}\right\| = e^{Re(\lambda_j)t}$.

If $p > \max_{1 \leq j \leq k} Re(\lambda_j)$, we may write

$$e^{Re(\lambda_j)t}\left|p_j(t)\right| = e^{pt}e^{-(p-Re(\lambda_j)t}\left|p_j(t)\right|$$

and since $p > Re(\lambda_j)$, we deduce that $\lim_{t\to\infty}$ Examplep$(-(p - Re(\lambda_j)t\left|p_j(t)\right| = 0$.

Hence, $\exists\ M > 0$ such that:

$$\sum_{j=1}^k \exp(-(p - Re(\lambda_j)t\left|p_j(t)\right| \leq M, \quad t \geq 0$$

Thus, we deduce that

$$\|e^{tA}\| \leq \sum_{j=1}^k \exp(Re(\lambda_j)t\left|p_j(t)\right| \leq M, \quad t \geq 0$$

Theorem 3.18. Every solution of (3.5) is asymptotically stable if the characteristic polynomial of $A = A(t)$ is stable.

Proof. Let $\exists$ positive constants M and p such that $\|X(t)\| \leq Me^{-pt}, t \geq 0$ (since the characteristic polynomial is stable).

We know that $t \mapsto Me^{-pt}$ is a decreasing function, so, given $\varepsilon > 0$, then $\|x_0 - y_0\| < \varepsilon M^{-1}e^{pt_0}$ implies that:

$$\|x(t,t_0,x_0) - y(t,t_0,y_0)\| \leq \|X(t)\|\|x_0 - y_0\|$$
$$\leq Me^{-pt}\|x_0 - x_1\|$$

The RHS is less than ε $\forall$ $t > t_0$ and tends to zero as t approaches 1.

Hence, all solutions of (3.5) are asymptotically stable.

Proposition 3.19 (Stability test using eigenvalues)

a) If each real portion of the eigenvalue of A is strictly less than zero, the constant-coefficient system (2.28) is said to be asymptotically stable.

b) (2.28) is stable if all eigenvalues of A have real parts zero and each eigenvalue is less than or equal to zero.

Next, we will consider a system of the form:

$$x' = Ax + f(t,x) \tag{3.6}$$

where $A = (a_{ij})_n$ is a constant matrix and f a vector-valued function defined as $f = (f_1, \ldots, f_n)$ satisfies:

i) $(t,x) \longmapsto f(t,x)$ is continuous for $\|x\| < a$ and $t \geq 0$,

ii) $\lim_{\|x\|\longmapsto 0} \frac{\|f(t,x)\|}{\|x\|} = 0$ with respect to t.

Theorem 3.20. If the characteristic polynomial of A in equation (3.6) is stable, then the solution $x(t) \equiv 0$ of (3.6) is asymptotically stable.

The solution $x(t) \equiv 0$ of (3.6) is asymptotically stable if the characteristic polynomial of A in (3.6) is stable.

It is important to introduce the ***Gronwall's inequality*** to prove theorem 3.20.

Theorem 3.21 (Gronwall's inequality). Let $K \geq 0$ and f, g be continuous nonnegative functions on the interval $a \leq t \leq b$. If

$$f(t) \leq K + \int_a^t f(s)g(s)ds \qquad \text{for } a \leq t \leq b.$$

Then

$$f(t) \leq K\, exp\left(\int_a^t g(s)ds\right) \qquad \text{for } a \leq t \leq b.$$

Proof. Let $h(t) = K + \int_a^t f(s)g(s)ds$ and observe that $h(a) = K$.

By hypothesis,

$$f(t) \leq h(t)$$

Recall that g is non-negative and by fundamental theoRemark of calculus.

$$h'(t) = f(t)g(t) \leq h(t)g(t) \;\; a \leq t \leq b$$

We multiply this inequality by $exp\left(-\int_a^t g(s)ds\right)$ and apply the identity:

$$h'(t)exp\left(-\int_a^t g(s)ds - h(t)g(t)exp\left(-\int_a^t g(s)ds\right)\right) = \frac{d}{dt}\left(h(t)exp - \int_a^t g(s)ds\right)$$

to obtain:

$$\frac{d}{dt}\left(h(t)\text{Examplep}(-\int_a^t g(s)ds)\right) \leq 0$$

Integrating from a to t gives:

$$h(t)exp\left(-\int_a^t g(s)ds\right) - h(a) \leq 0$$

Since $f(t) \leq h(t)$ and $h(a) = K$,

$$f(t) \leq h(t) \leq Kexp\left(\int_a^t g(s)ds\right) \qquad a \leq t \leq b$$

We will now proceed to prove theorem 3.20

Proof. First, we will prove that the solution $x(t) = x(t, 0, x_0)$ is defined on $t \geq 0$ when $x_0 \to 0$. If X is the fundamental matrix of $x^I = Ax$ with $X(0) = I$, then $\exists\ R > 0$ and $\alpha > 0$ such that:

$$\|X(t)\| \leq Re^{-\alpha t}\ \forall\ t \geq 0$$

Since matrix A is constant, x satisfies the relation:

$$x(t) = X(t)x_0 + \int_0^t X(t-s)f\big(s, x(s)\big)ds$$

Therefore,

$$\|x(t)\|e^{\alpha t} \leq R\|x_0\| + \int_0^t Re^{\alpha s}\big\|f\big(s, x(s)\big)\big\|ds$$

The first and second relations are definitely valid for t in the interval $[0, T)$ if $\|x(t)\| < a$ and $\|x_0\| < a$ on assumption.

It follows from condition (ii) that given any $m > 0$, $\exists\ d > 0$ such that for $\|x\| < d$ and $t \geq 0$, we have $\|f(t, x)\| \leq m\|x\|$. If we assume $\|x_0\| < d$, then by continuity of $t \longmapsto x(t)$, $\exists\ t_1 > 0 : \|x(t)\| < d\ \forall\ 0 \leq t < t_1$.

Therefore,

$$\|x(t)\|e^{\alpha t} \leq R\|x_0\| + \int_0^t mRe^{\alpha t}\|x(s)\|ds$$

for $0 \leq t < t_1$.

From the Gronwall's inequality, it implies that:

$$\|x(t)\| \leq R\|x_0\|e^{(mR-\alpha)t}; \qquad 0 \leq t < t_1$$

Choosing m such that $mR < \alpha$ and $x(0) = x_0$ since x_0 and m are at our disposal, so that $\|x_0\| < d/2R \Longrightarrow \|x(t)\| < d/2$ for $0 \leq t < t_1$.

Solution x, which exists locally at every point $(t, x), t > 0, \|x\| < a$ can be extended interval by interval once f is defined for $\|x\| < a$ and $t \geq 0$.

Therefore, for any solution x with $\|x_0\| < d/2R$, it is defined for $t \geq 0$ and satisfies $\|x(t)\| \leq d/2$. Obviously, we can make d small and by doing so, $x(t) \equiv 0$ is stable, hence $mR < \alpha$ is asymptotically stable.

CHAPTER IV

Floquet Theory: Presentation of Periodic Solutions and their Stability

The Floquet theory is a fundamental subject in the qualitative theory of ordinary differential equation (ODE). By using the Floquet theory, periodic linear systems can be represented and periodic solutions' stability can be analyzed. The purpose of this chapter is to first present main results concerning Floquet theory, then we will finish this up by applying this theory in the next and final chapter to the problem of the stability of periodic solutions. The Hill's equation's stability will then be concluded by using the Floquet's theory.

Definition 4.1. If $A(t + T) = A(t)\ \forall\ t \in R$, then an $n \times n$ matrix-valued function $t \longmapsto A(t)$ is said to be T-periodic.

4.1 Linear systems with periodic coefficients

We will take the $n \times n$ homogeneous linear systems below:

$$x' = A(t)x \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \tag{4.1}$$

where $t \longmapsto A(t)$ is a continuous, T-periodic matrix-valued function defined thus:

$$A(t + T) = A(t) \forall\ t \in \mathbb{R} \tag{4.2}$$

Example 4.2. The system $x' = A(t)x$ where

$$A(t) = \begin{pmatrix} 1 + \cos(t) & 0 \\ 1 & -1 \end{pmatrix}$$

is 2π-periodic.

Theorem 4.3. Let A be a matrix with period T. If $t \longmapsto X(t)$ is a fundamental matrix of (4.1), then so is $t \longmapsto X(t + T)$ and $\exists$ a nonsingular constant matrix B such that:

i) $X(t + T) = X(t)B\ \forall\ t \in \mathbb{R}$

ii) $det(B) = exp\left(\int_0^T tr(A(s)\right) ds$

Proof. Since $t \longmapsto A(t)$ is periodic, it is defined $\forall\ t \in \mathbb{R}$. As a result, the system's solutions are all defined for $t \in \mathbb{R}$.

If we set $Y(t) = X(t + T)$, then

$$Y'(t) = X'(t + T) = A(t + T)X(t + T) = A(t)X(t + T) = A(t)Y\,(t)$$

which shows that Y is the linear system's solution matrix of (4.1).

Furthermore, as $det(X(t+T)) \neq 0 \ \forall \ t \in \mathbb{R}$, we have that $detY(t) \neq 0 \ \forall \ t \in \mathbb{R}$. Consequently, Y is a fundamental matrix of (4.1).

i) Define $B(t) = X^{-1}(t)Y(t)$, B is nonsingular and a product of two nonsingular matrices.

Moreover, $Y(t) = X(t)X^{-1}(t)Y(t)$. The proof that B is a constant matrix is still needed.

Let $B_0 = B(t_0)$, we know by Chapter 2 that $Y_0(t) = X(t)B_0$ is a fundamental matrix, but by definition, $Y_0(t_0) = Y(t_0)$. Since both are solutions to (4.1), we must have

$Y_0(t) = Y(t)$ for all time by the uniqueness of solution.

So, $B_0 = B(t)$, and thus B is independent of time.

ii) By using Abel's formula, we get:

$$W(t) = W(t_0)exp\left(\int_{t_0}^{t} tr\big(A(s)\big)ds\right)$$

$$W(t+T) = W(t_0)exp\left(\int_{t_0}^{t} tr\big(A(s)\big)ds + \int_{t}^{t+T} tr\big(A(s)\big)ds\right)$$

$$W(t+T) = W(t)exp\left(\int_{t}^{t+T} tr\big(A(s)\big)ds\right)$$

$$W(t+T) = W(t)exp\left(\int_{0}^{T} tr\big(A(s)\big)ds\right)$$

We know that:

$$X(t+T) = X(t)B$$

$$det(X(t+T)) = det(X(t))det(B)$$

$$W(t+T) = W(t)det(B) \quad \forall \ t \in \mathbb{R}$$

taking $t = 0$, we get

$$\det(B) = exp\left(\int_{0}^{T} tr\big(A(s)\big)ds\right)$$

Remark 4.4. Since the matrix B is independent of time, B can be calculated by setting $t = 0$, so that $B = X^{-1}(0)X(T)$. By taking the initial condition $X(0) = I$, then $B = X(T)$.

Definition 4.5. Matrix $B = X(T)$ from remark 4.4 above is known as a ***monodromy matrix***.

Theorem 4.6 (Floquet's Theorem). Let X be any fundamental matrix of (4.1) and A be a continuous periodic matrix with period T. Then

$$X(t) = P(t)exp(tR), \tag{4.3}$$

where P is a periodic nonsingular matrix with period T and R is a constant matrix.

The representation $X(t) = P(t)\ exp(tR)$ is a Floquet normal form for X.

Proof. Let X be an arbitrary fundamental matrix of (4.1) and from theorem (4.3), matrix B is non-singular. By theorem (1.47), $\exists$ a matrix R such that:

$$e^{TR} = B$$

Let us define $P(t) = X(t)e^{-tR}\ \forall\ t \in \mathbb{R}$. P is obviously the product of two nonsingular matrices.

Furthermore,

$$P(t+T) = X(t+T)\ exp(-(t+T)R)$$

$$= X(t)\ exp(TR)exp(-(t+T)R)$$

$$= X(t)exp(-tR) = P(t) \qquad -\infty < t < \infty$$

Thus P has period T and solving $P(t) = X(t)e^{tR}$ for X, (4.3) is obtained.

4.2 Characteristic multipliers and exponents

Definition 4.7 (Characteristic multipliers). The eigenvalues $\lambda_1, \dots, \lambda_n$ of matrix B are known as the characteristic or Floquet multipliers of (4.1).

Definition 4.8 (Characteristic exponents). The number $r_1, \dots, r_n$ defined by the relations $\lambda_j = e^{r_j T}, j = 1, \dots, n$ are known as the characteristic or Floquet exponents of (4.1).

Floquet exponents have real parts called ***Lyapunov exponents***.

Proposition 4.9. The characteristic multipliers and exponents properties are stated below:

i) The trace of matrix B denoted by $tr(B)$ is given as:

$$tr(B) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

ii) The characteristic multipliers $\lambda_1, \dots, \lambda_n$ of matrix B satisfy

$$det(B) = \lambda_1 \lambda_2 \dots \lambda_n = exp\left(\int_0^T tr\big(A(s)\big)ds\right)$$

with $X(0) = I$.

iii) There is no dependence between fundamental matrix and characteristic multipliers.

iv) Characteristic exponents are not all the same (unique).

v) The Floquet exponents r_i that correspond to λ_i satisfy the equation:

$$r_1 + r_2 + \cdots + r_n = \frac{1}{T}\int_0^T tr(A(t)dt \left(mod\, \frac{2\pi}{T} i\right)$$

Corollary 4.10. The periodic system (4.1), where P is the periodic matrix of the Floquet's theorem, is changed to the system with constant coefficients by the variable change $x = P(t)y$.

Proof. As $X'(t) = A(t)X(t)$, (3.1.3) will give:

$$(P(t)\, exp(tR))' = P'(t)\, exp(tR) + P(t)Rexp(tR)$$
$$= A(t)P(t)\, exp(tR)$$

It follows that

$$P'(t) = A(t)P(t) - P(t)R$$

Thus,

$$x'(t) = P(t)y' + P'(t)y = P(t)y' + (A(t)P(t) - P(t)R)y$$
$$= A(t)P(t)y$$

and therefore

$$P(t)y' - P(t)Ry = 0$$

or

$y' = Ry$ (whose system is linear system and has constant coefficients).

Theorem 4.11. Let λ be a characteristic multiplier and r a corresponding characteristic exponent of the homogeneous linear T-periodic system (3.1.1), so that $\lambda = e^{rT}$. Then ∃ is a nontrivial solution *x* of (4.1) such that:

i) $x(t+T) = \lambda x(t)$

ii) ∃ a *T*-periodic function *p* such that $x(t) = e^{rt}p(t)$.

Proof.

i) Let $v \neq 0$ be an eigenvector of matrix *B* that corresponds to the eigenvalue λ.

Also, we will let

$$x(t) = X(t)v$$

Then,

$$x'(t) = Ax$$

and

$$x(t+T) = X(t+T)v$$
$$= X(t)Bv = \lambda X(t)v$$
$$= \lambda x(t)$$

so that

$$x(t+T) = \lambda x(t)$$

ii) Let $p(t) = x(t)e^{-rt}$. We are required to show that p is T-periodic.

$$p(t+T) = x(t+T)e^{-r(t+T)}$$
$$= \lambda x(t)e^{-(t+T)}$$
$$= \frac{\lambda}{e^{rT}} x(t)e^{-rt}$$
$$= x(t)e^{-rt} = p(t)$$

Thus, we get a solution $x(t) = e^{rt}p(t)$ where p real valued function of period T.

Corollary 4.12. The system (4.1) has a periodic solution with period T if at least one characteristic multiplier is equal to unity.

Theorem 4.13. System (4.1) has a periodic solution with period kT if at least one of the characteristic multipliers is the k-th root of unity.

Proof. Consider $(\lambda_1, \dots, \lambda_n)$ and $x = (x_1, \dots, x_n)$ be characteristic multipliers and solution of (4.1) respectively. It is enough to show that $x_j(t+kT) = \lambda_j^k x_j(t)$.

We need to go by induction. If $k = 1$, from proposition (4.9i), we have

$$x_j(t+T) = \lambda_j x_j(t)$$

If this is true for $k \geq 1$, we can now prove for $k = k+1$.

We get $x_j(t+(k+1)T) = x_j(t+T+kT) = \lambda_j^k x_j(t+T)$ induction at step k.

Consequently, $x_j(t+(k+1)T) = \lambda_j^{k+1} x_j \ \forall \ t \in \mathbb{R}$ and as λ_j is the k-th root of unity, it implies that $\lambda_j^k = 1$ so that $x_j(t+kT) = x_j(t),\ j = 1, \dots, n \ \forall \ t \in \mathbb{R}$. Hence, the proof.

4.3 Nonhomogeneous linear systems

Here, we shall take the following linear system under consideration:

$$x^I = A(t)x + b(t) \tag{4.4}$$

where $n \times n$ matrix function $t \longmapsto A(t)$ and $t \longmapsto b(t)$ are both periodic and continuous and have period T.

Theorem 4.14. A solution *x* of (4.4) is periodic that has a period *T* iff $x(T) = x(0)$.

Proof. It is obvious that $x(T) = x(0)$ if *x* is periodic of period *T*.

In converse, we will suppose *x* as a solution of (4.4) where $x(t) = x(0)$.

If we define $y(t) = x(t + T)$, then both x and *y* are solutions of (4.4) and $y(0) = x(T) = x(0)$. As a result, x and y have the same initial values, thus $x(t) = y(t) = y(t + T), -\infty < t < \infty$, from the uniqueness theorem in Chapter 2.

This shows that *x* is periodic.

A superior property for periodicity of solutions can be found in the next theorem.

Theorem 4.15. For every periodic vector b of period T, the system (4.4) will have a periodic solution with period T iff the corresponding homogeneous system does not have a nontrivial solution with period T

Proof. Consider the following homogeneous system with X as the fundamental matrix where, $X(0) = I$.

We will assume that *x* is a solution of (4.4) that satisfies $x(0) = x_0$,

From Chapter 2, we know that each solution *x* of (4.4) is of the form:

$$x(t) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)b(s)ds\,, \qquad t \in \mathbb{R}$$

By theorem 4.3, the solution *x* is periodic iff $x(0) = x_0 = x(T)$.

But

$$x(T) = X(T)x_0 + X(t)\int_0^T X^{-1}(s)b(s)ds$$

and the periodicity condition $x(T) = x_0$ becomes

$$[I - X(T)]x_0 = X(T)\int_0^T X^{-1}(s)b(s)ds$$

This system is composed of linear nonhomogeneous algebraic equations for components of the vector x_0, solvable for each periodic vector b. This becomes possible iff the determinant of

$(I - X(T))$ is not equal to zero i.e. $[det(I - X(T)) \neq 0]$, which is equivalent to our earlier assertion that $X(T)x_0 = x_0$ has only x_0 as its trivial solution.

Moreover, the solution of this homogeneous system can be expressed as:

$$u(t) = X(t)x_0$$

Therefore the relation

$$u(T) = X(T)x_0 = u(0) = x_0$$

can only be satisfied by the trivial solution $u(t) = 0$.

It follows that from corollary 4.11, there is only a trivial solution which is a periodic solution of period T for the homogeneous system. Hence the proof.

4.4 Periodic solution stability

Here, we will examine the periodic solutions' stability based on the Floquet theory. We will first look at the linear system in (4.1) from the Floquet's theorem and theorem 4.10. The behavior of this system is then determined by the distribution of Floquet multipliers.

Lemma 4.16. Let r and λ be Floquet exponent and characteristic multiplier respectively and correspond to (4.1). Then;

i) $Re(r) = 0$ if $|\lambda| = 1$

ii) $Re(r) < 0$ if $|\lambda| > 1$, and so $\lim_{t\to\infty} x(t) = \infty$

iii) $Re(r) < 0$ if $|\lambda| < 1$, and so $\lim_{t\to\infty} x(t) = 0$

Theorem 4.17 (Criterion of Stability). The periodic linear system (4.1) is considered:

a) stable iff $|\lambda| \leq 1$ ($Re(r_j) \leq 0$ respectively) $\forall$ characteristic multipliers λ_j (characteristic exponents r_j) of (4.1), and for $|\lambda_j| = 1$ (respectively $r_j = 0$) the associated Jordan block of e^{TR} (respectively. R), its eigenvalue is ***semisimple*** and of 1×1 dimension.

b) asymptotically stable iff the real component of each characteristic exponent is strictly less than zero (0) or the moduli of each characteristic multiplier is strictly less than one (1).

4.5 Autonomous systems

We shall look at a system of the form:

$$x' = f(x), \;\; x \in \mathbb{R}^n \tag{4.5}$$

where f is of class C^∞ on $\mathbb{R}^n$.

Theorem 4.18. If u is a T-periodic solution, then one characteristic multiplier of this problem related to the linearization of the solution around u will be unity.

Proof. Assume that u is a solution of period T. The solution can be linearized about u if we write $x = u + v$, and we get $v' = A(t)v$, where the Jacobian of f is $A = A(t)$ i.e.

$$A(t) = \left(\frac{\delta f}{\delta x}(u(t))\right)$$

So, u and A are both T-periodic.

Then a fundamental matrix X with $X(0) = I$ can be defined, so that $X(T) = B$.

As u is the solution of (4.5),

$$u'(t) = f(u(t))$$

Differentiating with respect to t again gives:

$$u''(t) = f'\big(u(t)\big).(u'(t))$$

that is

$$u'(t) = A(t)u'(t)$$

u' satisfies the linear system $v' = A(t)v$.

As assumed before, u is T-periodic, so, $u'(t) = u'(t+T)$ and the corresponding characteristic multiplier is 1.

The consequence is that the periodic solution of (4.5) has a local stability around u that depends on the linear stability problem $v' = A(t)v$.

Generalized result at $\boldsymbol{n = 2}$

We will take a problem that has the form $x' = f(x)$ and $x \in \mathbb{R}^2$ which has a periodic solution $x \coloneqq \emptyset(t)$ with period T.

Theorem 4.19. $\emptyset$ is a stable solution if

$$\int_0^T \nabla . f|_{x=\emptyset} ds = \int_0^T \left(\frac{\delta f_1}{\delta x_1} + \frac{\delta f_2}{\delta x_2}\right)\Big|_{\emptyset(s)} ds < 0\,,$$

where $\nabla = \left(\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}\right)$.

Proof. Let $\emptyset$ be a periodic solution of $x' = f(x)$, with period *T*. So the periodic matrix will be:

$$A(t) = Df(\emptyset(t))$$

We know from the above result that we must have $\lambda_1 = 1$ and from earlier,

$$\left.\begin{aligned} \lambda_1\lambda_2 = exp\left(\int_0^T tr\big(A(s)\big)ds\right) \\ \lambda_2 = exp\left(\int_0^T tr\big(A(s)\big)ds\right) \end{aligned}\right\} \tag{4.6}$$

From the criterion of stability, we know that to have bounded perturbation, $\lambda_1 \le 1$ and $\lambda_2 \le 1$ and hence it is known that $\lambda_1 = 1$ and we want λ_1 and λ_2 to be distinct, $\lambda_2 < 1$. So,

$$0 > \int_0^T tr\big(A(s)\big)ds = \int_0^T tr\left(\frac{\delta f_i}{\delta x_j}\right)ds, \;\; i,j = 1,2$$

$$0 > \int_0^T \left(\frac{\delta f_1}{\delta x_1} + \frac{\delta f_2}{\delta x_2}\right)\Big|_{\emptyset(s)} ds \tag{4.7}$$

$$0 > \int_0^T \nabla . f|_{x=\emptyset} ds \qquad \text{as desired.}$$

Instability is achieved when:

$$\int_0^T \nabla . f|_{x=\emptyset} ds > 0$$

Example 4.20. Consider the system below:

$$\begin{cases} x' = x - y - x(x^2 + y^2) \\ y' = x + y - y(x^2 + y^2) \end{cases} \tag{4.8}$$

Let $x = r(t)cos\big(\theta(t)\big), y = r(t)sin(\theta(t))$ and where (r, θ) is a polar coordinate.

Problem (4.8) then becomes:

$$\begin{gathered} \sin(\theta)\,(r - r\theta') = \cos(\theta)(r - r^3 - r') \\ \cos(\theta)\,(r - r\theta') = -\sin(\theta)(r - r^3 - r') \end{gathered} \tag{4.9}$$

Squaring and adding (4.9), we obtain:

$$(r - r\theta')^2 = (r - r^3 - r')^2 \tag{4.10}$$

Again, setting $a = r - r\theta'$, we have $sa = r - r^3 - r'$ with $s_1 = 1$ and $s_2 = -1$.

This gives:

$$\begin{aligned} \mathrm{asin}(\theta) &= sa\cos(\theta) \\ a\cos(\theta) &= -sa\sin(\theta) \end{aligned} \tag{4.11}$$

which can be rewritten as

$$\begin{aligned} \mathrm{asin}(\theta) &= sa\cos(\theta) \\ -s^2 asin(\theta) &= sa\cos(\theta) \end{aligned} \tag{4.12}$$

so that we must have

$$\begin{aligned} \mathrm{asin}(\theta) &= -a\sin(\theta) \\ asin(\theta) &= 0 \end{aligned} \tag{4.13}$$

As a result, we have

$$asin(\theta) = sa\cos(\theta) = 0$$

so that we must have $a = 0$.

This means that:

$$r - r\theta' = r - r^3 - r' = 0$$

We have that:

$$r' = r(1 - r^2)$$

which gives radius $r = \pm 1$ as non-trivial solutions

We will consider $r = 1$ of course without losing generality,

As $r\theta' = r$, we get $\theta' = 1$, so that $\theta = t + c$, where c is a constant. This results in a solution with period $T = 2\pi$.

Furthermore, we have:

$$\begin{aligned} \nabla . f|_{r=1} &= \left[\frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y}\right] \\ &= [(1 - 3x^2 - y^2) + (1 - x^2 - 3y^2)]_{r=1} \\ &= -2 \end{aligned} \tag{4.14}$$

so that

$$\begin{aligned} \lambda_2 &= exp\left(\int_0^T tr\big(A(s)\big)ds\right) \\ &= exp\left(\int_0^{2\pi} -2ds\right) \\ &= \mathrm{Examplep}(-4\pi) < 1 \end{aligned} \tag{4.15}$$

The periodic orbit is thus stable at radius $r = 1$.

4.6 Non-autonomous systems

Let us consider the n component linear system below:

$$x' = f(t, x) \tag{4.16}$$

with f being periodic of period T in t. Assume that f continuous in (t, y) and has continuous second partial derivatives wrt the x-components in the domain $D = \{(t, x): 0 \leq t \leq T, \|x\| < r\}$, where $r > 0$ is some constant.

To test the stability of the periodic solution x, let's assume that (4.16) has a periodic solution u with period T.

Consider v as the solution of (4.16) which is:

$$v(t) = u(t) + y(t) \tag{4.17}$$

So,

$$v'(t) = f(t, u(t) + y(t)) = u'(t) + y'(t)$$

But

$$u'(t) = f(t, u(t))$$

Hence, y satisfies the equation:

$$y'(t) = f\big(t, u(t) + y(t)\big) - f(t, u(t)) \tag{4.18}$$

By Taylor expansion, we have:

$$f\big(t, u(t) + y(t)\big)$$
$$= f\big(t, u(t)\big) + \frac{\delta f}{\delta x}\big(t, u(t)\big)y(t)$$
$$+ \int_0^1 (1 - s)\frac{\delta^2 f}{\delta x^2}\big(t, u(t) + sy(t)\big)\big(y(t), y(t)\big)ds$$

That is:

$$y' = Df\big(t, u(t)\big).y + g(t, y)$$

where

$$g(t, y) = \int_0^1 (1 - s)\frac{\delta^2 f}{\delta x^2}(t, u(t) + sy)(y, y)ds$$

It is clear that g is continuous in (t, y) $\forall$ t, periodic of period T in t and for $\|y\|$ small and

$$\lim_{\|y\| \to 0} \frac{\|g(t, y)\|}{\|y\|} = 0$$

Moreover, the matrix $Df(t, u(t))$ is periodic and continuous in t for period T.

It is obvious that $y = 0$ is a solution of (4.19) and that the periodic solution $u(t)$ of the system (4.16) is stable or asymptotically stable iff $y \coloneqq 0$ is respectively a stable or asymptotically stable solution of (4.11).

This leads us to study the stability of the system with the form:

$$y^I = A(t)y + g(t, y) \quad (4.19)$$

where $A(t) = Df(t, u(t))$ is a periodic matrix that has a period T and g a continuous vector-valued function in (t, y), periodic wrt t of period T and small by definition. The system associated to (4.19) which is homogeneous has periodic coefficients of period T. From Floquet's theorem, $\exists$ a T-periodic matrix P and a nonsingular constant matrix R, such that a change of variable $y = Pz$ changes (4.19) to a suitable system.

Hence,

$$z' = Rz + P^{-1}(t)g(t, P(t)z) \quad (4.20)$$

which has constant coefficients.

CHAPTER V

Application and conclusion

5.1 Application

A very important and common application of the Floquet theory is the stability of Hill's equation. We will now apply this theory in studying the Hill's equation shown below:

$$x'' + a(t)x = 0 \tag{5.1}$$

where $t \longmapsto a(t)$ is continuous, real-valued and periodic with period T.

If we set $x_1 = x$ and $x_2 = x_1'$, ODE (3.3.1) can be rewritten as

$$\begin{cases} x_1' = x_2 \\ x_2 = -a(t)x_1 \end{cases}$$

which is of the form of the first order linear ODE:

$$x' = A(t)x \tag{5.2}$$

where $A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}$ is periodic with period T and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

If X is the fundamental matrix associated to (5.2), then ∃ constant B (a non-singular matrix) such that:

$$X(t+T) = X(t)B, t \in \mathbb{R} \text{ (from theorem (4.3)).}$$

We choose a fundamental system of solution $x_1(t), x_2(t)$ that satisfies the IVP:

$$\begin{cases} x_1(0) = 1 \quad x_1'(0) = 0 \\ x_2(0) = 0 \quad x_2'(0) = 1 \end{cases}$$

and so $X(0) = I$. This implies that

$$B = \begin{pmatrix} x_1(T) & x_2(T) \\ x_1'(T) & x_2(T) \end{pmatrix}$$

Let λ_1, λ_2 be the characteristic multipliers of (5.2) and r_1, r_2 the corresponding characteristic exponents.

We have from property (i) of (4.6) that:

$$\lambda_1\lambda_2 = exp\int_0^T tr\big(A(s)\big)ds = 1$$

and from (ii) of same lemma, we have:

$$\lambda_1 + \lambda_2 = tr(B) = x_1'(T) + x_2'(T)$$

Let's set $K = \frac{tr(B)}{2}$, so that

$$\begin{cases} \lambda_1\lambda_2 = 1 \\ \lambda_1 + \lambda_2 = 2K \end{cases}$$

Therefore, λ_1, λ_2 are roots of equation

$$\lambda - 2K\lambda + 1 = 0$$

whose solution is given by $\lambda_1 = K - \sqrt{K^2 - 1}$ or $\lambda_2 = K + \sqrt{K^2 - 1}$. We know that

$\lambda_j = exp(r_j T)$ where $j = 1,2$.

Thus, for stability of (5.2), depending on the value of K, we will consider the following cases:

Case 1. If $K > 1$, then $\lambda_1, \lambda_2 \in \mathbb{R}^+ : \lambda_1\lambda_2 = 1$.

We may assume thus: $0 < \lambda_1 < 1 < \lambda_2$; $\lambda_1 = \frac{1}{\lambda_2}$ and there is $r > 0 : \lambda_2 = e^{rT}$ and $\lambda_2 = e^{-rT}$. A basic set of solutions of the form $e^{-rt}p_1(t), e^{rt}p_2(t)$, where the real valued functions p_1 and p_2 are T-periodic, are given by (4.20).

As a result, since at least one of these solutions is unbounded, it is unstable. Therefore, the zero solution is unstable.

Case 2. If $|K| < 1$, then $\lambda_1 = K - i\sqrt{1 - K^2}$ or $\lambda_2 = K + i\sqrt{1 - K^2}$

We know that $|\lambda_1| = 1$ because $\lambda_1\lambda_2 = 1$. As a result, both characteristic multipliers are located on the complex plane's unit circle. Since the imaginary parts of λ_1 and λ_2 are nonzero, one of the characteristic multipliers, say λ_2 is located in the upper half plane. As a result, there is a real number θ with $0 < \theta T < \pi$ and $\lambda_2 = e^{i\theta T}$. In fact, a solution of the type $e^{i\theta t}(u(t) + is(t))$ exists, where u and s are both T-periodic functions. Hence, there is a fundamental set of solutions of the form:

$$u(t)cos\,\theta t - s(t)sin\,\theta t, \qquad u(t)\,sin\,\theta t + s(t)\,cos\,\theta t$$

In this case, (5.2) is stable.

Case 3. If $K = 1$, hence $\lambda_1 = \lambda_2 = 1$. Theorem (4.16) only ensures that we have one solution $x(t) = e^{rt}p(t)$, where p is a T-periodic real-valued function. If $B = X(T) = I$, then $X(t) = P(t)$, where P is a T-periodic and invertible matrix, has a Floquet normal form. As a result, there is a basic set of solutions, and the stable zero solution is one of them. There exists a nonsingular matrix C such that $CX(T)C^{-1} = I + N = e^N$ and so $X(T) = e^{C^{-1}}NC$, where $N \neq 0$ is nilpotent, if $B = X(T)$ is not the identity matrix. Thus,

$X(t) = P(t)e^{tR}$ and $X(T) = B = e^{TR}$, where $R = C^{-1}\left(\frac{N}{T}\right)C$, X has a Floquet normal form.

Hence, the zero solution is unstable since the matrix function $t \to e^{tR}$ is unbounded.

Case 4. If $K = -1$, we have a similar situation as case 3, with the exception that the fundamental matrix is represented by $Q(t)e^{tR}$ where Q is a *2T*-periodic matrix function.

5.2 Conclusion

As an effective method of solving linear systems with periodic coefficients and obtaining stability properties of periodic solutions of linear and nonlinear systems with periodic coefficients, we present in this work the Floquet theory, which relies on the computation of monodromy matrices for solving linear and nonlinear systems with periodic coefficients.

In perspective, other tools such as Poincaré map $p_{n+1} = \varphi(p_n)$, could be used to study the existence of periodic solutions and their qualitative properties. Research can also be carried out on Floquet Theory for Partial Differential Equations, a subject that has recently attracted interest due to advancement of Elasticity Theory and Parametric Resonance Theory.

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APPENDIX

Some notations and abbreviations used in this research

S/N	Notation/Abbreviation	Meaning
1.	$\lvert x \rvert$	Absolute value of x
2.	$\lVert x \rVert$	Norm of x
3.	$\Rightarrow$	Implies that
4.	$\equiv$	Equivalent to
5.	wrt	With respect to
6.	$\forall$	For all
7.	$\exists$	There exists
8.		There does not exist
9.	iff	If and only if
10.	: or |	Such that
11.	$\coloneqq$	Definitionined as
12.	$\mathbb{R}$	Real number
13.	$\mathbb{C}$	ComplExample number
14.	det	Determinant of
15.	$tr(A)$	Trace of A
16.	$int(A)$	Interior of A
17.	$cl(A)$	Closure of A
18.	$\neq$	Not equal to
19.	$A \sim B$	A is similar to B
20.	I	Identity of a matrix
21.	IVP	Initial value problem