SAADU BELLO MUAZU

BOUNDED SOLUTIONS OF SEMI-LINEAR DELAY PARABOLIC EQUATIONS

PhD THESIS

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NEAR EAST UNIVERSITY

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Saadu Bello MUAZU

Supervisor

Prof. Dr. Allaberen ASHYRALYEV

Nicosia

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Approval

We certify that we have read the thesis submitted by Saadu Bello Muazu titled **"Bounded Solutions of Semi-Linear Delay Parabolic Equations"** and that in our combined opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy of Sciences.

Examining Committee	Name-Surname	Signature
Head of the Committee:	Prof. Dr. Deniz Ağırseven	....
Supervisor:	Prof. Dr. Allaberen Asyhralyev	....
Committee Member*:	Prof. Dr. Evren Hınçal	....
Committee Member*:	Assoc. Prof. Dr. Nuriye Sancar	....
Committee Member*:	Assoc. Prof. Dr. Okan Gerçek	....

Approved by the Head of the Department

13/09/2023

....

Prof. Dr. Evren Hınçal

Head of Department

Approved by the Institute of Graduate Studies

.../.../2023

Prof. Dr. Kemal Hüsnü Can Başer

Head of the Institute

Declaration

I hereby certify that all data, materials, research, and findings in this thesis were obtained and presented in accordance with the academic guidelines and ethical principles of the Institute of Graduate Studies, Near East University. I further affirm that I have properly cited and referenced any information and data that are not originated from this work, as required by the rules and conduct.

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Saadu Bello Muazu

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To my parents...

Abstract

Bounded Solutions of Semi-linear Delay Parabolic Equations
Saadu Bello Muazu
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In the present thesis, the initial-boundary value problems for the semi-linear delay differential equations in a Banach space with strongly unbounded operators are considered. The main theorems on the existence and uniqueness of a bounded solution to these problems are established. The application of the main theorems to four different semi-linear and three different types of nonlinear delay parabolic equations is presented. Analytic solutions of several two-dimensional delay parabolic equations are obtained by using classical methods. The first and second-order accuracy difference schemes for the solution of a one-dimensional semi-linear parabolic equation with time delay are presented. Finally, certain numerical experiments are given to confirm the agreement between experimental and theoretical results and to make clear how effective the proposed approach is. Numerical results are found, and error analysis is given in the tables.

Keywords: bounded solution; Banach and Hilbert spaces; unbounded operators; semi-linear parabolic equations; existence and uniqueness.

Özet

Yarı Doğrusal Gecikmeli Parabolik Denklemlerin Sınırlı Çözümleri

Saadu Bello Muazu

Doktora Tezi, Matematik Bölümü

Danışman: Prof. Dr. Allaberen Ashyralyev

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Bu tezde, güçlü sınırsız operatörlere sahip bir Banach uzayında yarı doğrusal gecikmeli diferansiyel denklemler için başlangıç-sınır değer problemleri ele alınmıştır. Bu problemlere yönelik sınırlı bir çözümün varlığı ve benzersizliğine ilişkin ana teoremler oluşturulmuştur. Ana teoremlerin dört farklı yarı doğrusal ve üç farklı türdeki doğrusal olmayan gecikmeli parabolik denklemlere uygulanması sunulmaktadır. Birkaç iki boyutlu gecikmeli parabolik denklemin analitik çözümleri klasik yöntemler kullanılarak elde edilir. Zaman gecikmeli tek boyutlu yarı doğrusal parabolik denklemin çözümü için birinci ve ikinci dereceden doğruluk farkı şemaları sunulmaktadır. Son olarak deneysel ve teorik sonuçlar arasındaki uyumu doğrulamak ve önerilen yaklaşımın ne kadar etkili olduğunu netleştirmek için bazı sayısal deneyler verilmiştir. Sayısal sonuçlar bulunmuş, hata analizleri tablolarda verilmiştir.

Anahtar Kelimeler: sınırlı çözüm; Banach ve Hilbert uzayları; sınırsız operatörler; yarı doğrusal parabolik denklemler; varlık ve teklik.

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List of Abbreviations

DEs	Differential Equations
DE	Differential Equation
BS	Bounded Solution
IBVP	Initial Boundary Value Problem
IBVPs	Initial Boundary Value Problems
IVP	Initial Value Problem
DPPDE	Delay Parabolic Partial Differential Equation
DPPDEs	Delay Parabolic Partial Differential Equations
SSFs	Sufficiently Smooth Functions
FSADSs	First- and Second-order Accuracy Difference Schemes
ES	Exact Solution
FADS	First-order Accuracy Difference Scheme
SADS	Second-order Accuracy Difference Scheme
AS	Approximate Solution
SEs	System of Equations
SLEs	System of Linear Equations

CHAPTER I

Introduction

Historical Note and Literature Survey

The theory of differential equations (DEs) plays an important role in many disciplines, such as biology, economics, engineering, medicine, and physics. modelling of almost any biological, technical, or physical process, such as interactions between neurons, bridge design, movement of celestial bodies, propagation of water, heat, and sound in the atmosphere, single and multidimensional dynamic systems, electrostatics, electrodynamics, fluid flow, elasticity, or various types of quantum mechanics. Ordinary and partial DEs are used to describe the phenomenon. For example, while modelling biological systems with differential equations, simplified models created to better understand very complex events may not reflect the rich diversity of dynamics observed in natural systems. To overcome this complexity, many possible approaches can be devised using systems of partial and ordinary DEs, i.e., the method involving a larger number of equations. Although these systems are quite good at approximating observed behaviour, problems arise because many parameters representing quantities that cannot be determined empirically are overlooked. Therefore, another approach that is gaining importance is to include time delay terms in DEs.

Often in the generated modelling, the future state of the system is independent of the past and is controlled only by the present. It should be borne in mind that this is only a first approximation to the real state. More realistic models will include some of the past states of these systems, so ideally, a real system should be modelled as a time-delayed differential equation(DE). Of course, delay is inevitable in routine life. In any system, there is always a delay, even for seconds. Science makes predictions about future events by watching some events, and while doing this, it aims to create a mathematical model of the event or system it studies. As a matter of fact, in many applications, the newly created model is established with the assumption that the past state of the event or system under consideration will not affect the future state. Not adding the situations that have occurred in the past to the problems encountered in the model invalidates the created system. When the perceived information from the outside world is taken into account, there is a reaction to every effect, and there is a slight delay in this process. Because the reaction to every action depends on a process. In physical phenomena, the current state of a system can also be determined by considering its past state. In order to predict how the mentioned system will behave, it is necessary to know the differential equation describing this system and its solutions.

Time delays occur very often in almost all cases, and to ignore them is to ignore reality. Delays, gestation periods, incubation periods, transportation delays, etc. can represent A simple example from nature is reforestation. Saplings in a forest need 20 years to reach maturity after reforestation. For certain types of trees, this time may be even longer. Thus, any mathematical model of forest harvesting and regeneration obviously involves and builds on time delays. Another example is that animals need time to digest their food before they can perform their activities. Delayed models have become widespread in many branches of biological modelling. These models are used to describe some topics such as infectious disease dynamics:(Ciupe et al., 2006; Nelson, Murray, and Perelson, 2000; Cooke, Kuang, and Bingtuan, 2005). Also, delays occur in studies of topics such as chemostat models (Zhao, T 1995); circadian rhythms (Smolen, Baxter, and Byrne, 2002); epidemiology (Cooke, van den Driessche, and Zou, 1999); the respiratory system (Vielle, and Chauvet, 1998); tumour growth (Villasana, Radunskaya, 2003); and neural networks (Campbell, Edwards, and Driessche, 2004). Statistical analysis of ecological data by (Turchin, P. 1990; Turchin, and Taylor, 1992) showed that there is evidence of delay effects in the population dynamics of many species.

Studies on delayed ordinary and partial DEs were carried out by many researchers (Ashyralyev and Akca, 1999, 2001; Ashyralyev, Akca and Guray, 1999; Yenicerioglu, 2007; Mohamad, Akca, and Covachev, 2009; Torelli, 1989; Ashyralyev, Akca, and Yeniçerioğlu, 2003; Li, Bohner, and Meng, 2008; Xu et al., 2001; Wolfgang, 1981; Liang, and Xiao, 2004; Ferreira, 2008); they generally focus on the properties of the solution, such as oscillation, stability, periodicity, and asymptoticity. In general, the inclusion of an unbounded delay term in DEs makes it difficult to analyse these types of equations. Additionally, there are a couple of works for which analytical solutions are provided. Because of this reason, the studies on numerical approaches compensate for the dearth of theoretical research. Particularly, one of the primary techniques employed in this field is the finite difference method. (Lu, X. 1998) investigates monotone iterative schemes for finite-difference solutions of reaction-diffusion systems with time delays and provides improved iterative schemes using the upper-lower solutions approach with the Gauss-Seidel or the Jacobi method. (Gu and Wang, 2014) constructed a linearized Crank-Nicolson difference scheme for the solution of a partial equation with variable coefficient delay and showed that this scheme is unconditionally stable and converges with a quadratic degree of convergence in both space and time variables.

(Berezansky and Braverman, 2006) examined stability for non-autonomous equations of the Carathéodory type, obtained new explicit stability conditions for linear differential equations with some delays, and reduced the stability problem for an equation with some delay to a stability problem for a specially constructed unique delay equation. They applied their results to study the local asymptotic stability of the Mackey-Glass equation with non-constant coefficients and delays. (Yenicerioglu and Yalçnbaş, 2004) established the necessary conditions for the stability of the solutions of second-order linear delay equations with variable coefficients.

In addition, (Ashyralyev and Sobolevskii, 2001) consider the initial value problem for the parabolic type linear delay differential equations; they provide a sufficient condition for the stability of the solution to this problem and obtain the stability estimates of solutions in Hölder norms. Various types of initial and boundary value problems for delay parabolic partial differential equations were investigated by (Ashyralyev, and Agirseven, 2014a, 2014b, 2014c, 2014d; Ashyralyev, Agirseven, and Agarwal, 2020; Ashyralyev, 2007; Agirseven, 2012; Ashyralyev, and Agirseven, 2013); they gave theorems on stability and convergence, found approximate solutions for the problems using first and second-order accuracy difference schemes, and performed error analysis.

Finally, the existence and uniqueness of a bounded solution (BS) of nonlinear delay parabolic equations were established by (Ashyralyev, Agirseven and Ceylan, 2017); they provide sufficient conditions for the existence of a unique BS of nonlinear delay parabolic equations. It should be noted that in past publications (Diagana, and Mbaye, 2015; Iasson, and Miroslav, 2014; Igbida, 2011; Kiguradze, and Kusano, 2005; Mavinga, and Nkashama, 2010; Nakao, 1977; Poorkarimi, and Wiener, 1986, 1989, 1999; Poorkarimi, Wiener, and Shah, 1989; Sadkowski, 1978; S. Shah, Poorkarimi, and Wiener, 1986; Sheng, and Agarwal, 1994; Smirnitskii, and Sobolevskii, 1981; Smirnitskii, 1993; Vyazmin, and Sorokin, 2017; Wiener, 1993; Youssfi, Benkirane, and Hadfi, 2016), bounded solutions of nonlinear parabolic and hyperbolic partial differential equations with or without delay have been investigated. However, due to the generality of the strategy used in this research, a larger class of semi-linear parabolic equations can be treated.

Layout of the Present Thesis

Semi-linear delay parabolic equations take an important place in applied sciences and engineering applications. The theory and applications of several problems for semi-linear delay parabolic equations have been studied in several works. Linear problems for delay parabolic equations can be solved by classical methods like Fourier transform method, Fourier series method and Laplace method. However, these classical methods can be used basically in the case when the differential equation has constant coefficients. It is well known that the most useful method for solving nonlinear delay parabolic equations with dependent coefficients in t and in the space variables is operator method.

In the Master Thesis (Burcu Ceylan, 2012); theorems on the existence and uniqueness of bounded solutions of nonlinear delay parabolic differential equations with undepended coefficients in t were studied. The book by (Ashyralyev and Sobolevskii, 2004); is devoted to the construction and investigation of the new high order of accuracy difference schemes of approximating the solutions of regular and singular pertubation boundary value problems for partial differential equations. The construction is based on the exact difference scheme and Taylor's decomposition on the two or three points. This approach permitted essentially to extend a class of problems where the theory of difference methods is applicable. Namely, now it is possible to study the existence and uniqueness of bounded solutions of semi-linear delay parabolic differential and difference equations.

In the present thesis, we investigate the abstract form of the initial value problems:

$$\begin{cases} \frac{dv}{dt} + Av(t) = f(t, B(t)v(t), B(t)v(t-d)), t \in [0, \infty), \\ v(t) = \varphi(t), t \in [-d, 0] \end{cases} \tag{1.1}$$

in an arbitrary Banach space E with linear unbounded operators A and $B(t)$ with dense domains $D(A) \subset D(B(t))$ and

$$\begin{cases} \frac{du}{dt} + A(t)u(t) = g(t, u(t), u(t-\omega)), t \in [0, \infty), \\ u(t) = \varphi(t), t \in [-\omega, 0] \end{cases} \tag{1.2}$$

in an arbitrary Banach space E with the unbounded operators $A(t)$ in E with dense domains $D(A(t)) \subset E$.

The main aim of this study is to provide the sufficient condition for the existence of a unique BS to problems (1.1) and (1.2).

The organization of this thesis is as follows:

The first chapter contains an introduction, a historical note and literature survey, definitions, and some basic concepts.

In the second chapter, we apply classical methods and obtain analytical solutions to several initial boundary value problems (IBVPs) for a two-dimensional delay parabolic partial differential equation (DPPDE).

In the third chapter, we study the theorem on the existence and uniqueness of the initial value problem (1.1). A semi-linear parabolic differential equation with an unbounded delay term is used to establish the theorem, and four different semi-linear DPPDEs are used to illustrate the main theorem's application. Numerical results are provided. (This chapter was published in an open access journal, MDPI; Mathematics 2023, Volume 11, Issue 16, 3470, and some part of the chapter is also accepted for publication in AIP Conference Proceedings, ICAAM 2022).

In the fourth chapter, the main theorem on the existence and uniqueness of a BS of problem (1.2) is established for a nonlinear DPPDE. The application of the main theorem to three types of nonlinear DPPDEs is illustrated. Numerical results are presented. (This chapter is also sent for publication in an open access journal, Filomat, under review.)

Finally, chapter five contains conclusion and future work. MATLAB programs made to find approximate solutions are given in the appendices.

Some Basic Concepts and Definitions

This section highlights some basic concepts and definitions on the theory of ordinary and partial DEs leading us to conduct and understand the works in this thesis

Sturm-Liouville problem (Arfken, Weber, 2005)

We denote the Sturm-Liouville operator as

$$L\left[y\right] = -\frac{d}{dx}\left[p(x)\frac{dx}{dy}\right] + q(x)y$$

and consider the Sturm-Liouville equation

$$L\left[y\right] + \lambda y = 0, \tag{1.3}$$

where $p > 0$, p and q are continous functions on the interval$[0, l]$ with local boundary conditions

$$\alpha_1 y(0) + \alpha_2 p(0)y'(0) = 0, \beta_1 y(l) + \beta_2 p(l)y'(l) = 0, \tag{1.4}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$ or nonlocal boundary conditions

$$y(0) - y(l) = 0, y'(0) = 0, y'(0) - y'(l) = 0. \tag{1.5}$$

The problem of finding a complex number $\lambda = \mu$ such that the boundary value problems (1.3), (1.4) or (1.3), (1.5) have a non trivial solution are called

Sturm-Liouville problems. The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(x, \mu)$ is called eigenfunction. We will consider three types of Sturm-Liouville problems

The Sturm-Liouville problem with Dirichlet Condition

$$-u^{''}(x) + \lambda u(x) = 0, 0 < x < l, u(0) = u(l) = 0 \tag{1.6}$$

has solution

$$u_k(x) = \sin \frac{k\pi x}{l} \text{ and } \lambda_k = -(\frac{k\pi}{l})^2, k = 1, 2, 3, ...$$

In the case when $l = \pi$, we have that

$$u_k(x) = \sin kx \text{ and } \lambda_k = -k^2, \; k = 1, 2, 3, ...$$

The Sturm-Liouville problem with Neumann Condition

$$-u^{''}(x) + \lambda u(x) = 0, 0 < x < l, u'(0) = u'(l) = 0 \tag{1.7}$$

has solution

$$u_k(x) = \cos \frac{k\pi x}{l} \text{ and } \lambda_k = -(\frac{k\pi}{l})^2, k = 0, 1, 2, ...$$

In the case when $l = \pi$, we have that

$$u_k(x) = \cos kx \text{ and } \lambda_k = -k^2, k = 0, 1, 2, ...$$

The Sturm-Liouville problem with Nonlocal Condition

$$-u^{''}(x) + \lambda u(x) = 0, 0 < x < l, u(0) = u(l), u'(0) = u'(l) \tag{1.8}$$

has solution

$$u_k(x) = \cos \frac{2k\pi x}{l} \;, \; k = 0, 1, 2, ...$$

$$u_k(x) = \sin \frac{2k\pi x}{l} \ , \ k = 1, 2, 3, ...$$

and

$$\lambda_k = -4(\frac{k\pi}{l})^2, k = 0, 1, 2, ...$$

In the case when $l = \pi$, we have that

$$u_k(x) = \cos 2kx \ , \ k = 0, 1, 2, ...$$

$$u_k(x) = \sin 2kx \ , \ k = 1, 2, 3, ...$$

and

$$\lambda_k = -4k^2, k = 0, 1, 2, ...$$

Fourier Series (Brown, Churchyll, 1993)

Let l be a fixed number and $f(x)$ be a periodic function with periodic $2l$, defined on$(-l, l)$. The Fourier Series of $f(x)$ is a way of expanding the function $f(x)$ into infinite series involving sines and cosines;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) \tag{1.9}$$

where the Fourier coefficients a_0 a_n and b_n are defined by the integrals

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx \tag{1.10}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos(\frac{n\pi x}{l} dx, n = 1, 2, 3, ... \tag{1.11}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l}dx, n = 1, 2, 3, ... \tag{1.12}$$

The Laplace transform (Franklyn, 1949)

The Laplace transform can be helpful in solving ordinary and partial differential equations because its can replace an ordinary differential equation with an algebraic equation or replace a partial differential equation with an ordinary differential equation. Another reason that the Laplace transform is useful is that it can be deal with the boundary conditions of a partial differential equation on an infinite domain.

Definition 1. Let f be a real valued function of the real variable t, defined for $t \rangle 0$. Let s be a variable that we will assume to be real, and consider the function F defined by

$$F(s) = \int_{0}^{\infty} e^{-st} f(t)dt \tag{1.13}$$

for all values of s for which this integral exists. The function F defined by the integral (1.13) is called the Laplace transform of the function f . we will denote the Laplace transform F of f by $L\{f\}$ and denote $F(s)$ by $L\{f(t)\}$. Note that for those $s \epsilon C$ for which the integral makes sense $F(s)$ is a complex-valued function of complex number.

The Fourier transform (Bracewell, 1999)

There are several ways to define the Fourier transform of a function $f : R \to C$.

Definition 1. Let f be a real valued function of the real variable x , defined for $x \in (-\infty, \infty)$.Let s be a variable and consider the function F defined by

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-ixs}dx, \tag{1.14}$$

for all values of s for which this integral exists. The function F defined by the integral (1.14) is called the Fourier transform of the function f. We will denote the

Fourier transform F of f by $F\{f\}$ and denote $F(s)$ by $F\{f(x)\}$. Note that for those $s \epsilon C$ for which the integral makes sense $F(s)$ is a complex-valued function of complex number.

CHAPTER II

Integral Transform Methods for Time-Delay Parabolic Differential Equations

Introduction

In this section, we study the analytical solutions of several two-dimensional delay parabolic differential equations by using classical methods such as Fourier series, Fourier transform, and Laplace transform, we obtain the exact solution of five initial boundary value problems.

Fourier Series Method

We consider Fourier series method for solution of two dimensional delay parabolic differential equations with Dirichlet, Neumann and nonlocal boundary conditions.

Problem 2.1. *Obtain the Fourier series solution of the following IBVP*

$$\begin{cases} u_t(t,x,y) - \frac{1}{2}u_{xx}(t,x,y) - \frac{1}{2}u_{yy}(t,x,y) + \frac{1}{4}u_{xx}(t-1,x,y) + \frac{1}{4}u_{yy}(t-1,x,y) \\ \ = -\frac{1}{2}e^{-t+1}\sin x \sin y, 0 < t < \infty, 0 < x, y < \pi, \\ u(t,x,y) = e^{-t}\sin x \sin y, -1 \leq \ t \leq \ 0, 0 \leq x, y \leq \pi, \\ u(t,0,y) = u(t,\pi,y) = 0, 0 \leq \ y \leq \ \pi, 0 \leq t < \infty, \\ u(t,x,0) = u(t,x,\pi) = 0, 0 \leq \ x \leq \ \pi, 0 \leq t < \infty. \end{cases} \tag{2.15}$$

Solution. In order to solve this problem, we cosider the Sturm-Liouville problem

$$-u''(x) + \lambda u(x) = 0, 0 \leq \ x \leq \ \pi, u(0) = u(\pi) = 0$$

generated by the space operator of problem (2.15). It is clear that the solution of this Sturm-Liouville problem is

$$u_k(x) = \sin kx, \lambda_k = -k^2, k = 1, 2, ...$$

Then, we will seek the Fourier series solution of problem (2.15) by the formula

$$u(t,x,y) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{k,n}(t) \sin kx \sin ny.$$

Here, $A_{k,n}(t), k, n = 1, 2, \ldots,$ are unknown functions. Applying this formula to the two dimensional delay heat equation and initial condition, we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}A'_{k,n}(t)\sin kx\sin ny+\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(k^2+n^2)A_{k,n}(t)\sin kx\sin ny$$
$$-\frac{1}{4}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(k^2+n^2)A_{k,n}(t-1)\sin kx\sin ny=-\frac{1}{2}e^{-t+1}\sin x\sin y,$$
$$0<t<\infty, 0<x,y<\pi,$$

and

$$u(t,x,y)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}A_{k,n}(t)\sin kx\sin ny$$
$$=e^{-t}\sin x\sin y, -1\le\ t\le\ 0, 0\le x,y\le\pi.$$

Equating coefficients of $\sin kx, k = 1, 2, ...$ and $\sin ny, n = 1, 2, ...$ to zero, we get

$$\begin{cases}A'_{1,1}(t)+A_{1,1}(t)-\frac{1}{2}A_{1,1}(t-1)=-\frac{1}{2}e^{-t+1}, t\ge 0,\\ A_{1,1}(t)=e^{-t}, -1\le\ t>0,\end{cases}$$

and for $k, n \neq 1$

$$\begin{cases}A'_{k,n}(t)+\frac{1}{2}(k^2+n^2)A_{k,n}(t)-\frac{1}{4}(k^2+n^2)A_{k,n}(t-1)=0, t\ge 0,\\ A_{k,n}(t)=0, -1\le\ t>0.\end{cases}$$

First, we will obtain $A_{1,1}(t)$. It is clear that $A_{1,1}(t)$ is the solution of the following initial value problem

$$\begin{cases}A'_{1,1}(t)+A_{1,1}(t)-\frac{1}{2}A_{1,1}(t-1)=-\frac{1}{2}e^{-t+1}, t\ge 0,\\ A_{1,1}(t)=e^{-t}, -1\le\ t\le\ 0\end{cases}$$

for the ordinary differential equation. We denote that

$$A_{1,1}(t)=\begin{cases}A_{1,1,0}(t)=e^{-t}, -1\le t\le 0,\\ A_{1,1,m}(t), m-1\le t\le m, m=1,2,\ldots.\end{cases}$$

Then,

$$\begin{cases}A'_{1,1,1}(t)+A_{1,1,1}(t)=0, 0<t<1,\\ A_{1,1,1}(0)=1.\end{cases}$$

Therefore,

$$A_{1,1,1}(t)=A_{1,1,1}(0)e^{-t}=e^{-t}.$$

Let $A_{1,1,m-1}(t) = e^{-t}, m-2 \leq t \leq m-1$, then $A_{1,1,m}(t)$ can be define by

$$\begin{cases} A'_{1,1,m}(t) + A_{1,1,m}(t) - \dfrac{1}{2}A_{1,1,m-1}(t-1) = -\dfrac{1}{2}e^{-t+1}, m-1 < t < m, \\ A_{1,1,m}(t-1) = A_{1,1,m-1}(t-1) = e^{-t+1}. \end{cases}$$

Then,

$$\begin{cases} A'_{1,1,m}(t) + A_{1,1,m}(t) = 0, m-1 < t < m, \\ A_{1,1,m}(m-1) = e^{-(m-1)}. \end{cases}$$

Therefore,

$$\begin{aligned} A_{1,1,m}(t) &= A_{1,1,m}(m-1)e^{-(t-m+1)} \\ &= e^{-(m-1)}e^{-(t-m+1)} = e^{-t}. \end{aligned}$$

So, by induction it is true for any m.

Hence,

$$A_{1,1}(t) = e^{-t}.$$

Recall that for $k, n \neq 1$ we have

$$\begin{cases} A'_{k,n}(t) + \dfrac{1}{2}(k^2+n^2)A_{k,n}(t) - \dfrac{1}{4}(k^2+n^2)A_{k,n}(t-1) = 0, t > 0 \\ A_{k,n}(t) = 0, -1 \leq \ t \leq \ 0. \end{cases}$$

It is easy, to see that

$$A_{k,n}(t) = 0.$$

Therefore, the exact solution for the initial boundary value problem (2.15) is

$$\begin{aligned} u(t,x,y) &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{k,n}(t)\sin kx \sin ny \\ &= A_{1,1}(t)\sin y \sin x = e^{-t}\sin x \sin y \end{aligned}$$

Note that using similar procedure one can obtain the solution of the following IBVP

$$\begin{cases} u_t\left(t,x\right)-\sum\limits_{r=1}^{n}(a_r u_{x_r}(t,x))_{x_r}=b\sum\limits_{r=1}^{n}(a_r u_{x_r}(t-w,x))_{x_r}+f\left(t,x\right), \\ 0<t<\infty, x=(x_1,...,x_n)\in\Omega, \\ u\left(t,x\right)=\varphi\left(t,x\right), x\in\overline{\Omega}, t\in[-w,0] \\ u(t,x)=0, x\in S, t\in[0,\infty) \end{cases} \tag{2.16}$$

for the multidimensional delay parabolic equation with Dirichlet boundary condition can be investigated. Here and in future $\Omega\subset R^n$ be a bounded open domain with smooth boundary S,$\bar{\Omega}=\Omega\cup S$. Under compatibility conditions problem (2.16) has a unique solution $u(t,x)$ for the smooth functions $f(t,x),(t,x)\in(0,\infty)\times\Omega, a_r>a>0,\ \varphi(t,x), x\in\overline{\Omega}, t\in[-w,0]$.

Problem 2.2. *Obtain the Fourier series solution of the following IBVP*

$$\begin{cases} u_t(t,x,y)-\frac{1}{2}u_{xx}(t,x,y)-\frac{1}{2}u_{yy}(t,x,y)+\frac{1}{4}u_{xx}(t-1,x,y)+\frac{1}{4}u_{yy}(t-1,x,y) \\ \ =-\frac{1}{2}e^{-t+1}\cos x\cos y, 0<t<\infty, 0<x,y<\pi, \\ u(t,x,y)=e^{-t}\cos x\cos y, -1\leq\ t\leq\ 0, 0\leq x,y\leq\pi, \\ u_x(t,0,y)=u_x(t,\pi,y)=0, 0\leq\ y\leq\ \pi, \\ u_y(t,x,0)=u_y(t,x,\pi)=0, 0\leq\ x\leq\ \pi, 0\leq t<\infty. \end{cases} \tag{2.17}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u^{''}(x)+\lambda u(x)=0, 0\leq\ x\leq\ \pi, u^{'}(0)=u^{'}(\pi)=0$$

generated by the space operator of problem (2.17). It is clear that the solution of this Sturm-Liouville problem is

$$u_k(x)=\cos kx, \lambda_k=-k^2, k=0,1,\ldots.$$

Then, we will seek the Fourier series solution of problem (2.17) by the formula

$$u(t,x,y)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_{k,n}(t)\cos kx\cos ny.$$

Here, $A_{k,n}(t), k, n = 0, 1, \ldots,$ are unknown functions. Applying this formula to the two dimensional delay heat equation and initial condition, we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A'_{k,n}(t)\cos kx \cos ny + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k^2+n^2)A_{k,n}(t)\cos kx\cos ny$$
$$-\frac{1}{4}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k^2+n^2)A_{k,n}(t-1)\cos kx\cos ny = -\frac{1}{2}e^{-t+1}\cos x\cos y,$$
$$0 < t < \infty, 0 < x, y < \pi,$$

and

$$u(t,x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_{k,n}(t)\cos kx\cos ny$$
$$= e^{-t}\cos x\cos y, -1 \le t \le 0, 0 \le x, y \le \pi.$$

Equating coefficients of $\cos kx, k = 0, 1, ...,$ and $\cos ny, n = 0, 1, ...,$ to zero, we get

$$\begin{cases} A'_{1,1}(t) + A_{1,1}(t) - \frac{1}{2}A_{1,1}(t-1) = -\frac{1}{2}e^{-t+1}, t > 0, \\ A_{1,1}(t) = e^{-t}, -1 \le t \le 0, \end{cases}$$

and for $k, n \neq 1$

$$\begin{cases} A'_{k,n}(t) + \frac{1}{2}(k^2+n^2)A_{k,n}(t) - \frac{1}{4}(k^2+n^2)A_{k,n}(t-1) = 0, t \ge 0, \\ A_{k,n}(t) = 0, -1 \le t \le 0. \end{cases}$$

First, we will obtain $A_{1,1}(t)$. It is clear that $A_{1,1}(t)$ is the solution of the following initial value problem

$$\begin{cases} A'_{1,1}(t) + A_{1,1}(t) - \frac{1}{2}A_{1,1}(t-1) = -\frac{1}{2}e^{-t+1}, t \ge 0, \\ A_{1,1}(t) = e^{-t}, -1 \le t \le 0 \end{cases}$$

for ordinary differential equation. We denote that

$$A_{1,1}(t) = \begin{cases} A_{1,1,0}(t) = e^{-t}, -1 \le t \le 0, \\ A_{1,1,m}(t), m-1 \le t \le m, m = 1, 2, \ldots. \end{cases}$$

Then,

$$\begin{cases} A'_{1,1,1}(t) + A_{1,1,1}(t) = 0, 0 < t < 1, \\ A_{1,1,1}(0) = 1. \end{cases}$$

Therefore,

$$A_{1,1,1}(t) = A_{1,1,1}(0)e^{-t} = e^{-t}.$$

Let $A_{1,1,m-1}(t) = e^{-t}, m-2 \le t \le m-1$, then $A_{1,1,m}(t)$ can be define by

$$\begin{cases} A'_{1,1,m}(t) + A_{1,1,m}(t) - \frac{1}{2}A_{1,1,m-1}(t-1) = -\frac{1}{2}e^{-t+1}, m-1 < t < m, \\ A_{1,1,m}(t-1) = A_{1,1,m-1}(t-1) = e^{-t+1}. \end{cases}$$

Then,

$$\begin{cases} A'_{1,1,m}(t) + A_{1,1,m}(t) = 0, m-1 < t < m, \\ A_{1,1,m}(m-1) = e^{-(m-1)}. \end{cases}$$

Therefore,

$$\begin{aligned} A_{1,1,m}(t) &= A_{1,1,m}(m-1)e^{-(t-m+1)} \\ &= e^{-(m-1)}e^{-(t-m+1)} = e^{-t}. \end{aligned}$$

So, by induction it is true for any m.

Hence,

$$A_{1,1}(t) = e^{-t}.$$

Recall that for $k, n \neq 1$ we have

$$\begin{cases} A'_{k,n}(t) + \frac{1}{2}(k^2+n^2)A_{k,n}(t) - \frac{1}{4}(k^2+n^2)A_{k,n}(t-1) = 0, t > 0 \\ A_{k,n}(t) = 0, -1 \le \ t \le \ 0. \end{cases}$$

It is easy, to see that

$$A_{k,n}(t) = 0.$$

Therefore, the exact solution for the initial boundary value problem (2.17) is

$$\begin{aligned} u(t,x,y) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_{k,n}(t)\cos kx \cos ny \\ &= A_{1,1}(t)\cos y \cos x = e^{-t}\cos x \cos y \end{aligned}$$

Note that using similar procedure one can obtain the solution of the following IBVP

$$\begin{cases} u_t\left(t,x\right)-\sum\limits_{r=1}^{n}(a_r u_{x_r}(t,x))_{x_r}=b\sum\limits_{r=1}^{n}(a_r u_{x_r}(t-w,x))_{x_r}+f\left(t,x\right), \\ 0<t<\infty, x=(x_1,...,x_n)\in\Omega, \\ u\left(t,x\right)=\varphi\left(t,x\right), x\in\overline{\Omega}, t\in[-w,0] \\ \frac{\partial u(t,x)}{\partial\bar{p}}=0, x\in S, t\in[0,\infty), \end{cases} \tag{2.18}$$

for the multidimensional delay parabolic equation with Neumann boundary condition can be investigated. Under compatibility conditions problem (2.18) has a unique solution $u(t,x)$ for the smooth functions $f(t,x),(t,x)\in(0,\infty)\times\Omega, a_r > a > 0$, $\varphi(t,x), x\in\overline{\Omega}, t\in[-w,0]$. Here, $\bar{p}$ is the normal vector to S.

Problem 2.3. *Obtain the Fourier series solution of the following IBVP*

$$\begin{cases} u_t(t,x,y)-u_{xx}(t,x,y)-u_{yy}(t,x,y)+\dfrac{1}{16}u_{xx}(t-1,x,y)+\dfrac{1}{16}u_{yy}(t-1,x,y) \\ \quad =-\dfrac{1}{2}e^{-8(t-1)}\sin 2x\cos 2y, 0<t<\infty, 0<x,y<\pi, \\ u(t,x,y)=e^{-8t}\sin 2x\cos 2y, -1\le\ t\le\ 0, 0\le x,y\le\pi, \\ u(t,0,y)=u(t,\pi,y), u_x(t,0,y)=u_x(t,\pi,y), 0\le y\le\pi, 0\le t<\infty, \\ u(t,x,0)=u(t,x,\pi), u_y(t,x,0)=u_y(t,x,\pi), 0\le x\le\pi, 0\le t<\infty. \end{cases} \tag{2.19}$$

Solution. In order to solve this problem, we consider the Sturm-Liouville problem

$$-u^{''}(x)+\lambda u(x)=0, 0\le\ x\le\ \pi, u(0)=u(\pi), u^{'}(0)=u^{'}(\pi)$$

generated by the space operator of problem (2.19). It is clear that the solution of this Sturm-Liouville problem is

$$u_n(x)=\cos 2nx, \lambda_n=-4n^2, n=0,1,..., u_k(x)=\sin 2kx, \lambda_k=-4k^2, n=1,2,....$$

Then, we will seek the Fourier series solution of problem (2.19) by the formula

$$u(t,x,y)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_{k,n}(t)\cos 2kx\cos 2ny+\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}B_{k,n}(t)\sin 2kx\cos 2ny$$
$$+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}D_{k,n}(t)\cos 2kx\sin 2ny+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}E_{k,n}(t)\sin 2kx\sin 2ny.$$

Here, $A_{k,n}(t), B_{k,n}(t), D_{k,n}(t)$ and $E_{k,n}(t)$ are unknown functions. Applying this formula to the two dimensional delay heat equation and initial condition, we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A'_{k,n}(t)\cos 2kx\cos 2ny+\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}B'_{k,n}(t)\sin 2kx\cos 2ny$$
$$+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}D'_{k,n}(t)\cos 2kx\sin 2ny+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}E'_{k,n}(t)\sin 2kx\sin 2ny$$
$$+\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(4k^2+4n^2)A_{k,n}(t)\cos 2kx\cos 2ny$$
$$+\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(4k^2+4n^2)B_{k,n}(t)\sin 2kx\cos 2ny$$
$$+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(4k^2+4n^2)D_{k,n}(t)\cos 2kx\sin 2ny$$
$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(4k^2+4n^2)E_{k,n}(t)\sin 2kx\sin 2ny$$
$$=-\frac{1}{2}e^{-8(t-1)}\sin 2x\cos 2y, 0<t<\infty, 0<x,y<\pi,$$

and

$$u(t,x,y)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_{k,n}(t)\cos 2kx\cos 2ny+\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}B_{k,n}(t)\sin 2kx\cos 2ny$$
$$+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}D_{k,n}(t)\cos 2kx\sin 2ny+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}E_{k,n}(t)\sin 2kx\sin 2ny$$
$$=e^{-8t}\sin 2x\cos 2y, -1\leq\ t\leq\ 0, 0\leq x,y\leq\pi.$$

Equating coefficients of $\cos 2kx\cos 2ny, \sin 2kx\cos 2ny, \cos 2kx\sin 2ny$ and $\sin 2kx\sin 2ny$, to zero, we get

$$\begin{cases} B'_{1,1}(t)+8B_{1,1}(t)-\dfrac{1}{2}B_{1,1}(t-1)=-\dfrac{1}{2}e^{-8(t-1)}, t>0, \\ B_{1,1}(t)=e^{-8t}, -1\leq\ t\leq\ 0, \end{cases}$$

and for $k,n\neq 1$

$$\begin{cases} A'_{k,n}(t)+(4k^2+4n^2)A_{k,n}(t)-\dfrac{1}{16}(4k^2+4n^2)A_{k,n}(t-1)=0, t>0, \\ A_{k,n}(t)=0, -1\leq\ t\leq\ 0, \end{cases}$$

$$\begin{cases} B'_{k,n}(t) + (4k^2+4n^2)B_{k,n}(t) - \dfrac{1}{16}(4k^2+4n^2)B_{k,n}(t-1) = 0, t > 0, \\ B_{k,n}(t) = 0, -1 \le t \le 0, \end{cases}$$

$$\begin{cases} D'_{k,n}(t) + (4k^2+4n^2)D_{k,n}(t) - \dfrac{1}{16}(4k^2+4n^2)D_{k,n}(t-1) = 0, t > 0, \\ D_{k,n}(t) = 0, -1 \le \ t \le \ 0, \end{cases}$$

$$\begin{cases} E'_{k,n}(t) + (4k^2+4n^2)E_{k,n}(t) - \dfrac{1}{16}(4k^2+4n^2)E_{k,n}(t-1) = 0, t > 0, \\ E_{k,n}(t) = 0, -1 \le \ t \le \ 0. \end{cases}$$

It is easy to see that; $A_{k,n}(t) = D_{k,n}(t) = E_{k,n}(t) = 0$ for all k, n and $B_{k,n}(t) = 0$ for $k, n \neq 1$.

Now, we will obtain $B_{1,1}(t)$, it is clear that $B_{1,1}(t)$ is the solution of the following problem

$$\begin{cases} B'_{1,1}(t) + 8B_{1,1}(t) - \dfrac{1}{2}B_{1,1}(t-1) = -\dfrac{1}{2}e^{-8(t-1)}, t \ge 0, \\ B_{1,1}(t) = e^{-8t}, -1 \le \ t \le \ 0 \end{cases}$$

for ordinary differential equation. We denote that

$$B_{1,1}(t) = \begin{cases} B_{1,1,0}(t) = e^{-8t}, -1 \le t \le 0, \\ B_{1,1,m}(t), m-1 \le t \le m, m = 1, 2, \ldots. \end{cases}$$

Then,

$$\begin{cases} B'_{1,1,1}(t) + B_{1,1,1}(t) = 0, 0 < t < 1, \\ B_{1,1,1}(0) = 1, \end{cases}$$

therefore,

$$B_{1,1,1}(t) = B_{1,1,1}(0)e^{-8t} = e^{-8t}.$$

Let $B_{1,1,m-1}(t) = e^{-8t}, m-2 \le t \le m-1$, then $B_{1,1,m}(t)$ can be define by

$$\begin{cases} B'_{1,1,m}(t) + 8B_{1,1,m}(t) - \dfrac{1}{2}B_{1,1,m-1}(t-1) = -\dfrac{1}{2}e^{-8(t-1)}, m-1 < t < m, \\ B_{1,1,m}(t-1) = B_{1,1,m-1}(t-1) = e^{-8(t-1)}. \end{cases}$$

Then,

$$\begin{cases} B'_{1,1,m}(t) + 8B_{1,1,m}(t) = 0, m-1 < t < m, \\ B_{1,1,m}(m-1) = e^{-8(m-1)}. \end{cases}$$

Therefore,

$$\begin{aligned} B_{1,1,m}(t) &= B_{1,1,m}(m-1)e^{-8(t-m+1)} \\ &= e^{-8(m-1)}e^{-8(t-m+1)} = e^{-8t}. \end{aligned}$$

So, by induction it is true for any m.

Hence,

$$B_{1,1}(t) = e^{-8t}.$$

Therefore, the exact solution for the initial boundary value problem (2.19) is

$$\begin{aligned} u(t,x,y) &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} B_{k,n}(t)\sin 2kx\cos 2ny \\ &= B_{1,1}(t)\cos 2y\sin 2x = e^{-8t}\sin 2x\cos 2y. \end{aligned}$$

Note that using similar procedure one can obtain the solution of the following IBVP

$$\begin{cases} u_t\left(t,x\right) - \sum\limits_{r=1}^{n}(a_r u_{x_r}(t,x))_{x_r} = b\sum\limits_{r=1}^{n}(a_r u_{x_r}(t-w,x))_{x_r} + f\left(t,x\right), \\ \\ 0<t<\infty, x=(x_1,...,x_n)\in\Omega, \\ \\ u\left(t,x\right) = \varphi\left(t,x\right), x\in\overline{\Omega}, t\in[-w,0] \\ \\ u(t,x)|_{S_1} = u(t,x)|_{S_2}, \frac{\partial u(t,x)}{\partial \bar{p}}|_{S_1} = \frac{\partial u(t,x)}{\partial \bar{p}}|_{S_2}, x\in S, t\in[0,\infty) \end{cases} \tag{2.20}$$

for the multidimensional delay parabolic differential equation with nonlocal boundary condition can be investigated. Under compatibility conditions problem (2.20) has a unique solution $u(t,x)$ for the smooth functions $f(t,x),(t,x)\in(0,\infty)\times\Omega, a_r>a>0,\ \varphi(t,x), x\in\overline{\Omega}, t\in[-w,0]$. Here,$S_1\cap S_2=\emptyset, S_1\cup S_2=S,\ \bar{p}$ is the normal vector to S.

The Laplace transform solution

We consider Laplace transform method for solution of the two dimensional delay semi-linear parabolic equation.

Problem 2.4. *Obtain the Laplace transform solution of the following IBVP*

$$\begin{cases} u_t(t,x,y) - u_{xx}(t,x,y) - u_{yy}(t,x,y) + \dfrac{1}{4}u_{xx}(t-1,x,y) + \dfrac{1}{4}u_{yy}(t-1,x,y) \\ \ + 3u(t,x,y) = -\dfrac{1}{2}e^{-t+1-x-y}, 0 < t < \infty, 0 < x, y < \infty, \\ u(t,x,y) = e^{-t-x-y}, -1 \le t \le 0, 0 \le x, y \le \infty, \\ u(t,0,y) = e^{-t-y}, u_x(t,0,y) = -e^{-t-y}, 0 \le y \le \infty, \\ u(t,x,0) = e^{-t-x}, u_y(t,x,0) = -e^{-t-x}, 0 \le x \le \infty, 0 \le t < \infty. \end{cases} \tag{2.21}$$

Solution. Using the formulas

$$L_x\{e^{-x}\} = \frac{1}{s+1}, L_y\{e^{-y}\} = \frac{1}{w+1}$$

and taking the Laplace transform for x of both sides of the parabolic equation and using condition, $u(t,0,y) = e^{-t-y}, u_x(t,0,y) = -e^{-t-y}$, we can write

$$L_x\left\{\frac{\partial u(t,x,y)}{\partial t}\right\} - L_x\left\{\frac{\partial^2 u(t,x,y)}{\partial x^2}\right\} - L_x\left\{\frac{\partial^2 u(t,x,y)}{\partial y^2}\right\} + \frac{1}{4}L_x\left\{\frac{\partial^2 u(t-1,x,y)}{\partial x^2}\right\}$$
$$+ \frac{1}{4}L_x\left\{\frac{\partial^2 u(t-1,x,y)}{\partial y^2}\right\} + 3L_x\left\{u(t,x,y)\right\} = -\frac{1}{2}L_x\left\{e^{-t+1-x-y}\right\}, 0 < t < \infty,$$

and

$$L_x\left\{u(t,x,y)\right\} = \frac{1}{s+1}e^{-t-y}, L_x\left\{u(t,x,0)\right\} = \frac{1}{s+1}e^{-t},$$
$$L_x\left\{u_y(t,x,0)\right\} = -\frac{1}{s+1}e^{-t}, -1 \le t \le 0, y \ge 0.$$

Then,

$$\begin{cases} u_t(t,s,y) - \{s^2 u(t,s,y) - se^{-t-y} + e^{-t-y}\} - u_{yy}(t,s,y) \\ + \dfrac{1}{4}\{s^2 u(t-1,s,y) - se^{-t+1-y} + e^{-t+1-y}\} + \dfrac{1}{4}u_{yy}(t-1,s,y) \\ + 3u(t,s,y) = -\dfrac{1}{2}e^{-t+1-y}\dfrac{1}{s+1}, t \ge 0, \\ u(t,s,y) = \dfrac{1}{s+1}e^{-t-y}, u(t,s,0) = \dfrac{1}{s+1}e^{-t}, u_y(t,s,0) = -\dfrac{1}{s+1}e^{-t}, -1 \le t \le 0. \end{cases}$$

Now, taking the Laplace transform with respect to y, we get

$$\begin{aligned} u_t(t,s,w) &- \left\{ s^2u(t,s,w) - se^{-t}\frac{1}{w+1} + e^{-t}\frac{1}{w+1} \right\} \\ &- \left\{ w^2u(t,s,w) - we^{-t}\frac{1}{s+1} + e^{-t}\frac{1}{s+1} \right\} \\ &+ \frac{1}{4}\left\{ s^2u(t-1,s,w) - se^{-t+1}\frac{1}{w+1} + e^{-t+1}\frac{1}{w+1} \right\} \\ &+ \frac{1}{4}\left\{ w^2u(t-1,s,w) - we^{-t+1}\frac{1}{s+1} + e^{-t+1}\frac{1}{s+1} \right\} \\ &+ 3u(t,s,w) = -\frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}, t \geq 0, \end{aligned}$$

and

$$u(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, -1 \leq t \leq 0.$$

So

$$\begin{cases} u_t(t,s,w) + (3-s^2-w^2)u(t,s,w) + \frac{1}{4}(s^2+w^2)u(t-1,s,w) \\ + (s-1)e^{-t}\frac{1}{w+1} + (w-1)e^{-t}\frac{1}{s+1} + (1-s)e^{-t+1}\frac{1}{w+1} \\ + (1-w)e^{-t+1}\frac{1}{s+1} = -\frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}, t \geq 0, \\ u(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, -1 \leq t \leq 0. \end{cases}$$

Now, we obtain $u(t,s,w)$. It is clear that $u(t,s,w)$ is a solution of the following initial boundary value problem,

$$\begin{cases} u_t(t,s,w) + (3-s^2-w^2)u(t,s,w) + \frac{1}{4}(s^2+w^2)u(t-1,s,w) \\ = (1-s)e^{-t}\frac{1}{w+1} + (1-w)e^{-t}\frac{1}{s+1} + (s-1)e^{-t+1}\frac{1}{w+1} \\ + (w-1)e^{-t+1}\frac{1}{s+1} - \frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}, t > 0, \\ u(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, -1 \leq t \leq 0. \end{cases}$$

We denote that

$$u(t,s,w) = \{u_m(t,s.w), m-1 \leq t \leq m, m = 1,2,\dots\}.$$

Since

$$u_1(t-1,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, -1 \leq t \leq 0,$$

we have that

$$\begin{cases} u_{1,t}(t,s,w) + (3 - s^2 - w^2)u_1(t,s,w) = -\frac{1}{4}(s^2 + w^2)e^{-t}\frac{1}{s+1}\frac{1}{w+1} \\ + (1-s)e^{-t}\frac{1}{w+1} + (1-w)e^{-t}\frac{1}{s+1} + (s-1)e^{-t+1}\frac{1}{w+1} \\ + (w-1)e^{-t+1}\frac{1}{s+1} - \frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}, t > 0, \\ u_1(0,s,w) = \frac{1}{s+1}\frac{1}{w+1}. \end{cases}$$

Solving this linear problem, we get

$$u_1(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, 0 \le t \le 1.$$

Let

$$u_{m-1}(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, m-2 \le t \le m-1.$$

Now, we obtain $u_m(t,s,w)$ as a solution of the problem

$$\begin{cases} u_{m,t}(t,s,w) + (3 - s^2 - w^2)u_m(t,s,w) + \frac{1}{4}(s^2 + w^2)u_m(t-1,s,w) \\ = (1-s)e^{-t}\frac{1}{w+1} + (1-w)e^{-t}\frac{1}{s+1} + (s-1)e^{-t+1}\frac{1}{w+1} \\ + (w-1)e^{-t+1}\frac{1}{s+1} - \frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}, m-1 \le t \le m, \\ u(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, m-2 \le t \le m-1. \end{cases}$$

Since

$$u_m(t-1,s,w) = u_{m-1}(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1},$$

we have that

$$\begin{cases} u_{m,t}(t,s,w) + (3 - s^2 - w^2)u_m(t,s,w) = -\frac{1}{4}(s^2 + w^2)e^{-t}\frac{1}{s+1}\frac{1}{w+1} \\ + (1-s)e^{-t}\frac{1}{w+1} + (1-w)e^{-t}\frac{1}{s+1} + (s-1)e^{-t+1}\frac{1}{w+1} \\ + (w-1)e^{-t+1}\frac{1}{s+1} - \frac{1}{2}e^{-t+1}\frac{1}{s+1}\frac{1}{w+1}.t \ge 0, \\ u_m(m-1,s,w) = \frac{1}{s+1}\frac{1}{w+1}e^{-(m-1)}. \end{cases}$$

Therefore,

$$u_m(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, m-1 \le t \le m.$$

By induction,

$$u_m(t,s,w) = e^{-t}\frac{1}{s+1}\frac{1}{w+1}, m-1 \le t \le m,$$

is true for any $m \geq 1$. Thus

$$u(t,s,w) = \left\{ e^{-t}\frac{1}{s+1}\frac{1}{w+1}, m-1 \leq t \leq m, m = 1,2,\dots \right\} = e^{-t}\frac{1}{s+1}\frac{1}{w+1}.$$

Taking the inverse Laplace transform, we obtain the following exact solution of the problem

$$u(t,x,y) = e^{-t-x-y}.$$

Note that using similar procedure one can obtain the solution of the following IBVP

$$\begin{cases} u_t(t,x) - \sum_{r=1}^{n} (a_r u_{x_r}(t,x))_{x_r} = b \sum_{r=1}^{n} (a_r u_{x_r}(t-w,x))_{x_r} + f(t,x), \\ \quad x = (x_1,...,x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\ u(0,x) = \varphi(x), x \in \overline{\Omega}^+, t \in [-w,0] \\ u(t,x) = \alpha(t,x), \quad u_{x_r}(t,x) = \beta(t,x), \\ 1 \leq r \leq n, 0 \leq t \leq T, x \in S^+ \end{cases} \tag{2.22}$$

for the multidimensional delay differential equation. Assume that $a_r(x) > a_0 > 0$ and $f(t,x), \left(t \in (0,T), x \in \overline{\Omega}^+\right), \varphi(x), \left(x \in \overline{\Omega}^+\right), \alpha(t,x), \beta(t,x) \ (t \in [0,T], x \in S^+)$ are given smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \leq k \leq n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However Laplace transform method described in solving (2.22) can be used only in the case when (2.22) has constant coefficients.

The Fourier transform solution

We consider Fourier transform method for solution of the two dimensional delay parabolic differential equation.

Problem 2.5. *Obtain the Fourier transform solution of the following IBVP*

$$\begin{cases} u_t(t,x,y) - u_{xx}(t,x,y) - u_{yy}(t,x,y) + \frac{1}{4}u_{xx}(t-1,x,y) + \frac{1}{4}u_{yy}(t-1,x,y) \\ \quad + u(t,x,y) = (2-4x^2-4y^2)e^{-t-x^2-y^2} - \frac{1}{2}e^{-t+1-x^2-y^2}, \\ 0 < t < \infty, -\infty < x, y < \infty, \\ \quad u(t,x,y) = e^{-t-x^2-y^2}, -1 \le t \le 0, -\infty < x, y < \infty. \end{cases} \tag{2.23}$$

Solution. We denote

$$F\{u(t,x,y)\} = u(t,s,w).$$

Then, we have that

$$F\left\{\frac{\partial u(t,x,y)}{\partial t}\right\} = u_t(t,s,w)$$
$$F\left\{\frac{\partial^2 u(t,x,y)}{\partial x^2}\right\} = -s^2 u(t,s,w)$$
$$F\left\{\frac{\partial^2 u(t,x,y)}{\partial y^2}\right\} = -w^2 u(t,s,w).$$

Taking the Fourier transform of both sides of the equation and using initial condition, we get

$$\begin{aligned} &u_t(t,s,w) + s^2 u(t,s,w) + w^2 u(t,s,w) - \frac{1}{4}s^2 u(t-1,s,w) \\ &- \frac{1}{4}w^2 u(t-1,s,w) + u(t,s,w) = F_x F_y \left\{(2-4x^2-4y^2)e^{-x^2-y^2}\right\} e^{-t} \\ &- \frac{1}{2}F_x F_y \left\{e^{-x^2-y^2}\right\} e^{-t+1}, 0 < t < \infty, \end{aligned}$$

and

$$u(t,s,w) = F_x F_y \left\{e^{-x^2-y^2}\right\} e^{-t}, -1 \le t \le 0.$$

Since

$$\begin{aligned} F_x \left\{(2-4x^2)e^{-x^2}\right\} &= -F_x \left\{(e^{-x^2})''\right\} = s^2 F_x \left\{e^{-x^2}\right\}, \\ F_y \left\{(2-4y^2)e^{-y^2}\right\} &= -F_y \left\{(e^{-y^2})''\right\} = w^2 F_y \left\{e^{-y^2}\right\}. \end{aligned}$$

We can write

$$\begin{aligned} &u_t(t,s,w) + (1+s^2+w^2)u(t,s,w) - \frac{1}{4}(s^2+w^2)u(t-1,s,w) \\ &= \left(-F_x F_y \{2\} + s^2 F_x \left\{e^{-x^2}\right\} + w^2 F_y \left\{e^{-y^2}\right\}\right) e^{-t} \\ &- \frac{1}{2}F_x F_y \left\{e^{-x^2-y^2}\right\} e^{-t+1}, 0 < t < \infty, \end{aligned}$$

and

$$u(t,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},-1\le t\le 0.$$

Now, we obtain $u(t,s,w)$. It is clear that $u(t,s,w)$ is the solution of the initial boundary value problem

$$\begin{cases} u_t(t,s,w)+(1+s^2+w^2)u(t,s,w)-\dfrac{1}{4}(s^2+w^2)u(t-1,s,w) \\ =\left(-F_xF_y\left\{2\right\}+s^2F_x\left\{e^{-x^2}\right\}+w^2F_y\left\{e^{-y^2}\right\}\right)e^{-t} \\ -\dfrac{1}{2}F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t+1},0<t<\infty, \\ u(t,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},-1\le t\le 0. \end{cases}$$

We denote that

$$u(t,s,w)=\left\{u_m(t,s,w),(m-1)\le t\le m,m=1,2,\dots\right\}$$

Since $u_1(t-1,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},-1\le t\le 0$, we have that

$$\begin{cases} u_{1,t}(t,s,w)+(1+s^2+w^2)u_1(t,s,w)=\dfrac{1}{4}(s^2+w^2)F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t} \\ +\left(-F_xF_y\left\{2\right\}+s^2F_x\left\{e^{-x^2}\right\}+w^2F_y\left\{e^{-y^2}\right\}\right)e^{-t} \\ -\dfrac{1}{2}F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t+1},0<t<\infty, \\ u_1(0,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}. \end{cases}$$

Solving this linear problem, we get

$$u_1(t,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},0\le t\le 1.$$

Let

$$u_{m-1}(t,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},m-2\le t\le m-1.$$

Now, we obtain $u_m(t,s,w)$ as the solution of problem

$$\begin{cases} u_{m,t}(t,s,w)+(1+s^2+w^2)u_m(t,s,w)-\dfrac{1}{4}(s^2+w^2)u_m(t-1,s,w) \\ =\left(-F_xF_y\left\{2\right\}+s^2F_x\left\{e^{-x^2}\right\}+w^2F_y\left\{e^{-y^2}\right\}\right)e^{-t} \\ -\dfrac{1}{2}F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t+1},m-1\le t\le m, \\ u_m(t,s,w)=F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t},m-2\le t\le m-1. \end{cases}$$

Since $u_m(t-1,s,w) = u_{m-1}(t,s,w) = F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t}$, we have that

$$\begin{cases} u_{m,t}(t,s,w) + (1+s^2+w^2)u_m(t,s,w) = \frac{1}{4}(s^2+w^2)F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t} \\ +\left(-F_xF_y\left\{2\right\} + s^2F_x\left\{e^{-x^2}\right\} + w^2F_y\left\{e^{-y^2}\right\}\right)e^{-t} \\ -\frac{1}{2}F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t+1}, m-1<t<m, \\ u_m(m-1,s,w) = F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-(m-1)}. \end{cases}$$

Therefore,

$$u_m(t,s,w) = F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t}, m-1 \le t \le m.$$

By induction,

$$u_m(t,s,w) = F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t}, m-1 \le t \le m,$$

is true for any $m \ge 1$. Thus,

$$\begin{aligned} u(t,s,w) &= \left\{F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t}, (m-1) \le t \le m, m = 1,2,\dots\right\} \\ &= F_xF_y\left\{e^{-x^2-y^2}\right\}e^{-t}. \end{aligned}$$

Taking the inverse Fourier transform, we obtain the following exact solution of the problem

$$u(t,x,y) = e^{-t-x^2-y^2}.$$

Note that using the same manner one obtain the solution of the following IBVP

$$\begin{cases} u_t\left(t,x\right) - \sum\limits_{r=1}^{n}(a_r u_{x_r}(t,x))_{x_r} = b\sum\limits_{r=1}^{n}(a_r u_{x_r}(t-w,x))_{x_r} + f\left(t,x\right), \\ 0<t<T, x, r \in \mathbb{R}^n, |r| = r_1 + ... + r_n, \\ u(0,x) = \varphi(x), x \in \mathbb{R}^n \end{cases} \quad (2.24)$$

for the multidimensional delay parabolic differential equations. Assume that $\alpha_r \ge \alpha \ge 0$ and $f\left(t,x\right), \left(t \in [0,T], x \in \mathbb{R}^n\right), \varphi(x), (x \in \mathbb{R}^n)$ are given smooth functions.

However Fourier transform method described in solving (2.24) can be used only in the case when (2.24) has constant coefficients.

CHAPTER III

Stability of the Time-Delay Parabolic Differential Equations

Introduction

In this section, the necessary conditions for the existence of unique bounded solutions of the semi-linear delay parabolic differential equation in an arbitrary Banach space E with strongly unbounded operators are established. In practice, theorems on stability estimation for the solution of the initial boundary value problem for four different semi-linear delay parabolic equations are obtained.

Auxiliary Statements

Necessary definitions, estimates, lemmas, and theorems by (Ashyralyev, 2014; Kreyszig, 1978; and Kolmogorov, 1965) are given below.

Banach and Hilbert Spaces

Let L be linear space.Then

$$x, y \in L, \exists\ x + y \in L \text{ and } \lambda x \in L,\ \ \lambda \text{ is a number.}$$

$E = (L, \|\cdot\|)$ be normed space

$$\begin{aligned} \forall x &\in L,\ \varphi(x) = \|x\|, \\ 1.\ \|x\| &\geq 0,\ \ \|x\| = 0 \Longleftrightarrow x = \widetilde{0}\ \ (\text{zero element}), \\ 2.\ \|\lambda x\| &= |\lambda|\, \|x\|, \\ 3.\ \|x + y\| &\leq \|x\| + \|y\| \ \text{ for any } x, y \in L. \end{aligned}$$

Then we say that, E is a Banach spaces if E- is normed space and E - is complete $\Longleftrightarrow$ Every Cauchy sequence is convergent $\Longleftrightarrow$ From $\|x_n - x_m\| \underset{n,m\to\infty}{\longrightarrow} 0 \ \Rightarrow \exists x \in E,\ \ \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$. We denote it by E, the all Banach spaces.

$H = (L, \langle\cdot\rangle)$ be inner product space

$$\begin{aligned}
1.\, \langle x, y\rangle &= \langle y, x\rangle\,, \\
2.\, \langle x_1 + x_2, y\rangle &= \langle x_1, y\rangle + \langle x_2, y\rangle\,, \\
3.\, \langle \lambda x, y\rangle &= \lambda \langle x, y\rangle\,, \\
4.\, \langle x, x\rangle &= 0 \Longleftrightarrow x = \widetilde{0}
\end{aligned}$$

$\|x\| = \sqrt{\langle x, x\rangle}$. So all inner product spaces are also normed spaces. We say that, H is a Hilbert space if H - is an inner product space and H - is a complete space.

Linear Operators: Boundedness, Norm of Operator

$A : E \to E_1$ is called the linear operator if $D(A)$ is the linear space and

$$\begin{aligned}
A(\alpha x + \beta y) &= \alpha Ax + \beta Ay \text{ for any } \alpha, \beta \text{ numbers, } x, y \in D(A)\,, \\
D(A) &= \{x \in E,\ \exists\ Ax\}\,, \\
R(A) &= \{z \in E_1,\ z = Ax \text{ for any } x \in D(A)\}\,.
\end{aligned}$$

E and E_1 be Banach spaces. In the case when $E_1 = (-\infty, \infty)$, $A : E \to (-\infty, \infty)$ is called the linear functional.

Definition 3.2.2.1. Let E and E_1 be Banach spaces. $A : E \to E_1$ is called the bounded operator if there is a real positive $M > 0$ such that

$$\|Ax\|_{E_1} \le M \|x\|_E \text{ for all } x \in D(A)\,.$$

$\inf M = \|A\|_{E \to E_1}$ is called norm of the operator A. If $E = E_1$,

$$\|A\|_{E \to E_1} = \|A\|_{E \to E} = \|A\|\,.$$

Theorem 3.1. *The following formulas are valid:*

$$\|A\| = \sup_{\|x\|_E \le 1} \|Ax\|_E = \sup_{\|x\|_E = 1} \|Ax\|_E = \sup_{\|x\|_E \ne \widetilde{0} \in E} \frac{\|Ax\|_E}{\|x\|_E}.$$

Linear Positive Operators in a Hilbert Space

Let $A : H \to H$ be a linearly bounded operator in a Hilbert Space H. Then $A^* : H \to H$ is defined to be the operator satisfying

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \text{ for any } x, y \in H.$$

A^* is called the Hilbert adjoint operator A^* to A. A is said to be self adjoint or Hamiltonian, if

$$A = A^* \Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle \text{ for any } x, y \in H.$$

Let $A : H \to H$ is said to be positive and written $A \geq \widetilde{0}$ if

$$\langle Ax, x \rangle \geq 0 \text{ for any } x \in H.$$

$A : H \to H$ is said to be positive definite and written $A \geq \delta I > \widetilde{0}$ if

$$\langle Ax, x \rangle \geq \delta \langle x, x \rangle \text{ for any } x \in H.$$

We consider some examples of positive operators in a Hilbert space

Let $L_2[0, l]$ be the space of all square integrable functions $\gamma(x)$ difened on $[0, l]$ equipped with the norm

$$||\gamma||_{L_2[0,l]} = \left(\int_0^l |\gamma(x)|^2 dx \right)^2.$$

First, we introduce the differential operator A defined by the formula

$$Au = -\frac{d}{dx} \left(a(x) \frac{du(x)}{dx} \right) + \delta u(x) \tag{3.25}$$

with the domain

$$D(A) = \{ u : u, u'' \in L_2[0, l], u(0) = u(l) = 0 \}.$$

Lemma 3.1. Let $a(x) \geq a \geq 0$ and A be a differential operator defined by formula (3.25). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0, l]$.

Proof of lemma 3.1. Assume that $u, v \in D(A)$. Applying the following formula

$$< u, v >= \int_0^l u(x)v(x)dx,$$

we get

$$\begin{aligned} < Au, v > &= \int_0^l Au(x)v(x)dx \\ &= \int_0^l \left(-\frac{d}{dx} \left(a(x) \frac{du(x)}{dx} \right) + \delta u(x) \right) v(x)dx \\ &= -a(l) u^{'}(l)v(l) + a(0) u^{'}(0)v(0) \\ &+ \int_0^l a(x) u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx, \end{aligned} \tag{3.26}$$

and

$$
\begin{aligned}
< u, Av > &= \int_0^l u(x)Av(x)dx \\
&= \int_0^l u(x)\left(-\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right)+\delta v(x)\right)dx \\
&= -a(l)\,v^{'}(l)u(l) + a(0)\,v^{'}(0)u(0) \\
&+ \int_0^l a(x)\,u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx.
\end{aligned}
\tag{3.27}
$$

From (3.26) and (3.27) it follows

$$
< Au, v >=< u, Av >= \int_0^l a(x)\,u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx. \tag{3.28}
$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.28), we get

$$
< Au, u >= \int_0^l a(x)\,u^{'}(x)u^{'}(x)dx + \int_0^l \delta u(x)u(x)dx.
$$

Moreover, using the condition $u(0) = 0$, we get

$$
u(y) = \int_0^y \frac{du(x)}{dx}dx = \int_0^y \frac{du(y-t)}{dt}dt.
$$

We will introduce the following function u_* defined by formula

$$
\frac{du_*(y-t)}{dt} = \begin{cases} \frac{du(y-t)}{dt}, \ 0 \le t \le y, \ y \in [0, l], \\ \\ \quad 0, \qquad \text{otherwise.} \end{cases}
$$

Then

$$
u(y) = \int_0^l \frac{du_*(y-t)}{dt}dt.
$$

Applying the Minkowsky inequality and the definition of the function $u_*(x)$, we get

$$\left(\int_0^l u^2(y)dy\right)^{\frac{1}{2}} \leq \int_0^l \left(\int_0^l \left(\frac{du_*(y-t)}{dt}\right)^2 dy\right)^{\frac{1}{2}} dt$$

$$\leq \int_0^l \left(\int_0^l \left(\frac{du(x)}{dx}\right)^2 dy\right)^{\frac{1}{2}} dt$$

$$= l\left(\int_0^l \left(\frac{du(x)}{dx}\right)^2 dy\right)^{\frac{1}{2}}.$$

Therefore,

$$\langle u,\ u\rangle = \int_0^l u^2(y)dy \leq l^2 \int_0^l \left(\frac{du(x)}{dx}\right)^2 dx \tag{3.29}$$

$$= l^2 \int_0^l \frac{du(x)}{dx}\frac{du(x)}{dx}dx = l^2 \langle u', u'\rangle .$$

Applying the estimate (3.29), we get

$$< Au, u >\geq \left(\frac{a}{l^2}+\delta\right) < u, u > .$$

That means A is a positive definite equation operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Now, we introduce the differential operator A defined by the formula (3.25) with the domain

$$D(A) = \{u : u, u'' \in L_2[0, l], u'(0) = u'(l) = 0\}. \tag{3.30}$$

Lemma 3.2. Let $a(x) \geq 0$ and A be a differential operator defined by formula (3.25) with the domain (3.30). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0, l]$.

Proof of lemma3.2. Assume that $u, v \in D(A)$. Then, we have formulas

(3.26) and (3.27). Applying these formulas, we get

$$\begin{aligned} < Au, v > &= \int_0^l Au(x)v(x)dx \\ &= \int_0^l \left(-\frac{d}{dx}\left(a\left(x\right) \frac{du\left(x\right)}{dx} \right) + \delta u(x) \right) v(x)dx \\ &= \int_0^l a\left(x\right) u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} < u, Av > &= \int_0^l \ u(x)Av(x)dx \\ &= \int_0^l u(x) \left(-\frac{d}{dx}\left(a\left(x\right) \frac{dv\left(x\right)}{dx} \right) + \delta v(x) \right) dx \\ &= \int_0^l a\left(x\right) u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx. \end{aligned} \tag{3.32}$$

From (3.31) and (3.32) it follows

$$< Au, v >=< u, Av >= \int_0^l a\left(x\right) u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx. \tag{3.33}$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.33), we get

$$< Au, u >= \int_0^l a\left(x\right) u^{'}(x)u^{'}(x)dx + \int_0^l \delta u(x)u(x)dx \geq \delta < u, u > .$$

That means A is a positive definite operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Next, we introduce the differential operator A defined by the formula (3.25) with the domain

$$D\left(A\right) = \left\{ u : u, u'' \in L_2\left[0, l\right], u\left(0\right) = u\left(l\right), u'\left(0\right) = u'\left(l\right) \right\}. \tag{3.34}$$

Lemma 3.3. Let $a\left(x\right) \geq 0$ and $a\left(0\right) = a\left(l\right)$ and A be a differential operator defined by formula (3.25) with the domain (3.34). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0, l]$.

Proof of lemma 3.3. Assume that $u, v \in D(A)$. Then, we have formulas (3.26) and (3.27). Applying these formulas, we get

$$\begin{aligned} < Au, v > &=< u, Av > \\ &= \int_0^l \left(-\frac{d}{dx}\left(a\left(x\right) \frac{du\left(x\right)}{dx} \right) + \delta u(x) \right) v(x)dx \\ &= \int_0^l a\left(x\right) u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx. \end{aligned} \tag{3.35}$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.35), we get

$$< Au, u >= \int_0^l a(x)\, u^{'}(x)u^{'}(x)dx + \int_0^l \delta u(x)u(x)dx \geq \delta < u, u > .$$

That means A is a positive definite operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Finally, we introduce the differential operator A defined by the formula (3.25) with the domain

$$D(A) = \{u : u, u'' \in L_2[0, l], u(0) = bu'(0), -u(l) = cu'(l)\}. \tag{3.36}$$

Lemma 3.4. Let $a(x) \geq 0$, $b, c > 0$ and A be a differential operator defined by formula (3.25) with the domain (3.36). Prove that A is the positive definite and self-adjoint operator in $H = L_2[0, l]$.

Proof of lemma 3.4. Assume that $u, v \in D(A)$. Then, we have formulas (3.26) and (3.27). Applying these formulas, we get

$$\begin{aligned}
< Au, v > &=< u, Av > \\
&= \int_0^l \left(-\frac{d}{dx} \left(a(x) \frac{du(x)}{dx} \right) + \delta u(x) \right) v(x)dx \\
&= ca(l)\, u^{'}(l)v^{'}(l) + a(0)\, bu^{'}(0)v^{'}(0) \\
&+ \int_0^l a(x)\, u^{'}(x)v^{'}(x)dx + \int_0^l \delta u(x)v(x)dx.
\end{aligned} \tag{3.37}$$

That means A is a self-adjoint operator. Putting $u = v$ in (3.37), we get

$$\begin{aligned}
< Au, u > &= ca(l) \left(u^{'}(l) \right)^2 + a(0)\, b \left(u^{'}(0) \right)^2 \\
&+ \int_0^l a(x)\, u^{'}(x)u^{'}(x)dx + \int_0^l \delta u(x)u(x)dx \geq \delta < u, u > .
\end{aligned}$$

That means A is a positive definite operator. Therefore A is a self-adjoint and positive operator in a Hilbert space $H = L_2[0, l]$.

Banach Fixed-Point Theorem and Its Applications

Definition 3.2.5.1. Let $E = (E, d)$ be a metric space. A fixed pointof a mapping $T : E \to E$ of a set E into itself is an element $x \in E$ which is mapped onto itself, that is, $Tx = x$, the image Tx coincides with x. Note that the Banach fixed-point theorem to be stated below is an existence and uniqueness theorem for fixed points of certain mappings, and it also gives a constructive procedure for obtaining better and better approximations to the solution of the equation

$$x = Tx. \tag{3.38}$$

Actually, we choose an arbitrary $x_0 \in E$ and determine successively a sequence $\{x_j\}_{n=0}^{\infty}$ defined by the relation

$$x_j = Tx_{j-1},\ j \in \mathbb{N}_1. \tag{3.39}$$

Here and in this Thesis we will put $_k = \{j \in \mathbb{Z};\ j \geq k\}$.

This procedure is called an iteration. Banach's fixed-point theorem gives sufficient conditions for the existence and uniqueness of a fixed point of a class of mappings, called contractions.

Definition 3.2.5.2. A mapping $T : E \to E$ is called a contraction on E, if there is a positive real number $\alpha < 1$ such that for all $x, y \in E$

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{3.40}$$

Theorem 3.2. *Assume that $E \neq \varnothing$ is complete and let T be a contraction mapping on E. Then, T has precisely one fixed point.*

Theorem 3.3. *Let T be a mapping of a complete metric space E into itself. Assume that T is a contraction on a closed ball $F = \{x | d(x, x_0) \leq r\}$, that is, T satisfies assumption (3.40) for all $x, y \in F$. Moreover, assume that*

$$d(x_0, Tx_0) < (1 - \alpha)r. \tag{3.41}$$

Then, the sequence $\{x_j\}_{j=0}^{\infty}$ defined by recursive formula (3.39) with arbitrary $x_0 \in E$ converges to an $x \in F$. This x is a fixed point of the mapping T and is the only fixed point of T in F. Now, we study the applications of the fixed-point theorem to integral equations.

Definition 3.2.5.3. An integral equation of the form

$$x(t) = \mu \int_a^b k(t, s; x(s))\, ds + f(t) \tag{3.42}$$

is called a Fredholm equation of the second kind. Here, $[a, b]$ is a given interval, μ is a given parameter, f is a given function defined on $[a, b]$, x is an unknown function defined on $[a, b]$. The kernel k of the equation is a given function defined on $[a, b]\times [a, b] \times \mathbb{R}^1$.

Integral equations can be considered on various function spaces. We consider equation (3.42) on $C[a, b]$, the space of all continuous functions defined on the interval $[a, b]$ with the metric d defined by

$$d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)| . \tag{3.43}$$

$C[a, b] = (C[a, b], d)$ is complete. We assume that $f \in C[a, b]$ and k is a continuous function defined on $[a, b]\times [a, b] \times R^1$. Moreover, k satisfies on $[a, b]\times [a, b] \times \mathbb{R}^1$ the Lipschitz condition of the form

$$|k (t, s; u_1) - k (t, s; u_2)| \leq l |u_1 - u_2| . \tag{3.44}$$

Obviously, equation (3.42) can be written $x = Tx$, where

$$Tx(t) = \mu \int_a^b k (t, s; x (s)) \, ds + f (t) . \tag{3.45}$$

Since f and k are continuous functions, formula (3.45) defines an operator T : $C[a, b] \rightarrow C[a, b]$. We now impose a restriction on μ such that T becomes a contraction. Applying formulas (3.43), (3.45), and condition (3.44), we get

$$\begin{aligned} d(Tx, Ty) &= \max_{t\in[a,b]} |Tx(t) - Ty(t)| \\ &= |\mu| \max_{t\in[a,b]} \left| \int_a^b (k (t, s; x (s)) - k (t, s; y (s))) \, ds \right| \\ &\leq l |\mu| \max_{t\in[a,b]} \int_a^b |x (s) - y (s)| \, ds \leq l |\mu| \max_{s\in[a,b]} |x (s) - y (s)| \int_a^b ds \\ &= l |\mu| (b - a)d(x, y). \end{aligned}$$

So, $d(Tx, Ty) \leq \alpha d(x, y)$, where $\alpha = l |\mu| (b - a)$. We see that T becomes a contraction if

$$|\mu| < \frac{1}{l(b - a)}. \tag{3.46}$$

Banach's fixed-point theorem now gives the following theorem.

Theorem 3.4. *Assume that k and f in equation (3.42) are continuous functions on $[a, b]\times [a, b] \times \mathbb{R}^1$ and $[a, b]$, respectively. Moreover, k satisfies on $[a, b]\times$*

$[a,b]\times\mathbb{R}^1$ *the Lipschitz condition (3.44). Suppose that* μ *satisfies condition (3.46). Then, equation (3.42) has a unique solution* x *defined on* $[a,b]$*. This function* x *is the limit of the iterative sequence* $\{x_j\}_{j=0}^{\infty}$ *defined by the recursive formula*

$$x_j(t) = \mu \int_a^b k(t, s; x_{j-1}(s))\, ds + f(t), \ \ j = 1, 2, ..., \tag{3.47}$$

$x_0(t)$ *is the given continuous function.*

Definition 3.2.5.4. An integral equation of the form

$$x(t) = \mu \int_a^t k(t, s; x(s))\, ds + f(t) \tag{3.48}$$

is called a Volterra equation of the second kind. Here, μ is a given parameter, f is a given function defined on $[a,b]$, x is an unknown function defined on $[a,b]$. The kernel k of the equation is a given function defined on $D\times\mathbb{R}^1$, where D is the triangular region in the $ts-$plane given by $a\le s\le t, a\le t\le b$.

The difference between (3.42) and (3.48) is that in (3.42) the upper limit of integration b is constant, whereas in (3.48) it is variable. This is essential. In fact, without any restriction on μ we now get the following existence and uniqueness theorem.

Theorem 3.5. *Assume that* k *and* f *in equation (3.48) are continuous functions on* $[a,b]\times$ $[a,t]\times\mathbb{R}^1$ *and* $[a,b]$*, respectively. Moreover,* k *satisfies on* $[a,b]\times$ $[a,t]\times\mathbb{R}^1$ *the Lipschitz condition (3.44). Then, equation (3.42) has a unique solution* x *defined on* $[a,b]$ *for every* μ*. This function* x *is the limit of the iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by the recursive formula*

$$x_j(t) = \mu \int_a^t k(t, s; x_{j-1}(s))\, ds + f(t), \ \ j = 1, 2, ..., \tag{3.49}$$

$x_0(t)$ *is a given continuous function.*

Proof of theorem 3.5. We consider equation (3.48) on $C^*[a,b]$, the space of all continuous functions defined on the interval $[a,b]$ with the metric d_* defined by

$$d_*(x,y) = \max_{t\in[a,b]} e^{-L(t-a)}\,|x(t)-y(t)|\,, \ \ L > l\,|\mu|\,. \tag{3.50}$$

Since $e^{-L(b-a)} \le e^{-L(t-a)} \le 1$, we have that

$$e^{-L(b-a)}d(x,y) \le d_*(x,y) \le d(x,y) \text{ for any } x,y\in C[a,b]. \tag{3.51}$$

$C^*[a,b] = (C^*[a,b], d)$ is complete. Obviously, equation (3.48) can be written as $x = Tx$, where

$$Tx(t) = \mu \int_a^t k\left(t, s; x\left(s\right)\right) ds + f\left(t\right). \tag{3.52}$$

Since f and k are continuous functions, formula (3.45) defines an operator $T : C^*[a,b] \to C^*[a,b]$. Applying formulas (3.52), (3.50), and condition (3.44), we get

$$\begin{aligned} d_*(Tx, Ty) &= \max_{t\in[a,b]} e^{-L(t-a)} \left|Tx(t) - Ty(t)\right| \\ &= |\mu| \max_{t\in[a,b]} e^{-L(t-a)} \left| \int_a^t \left(k\left(t, s; x\left(s\right)\right) - k\left(t, s; y\left(s\right)\right)\right) ds \right| \\ &\le l\left|\mu\right| \max_{t\in[a,b]} \int_a^t e^{-L(t-s)} e^{-L(s-a)} \left|x\left(s\right) - y\left(s\right)\right| ds \\ &\le l\left|\mu\right| \max_{s\in[a,t]} e^{-L(s-a)} \left|x\left(s\right) - y\left(s\right)\right| \max_{t\in[a,b]} \int_a^t e^{-L(t-s)} ds \\ &= \max_{t\in[a,b]} \frac{l\left|\mu\right|}{L} (1 - e^{-L(t-a)}) d_*(x, y) \le \frac{l\left|\mu\right|}{L} d_*(x, y). \end{aligned}$$

So, $d(Tx, Ty) \le \alpha d(x, y)$, where $\alpha = \frac{l|\mu|}{L}$. Since $L > l\left|\mu\right|$, we have that $\alpha < 1$. That means T is a contraction mapping on $C^*[a,b]$. Then, equation (3.42) has a unique solution x defined on $[a,b]$ for every μ. This function x is the limit of the iterative sequence $\{x_j\}_{j=0}^{\infty}$ defined by recursive formula (3.42). Theorem 3.2.5.4 is proved.

The Main Theorem on Existence and Uniqueness

First, We consider the IVP

$$\begin{cases} \frac{dv}{dt} + Av(t) = f(t, B(t)v(t), B(t)v(t-d)), t \in [0, \infty), \\ v(t) = \varphi(t), t \in [-d, 0]. \end{cases} \tag{3.53}$$

for the semi-linear differential equation in a Banach space E with linear unbounded operators A and $B(t)$ with dense domains $D(A) \subset D(B(t))$. Assume that A is a very positive operator in E. That means $-A$ is the generator of the analytic semigroup $\exp\{-tA\}t \in [0, \infty)$ of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. The following estimates are valid:

$$\|\exp\{-tA\}\|_{E\to E} \le Pe^{-\delta t}, \quad \|tA\exp\{-tA\}\|_{E\to E} \le P, t \in (0, \infty) \tag{3.54}$$

for some $P > 0$, $\delta > 0$. Let $B(t)$ be closed operators. The operator function $B(t)$ is strongly continuous on $D(A)$ and $\left\|B(t)A^{-1/2}\right\|_{E\to E} \leq H$.

A function $v(t)$ is called a solution to problem (3.53) if it satisfied the following conditions:

1. $v(t)$ is a continuously differentiable function on $[-d, \infty)$.
2. The element $v(t) \in D(A)$ $\forall t \in [-d, \infty)$, and the function $Av(t)$ is continuous on $[-d, \infty)$.
3. $v(t)$ satisfies the equation and the initial condition (3.53).

We reduced problem (3.53) into an integral equation of the form

$$v(t) = e^{-A(t-(m-1)\theta)}v((m-1)d) + \int_{(m-1)d}^{t} e^{-A(t-s)}f(s, B(s)v(s), B(s)v(s-d))ds,$$

$$t \in [(m-1)d, md], m \in N, v(t) = \varphi(t), t \in [-d, 0]$$

in $[0, \infty) \times E$, and the recursive formula for the solution of problem (3.53) by using successive approximations is

$$\begin{aligned} v_i(t) &= e^{-A(t-(m-1)d)}v_i((m-1)d) + \int_{(m-1)d}^{t} e^{-A(t-s)}f(s, B(s)v_{i-1}(s), B(s)v_i(s-d))ds, \\ v_0(t) &= e^{-A(t-(m-1)d)}v((m-1)d), t \in [(m-1)d, md], m \in N, i \in N, \end{aligned}$$

$$v(t) = \varphi(t), t \in [-d, 0]. \tag{3.55}$$

Here, N is the set of natural numbers.

Theorem 3.6. *Assume that the hypotheses below are fulfilled:*

1. $\varphi : [-d, 0] \times D\left(A^{\frac{1}{2}}\right) \longrightarrow E$ *be continuous function and*

$$\|\varphi(t)\|_{D\left(A^{\frac{1}{2}}\right)} \leq H. \tag{3.56}$$

2. $f : [0, \infty) \times D\left(A^{\frac{1}{2}}\right) \times D\left(A^{\frac{1}{2}}\right) \longrightarrow E$ *is a bounded and continuous function, i.e.,*

$$\|f(A^{\frac{1}{2}}v, A^{\frac{1}{2}}u)\|_E \leq \bar{H} \tag{3.57}$$

and with respect to z, the Lipschitz condition holds:

$$\|f(A^{\frac{1}{2}}v, A^{\frac{1}{2}}z) - f(A^{\frac{1}{2}}u, A^{\frac{1}{2}}z)\|_E \leq L\|A^{\frac{1}{2}}v - A^{\frac{1}{2}}u\|_E. \tag{3.58}$$

Here, $H, \bar{H}, L$ are positive constants and $L < \frac{1}{2Pd^{\frac{1}{2}}}$. Then, the problem (3.53) has a unique BS in $[0, \infty) \times E$.

Proof of theorem 3.6. Using the interval $t \in [0, d]$, we can written problem (3.53) as

$$\frac{dv}{dt} + Av(t) = f(t, B(t)v(t), B(t)\varphi(t-d)), v(0) = \varphi(0)$$

which in an equivalent integral form, becomes

$$v(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} f(s, B(s)v(s), B(s)\varphi(s-d))ds. \tag{3.59}$$

In accordance with the recursive approximation approach (3.55), we obtain

$$v_i(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} f(s, B(s)v_{i-1}(s), B(s)\varphi(s-d))ds, i = 1, 2, \ldots \tag{3.60}$$

Consequently,

$$v(t) = v_0(t) + \sum_{i=0}^{\infty}(v_{i+1}(t) - v_i(t)), \tag{3.61}$$

where

$$v_0(t) = e^{-At}\varphi(0).$$

From (3.54) and (3.56), it follows that

$$\|A^{\frac{1}{2}}v_0(t)\|_E = \|e^{-At}\|_{E\to E}\|A^{1/2}\varphi(0)\|_E \leq HP.$$

Using Equation (3.60) along with estimates (3.54) and (3.57), we obtain

$$\|A^{\frac{1}{2}}v_1(t) - A^{\frac{1}{2}}v_0(t)\|_E$$

$$\leq \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(s, B(s)A^{-\frac{1}{2}}A^{\frac{1}{2}}v_0, B(s)A^{-\frac{1}{2}}A^{\frac{1}{2}}\varphi(s-d))\|_E ds \leq 2\bar{H}Pt^{\frac{1}{2}}.$$

By triangle inequality, we have

$$\|A^{\frac{1}{2}}v_1(t)\|_E \leq HP + 2\bar{H}Pt^{\frac{1}{2}}.$$

Using Formula (3.60) along with estimates (3.54), (3.57), and (3.58), we obtain

$$\|A^{\frac{1}{2}}v_2(t) - A^{\frac{1}{2}}v_1(t)\|_E$$

$$\leq \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(s, B(s)v_1, B(s)\varphi(s-d)) - f(s, B(s)v_0, B(s)\varphi(s-d))\|_E ds$$

$$\leq LP \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|B(s)v_1(s) - B(s)v_0(s)\|_E ds \leq 2LP^2\bar{H}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}s^{\frac{1}{2}}ds$$

$$\leq 4LP^2\bar{H}t.$$

Then,

$$\|A^{\frac{1}{2}}v_2(t)\|_E \leq HP + 2\bar{H}Pt^{\frac{1}{2}} + 4LP^2\bar{H}t.$$

Let

$$\|A^{\frac{1}{2}}v_n(t) - A^{\frac{1}{2}}v_{n-1}(t)\|_E \leq \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^n.$$

Therefore, we obtain

$$\|A^{\frac{1}{2}}v_{n+1}(t) - A^{\frac{1}{2}}v_n(t)\|_E$$

$$\leq \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(B(s)v_n, B(s)\varphi(s-d)) - f(B(s)v_{n-1}, B(s)\varphi(s-d))\|_E ds$$

$$\leq P\int_0^t L\|B(s)v_n(s) - B(s)v_{n-1}(s)\|_E ds \leq P\int_0^t L\frac{\bar{H}}{L}(2LPs^{\frac{1}{2}})^n ds$$

$$\leq \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{n+1}.$$

Henceforth, for any $n, n \geq 1$, we obtain

$$\|A^{\frac{1}{2}}v_{n+1}(t) - A^{\frac{1}{2}}v_n(t)\|_E \leq \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{n+1}$$

and

$$\|A^{\frac{1}{2}}v_{n+1}(t)\|_E \leq HP + 2\bar{H}Pt^{\frac{1}{2}} + \ldots + \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{n+1}$$

by mathematical induction. It is implied by that equation and Equation (3.61) that

$$\|A^{\frac{1}{2}}v(t)\|_E \leq \|A^{\frac{1}{2}}v_0(t)\|_E + \sum_{i=0}^{\infty}\|A^{\frac{1}{2}}v_{i+1}(t) - A^{\frac{1}{2}}v_i(t)\|_E$$

$$\leq HP + \sum_{i=0}^{\infty}\frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{i+1} < \infty, t \in [0, d].$$

This shows that problem (3.53) solution exists and is bounded in $[0, d] \times E$.

From $t \in [d, 2d]$, it follows that $0 \leq t - d \leq d$. We denote that

$$\varphi_1(t) = v(t-d), t \in [d, 2d],$$

and suppose that problem (3.53) has a BS in $[d, 2d] \times E$. Replacing t and $t - d$, we can write

$$\|A^{\frac{1}{2}}\varphi_1(t)\| \leq H_1$$

and

$$\|f(A^{\frac{1}{2}}v_0(t), A^{\frac{1}{2}}\varphi_1(t))\|_E \leq \bar{H}_1.$$

According to successive approximation of Formula (3.55), we can write

$$v_0(t) = e^{-A(t-d)}\varphi_1(d)$$

$$v_i(t) = e^{-A(t-d)}\varphi_1(d) + \int_d^t e^{-A(t-s)} f(B(s)v_{i-1}(s), B(s)\varphi_1(s))ds, i = 1, 2, \dots$$

In the same way, for any $r, r \geq 1$, we obtain

$$\|A^{\frac{1}{2}}v_{r+1}(t) - A^{\frac{1}{2}}v_r(t)\|_E \leq \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{r+1},$$

and

$$\|A^{\frac{1}{2}}v_{r+1}(t)\|_E \leq H_1P + 2\bar{H}_1Pt^{\frac{1}{2}} + \dots + \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{r+1}.$$

From that, it implies that

$$\begin{aligned}\|A^{\frac{1}{2}}v(t)\|_E &\leq \|A^{\frac{1}{2}}v_0(t)\|_E + \sum_{i=0}^{\infty}\|A^{\frac{1}{2}}v_{i+1}(t) - A^{\frac{1}{2}}v_i(t)\|_E \\ &\leq H_1P + \sum_{i=0}^{\infty}\frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{i+1} < \infty, t \in [d, 2d].\end{aligned}$$

This proves that problem (3.53)'s solution exists, and it is bounded in $[d, 2d] \times E$. In the same procedure one, can establish that

$$\|A^{\frac{1}{2}}v(t)\|_E \leq H_1P + \sum_{i=0}^{\infty}\frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{i+1}, t \in [nd, (n+1)d],$$

where H_n and $\bar{H}_n$ are bounded. This shows that problem (3.53)'s solution exists and is bounded in $[nd, (n+1)\,d] \times E$. Overall, the constructed function $v(t)$ of problem (3.53) is a BS in $[0, \infty) \times E$.

We shall now show that this solution to problem (3.53) is unique. Suppose that problem (3.53) has a BS solution $u(t)$ and that $u(t) \neq v(t)$. We write down $z(t) = u(t) - v(t)$. Hence, for $z(t)$, we obtain that

$$\begin{cases} \frac{dz}{dt} + Az(t) = f(B(s)u(t), B(s)u(t-d)) - f(B(s)v(t), B(s)v(t-d)), \\ t \in (0, \infty), \\ \\ z(t) = 0, t \in [-d, 0]. \end{cases}$$

We consider $t \in [0, d]$. As $u(t-d) = u(t-d) = \varphi(t-d)$, we obtain

$$\begin{cases} \frac{dz}{dt} + Az(t) = f(B(s)u(t), B(s)\varphi(t-d)) - f(B(s)v(t), B(s)\varphi(t-d)), \\ t \in (0, \infty), \\ \\ z(t) = 0, t \in [-d, 0]. \end{cases}$$

Henceforth,

$$z(t) = e^{-At}z(0) + \int_0^t e^{-A(t-s)}\left[f(B(s)v(s), B(s)\varphi(s-d)) - f(B(s)u(s), B(s)\varphi(s-d))\right] ds.$$

Using (3.54) and (3.57), we obtain

$$\begin{aligned}\|A^{\frac{1}{2}}z(t)\|_E \le \int_0^t &\|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(B(s)v(s), B(s)\varphi(s-d))\\ &- f(B(s)u(s), \varphi(s-d))\|_E ds\\ &\le PL\int_0^t \|B(s)v(s) - B(s)u(s)\|_E ds \le PL\int_0^t \|A^{\frac{1}{2}}z(s)\|_E ds.\end{aligned}$$

By means of integral inequality, we can write

$$\|A^{\frac{1}{2}}z(t)\|_E \le 0.$$

This implies that $A^{\frac{1}{2}}z(t) = 0$, which proves that problem (3.53)'s solution is unique and bounded in $[0, d] \times E$.

Using a similar procedure and mathematical induction, we can show that problem (3.53)'s solution is unique and bounded in $[0, \infty) \times E$. Hence the proved.

Remark 3.1. The approach used in the current study also makes it possible to demonstrate, under certain presumptions, that there exists a unique BS to the IVP for semi-linear parabolic equations

$$\begin{cases} \frac{dv}{dt} + A(t)v(t) = f(t, B(t)v(t), B(t)v([t])), 0 < t < \infty, \\ v(0) = \varphi \end{cases} \tag{3.62}$$

in a Banach space E with unbounded operators $A(t)$ and $B(t)$.

Remark 3.2. It is known that various problems in fluid mechanics dynamics, elasticity and other areas of physics lead to fractional parabolic-type differential equations. Methods of solutions of problems for linear fractional differential equations have been studied extensively by many researchers (see, e.g., (Podlubny, 1999; Samko, Kilbas, and Marichev, 1993; Lavoie, Osler, and Trembly, 1976; Tarasov, 2007; El-Mesiry et al., 2005; El-Sayed, and Gaafar, 2001; Gorenflo, and Mainardi, 2008; Ashyralyev, 2009) and the references given therein). The approach used in the current study also makes it possible to demonstrate, under certain presumptions, that there exists a unique BS to the IVP for semi-linear fractional parabolic equations

$$\begin{cases} \frac{dv}{dt} + Av(t) + D^{\alpha}v(t) = f(t, B(t)v(t), B(t)v(t-d)), t \in [0, \infty), \\ v(t) = \varphi(t), t \in [-d, 0] \end{cases} \tag{3.63}$$

in a Banach space E with unbounded operators A and $B(t)$. Here, $\alpha \in [0, 1)$.

Applications

We begin by considering an IBVP for semi-linear one-dimensional DPPDEs with the Dirichlet condition:

$$\begin{cases} v_t(t,x) - a(x)v_{xx}(t,x) + \delta v(t,x) = f(t, x, v_x(t,x), v_x(t-d,x)), \\ t \in (0, \infty), x \in (0, l) \\ v(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,l) = 0, t \in [-d, 0], x \in [0, l], \\ v(t,0) = v(t,l) = 0, t \in [-d, \infty), \end{cases} \tag{3.64}$$

where $\varphi(t,x), a(x)$ are given sufficiently smooth functions (SSFs) and a delta greater than zero is a significant enough number. Suppose that $a(x) \geq a > 0$.

We can reduce the IBVP (3.64) to IVP (3.53) in $E = C[0,l]$ with the strong positive operator A^x in $C[0,l]$ according to the following formula:

$$A^x v = -a(x)\frac{d^2v}{dx^2} + \delta v \tag{3.65}$$

with domain $D(A^x) = \left\{v \in C^{(2)}[0,l] : v(0) = v(l) = 0\right\}$ (Bazarov, 1989). Moreover, we have the following estimates:

$$\|\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \leq P,\ t \in [0, \infty),$$

$$\|tA^x \exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \leq P, t \in (0, \infty).$$

Therefore, from that and abstracting Theorem 3.6, we have the following:

Theorem 3.7. *Suppose the hypotheses below:*

1. $\varphi : [-d, 0] \times [0, l] \times C^{(1)}[0, l] \to C[0, l]$ *is a continuous function and*

$$\|\varphi_x(t, .)\|_{C[0,l]} \leq H. \tag{3.66}$$

2. *$f : [0,\infty)\times(0,l)\times C^{(1)}[0,l]\times C^{(1)}[0,l] \to C[0,l]$ is a bounded and continuous function, i.e.,*

$$\|f(t,.,v_x,u_x))\|_{C[0,l]} \le \overline{H} \tag{3.67}$$

and with respect to z, the Lipschitz condition holds:

$$\|f(t,.,v_x,z_x)-f(t,.,u_x,z_x)\|_{C[0,l]} \le L\,\|v_x-u_x\|_{C[0,l]}\,, \tag{3.68}$$

where $L, H, \overline{H}$, are positive constants and $L < \frac{1}{2Pd^{\frac{1}{2}}}$. Then, problem (3.64) has a unique BS in $[0,\infty)\times C[0,l]$.

In addition, we consider the IBVP for semi-linear one-dimensional DPPDEs with the Neumann condition:

$$\begin{cases} v_t(t,x)-a(x)v_{xx}(t,x)+\delta v(t,x)=f(t,x,v_x(t,x),v_x(t-d,x)), \\ t\in(0,\infty), x\in(0,l) \\ v(t,x)=\varphi(t,x), \varphi_x(t,0)=\varphi_x(t,l)=0, t\in[-d,0], x\in[0,l]\,, \\ v_x(t,0)=v_x(t,l)=0, t\in[-d,\infty), \end{cases} \tag{3.69}$$

where $\varphi(t,x), a(x)$ are given SSFs and delta greater than zero is a significant enough number. We suppose that $a(x)\ge a>0$.

We can reduce the IBVP (3.69) to IVP (3.53) in $E=C[0,l]$ with the strong positive operator A^x in $C[0,l]$ according to the formula (3.65) with domain:

$$D(A^x)=\left\{v\in C^{(2)}[0,l] : v'(0)=v'(l)=0\right\} (Bazarov, 1989).$$

Moreover, we have the following estimates:

$$\|\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P,\ t\in[0,\infty),$$

$$\|tA^x\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P, t\in(0,\infty).$$

Therefore, from that and abstracting Theorem 3.6, we have the following:

Theorem 3.8. *Suppose that assumptions (3.66)–(3.68) hold. Then, problem (3.69) has a unique BS in $[0,\infty)\times C[0,l]$.*

Furthermore, we consider the IBVP for semi-linear one-dimensional DPPDEs with nonlocal conditions:

$$\begin{cases} v_t(t,x) - a(x)v_{xx}(t,x) + \delta v(t,x) = f(x, v_x(t,x), v_x(t-d,x)), \\ t \in (0,\infty), x \in (0,l)\,, \\ v(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,l), \varphi_x(t,0) = \varphi_x(t,l), \\ t \in [-d,0], x \in [0,l]\,, \\ v(t,0) = v(t,l), v_x(t,0) = v_x(t,l), t \in [-d,\infty), \end{cases} \tag{3.70}$$

where $\varphi(t,x), a(x)$ are given SSFs and a delta greater than zero is a significant enough number. We suppose that $a(x) \geq a > 0$.

We can reduce the IBVP (3.70) to IVP (3.53) in $E = C\,[0,l]$ with the strong positive operator A^x in $C\,[0,l]$ according to the formula (3.65) with domain:

$$D(A^x) = \left\{ v \in C^{(2)}\,[0,l] : v\,(0) = v\,(l)\,, v'\,(0) = v'\,(l) \right\} (Bazarov, 1989).$$

Moreover, we have the following estimates:

$$\| \exp\{-tA^x\} \|_{C[0,l] \to C[0,l]} \leq P, t \in [0,\infty),$$

$$\| tA^x \exp\{-tA^x\} \|_{C[0,l] \to C[0,l]} \leq P, t \in (0,\infty).$$

Therefore, from that and abstracting Theorem 3.6, we have the following:

Theorem 3.9. *Suppose that assumptions (3.66)–(3.68) hold. Then, problem (3.70) has a unique BS in* $[0,\infty) \times C\,[0,l]$.

Finally, we consider the IBVP for semi-linear one-dimensional DPPDEs with

Robin condition:

$$\begin{cases} v_t(t,x) - (a(x)v_x(t,x))_x + \delta v(t,x) = f(x, v_x(t,x), v_x(t-d,x)), \\ t \in (0,\infty), x \in (0,l), \\ v(t,x) = \varphi(t,x), \varphi(t,0) = b\varphi_x(t,0), \quad -\varphi(t,l) = c\varphi_x(t,l), \\ t \in [-d,0], x \in [0,l], \\ v(t,0) = bv_x(t,0), \quad\quad -v(t,l) = cv_x(t,l), t \in [-d,0], \end{cases} \tag{3.71}$$

where $\varphi(t,x), a(x)$ are given SSFs. Here, $a(x) \geq a > 0$ and b, c, δ are positive constants.

We can reduce the IBVP (3.71) to IVP (3.53) in $E = L_2[0,l]$ with the self-adjoint positive-definite operator A^x in $L_2[0,l]$ according to the following formula:

$$Az = -\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right) + \delta v(x) \tag{3.72}$$

with domain $D(A^x) = \{v : v, v_2'[0,l], v(0) = bv'(0), -v(l) = cv'(l)\}$ (Ashyralyev, Urun, & Parmaksizoglu, 2022). Moreover, we have the following estimates:

$$\|\exp\{-tA^x\}\|_{L_2[0,l]\to L_2[0,l]} \leq 1, \ t \in [0,\infty),$$

$$\|tA^x \exp\{-tA^x\}\|_{L_2[0,l]\to L_2[0,l]} \leq 1, t \in (0,\infty).$$

Therefore, from that and abstracting Theorem 3.6, we have the following:

Theorem 3.10. *Suppose the hypotheses below:*

1. $\varphi : [-d,0] \times [0,l] \times L_2[0,l] \to C[0,l]$ *is a continuous function and*

$$\|\varphi_x(t,.)\|_{W_2^1[0,l]} \leq H. \tag{3.73}$$

2. $f : [0,\infty) \times (0,l) \times W_2^1[0,l] \times W_2^1[0,l] \to L_2[0,l]$ *is a bounded and continuous function, i.e.,*

$$\|f(t,.,v_x,u_x))\|_{L_2[0,l]} \leq \overline{H} \tag{3.74}$$

and with respect to z, the Lipschitz condition holds:

$$\|f(t,.,v_x,z_x) - f(t,.,u_x,z_x)\|_{L_2[0,l]} \leq L\,\|v_x - u_x\|_{L_2[0,l]}\,, \tag{3.75}$$

where $L, H, \overline{H}$, are positive constants and $L < \frac{1}{2Pd^{\frac{1}{2}}}$. Then, problem (3.71) has a unique BS in $[0,\infty) \times L_2[0,l]$.

Numerical Results

Generally speaking, semi-linear problems cannot be solved precisely. The numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We need numerical experiments to confirm the agreement between theoretical and experimental results and to make clear how effective the approach is, especially when one cannot know the concrete values of constants in stability estimates. In the present section, we obtain the numerical algorithms for the approximate solution of one-dimensional delay parabolic differential equations. Henceforth, the iterative first-order and second-order accuracy difference schemes (FSADSs) for the approximate solution of semi-linear one-dimensional delay parabolic equations are described, numerical results were obtained, and error analysis was given in tables.

Problem 3.6. *Consider the IBVP*

$$\begin{cases} v_t\,(t,x) - v_{xx}\,(t,x) = v_x\,(t,x)\,\{v\,([t-1]\,,x)\cos x - v_x\,([t-1]\,,x)\sin x\}\,, \\ \\ t \in (0,\infty),\ \ x \in (0,\pi), \\ \\ v\,(0,x) = \sin x, x \in [0,\pi], \\ \\ v\,(t,0) = v\,(t,\pi) = 0,\ \ t \in [0,\infty) \end{cases} \tag{3.76}$$

for the semi-linear DPPDE. Here, $[\cdot]$ is notation of an integer function. The ES of this problem is $v\,(t,x) = e^{-t}\sin x$.

We obtain the following iterative FADS for the approximate solution of the

IBVP (3.76)

$$\begin{cases} \frac{{}_rv_n^k - {}_rv_n^{k-1}}{\tau} - \frac{{}_rv_{n+1}^k - 2{}_rv_n^k + {}_rv_{n-1}^k}{h^2} \\ = \frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h} \left\{ {}_rv_n^{[k-N]} \cos x_n - \frac{{}_rv_{n+1}^{[k-N]} - {}_rv_{n-1}^{[k-N]}}{2h} \sin x_n \right\}, \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1, M-1}, \\ {}_rv_n^0 = \sin x_n, x_n = nh, n \in \overline{0, M}, \\ {}_rv_0^k = {}_rv_M^k = 0, k \in \overline{0,\infty} \end{cases} \tag{3.77}$$

for the numerical solution of the semi-linear delay parabolic equation.

Here, r stands for the iteration number, ${}_0v_n^k$, $k \in \overline{0, N}$, and $n \in \overline{0, M}$ is the initial starting value. Numerically, we use the steps listed below to solve the difference scheme (3.77). For $k \in \overline{0, N}$, $n \in \overline{0, M}$

- $r = 1$
- ${}_{r-1}v_n^k$ is known;
- ${}_rv_n^k$ is determined;
- $r = r + 1$ is taken, and we proceed to step 2 if the maximum absolute error between ${}_{r-1}v_n^k$ and ${}_rv_n^k$ is more than the specified tolerance value. If not, stop the iteration process and use ${}_rv_n^k$ as the solution to the given problem.

We write (3.77) in matrix form:

$$\begin{gathered} A_rV^k + B_rV^{k-1} = R\varphi({}_{r-1}v^k, {}_rv^{k-N}), k \in \overline{1, N}, \\ {}_rV^0 = \{\sin x_n\}_{n=0}^M, n \in \overline{0, M}, \end{gathered} \tag{3.78}$$

Additionally, using the SADS for the AS of problem (3.76), we have the following SEs:

$$\begin{cases} \frac{{}_{r}v_{n}^{k}-{}_{r}v_{n}^{k-1}}{\tau}-\frac{{}_{r}v_{n+1}^{k}-2{}_{r}v_{n}^{k}+{}_{r}v_{n-1}^{k}}{h^{2}}+\tau\frac{{}_{r}v_{n+2}^{k}-4{}_{r}v_{n+1}^{k}+6{}_{r}v_{n}^{k}-4{}_{r}v_{n-1}^{k}+{}_{r}v_{n-2}^{k}}{2h^{4}} \\ =\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^{k}-{}_{r-1}v_{n-1}^{k}}{2h}\right\}\left\{{}_{r}v_{n}^{k-N}\cos x_{n}-\frac{{}_{r}v_{n+1}^{k-N}-{}_{r}v_{n-1}^{k-N}}{2h}\sin x_{n}\right\} \\ +\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^{k-1}-{}_{r-1}v_{n-1}^{k-1}}{2h}\right\}\left\{{}_{r}v_{n}^{k-1-N}\cos x_{n}-\frac{{}_{r}v_{n+1}^{k-1-N}-{}_{r}v_{n-1}^{k-1-N}}{2h}\sin x_{n}\right\} \\ -\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+2}^{k}-{}_{r-1}v_{n}^{k}}{2h}\right\}\frac{{}_{r}v_{n+1}^{k-N}\cos x_{n+1}-\frac{{}_{r}v_{n+2}^{k-N}-{}_{r}v_{n}^{k-N}}{2h}\sin x_{n+1}}{h^{2}} \\ -\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+1}^{k}-{}_{r-1}v_{n-1}^{k}}{2h}\right\}\frac{-2{}_{r}v_{n}^{k-N}\cos x_{n}+2\frac{{}_{r}v_{n+1}^{k-N}-{}_{r}v_{n-1}^{k-N}}{2h}\sin x_{n}}{h^{2}} \\ -\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n}^{k}-{}_{r-1}v_{n-2}^{k}}{2h}\right\}\frac{{}_{r}v_{n-1}^{k-N}\cos x_{n-1}-\frac{{}_{r}v_{n}^{k-N}-{}_{r}v_{n-2}^{k-N}}{2h}\sin x_{n-1}}{h^{2}} \\ -\frac{\tau}{4}\left\{\frac{{}_{r}v_{n+2}^{k-1}-{}_{r}v_{n}^{k-1}}{2h}\right\}\frac{{}_{r-1}v_{n+1}^{k-1-N}\cos x_{n+1}-\frac{{}_{r-1}v_{n+2}^{k-1-N}-{}_{r-1}v_{n}^{k-1-N}}{2h}\sin x_{n+1}}{h^{2}} \\ -\frac{\tau}{4}\left\{\frac{{}_{r}v_{n+1}^{k-1}-{}_{r}v_{n-1}^{k-1}}{2h}\right\}\frac{-2{}_{r-1}v_{n}^{k-1-N}\cos x_{n}+2\frac{{}_{r-1}v_{n+1}^{k-1-N}-{}_{r-1}v_{n-1}^{k-1-N}}{2h}\sin x_{n}}{h^{2}} \\ -\frac{\tau}{4}\left\{\frac{{}_{r}v_{n}^{k-1}-{}_{r}v_{n-2}^{k-1}}{2h}\right\}\frac{{}_{r-1}v_{n-1}^{k-1-N}\cos x_{n-1}-\frac{{}_{r-1}v_{n}^{k-1-N}-{}_{r-1}v_{n-2}^{k-1-N}}{2h}\sin x_{n-1}}{h^{2}}, \\ t_{k}=k\tau,x_{n}=nh,k\in\overline{1,N},n\in\overline{2,M-2}, \\ {}_{r}v_{n}^{0}=\varphi(x_{n})=\sin x_{n},n\in\overline{0,M},{}_{r}v_{0}^{k}={}_{r}v_{M}^{k}=0,\ \ k\in\overline{0,N}, \\ {}_{r}v_{3}^{k}=4{}_{r}v_{2}^{k}-5{}_{r}v_{1}^{k},{}_{r}v_{M-3}^{k}=4{}_{r}v_{M-2}^{k}-5{}_{r}v_{M-1}^{k},\ \ k\in\overline{0,N}. \end{cases} \tag{3.79}$$

We obtain again $(M+1)\times(M+1)$ SLEs, and we reformat them into matrix form (3.78).

Consequently, we obtain a second-order difference equation with respect to k matrix coefficients. Using (3.78), we can obtain this difference scheme's solution. The initial guess in computations for both FSADSs is set as ${}_{0}v_{n}^{k}=e^{-t_{k}}\sin x_{n}$, and the iterative procedure is stopped when the maximum errors between two successive outcomes of the difference schemes (3.77) and (3.79) become less than 10^{-8}.

For various values of M and N, we provide numerical results and ${}_{r}v_{n}^{k}$ represents the numerical solutions of these difference schemes at (t_{k},x_{n}). Tables 1-3 are constructed for $M=N=30,\ 60,\ 120$ in that order for $t\in[r,r+1]$, $r=0,1,2$ and the errors are calculated using the following formula:

$${}_{r}\left(E_{M}^{N}\right)_{p}=\max_{\substack{pN+1\leq k\leq(p+1)N,p=0,1,\ldots\\ 1\leq n\leq M-1}}\left|v\left(t_{k},x_{n}\right)-{}_{r}v_{n}^{k}\right|. \tag{3.80}$$

To finish iteration process, we used the following condition in each sub-interval:

$$\max_{\substack{pN+1\le k\le (p+1)N, p=0,1,\ldots \\ 1\le n\le M-1}} \left| {}_{r}v_{n}^{k} - {}_{r-1}v_{n}^{k},\right| < 10^{-8} \tag{3.81}$$

Table 1.

Error comparison between difference schemes (3.77) and (3.79)

in $t \in [0,1]$ (Number of iterations = r)

Method	$M=N=30$	$M=N=60$	$M=N=120$
(3.77)	$6.3783 \times 10^{-3}, r=2$	$3.1279 \times 10^{-2}, r=2$	$1.5485 \times 10^{-3}, r=2$
(3.79)	$4.5864 \times 10^{-4}, r=3$	$1.1212 \times 10^{-4}, r=3$	$2.7577 \times 10^{-5}, r=2$

Table 2.

Error comparison between difference schemes (3.77) and (3.79)

in $t \in [1,2]$ (Number of iterations = r)

Method	$M=N=30$	$M=N=60$	$M=N=120$
(3.77)	$2.3464 \times 10^{-3}, r=3$	$1.5070 \times 10^{-3}, r=3$	$5.6964 \times 10^{-4}, r=2$
(3.79)	$1.6358 \times 10^{-4}, r=3$	$4.2149 \times 10^{-5}, r=2$	$1.0698 \times 10^{-5}, r=2$

Table 3.

Error comparison between difference schemes (3.77) and (3.79) in $t \in [2,3]$ (Number of iterations = r)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(3.77)	$8.6321 \times 10^{-4}, r = 3$	$4.2332 \times 10^{-4}, r = 2$	$2.0956 \times 10^{-4}, r = 2$
(3.79)	$5.3201 \times 10^{-5}, r = 2$	$1.3581 \times 10^{-5}, r = 2$	$3.4122 \times 10^{-6}, r = 2$

Problem 3.7. *We also consider the IBVP*

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + \sin(v(t,x)) \\ = v_x(t,x)\{2v([t-1],x)\cos 2x - v_x([t-1],x)\sin 2x\} + f(t,x), \\ t \in (0,\infty),\ x \in (0,\pi), \\ v(0,x) = \sin 2x,\ x \in [0,\pi], \\ v(t,0) = v(t,\pi), v_x(t,0) = v_x(t,\pi),\ t \in [0,\infty) \end{cases} \tag{3.82}$$

for the semi-linear DPPDE. The ES of this problem is $v(t,x) = e^{-4t}\sin 2x$ *and* $f(t,x) = \sin(e^{-4t}\sin 2x)$.

We obtain the following FADS for the approximate solution of the IBVP (3.82)

$$\begin{cases} \frac{{}_rv_n^k - {}_rv_n^{k-1}}{\tau} - \frac{{}_rv_{n+1}^k - 2{}_rv_n^k + {}_rv_{n-1}^k}{h^2} = 2\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\} {}_rv_n^{[k-N]}\cos 2x_n \\ -\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\} \frac{{}_rv_{n+1}^{[k-N]} - {}_rv_{n-1}^{[k-N]}}{2h}\sin 2x_n - \sin\left({}_{r-1}v_n^k\right) + f(t_k, x_n), \\ t_k = k\tau, x_n = nh, k \in \overline{1,N}, n \in \overline{1,M-1}, \\ {}_rv_n^0 = \sin 2x_n, x_n = nh, n \in \overline{0,M}, \\ {}_rv_0^k = {}_rv_M^k, {}_rv_1^k - {}_rr_0^k = {}_rv_M^k - {}_rv_{M-1}^k, \\ pN+1 \le k \le (p+1)N, p = 0,1,... \end{cases} \tag{3.83}$$

for the numerical solution of the delay parabolic equation with nonlocal conditions.

We write (3.83) in matrix form:

$$
\begin{aligned}
&A_r V^k + B_r V^{k-1} = R_{r-1}\theta^k, k \in \overline{p(N+1),(p+1)N}, p = 0,1,...,\\
&\quad {}_rV^0 = \{\sin 2x_n\}_{n=0}^{M},
\end{aligned}
\tag{3.84}
$$

where

$$
\begin{aligned}
&{}_rV^k = \left\{{}_rv_n^k\right\}_{n=0}^{M},\\
&{}_{r-1}\theta_n^k = -\sin\left({}_{r-1}v_n^k\right) + f(t_k,x_n) + 2\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}{}_rv_n^{[k-N]}\cos 2x_n\\
&\quad -\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}\frac{{}_rv_{n+1}^{[k-N]} - {}_rv_{n-1}^{[k-N]}}{2h}\sin 2x_n,\\
&\quad n = 0,...,M, k \in \overline{p(N+1),(p+1)N}, p = 0,1,...,
\end{aligned}
$$

Furthermore, using the SADS for the AS of problem (3.82), we obtain the following SEs:

$$\begin{cases}
\frac{{}_rv_n^k-{}_rv_n^{k-1}}{\tau}-\frac{{}_rv_{n+1}^k-2{}_rv_n^k+{}_rv_{n-1}^k}{h^2}+\tau\frac{{}_rv_{n+2}^k-4{}_rv_{n+1}^k+6{}_rv_n^k-4{}_rv_{n-1}^k+{}_rv_{n-2}^k}{2h^4}\\
-\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^k-{}_{r-1}v_{n-1}^k}{2h}\right\}\left[{}_rv_n^{[k-N]}\cos 2x_n-\frac{{}_rv_{n+1}^{[k-N]}-{}_rv_{n-1}^{[k-N]}}{2h}\sin 2x_n\right]\\
-\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^{k-1}-{}_{r-1}v_{n-1}^{k-1}}{2h}\right\}\left[{}_rv_n^{[k-1-N]}\cos 2x_n-\frac{{}_rv_{n+1}^{[k-1-N]}-{}_rv_{n-1}^{[k-1-N]}}{2h}\sin 2x_n\right]\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+2}^k-{}_{r-1}v_n^k}{2h}\right\}\frac{{}_rv_{n+1}^{[k-N]}\cos 2x_{n+1}-\frac{{}_rv_{n+2}^{[k-N]}-{}_rv_n^{[k-N]}}{2h}\sin 2x_{n+1}}{h^2}\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+1}^k-{}_{r-1}v_{n-1}^k}{2h}\right\}\frac{-2{}_rv_n^{[k-N]}\cos 2x_n+2\frac{{}_rv_{n+1}^{[k-N]}-{}_rv_{n-1}^{[k-N]}}{2h}\sin 2x_n}{h^2}\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_n^k-{}_{r-1}v_{n-2}^k}{2h}\right\}\frac{{}_rv_{n-1}^{[k-N]}\cos 2x_{n-1}-\frac{{}_rv_n^{[k-N]}-{}_rv_{n-2}^{[k-N]}}{2h}\sin 2x_{n-1}}{h^2}\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+2}^{k-1}-{}_{r-1}v_n^{k-1}}{2h}\right\}\frac{{}_rv_{n+1}^{[k-1-N]}\cos 2x_{n+1}-\frac{{}_rv_{n+2}^{[k-1-N]}-{}_rv_n^{[k-1-N]}}{2h}\sin 2x_{n+1}}{h^2}\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+1}^{k-1}-{}_{r-1}v_{n-1}^{k-1}}{2h}\right\}\frac{-2{}_rv_n^{[k-1-N]}\cos 2x_n+2\frac{{}_rv_{n+1}^{[k-1-N]}-{}_rv_{n-1}^{[k-1-N]}}{2h}\sin 2x_n}{h^2}\\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_n^{k-1}-{}_{r-1}v_{n-2}^{k-1}}{2h}\right\}\frac{{}_rv_{n-1}^{[k-1-N]}\cos 2x_{n-1}-\frac{{}_rv_n^{[k-1-N]}-{}_rv_{n-2}^{[k-1-N]}}{2h}\sin 2x_{n-1}}{h^2}\\
+\sin\left({}_{r-1}v_n^k\right)=f(t_k,x_n),\\
\\
t_k=k\tau, x_n=nh, k\in\overline{1,N}, n\in\overline{2,M-2},\\
{}_rv_n^0=\sin 2x_n, 0\le n\le M,\\
{}_rv_0^k={}_rr_M^k, -{}_rv_2^k+4{}_rv_1^k-3{}_rv_0^k=3{}_rv_M^k-4{}_rv_{M-1}^k+{}_rv_{M-2}^k,\\
-{}_rv_3^k+4{}_rv_2^k-5{}_rv_1^k+2{}_rv_0^k\\
=2{}_rv_M^k-5{}_rv_{M-1}^k+4{}_rv_{M-2}^k-{}_rv_{M-3}^k,\\
-3{}_rv_4^k+14{}_rv_3^k-24{}_rv_2^k+18{}_rv_1^k-5{}_rv_0^k\\
=5{}_rv_M^k-18{}_rv_{M-1}^k+24{}_rv_{M-2}^k-14{}_rv_{M-3}^k+3{}_rv_{M-4}^k,\\
k\in\overline{p(N+1),(p+1)N}, p=0,1,....
\end{cases}\tag{3.85}$$

We obtain another $(M+1)\times(M+1)$ SLE; they are then rewritten in matrix form (3.84).

For a range of M and N values, we provide numerical results, and ${}_rv_n^k$ represents the numerical solutions of these difference schemes at (t_k,x_n). Tables 4–6 are constructed for $M=N=30,60,120$ in that order for $t\in[r,r+1]$, $r=0,1,2$, and the errors are calculated using Formulas (3.80) and (3.81).

Table 4.

Error comparison between difference schemes (3.83) and (3.85) in $t \in [0, 1]$ (Number of iterations = r)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(3.83)	$2.4431 \times 10^{-2}, r = 2$	$1.2259 \times 10^{-2}, r = 2$	$6.1304 \times 10^{-3}, r = 2$
(3.85)	$2.0589 \times 10^{-3}, r = 8$	$5.4628 \times 10^{-4}, r = 8$	$1.3865 \times 10^{-4}, r = 7$

Table 5.

Error comparison between difference schemes (3.83) and (3.85) in $t \in [1, 2]$ (Number of iterations = r)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(3.83)	$5.3731 \times 10^{-3}, r = 9$	$2.5664 \times 10^{-3}, r = 8$	$1.2517 \times 10^{-3}, r = 8$
(3.85)	$3.0514 \times 10^{-4}, r = 8$	$7.5756 \times 10^{-5}, r = 7$	$1.9241 \times 10^{-5}, r = 6$

Table 6.

Error comparison between difference schemes (3.83) and (3.85) in $t \in [2, 3]$ (Number of iterations = r)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(3.83)	$1.0838 \times 10^{-4}, r = 7$	$4.9176 \times 10^{-5}, r = 6$	$2.3435 \times 10^{-5}, r = 6$
(3.85)	$1.5588 \times 10^{-5}, r = 7$	$2.0085 \times 10^{-6}, r = 5$	$4.8130 \times 10^{-7}, r = 3$

These numerical experiments back up the theoretical claims, as shown in Tables 1–6. With more grid points, the maximum errors and the number of iterations are reduced. As we doubled the values of N and M each time, beginning with $M = N = 30$. In the FADSs (3.77) and (3.83) in Tables 1-6 respectively, the errors decrease roughly by a proportion of 1/2, while in the SADSs (3.79) and (3.85) in Tables 1-6 respectively, the errors decrease roughly by a proportion of 1/4. Errors shown in the tables demonstrate the consistency of the different schemes and the reliability of the findings. Accordingly, the SADS increases faster than the FADS. These numerical experiments back up the theoretical claims as shown in the tables. With more grid points, the maximum errors can be reduced.

CHAPTER IV

Stability of the Time-Delay Parabolic Differential Equations with Dependent Coefficients

Introduction

In this section, the necessary conditions for the existence of unique bounded solutions of nonlinear delay parabolic differential equations in an arbitrary Banach space with strongly unbounded operators dependent in t are established. In practise, theorems on stability estimation for the solution of the initial boundary value problem for three different types of nonlinear delay parabolic equations are obtained.

The Main Theorem on Existence and Uniqueness

We consider the IVP

$$\begin{cases} \frac{du}{dt} + A(t)u(t) = g(t, u(t), u(t-\omega)), t \in [0, \infty), \\ u(t) = \varphi(t), t \in [-\omega, 0] \end{cases} \tag{4.86}$$

in an arbitrary Banach space E with the unbounded operators $A(t)$ in E with dense domains $D(A(t)) \subset E$. Suppose that for each $t \in [0, \infty)$ the operator $-A(t)$ generates an analytic semi-group $\exp\{-sA(t)\}(s \geq 0)$ with exponentially decreasing norm, when $s \to +\infty$, i.e. the following estimates

$$\|\exp(-sA(t))\|_{E \to E}, \|sA(t)\exp(-sA(t))\|_{E \to E} \leq Me^{-\delta s}(s > 0) \tag{4.87}$$

hold for some $M \in [1, +\infty)$, $\delta \in (0, +\infty)$. From this inequality it follows the operator $A^{-1}(t)$ exists and bounded and hence $A(t)$ is closed in $E_1 \subset E$, such that $A(t) : D(A(t)) \to E$ and $D(A(t)) = D(A(0))$ for $0 \leq t < \infty$.

Assume that the operator $A(t)A^{-1}(s)$ is Holder continuous in t in the uniform operator topology for each fixed s, that is,

$$\left\|[A(t) - A(\tau)]A^{-1}(s)\right\|_{E \to E} \leq M|t - \tau|^{\varepsilon}, 0 < \varepsilon \leq 1, \tag{4.88}$$

where M and ε are positive constants independent of t, s and τ for $0 \leq t, s, \tau < \infty$.

An operator-valued function $V(t, y)$, defined and strongly continuous jointly in t, y for $0 \leq y < t < \infty$, is called a fundamental solution of (4.86) if

1) the operator $V(t, y)$ is strongly continuous in t and y for $0 \leq y < t < \infty$,

2) the following identity holds:

$$V(t, y) = V(t, \tau)V(\tau, y), V(y, y) = I \tag{4.89}$$

   for $0 \leq y \leq \tau \leq t < \infty$, where, I is the identity operator,

3) the operator $V(t, y)$ maps the region D into itself. The operator $U(t, y) = A(t)V(t, y)A^{-1}(y)$ is bounded and strongly continuous in t and y for $0 \leq y < t < \infty$,

4) on the region D the operator $V(t, y)$ is differentiable relative to t and y, while

$$V_t(t, y) + A(t)V(t, y) = 0, \tag{4.90}$$

   and

$$V_y(t, y) - V(t, y)A(y) = 0, \tag{4.91}$$

5) the subsequent estimates hold:

$$\|V(t, y)\|_{E \to E} \leq Pe^{-\delta(t-y)}, \quad t \geq y \geq 0 \tag{4.92}$$

   for some $\delta \in [0, \infty)$ and $P \in [1, \infty)$.

A function $u(t)$ is called a solution of problem (4.86) if the conditions below are satisfied:

1. $u(t)$ is continuously differentiable on $[-\omega, \infty)$.
2. The element $u(t) \in D(A(t))$, $\forall t \in [-\omega, \infty)$, and the function $A(t)u(t)$ is continuous on $[-\omega, \infty)$.
3. $u(t)$ satisfies the equation and the initial condition (4.86).

We reduced problem (4.86) into an integral equation of the form

$$u(t) = V(t, m\omega)u(m\omega) + \int_{m\omega}^{t} V(t, y)g(y, u(y), u(y-\omega))dy,$$

$$m\omega \leq t \leq (m+1)\,\omega, m = 0, 1, ..., u(t) = \varphi(t), -\omega \leq t \leq 0$$

in $[-\omega, \infty) \times E$ and using successive approximations, obtained recursive formula for the solution of problem (4.86) is

$$\begin{aligned} u_i(t) &= V(t, m\omega)u_i(m\omega) + \int_{m\omega}^{t} V(t, y)g(y, u_{i-1}(y), u_i(y-\omega))dy, \\ u_0(t) &= V(t, m\omega)u(m\omega), m\omega \leq t \leq (m+1)\omega, m = 0, 1, ..., \end{aligned}$$

$$i = 1, 2, ..., u(t) = \varphi(t), -\omega \leq t \leq 0. \tag{4.93}$$

Theorem 4.11. *Assume the hypotheses below:*

1. $\varphi : [-\omega, 0] \times E \longrightarrow E$ *be continuous function and*

$$\|\varphi(t)\|_E \leq M. \tag{4.94}$$

2. $g : [0, \infty) \times E \times E \longrightarrow E$ *be bounded and continuous function, i.e.;*

$$\|g(t, u, v)\|_E \leq \bar{M} \tag{4.95}$$

and with respect to z*, the Lipschitz condition holds uniformly*

$$\|g(t, v, z) - g(t, u, z)\|_E \leq L\|v - u\|_E \tag{4.96}$$

where $L, M, \bar{M}$ *are positive constants. Then problem (4.86) has a unique bounded solution in* $[0, \infty) \times E$*.*

Proof of theorem 4.11. Using the interval $t \in [0, \omega]$, problem (4.86) can be written as

$$\frac{du}{dt} + A(t)u(t) = g(t, u(t), \varphi(t-\omega)), u(0) = \varphi(0)$$

which in an equivalent integral form, becomes

$$u(t) = V(t, 0)\varphi(0) + \int_{0}^{t} V(t, y)g(y, u(y), \varphi(y-\omega))dy. \tag{4.97}$$

In accordance with the recursive approximation approach (4.93), we get

$$u_i(t) = V(t, 0)\varphi(0) + \int_{0}^{t} V(t, y)g(y, u_{i-1}(y), \varphi(y-\omega))dy, i = 1, 2, ... \tag{4.98}$$

Therefore,

$$u(t) = u_0(t) + \sum_{i=1}^{\infty}(u_i(t) - u_{i-1}(t)), \tag{4.99}$$

where

$$u_0(t) = V(t,0)\varphi(0).$$

From (4.92) and (4.94), we obtain

$$\|u_0(t)\|_E = \|V(t,0)\|\|\varphi(0)\|_E \leq MP.$$

Using formula (4.98) along with estimates (4.92) and (4.95), we get

$$\|u_1(t) - u_0(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_0,\varphi(y-\omega))\|_E dy \leq \bar{M}Pt.$$

By the triangle inequality, we have

$$\|u_1(t)\|_E \leq MP + \bar{M}Pt.$$

Applying formula (4.98) along with estimates (4.96),(4.92) and (4.95), we obtain

$$\|u_2(t) - u_1(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_1(y),\varphi(y-\omega)) - g(y,u_0(y),\varphi(y-\omega))\|_E dy$$

$$\leq LP\int_0^t \|u_1(y) - u_0(y)\|_E dy \leq LP^2\bar{M}\int_0^t y dy = \frac{\bar{M}}{L}\frac{(PLt)^2}{2!}.$$

Then, by the triangle inequality, we have

$$\|u_2(t)\|_E \leq MP + \bar{M}Pt + \frac{\bar{M}}{L}\frac{(PLt)^2}{2!}.$$

Let

$$\|u_i(t) - u_{i-1}(t)\|_E \leq \frac{\bar{M}}{L}\frac{(LPt)^i}{i!}.$$

Then, we obtain

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \int_0^t \|V(t,y)\|\|g(y,u_i(y),\varphi(y-\omega)) - g(y,u_{i-1}(y),\varphi(y-\omega))\|_E dy$$

$$\leq P\int_0^t L\|u_i(y) - u_{i-1}(y)\|_E dy \leq P\int_0^t L\frac{\bar{M}}{L}\frac{(LPy)^i}{i!}dy$$

$$= \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}.$$

Consequently, for any $i, i \geq 1$, we have that

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}$$

and

$$\|u_{i+1}(t)\|_E \leq PM + \bar{M}Pt + \frac{\bar{M}}{L}\frac{(PLt)^2}{2!} + ... + \frac{\bar{M}}{L}\frac{(LPt)^{i+1}}{(i+1)!}$$

by mathematical induction. It is implied by that and formula (4.99) that

$$\|u(t)\|_E \leq \|u_0(t)\|_E + \sum_{i=1}^{\infty} \|u_i(t) - u_{i-1}(t)\|_E \leq MP + \sum_{i=1}^{\infty} \frac{\bar{M}}{L}\frac{(LPt)^i}{i!}$$

$$\leq MP + \frac{\bar{M}}{L}e^{LPt}, 0 \leq t \leq \omega$$

which shows that solution of problem (4.86) exists and is bounded in $[0, \omega] \times E$.

Next, for $t \in [\omega, 2\omega]$, note that $0 \leq t - \omega \leq \omega$. We denote that

$$\varphi_1(t) = u(t - \omega), t \in [\omega, 2\omega].$$

and suppose that problem (4.86) has a BS in $[\omega, 2\omega] \times E$. Replacing t and $t - \omega$ and assuming that

$$\|g(t, u_0(t), \varphi_1(t))\|_E \leq \bar{M}_1$$

and

$$\|\varphi_1(t)\|_E \leq M_1.$$

Hence,

$$u_0(t) = V(t, \omega)\varphi_1(\omega)$$

$$u_i(t) = V(t, \omega)\varphi_1(\omega) + \int_{\omega}^{t} V(t, y)g(y, u_{i-1}(y), \varphi_1(y))dy, i = 1, 2, ...$$

In the same way, for any $i, i \geq 1$, we have

$$\|u_{i+1}(t) - u_i(t)\|_E \leq \frac{\bar{M}_1}{L}\frac{(LPt)^{i+1}}{(i+1)!}$$

and

$$\|u_{i+1}(t)\|_E \leq PM_1 + \bar{M}_1Pt + \frac{\bar{M}_1}{L}\frac{(LPt)^2}{2!} + ... + \frac{\bar{M}_1}{L}\frac{(LPt)^{i+1}}{(i+1)!}.$$

Then it follows that

$$\|u(t)\|_E \leq M_1P + \frac{\bar{M}_1}{L}e^{LP(t-\omega)}, \omega \leq t \leq 2\omega$$

this proves that solution of problem (4.86) exists and is bounded in $[\omega, 2\omega] \times E$. In the same procedure, we can obtain that

$$\|u(t)\|_E \leq M_mP + \frac{\bar{M}_m}{L}e^{LP(t-m\omega)}, m\omega \leq t \leq (m+1)\omega,$$

where M_m and $\bar{M}_m$ are bounded. This proves the existence of a BS of problem (4.86) in $[m\omega, (m+1)\,\omega] \times E$. The function $u(t)$ constructed for problem (4.86) has a BS in $[0, \infty) \times E$.

We shall now prove that this solution of problem (4.86) is unique. Assume that problem (4.86) has a BS $v(t)$ and that $v(t) \neq u(t)$. We denote $w(t) = v(t) - u(t)$. Hence for $w(t)$, we obtain that

$$\begin{cases} \frac{dw}{dt} + A(t)w(t) = g(t, v(t), v(t-\omega)) - g(t, u(t), u(t-\omega)), t \in (0, \infty), \\ w(t) = 0, t \in [-\omega, 0]. \end{cases}$$

We consider $0 \leq t \leq \omega$. As $v(t-\omega) = u(t-\omega) = \varphi(t-\omega)$, we get

$$\begin{cases} \frac{dw}{dt} + A(t)w(t) = g(t, v(t), \varphi(t-\omega)) - g(t, u(t), \varphi(t-\omega)), t \in (0, \infty), \\ w(t) = 0, t \in [-\omega, 0]. \end{cases}$$

Therefore,

$$w(t) = \int_0^t V(t, y) \left[g(y, v(y), \varphi(y-\omega)) - g(y, u(y), \varphi(y-\omega)) \right] dy$$

Applying estimates (4.92) and (4.95), we get

$$\begin{aligned} \|w(t)\|_E &\leq \int_0^t \|V(t, s)\| \, \|g(y, v(y), \varphi(y-\omega)) - g(y, u(\omega), \varphi(y-\omega))\|_E dy \\ &\leq PL \int_0^t \|v(y) - u(y)\|_E dy \leq PL \int_0^t \|w(y)\|_E dy. \end{aligned}$$

By means of integral inequality, we obtain

$$\|w(t)\|_E \leq 0.$$

This implies that, $w(t) = 0$ which proves that solution of problem (4.86) is unique and bounded in $[0, \omega] \times E$.

Using similar procedure and mathematical induction, we can prove that solution of problem (4.86)is unique and bounded in $[0, \infty) \times E$.

Remark 4.3. The approach used in the current study also makes it possible to prove, under certain assumptions, that there exists a unique bounded solution of the IVP for semi-linear parabolic equations

$$\begin{cases} \frac{du}{dt} + A(t)u(t) = g(t, B(t)u(t), B(t)u(t-\omega)), t \in [0, \infty), \\ u(t) = \varphi(t), t \in [-\omega, 0]. \end{cases} \tag{4.100}$$

in an arbitrary Banach space E with unbounded operators $A(t)$ and $B(t)$ with dense domains $D(A(t)) \subset D(B(t))$.

Applications

First, we consider the IBVP for nonlinear one dimensional DPPDEs with Dirichlet condition

$$\begin{cases} u_t(t,x) - a(t,x)u_{xx}(t,x) + \delta u(t,x) = g(t,x,u(t,x),u(t-\omega,x)), \\ t \in (0,\infty), x \in (0,b)\,, \\ u(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,b) = 0, t \in [-\omega,0], x \in [0,b]\,, \\ u(t,0) = u(t,b) = 0, t \in [0,\infty), \end{cases} \tag{4.101}$$

where $\varphi(t,x), a(t,x)$ are given SSFs and $\delta > 0$ is the sufficiently large number. Assume that $a(t,x) \geq a > 0$.

Theorem 4.12. *Assume the hypotheses below:*

i $\varphi : [-\omega,0] \times C\,[0,b] \to C\,[0,b]$ *be continuous function and*

$$\|\varphi(t,.)\|_{C[0,b]} \leq M. \tag{4.102}$$

ii $g : (0,\infty) \times (0,b) \times C\,[0,b] \times C\,[0,b] \to C\,[0,b]$ *be bounded and continuous function, i.e.;*

$$\|g(t,.,u,v))\|_{C[0,b]} \leq \overline{M} \tag{4.103}$$

and with respect to z*, the Lipschitz condition holds uniformly.*

$$\|g(t,.,u,z) - g(t,.,v,z)\|_{C[0,b]} \leq L\,\|u - v\|_{C[0,b]}\,, \tag{4.104}$$

where, $L, M, \overline{M}$ *are positive constants. Then problem (4.101) has a unique BS in* $[0,\infty) \times C\,[0,b]$.

The proof of the Theorem 4.12 is based on the abstract Theorem 4.11, on the strong positivity of a differential operator A^x in $C\,[0,b]$ according to the following formula:

$$A^x(t)v(x) = -a(t,x)\frac{d^2v(x)}{dx^2} + \delta v(x) \tag{4.105}$$

with domain $D(A^x(0)) = \left\{ v \in C^{(2)}[0,b] : v(0) = v(b) = 0 \right\}$ (Poorkarimi, and Wiener, 1989) and on the estimate

$$\|V(t,y)\|_{C[0,b] \to C[0,b]} \le M_1, \quad t \ge y \ge 0. \tag{4.106}$$

Second, we consider the IBVP for nonlinear one dimensional DPPDEs with nonlocal conditions

$$\begin{cases} u_t(t,x) - a(t,x)u_{xx}(t,x) + \delta u(t,x) = g(t,x,u(t,x),u(t-\omega,x)), \\ t \in (0,\infty), x \in (0,b), \\ u(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,b), \varphi_x(t,0) = \varphi_x(t,b), \\ t \in [-\omega,0], x \in [0,b], \\ u(t,0) = u(t,b), u_x(t,0) = u_x(t,b), t \in [0,\infty), \end{cases} \tag{4.107}$$

where $\varphi(t,x), a(t,x)$ are SSFs given and $\delta > 0$ is the sufficiently large number. Assume that $a(t,x) \ge a > 0$.

Theorem 4.13. *Suppose that the assumptions (4.102), (4.103) and (4.104) hold. Then problem (4.107) has a unique BS in* $[0,\infty) \times C[0,b]$.

The proof of the Theorem 4.13 is based on the abstract Theorem 4.11, on the strong positivity of a differential operator A^x in $C[0,b]$ according to the following formula:

$$A^x(t)v(x) = -a(t,x)\frac{d^2v(x)}{dx^2} + \delta v(x) \tag{4.108}$$

with domain $D(A^x(0)) = \left\{ v \in C^{(2)}[0,b] : v(0) = v(b), v'(0) = v'(b) \right\}$(Ashyralyev, 2007) and on estimate (4.106).

Third, we consider the initial value problem on the range

$$\{0 \le t < \infty, x = (x_1, \cdots, x_n) \in R^n, r = (r_1, \cdots, r_n)\}$$

for $2m$-th order multidimensional nonlinear DPPDEs

$$\begin{cases} u_t(t,x) + \sum\limits_{|r|=2m} a_r(t,x) u_{x_1^{r_1}\ldots x_n^{r_n}}(t,x) + \delta u(t,x) \\ \\ = g(t,x,u(t,x),u(t-\omega,x)), t \in (0,\infty), x \in R^n, \\ \\ u(t,x) = \varphi(t,x), t \in [-\omega,0], x \in R^n, \end{cases} \tag{4.109}$$

where $a_r(t,x)$ and $\varphi(t,x)$ are given SSFs and $\delta > 0$ is the sufficiently large number. We will suppose that the symbol $[\xi = (\xi_1, \cdots, \xi_n) \in R^n]$ and $|r| = r_1 + ... + r_n$,

$$A^x(t,\xi) = \sum_{|r|=2m} a_r(t,x)\,(i\xi_1)^{r_1} \ldots (i\xi_n)^{r_n}$$

of the differential operator of the form

$$A_1^x(t) = \sum_{|r|=2m} a_r(t,x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{4.110}$$

acting on functions defined on the space R^n, the inequalities are satisfied.

$$0 < M_1|\xi|^{2m} \le (-1)^m A^x(t,\xi) \le M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$, where $|\xi| = (|\xi_1|^2 + \cdots + |\xi_n|^2)^{\frac{1}{2}}$. We can reduce the initial value problem (4.109) to the initial value problem (4.86) in Banach space $E = C(R^n)$ with a strongly positive operator $A^x(t) = A_1^x(t) + \delta I$ defined by (4.110) (Smirnitskii, and Sobolevskii, 1981; Smirnitskii, 1993). The corollary below follows from the abstract Theorem 4.1.

Theorem 4.14. *Assume the hypotheses below:*

i $\varphi : [-\omega, 0] \times C(R^n) \to C(R^n)$ *be bounded and continuous function and*

$$\|\varphi(t,.)\|_{C(R^n)} \le M.$$

ii $g : (0,\infty) \times C(R^n) \times C(R^n) \to C(R^n)$ *be bounded and continuous function, i.e.;*

$$\|g(t,.,u,v))\|_{C(R^n)} \le \overline{M}$$

and with respect to z*, the Lipschitz condition holds uniformly.*

$$\|g(t,.,v,z) - g(t,.,u,z)\|_{C(R^n)} \le L\,\|v-u\|_{C(R^n)}\,,$$

where $L, M, \overline{M}$ *are positive constants. Then problem (4.109) has a unique bounded solution in* $[0,\infty) \times C(R^n)$.

The proof of Theorem 4.14 is based on the abstract Theorem 4.11, on the strong positivity of a differential operator $A^x(t)$ in $C(R^n)$ according to the formula (4.110), and on the estimate

$$\|V(t,y)\|_{C(R^n)\to C(R^n)} \leq M_3, \quad t \geq y \geq 0.$$

Numerical Results

Generally speaking, nonlinear problems cannot be solved precisely. Therefore the FSADSs for the solution of nonlinear one-dimensional DPPDE are presented. Numerical results are given.

Problem 4.8. *Consider the IBVP*

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = u(t,x)\left[u([t-1],x)\cos x - \frac{\partial u([t-1],x)}{\partial x}\sin x\right], \\ t \in (0,\infty), \ \ x \in (0,\pi), \\ u(0,x) = \sin x, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \ \ t \in [0,\infty) \end{cases} \tag{4.111}$$

for the nonlinear delay parabolic equation.Here $[\cdot]$ *is notation of integer function. The exact solution (ES) of this test example is* $u(t,x) = e^{-t}\sin x$.

We get the following iterative FADS for the approximate solution (AS) of the IBVP (4.111)

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} - {}_{m-1}u_n^k {}_m u_n^{[k-N]} \cos x_n \\ + {}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h} \sin x_n = 0, \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1, M-1}, \\ {}_m u_n^0 = \sin x_n, x_n = nh, n \in \overline{0, M}, \\ {}_m u_0^k = {}_m u_M^k = 0, k \in \overline{0,\infty} \end{cases} \tag{4.112}$$

for the nonlinear delay parabolic equation.
Here m denotes the iteration number and an initial guess ${}_0u_n^k, k \in \overline{0,N}, n \in \overline{0,M}$ is to be made. For solving difference scheme (4.112), we follow the numerical steps given below. The algorithm is as follows for $k \in \overline{0,N}$,$n \in \overline{0,M}$:

1. $m = 1$
2. ${}_{m-1}u_n^k$ is known
3. ${}_m u_n^k$ is calculated
4. If the max absolute error between ${}_{m-1}u_n^k$ and ${}_m u_n^k$ is greater than the given tolerance value, take $m = m + 1$ and go to step 2. Otherwise, terminate the iteration process and take ${}_m u_n^k$ as the result of the given problem.

We write (4.112) in matrix form

$$A_m U^k + B_m U^{k-1} = R\varphi({}_{m-1}u^k, {}_m u^{k-N}), k \in \overline{1,N},$$

$${}_m U^0 = \{\sin x_n\}_{n=0}^M, \tag{4.113}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ a & c & a & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & a & c & a & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c & a & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & c & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & & a & c & a \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$a = -\frac{1}{h^2}, \ , b = \frac{1}{\tau}, c = \frac{1}{\tau} + \frac{2}{h^2},$$

R is identity matrix of size $M+1$, ${}_m u_n^k = e^{-t_k} \sin x_n$ for $k \in \overline{-N, 0}$ and $\varphi({}_{m-1}u^k, {}_m u^{k-N})$,U^s are $(M+1) \times 1$ column vectors as

$$\varphi({}_{m-1}u^k, {}_m u^{k-N}) = \begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} , {}_m U^s = \begin{bmatrix} {}_m U_0^s \\ {}_m U_1^s \\ \vdots \\ {}_m U_{M-1}^s \\ {}_m U_M^s \end{bmatrix}_{(M+1)\times 1}, \ s = k, \ k \pm 1,$$

where

$$\varphi_n^k = {}_{m-1}u_n^k {}_m u_n^{[k-N]} \cos x_n - {}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h} \sin x_n, \; n \in \overline{1, M-1}.$$

So, we have the first order difference equation with respect to k with matrix coefficients. From (4.113) it follows that

$$ {}_mU^k = -A^{-1}B_m U^{k-1} + A^{-1}R\varphi^k, k \in \overline{pN+1, (p+1)N}, p = 0, 1, 2, ...,$$

$$ {}_mU^0 = \{\sin x_n\}_{n=0}^M . \tag{4.114}$$

Additionally, using the SADS for the AS of problem (4.111), we obtain the following system of equations

$$\begin{cases} \frac{{}_mu_n^k - {}_mu_n^{k-1}}{\tau} - \frac{{}_mu_{n+1}^k - 2{}_mu_n^k + {}_mu_{n-1}^k}{h^2} + \tau \frac{{}_mu_{n+2}^k - 4{}_mu_{n+1}^k + 6{}_mu_n^k - 4{}_mu_{n-1}^k + {}_mu_{n-2}^k}{2h^4} \\ = \frac{1}{2}{}_mu_n^k \left[{}_{m-1}u_n^{k-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n\right] \\ + \frac{1}{2}{}_mu_n^{k-1} \left[{}_{m-1}u_n^{k-1-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n\right] \\ -\frac{\tau}{4}{}_mu_{n+1}^k \frac{{}_{m-1}u_{n+1}^{k-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-N} - {}_{m-1}u_n^{k-N}}{2h} \sin x_{n+1}}{h^2} \\ -\frac{\tau}{4}{}_mu_n^k \frac{-2{}_{m-1}u_n^{k-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n}{h^2} \\ -\frac{\tau}{4}{}_mu_{n-1}^k \frac{{}_{m-1}u_{n-1}^{k-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-N} - {}_{m-1}u_{n-2}^{k-N}}{2h} \sin x_{n-1}}{h^2} \\ -\frac{\tau}{4}{}_mu_{n+1}^{k-1} \frac{{}_{m-1}u_{n+1}^{k-1-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-1-N} - {}_{m-1}u_n^{k-1-N}}{2h} \sin x_{n+1}}{h^2} \\ -\frac{\tau}{4}{}_mu_n^{k-1} \frac{-2{}_{m-1}u_n^{k-1-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n}{h^2} \\ -\frac{\tau}{4}{}_mu_{n-1}^{k-1} \frac{{}_{m-1}u_{n-1}^{k-1-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-1-N} - {}_{m-1}u_{n-2}^{k-1-N}}{2h} \sin x_{n-1}}{h^2}, \\ t_k = k\tau, x_n = nh, k \in \overline{1, N-1}, n \in \overline{2, M-2}, \\ {}_mu_n^0 = \sin x_n, n \in \overline{0, M}, {}_mu_0^k = {}_mu_M^k = 0, k \in \overline{0, \infty}, \\ {}_mu_3^k = 4{}_mu_2^k - 5{}_mu_1^k, {}_mu_{M-3}^k = 4{}_mu_{M-2}^k - 5{}_mu_{M-1}^k, k \in \overline{0, \infty}. \end{cases} \tag{4.115}$$

We obtain again $(M+1) \times (M+1)$ SLE and we reformat them into matrix form

(4.113), where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 1 \\ e & f & g & f & e & . & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & g & f & e & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . & . & e & f & g & f & e \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & -1 & 4 & -5 & 0 \\ 0 & -5 & 4 & -1 & 0 & 0 & . & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$e = \frac{\tau}{2h^4}, f = -\frac{1}{h^2} - \frac{2\tau}{h^4},$$
$$g = \frac{1}{\tau} + \frac{2}{h^2} + \frac{3\tau}{h^4}, l = -\frac{1}{\tau},$$

$$\varphi({}_{m-1}u^k, {}_m u^{k-N}) = \begin{bmatrix} 0 \\ 0 \\ \varphi_2^k \\ \vdots \\ \varphi_{M-2}^k \\ 0 \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

where

$$\begin{aligned}
\varphi_n^k =& \frac{1}{2}{}_m u_n^k \left[{}_{m-1}u_n^{k-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n \right] \\
&+ \frac{1}{2}{}_m u_n^{k-1} \left[{}_{m-1}u_n^{k-1-N} \cos x_n - \frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n \right] \\
&- \frac{\tau}{4}{}_m u_{n+1}^k \frac{{}_{m-1}u_{n+1}^{k-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-N} - {}_{m-1}u_n^{k-N}}{2h} \sin x_{n+1}}{h^2} \\
&- \frac{\tau}{4}{}_m u_n^k \frac{-2{}_{m-1}u_n^{k-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-N} - {}_{m-1}u_{n-1}^{k-N}}{2h} \sin x_n}{h^2} \\
&- \frac{\tau}{4}{}_m u_{n-1}^k \frac{{}_{m-1}u_{n-1}^{k-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-N} - {}_{m-1}u_{n-2}^{k-N}}{2h} \sin x_{n-1}}{h^2} \\
&- \frac{\tau}{4}{}_m u_{n+1}^{k-1} \frac{{}_{m-1}u_{n+1}^{k-1-N} \cos x_{n+1} - \frac{{}_{m-1}u_{n+2}^{k-1-N} - {}_{m-1}u_n^{k-1-N}}{2h} \sin x_{n+1}}{h^2} \\
&- \frac{\tau}{4}{}_m u_n^{k-1} \frac{-2{}_{m-1}u_n^{k-1-N} \cos x_n + 2\frac{{}_{m-1}u_{n+1}^{k-1-N} - {}_{m-1}u_{n-1}^{k-1-N}}{2h} \sin x_n}{h^2} \\
&- \frac{\tau}{4}{}_m u_{n-1}^{k-1} \frac{{}_{m-1}u_{n-1}^{k-1-N} \cos x_{n-1} - \frac{{}_{m-1}u_n^{k-1-N} - {}_{m-1}u_{n-2}^{k-1-N}}{2h} \sin x_{n-1}}{h^2}
\end{aligned}$$

for $n \in \overline{2, M-2}$. Hence, we have a second order difference equation with respect to k matrix coefficients. Applying (4.113), we can obtain the solution of this difference scheme. In computations for both first and second order of accuracy difference schemes, the initial guess is chosen as ${}_0u_n^k = e^{-t_k} \sin x_n$ and when the maximum errors between two consecutive results of iterative difference schemes (4.112) and (4.115) become less than 10^{-8}, the iterative process is terminated.

We provide numerical results for various values of M and N and the numerical solutions of these difference schemes are represented by ${}_{m}u_{n}^{k}$ at (t_k, x_n). Tables 7-9 are constructed for $N = M = 30,\ 60,\ 120$ in $t \in [n, n+1],\ n = 0, 1, 2,$ respectively. The errors are calculated using the following formula.

$$_{m}\left(E_{M}^{N}\right)_{p} = \max_{\substack{pN+1 \leq k \leq (p+1)N, p=0,1,\ldots. \\ 1 \leq n \leq M-1}} \left| u\left(t_k, x_n\right) - {}_{m}u_{n}^{k} \right|. \tag{4.116}$$

To finish iteration process it was used condition

$$\max_{\substack{pN+1 \leq k \leq (p+1)N, p=0,1,\ldots. \\ 1 \leq n \leq M-1}} \left| {}_{m}u_{n}^{k} - {}_{m-1}u_{n}^{k,} \right| < 10^{-8} \tag{4.117}$$

in each sub-interval.

Table 7.

Error comparison between difference schemes (4.112) and (4.115) in $t \in [0,1]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.112)	$6.3783 \times 10^{-3}, m = 2$	$3.1279 \times 10^{-2}, m = 2$	$1.5485 \times 10^{-3}, m = 2$
(4.115)	$4.5864 \times 10^{-4}, m = 3$	$1.1212 \times 10^{-4}, m = 3$	$2.7577 \times 10^{-5}, m = 2$

Table 8.

Error comparison between difference schemes (4.112) and (4.115) in $t \in [1,2]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.112)	$2.3464 \times 10^{-3}, m = 3$	$1.5070 \times 10^{-3}, m = 3$	$5.6964 \times 10^{-4}, m = 2$
(4.115)	$1.6358 \times 10^{-4}, m = 3$	$4.2149 \times 10^{-5}, m = 2$	$1.0698 \times 10^{-5}, m = 2$

Table 9.

Error comparison between difference schemes (4.112) and (4.115) in $t \in [2,3]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.112)	$8.6321 \times 10^{-4}, m = 3$	$4.2332 \times 10^{-4}, m = 2$	$2.0956 \times 10^{-4}, m = 2$
(4.115)	$5.3201 \times 10^{-5}, m = 2$	$1.3581 \times 10^{-5}, m = 2$	$3.4122 \times 10^{-6}, m = 2$

Problem 4.9. *We also consider the IBVP*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + \sin(u(t,x)) \\ = u(t,x)\left[2u([t-1],x)\cos 2x - \frac{\partial u([t-1],x)}{\partial x}\sin 2x\right] + f(t,x), \\ u(0,x) = \sin 2x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \ t \in [0,\infty) \end{cases} \tag{4.118}$$

for the nonlinear delay parabolic differential equation. The ES of this test example is $u(t,x) = e^{-4t}\sin 2x$ and $f(t,x) = \sin(e^{-4t}\sin 2x)$.

We get the following FASD for the AS of the IBVP (4.118)

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} - 2{}_{m-1}u_n^k {}_m u_n^{[k-N]} \cos 2x_n \\ + {}_{m-1}u_n^k \frac{{}_m u_{n+1}^{[k-N]} - {}_m u_{n-1}^{[k-N]}}{2h} \sin 2x_n = \sin\left({}_m u_n^k\right) + f(t_k, x_n), \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1, M-1}, \\ {}_m u_n^0 = \sin 2x_n, x_n = nh, n \in \overline{0, M}, \\ {}_m u_0^k = {}_m u_M^k, {}_m u_1^k - {}_m u_0^k = {}_m u_M^k - {}_m u_{M-1}^k, k \in \overline{pN+1, (p+1)N}, p = 0, 1, ... \end{cases} \tag{4.119}$$

for the nonlinear delay parabolic equation with nonlocal conditions. We write (4.119) in matrix form

$$A_m U^k + B_m U^{k-1} = R\theta, k \in \overline{pN+1, (p+1)N}, p = 0, 1, ...,$$

$${}_m U^0 = \left\{\sin 2x_n\right\}_{n=0}^{M}, \tag{4.120}$$

where

$${}_m U^k = \left\{{}_m u_n^k\right\}_{n=0}^{M}, \theta_n^k = \sin\left({}_m u_n^k\right) + f(t_k, x_n),$$

$$n = 0, ..., M, k \in \overline{pN+1, (p+1)N}, p = 0, 1, ...,$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 1 \\ a & c_1^k & a & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & a & c_2^k & a & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c_3^k & a & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & c_4^k & a & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & a & c_{M-1}^k & a \\ 1 & -1 & 0 & 0 & 0 & . & . & . & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & l & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$a = -\tfrac{1}{h^2}, l = -\tfrac{1}{\tau},$$
$$c_n^k = \tfrac{1}{\tau} + \tfrac{2}{h^2} - 2u_n^{[k-N]} \cos 2x_n + \tfrac{u_{n+1}^{[k-N]} - u_{n-1}^{[k-N]}}{2h} \sin 2x_n$$

and R is identity matrix of size $M+1$, θ is zero matrix with $(M+1)\times 1$ dimension. So, we have the first order difference equation with respect to k with matrix coefficients. From (4.120) it follows that

$${}_mU^k = -A^{-1}B_mU^{k-1} + A^{-1}R\theta^k, k \in \overline{pN+1,(p+1)N}, p = 0, 1, ...,$$

$${}_mU^0 = \{\sin 2x_n\}_{n=0}^{M}. \tag{4.121}$$

Furthermore, using the SADS for the AS of problem (4.118), we obtain the following system of equations

$$\begin{cases} \frac{{}_m u_n^k - {}_m u_n^{k-1}}{\tau} - \frac{{}_m u_{n+1}^k - 2{}_m u_n^k + {}_m u_{n-1}^k}{h^2} + \tau \frac{{}_m u_{n+2}^k - 4{}_m u_{n+1}^k + 6{}_m u_n^k - 4{}_m u_{n-1}^k + {}_m u_{n-2}^k}{2h^4} \\ -\frac{1}{2}\left[{}_m u_n^k {}_{m-1}u_n^{[k-N]} \cos 2x_n - {}_m u_n^k \frac{{}_{m-1}u_{n+1}^{[k-N]} - {}_{m-1}u_{n-1}^{[k-N]}}{2h} \sin 2x_n\right] \\ -\frac{1}{2}\left[{}_m u_n^{k-1} {}_{m-1}u_n^{[k-1-N]} \cos 2x_n - {}_m u_n^{k-1} \frac{{}_{m-1}u_{n+1}^{[k-1-N]} - {}_{m-1}u_{n-1}^{[k-1-N]}}{2h} \sin 2x_n\right] \\ +\frac{\tau}{4} \frac{{}_m u_{n+1}^k {}_{m-1}u_{n+1}^{[k-N]} \cos 2x_{n+1} - {}_m u_{n+1}^k \frac{{}_{m-1}u_{n+2}^{[k-N]} - {}_{m-1}u_n^{[k-N]}}{2h} \sin 2x_{n+1}}{h^2} \\ +\frac{\tau}{4} \frac{-2{}_m u_n^k {}_{m-1}u_n^{[k-N]} \cos 2x_n + 2{}_m u_n^k \frac{{}_{m-1}u_{n+1}^{[k-N]} - {}_{m-1}u_{n-1}^{[k-N]}}{2h} \sin 2x_n}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^k {}_{m-1}u_{n-1}^{[k-N]} \cos 2x_{n-1} - {}_m u_{n-1}^k \frac{{}_{m-1}u_n^{[k-N]} - {}_{m-1}u_{n-2}^{[k-N]}}{2h} \sin 2x_{n-1}}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_{n+1} - {}_m u_{n+1}^{k-1} \frac{{}_{m-1}u_{n+2}^{[k-1-N]} - {}_{m-1}u_n^{[k-1-N]}}{2h} \sin 2x_{n+1}}{h^2} \\ +\frac{\tau}{4} \frac{-2{}_m u_n^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_n + 2{}_m u_n^{k-1} \frac{{}_{m-1}u_{n+1}^{[k-1-N]} - {}_{m-1}u_{n-1}^{[k-1-N]}}{2h} \sin 2x_{n-1}}{h^2} \\ +\frac{\tau}{4} \frac{{}_m u_{n-1}^{k-1} {}_{m-1}u_{n+1}^{[k-1-N]} \cos 2x_{n-1} - {}_m u_{n-1}^{k-1} \frac{{}_{m-1}u_n^{[k-1-N]} - {}_{m-1}u_{n-2}^{[k-1-N]}}{2h} \sin 2x_{n-1}}{h^2} \\ +\sin\left({}_m u_n^k\right) = f(t_k, x_n), \\ \\ t_k = k\tau, x_n = nh, k \in \overline{1, N-1}, n \in \overline{2, M-2}, \\ {}_m u_n^0 = \sin 2x_n, n \in \overline{0, M}, \\ {}_m u_0^k = {}_m u_M^k, -{}_m u_2^k + 4{}_m u_1^k - 3{}_m u_0^k = 3{}_m u_M^k - 4{}_m u_{M-1}^k + {}_m u_{M-2}^k, \\ -{}_m u_3^k + 4{}_m u_2^k - 5{}_m u_1^k + 2{}_m u_0^k = 2{}_m u_M^k - 5{}_m u_{M-1}^k + 4{}_m u_{M-2}^k - {}_m u_{M-3}^k, \\ \\ -3{}_m u_4^k + 14{}_m u_3^k - 24{}_m u_2^k + 18{}_m u_1^k - 5{}_m u_0^k \\ = 5{}_m u_M^k - 18{}_m u_{M-1}^k + 24{}_m u_{M-2}^k - 14{}_m u_{M-3}^k + 3{}_m u_{M-4}^k, \\ k \in \overline{pN+1, (p+1)N}, p = 0, 1, ... \end{cases} \tag{4.122}$$

We obtain another $(M+1) \times (M+1)$ SLE they are then rewritten in matrix form (4.120), where

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & -1 \\ 2 & -5 & 4 & -1 & 0 & . & 0 & 1 & -4 & 5 & -2 \\ e & f_3^k & g_3^k & w_3^k & e & . & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f_4^k & g_4^k & w_4^k & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & f_{M-2}^k & g_{M-2}^k & w_{M-2}^k & e & 0 \\ 0 & 0 & 0 & 0 & 0 & . & e & f_{M-1}^k & g_{M-1}^k & w_{M-1}^k & e \\ -3 & 4 & -1 & 0 & 0 & . & 0 & 0 & -1 & 4 & -3 \\ -5 & 18 & -24 & 14 & -3 & . & 3 & -14 & 24 & -18 & 5 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & z_3^k & l_3^k & m_3^k & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3^k & l_3^k & m_3^k & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4^k & l_4^k & m_4^k & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & z_{M-2}^k & l_{M-2}^k & m_{M-2}^k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & z_{M-1}^k & l_{M-1}^k & m_{M-1}^k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$f_n^k=-\tfrac{1}{h^2}-\tfrac{2\tau}{h^4}+\tfrac{\tau}{2h^2}u_{n-1}^{[k-N]}\cos 2x_{n-1}-\tfrac{\tau}{8h^3}\ u_n^{[k-N]}\sin 2x_{n-1}+\tfrac{\tau}{8h^3}\ u_{n-2}^{[k-N]}\sin 2x_{n-1},$$

$$e=\tfrac{\tau}{2h^4},g_n^k=\tfrac{1}{\tau}+\tfrac{2}{h^2}+\ \tfrac{3\tau}{h^4}-u_n^{[k-N]}\cos 2x_n+\tfrac{1}{4h}u_{n+1}^{[k-N]}\sin 2x_n$$

$$-\tfrac{1}{4h}u_{n-1}^{[k-N]}\sin 2x_n-\tfrac{\tau}{h^2}u_n^{[k-N]}\cos 2x_n+\tfrac{\tau}{4h^3}u_{n+1}^{[k-N]}\sin 2x_n-\tfrac{\tau}{4h^3}u_{n-1}^{[k-N]}\sin 2x_n,$$

$$w_n^k=-\tfrac{1}{h^2}-\tfrac{2\tau}{h^4}+\tfrac{\tau}{2\ h^2}u_{n+1}^{[k-N]}\cos 2x_{n+1}-\tfrac{\tau}{8h^3}u_{n+2}^{[k-N]}\sin 2x_{n+1}+\tfrac{\tau}{8h^3}u_n^{[k-N]}\sin 2x_{n+1},$$

$$z_n^k=\tfrac{\tau}{2h^2}u_{n-1}^{[k-N]}\cos 2x_{n-1}-\tfrac{\tau}{8h^3}u_n^{[k-N]}\sin 2x_{n-1}+\tfrac{\tau}{8h^3}u_{n-2}^{[k-N]}\sin 2x_{n-1},$$

$$l_n^k=-\tfrac{1}{\tau}-\ u_n^{[k-N]}\cos 2x_n+\tfrac{1}{4h}u_{n+1}^{[k-N]}\ \sin 2x_n-\tfrac{1}{4h}u_{n-1}^{[k-N]}\ \sin 2x_n$$

$$-\tfrac{\tau}{h^2}u_n^{[k-N]}\cos 2x_n+\tfrac{\tau}{4h^3}u_{n+1}^{[k-N]}\sin 2x_n-\tfrac{\tau}{4h^3}u_{n-1}^{[k-N]}\sin 2x_n,$$

$$m_n^k=\tfrac{\tau}{2h^2}u_{n+1}^{[k-N]}\cos 2x_{n+1}-\tfrac{\tau}{8h^3}u_{n+2}^{[k-N]}\sin 2x_{n+1}+\tfrac{\tau}{8h^3}u_n^{[k-N]}\sin 2x_{n+1}.$$

We provide numerical results for a range of values of M and N and ${}_mu_n^k$ represent the numerical solutions of these difference schemes at (t_k,x_n) . Table 10,

Table 11 and Table 12 are constructed for $M = N = 30$, 60, 120 in $t \in [p, p+1]$, $p = 0, 1, 2$, and the errors are computed by the formulas (4.116) and (4.117).

Table 10.

Error comparison between difference schemes (4.119) and (4.122) in $t \in [0,1]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.119)	$2.4431 \times 10^{-2}, m = 2$	$1.2259 \times 10^{-2}, m = 2$	$6.1304 \times 10^{-3}, m = 2$
(4.122)	$2.0589 \times 10^{-3}, m = 8$	$5.4628 \times 10^{-4}, m = 8$	$1.3865 \times 10^{-4}, m = 7$

Table 11.

Error comparison between difference schemes (4.119) and (4.122) in $t \in [1,2]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.119)	$5.3731 \times 10^{-3}, m = 9$	$2.5664 \times 10^{-3}, m = 8$	$1.2517 \times 10^{-3}, m = 8$
(4.122)	$3.0514 \times 10^{-4}, m = 8$	$7.5756 \times 10^{-5}, m = 7$	$1.9241 \times 10^{-5}, m = 6$

Table 12.

Error comparison between difference schemes (4.119) and (4.122) in $t \in [2,3]$ (Number of iterations = m)

Method	$M = N = 30$	$M = N = 60$	$M = N = 120$
(4.119)	$1.0838 \times 10^{-4}, m = 7$	$4.9176 \times 10^{-5}, m = 6$	$2.3435 \times 10^{-5}, m = 6$
(4.122)	$1.5588 \times 10^{-5}, m = 7$	$2.0085 \times 10^{-6}, m = 5$	$4.8130 \times 10^{-7}, m = 3$

As we doubled the values of N and M each time, beginning with $M = N = 30$. In the FADSs (4.112) and (4.119) in Tables 1-6 respectively, the errors decrease roughly by a proportion of 1/2, while in the SADSs (4.115) and (4.122) in Tables 7-12 respectively, the errors decrease roughly by a proportion of 1/4. Errors shown in the tables demonstrate the consistency of the different schemes and the reliability of the findings. Accordingly, the SADS increases faster than the FADS. These numerical experiments back up the theoretical claims as shown in the tables. With more grid points, the maximum errors can be reduced.

CHAPTER V
Conclusion

This thesis is devoted to constructing the necessary conditions for the existence of a unique bounded solution of semi-linear delay parabolic differential equations. The following results were achieved:

- Historical note and relevant literature are studied.
- The classical methods such as the Fourier series, Laplace transform, and Fourier transform are used to get the exact solution of five semi-linear two-dimensional delay parabolic equations.
- The initial-boundary value problems for the parabolic delay differential equations in a Banach space with strongly unbounded operators are investigated.
- The main theorems on the existence and uniqueness of a bounded solution to these problems are established.
- The application of the main theorems to four different semi-linear and three different types of nonlinear delay parabolic equations is presented.
- The first and second-order iterative accuracy difference schemes for the approximate solution of one-dimensional delay parabolic differential equations are obtained.
- Numerical experiments and error analysis are performed; results are provided in the tables.
- The Matlap implementation of these iterative difference schemes is presented.

References

Ciupe, M., Bivort, B.L., Bortz, D.M., & Nelson, P.W. (2006). Estimating kinetic parameters from HIV primary infection data through the eyes of three different mathematical models. *Mathematical biosciences, 200 1,* 1-27

Nelson, P.W., Murray, J.D., & Perelson, A.S. (2000). A model of HIV-1 pathogenesis that includes an intracellular delay. *Mathematical biosciences, 163 2,* 201-15.

Cooke, K.L., Kuang, Y., & Bingtuan, L. (2005). Analyses of an Antiviral Immune Response Model with Time Delays

Zhao, T. (1995). Global periodic solutions for a differential delay system modeling a microbial population in the chemostat, *Journal of mathematical analysis and applications, 193*, 329–352.

Smolen, P., Baxter, D. A., & Byrne, J. H. (2002). A reduced model clarifies the role of feedback loops and time delays in the Drosophila circadian oscillator. *Biophysical journal, 83*(5), 2349–2359. https://doi.org/10.1016/S0006-3495(02)75249-1

Cooke, K., van den Driessche, P., & Zou, X. (1999). Interaction of maturation delay and nonlinear birth in population and epidemic models. *Journal of mathematical biology, 39*(4), 332–352. https://doi.org/10.1007/s002850050194

Vielle, B., & Chauvet, G. (1998). Delay equation analysis of human respiratory stability *Mathematical biosciences, 152*(2), 105–122.

Villasana, M., & Radunskaya, A, (2003). A delay differential equation model for tumor growth. *journal of mathematical biology, 47*, 270–294. https://doi.org/10.1007/s00285-003-0211-0.

Campbell, S.A., Edwards, R., & Driessche, P.V. (2004). Delayed Coupling Between Two Neural Network Loops. *SIAM J. Appl. Math., 65*, 316-335.

Turchin, P. (1990). Rarity of density dependence or population regulation with lags?. *Nature 344*, 660–663. https://doi.org/10.1038/344660a0

Turchin, P., & Taylor, A.D. (1992). Complex dynamics in ecological time series, *Ecology, 73*, 289–305.

Ashyralyev, A., & Akca, H., (1999). On difference schemes for semilinear delay differential equations with constant delay, *Proceeding of the Conference TSU: Actual Problems of Applied Mathematics*, Ashgabat, 18-27.

Ashyralyev, A., Akca, H., & Guray, U. (1999). Second order of accuracy difference scheme for approximate solutions of delay differential equations, *Functional Differential Equations, vol. 6*, 223-231.

Yenicerioglu, A.F. (2007), The behavior of solutions of second order delay differential equations, *Journal of Mathematical Analysis and Applications, 332* (2), 1278-1290.

Mohamad, S., Akca, H., & Covachev, V. (2009). Discrete-time Cohen-Grossberg neural networks with transmission delays and impulses, Differential and Difference Equations and Applications Book Series: *Tatra Mountains Mathematical Publications, 43*, 145-161.

Torelli, L. (1989). Stability of numerical methods for delay differential equations. *Journal of Computational and Applied Mathematics, 25*, 11-26.

Ashyralyev, A., & Akça, H. (2001). Stability estimates of difference schemes for neutral delay differential equations.*A Nonlinear Analysis-theory Methods & Applications, 44*, 443-452.

Ashyralyev, A., Akça, H., & Yenicerioglu, A.F., (2003). Stability properties of difference schemes for neutral differential equations . Differential equations and Applications *(2004 Nova Science Publishers, Inc.)* 27–34.

Li, B., Bohner, M., & Meng, F. (2008). Periodic solutions of functional dynamic equations with infinite delay. *Nonlinear Analysis, 68*, 1226-1245

Xu, D. Y., Li, S.Y., Zhou, X.P., & Pu, Z.L. (2001). Invariant set and stable region of a class of partial differential equations with time delays. *Nonlinear Analysis, 2*, 161-169.

Wolfgang, V. (1981). Nonlinear parabolic differential-functional inequalities with boundary-functional conditions. *Beitrge Anal. 18*, 85-89.

Liang, J., & Xiao, T.J. (2004). Solvability of the Cauchy problem for infinite delay equations. *Nonlinear Analysis, 58*, 271-297

Ferreira, J.A. (2008). Energy estimates for delay diffusion-reaction equations. *Journal of Computational Mathematics, 26*, 536-553

Lu, X. (1998). Combined iterative methods for numerical solutions of parabolic problems with time delays. *Applied Mathematics and Computations, 89*, 213-224

Gu, W., & Wang, P. (2014). A Crank-Nicolson Difference Scheme for Solving a Type of Variable Coefficient Delay Partial Differential Equations. *Journal of Applied Mathematics*, Article ID 560567 — https://doi.org/10.1155/2014/560567

Berezansky, L., & Braverman, E. (2006). On stability of some linear and non-linear delay differential equations. *Journal of Mathematical Analysis and Applications, 314*(2), 391-411.

Yenicerioglu, A.F., & Yalcinbas, S. (2004). On the stability of the second-order delay differential equations with variable coefficients *Applied Mathematics and Computation, 152*(3), 667–673.

Ashyralyev, A. & Sobolevskii, P.E. (2001). On the stability of the linear delay differential and difference equations. *Abstract and Applied Analysis,6*(5), 267-297.

Ashyralyev, A., & Agirseven, D. (2014). Stability of Parabolic Equations with

Unbounded Operators Acting on Delay Terms. *Electronic Journal of Differential Equations,* 160, 1-13.

Ashyralyev, A., & Agirseven, D. (2014). Stability of Delay Parabolic Difference Equations, *Filomat, 28*(5), 995–1006.

Ashyralyev, A., & Agirseven, D. (2014). On source identification problem for a delay parabolic equation, *Nonlinear Analysis, 19*(3), 335-349.

Ashyralyev, A., & Agirseven, D. (2014). Well-posedness of delay parabolic equations with unbounded operators acting on delay terms *Boundary Value Problems, 126.*

Ashyralyev, A., Agirseven, D. & Agarwal, R.P. (2020). Stability estimates for delay parabolic differential and difference equations. *J. Comput. Appl. Math. 19*, 175–204

Ashyralyev, A., (2007). Fractional spaces generated by the positivite differential and difference operator in a Banach space.In: Tas, K, Tenreiro Machado, JA, Baleanu, D, (eds.) *Proceedings of the Conference Mathematical Methods and Engineering.* Springer, Netherlands, 1322.

Agirseven, D. (2012). Approximate solutions of delay parabolic equations with the Dirichlet condition. *Abstract and Applied Analysis*,(682752)

Ashyralyev, A., & Agirseven, D. (2013). On convergence of difference schemes for delay parabolic equations. *Computers & Mathematics with Applications, 66*(7), 1232-1244.

Ashyralyev, A., Agirseven, D., & Ceylan, B. (2017). Bounded solutions of delay nonlinear evolutionary equations. *Journal of Computational and Applied Mathematics, 318*, 69–78.

Diagana, T., & Mbaye, M.M. (2015). Existence of bounded solutions for nonlinear hyperbolic partial differential equations *Electronic Journal of Differential Equations, 241*, 1–10

Iasson, K., & Miroslav, K. (2014). On the relation of delay equations to first-order hyperbolic partial differential equations, *ESAIM: Control, Optimisation and Calculus of Variations 20*, 894–923.

Igbida, J. (2011). Bounded Solutions to Nonlinear Parabolic Equations,. *International Scholarly Research Notices*, (Article ID 574382).

Kiguradze, T., & Kusano, T. (2005). Bounded and Periodic in a Strip Solutions of Nonlinear Hyperbolic Systems with Two Independent Variables, *Computers and Mathematics with Applications 49*, 335–364.

Mavinga, N., & Nkashama, M.N, (2010). Bounded Solutions of Nonlinear Parabolic Equations. *Electronic Journal of Differential Equations Conf. 19*, 207–220.

Nakao, M. (1977). Bounded, Periodic or Almost-Periodic Solutions of Nonlinear Hyperbolic Partial Differential Equations, *Journal of Differential Equations 23*(3), 368–386.

Poorkarimi, H., & Wiener, J. (1986). Bounded solutions of non-linear hyperbolic equations with delay, *Proceedings of the VII International Conference on Non-Linear Analysis*, Lakshmikantham, Ed. 1, 471–478.

Poorkarimi, H., & Wiener, J. (1989). On the structure of fractional spaces, *Proceedings of the XXVII All-Union Scientific Student Conference*,Novosibirsk. Gos. Univ. 3-7.

Poorkarimi, H., & Wiener, J. (1999). Bounded Solutions of Nonlinear Parabolic Equations with Time Delay, Conference of Applied Mathematics, Univ. of Central Oklahoma,*Electronic Journal of Differential Equations 02l*, 87–91.

Poorkarimi, H.; Wiener, J., & Shah, S.M. (1989). On the exponential growth of solutions to non-linear hyperbolic equations. *Internat. Joun. Math. Sci. 12*, 539–546.

Sadkowski, M. (1978). Boundedness of Solutions of some non-linear hyperbolic equations, *Demonstratio Mathematica XI*(1), 127–138

Shah, S.M., Poorkarimi, H., & Wiener, J. (1986). Bounded solutions of retarded nonlinear hyperbolic equations. *Bull. Allahabad Math. Soc. 1*, 1–14.

Sheng, Q., & Agarwal, R.P. (1994). Existence and uniqueness of periodic solutions for higher order hyperbolic partial differential equations. *J. Math. Anal. Appl. 181*(2), 392–406.

Smirnitskii, Yu.A., & Sobolevskii, P.E. (1981). Positivity of multidimensional difference operators in the Cnorm. *Usp. Mat.Nauk, 36*(4), 202–203.

Smirnitskii, Yu.A. (1993). Fractional powers of elliptic difference operators, PhD Thesis, *Voronezh State University, Voronezh*, (Russian)

Vyazmin, A. V., & Sorokin, V. G. (2017). Exact solutions to nonlinear delay differential equations of hyperbolic type. *Journal of Physics, 788* (012037)

Wiener, J. (1993). Generalized Solutions of Functional Differential Equations. *World Scientific,* Singapore.

Youssfi, A., Benkirane, A. & Hadfi, Y.E. (2016). On Bounded Solutions for Nonlinear Parabolic Equations with Degenerate Coercivity. *Mediterranean Journal of Mathematics, 13*, 3029–3040.

Ceylan, B. (2012). Bounded Solutions of Nonlinear Parabolic Equations With Time Delay. , Master's Thesis *Trakya University, Edirne,*, Turkey.

Ashyralyev, A. & Sobolevskii, P.E. (2004). New Difference Schemes for Partial Differential Equations. *Birkhauser Verlag: Basel,* Boston, Berlin

Podlubny, I. (1999). Fractional Differential Equations Vol. 198 of Mathematics in Science and Engineering. *Academic Press: San Diego,* CA, USA.

Samko,S.G., Kilbas, A.A., & Marichev, O.I. (1993). Fractional Integrals and Derivatives; *Gordon and Breach: Yverdon,* Switzerland.

Lavoie, J.L., Osler, T.J., & Trembly, R. (1976). Fractional derivatives and special functions. *SIAM Rev. 18*, 240–268.

Tarasov, V.E. (2007). Fractional derivative as fractional power of derivative. *Int. J. Math. 18*, 281–299.

El-Mesiry, A.E.M.; El-Sayed, A.M.A.; & El-Saka, H.A.A. (2005). Numerical methods for multi-term fractional arbitrary orders differential equations. *Applied Mathematics and Computations, 160*, 683–699.

El-Sayed, A.M.A.; & Gaafar, F.M. (2001). Fractional-order differential equations with memory and fractional-order relaxation-oscillation model. *Pure Math. Appl. Ser. A, 12*, 296–310.

Gorenflo, R.; & Mainardi, F. (2008). Fractional calculus: Integral and differential equations of fractional order. *J. Math. Phys.* 223–276.

Ashyralyev, A. (2009). A note on fractional derivatives and fractional powers of operators. *Journal of Mathematical Analysis and Applications, 357*, 232–236.

Bazarov, A. (1989). On the structure of fractional spaces. In Proceedings of the XXVII All-Union Scientific Student Conference "The Student and Scientific-Technological Progress", *Gos. Univ. Novosibirsk:* Novosibirsk, Russian, 3–7.

Ashyralyev, A.; Urun, M.; & Parmaksizoglu, I. (2009). Mathematical modeling of the energy consumption problem. *Bull. Karaganda Univ.* Math. 105, 13–24.

Ashyralyev, A. (2007).Fractional spaces generated by the positivite differential and difference operator in a Banach space, In: Tas, K, Tenreiro Machado, JA, Baleanu, D, (eds.) *Proceedings of the Conference Mathematical Methods and Engineering, 1322*, Springer, Netherlands.

Appendices

Appendix A

Matlab Implementation for the Approximate Solution of the One-Dimensional First Order of Accuracy Difference Scheme (3.77)

function (parabolic Dirichlet(N,M))

h=pi/M; tau=1/N;

u(:,:,1)=zeros(N+1, M+1);

for k=1:N+1;

for n=1:M+1;

u(n,k,1)=exp(-(k-1)*tau)*sin((n-1)*h);

end;

end;

a=-1/(h^2); $b = -1/tau;$

$c = 1/tau + 2/(h^2);$

$A = zeros(M+1, M+1); B = A; C = A;$

$A(1,1) = 1; A(M+1, M+1) = 1;$

$forn = 2 : M;$

$A(n, n-1) = a; A(n, n) = c; A(n, n+1) = a;$

$B(n, n) = b;$

$end;$

$G = inv(A);$

$R = eye(M+1, M+1);$

$m = 0; difference = 1; tolerance = 10^-8$

$; disp('firstinterval')$

$whiledifference > tolerance,$

$m = m+1;$

$fii = zeros(M+1, M+1); psi = fii; psi2 = fii;$

$fork = 1 : N+1;$

$fii(1, k) = 0; fii(M+1, k) = 0;$

$forn = 2 : M;$

$fii(n, k) = 1/(2*h)*(u(n+1, k, m) - u(n-1, k, m))*(exp(-(k-1-N)*$

$tau) * sin((n-1)*h)*cos((n-1)*h) - (exp(-(k-1-N)*tau)*sin(n*h) -$

$exp(-(k-1-N)*tau)*sin((n-2)*h))/(2*h)*sin((n-1)*h));$

$end;$

$end;$

$fii = zeros(M+1, M+1); psi = fii; psi2 = fii;$

$forn = 1 : M+1;$

$u(n, 1, m+1) = sin((n-1)*h);$

$end;$

$fork = 2 : N+1;$

$u(:, k, m+1) = G*(R*fii(:, k-1) - B*u(:, k-1, m+1));$

$end;$

$forj = 1 : M+1;$

$fork = 1 : N+1;$

$t(k) = (k-1)*tau;$

$x(j) = (j-1)*h;$

$es(j, k) = exp(-t(k))*sin(x(j));$

$end;$

$end;$

$difference = max(max(abs(u(:, :, m+1) - u(:, :, m))));$

$maxerror = max(max(abs(es - u(:, :, m+1))));$

$str1 = strcat('m =', num2str(m), 'error =', num2str(maxerror), '$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

$v(:, :, 1) = zeros(N+1, M+1);$

$fork = N+1 : 2*N+1;$

$forn = 1 : M+1;$

$v(n, k, 1) = exp(-(k-1)*tau)*sin((n-1)*h);$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('secondinterval')$

$while difference > tolerance,$

$m = m + 1;$

$for k = N + 1 : 2 * N + 1;$

$psi(1, k) = 0; psi(M + 1, k) = 0;$

$for n = 2 : M;$

$t = (k - 1) * tau;$

$psi(n, k) = 1/(2 * h) * (v(n + 1, k, m) - v(n - 1, k, m)) * (u(n, k - N) * cos((n -$
$1) * h) - (u(n + 1, k - N) - u(n - 1, k - N))/(2 * h) * sin((n - 1) * h));$

$end;$

$end;$

$for n = 1 : M + 1;$

$for k = 1 + N : 2 * N + 1;$

$v(n, N + 1, m + 1) = exp(-N * tau) * sin((n - 1) * h);$

$end;$

$end;$

$for k = 2 + N : 2 * N + 1;$

$v(:, k, m + 1) = G * (R * psi(:, k - 1) - B * v(:, k - 1, m + 1));$

$end;$

$for j = 1 : M + 1;$

$for k = N + 1 : 2 * N + 1;$

$t(k) = (k - 1) * tau;$

$x(j) = (j - 1) * h;$

$es2(j, k) = exp(-t(k)) * sin(x(j));$

$end;$

$end;$

$difference = max(max(abs(v(:, :, m + 1) - v(:, :, m))));$

$maxerror = max(max(abs(es2 - v(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$
$dif =', num2str(difference));$

$end;$

$disp(str1)$

$w(:, :, 1) = zeros(N + 1, M + 1);$

$for k = 2 * N + 1 : 3 * N + 1;$

$for n = 1 : M + 1;$

$w(n, k, 1) = exp(-(k - 1) * tau) * sin((n - 1) * h);$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('thirdinterval')$

$while difference > tolerance,$

$m = m + 1;$

$for k = 2 * N + 1 : 3 * N + 1;$

$psi2(1, k) = 0; psi2(M + 1, k) = 0;$

$for n = 2 : M;$

$t = (k - 1) * tau;$

$psi2(n, k) = 1/(2 * h) * (w(n + 1, k, m) - w(n - 1, k, m)) * (v(n, k - N) * cos((n - 1) * h) - (v(n + 1, k - N) - v(n - 1, k - N))/(2 * h) * sin((n - 1) * h));$

$end;$

$end;$

$for n = 1 : M + 1;$

$for k = 1 + N : 2 * N + 1;$

$w(n, 2 * N + 1, m + 1) = exp(-2 * N * tau) * sin((n - 1) * h);$

$end;$

$end;$

$for k = 2 + 2 * N : 3 * N + 1;$

$w(:, k, m + 1) = G * (R * psi2(:, k - 1) - B * w(:, k - 1, m + 1));$

$end;$

$for j = 1 : M + 1;$

$for k = 2 * N + 1 : 3 * N + 1;$

$t(k) = (k - 1) * tau;$

$x(j) = (j - 1) * h;$

$es3(j, k) = exp(-t(k)) * sin(x(j));$

$end;$

$end;$

$difference = max(max(abs(w(:,:,m+1) - w(:,:,m))));$

$maxerror = max(max(abs(es3 - w(:,:,m+1))));$

$str1 = strcat('m =', num2str(m), ' error =', num2str(maxerror), '$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

Appendix B

Matlab Implementation for the Approximate Solution of the One-Dimensional Second Order of Accuracy Difference Scheme (3.79)

function (parabolic Dirichlet second(N,M))

h=pi/M; tau=1/N;

u(:,:,1)=zeros(N+1, M+1);

for k=1:N+1;

for n=1:M+1;

u(n,k,1)=exp(-(k-1)*tau)*sin((n-1)*h);

end;

end;

a=tau/(2*h^4); $b = -1/h^2 - 2 * tau/h^4;$

$c = 1/tau + 2/h^2 + 3 * tau/h^4; e = -1/tau;$

$A = zeros(M + 1, M + 1); B = A;$

$A(1, 1) = 1; A(2, M + 1) = 1; A(M, M) = -5; A(M, M - 1) = 4; A(M, M - 2) = -1;$

$A(M + 1, 2) = -5; A(M + 1, 3) = 4; A(M + 1, 4) = -1;$

$forn = 3 : M - 1;$

$fork = 1 : N + 1;$

$A(n, n - 2) = a; A(n, n - 1) = b; A(n, n) = c;$

$A(n, n + 1) = b; A(n, n + 2) = a; B(n, n) = e;$

$end;$

$end;$

$G = inv(A);$

$R = eye(M + 1, M + 1);$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('firstinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fii = zeros(M + 1, M + 1); psi = fii; psi2 = fii;$

$fork = 2 : N;$

$fii(1,k)=0; fii(2,k)=0;$

$fii(M,k)=0; fii(M+1,k)=0;$

$forn=3:M-1;$

$t=(k-1)*tau;$

$fii(n,k-1)=1/(4*h)*(u(n+1,k,m)-u(n-1,k,m))*(cos(n*h)*exp(-(k-N-1)*tau)*sin(n*h)-(sin((n+1)*h)*exp(-(k-N-1)*tau)-exp(-(k-1-N)*tau)*sin((n-1)*h))/(2*h)*sin(n*h))+1/(4*h)*(u(n+1,k-1,m)-u(n-1,k-1,m))*(cos(n*h)*exp(-(k-N-2)*tau)*sin(n*h)-(sin((n+1)*h)*exp(-(k-N-2)*tau)-exp(-(k-2-N)*tau)*sin((n-1)*h))/(2*h)*sin(n*h))-tau/(8*h^3)*(u(n+2,k,m)-u(n,k,m))*(cos((n+1)*h)*sin((n+1)*h)*exp(-(k-1-N)*tau)-(sin((n+2)*h)*exp(-(k-1-N)*tau)-sin(n*h)*exp(-(k-1-N)*tau))/(2*h)*sin((n+1)*h))-tau/(8*h^3)*(u(n+1,k,m)-u(n-1,k,m))*(-2*exp(-(k-1-N)*tau)*sin(n*h)*cos(n*h)+sin(n*h)*(exp(-(k-1-N)*tau)*sin((n+1)*h)-exp(-(k-1-N)*tau)*sin((n-1)*h))/h)-tau/(8*h^3)*(u(n,k,m)-u(n-2,k,m))*(cos((n-1)*h)*sin((n-1)*h)*exp(-(k-1-N)*tau)-sin((n-1)*h)*(sin(n*h)*exp(-(k-1-N)*tau)-sin((n-2)*h)*exp(-(k-1-N)*tau))/(2*h))-tau/(8*h^3)*(u(n+2,k-1,m)-u(n,k-1,m))*(cos((n+1)*h)*sin((n+1)*h)*exp(-(k-2-N)*tau)-(sin((n+2)*h)*exp(-(k-2-N)*tau)-sin(n*h)*exp(-(k-2-N)*tau))/(2*h)*sin((n+1)*h))-tau/(8*h^3)*(u(n+1,k-1,m)-u(n-1,k-1,m))*(-2*exp(-(k-2-N)*tau)*sin(n*h)*cos(n*h)+sin(n*h)*(exp(-(k-2-N)*tau)*sin((n+1)*h)-exp(-(k-2-N)*tau)*sin((n-1)*h))/h)-tau/(8*h^3)*(u(n,k-1,m)-u(n-2,k-1,m))*(cos((n-1)*h)*sin((n-1)*h)*exp(-(k-2-N)*tau)-sin((n-1)*h)*(sin(n*h)*exp(-(k-2-N)*tau)-sin((n-2)*h)*exp(-(k-2-N)*tau))/(2*h));$

$end;$

$end;$

$forn=1:M+1;$

$u(n,1,m+1)=sin((n-1)*h);$

$end;$

$fork=2:N+1;$

$u(:,k,m+1)=G*(R*fii(:,k-1)-B*u(:,k-1,m+1));$

$end;$

$forj = 1 : M + 1;$

$fork = 1 : N + 1;$

$t = (k - 1) * tau;$

$x = (j - 1) * h;$

$es(j, k) = exp(-t) * sin(x);$

$end;$

$end;$

$difference = max(max(abs(u(:, :, m + 1) - u(:, :, m))));$

$maxerror = max(max(abs(es - u(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$
$dif =', num2str(difference));$

$end;$

$disp(str1)$

$v(:, :, 1) = zeros(N + 1, M + 1);$

$fork = N + 1 : 2 * N + 1;$

$forn = 1 : M + 1;$

$v(n, k, 1) = exp(-(k - 1) * tau) * sin((n - 1) * h);$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('secondinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fork = N + 2 : 2 * N + 1;$

$forn = 3 : M - 1;$

$psi(1, k) = 0; psi(2, k) = 0;$

$psi(M + 1, k) = 0; psi(M, k) = 0;$

$t = (k - 1) * tau;$

$psi(n, k-N) = 1/(4*h)*(v(n+1, k, m)-v(n-1, k, m))*(cos(n*h)*u(n, k-N)-(u(n+1, k-N)-u(n-1, k-N))/(2*h)*sin(n*h))+1/(4*h)*(v(n+1, k-1, m)-v(n-1, k-1, m))*(cos(n*h)*u(n, k-N-1)-sin(n*h)*(u(n+1, k-N-1)-u(n-1, k-N-1))/(2*h))-tau/(8*h^3)*(v(n+2, k, m)-v(n, k, m))*(cos((n+1)*h)*$

$u(n{+}1,k{-}N){-}sin((n{+}1){*}h){*}(u(n{+}2,k{-}N){-}u(n,k{-}N))/(2{*}h)){-}tau/(8{*}h^3){*}$
$(v(n{+}1,k,m){-}v(n{-}1,k,m)){*}({-}2{*}u(n,k{-}N){*}cos(n{*}h){+}sin(n{*}h){*}(u(n{+}1,k{-}$
$N){-}u(n{-}1,k{-}N))/h){-}tau/(8{*}h^3){*}(v(n,k,m){-}v(n{-}2,k,m)){*}(cos((n{-}1){*}h){*}$
$u(n{-}1,k{-}N){-}sin((n{-}1){*}h){*}(u(n,k{-}N){-}u(n{-}2,k{-}N))/(2{*}h)){-}tau/(8{*}h^3){*}$
$(v(n{+}2,k{-}1,m){-}v(n,k{-}1,m)){*}(cos((n{+}1){*}h){*}u(n{+}1,k{-}N{-}1){-}sin((n{+}1){*}$
$h){*}(u(n{+}2,k{-}N{-}1){-}u(n,k{-}N{-}1))/(2{*}h)){-}tau/(8{*}h^3){*}(v(n{+}1,k{-}1,m){-}$
$v(n{-}1,k{-}1,m)){*}({-}2{*}u(n,k{-}N{-}1){*}cos(n{*}h){+}sin(n{*}h){*}(u(n{+}1,k{-}N{-}1){-}$
$u(n{-}1,k{-}N{-}1))/h){-}tau/(8{*}h^3){*}(v(n,k{-}1,m){-}v(n{-}2,k{-}1,m)){*}(cos((n{-}1){*}$
$h){*}u(n{-}1,k{-}N{-}1){-}sin((n{-}1){*}h){*}(u(n,k{-}N{-}1){-}u(n{-}2,k{-}N{-}1))/(2{*}h));$
$end;$
$end;$
$forn = 1 : M + 1;$
$v(n, N + 1, m + 1) = exp(-N * tau) * sin((n - 1) * h);$
$end;$
$fork = 2 + N : 2 * N + 1;$
$v(:, k, m + 1) = G * (R * psi(:, k - 1) - B * v(:, k - 1, m + 1));$
$end;$
$forj = 1 : M + 1;$
$fork = N + 1 : 2 * N + 1;$
$t = (k - 1) * tau;$
$x = (j - 1) * h;$
$es2(j, k) = exp(-t) * sin(j);$
$end;$
$end;$
$difference = max(max(abs(v(:, :, m + 1) - v(:, :, m))));$
$maxerror = max(max(abs(es2 - v(:, :, m + 1))));$
$str1 = strcat('m =', num2str(m), 'error =', num2str(maxerror), '$
$dif =', num2str(difference));$
$end;$
$disp(str1)$
$w(:, :, 1) = zeros(N + 1, M + 1);$
$fork = 2 * N + 1 : 3 * N + 1;$

$forn = 1 : M + 1;$

$w(n, k, 1) = exp(-(k - 1) * tau) * sin((n - 1) * h);$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('thirdinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fork = 2 * N + 2 : 3 * N + 1;$

$psi2(1, k) = 0; psi2(2, k) = 0;$

$psi2(M, k) = 0; psi2(M + 1, k) = 0;$

$forn = 3 : M - 1;$

$t = (k - 1) * tau;$

$psi2(n, k) = 1/(4*h)*(w(n+1, k, m)-w(n-1, k, m))*(cos(n*h)*v(n, k-N)-(v(n+1, k-N)-v(n-1, k-N))/(2*h)*sin(n*h))+1/(4*h)*(w(n+1, k-1, m)-w(n-1, k-1, m))*(cos(n*h)*v(n, k-N-1)-sin(n*h)*(v(n+1, k-N-1)-v(n-1, k-N-1))/(2*h))-tau/(8*h^3)*(w(n+2, k, m)-w(n, k, m))*(cos((n+1)*h)*v(n+1, k-N)-sin((n+1)*h)*(v(n+2, k-N)-v(n, k-N))/(2*h))-tau/(8*h^3)*(w(n+1, k, m)-w(n-1, k, m))*(-2*v(n, k-N)*cos(n*h)+sin(n*h)*(v(n+1, k-N)-v(n-1, k-N))/h)-tau/(8*h^3)*(w(n, k, m)-w(n-2, k, m))*(cos((n-1)*h)*v(n-1, k-N)-sin((n-1)*h)*(v(n, k-N)-v(n-2, k-N))/(2*h))-tau/(8*h^3)*(w(n+2, k-1, m)-w(n, k-1, m))*(cos((n+1)*h)*v(n+1, k-N-1)-sin((n+1)*h)*(v(n+2, k-N-1)-v(n, k-N-1))/(2*h))-tau/(8*h^3)*(w(n+1, k-1, m)-w(n-1, k-1, m))*(-2*v(n, k-N-1)*cos(n*h)+sin(n*h)*(v(n+1, k-N-1)-v(n-1, k-N-1))/h)-tau/(8*h^3)*(w(n, k-1, m)-w(n-2, k-1, m))*(cos((n-1)*h)*v(n-1, k-N-1)-sin((n-1)*h)*(v(n, k-N-1)-v(n-2, k-N-1))/(2*h));$

$end;$

$end;$

$forn = 1 : M + 1;$

$w(n, 2 * N + 1, m + 1) = exp(-2 * N * tau) * sin((n - 1) * h);$

$end;$

$for k = 2 + 2 * N : 3 * N + 1;$

$w(:, k, m + 1) = G * (R * psi2(:, k - 1) - B * w(:, k - 1, m + 1));$

$end;$

$for j = 1 : M + 1;$

$for k = 2 * N + 1 : 3 * N + 1;$

$t = (k - 1) * tau;$

$x = (j - 1) * h;$

$es3(j, k) = exp(-t) * sin(x);$

$end;$

$end;$

$difference = max(max(abs(w(:, :, m + 1) - w(:, :, m))));$

$maxerror = max(max(abs(es3 - w(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m), ' error =', num2str(maxerror), '$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

Appendix C

Matlab Implementation for the Approximate Solution of the One-Dimensional First Order of Accuracy Difference Scheme (3.83)

```
function (parabolic Nonlocal(N,M))
h=pi/M;tau=1/N;
u(:,:,1)=zeros(N+1, M+1);
for k=1:3*N+1;
t(k)=(k-1)*tau;
end;
for n=1:M+1;
x(n)=(n-1)*h;
end;
for n=1:M+1;
for k=1:N+1;
u(n,k,1)=exp(-4*t(k))*sin(2*x(n));
end;
uN(n)=u(n,1,1);
end;
```

a=-1/(h^2); $b = -1/tau;$

$A = zeros(M+1, M+1); B = A; C = A;$

$A(1,1) = 1; A(1, M+1) = -1;$

$A(M+1,1) = -1; A(M+1,2) = 1; A(M+1,M) = 1; A(M+1,M+1) = -1;$

$forn = 2 : M;$

$A(n, n-1) = a; A(n, n+1) = a; B(n, n) = b;$

$c = 1/tau + 2/h^2 - 2 * uN(n) * cos(2 * x(n)) + (uN(n+1) - uN(n-1))/(2 * h) * sin(2 * x(n));$

$A(n, n) = c;$

$end;$

$G = inv(A);$

$R = eye(M+1, M+1);$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('firstinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fii = zeros(M + 1, M + 1); psi = fii; psi2 = fii; psi3 = fii;$

$fork = 1 : N + 1;$

$fii(1, k) = 0; fii(2, k) = 0;$

$fii(M, k) = 0; fii(M + 1, k) = 0;$

$forn = 3 : M - 1;$

$fii(n, k) = -sin(u(n, k, m)) + sin(exp(-4 * t(k)) * sin(2 * x(n)));$

$end;$

$end;$

$fii = zeros(M + 1, M + 1); psi = fii; psi2 = fii; psi3 = fii;$

$forn = 1 : M + 1;$

$u(n, 1, m + 1) = sin(2 * x(n));$

$end;$

$fork = 1 : N;$

$u(:, k + 1, m + 1) = G * (R * fii(:, k) - B * u(:, k, m + 1));$

$end;$

$forj = 1 : M + 1;$

$fork = 1 : N + 1;$

$es(j, k) = exp(-4 * t(k)) * sin(2 * x(j));$

$end;$

$end;$

$difference = max(max(abs(u(:, :, m + 1) - u(:, :, m))));$

$maxerror = max(max(abs(es - u(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

$v(:, :, 1) = zeros(N + 1, M + 1);$

$forn = 1 : M + 1;$

$fork = N + 1 : 2 * N + 1;$

$v(n,k,1) = exp(-4*t(k))*sin(2*x(n));$

$end;$

$end;$

$uN(:) = u(:,N+1,m);$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('secondinterval')$

$whiledifference > tolerance,$

$m = m+1;$

$fork = N+1 : 2*N+1;$

$psi(1,k) = 0; psi(2,k) = 0;$

$psi(M,k) = 0; psi(M+1,k) = 0;$

$forn = 3 : M-1;$

$psi(n,k) = -sin(v(n,k,m)) + sin(exp(-4*t(k))*sin(2*x(n)));$

$end;$

$end;$

$forn = 1 : M+1;$

$v(n,N+1,m+1) = uN(n);$

$end;$

$fork = 1+N : 2*N;$

$forn = 2 : M;$

$c = 1/tau + 2/h^2 - 2*uN(n)*cos(2*x(n)) + (uN(n+1) - uN(n-1))/(2*h)*$
$sin(2*x(n));$

$A(n,n) = c;$

$end;$

$G = inv(A);$

$v(:,k+1,m+1) = G*(R*psi(:,k) - B*v(:,k,m+1));$

$end;$

$forj = 1 : M+1;$

$fork = N+1 : 2*N+1;$

$es2(j,k) = exp(-4*t(k))*sin(2*x(j));$

$end;$

$end;$

$difference = max(max(abs(v(:,:,m+1) - v(:,:,m))));$

$maxerror = max(max(abs(es2 - v(:,:,m+1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

$w(:,:,1) = zeros(N+1, M+1);$

$uN(:) = v(:, 2*N+1, m);$

$fork = 2*N+1 : 3*N+1;$

$forn = 1 : M+1;$

$w(n,k,1) = exp(-4*t(k))*sin(2*x(n));$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('thirdinterval')$

$whiledifference > tolerance,$

$m = m+1;$

$fork = 2*N+1 : 3*N+1;$

$psi2(1,k) = 0; psi2(2,k) = 0;$

$psi2(M,k) = 0; psi2(M+1,k) = 0;$

$forn = 3 : M-1;$

$psi2(n,k) = -sin(w(n,k,m)) + sin(exp(-4*t(k))*sin(2*x(n)));$

$end;$

$end;$

$forn = 2 : M;$

$c = 1/tau + 2/h^2 - 2*uN(n)*cos(2*x(n)) + (uN(n+1) - uN(n-1))/(2*h)*$

$sin(2*x(n));$

$A(n,n) = c;$

$end;$

$G = inv(A);$

$forn = 1 : M+1;$

$w(n, 2*N+1, m+1) = uN(n);$

$end;$

$fork = 1 + 2 * N : 3 * N;$

```
    w(:,k+1,m+1)=G*(R*psi2(:,k)-B*w(:,k,m+1));
end;
for j=1:M+1;
for k=2*N+1:3*N+1;
t(k)=(k-1)*tau;
es3(j,k) = exp(-4*t(k))*sin(2*x(j));
end;
end;
difference=max(max(abs(w(:,:,m+1)-w(:,:,m))));
maxerror=max(max(abs(es3-w(:,:,m+1))));
str1=strcat('m=',num2str(m),' error=',num2str(maxerror),'
dif=',num2str(difference));
end;
disp(str1)
```

Appendix D

Matlab Implementation for the Approximate Solution of the One-Dimensional Second Order of Accuracy Difference Scheme (3.85)

function (parabolic Nonlocal seconds(N,M))

h=pi/M;tau=1/N;

u(:,:,1)=zeros(N+1, M+1);

for k=1:3*N+1;

t(k)=(k-1)*tau;

end;

for n=1:M+1;

x(n)=(n-1)*h;

end;

for n=1:M+1;

for k=1:N+1;

u(n,k,1)=exp(-4*t(k))*sin(2*x(n));

end;

uN(n)=sin(2*x(n));

end;

a=tau/(2*h^4);

$A = zeros(M+1, M+1); B = A;$

$A(1,1) = 1; A(1, M+1) = -1;$

$A(2,1) = 2; A(2,2) = -5; A(2,3) = 4; A(2,4) = -1;$

$A(2, M+1) = -2; A(2, M) = 5; A(2, M-1) = -4; A(2, M-2) = 1;$

$A(M,1) = -3; A(M,2) = 4; A(M,3) = -1;$

$A(M, M+1) = -3; A(M, M) = 4; A(M, M-1) = -1;$

$A(M+1,1) = -5; A(M+1,2) = 18; A(M+1,3) = -24; A(M+1,4) = 14; A(M+1,5) = -3;$

$A(M+1, M+1) = -5; A(M+1, M) = 18; A(M+1, M-1) = -24; A(M+1, M-2) = 14; A(M+1, M-3) = -3;$

$forn = 3 : M-1;$

$A(n, n-2) = a; A(n, n+2) = a;$

$end;$

$forn = 3 : M - 1;$

$b = -1/h^2 - 2 * tau/h^4 + tau/(2 * h^2) * uN(n-1) * cos(2 * x(n-1)) - tau/(8 * h^3) * uN(n) * sin(2 * x(n-1)) + tau/(8 * h^3) * uN(n-2) * sin(2 * x(n-1));$

$c = 1/tau + 2/h^2 + 3 * tau/h^4 - uN(n) * cos(2 * x(n)) + 1/(4 * h) * uN(n+1) * sin(2*x(n)) - 1/(4*h) * uN(n-1) * sin(2*x(n)) - tau/h^2 * uN(n) * cos(2*x(n)) + tau/(4 * h^3) * uN(n+1) * sin(2 * x(n)) - tau/(4 * h^3) * uN(n-1) * sin(2 * x(n));$

$d = -1/h^2 - 2 * tau/h^4 + tau/(2 * h^2) * uN(n+1) * cos(2 * x(n+1)) - tau/(8 * h^3) * uN(n+2) * sin(2 * x(n+1)) + tau/(8 * h^3) * uN(n) * sin(2 * x(n+1));$

$e = tau/(2 * h^2) * uN(n-1) * cos(2 * x(n-1)) - tau/(8 * h^3) * uN(n) * sin(2 * x(n-1)) + tau/(8 * h^3) * uN(n-2) * sin(2 * x(n-1));$

$f = -1/tau - uN(n) * cos(2 * x(n)) + 1/(4 * h) * uN(n+1) * sin(2 * x(n)) - 1/(4 * h) * uN(n-1) * sin(2 * x(n)) - tau/h^2 * uN(n) * cos(2 * x(n)) + tau/(4 * h^3) * uN(n+1) * sin(2 * x(n)) - tau/(4 * h^3) * uN(n-1) * sin(2 * x(n));$

$g = tau/(2 * h^2) * uN(n+1) * cos(2 * x(n+1)) - tau/(8 * h^3) * uN(n+2) * sin(2 * x(n+1)) + tau/(8 * h^3) * uN(n) * sin(2 * x(n+1));$

$A(n, n-1) = b; A(n, n) = c; A(n, n+1) = d;$

$B(n, n-1) = e; B(n, n) = f; B(n, n+1) = g;$

$end;$

$G = inv(A);$

$R = eye(M+1, M+1);$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('firstinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fii = zeros(M+1, M+1); psi = fii; psi2 = fii; psi3 = fii;$

$fork = 1 : N + 1;$

$fii(1, k) = 0; fii(2, k) = 0;$

$fii(M, k) = 0; fii(M+1, k) = 0;$

$forn = 3 : M - 1;$

$fii(n, k) = -sin(u(n, k, m)) + sin(exp(-4 * t(k)) * sin(2 * x(n)));$

$end;$

$end;$

$forn = 1 : M + 1;$

$u(n, 1, m + 1) = sin(2 * x(n));$

$end;$

$fork = 1 : N;$

$u(:, k + 1, m + 1) = G * (R * fii(:, k) - B * u(:, k, m + 1));$

$end;$

$forj = 1 : M + 1;$

$fork = 1 : N + 1;$

$es(j, k) = exp(-4 * t(k)) * sin(2 * x(j));$

$end;$

$end;$

$difference = max(max(abs(u(:, :, m + 1) - u(:, :, m))));$

$maxerror = max(max(abs(es - u(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

$v(:, :, 1) = zeros(N + 1, M + 1);$

$forn = 1 : M + 1;$

$fork = N + 1 : 2 * N + 1;$

$v(n, k, 1) = exp(-4 * t(k)) * sin(2 * x(n));$

$end;$

$end;$

$uN(:) = u(:, N + 1, m);$

$forn = 3 : M - 1;$

$b = -1/h^2 - 2 * tau/h^4 + tau/(2 * h^2) * uN(n - 1) * cos(2 * x(n - 1)) - tau/(8 * h^3) * uN(n) * sin(2 * x(n - 1)) + tau/(8 * h^3) * uN(n - 2) * sin(2 * x(n - 1));$

$c = 1/tau + 2/h^2 + 3 * tau/h^4 - uN(n) * cos(2 * x(n)) + 1/(4 * h) * uN(n + 1) * sin(2*x(n)) - 1/(4*h)*uN(n-1)*sin(2*x(n)) - tau/h^2*uN(n)*cos(2*x(n)) + tau/(4 * h^3) * uN(n + 1) * sin(2 * x(n)) - tau/(4 * h^3) * uN(n - 1) * sin(2 * x(n));$

$d = -1/h^2 - 2 * tau/h^4 + tau/(2 * h^2) * uN(n + 1) * cos(2 * x(n + 1)) - tau/(8 *$

$h^3) * uN(n+2) * sin(2*x(n+1)) + tau/(8*h^3) * uN(n) * sin(2*x(n+1));$

$e = tau/(2*h^2) * uN(n-1) * cos(2*x(n-1)) - tau/(8*h^3) * uN(n) * sin(2*$
$x(n-1)) + tau/(8*h^3) * uN(n-2) * sin(2*x(n-1));$

$f = -1/tau - uN(n) * cos(2*x(n)) + 1/(4*h) * uN(n+1) * sin(2*x(n)) - 1/(4*$
$h) * uN(n-1) * sin(2*x(n)) - tau/h^2 * uN(n) * cos(2*x(n)) + tau/(4*h^3) *$
$uN(n+1) * sin(2*x(n)) - tau/(4*h^3) * uN(n-1) * sin(2*x(n));$

$g = tau/(2*h^2) * uN(n+1) * cos(2*x(n+1)) - tau/(8*h^3) * uN(n+2) * sin(2*$
$x(n+1)) + tau/(8*h^3) * uN(n) * sin(2*x(n+1));$

$A(n,n-1) = b; A(n,n) = c; A(n,n+1) = d;$

$B(n,n-1) = e; B(n,n) = f; B(n,n+1) = g;$

$end;$

$G = inv(A);$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('secondinterval')$

$whiledifference > tolerance,$

$m = m+1;$

$fork = N+1 : 2*N+1;$

$psi(1,k) = 0; psi(2,k) = 0;$

$psi(M,k) = 0; psi(M+1,k) = 0;$

$forn = 3 : M-1;$

$psi(n,k) = -sin(v(n,k,m)) + sin(exp(-4*t(k)) * sin(2*x(n)));$

$end;$

$end;$

$forn = 1 : M+1;$

$v(n,N+1,m+1) = uN(n);$

$end;$

$fork = 1+N : 2*N;$

$v(:,k+1,m+1) = G * (R * psi(:,k) - B * v(:,k,m+1));$

$end;$

$forj = 1 : M+1;$

$fork = N+1 : 2*N+1;$

$es2(j,k) = exp(-4*t(k)) * sin(2*x(j));$

$end;$

$end;$

$difference = max(max(abs(v(:,:,m+1) - v(:,:,m))));$

$maxerror = max(max(abs(es2 - v(:,:,m+1))));$

$str1 = strcat('m =', num2str(m), 'error =', num2str(maxerror), '$

$dif =', num2str(difference));$

$end;$

$disp(str1)$

$w(:,:,1) = zeros(N+1, M+1);$

$uN(:) = v(:, 2*N+1, m);$

$forn = 3 : M-1;$

$b = -1/h^2 - 2*tau/h^4 + tau/(2*h^2)*uN(n-1)*cos(2*x(n-1)) - tau/(8*$

$h^3)*uN(n)*sin(2*x(n-1)) + tau/(8*h^3)*uN(n-2)*sin(2*x(n-1));$

$c = 1/tau + 2/h^2 + 3*tau/h^4 - uN(n)*cos(2*x(n)) + 1/(4*h)*uN(n+1)*$

$sin(2*x(n)) - 1/(4*h)*uN(n-1)*sin(2*x(n)) - tau/h^2*uN(n)*cos(2*x(n)) +$

$tau/(4*h^3)*uN(n+1)*sin(2*x(n)) - tau/(4*h^3)*uN(n-1)*sin(2*x(n));$

$d = -1/h^2 - 2*tau/h^4 + tau/(2*h^2)*uN(n+1)*cos(2*x(n+1)) - tau/(8*$

$h^3)*uN(n+2)*sin(2*x(n+1)) + tau/(8*h^3)*uN(n)*sin(2*x(n+1));$

$e = tau/(2*h^2)*uN(n-1)*cos(2*x(n-1)) - tau/(8*h^3)*uN(n)*sin(2*$

$x(n-1)) + tau/(8*h^3)*uN(n-2)*sin(2*x(n-1));$

$f = -1/tau - uN(n)*cos(2*x(n)) + 1/(4*h)*uN(n+1)*sin(2*x(n)) - 1/(4*$

$h)*uN(n-1)*sin(2*x(n)) - tau/h^2*uN(n)*cos(2*x(n)) + tau/(4*h^3)*$

$uN(n+1)*sin(2*x(n)) - tau/(4*h^3)*uN(n-1)*sin(2*x(n));$

$g = tau/(2*h^2)*uN(n+1)*cos(2*x(n+1)) - tau/(8*h^3)*uN(n+2)*sin(2*$

$x(n+1)) + tau/(8*h^3)*uN(n)*sin(2*x(n+1));$

$A(n, n-1) = b; A(n, n) = c; A(n, n+1) = d;$

$B(n, n-1) = e; B(n, n) = f; B(n, n+1) = g;$

$end;$

$G = inv(A);$

$fork = 2*N+1 : 3*N+1;$

$forn = 1 : M+1;$

$w(n, k, 1) = exp(-4*t(k))*sin(2*x(n));$

$end;$

$end;$

$m = 0; difference = 1; tolerance = 10^-8;$

$disp('thirdinterval')$

$whiledifference > tolerance,$

$m = m + 1;$

$fork = 2 * N + 1 : 3 * N + 1;$

$psi2(1, k) = 0; psi2(2, k) = 0;$

$psi2(M, k) = 0; psi2(M + 1, k) = 0;$

$forn = 3 : M - 1;$

$psi2(n, k) = -sin(w(n, k, m)) + sin(exp(-4 * t(k)) * sin(2 * x(n)));$

$end;$

$end;$

$forn = 1 : M + 1;$

$w(n, 2 * N + 1, m + 1) = uN(n);$

$end;$

$fork = 1 + 2 * N : 3 * N;$

$w(:, k + 1, m + 1) = G * (R * psi2(:, k) - B * w(:, k, m + 1));$

$end;$

$forj = 1 : M + 1;$

$fork = 2 * N + 1 : 3 * N + 1;$

$t(k) = (k - 1) * tau;$

$es3(j, k) = exp(-4 * t(k)) * sin(2 * x(j));$

$end;$

$end;$

$difference = max(max(abs(w(:, :, m + 1) - w(:, :, m))));$

$maxerror = max(max(abs(es3 - w(:, :, m + 1))));$

$str1 = strcat('m =', num2str(m),' error =', num2str(maxerror),'$

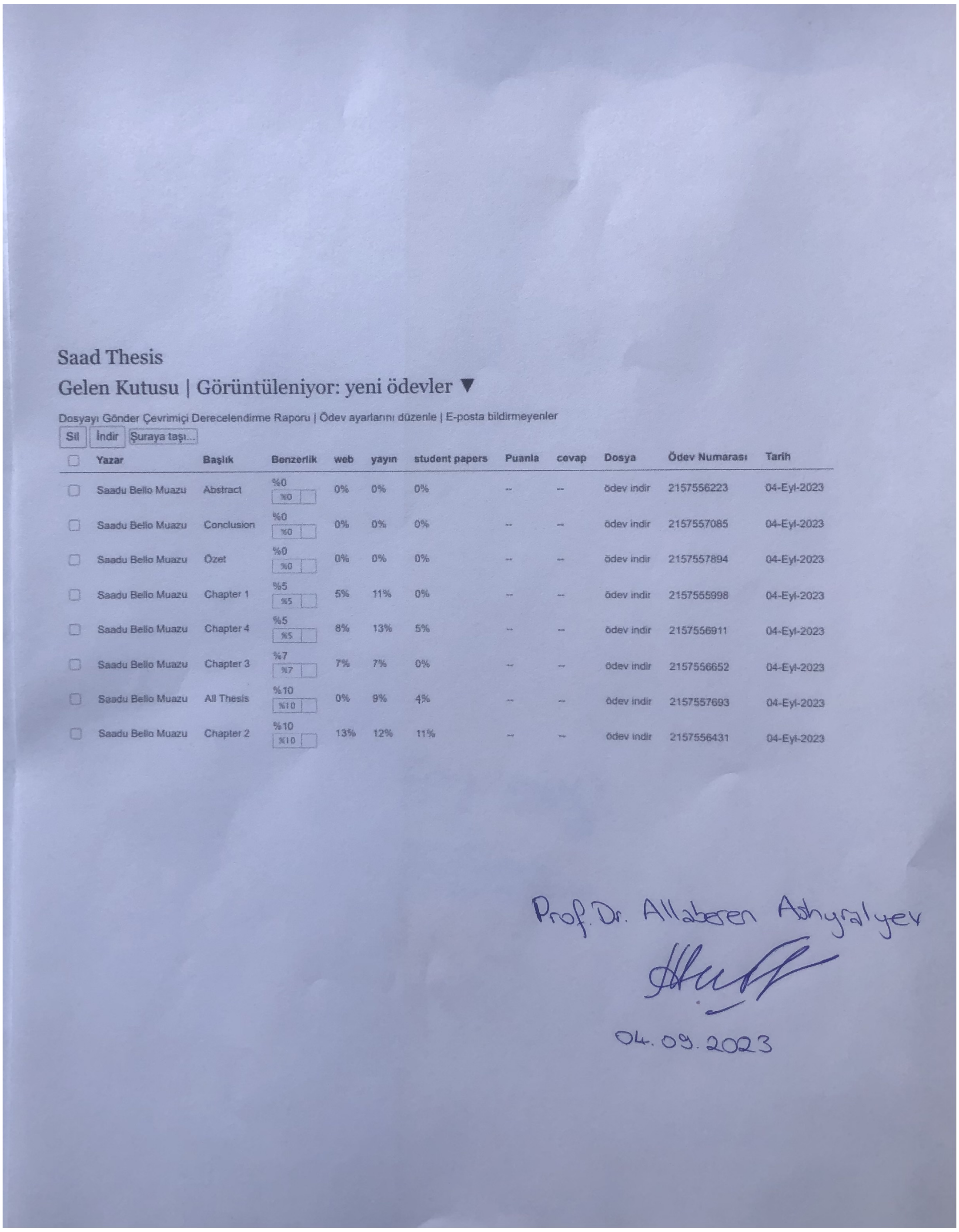
$dif =', num2str(difference));$

$end;$

$disp(str1)$

Appendix E

Turnitin Similarity Report



Saad Thesis

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

Sil | İndir | Şuraya taşı...

Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
Saadu Bello Muazu	Abstract	%0	0%	0%	0%	--	--	ödev indir	2157556223	04-Eyl-2023
Saadu Bello Muazu	Conclusion	%0	0%	0%	0%	--	--	ödev indir	2157557085	04-Eyl-2023
Saadu Bello Muazu	Özet	%0	0%	0%	0%	--	--	ödev indir	2157557894	04-Eyl-2023
Saadu Bello Muazu	Chapter 1	%5	5%	11%	0%	--	--	ödev indir	2157555998	04-Eyl-2023
Saadu Bello Muazu	Chapter 4	%5	8%	13%	5%	--	--	ödev indir	2157556911	04-Eyl-2023
Saadu Bello Muazu	Chapter 3	%7	7%	7%	0%	--	--	ödev indir	2157556652	04-Eyl-2023
Saadu Bello Muazu	All Thesis	%10	0%	9%	4%	--	--	ödev indir	2157557693	04-Eyl-2023
Saadu Bello Muazu	Chapter 2	%10	13%	12%	11%	--	--	ödev indir	2157556431	04-Eyl-2023

Prof. Dr. Allaberen Ashyralyev

04.09.2023

Appendix F
Curriculum Vitae

Personal Information:

Full Name: Sa'adu Bello Mu'azu
Current Contact Address: Department of Mathematics, Faculty of Physical Sciences, Kebbi State University of Science and Technology, Aliero P.M.B. 1144
Mobile: +2347032159261, +905338679371
E-mail: saadbm13.sbm@gmail.com

Objective:

To Strive for Excellence and Precision at all Times, in all Positions and Circumstances, Attaining Professional Distinction and Proficiency.

Work Experience:

- Lecturer I, Kebbi State University of Science and Technology, Aliero 2020 to Date
- Lecturer II, Kebbi State University of Science and Technology, Aliero 2016 - 2019
- Assistant Lecturer, Kebbi State University of Science and Technology, Aliero 2013 - 2016
- Graduate Assistant, Kebbi State University of Science and Technology, Aliero 2009 - 2013
- National Youth Service Corps (NYSC) 2007 - 2008

Educational; Institutions Attended and Qualifications Obtained with Date:

- Near East University, Cyprus
  Ph.D. Mathematics 2019 - 2023
- Usmanu Danfodiyo University, Sokoto
  M.Sc. Mathematics 2009 - 2012
- Usmanu Danfodiyo University, Sokoto
  B.Sc. Mathematics 2002 - 2006

- Nagarta College, Sokoto
  Senior.Secondary Certificate Examination (SSCE) 1994 - 2000
- Waziri Model Primary School, Sokoto
  First School Leaving Certificate(FSLC) 2009 - 2012

Courses Thought:

- Elementary Mathematics I, II, and III
- Linear Algebra I and II
- Abstract Algebra I and II
- Real Analysis I and II
- Metric Space and Topology

Publications:

- Ashyralyev, A.; Mu'azu, S.B. Bounded Solutions of Semi-Linear Parabolic Differential Equations with Unbounded Delay Terms. Mathematics 2023, 11, 3470
- Tanko Adamu, Sa'adu Bello, Sudi Balimuttajjo, Micheal Byamukama, Characterization of Eigenvalues of H-Laolace Operator with Boundary Condition. V. International Scientific and Vocational Studies Congress - Science and Health (BILMES SM 2020)
- Alternative Method of Constructing Block Design From (132)-Avoiding Class of Aunu Number. International Journal of Science & Engineering Technology, Vol.12, No.1 pp:69-74. 2016
- Cephas Iko-Ojo Gabriel, Bello Saadu Muazu, Effective Mathematics Approach for Balancing Chemical Equations in School. Academic Journal of Applied Mathematical Sciences. Vol.1, No.1. pp:27-38. 2015

Conference Proceeding:

- Ashyralyev, A., Mu'azu, S¿B., Ashyralyyev, C. Numerical Solution of Delay Nonlinear Parabolic Differential Equations with Nonlocal Conditions. AIP Conference Proceedings, (ICAAM2022), Antalya.

- Construction of Block Design From (132)-Avoiding Class of Aunu Number. In Proceedings of the Annual National Conference of Mathematical Association of Nigeria. pp:632-636, 2016

Papers under review:

- Allaberen Ashyralyev, Deniz Agirseven, and Sa'adu Bello Mu'azu. A Note on the Delay Nonlinear Parabolic Differential Equations

Skills:

- Languages: Fluent in English and Hausa

- Computer: LaTeX, Word, Excel, PowerPoint, MATLAB

Interest

To Speak International Languages Conversationally and learn from others positively.