Ülker Okur

Operator approach for the solution of Stochastic differential equations

PhD THESIS

2023

NEAR EAST UNIVERSITY
INSTITUTE OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

OPERATOR APPROACH FOR THE SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Approval

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NEAR EAST UNIVERSITY INSTITUTE OF GRADUATE STUDIES

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Prof. Dr. Kemal Hüsnü Can Başer

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Declaration

I hereby declare that all information, documents, analysis and results in this thesis have been collected and presented according to the academic rules and ethical guidelines of Institute of Graduate Studies, Near East University. I also declare that as required by these rules and conduct, I have fully cited and referenced information and data that are not original to this study.

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Abstract

Operator Approach for the solution of the stochastic differential equations

Okur, Ülker

PhD, Department of Mathematics

Supervisor: Prof. Dr. Allaberen Ashyralyev

September, 2023, 140 pages

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Key Words**: **difference scheme, stochastic parabolic equation, convergence estimates, stability, positive operator, Hilbert spac

ÖZET

Stokastik diferansiyel denklemlerin çözümü için Operatör Yaklaşımı

Okur, Ülker

Doktora Tezi, Matematik Anabilim Dalı

Danışman: Prof. Dr. Allaberen Ashyralyev

Eylül, 2023, 140 sayfa

Bu tezde, Hilbert uzayında zamana bağlı pozitif operatörlerle parabolik tipte stokastik diferansiyel denklem için soyut bir Cauchy problemi ele alınmıştır. Parabolik tipte bir diferansiyel denklem için soyut bir Cauchy probleminin kararlılığı belirlenir. Uygulamada kat sayılari, t'ye bağımlı olan tek boyutlu ve çok boyutlu stokastik parabolik denklem için başlangıç sınır değer problemi için kararlılık tahminlerine ilişkin teoremler kanıtlanmıştır. Bu soyut Cauchy probleminin parabolik tipteki diferansiyel denklemin yaklaşık çözümleri için fark şemalarının yakınsaklığının ana teoremleri sunulmuştur. Uygulamalarda, dört tür stokastik diferansiyel denklemlerin yaklaşık çözümleri için fark şemalarının çözümü ve yakınsaklık tahminleri elde edilir. Dirichlet, Neumann koşulları ile stokastik diferansiyel denklemler için Cauchy probleminin yaklaşık çözümünün doğruluk farkı şemaları için sayısal sonuçlar verilmiştir.

Anahtar Kelimeler***:*** farkşeması; stokastik parabolik denklem; yakınsama tahmin etmek; kararlılık; pozitif, operatör; Hilbert uzayı

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List of Abbreviations

SDE	Stochastic Differential Equation
SPDE	Stochastic Parabolic Differential Equation
IBVP	Initial-Boundary Value Problem
DS	Difference Scheme
RDS	Rothe Difference Scheme
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CHAPTER I

Introduction

Historical Note and Literature Survey

Initial value and the boundary value problems for stochastic ordinary and partial differential equations take an important place in applied sciences and engineering applications. Methods for numerically solving the initial value and the boundary value problems for stochastic ordinary differential equations have been studied and developed over the last three decades. It is known that most problems in heat flow, fusion process, modelling financial instruments like options, bonds and interest rates and other areas which are involved with uncertainty correspond to SDE's. The stochastic partial equations appear in several different applications. One of them is the random evolution of systems with a spatial extension such as random interface growth, random evolution of surfaces and fluids subject to random forcing. In particular, for mathematical finance they have been used to model term structure of finance, term structure of interest rates or volatility surfaces.

When we study SDE's, then we commonly uses the following shorthand notation:

$$\begin{cases} dx_t = f(t, x_t)dt + g(t, x_t)dw_t \\ x(0) = x_0, \end{cases}$$

where $w \in \mathbb{R}^d$ is a Wiener process and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the drift function. Let $x_0 \in \mathbb{R}^n$ be the initial condition and let $g(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ the diffusion function. The stochastic process x_t is called an Itô process. So, the SDE has a deterministic part and a random part. For a better understanding, we can describe the equation as

$$dx_t = deterministic\, part\, dt + random\, part\, dw_t$$

The random part is given by the Wiener process.

Methods of the solutions of stochastic partial differential equations (SPDE's) have been studied extensively. A number of stochastic processes is given by

(SDE), i.e. they are solutions of the corresponding integral equations. A function, which is dependent of these stochastic processes for example a portfolio of shares can also be represented by means of a stochastic integral equation. The stochastic integrals, which possess in the stochastics of the financial markets are of paramount importance. The most algorithms that are used for the solution of ordinary differential equations will work very poorly for SDEs, which have very poor numerical convergence. Many stochastic partial differential equations are popular to investigate the solutions to stochastic equations in Hilbert and Banach spaces with the method of operators as a tool. Also estimate of the convergence by the difference between the exact and approximate solution have been studied extensively by some researches.

SDE can be derived as models of indeterministic systems and considered as methods for solving boundary value problems and have been studied by many authors (E. Pardoux, 1979; S. Peszat and J Zabczyk, 2000). The stochastic partial differential equations have been studied by many authors (A. Yurtsever and A. Yazliyev, 2000; G. Da Prato and A. Lunardi, 1998; Ashyralyev A. and G. Michaletzky, 1993; A. Ashyralyev and I. Hasgur, 1995; A. Yurtsever and A. Yazliyev, 2000; A. Ashyralyev and M.E. San, 2012). SDE can be derived as models of indeterministic systems and considered as methods for solving boundary value problems and have been studied by (E. Pardoux, 1979; and S. Peszat and J. Zabczyk, 2000). Numerical solutions of stochastic differential equations and stochastic partial differential equations have been studied of many different algorithms by (P. Kloeden and E. Platen, 1995) The partial differential equations of parabolic type in the Banach space have been extensively studied by many researchers (A. Ashyralyev, P.E. Sobolevski, 1994; A. Ashyralyev and M. Akat, 2011; A. Ashyralyev and P.E. Sobolevskii, 2004). Many various properties of boundary value problems for partial differential equations, of stability of DS's for partial differential equations, and of summation of Fourier series is studied in various papers (P.E. Sobolevskii, 2005; Krein, S.G., 1966; Prato G. Da, Grisvard, P., 1975; H.O. Fattorini, 1985). The operator approach of permitted essentially extends a class of problems where the theory of difference methods is applicable are studied in many parers by (Ashyralyev, A., Michaletzky,G.,1993; A. Ashyralyev and I. Hasgur, 1995; A. Yurtsever and A. Yazliyev, 2000; A. Ashyralyev, 2008; A. Ashyralyev and M.E. San, 2012; . E. San, 2012; A. Ashyralyev, and M. Akat, 2011; N. Aggez, N., and M. Ashyralyyewa, 2012; Hausenblas, E., 2002). The optimal regularity of the stochastic convolution is studied by (G. Da Prato and A. Lunardi, 1998). A proof of the convergence of finite difference approximations of the solution of initial value problem for the nonlinear stochastic partial differential equation of the form is presented by (T. Shardlow, 1998). Nevertheless the 3/2-th order of approximation of implicit and CNDS's for the solution of the initial value Cauchy problem are presented (A. Ashyralyev and I. Hasgur, 1995; A. Yurtsever and A. Yazliyev, 2000). The modified CNDS for the approximate solution of the initial value Cauchy problem was studied in (A. Ashyralyev, 2008). The multipoint nonlocal-boundary value problem for stochastic parabolic differential equations (SPDE's) in Hilbert spaceis studied by (A. Ashyralyev and M.E. San, 2012). E.J. Allen, S.J. Novosel, and Z. Zhang.

In the book (A. Ashyralyev and P.E. Sobolevskii, 1994), the well-posedness of an abstract Cauchy problem for differential equation of parabolic type

$$v'(t) + A(t)v(t) = f(t), 0 < t < T, v(0) = v_0 \tag{1}$$

in an arbitrary Banach space E with the dependent positive operators $A(t)$ was established. Theorems on well-posedness of initial-boundary value problems for parabolic equations in various Banach spaces were proved. The high order of accuracy DS's generated by an exact difference scheme or by the Taylor's decomposition on the two points for the numerical solutions of the problem (1) were presented. The well-posedness of these SD's in various Banach spaces was studied. The stability and coercive stability estimates in various Banach norms for the solutions of the high order of accuracy DS's of the mixed type boundary value problems for parabolic equations were obtained.

The exact DS approach permitted essentially to extend a class of problems where the theory of difference methods is applicable. Namely, now it is possible to investigate the single step DS's of numerical solutions for stochastic parabolic equations with depended coeffcients $A(t)$ in t and the space variables.

IBVP's for stochastic partial differential equations of parabolic type with time-dependent coefficients and with white noise in the right-hand side have been investigated by many scientists (see, for examples, (I. I. Gikhman, 1980; A. Ya. Dorogovtsev, S.D. Īvasishen, A.G. Kukush, 1985;H. M. Perun,2008;A. Ichikawa, 1978;I. I. Gikhman, 1979;R. F. Curtain and A. J. Pritchard, 1976;G. Da prato and A. Lunardi, 2007), and the references given therein).

In the papers (I. I. Gikhman, 1980; A. Ya. Dorogovtsev, S.D. Īvasishen, A.G. Kukush, 1985;H. M. Perun, 2008) boundary-value problems for second-order parabolic equations with white noise were investigated by different methods. In the paper (H. M. Perun, 2008) a theorem on the well-posedness of the Cauchy problem for a linear higher-order stochastic equation of parabolic type with time-dependent coefficients and continuous perturbations whose solutions are subjected to pulse action at fixed times was proved.

In the paper (A. Ichikawa, 1978) linear stochastic integral evolution equations were studied. They were associated with formal stochastic partial differential equations as well as stochastic delay differential equations. The existence and uniqueness of a solution was established for systems with disturbances depending on the state, both current and past, using semigroups or more generally evolution operators and known properties of such operators. In future, we will consider integral equations described by mild evolution operators which were introduced in (G. Da prato and A. Lunardi; 2007).

In the PhD Thesis M.E. San, application of semigroups method for stochastic parabolic equations with depended in the space variables operator $A(t) \equiv A$ was considered. The single step DS's for numerical solutions for the local and nonlocal problems for stochastic parabolic equations with depended in the space variables coeficients were studied.

1.1 Layout of the Present Thesis

In the present chapter we consider the initial value problem for the stochastic partial differential equation of parabolic type and the single-step exact DS for the solution of this problem.

It is well known that various IBVP's for stochastic parabolic equations can be reduced to the Cauchy problem for the first order SDE

$$dv(t) + A(t)v(t)dt = f(t, w(t))dt + g(t, w(t))dw_t,\ 0 < t < T,\ v(0) = \varphi \qquad (2)$$

in a Hilbert space H with a self adjoint PD operator $A(t)$ dependent in t with the closed domain $D(A(t)) \subset H$ and the restriction $A(t) \geq \delta I$, where $\delta > 0$ and I is the identity operator. Let $w_t = \sqrt{t}\xi$ be the standard Wiener process on the probability space (Π, F, P) and $\xi \in \mathcal{N}(0, 1)$ be the standard normal distribution with mean $\mu = 0$ and variance $\sigma = 1$. Here $v(t)$ and $f(t, w_t)$ are the unknown and the given abstract functions.

The main goal of this study is to construct and investigate the stable DS's for the approximate solution of problem (2). The RDS and CNDS generated by the single step DS for the solution of problem (2) are presented. The convergence estimates for the solution of these difference schemes are established. In applications, the convergence estimates for the solution of DS's for stochastic parabolic problems are established. For the numerical study, procedure of modified Gauss elimination method is used to solve these DS's.

Let us briefly describe the contents of the work. This study consists of introduction and five chapters.

The first chapter a historical note and literature survey.

The second chapter is to study o the linear stochastic parabolic equations. Applying results of Chapter One and Fourier series, Laplace and Fourier transform methods, we obtain the exact solution of several stochastic parabolic equations with dependent coefficients.

In the third chapter, the main theorem on stability of the Linear stochastic parabolic equations is established. In applications of the main theorem, stability estimates for the solutions of four problems of the SDE's with local and nonlocal conditions are obtained.

In the fourth chapter, the initial value problem (2) for the stochastic partial differential equation of parabolic type is considered. The single-step exact difference scheme

$$v(t_k) - v(t_{k-1}) + \big(I - v(t_k, t_{k-1})\big)v(t_{k-1}) = \int_{t_{k-1}}^{t_k} v(t_k, s)f(s, w_s)dw_s,$$

$$t_k = k\tau, 1 \leq k \leq N, N\tau = T, v(0) = 0$$

for the solution of the Cauchy problem (2) is presented.

Also, in the fourth chapter, approximate formulas for $v(t_k, t_{k-1})$ and $\int_{t_{k-1}}^{t_k} v(t_k, s)f(s, w_s)dw_s$ are presented. Applying these formulas, 1/2-th order of accuracy in t RDS for the approximate solution of problem (2) generated by the single step DS for the approximate solution of problem (2) is presented. The main theorem on convergence of RDS in a Hilbert space is established. Applications, the convergence estimates for the solution of initial-boundary problems for stochastic parabolic equations with dependent coefficients are obtained.

Furthermore, the fourth chapter also contains, applying more accurate approximate formulas for $v(t_k, t_{k-1})$ and
$\int_{t_{k-1}}^{t_k} v(t_k, s)f(s, w_s)dw_s$ the CNDS is constructed. The main theorem on convergence of CNDS in a Hilbert space is established. The fifth chapter is devoted to the numerical analysis. The sixth chapter is conclusion.
Three extended abstracts are published in AIP Conferences. First paper was published in AIP of MPDSIDA 2023, the second ICMS 2022, third paper was published in ICAAM 2022. One paper was published in MDPI Journal AXIOM in July 2023.

Basic Concept and Definions

This section highlights basic concepts and definitions on the theory of ordinary and partial differential equations leading us to conduct and understand the works in this thesis.

Sturm-Liouville Problem

(Arfken, Weber, 2005)

We denote the Sturm Liouville operator as

$$L[v] = -\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + q(x)v$$

and consider the Sturm Liouville equation

$$L[v] + \lambda v = 0, \tag{3}$$

where $p > 0$ and p and q are continuous functions on the interval $[0, l]$ with local boundary conditions

$$\alpha_1 v(0) + \alpha_2 p(0) v^{'}(0) = 0; \beta_1 v(l) + \beta_2 p(l) v^{'}(l) = 0, \tag{4}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$ or nonlocal boundary conditions

$$v(0) - v(l) = 0, v'(0) - v'(l) = 0, \tag{5}$$

The problem of finding a complex number $\lambda = \mu$ such that the boundary value problems (3), (4) or (3), (5) have a non trivial solution are called Sturm-Liouville problems.

The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(x, \mu)$ is called an eigenfunction.

We will consider three types of Sturm-Liouville problem.

The Sturm-Liouville Problem with Dirichlet Condition

$$-u^{''}(x) + \lambda u(x) = 0, 0 < x < l, u(0) = u(l) = 0$$

has solution

$$u_k(x) = \sin \frac{kx}{l}$$

and

$$\lambda_k = -\left(\frac{k\pi}{l}\right)^2, k = 1, 2, ...$$

In the case when $l = \pi$

$$u_k(x) = \sin kx$$

and

$$\lambda_k = -k^2, k = 1, 2, ...$$

The Sturm-Liouville Problem with Neumann Condition

$$-u^{''}(x) + \lambda u(x) = 0, 0 < x < l, u'(0) = u'(l) = 0$$

has solution

$$u_k(x) = \cos\frac{kx}{l}$$

and

$$\lambda_k = (\frac{k\pi}{l}), k = 0, 1, 2, ...$$

In the case when $l = \pi$

$$u_k(x) = \cos kx$$

and

$$\lambda_k = -k^2, k = 0, 1, 2, ...$$

The Sturm-Liouville Problem with Nonlocal Conditions

$$-u^{''}(x) - \lambda u(x) = 0, 0 < x < l, u(0) = u(l), u^{'}(0) = u^{'}(l)$$

has solution

$$u_k(x) = \cos 2kx, k = 0, 1, 2, ...$$

$$u_k(x) = \sin 2kx, k = 1, 2, ...$$

and

$$\lambda_k = 4k^2, k = 0, 1, 2, ...$$

Fourier Series (Brown, Churchyll, 1993)

Let l be a fixed number and $f(x)$ be a periodic function with periodic $2l$, defined on $(-l, l)$. The Fourier series of $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sins and cosines :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}), \tag{6}$$

where a_0, a_n and b_n called the Fourier coefficientes of $f(x)$, are given by these formulas

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(\frac{n\pi x}{p})dx, n = 1, 2, ...$$

and

$$b_n = \frac{1}{l}\int_{-l}^{i} \sin(\frac{n\pi x}{p})dx, n = 1, 2, ...$$

Laplace Transform (Franklyn,1 949)

Let $f(t)$ be defined for $t \geqslant 0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or $L\{f(t)\}$, is an integral transform given by the integral

$$F(s) = L\{f(t)\} = \int\limits_{0}^{\infty} f(t)e^{-st}dt$$

provided that this (improper) integral exsists i.e that this integral is convergent.

The Laplace transform is operation that transforms a function of t (i.e a function of time domain), defined on $[0, \infty]$ to a function of s (i.e of frequency domain). The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. $F(s)$ is Laplace transform or simply transform of $f(t)$. Together the two functions $f(t)$ and $F(s)$ are called a Laplace transform pair.

Fourier Transform (Bracewell, 1999)

The Fourier transform of a function $f = f(x)$ denoted by $F(s)$ or $F\{f(x)\}$, is an integral transform given by the integral

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-xs}dx.$$

Basic Formulas

We will need an estimate for stochastic integrals , that is very paricular case of the Burkholder-Davis-Gundy intequality.

Burkholder-Davis-Gundy Intequality

Let (Π, F, F_t, P) be as stochastic basis. H a hilbert space. Then for any $p[\in 1, \infty[$ there exists a constant $M > 0$ such that for any F_t-adapted measurable stochastic process $G : [0, T] \times \Pi \to H$ satisfying $\int_0^T \|G(t)\|^2 dt < \infty$ almost surely one has

$$E \sup_{0 \le t \le T} \| \int_0^t G(s) dw_s \|^p \le M\, E\big(\int_0^t \|G(s)\|^p dw_s \big)^{\frac{p}{2}}.$$

Paley-Wiener-Zygmund

By theorem of Paley-Wiener-Zygmund(1933) the path of Wiener process is almost nowhere differentiable, which is shown in the following theorem.
The Wiener process or the standard Wiener process is defined by the following coditions:

- $w_0 = 0$.
- The wiener process w has independent increments $0 < t_1 < t_2 < t_3 < t_4 < T$, such that $w_{t_4} - w_{t_3}$ and $w_{t_2} - w_{t_1}$ are independent stochastic variables.
- For $t_1 < t_2$ the stochastic variable $w_{t_2} - w_{t_1}$ has the normal distribution $N(0, \sqrt{t_2 > t_1})$

We denote that $(dw_t)^2 = dt$ and $dw_t = \sqrt{t} \xi dt$, where dw_t denotes the differential form of the wiener process. By applying the stochastic integral of the random part is also based on the Wiener process and on the Itô formula.
For more information see in (R.F. Curtain and P.L. Falb, 1971) and (A. Karczewska, 2005).

Now let us describe more about the wiener process. Let w_t be an H-valued random process on T. Then w_t is a wiener process if

- $E\{w_t - w_s\} = 0$ for all s, t in $[0, T]$,
- w_t is continuous in t if $P(w_t) = 1$,
- $E\{[w_t - w_s]\} \circ \{[w_t - w_s]\}^2 = (t - s)w$ for all s, t in $[0, T]$, where w is compact, positive, bounded trace class operator mapping H into itself,

- $E\{||w_t - w_s||^2\} < \infty$ for all s, t in $[0, T]$.

The operator w has countable eigenvalues $\lambda_i > 0$ for all i such that $Tr(w) = \sum_{i=0}^{\infty} \lambda_i$. There is also a complete orthonormal basis $\{e_i\}$ of H, consequently $we_i = \lambda_i$. Curtain, Ruth F., and Peter L. Falb studied in the paper (R.F. Curtain and P.L. Falb, 1971) over the wiener process and devolep stochastic integral in a Hilbert space H.

Theorem 1.4.2.1 Let $(w_t)_{t\geq 0}$ be a standard Wiener process given on the probability space (Π, F, P). Then, we get

$$P\{w : t \to w_t \textit{ is nowhere differentiable}\} = 1.$$

For the proof see (N. Gantert, 2012) in Satz 21.17, . Moreover, the solution of this problem requires non smoothness function $f(t, w_t)$ for $t > 0$, since the function $f(t, w_t)$ depends on the Wiener process w_t. The equation is definite in stochastic. A probability space (Π, F, P) is equipped a right-continuous filtration $\{F_t\}_{t\geq 0} = F$ such that F_0 contains all sets of $\mathcal{P}$-measures zero. The wiener process is assumed to be adapted to $\{F_t\}_{t\geq 0}$ and for every $t > s$ the increments $w_t - w_s$ are independent of F_t.

Likewise we regard the stochastic parabolic equation with the smooth function $f(t)$, which is deterministic. This means $f(t, w_t) = f(t) \in L_2([0, T])$ map to $f(t) : [0, T] \to \mathbb{R}$. The stochastic integral $\int_0^T f(t) dw_t$ has a centered distribution $\int_0^T f(t) dw_t \sim \mathcal{N}(0, \int_0^T |f(t)|^2 dt)$ with mean $\mathbb{E}[\int_0^T f(t) dw_t] = 0$. For more information see (N. Privault, 2013) in Propostition 4.6.

Itô Integral

SDEs contain a variable which has random white noise calculated as the derivative of the Wiener process or Brownion motion. The wiener process is almost surely nowhere differentiable. So it requires its own rules of calculus. These are the Ito stochastic calculus and the Stratonovich stochastic calculus.

Next we will use a definition that provides the Itô integral, in order to understand it, however, we need to look at two other definitions. The following definition describes the adapted process.

Definition 1.4.3.1 Let (Π, F, P) be a probability space. Furthermore let the segment interval $[0, T]$ be partitioned in $0 = t_1 \leq t_2 \leq \ldots \leq t_N = T$ and $t_i - t_{i-1} = \tau_i$. Assume that $F = (\mathcal{F}_{\tau_i})_{\tau i \in [0,T]}$ be a filtration (filtered Algebra) such that $F = \cup_{t_i \in [0,T]} \mathcal{F}_{\tau_i}$ is the sigma Algebra $\mathcal{F}$. Additionally let $(S, \mathcal{E})$ be a measurable space and $X : \tau_i \to S$ be a stochastic process. The process X is adapted to the filtration $(\mathcal{F}_{\tau_i})_{\tau i \in [0,T]}$, if the random variable $X_i : \tau_i \times \Pi \to S$ is a $(\mathbb{F}_t, \mathcal{E})$-measurable function for each $t_i \in [0, T]$.

We further consider a subspace in the next definition.

Definition 1.4.3.2 Let $M_w^2([0, T], H_1)$ be a subspace of piecewise continuous functions in H. Additionally let the function $f(t, w_t)$ be in the space of $M_w^2([0, T], H_1)$. Assume that $X_i \in (\mathcal{F}_\tau)_{\tau \in [0,T]}$. For the partitioned segment $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$ and $t_i - t_{i-1} = \tau_i$, we have the stochastic definite integral of

$$f(t, w_t) = \sum_{i=1}^{N} X_i(w_t) 1_{t_{i-1} \leq t \leq t_i}$$

over the interval $[t_{i-1}, t_i] \subset [0, T]$ is defined as $\int_0^T f(t, w_t) dw_t = \int_0^T 1_{[t_{i-1}, t_i]}(t) dw_t$ with in particular $\int_{t_i}^{t_{i-1}} dw_t = \int_0^T 1_{[t_{i-1}, t_i]}(t) dw_t = w_{t_{i-1}} - w_{t_i}$.

Finally, we can proceed to define the Itô integral.

Definition 1.4.3.3 Let the definition of Adapted process and the function $\mathbb{E} \int_0^T |f(t)|^2 dt$ be satisfied. Additionally, let be $X \in \mathcal{F}_{t_i}$ and $E[X^2]$ for all $t_i \in [0, T]$. For $f(t, w_t) \in M_w^2$ we have the Itô integral

$$I(f)(w_t) = \int_0^T f(t, w_t) dw_t = \sum_{i=1}^{N} X_i(w_t)(w_{t_i} - w_{t_{i-1}}).$$

Itô Isometrie

The next lemma is a useful result concerning Itô Isometrie on $M_w^2([0, T], H_1)$ for our purposes. Before we analyze this lemma, we recall some well-known properties in the following list, which we need it later in the announced lemma.

(1) Let $Y_1, \ldots, Y_n$ be independently random variables with $\mathbb{E}(Y_t) = 0$ for every $t = 1, \ldots, n$. Assume that the filtration $\mathcal{F}_t = \sigma(Y_1, \ldots, Y_n)$ and $X_t = \sum_{s=1}^{t} Y_s$. Then for $r < s$ and $s < r$, we have

$$\mathbb{E}(Y_r | \mathcal{F}_t) = 0.$$

(2) For finite sums the expectation is linear such that

$$\mathbb{E}\big(\sum_{i=1}^{n} X_i\big) = \sum_{i=1}^{n} \mathbb{E}(X_i).$$

Now we can establish the announced lemma.

Lemma 1.4.4.1 Let the definition 1.4.3.3 be satisfied. The Itô integral I is an isometrie of $M_w^2 \subset L^2(dw_t \times dt)$ in $L^2(dw_t)$, such that

$$\|I(f(t,w_t))\|_{L^2(\Pi)} = \|f(t,w_t)\|_{L^2(\Pi\times[0,T])}, \qquad \forall f \in M_w^2.$$

Proof For L_2-Norm of $I(f(t,w_t))$, we have

$$\|I(f(t,w_t))\|^2_{L^2(\Pi)} = \mathbb{E}\Big[\big(\int_0^T X(w)dw_t\big)^2\Big] = \mathbb{E}\Big[\big(\sum_{i=1}^{N} X_i(w)(w_{t_i} - w_{t_{i-1}})\big)^2\Big]$$

$$= \mathbb{E}\Big[\sum_{i<j}^{N} X_i X_j \big(w_{t_i} - w_{t_{i-1}}\big)\big(w_{t_j} - w_{t_{j-1}}\big)\Big] + \Big[\mathbb{E}\sum_{i=j}^{N} \big(X_i(w)\big)^2 \big(w_{t_i} - w_{t_{i-1}}\big)^2\Big]$$

$$= \mathbb{E}\big[\sum_{i<j}^{N} X_i X_j \big(w_{t_i} - w_{t_{i-1}}\big)\big]\mathbb{E}\big[\big(w_{t_j} - w_{t_{j-1}}\big)|\mathcal{F}t_j\big] + \mathbb{E}\Big[\sum_{i=j}^{N} \big(X_i(w)\big)^2 \big(w_{t_i} - w_{t_{i-1}}\big)^2\Big]$$

$$\sum_{i=j}^{N} \mathbb{E}\big(X_i(w)\big)^2 \mathbb{E}\big(w_{t_i} - w_{t_{i-1}}\big)^2 = \sum_{i=j}^{N} \mathbb{E}X_i^2(w)\big(t_i - t_{i-1}\big)$$

$$= \sum_{i=j}^{N} \mathbb{E}X_i^2(w) 1_{t_i<t<t_{i-1}}\big(t_i - t_{i-1}\big)$$

$$= \mathbb{E}\int_0^T \|f(t,w_t)\|^2 dt = \|f(t,w_t)\|^2_{L^2(\Pi\times[0,T])}.$$

In the fourth equation we use the properties (1) and (2) above. We apply the property (3) in the fifth equation. This is proved.

As an application of the Ito isometry, we note that

$$\mathbb{E}[\big(\int_0^T f(t,w_t)dw_t\big)^2] = \mathbb{E}[\int_0^T \|f(t,w_t)\|^2 dt] = \int_0^T \mathbb{E}[\|f(t,w_t)\|^2]dt$$

and analogously, we note that

$$\mathbb{E}[(\int_0^T w_t dw_t)^2] = \mathbb{E}[\int_0^T |w_t|^2 dt] = \int_0^T \mathbb{E}[|w_t|^2]dt = \int_0^T t dt = \frac{1}{2}T^2.$$

The above lemma show that the map is an isometry, so it is continuous.

Here we summerize the Itô formula by integration with respect to the Brownion motion, which is the process

$$I(f)(w_t) = \int_0^T f(t, w_t) dw_t = \sum_{i=1}^{N} X_i(w_t)(w_{t_i} - w_{t_{i-1}}),$$

where the smooth function $f(t, w_t) = I_{[0,T]}(t)$ is measurable with respect to the filtration $\mathcal{F}$, if for every $t_i \in [0, T]$ the restriction $f : [0, T] \times \Pi \to \mathbb{R}$ is a measurable funciton for every t_i. Then every step is measurable. Furthermore every continuous and adapted proces is measurable. Third, the process is $f(t, w_t)$ is measurable. For more information about the Itô process, please see in the introduction.

CHAPTER II

Methods of Solution of Linear Stochastic Parabolic Differential Equations

Introduction

Linear stochastic parabolic equations have the significant role in natural and applied sciences. Therefore, it is important to study stochastic parabolic equations. Therefore, the main aim of this chapter is to study of the stochastic parabolic equations with dependent coefficients. Applying results of Chapter One and Fourier series, Laplace and Fourier transform methods, we obtain the exact solution of several stochastic parabolic equations with dependent in t coefficients.

The Fourier Series Method

First, we consider initial boundary value problem

$$\begin{cases} du\left(t,x\right)-e^{-t}u_{xx}\left(t,x\right)dt=e^{-t}\sin xdw_{t}+(-e^{-t}+e^{-2t})\sin xw_{t}dt, \\ \\ x\in\left(0,\pi\right),\ 0<t<T, \\ \\ u(0,x)=0,\ x\in[0,\pi], \\ \\ u(t,0)=u(t,\pi)=0,\ t\in[0,T] \end{cases} \tag{7}$$

for one dimensional stochastic parabolic equations with dependent coefficients. For solving this problem, we consider the Sturm-Liouville problem

$$-u''(x)-\lambda u(x)=0,\ u(\pi)=u(0)=0,0<x<\pi$$

generated by the space operator of problem (7). As noted in Chapter 1 the solution of this Sturm-Liouville problem is

$$\lambda_{k}=k^{2},\ u_{k}(x)=\sin kx,\ k=1,2,....$$

Then, we obtain Fourier series solution of mixed problem (7) by the formula

$$u(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx, \tag{8}$$

where $A_k(t)$ are unknown functions. Applying this equation and initial condition, we get

$$\sum_{k=1}^{\infty}dA_k(t)\sin kx+e^{-t}\sum_{k=1}^{\infty}k^2A_k(t)\sin kxdt$$
$$=e^{-t}\sin xdw_t+\left(-e^{-t}+e^{-2t}\right)\sin xdt,\ \ x\in(0,\pi)\,,\ \ 0<t<T$$

and

$$\sum_{k=1}^{\infty}A_k(0)\sin(kx)=0,\ \ 0<x<\pi.$$

Equating coefficients $\sin kx,\ k=1,2,...$to zero, we get the initial value problems

$$\begin{cases} dA_1(t)+e^{-t}A_1(t)dt=e^{-t}dw_t+\left(-e^{-t}+e^{-2t}\right)w_tdt,\ \ 0<t<T, \\ \\ A_1(0)=0, \end{cases} \tag{9}$$

$$\begin{cases} dA_k(t)+k^2e^{-t}A_k(t)dt=0,\ \ 0<t<T, \\ \\ A_k(0)=0, k=2,3,\cdots \end{cases} \tag{10}$$

for the ordinary differential equations. First, we obtain $A_1(t)$. Putting $t=s$,we get

$$dA_1(s)+e^{-s}A_1(s)ds=e^{-s}dw_s+\left(-e^{-s}+e^{-2s}\right)w_sds.$$

Multiplying by $e^{-e^{-s}}$, we get

$$e^{-e^{-s}}dA_1(s)+e^{-e^{-s}}e^{-s}A_1(s)ds=e^{-e^{-s}}\left[e^{-s}dw_s+\left(-e^{-s}+e^{-2s}\right)w_sds\right].$$

We have that

$$e^{-e^{-s}}dA_1(s)+e^{-e^{-s}}e^{-s}A_1(s)ds=d\left(e^{-e^{-s}}A_1(s)\right).$$

Therefore, taking the integral with respect to s from o to t, we get

$$\int_0^t d\left(e^{-e^{-s}}A_1(s)\right)=\int_0^t e^{-e^{-s}}\left[e^{-s}dw_s+\left(-e^{-s}+e^{-2s}\right)w_sds\right].$$

Then

$$e^{-e^{-t}}A_1(t) - e^{-1}A_1(0) = \int\limits_0^t e^{-e^{-s}}\left[e^{-s}dw_s + \left(-e^{-s} + e^{-2s}\right)w_s ds\right]$$

or

$$e^{-e^{-t}}A_1(t) = \int\limits_0^t e^{-e^{-s}}\left[e^{-s}dw_s + \left(-e^{-s} + e^{-2s}\right)w_s ds\right].$$

Since

$$\int\limits_0^t e^{-e^{-s}}e^{-s}dw_s = e^{-e^{-t}}e^{-t}w_t - \int\limits_0^t w_s d\left(e^{-e^{-s}}e^{-s}\right)$$

$$= e^{-e^{-t}}e^{-t}w_t - \int\limits_0^t e^{-e^{-s}}\left(-e^{-s} + e^{-2s}\right)w_s ds,$$

we can write

$$A_1(t) = e^{e^{-t}}e^{-e^{-t}}e^{-t}w_t = e^{-t}w_t.$$

Second, we obtain $A_k(t), k \neq 1$. Putting $t = s$,we get

$$dA_k(s) + k^2e^{-s}A_k(s)ds = 0.$$

Multiplying by $e^{-k^2e^{-s}}$, we get

$$e^{-k^2e^{-s}}dA_k(s) + k^2e^{-k^2e^{-s}}e^{-s}A_k(s)ds = 0.$$

We have that

$$e^{-k^2e^{-s}}dA_k(s) + k^2e^{-k^2e^{-s}}e^{-s}A_k(s)ds = d\left(e^{-k^2e^{-s}}A_k(s)\right).$$

Therefore, taking the integral with respect to s from o to t, we get

$$\int\limits_0^t d\left(e^{-k^2e^{-s}}A_k(s)\right) = 0.$$

Then

$$e^{-k^2e^{-t}}A_k(t) - e^{-k^2}A_k(0) = 0$$

or

$$A_k(t) = 0.$$

Then, applying formula (8), we can obtain Fourier series solution of mixed problem (7) by the following formula

$$u(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx=A_1(t)\sin x=e^{-t}w_t\sin x.$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} du(t,x)-\sum\limits_{r=1}^{n}\alpha_r(t)\frac{\partial^2u(t,x)}{\partial x_r^2}dt=g(t,x)dw_t+f(t,x)dt, \\ \\ x=(x_1,...,x_n)\in\overline{\Omega},\ 0<t<T, \\ \\ u(0,x)=\varphi(x),\ x\in\overline{\Omega}, \\ \\ u(t,x)=0,\ x\in S,\ t\in[0,T] \end{cases} \tag{11}$$

for the multidimensional stochastic parabolic equations with dependent coefficients in t. Suppose that $\alpha_r(t)>\alpha>0$ and $g\left(t,x\right),f\left(t,x\right),\left(t,x\right)\in(0,T]\times\Omega$, $\varphi(x)\left(x\in\overline{\Omega}\right)$ are given smooth functions in x. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1, 1\le k\le n)$ with the boundary

$$S,\overline{\Omega}=\Omega\cup S.$$

However Fourier series method described in solving (11) can be used only in the case when (11) has constant coefficients in x.

Second, we consider initial boundary value problem

$$\begin{cases} du\left(t,x\right)-e^{-t}u_{xx}\left(t,x\right)dt \\ =e^{-4t}\cos 2xdw_t+(-4e^{-4t}+4e^{-5t})\cos 2xw_tdt, \\ \\ x\in\left(0,\pi\right),\ 0<t<T, \\ \\ u(0,x)=0,\ x\in[0,\pi], \\ \\ u_x(t,0)=u_x(t,\pi)=0,\ t\in[0,T] \end{cases} \tag{12}$$

for one dimensional stochastic parabolic equation with dependent in t coefficients. For solving this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0,\ u'(\pi) = u'(0) = 0, 0 < x < \pi$$

generated by the space operator of problem (12). As noted in Chapter 2 the solution of this Sturm-Liouville problem is

$$\lambda_k = -k^2,\ u_k(x) = \cos(kx),\ k = 0, 1, 2, ...$$

Thus we will seek the Fourier series solution of (12) by the formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos(kx), \tag{13}$$

where $A_k(t), k = 0, 1, ...$ are unknown functions. Putting $u(t, x)$ into the equation (12) and using the given initial condition, we obtain

$$\sum_{k=0}^{\infty} dA_k(t)\cos(kx) + e^{-t}\sum_{k=1}^{\infty} k^2 A_k(t)\cos(kx)dt$$
$$= e^{-4t}\cos 2x dw_t + \left(-4e^{-4t} + 4e^{-5t}\right)\cos 2x dt,\ 0 < x < \pi$$

and

$$u(0,x) = \sum_{k=0}^{\infty} A_k(0)\cos(kx) = 0,\ 0 \le x \le \pi.$$

Equating coeficients $\cos kx, k = 0, 1, 2, ...$ to zero, we get the initial value problems

$$\begin{cases} dA_2(t) + 4e^{-t}A_2(t)dt \\ = e^{-4t}dw_t + \left(-4e^{-4t} + 4e^{-5t}\right) w_t dt,\ 0 < t < T, \\ A_2(0) = 0, \end{cases}$$

$$\begin{cases} dA_k(t) + k^2 e^{-t}A_k(t)dt = 0,\ 0 < t < T, \\ A_k(0) = 0,\ k = 0, 1, 3, ... \end{cases}$$

for the ordinary differential equations. First, we obtain $A_2(t)$. Putting $t = s$,we get

$$dA_2(s) + 4e^{-s}A_2(s)ds = e^{-4s}dw + \left(-4e^{-4s} + 4e^{-5s}\right) w_s ds.$$

Multiplying by $e^{-4e^{-s}}$, we get

$$e^{-4e^{-s}}dA_2(s) + e^{-4e^{-s}}4e^{-s}A_2(s)ds$$

$$= e^{-4e^{-s}}[e^{-4s}dw_s + \left(-4e^{-4s} + 4e^{-5s}\right) w_s ds].$$

We have that

$$e^{-4e^{-s}}dA_2(s) + 4e^{-4e^{-s}}e^{-s}A_2(s)ds = d(e^{-4e^{-s}}A_2(s)).$$

Therefore, taking the integral with respect to s from 0 to t, we get

$$\int_0^t d\left(e^{-4e^{-s}}A_2(s)\right) = \int_0^t e^{-4e^{-s}}[e^{-4s}dw_s + \left(-4e^{-4s} + 4e^{-5s}\right) w_s ds].$$

Then

$$e^{-4e^{-t}}A_2(t) - 4e^{-4}A_2(0) = \int_0^t e^{-4e^{-s}}[e^{-4s}dw_s + \left(-4e^{-4s} + 4e^{-5s}\right) w_s ds]$$

or

$$e^{-4e^{-t}}A_2(t) = \int_0^t e^{-4e^{-s}}[e^{-4s}dw_s + \left(-4e^{-4s} + 4e^{-5s}\right) w_s ds].$$

Since

$$\int_0^t e^{-4e^{-s}}e^{-4s}dw_s = e^{-4e^{-t}}e^{-4t}w_t - \int_0^t w_s d\left(e^{-4e^{-s}}e^{-4s}\right)$$

$$= e^{-4e^{-t}}e^{-4t}w_t - \int_0^t e^{-4e^{-s}}\left(-4e^{-4s} + 4e^{-5s}\right) w_s ds,$$

we can write

$$A_2(t) = e^{-4e^{-t}}e^{-4t}w_t e^{4e^{-t}} = e^{-4t}w_t.$$

Second, we obtain $A_k(t), k \neq 2$. Putting $t = s$,we get

$$dA_k(s) + k^2 e^{-s}A_k(s)ds = 0.$$

Multiplying by $e^{-k^2e^{-s}}$, we get

$$e^{-k^2e^{-s}}dA_k(s) + k^2 e^{-k^2e^{-s}}e^{-4s}A_k(s)ds = 0.$$

We have that

$$e^{-k^2e^{-s}}dA_k(s) + k^2 e^{-k^2e^{-s}}e^{-s}A_k(s)ds = d\left(e^{-k^2e^{-s}}A_k(s)\right).$$

Therefore, taking the integral with respect to s from o to t, we get

$$\int_0^t d\left(e^{-k^2e^{-s}}A_k(s)\right) = 0.$$

Then

$$e^{-k^2e^{-t}}A_k(t) - e^{-k^2}A_k(0) = 0$$

or

$$A_k(t) = 0.$$

Then, applying formula (13), we can obtain Fourier series solution of mixed problem (12) by the following formula

$$u(t,x) = \sum_{k=1}^{\infty} A_k(t)\cos k2x = A_2(t)\cos 2x = e^{-4t}w_t \cos 2x$$

$$u(t,x) = e^{-4t}\cos 2xw_t.$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} du(t,x) - \sum\limits_{r=1}^{n} \alpha_r(t)\frac{\partial^2 u(t,x)}{\partial x_r^2}dt = g(t,x)dw_t + f(t,x)dt, \\ x = (x_1,...,x_n) \in \overline{\Omega},\ 0 < t < T, \\ u(0,x) = \varphi(x),\ x \in \overline{\Omega}, \\ \frac{\partial u}{\partial \overline{m}}(t,x) = 0,\ x \in S,\ t \in [0,T] \end{cases} \tag{14}$$

for the multidimensional stochastic parabolic equations with dependent coefficients in t. Suppose that $\alpha_r(t) > \alpha > 0$ and $f(t,x), g(t,x), (t,x) \in (0,T] \times \Omega$, $\varphi(x)\ \left(x \in \overline{\Omega}\right)$ are given smooth functions in x. Here and in future $\overline{m}$ is the normal to S. However Fourier series method described in solving (14) can be used only in the case when (14) has constant coefficients in x.

Third, we consider initial boundary value problem

$$\begin{cases} du\left(t,x\right)-e^{-4t}u_{xx}\left(t,x\right)dt \\ =e^{-4t}\sin 2xdw_{t}+(-4e^{-4t}+4e^{-8t})\sin 2xw_{t}dt, \\ \\ x\in\left(0,\pi\right),\ 0<t<T, \\ \\ u(0,x)=0,\ x\in[0,\pi], \\ \\ u(t,0)=u(t,\pi),u_{x}(t,0)=u_{x}(t,\pi),\ t\in[0,T] \end{cases} \tag{15}$$

for one dimensional stochastic parabolic equations with dependent coefficients.

For solving this problem, we consider the Sturm Liouville problem

$$-u''(x)-\lambda u(x)=0,\ 0<x<\pi$$

$u(\pi)=u(0),\ u'(\pi)=u'(0)$ generated by the space operator of problem (15). As noted in Chapter 1 the solution of this Sturm-Liouville problem is

$$\lambda_{k}=-4k^{2},\ k=0,1,2,...,$$

$$u_{k}(x)=\sin(2kx),\ k=1,2,....,\ u_{k}(x)=\sin(2kx),k=1,2,...$$

Then, applying formulas

$$u(t,x)=\sum_{k=1}^{\infty}A_{k}(t)\sin(2kx)+\sum_{k=0}^{\infty}B_{k}(t)\cos(2kx), \tag{16}$$

where $A_{k}(t),1,2,...$ and $B_{k}(t),k=0,1,...$ are unknown functions. Putting $u(t,x)$ into the equation (15) and using the given initial condition, we obtain

$$\sum_{k=1}^{\infty}dA_{k}(t)\cos(2kx)+\sum_{k=0}^{\infty}dB_{k}(t)\sin(2kx)$$

$$+e^{-4t}\sum_{k=1}^{\infty}4k^{2}A_{k}(t)\cos(2kx)+e^{-4t}\sum_{k=0}^{\infty}4k^{2}B_{k}(t)\sin(2kx)$$

$$=e^{-4t}\sin(2x)dw_{t}+\left(-4e^{-4t}+4e^{-8t}\right)\sin(2x)w_{t}dt,x\in(0,\pi),0<t<1$$

and

$$u(0,x)=\sum_{k=1}^{\infty}A_{k}(0)\cos(2kx)+\sum_{k=0}^{\infty}B_{k}(0)\sin(2kx)=0,\ 0\leq x\leq\pi.$$

Equating coefficients of $\cos(2kx), k = 0, 1, ...$ and $\sin(2kx), k = 1, 2, ...$ to zero, we get

$$\begin{cases} dB_1(t) + 4e^{-4t} B_1(t)dt \\ = e^{-4t}\sin(2x)dw_t + (-4e^{-4t} + 4e^{-8t})\sin(2x)dt, \ 0 < t < T \\ B_1(0) = 0, \end{cases}$$

$$\begin{cases} dB_k(t) + 4k^2 e^{-4t} B_k(t)dt = 0, \ 0 < t < T \\ B_k(0) = 0, k = 2, 3, ... \end{cases}$$

and

$$\begin{cases} dA_k(t) + 4k^2 e^{-4t} A_k(t)dt = 0, \ 0 < t < T \\ A_k(0) = 0, k = 0, 1, ... \end{cases}$$

We obtain $A_k(t)$ for the ordinary differential equations. Putting $t = s$,we get

$$dA_k(s) + 4k^2 e^{-4s} A_k(s)ds = 0.$$

Multiplying by $e^{-4k^2 e^{-4s}}$, we get

$$e^{-4k^2 e^{-4s}} dA_k(s) + 4k^2 e^{-4k^2 e^{-4s}} e^{-4s} A_k(s)ds = 0.$$

We have that

$$e^{-4k^2 e^{-4s}} dA_k(s) + 4k^2 e^{-4k^2 e^{-4s}} e^{-4s} A_k(s)ds = d\left(e^{-4k^2 e^{-4s}} A_k(s)\right).$$

Therefore, taking the integral with respect to s from 0 to t, we get

$$\int_0^t d\left(e^{-4k^2 e^{-4s}} A_k(s)\right) = 0.$$

Then

$$e^{-4k^2 e^{-4t}} A_k(t) - e^{-4k^2} A_k(0) = 0$$

or

$$A_k(t) = 0.$$

For $B_k(t), t = 2, 3, ...$, we have

$$dB_k(t) + 4k^2 e^{-4t} B_k(t)ds = 0.$$

Multiplying with $e^{-4k^2e^{-4s}}$ an set $t=s$, we get

$$e^{-4k^2e^{-4s}}dB_k(s)+4k^2e^{-4k^2e^{-4s}}e^{-4s}B_k(s)ds=\int_0^t d(e^{-4k^2e^{-4s}}B_k(s)).$$

$$\int_0^t d(e^{-4k^2e^{-4s}}B_k(s))=e^{-4k^2e^{-4t}}B_k(t)-e^{-4k^2}B_k(0).$$

Then

$$B_k(t)=0.$$

For $B_1(t)$, we have

$$dB_1(s)+4e^{-4s}B_1(s)ds=e^{-4t}dw_s+\left(-4e^{-4s}+4e^{-8s}\right)w_sds.$$

Multiplying by $e^{-e^{-4s}}$, we get

$$\begin{aligned}&e^{-e^{-4s}}dB_1(s)+e^{-e^{-4s}}4e^{-4s}B_1(s)ds\\&=e^{-e^{-4s}}[e^{-4t}dw_s+\left(-4e^{-4s}+4e^{-8s}\right)w_sds].\end{aligned}$$

We have that

$$e^{-e^{-4s}}dB_1(s)+4e^{-e^{-4s}}e^{-4s}B_1(s)ds=d(e^{-e^{-4s}}B_1(s)).$$

Therefore, taking the integral with respect to s from 0 to t, we get

$$\int_0^t d\left(e^{-e^{-4s}}B_1(s)\right)=\int_0^t e^{-e^{-4s}}[e^{-4t}dw_s+\left(-4e^{-4s}+4e^{-8s}\right)w_sds].$$

Then

$$e^{-e^{-4t}}B_1(t)-e^{-4}B_1(0)=\int_0^t e^{-e^{-4s}}[e^{-4t}dw_s+\left(-4e^{-4s}+4e^{-8s}\right)w_sds]$$

or

$$e^{-e^{-4t}}B_1(t)=\int_0^t e^{-e^{-4s}}[e^{-4s}dw_s+\left(-4e^{-4s}+4e^{-8s}\right)w_sds].$$

Since

$$\int_0^t e^{-e^{-4s}}e^{-4s}dw_s=e^{-e^{-4t}}e^{-4t}w_t-\int_0^t w_sd\left(e^{-4e^{-4s}}e^{-4s}\right)$$

$$= e^{-e^{-4t}} e^{-4t} w_t - \int_0^t e^{-e^{-4s}} \left(-4e^{-4s} + 4e^{-8s}\right) w_s ds,$$

we can write

$$B_1(t) = (e^{-e^{-4t}} e^{-4t} w_t) e^{e^{-4t}} = e^{-4t} w_t.$$

Then, applying formula (16), we can obtain Fourier series solution of mixed problem (15) by the following formula

$$u(t,x) = \sum_{k=0}^{\infty} A_k(t)\cos(2kx) + \sum_{k=1}^{\infty} B_k(t)\sin(2kx)$$

$$= B_1(t)\sin(2x) = e^{-4t} w_t \sin(2x).$$

Note that using similar procedure one can obtain the solution of following initial boundary value problem

$$\begin{cases} du(t,x) - \sum\limits_{r=1}^{n} a_r(t) \frac{\partial^2 u(t,x)}{\partial x_r^2} dt = g(t,x)dt + f(t,x)dw_t, \\ x = (x_1, ..., x_n) \in \Omega, \ 0 < t < T, \\ u(0,x) = \varphi(x), \quad x \in \overline{\Omega}, \\ u(t,x)|_{S_1} = u(t,\pi)|_{S_2}, \frac{\partial u(t,x)}{\partial m}|_{S_1} = \frac{\partial u(t,x)}{\partial m}|_{S_2}, t \in [0,T] \end{cases} \tag{17}$$

for the multididimensional SDE. Assume that $a_r(t) > a_0 > 0$ and $g\left(t,x\right), f\left(t,x\right), (t,x) \in (0,T)\times\Omega$, $\varphi(x)$, $\left(x\in\overline{\Omega}\right)$ are smooth functions. Here $S = S_1 \cup S_2$, $\varnothing = S_1 \cap S_2$. However Fourier series method described in solving (21) can be used only in the case when (21) has constant coefficients.

The Laplace Transform Solution

First, we consider the initial boundary value problem

$$\begin{cases} du\,(t,x) - e^{-t}u_{xx}\,(t,x)\,dt = e^{-t}e^{-x}dw_t + \left(-e^{-t} - e^{-2t}\right)e^{-x}w_t dt, \\ \\ x \in (0,\infty)\,,\ 0 < t < T, \\ \\ u(0,x) = 0,\ x \in [0,\infty), \\ \\ u(t,0) = e^{-t}w_t, u_x(t,0) = -e^{-t}w_t,\ t \in [0,T] \end{cases} \tag{18}$$

for the one dimensional SDE.

For solving this problem, we consider Laplace transform. Using formula,

$$L\{e^{-x}\} = \frac{1}{s+1} \tag{19}$$

and taking the Laplace transform of both sides of the differential equation and using conditions $u(t,0) = e^{-t}w_t, u_x(t,0) = -e^{-t}w_t$, we can write

$$L\{du(t,x)\} - L\{e^{-t}u_{xx}dt\} = L\{e^{-t}e^{-x}dw_s\} + L\{(-e^{-t} - e^{-2t})e^{-x}w_t dt\}$$

$$0 < t < T, L\{u(0,x)\} = 0$$

or

$$du(t,s)-e^{-t}(s^2u(t,s)-su(t,0)-u_x(t,0))dt = \frac{1}{1+s}e^{-t}dw_t+(-e^{-t}-e^{-2t})\frac{1}{1+s}w_t dt.$$

Then

$$du(t,s) - (e^{-t}s^2u(t,s) - se^{-2t}w_t + e^{-2t}w_t)dt$$
$$= \frac{1}{1+s}e^{-t}dw_t + (-e^{-t} - e^{-2t})\frac{1}{1+s}w_t dt, u(0,s) = 0.$$

Therefore, we can write

$$du(t,s)-s^2e^{-t}u(t,s)dt = \frac{1}{1+s}e^{-t}dw_t+(-e^{-t}-e^{-2t})\frac{1}{1+s}w_t dt-((s-1)e^{-2t}w_t)dt$$

Multiplying by $e^{s^2e^{-t}}$, we get

$$e^{s^2e^{-t}}du(t,s) - e^{s^2e^{-t}}s^2e^{-t}u(t,s)dt = e^{s^2e^{-t}}\left[\frac{1}{1+s}e^{-t}dw_t\right.$$

$$+(-e^{-t}(1+e^{-t}))\frac{1}{1+s}w_t dt + ((1-s)e^{-2t}w_t)dt\Bigg].$$

$$= e^{s^2e^{-t}}e^{-t}\left[\frac{1}{1+s}dw_t + w_t\left(\frac{1-s^2}{1+s} - (1+e^{-2t})\frac{1}{1+s}\right)dt\right]$$

$$= e^{s^2e^{-t}}e^{-t}\frac{1}{1+s}\left[dw_t - w_t(s^2e^{-t}+1))dt\right]$$

Putting $t = v$, we get

$$e^{s^2e^{-v}}du(v,s) - e^{s^2e^{-v}}s^2e^{-v}u(v,s)dv = e^{s^2e^{-v}}e^{-v}\frac{1}{1+s}[dw_v - w_v(s^2e^{-v}+1))dv]$$

We have that

$$d\left(e^{s^2e^{-v}}u(v,s)\right) = e^{s^2e^{-v}}du(v,s) - e^{s^2e^{-2v}}s^2e^{-v}u(v,s)dv \tag{20}$$

Therefore, taking the integral with respect to v from 0 to t, we get

$$\int_0^t d\left(e^{s^2e^{-v}}u(v,s)\right) = \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-t}+1)dv]$$

Then

$$e^{s^2e^{-v}}u(v,s) - e^{s^2}u(0,s) = \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-t}+1)dv]$$

or

$$e^{s^2e^{-t}}u(t,s) = \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-t}+1)dv]$$

Since

$$\frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}dw_v = e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t - \frac{1}{1+s}\int_0^t w_v d(e^{s^2e^{-v}}e^{-v})$$

$$= e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t - \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}(-s^2e^{-t}-1)w_v dv,$$

we can write

$$u(t,s) = e^{-s^2e^{-t}}e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t = \frac{1}{1+s}e^{-t}w_t.$$

Taking the inverse Laplace transform with respect to x, we obtain

$$u(t,x) = e^{-x-t}w_t.$$

Note that using same manner one can obtain the solution of following initial boundary value problem

$$\begin{cases} du(t,x) - \sum\limits_{r=1}^{n} a_r(t) \frac{\partial^2 u(t,x)}{\partial x_r^2} dt = g(t,x)dt + f(t,x)dw_t, \\ x = (x_1, ..., x_n) \in \Omega^+, \ 0 < t < T, \\ u(0,x) = \varphi(x), \quad x \in \overline{\Omega}^+, \\ u(t,x) = \alpha\left(t,x\right), \ u_{x_r}(t,x) = \beta_r\left(t,x\right), \\ 1 \le r \le n, \ (t,x) \in [0,T] \times S^+ \end{cases} \tag{21}$$

for the multidimensional SDE. Assume that $a_r(t) > a_0 > 0$ and $g\left(t,x\right), f\left(t,x\right), (t,x) \in (0,T) \times \Omega^+$, $\varphi(x)$, $\left(x \in \overline{\Omega}^+\right)$, $\alpha\left(t,x\right)$,
$\beta_r\left(t,x\right)\left(t,x\right) \in [0,T] \times S^+)$ are smooth functions. Here and in future Ω^+ is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with the boundary S^+ and

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However the Laplace transform method described in solving (21) can be used only in the case when (21) has constant coefficients.

Second, we consider the initial boundary value problem

$$\begin{cases} du\left(t,x\right) - e^{-t} u_{xx}\left(t,x\right)dt = e^{-t}e^{-x}dw_t + \left(-e^{-t} - e^{-2t}\right)e^{-x}w_t dt, \\ x \in (0,\infty), \ 0 < t < T, \\ u(0,x) = 0, \ x \in [0,\infty), \\ u(t,0) = e^{-t}w_t, u(t,\infty) = 0, \ t \in [0,T] \end{cases} \tag{22}$$

Applying formula ((19)) and taking the Laplace transform of both sides of the differential equation and noting $\gamma(t) = u_x(t,0)$ and using condition $u(t,0) =$

$e^{-t}w_t$, we can write

$$L\{du(t,x)\} - L\{e^{-t}u_{xx}dt\} = L\{e^{-t}e^{-x}dw_t\} + L\{(-e^{-t} - e^{-2t})e^{-x}w_t dt\}$$

$$0 < t < T, L\{u(0,x)\} = 0$$

or

$$du(t,s) - (s^2 e^{-t}u(t,s) - se^{-2t}w_t + \gamma(t)e^{-t})dt$$
$$= e^{-t}\frac{1}{s+1}dw_t + (-e^{-t} - e^{-2t})\frac{1}{s+1}w_t dt,\ u(0,s) = 0.$$

Then

$$du(t,s) - (e^{-t}s^2 u(t,s) - se^{-2t}w_t + e^{-2t}w_t)dt + \left(-e^{-t}w_t - \gamma(t)\right)dt$$
$$= \frac{1}{1+s}e^{-t}dw_t + (-e^{-t} - e^{-2t})\frac{1}{1+s}w_t dt, u(0,s) = 0.$$

Therefore, we can write

$$du(t,s) - s^2 e^{-t}u(t,s)dt$$

$$= \frac{1}{1+s}e^{-t}dw_t + (-e^{-t} - e^{-2t})\frac{1}{1+s}w_t dt - ((s-1)e^{-2t}w_t)dt + \left(e^{-t}w_t + \gamma(t)\right)dt.$$

Multiplying by $e^{s^2e^{-t}}$, we get

$$e^{s^2e^{-t}}du(t,s) - e^{s^2e^{-t}}s^2e^{-t}u(t,s)dt = e^{s^2e^{-t}}\left[\frac{1}{1+s}e^{-t}dw_t\right.$$
$$\left.+(-e^{-t}(1+e^{-t}))\frac{1}{1+s}w_t dt + ((1-s)e^{-2t}w_t)dt + \left(e^{-t}w_t + \gamma(t)\right)dt\right].$$

$$= e^{s^2e^{-t}}\left[e^{-t}\left[\frac{1}{1+s}dw_t + w_t\left(\frac{1-s^2}{1+s} - (1+e^{-2t})\frac{1}{1+s}\right)dt\right] + \left(e^{-t}w_t + \gamma(t)\right)dt\right]$$
$$= e^{s^2e^{-t}}e^{-t}\frac{1}{1+s}\left[dw_t - w_t(s^2e^{-t}+1))dt\right] + e^{s^2e^{-t}}\left(e^{-t}w_t + \gamma(t)\right)dt$$

Putting $t = v$, we get

$$e^{s^2e^{-v}}du(v,s) - e^{s^2e^{-v}}s^2e^{-v}u(v,s)dv$$

$$= e^{s^2e^{-v}}e^{-v}\frac{1}{1+s}[dw_v - w_v(s^2e^{-v}+1))dv] + e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv$$

We have that

$$d\left(e^{s^2e^{-v}}u(v,s)\right) = e^{s^2e^{-v}}du(v,s) - e^{s^2e^{-2v}}s^2e^{-v}u(v,s)dv$$

Therefore, taking the integral with respect to v from 0 to t, we get

$$\int_0^t d\left(e^{s^2e^{-v}}u(v,s)\right) = \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-t}+1)dv]$$

$$+\int_0^t e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv$$

Then

$$e^{s^2e^{-t}}u(v,s) - e^{s^2}u(0,s)$$

$$= \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-v}+1)dv]$$

$$+\int_0^t e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv$$

or

$$e^{s^2e^{-t}}u(t,s) = \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}[dw_v - w_v(s^2e^{-t}+1)dv]$$

$$+\int_0^t e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv$$

Since

$$\frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}dw_v = e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t - \frac{1}{1+s}\int_0^t w_v d(e^{s^2e^{-v}}e^{-v})$$

$$= e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t - \frac{1}{1+s}\int_0^t e^{s^2e^{-v}}e^{-v}(-s^2e^{-t}-1)w_v dv,$$

we can write

$$u(t,s) = e^{-s^2e^{-t}}e^{s^2e^{-t}}\frac{1}{1+s}e^{-t}w_t + e^{-s^2e^{-t}}\int_0^t e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv$$

$$= \frac{1}{1+s}e^{-t}w_t + e^{-s^2e^{-t}}\int_0^t e^{s^2e^{-v}}\left(e^{-v}w_v + \gamma(v)\right)dv.$$

Taking the inverse Laplace transform with respect to x, we obtain

$$u(t,x) = e^{-x-t}w_t + L^{-1}\left\{ e^{-s^2e^{-t}} \int\limits_0^t e^{s^2e^{-v}} \left(e^{-v}w_v + \gamma(v)\right) dv \right\}$$

$$= e^{-x-t}w_t + \int\limits_0^t \left(e^{-v}w_v + \gamma(v)\right) L^{-1}\left\{ e^{-s^2e^{-t}} e^{s^2e^{-v}} \right\} dv$$

Therefore, passing to limit when $x \to \infty$, we get

$$u(t,\infty) = e^{-t}w_t \lim_{x\to\infty} e^{-x} + \int\limits_0^t \left(e^{-v}w_v + \gamma(v)\right) \lim_{x\to\infty} L^{-1}\left\{ e^{-s^2e^{-t}} e^{s^2e^{-v}} \right\} dv.$$

From that it follows

$$e^{-v}w_v + \gamma(v) = 0.$$

Therefore, $u(t,x) = e^{-x-t}w_t$ is the solution of the given initial boundary value problem.

Note that using same manner one can obtain the solution of following initial boundary value problem

$$\begin{cases} du(t,x) - \sum\limits_{r=1}^{n} \alpha_r(t) \frac{\partial^2 u(t,x)}{\partial x_r^2} dt = g(t,x)dw_t + f(t,x)dt, \\ \\ x = (x_1, ..., x_n) \in \Omega^+, \ 0 < t < T, \\ \\ u(0,x) = \varphi(x), \quad x \in \overline{\Omega}^+, \\ \\ u(t,x) = \alpha\left(t,x\right), \ 1 \le r \le n, \ (t,x) \in [0,T] \times S^+ \end{cases} \tag{23}$$

for the multidimensional stochastic parabolic equations with dependent coefficients in t. Assume that $a_r(t) > a_0 > 0$ and $g\left(t,x\right), f\left(t,x\right), (t,x) \in (0,T) \times \Omega^+$, $\varphi(x)$, $\left(x \in \overline{\Omega}^+\right)$, $\alpha\left(t,x\right)$, $(t,x) \in [0,T] \times S^+)$ are smooth functions. However the Laplace transform method described in solving (23) can be used only in the case when (23) has constant coefficients in x.

The Fourier Transform Solution

We consider the initial-value problem

$$\begin{cases} du\,(t,x) - e^{-t}u_{xx}dt = e^{-t}e^{-x^2}dw_t \\ +(-e^{-t} - (-2+4x^2)\,e^{-2t})e^{-x^2}w_t dt, \\ x \in (-\infty,\infty)\,,\ 0<t<T, \\ u(0,x)=0,\ x\in(-\infty,\infty) \end{cases} \tag{24}$$

for the one dimensional stochastic partial differential equation.

For solving this problem, we consider Fourier transform method. We denote

$$u\,(t,s) = F\,\{u\,(t,x)\}\,,\ q(s) = F\left\{e^{-x^2}\right\}.$$

Taking the Fourier transform, we get the following initial value problem

$$\begin{cases} du\,(t,s) - s^2e^{-t}u(t,s)dt = e^{-t}q(s)dw_t \\ +\left[-e^{-t}q(s) - e^{-2t}F\left\{(-2+4x^2)\,e^{-x^2}\right\}\right]w_t dt, \\ 0<t<T, u(0,s)=0 \end{cases}$$

for the one stochastic ordinary differential equation.
Since $F\{(-2+4x^2)e^{-x^2}\} = F\{\left(e^{-x^2}\right)^{\prime\prime 2} F\{e^{-x^2}\}$, we can write

$$\begin{cases} du\,(t,s) - s^2e^{-t}u(t,s)dt = e^{-t}q(s)dw_t \\ +\left[-e^{-t}q(s) - e^{-2t}s^2F\{e^{-x^2}\}\right]w_t dt, \\ 0<t<T, u(0,s)=0 \end{cases}$$

or

$$\begin{cases} du\,(t,s) - s^2e^{-t}u(t,s)dt = e^{-t}q(s)dw_t \\ +\left[-e^{-t}q(s) - e^{-2t}s^2q(s)\right]w_t dt, \\ 0<t<T, u(0,s)=0. \end{cases}$$

Multiplying both sides by $e^{s^2te^{-t}}$, we get

$$e^{s^2te^{-t}}(du\,(t,s)-s^2e^{-t}u(t,s)dt) = e^{s^2te^{-t}}e^{-t}q(s)dw_t+e^{s^2te^{-t}}\left[-e^{-t}q(s)-e^{-2t}s^2q(s)\right]w_tdt.$$

Putting $t=y$, we get

$$e^{s^2ye^{-y}}(du\,(y,s)-s^2e^{-y}u(y,s)dy) = e^{s^2ye^{-y}}e^{-y}q(s)dw_y+e^{s^2ye^{-y}}\left[-e^{-y}q(s)-e^{-2y}s^2q(s)\right]w_ydy.$$

We have that

$$d\left(e^{s^2ye^{-y}}u(y,s)\right) = e^{s^2ye^{-y}}du(y,s) - e^{s^2ye^{-y}}e^{-y}s^2u(y,s)dy.$$

Therefore, taking the integral with respect to y from 0 to t, we get

$$\int_0^t d\left(e^{s^2ye^{-y}}u(y,s)\right) = \int_0^t e^{s^2ye^{-y}}e^{-y}q(s)dw_y+\int_0^t e^{s^2ye^{-y}}\left[-e^{-y}q(s)-e^{-2y}s^2q(s)\right]w_ydy$$

or

$$e^{s^2ye^{-y}}u(t,s)-u(0,s) = \int_0^t e^{s^2ye^{-y}}e^{-y}q(s)dw_y+\int_0^t e^{s^2ye^{-y}}\left[-e^{-y}q(s)-e^{-2y}s^2q(s)\right]w_ydy.$$

Then,

$$e^{s^2ye^{-y}}u(t,s) = \int_0^t e^{s^2ye^{-y}}e^{-y}q(s)dw_y + q(s)\int_0^t e^{s^2ye^{-y}}e^{-y}\left[-1-e^{-y}s^2\right]w_ydy.$$

Since

$$\int_0^t e^{s^2ye^{-y}}e^{-y}q(s)dw_y = e^{s^2ye^{-y}}e^{-t}q(s)dw_t - q(s)\int_0^t w_yd(e^{s^2ye^{-y}}e^{-y})$$

$$= e^{s^2te^{-t}}e^{-t}q(s)dw_t - q(s)\int_0^t e^{s^2e^{-y}}e^{-y}(-s^2e^{-y}-1)w_ydy,$$

we have that

$$u(t,s) = e^{s^2e^{-t}}e^{-s^2e^{-t}}e^{-t}q(s)w_t = e^{-t}q(s)w_t.$$

Therefore using the inverse Fourier transform, we get

$$u(t,x) = e^{-t}F^{-1}\left\{q(s)\right\}w_t = e^{-t}e^{-x^2}w_t$$

is the exact solution of the given initial boundary value problem.

Note that using similar procedure we can get the solution of following initial boundary value problem

$$\begin{cases} du(t,x) - \sum\limits_{r=1}^{n} \alpha_r(t) \frac{\partial^2 u(t,x)}{\partial x_r^2} dt = g(t,x)dw_t + f(t,x)dt, \\ \\ x = (x_1, ..., x_n) \in R^n, \ 0 < t < T, |r| = r_1 + ... + r_n, \\ \\ u(0,x) = \varphi(x), \quad x \in R^n \end{cases} \tag{25}$$

for the multidimensional stochastic parabolic equations with dependent coefficients in t. Assume that $a_r(t) > a_0 > 0$ and $g(t,x), f(t,x), (t,x) \in (0,T) \times R^n$, $\varphi(x), (x \in R^n)$ are smooth functions. However the Fourier transform method described in solving (25) can be used only in the case when (25) has constant coefficients in x.

CHAPTER III

The Abstract Cauchy Problem for the SDE in Hilbert Spaces with the Time-Dependent Positive Operator

The Main Theorem on Stability

It is well known that various IBVP's for stochastic parabolic equations can be reduced to the Cauchy problem for the SDE

$$dv(t) + A(t)v(t)dt = f(t, w(t))dt + g(t, w(t))dw_t,\ 0 < t < T,\ v(0) = \varphi \tag{26}$$

in a Hilbert space H with the unbounded operators $A(t)$. Here $w_t = \sqrt{t}\xi$ is a standard Wiener process given on the probability space (Π, F, P) and $\xi \in \mathcal{N}(0,1)$ is the standard normal distribution. Moreover, $v(t)$, $f(t, w(t))$ and $g(t, w(t))$ are the unknown and given functions, respectively, defined on $(0,T) \times \Pi$ with values in H. Furthermore, assume that $f(t, w(t))$ and $g(t, w(t))$ are elements of space $M_w^2([0,T] \times \Pi, H_1)$, which consists of H_1-value process for which the conditions

$$E\int_0^T \|f(t, w(t))\|_{H_1}^2 dt, E\int_0^T \|g(t, w(t))\|_{H_1}^2 dt < \infty \tag{27}$$

are satisfied. Here $H_1 \subset H$ and E is the expectation and the integrals are understood in the sense of Bochner.

Suppose that for each $t \in [0,T]$ the operator $-A(t)$ generates an analytic semigroup $\exp\{-sA(t)\}(s \geq 0)$ with exponentially decreasing norm, when $s \to +\infty$, i.e. the following estimates

$$\|\exp(-sA(t))\|_{H\to H}, \|sA(t)\exp(-sA(t))\|_{H\to H} \leq Me^{-\delta s}(s > 0) \tag{28}$$

hold for some $M \in [1, +\infty)$, $\delta \in (0, +\infty)$. From this inequality it follows the operator $A^{-1}(t)$ exists and bounded and hence $A(t)$ is closed in $H_1 \subset H$, such that $A(t):\ D(A(t)) \to H$ and $D(A(t)) = D(A(0))$ for $0 \leq t \leq T$.

Suppose that the operator $A(t)A^{-1}(s)$ is Holder continuous in t in the uniform operator topology for each fixed s, that is,

$$\left\|[A(t) - A(\tau)]A^{-1}(s)\right\|_{H\to H} \leq M|t-\tau|^{\varepsilon}, 0 < \varepsilon \leq 1, \tag{29}$$

where M and ε are positive constants independent of t, s and τ for $0 \le t, s, \tau \le T$.

An operator-valued function $v(t,s)$, defined and strongly continuous jointly in t,s for $0 \le s < t \le T$, is called a fundamental solution of (26) if

1) the operator $v(t,s)$ is strongly continuous in t and s for $0 \le s < t \le T$,

2) the following identity holds:

$$v(t,s) = v(t,\tau)v(\tau,s), v(t,t) = I \tag{30}$$

for $0 \le s \le \tau \le t \le T$, where, I is the identity operator,

3) the operator $v(t,s)$ maps the region D into itself. The operator $u(t,s) = A(t)v(t,s)A^{-1}(s)$ is bounded and strongly strongly continuous in t and s for $0 \le s < t \le T$,

4) on the region D the operator $v(t,s)$ is differentiable relative to t and s, while

$$\frac{\partial v(t,s)}{\partial t} + A(t)v(t,s) = 0, \tag{31}$$

and

$$\frac{\partial v(t,s)}{\partial s} - v(t,s)A(s) = 0. \tag{32}$$

Applying (30) and (31), we get the following formula, (see, (R.F. Curtain and P.L. Falb, 1971) and (D.A. Dawson, 1975)).

$$v(t) = v(t,0)v(0) + \int_0^t v(t,y)\{f(y,w(y))dy + g(y,w(y))dw_y\} \tag{33}$$

for a mild solution of the problem (26) under the assumptions (27).

Lemma 3.1.1 For any $0 \le s \le t \le T$ and $\beta \le \alpha \in (0,1]$ the following estimates hold (A. Ashyralyev and P.E.Sobolevskii, 2004; P.E. Sobolevskii, 1964)

$$\|v(t,s)\|_{H\to H} \le M, \tag{34}$$

$$\|A^{\alpha}(t)v(t,s)A^{-\beta}(p)\|_{H\to H} \le \frac{M}{(t-s)^{\alpha-\beta}}, \tag{35}$$

$$\|A(t)A^{-1}(s)\|_{H\to H} \le M. \tag{36}$$

The following Tubaro theorem was established in the paper (L. Tubaro; 1984).

Theorem 3.1.2 Suppose that e^{-tA} is a C_0-semigroup on H satisfying

$$\|e^{-tA}\|_{H\to H} \leq e^{-\delta t}$$

for some $\delta \geq 0$ and all $t \geq 0$. Then for every $p \in]0,\infty[$ there exists a constant $M_p(\delta) < \infty$ such that

$$E \sup_{0\leq t\leq T} \left\| \int_0^t e^{-A(t-s)} f(y,w(y))dw_y \right\|^p$$

$$\leq M_p(\delta)\, E \left(\int_0^T \|f(y,w(y))dw_y\|_H^2 dy \right)^{\frac{p}{2}} \tag{37}$$

for $f \in M_w^2([0,T]\times\Pi, H_1)$.

We have that

Theorem 3.1.3 Suppose that

$$E\|v(0)\|_H, E\int_0^T \|f(y,w(y))\|_H dy, E\int_0^T \|g(y,w(y))\|_H^2 dy < \infty. \tag{38}$$

Then, for the solution of problem (26) the following estimates hold

$$\max_{t\in[0,T]} E\|v(t)\|_H \leq M\,[E\|v(0)\|_H$$

$$+E\int_0^T \|f(y,w(y))\|_H dy + \left(E\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}\Bigg]. \tag{39}$$

Proof. Using formula (33) and the triangle inequality, we get

$$\max_{t\in[0,T]} E\|v(t)\|_H^2 \leq \max_{t\in[0,T]} E\|v(t,0)v(0)\|_H$$

$$+\max_{t\in[0,T]} E\left\|\int_0^t v(t,y)f(y,w(y))dy\right\|_H + \max_{t\in[0,T]} E\left\|\int_0^t v(t,y)g(y,w(y))dw_y\right\|_H$$

$$= I_1 + I_2 + I_3.$$

Here,

$$I_1 = \max_{t\in[0,T]} E\|v(t,0)v(0)\|_H,$$

$$I_2 = \max_{t\in[0,T]} E\left\|\int_0^t v(t,y)f(y,w(y))dy\right\|_H,$$

$$I_3 = \max_{t\in[0,T]} E\left\|\int_0^t v(t,y)g(y,w(y))dw_y\right\|_H.$$

We will estimate I_r for all $r = 1, 2, 3$, separately. We start with I_1. Applying estimate (34), we can write

$$I_1 = \max_{t\in[0,T]} E\|v(t,0)\|_{H\to H}\|v(0)\|_H \le M_3\|v(0)\|_H.$$

Now let us estimate I_2. Using estimate (34), we get

$$I_2 \le \max_{t\in[0,T]} E\int_0^t \|v(t,y)\|_{H\to H}\,\|f(y,w(y))\|_H dy \le M_4 E\int_0^t \|f(y,w(y))\|_H dy.$$

Finally, let us estimate I_3. Using estimate (34), we get

$$I_3 \le \max_{t\in[0,T]} E\left(\int_0^t \|v(t,y)g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}$$

$$\le \max_{t\in[0,T]} E\left(\int_0^t \|v(t,y)\|_{H\to H}^2\|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} \le M_5 E\left(\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

Combining the estimates for I_r for all $r = 1, 2$ and 3, we get (39). Theorem 3.1.3 is established.

Theorem 3.1.4 Suppose that

$$E\|v(0)\|_H^2, E\int_0^T \|f(y,w(y))\|_H^2 dy, E\int_0^T \|g(y,w(y))\|_H^2 dy < \infty. \tag{40}$$

Then, for the solution of problem (26) the following estimates hold

$$\left(E\int_0^T \|v(t)\|_H^2 dt\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\|v(0)\|_H^2\right)^{\frac{1}{2}}\right. \tag{41}$$

$$\left. + \left(E\int_0^T \|f(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}\right].$$

Proof. Using formula (33) and the triangle inequality, we get

$$\left(E\int_0^T \|v(t)\|_H^2\,dt\right)^{\frac{1}{2}} \leq \left(E\int_0^T \|v(t,0)v(0)\|_H^2 dt\right)^{\frac{1}{2}}$$

$$+\left(E\int_0^T \left\|\int_0^t v(t,y)f(y,w(y))dy\right\|_H^2 dt\right)^{\frac{1}{2}} + \left(E\int_0^T \left\|\int_0^t v(t,y)g(y,w(y))dw_y\right\|_H^2 dt\right)^{\frac{1}{2}}$$
$$= J_1 + J_2 + J_3.$$

Here,

$$J_1 = \left(E\int_0^T \|v(t,0)v(0)\|_H^2 dt\right)^{\frac{1}{2}},$$

$$J_2 = \left(E\int_0^T \left\|\int_0^t v(t,y)f(y,w(y))dy\right\|_H^2 dt\right)^{\frac{1}{2}},$$

$$J_3 = \left(E\int_0^T \left\|\int_0^t v(t,y)g(y,w(y))dw_y\right\|_H^2 dt\right)^{\frac{1}{2}}.$$

We will estimate J_r for all $r = 1, 2, 3$, separately. We start with J_1. Applying estimate (34), we can write

$$J_1 \leq \left(E\int_0^T \|v(t,0)\|_{H\to H}^2 \|v(0)\|_H^2 dt\right)^{\frac{1}{2}} \leq M_1 T E\left(\|v(0)\|_H^2\right)^{\frac{1}{2}}.$$

Now let us estimate J_2. Making the substitution $s = t - y$, we get

$$\int_0^t v(t,y)f(y,w(y))dy = \int_0^t v(t,t-s)f(t-s,w(t-s))ds$$

$$= \int_0^T v(t,t-s)f_*(t-s,w(t-s))ds.$$

Using the Minkowski inequality and estimate (34), we get

$$J_2 \leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|f_*(t-s,w(t-s))\|_H^2\,dt\right)^{\frac{1}{2}} ds$$

$$\leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|f(y,w(y))\|_H^2\,dy\right)^{\frac{1}{2}} ds$$

$$= M_4\left(\delta\right)\left(E\int\limits_0^T \|f(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

Finally, let us estimate J_3. Making the substitution $s = t - y$, we get

$$\int\limits_0^t v(t,y)g(y,w(y))dw_y = \int\limits_0^t v(t,t-s)g(t-s,w(t-s))dw_{t-s}$$

$$= \int\limits_0^T v(t,t-s)g_*(t-s,w(t-s))dw_{t-s}.$$

Here

$$g_*(t-s,w(t-s)) = \begin{cases} g_*(t-s,w(t-s)), 0 \le s \le t, \\ \\ 0, t-s \notin [0,T]. \end{cases}$$

Using the Minkowski inequality and estimate (34) and the estimate from Theorem 1.1 it follows that

$$J_2 \le E\int\limits_0^T Me^{-\delta s}\left(\int\limits_0^T \|g_*(t-s,w(t-s))\|_H^2 \left(dw_{t-s}\right)^2\right)^{\frac{1}{2}} ds$$

$$\le E\int\limits_0^T Me^{-\delta s}\left(\int\limits_0^T \|g(y,w(y))\|_H^2 \, dy\right)^{\frac{1}{2}} ds$$

$$= M_4\left(\delta\right)\left(E\int\limits_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

Combining the estimates for J_r for all $r = 1, 2$ and 3, we get (41). Theorem 3.1.4 is proved.

Applications

Now, consider the applications of the abstract theorems. First, we consider one dimensional stochastic parabolic equations

$$\begin{cases} du\left(t,x,w(t)\right)+\left(-\left(a\left(t,x\right)u_x\left(t,x,w(t)\right)\right)_x+\delta u\left(t,x,w(t)\right)\right)dt \\ \\ =f\left(t,x,w(t)\right)dt+g\left(t,x,w(t)\right)dw(t),0<t<T,x\in\left(0,l\right), \\ \\ u\left(0,x,w(0)\right)=\varphi\left(x\right),x\in\left[0,l\right], \\ \\ u\left(t,0,w(t)\right)=u\left(t,l,w(t)\right),u_x\left(t,0,w(t)\right)=u_x\left(t,l,w(t)\right),t\in\left[0,T\right] \end{cases} \tag{42}$$

with nonlocal conditions. Under compatibility conditions problem (42) has as weak unique solution $u(t,x,w(t))$ for the smooth in x functions $f(t,x,w(t))$ and $g\left(t,x,w(t)\right)$

$(t\in(0,T)\times\Pi\times(0,l))$, $\varphi(x),a\left(t,x\right)\geq a>0,x\in[0,l],a\left(t,l\right)=a\left(t,0\right),t\in(0,T)$.

Problem (42) can be written as the Cauchy problem (26) in a Hilbert space $H=L_2[0,l]$ with self-adjoint PD operator $A(t)=A^x(t)$ defined by the formula

$$A^x(t)v(x)=-\left(a(t,x)v_x(x)\right)_x+\delta v\left(x\right) \tag{43}$$

with the domain $D(A^x)=\left\{v\in W_2^2\left[0,l\right]:v\left(0\right)=v\left(l\right),v_x\left(0\right)=v_x\left(l\right)\right\}$.Here, the Sobolev space $W_2^2\left[0,l\right]$ is defined as the set of all functions $v(x)$ defined on $[0,l]$ such that $v(x)$ and the second order derivative function $v''(x)$ are all locally integrable in $L_2[0,l]$, equipped the norm

$$\|v\|_{W_2^2[0,l]}=\left(\int_0^l\left|v(x)\right|^2dx\right)^{\frac{1}{2}}+\left(\int_0^l\left|v'(x)\right|^2dx\right)^{\frac{1}{2}}+\left(\int_0^l\left|v''(x)\right|^2dx\right)^{\frac{1}{2}}.$$

Therefore the abstract theorems 1.1 and 1.1 permit to get the following result on the stability of problem (42).

Theorem 3.2.1 Suppose that

$$E||\varphi||_{L_2[0,l]}^2,E\int_0^T\|f(t,\cdot,w_t)\|_{L_2[0,l]}^2\,dt,E\int_0^T\|g(t,.,w_t)\|_{L_2[0,l]}^2\,dt<\infty. \tag{44}$$

Then, for the solution of problem (42) the following estimates hold

$$\max_{t\in[0,T]} E||u(t)||_{L_2[0,l]} \leq M \left[E||\varphi||_{L_2[0,l]}\right.$$

$$\left. +E\int_0^T ||f(y,w(y))||_{L_2[0,l]}dy + \left(E\int_0^T ||g(y,w(y))||^2_{L_2[0,l]}dy\right)^{\frac{1}{2}}\right],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2[0,l]}\,dt\right)^{\frac{1}{2}} \leq M(\delta)\left[\left(E\|\varphi\|^2_{L_2[0,l]}\right)^{\frac{1}{2}}\right.$$

$$\left. +\left(E\int_0^T \|f(y,w(y))\|^2_{L_2[0,l]}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2[0,l]}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof of Theorem 3.2.1 is based on the Theorems 3.1.3 and 3.1.4, on the self-adjointness and positivity of operator $A(t) = A^x(t)$ defined by the formula (43).

Second, we consider one dimensional stochastic parabolic equations

$$\begin{cases} du\,(t,x,w(t)) + \left(-\left(a\,(t,x)\,u_x\,(t,x,w(t))\right)_x + \delta u\,(t,x,w(t))\right)dt \\ -\beta\left(a(t,-x)u(t,-x,w(t)) + \delta u(t,-x,w(t)\right)dt \\ = f\,(t,x,w(t))\,dt + g\,(t,x,w(t))\,dw(t), 0<t<T, x\in(-l,l)\,, \\ u\,(0,x,w(0)) = \varphi\,(x)\,, x\in[-l,l]\,, \\ u\,(t,-l,w(t)) = u\,(t,l,w(t)) = 0, t\in[0,T] \end{cases} \tag{45}$$

with involution and Dirichlet conditions. Under compatibility conditions problem (45) has as weak unique solution $u(t,x,w(t))$ for the smooth in x functions $f(t,x,w(t))$ and $g\,(t,x,w(t))\,(t\in(0,T)\times\Pi\times(-l,l))$, $\varphi(x), a\,(t,x)\,, x\in[-l,l]$,

$a \geq a\,(t,x) = a\,(t,-x) \geq \delta > 0, \delta - a\,|\beta| \geq 0, t\in(0,T)\,.$

Problem (45) can be written as the Cauchy problem (26) in a Hilbert space

$H = L_2[-l,l]$ with self-adjoint PD operator $A(t) = A^x(t)$ defined by the formula

$$A(t)v(x) = -\left(a(t,x)v_x(x)\right)_x - \beta\left(a(t,-x)v_x\left(-x\right)\right)_x + \delta v\left(x\right) \tag{46}$$

with the domain $D(A(t)) = \{v \in W_2^2\left[-l,l\right] : v\left(-l\right) = v\left(l\right) = 0\}$ .Here, the Sobolev space $W_2^2\left[-l,l\right]$ is defined as the set of all functions $v(x)$ defined on $[-l,l]$ such that $v(x)$ and the second order derivative function $v''(x)$ are all locally integrable in $L_2[-l,l]$, equipped the norm

$$\|v\|_{W_2^2[-l,l]} = \left(\int_{-l}^{l} |v(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{-l}^{l} |v(x)''|^2\,dx\right)^{\frac{1}{2}}$$

We have that

Lemma 3.2.2 Let $a \geq a\left(t,x\right) = a\left(-x,t\right) \geq \sigma > 0$ and $\delta - a\left|\beta\right| \geq 0$. Then, the operator $A(t)$ defined by formula (46) is the self-adjoint and PD operator in $L_2[-l,l]$ space. Therefore the abstract theorems permit to get the following result on the stability of problem (45).

Theorem 3.2.3 Suppose that

$$E||\varphi||^2_{L_2[-l,l]}, E\int_0^T \|f(t,\cdot,w_t)\|^2_{L_2[-l,l]}\,dt, E\int_0^T \|g(t,.,w_t)\|^2_{L_2[-l,l]}\,dt < \infty. \tag{47}$$

Then, for the solution of problem (45) the following estimates hold

$$\max_{t\in[0,T]} E||u(t)||_{L_2[-l,l]} \leq M\left[E||\varphi||_{L_2[-l,l]}\right.$$

$$\left. +E\int_0^T ||f(y,w(y))||_{L_2[-l,l]}dy + \left(E\int_0^T ||g(y,w(y))||^2_{L_2[-l,l]}dy\right)^{\frac{1}{2}}\right],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2[-l,l]}\,dt\right)^{\frac{1}{2}} \leq M(\delta)\left[\left(E\|\varphi\|^2_{L_2[-l,l]}\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(E\int_0^T \|f(y,w(y))\|^2_{L_2[-l,l]}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2[-l,l]}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof of Theorem 3.2.3 is based on the Theorems 3.1.3 and 3.1.4 and Lemma 3.2.2 on the self-adjointness and positivity of operator $A(t) = A^x(t)$ defined by the formula (46).

Third, let Ω be the unit open cube in the n-dimensional Euclidean space

$\mathbb{R}^n = \{x = (x_1, \cdots, , x_n) : 0 < x_i < 1, i = 1, \cdots, n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0, T] \times \Omega$, the mixed problem for the multidimensional stochastic parabolic equation

$$\begin{cases} du(t,x,w(t)) + \left[-\sum\limits_{r=1}^{n} (a_r(t,x) u_{x_r})_{x_r} + \delta u(t,x,w(t)) \right] dt \\ = g(t,x,w(t))dt + f(t,x,w(t))dw_t, \\ \\ 0 < t < T, x = (x_1, \ldots, \ x_n) \in \Omega, \\ \\ u(0,x,w(0)) = \varphi(x), \ x \in \overline{\Omega}, \\ \\ u(t,x,w(t)) = 0, \ x \in S, t \in [0,T] \end{cases} \tag{48}$$

with the Dirichlet condition is considered. Under compatibility conditions problem (48) has as weak unique solution $u(t,x,w(t))$ for the smooth in x functions $f(t,x,w(t))$ and $g\,(t,x,w(t))\,(t \in (0,T) \times \Pi \times \Omega,\ \varphi(x) \in \overline{\Omega}, a\,(t,x) \geq \delta > 0, x \in (0,T) \times \Omega$.

Problem (48) can be written as the Cauchy problem (26) in a Hilbert space

$H = L_2\left(\overline{\Omega}\right)$ with self-adjoint-positive definite operator $A(t) = A^x(t)$ defined by the formula

$$A(t)v(x) = -\sum_{r=1}^{n} (a_r(t,x) v_{x_r})_{x_r} + \delta v(x) \tag{49}$$

with domain

$$D(A(t)) = \left\{ v(x) : v(x), (a_r(x) v_{x_r})_{x_r} \in L_2(\Omega), 1 \leq r \leq n, u(x) = 0, x \in S \right\}.$$

Theorem 3.2.4 Suppose that

$$E\|\varphi\|^2_{L_2(\Omega)}, E\int\limits_0^T \|f(t,\cdot,w_t)\|^2_{L_2(\Omega)}\,dt, E\int\limits_0^T \|g(t,.,w_t)\|^2_{L_2(\Omega)}\,dt < \infty. \tag{50}$$

Then, for the solution of problem (48) the following estimates hold

$$\max_{t\in[0,T]} E\|u(t)\|_{L_2(\Omega)} \leq M \left[E\|\varphi\|_{L_2(\Omega)} \right.$$

$$+E\int_0^T \|f(y,w(y))\|_{L_2(\Omega)}dy + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\Bigg],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2(\Omega)}\,dt\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\|\varphi\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(E\int_0^T \|f(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof of Theorem 3.2.4 is based on the Theorems 3.1.3 and 3.1.4, on the self-adjointness and positivity of operator $A(t) = A^x(t)$ defined by the formula (49) and the theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$.

Fourth, in $[0,T]\times\Omega$, the mixed problem for the multidimensional stochastic parabolic equation

$$\begin{cases} du(t,x,w(t)) + \left(-\sum\limits_{r=1}^{n}(a_r(t,x)u_{x_r})_{x_r} + \delta u(t,x,w(t))\right)dt \\ = g(t,x,w(t))dt + f(t,x,w(t))dw_t, \\ \\ 0<t<T, x=(x_1,\ldots,\ x_n)\in\Omega, \\ \\ u(0,x,w(0)) = \varphi(x),\ \ x\in\overline{\Omega}, \\ \\ \frac{\partial}{\partial\mu}u(t,x,w(t)) = 0,\ \ x\in S, t\in[0,T] \end{cases} \tag{51}$$

with the Neumann condition is considered. Here, μ is the normal vector to Ω. Under compatibility conditions problem (51) has as weak unique solution $u(t,x,w(t))$ for the smooth in x functions $f(t,x,w(t))$ and

$g\,(t,x,w(t))\,(t\in(0,T)\times\Pi\times\Omega,\ \varphi(x)\in\overline{\Omega},$

$a\,(t,x)\ge\delta>0, x\in(0,T)\times\Omega.$

Problem (51) can be written as the Cauchy problem (26) in a Hilbert space

$H = L_2\left(\overline{\Omega}\right)$ with self-adjoint positive definite operator $A(t) = A^x(t)$ defined by the formula

$$A(t)v(x) = -\sum_{r=1}^{n}(a_r(t,x)v_{x_r})_{x_r} + \delta v(x) \tag{52}$$

with domain

$$D(A(t)) = \{v(x) : v(x), (a_r(x)v_{x_r})_{x_r} \in L_2(\Omega), 1 \le r \le n, u(x) = 0, x \in S\}.$$

Theorem 3.2.5 Suppose that all assumptions of Theorem 3.1 are satisfied. Then, for the solution of problem (51) the following estimates hold

$$\max_{t\in[0,T]} E\|u(t)\|_{L_2(\Omega)} \le M \left[E\|\varphi\|_{L_2(\Omega)}\right.$$

$$\left. +E\int_0^T \|f(y,w(y))\|_{L_2(\Omega)}dy + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2(\Omega)}\,dt\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\|\varphi\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(E\int_0^T \|f(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof of Theorem 3.2.5 is based on the Theorems 3.1.3 and 3.1.4, on the self-adjointness and positivity of operator $A(t) = A^x(t)$ defined by the formula (52) and the theorem on coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$[35].

CHAPTER IV

The Single Step Stable Difference Scheme

Introduction

In this section, we consider the single-step DS's generated by an exact DS for the approximate solution of problem (26) and their applications.Namely 1/2-th and 3/2-th order of accuracy in t DS's generated by the single-step exact DS for the approximate solution of problem (26) for stochastic equation in Hilbert spaces are presented.The main theorems on convergence of these DS in Hilbert spaces are established. Applications, the convergence estimates for the solution of initial-boundary problems for stochastic parabolic equations with dependent coefficients are obtained.

Rothe Difference Scheme with the Standard Wiener Process

On the segment $[0,T]$, we consider the uniform grid space

$$[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N\tau = T\}$$

with step $\tau > 0$. Here N is a fixed positive integer. On the grid space $[0,T]_\tau$ we define the grid function $\{v(t_k)\}_{k=0}^N$. In the following theorem we consider the single-step exact DS for the solution of problem (26) on grid points $t_k \in [0,T]_\tau$.

Theorem 4.2.1 Let $v(t)$ of (33) be the solution of (26) at the grid points $t = t_k$. Then $\{v(t_k)\}_{k=0}^N$ is the solution of the initial-value problem for the first-order difference equation

$$\begin{cases} v(t_k) - v(t_{k-1}) + \big(I - v(t_k, t_{k-1})\big)v(t_{k-1}) \\ \\ = f_k + g_k, 1 \le k \le N, v(0) = \varphi, \end{cases} \tag{53}$$

where

$$f_k = \int_{t_{k-1}}^{t_k} v(t_k, s) f(s, w_s) ds, \qquad g_k = \int_{t_{k-1}}^{t_k} v(t_k, s) g(s, w_s) dw_s.$$

Proof. Putting in (33) $t = t_k,\ t_{k-1}$, we get

$$v(t_k) = \sum_{i=1}^{k} v(t_k, t_i) \int\limits_{t_{i-1}}^{t_i} v(t_i, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big),$$

$$v(t_{k-1}) = \sum_{i=1}^{k-1} v(t_{k-1}, t_i) \int\limits_{t_{i-1}}^{t_i} v(t_i, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big).$$

Multiplying the last equation by $v(t_k, t_{k-1})$ and using semigroup property of $v(t_k, s)$, we get

$$v(t_k, t_{k-1})v(t_{k-1}) = \sum_{i=1}^{k-1} v(t_k, t_i) \int_{t_{i-1}}^{t_i} v(t_i, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big).$$

Taking the difference, we get

$$v(t_k) - v(t_k, t_{k-1})v(t_{k-1})$$

$$= \int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big),\ 1 \le k \le N.$$

From that it follows (53).

Note that the solution $v(t)$ of problem (26) at the grid points t_k is the solution of difference problem (53). Therefore, the initial-value problem (53) is called the single-step exact DS for the solution of problem (26) on grid points $t_k \in [0, T]_\tau$.

Further, we will consider the applications of the exact difference scheme (53) for construction of single-step DS's in time for the approximate solutions of problem (26). From the mentioned DS (53) it is clear that for the approximate solutions of problem (26) it is necessary to approximate the expressions

$$v(t_k, t_{k-1}) \quad \text{and} \quad \int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big).$$

Now, we will present approximate formulas for these expressions. First of all let us establish a lemma which we will be need later.

Lemma 4.2.2 (A. Ashyralyev and P.E. Sobolevskii, 2004; P.E. Sobolevskii, 1965). For any $0 \le s \le p \le t \le T$ and $u \in D$ the following identities hold:

$$v(t, s)u = e^{-(t-s)A(s)}u$$

$$+ \int\limits_{s}^{t} v(t, z)[A(z) - A(s)]A^{-1}(s)e^{-(z-s)A(s)}A(s)udz, \tag{54}$$

$$v(t,s)u = e^{-(t-s)A(t)}u + \int\limits_s^t e^{-(t-z)A(t)}[A(z) - A(t)]v(z,s)udz. \tag{55}$$

According the Lemma 4.2.2, we get

$$v(t_k, t_{k-1})u = e^{-A_k\tau}u + o(\tau)$$

for all elements $u \in H$ and

$$v(t_k, t_{k-1})u = e^{-A_k\tau}u + o(\tau^2)$$

for all elements $u \in D$. It is easy to show that

$$e^{-A_k\tau}u = R(\tau A_k)u + o(\tau) \tag{56}$$

for all elements $u \in D$ and

$$e^{-A_k\tau}u = R(\tau A_k)u + o(\tau^2)$$

for all elements $u \in D(A_k^2)$. Therefore,

$$v(t_k, t_{k-1})u = R(\tau A_k)u + o(\tau) \tag{57}$$

for all elements $u \in D$ and

$$v(t_k, t_{k-1})u = R(\tau A_k)u + o(\tau^2)$$

for all elements $u \in D(A_k^2)$. Here $R(\tau A_k) = \big(I + \tau A_k\big)^{-1}$. In the future, we will put $u_\tau(k, k-1) = (I + \tau A_k)^{-1} = R(\tau A_k)$.

Now, we consider the expression $\int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big)$. We will present the approximate formula for the expression $v(t_k, s)$ for all $t_{k-1} \leq s \leq t_k$. Applying formula (55) and putting $t = t_k$, we get

$$v(t_k, s) = e^{-A(s)(t_k - s)} + \int\limits_s^{t_k} e^{-A(s)(t_k - p)}\big[A(s) - A(p)\big]v(p, s)dp.$$

Therefore,

$$v(t_k, s)u = e^{-A(s)(t_k - s)}u + o(\tau)$$

for all elements $u \in H$ and

$$v(t_k, s)u = e^{-A(s)(t_k - s)}u + o(\tau^2)$$

for all elements $u \in D$. We have that

$$e^{-A(s)(t_k - s)} u = e^{-A_k \tau} u + o(\tau) \tag{58}$$

for all elements $u \in D$. Therefore,

$$v(t_k, s)u = R(\tau A_k)u + o(\tau)$$

for all elements $u \in D$ and $t_{k-1} \leq s \leq t_k, 1 \leq k \leq N$. Thus

$$R(\tau A_k)\varphi_k = R(\tau A_k) \int_{t_{k-1}}^{t_k} \big(f(s, w_s)ds + g(s, w_s)dw_s\big)$$

is the approximation of the expression $\int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big)$

$$\int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big) = R(\tau A_k)\varphi_k + o(\tau^{\frac{1}{2}}).$$

Replacing $v(t_k, t_{k-1})$ by $R(\tau A_k) = u_\tau(k, k-1)$, $v(t_k)$ by u_k and elements f_k by elements $R(\tau A_k)\varphi_k$, we get RDS

$$u_k - u_{k-1} + \big(I - u_\tau(k, k-1)\big)u_{k-1} = R(\tau A_k)\varphi_k,\ 1 \leq k \leq N,\ u_0 = \varphi$$

for the approximate solution of (26). From the DS above it follows

$$u_k - u_{k-1} + \tau A_k u_k = \varphi_k,\ 1 \leq k \leq N,\ u_0 = \varphi. \tag{59}$$

It is clear that RDS (59) is uniquely solvable and the following formula holds

$$u_k = u_\tau(k, 0)\varphi + \sum_{i=1}^{k} u_\tau(k, i)R(\tau A_i)\varphi_i$$

$$= u_\tau(k, 0)\varphi + \sum_{i=1}^{k} u_\tau(k, i-1) \int_{t_{i-1}}^{t_i} \big(f(s, w_s)ds + g(s, w_s)dw_s\big), \tag{60}$$

where

$$u_\tau(k, i) = \begin{cases} u_\tau(k, k-1) \cdots u_\tau(i+1, i), & k > i, \\ I, & k = i. \end{cases}$$

The Main Theorem on Stability and Convergence

Now, we will investigate the convergence of RDS (59). Note that $u_\tau(k, i)$ is the approximation of $v(t_k, t_i)$. Therefore, we have that

Lemma 4.2.1 (A. Ashyralyev and P.E. Sobolevskii, 2004; P.E. Sobolevskii; 1977), For any $0 \leq t_i < t_k \leq T$ and $\alpha \in (0,1]$, the following estimates hold:

$$\|u_\tau(k,i)\|_{H\to H} \leq M, \tag{61}$$

$$\|A_k^\alpha u_\tau(k,i)A_i^{-\alpha}\|_{H\to H} \leq M, \tag{62}$$

$$\|A_k^\alpha u_\tau(k,i)\|_{H\to H} \leq M\frac{1}{((k-i)\tau)^\alpha}, \tag{63}$$

where M does not depend on τ, k and i.

Lemma 4.2.2 (A. Ashyralyev and P.E. Sobolevskii, 2004; P.E. Sobolevskii, 1975; P.E. Sobolevskii, 1971) For any $0 \leq t_{i-1} \leq s \leq t_i < t_k \leq T$, the following estimates hold:

$$\left\|\left[v(t_k,t_i) - u_\tau(k,i)\right]A_i^{-1}\right\|_{H\to H} \leq M\tau, \tag{64}$$

$$\left\|u_\tau(k,i)\big(v(t_i,s) - u_\tau(i,i-1)\big)A_i^{-1}\right\|_{H\to H} \leq M\tau, \tag{65}$$

where M does not depend on τ, k, s and i.

We have the following main theorem on stability of difference scheme (59).

Theorem 4.2.3 Suppose that

$$E\|\varphi\|_H < \infty, E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H < \infty, E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2 < \infty, \tag{66}$$

where $\varphi_i^1 = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} f(s,w_s)ds, \varphi_i^2 = \frac{1}{\sqrt{\tau}}\int_{t_{i-1}}^{t_i} g(s,w_s)dw_s$. Then, for the solution of DS (59) the following estimate holds

$$\max_{1\leq k\leq N} E\|u_k\|_H \leq M\left[E\|\varphi\|_H + E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H \tau + \left(E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2 \tau\right)^{\frac{1}{2}}\right]. \tag{67}$$

Proof. Using formula (60) and the triangle inequality, we get

$$\max_{1\leq k\leq N} E\|u_k\|_H \leq \max_{1\leq k\leq N} E\|u_\tau(k,0)\varphi\|_H$$

$$+\max_{1\leq k\leq N} E\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^1\|_H\tau + \max_{1\leq k\leq N} E\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H\tau$$

$$= P_1 + P_2 + P_3.$$

Here,

$$P_1 = \max_{1\le k\le N} E\|u_\tau(k,0)\varphi\|_H,$$

$$P_2 = \max_{1\le k\le N} E\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^1\|_H\tau,$$

$$P_3 = \max_{1\le k\le N} E\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H\tau.$$

We will estimate P_r for all $r = 1, 2, 3$, separately. We start with P_1. Applying estimate (61), we can write

$$P_1 = \max_{1\le k\le N}\|u_\tau(k,0)\|_{H\to H}E\|\varphi\|_H \le M_3 E\|\varphi\|_H.$$

Now let us estimate P_2. Using estimate (61), we get

$$P_2 \le \max_{1\le k\le N} E\sum_{i=1}^{k}\|u_\tau(k,i)\|_{H\to H}\|\varphi_i^1\|_H\tau \le M_4 E\sum_{i=1}^{N}\|\varphi_i^1\|_H\tau.$$

Finally, let us estimate P_3. Using estimate (61), we get

$$I_3 \le \max_{1\le k\le N} E\left(\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H\tau\right)^{\frac{1}{2}}$$

$$\le \max_{1\le k\le N} E\left(\sum_{i=1}^{k}\|u_\tau(k,i)\|_{H\to H}^2\left\|\varphi_i^2\right\|_H\tau^2\right)^{\frac{1}{2}} \le M_5 E\left(\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau\right)^{\frac{1}{2}}.$$

Combining the estimates for P_r for all $r = 1, 2$ and 3, we get (67). Theorem 4.2.3 is established.

Theorem 4.2.4 Suppose that

$$E\|\varphi\|_H^2, E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H^2\tau, E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau < \infty. \tag{68}$$

Then, for the solution of DS (59) the following estimate holds

$$\left(E\sum_{i=1}^{N}\|u_k\|_H\tau\right)^{\frac{2}{2}} \le M\left[(E\|\varphi\|_H)^{\frac{1}{2}}\right.$$

$$\left.+\left(E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H^2\tau\right)^{\frac{1}{2}} + \left(E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau\right)^{\frac{1}{2}}\right]. \tag{69}$$

Proof. Using formula (60) and the triangle inequality, we get

$$\left(E\sum_{i=1}^{N}\|u_k\|_H\tau\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\sum_{i=1}^{N}\|u_\tau(k,0)\varphi\|_H^2\tau\right)^{\frac{1}{2}}\right. \tag{70}$$

$$\left.+\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^1\|_H^2\tau\right)^{\frac{1}{2}}+\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H^2\tau\right)^{\frac{1}{2}}\right].$$

$$= R_1 + R_2 + R_3.$$

Here,

$$R_1 = \left(E\sum_{i=1}^{N}\|u_\tau(k,0)\varphi\|_H^2\tau\right)^{\frac{1}{2}},$$

$$R_2 = \left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^1\|_H^2\tau\right)^{\frac{1}{2}},$$

$$R_3 = \left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H^2\tau\right)^{\frac{1}{2}}.$$

We will estimate R_r for all $r = 1, 2, 3$, separately. We start with R_1. Applying estimate (34), we can write

$$R_1 \le \left(E\sum_{i=1}^{N}\|u_\tau(k,0)\|_{H\to H}^2\|\varphi\|_H^2\tau\right)^{\frac{1}{2}} \le M_1 TE\left(\|\varphi\|_H^2\right)^{\frac{1}{2}}.$$

Now let us estimate R_2. Making the substitution $m = k - i + 1$, we get

$$\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^1\|_H^2\tau = \sum_{m=1}^{k}\|u_\tau(k,k-m+1)\varphi_{k-m+1}^1\|_H^2\tau$$

$$= \sum_{m=1}^{N}\|u_\tau(k,k-m+1)\varphi_{k-m+1}^{*1}\|_H^2\tau.$$

Here

$$\varphi_{k-m+1}^{*1} = \begin{cases} \varphi_{k-m+1}^1, 1 \le m \le k, \\ \\ 0, (k-m)\tau \notin [0,T]_\tau. \end{cases}$$

Using the Minkowski inequality and estimate (61), we get

$$R_2 \le E\sum_{m=1}^{N}Me^{-\delta m\tau}\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{*1}\right\|_H^2\tau\right)^{\frac{1}{2}}\tau$$

$$\leq MTE\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{1}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}.$$

Finally, let us estimate R_3. Making the substitution $m = k - i + 1$, we get

$$\sum_{i=1}^{k}\|u_\tau(k,i)\varphi_i^2\|_H^2\tau = \sum_{m=1}^{k}\|u_\tau(k,k-m+1)\varphi_{k-m+1}^2\|_H^2\tau$$

$$= \sum_{m=1}^{N}\|u_\tau(k,k-m+1)\varphi_{k-m+1}^{*2}\|_H^2\tau.$$

Here

$$\varphi_{k-m+1}^{*2} = \begin{cases} \varphi_{k-m+1}^2, 1 \leq m \leq k, \\ \\ 0, (k-m)\,\tau \notin [0,T]_\tau . \end{cases}$$

Using the Minkowski inequality and estimate (61), we get

$$R_3 \leq E\sum_{m=1}^{N} Me^{-\delta m\tau}\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{*2}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}\tau$$

$$\leq MTE\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{2}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}.$$

Combining the estimates for R_r for all $r = 1, 2$ and 3, we get (69). Theorem 4.2.4 is proved.

We say that the difference problem has an m-th order of accuracy on the solution $v(t)$ of the Cauchy problem (26), if the error vector

$$\{v(t_k) - u_k\}_{k=0}^{N}$$

satisfies the estimate

$$\left(\sum_{k=0}^{N} E\left\|v(t_k) - u_k\right\|_H^2\right)^{\frac{1}{2}} \leq M\tau^m, \tag{71}$$

where M does not dependent on τ.

The estimate of convergence for the solution of the DS (59) is shown in the following main theorem.

Theorem 4.2.5 Assume that

$$E\left\|A(0)\varphi\right\|_H^2, E\int_0^T \left\|A(t)f(t,w_t)\right\|_H^2 dt, E\int_0^T \left\|A(t)g(t,w_t)\right\|_H^2 dt < \infty, \tag{72}$$

then the following convergence estimate is valid

$$\left(\sum_{k=0}^{N} E\left\|v(t_k)-u_k\right\|_H^2\right)^{\frac{1}{2}} \leq \tilde{M}(\delta)\tau^{\frac{1}{2}}. \tag{73}$$

Here, M and $\tilde{M}(\delta)$ do not depend on τ.

Proof. Using formulas (60) and (27), we get

$$v(t_k)-u_k=\sum_{i=1}^{k-1}\left[v(t_k,t_i)-u_\tau(k,i)\right]\int_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big) \tag{74}$$

$$+\sum_{i=1}^{k} u_\tau(k,i)\int_{t_{i-1}}^{t_i}\left[v(t_i,s)-R(\tau A_i)\right]\big(f(s,w_s)ds+g(s,w_s)dw_s\big)=D_{1,k}+D_{2,k},$$

where

$$D_{1,k} = \sum_{i=1}^{k-1}\left[v(t_k,t_i)-u_\tau(k,i)\right]\int_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big),$$

$$D_{2,k} = \sum_{i=1}^{k} u_\tau(k,i)\int_{t_{i-1}}^{t_i}\left[v(t_i,s)-R(\tau A_i)\right]\big(f(s,w_s)ds+g(s,w_s)dw_s\big).$$

We will estimate $\left(\sum\limits_{k=0}^{N} E\|D_{r,k}\|_H^2\right)^{\frac{1}{2}}$ for $r=1$ and 2, separately. First, let us estimate $\left(\sum\limits_{k=0}^{N} E\left\|D_{1,k}\right\|_H^2\right)^{\frac{1}{2}}$. Using formula (60) , estimates (64), (35) and the triangle inequality, we obtain

$$\left(\sum_{k=0}^{N} E\left\|D_{1,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$=\Bigg(\sum_{k=0}^{N} E\Bigg\|\sum_{i=1}^{k-1}\big(v(t_k,t_i)-u_\tau(k,i)\big)A_i^{-1}$$

$$\times\int_{t_{i-1}}^{t_i} A_i v(t_i,s)A^{-1}(s)A(s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big)\Bigg\|_H^2\Bigg)^{\frac{1}{2}}$$

$$\leq\Bigg(\sum_{k=0}^{N}\sum_{i=1}^{k-1}\Big\|\big(v(t_k,t_i)-u_\tau(k,i)\big)A_i^{-1}\Big\|_{H\to H}^2$$

$$\times\int_{t_{i-1}}^{t_i}\Big\|A_i v(t_i,s)A^{-1}(s)\Big\|_{H\to H}^2 E\Big\|A(s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big)\Big\|_H^2 ds\Bigg)^{\frac{1}{2}}$$

$$\leq \quad M_0\tau\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}M_1^2\Big[\int\limits_{t_{i-1}}^{t_i}E\Big\|A(s)f(s,w_s)\Big\|_H^2ds+\int\limits_{t_{i-1}}^{t_i}E\Big\|A(s)g(s,w_s)\Big\|_H^2ds\Big]\Big)^{\frac{1}{2}}$$

$$\leq \quad M_0\tau M_1\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}\Big[\int\limits_{t_{i-1}}^{t_i}E\Big\|A(s)f(s,w_s)\Big\|_H^2ds+\int\limits_{t_{i-1}}^{t_i}E\Big\|A(s)g(s,w_s)\Big\|_H^2ds\Big]\Big)^{\frac{1}{2}}$$

$$\leq \quad M_2\tau\Big(\sum_{k=0}^{N}\Big[\int\limits_{0}^{T}E\Big\|A(s)f(s,w_s)\Big\|_H^2ds+\int\limits_{0}^{T}E\Big\|A(s)g(s,w_s)\Big\|_H^2ds\Big]\Big)^{\frac{1}{2}}$$

$$\leq \quad M_2M\tau^{\frac{1}{2}}.$$

Second, let us estimate $\left(\sum\limits_{k=0}^{N}E\left\|D_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$. Using the triangle inequality, formula (60) and estimates (65), we get

$$\left(\sum_{k=0}^{N}E\left\|D_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$=\Big(\sum_{k=0}^{N}E\Big\|\sum_{i=1}^{k}u_\tau(k,i)\int_{t_{i-1}}^{t_i}\big(v(t_i,s)-u_\tau(i,i-1)\big)A^{-1}(s)A(s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big)\Big\|_H^2\Big)^{\frac{1}{2}}$$

$$\leq \quad \Big(\sum_{k=0}^{N}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\Big\|u_\tau(k,i)\big(v(t_i,s)-u_\tau(i,i-1)\big)A^{-1}(s)\Big\|_{H\to H}^2$$

$$\times E\Big\|A(s)\big(f(s,w_s)ds+g(s,w_s)dw_s\big)\Big\|_H^2ds\Big)^{\frac{1}{2}}$$

$$\leq \quad \Big(\sum_{k=0}^{N}\sum_{i=1}^{k}M_3^2\tau^2\Big[\int_{t_{i-1}}^{t_i}E\Big\|A(s)f(s,w_s)\Big\|_H^2ds+\int\limits_{t_{i-1}}^{t_i}E\Big\|A(s)g(s,w_s)\Big\|_H^2ds\Big]\Big)^{\frac{1}{2}}$$

$$\leq \quad M_4\tau\Big(\sum_{k=0}^{N}\Big[\int\limits_{0}^{T}E\Big\|A(s)f(s,w_s)\Big\|_H^2ds+\int\limits_{0}^{T}E\Big\|A(s)g(s,w_s)\Big\|_H^2ds\Big]\Big)^{\frac{1}{2}}$$

$$\leq \quad M_4M\tau^{\frac{1}{2}}.$$

Then combining both estimates, we get

$$\left(\sum_{k=0}^{N}E\left\|D_{1,k}+D_{2,k}\right\|_H^2\right)^{\frac{1}{2}}\leq\tilde{M}(\delta)\tau^{\frac{1}{2}}.$$

From that it follows (73) . Theorem 4.2.5 is proved.

Applications

We consider the applications of Theorem 4.2.5 to stochastic parabolic equations. First, let us consider the IBVP(26) for one dimensional stochastic parabolic equation with nonlocal conditions.

The discretization of problem (26) is carried out in two steps. In the first step, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh,\ 0 \le n \le M,\ Mh = l\}.$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [0,l]_h} |\varphi(x)|^2 h \right)^{1/2}.$$

To the differential operator $A^x(t)$ generated by problem (1), we assign the difference operator $A_h^x(t)$ by the formula

$$A_h^x(t)\varphi^h(x) = \{-(a(t,x)\varphi_{\overline{x}})_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{75}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well-known that $A_h^x(t)$ is a self-adjoint PD operator in L_{2h}. With the help of $A_h^x(t)$, we arrive at the initial value problem

$$\begin{cases} du^h(t,x,w_t) + A_h^x(t)u^h(t,x,w_t)dt \\ = f^h(t,x,w_t)dt + g^h(t,x,w_t)dw_t,\ 0 < t < T,\ x \in [0,l]_h, \\ u^h(0,x,0) = \varphi^h(x), x \in [0,l]_h \end{cases} \tag{76}$$

for the stochastic ordinary differential equation. In the second step, we replace (76) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) = R(\tau A_{h,k}^x)\varphi_k^h, \\ \varphi_k^h = \int_{t_{k-1}}^{t_k} \left(f(s,w_s)ds + g(s,w_s)dw_s\right), \\ A_{h,k}^x = A_h^x(t_k),\ t_k = k\tau,\ 1 \le k \le N,\ x \in [0,l]_h, \\ u_0^h(x) = \varphi^h(x),\ x \in [0,l]_h. \end{cases} \tag{77}$$

Theorem 4.2.2.1 Assume that

$$E\left\|\varphi\right\|^2_{W_2^4[0,l]}, E\int\limits_0^T \left\|f(t,w_t)\right\|^2_{W_2^4[0,l]}dt, E\int\limits_0^T \left\|g(t,w_t)\right\|^2_{W_2^4[0,l]}dt < \infty. \tag{78}$$

Then, the solutions of DS (77) satisfy the following convergence estimate:

$$\left(\sum_{k=0}^{N} E\left\|v^h(t_k)-u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \leq C(\delta)\left(\tau^{\frac{1}{2}}+h\right),$$

where $C(\delta)$ do not depend on τ and h.

The proof of Theorem 4.2.2.1 is based on the abstract Theorem 4.2.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (75).

Second, let us consider the IBVP (45) for one dimensional stochastic parabolic equation with involution and Dirichlet conditions.

The discretization of problem (45) is carried out in two steps. In the first step, we define the grid space

$$[-l,l]_h = \{x = x_n : x_n = nh, -M \leq n \leq M,\ Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([-l,l]_h)$ and $W_{2h}^2 = W_2^2([-l,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_j\}_{-M}^{M}$ defined on $[-l,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[-l,l]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[-l,l]_h} \left|\left(\varphi^h\right)_{x\overline{x},j}\right|^2 h\right)^{1/2},$$

respectively. To the differential operator $A^x(t)$ generated by problem (45), we assign the difference operator $A_h^x(t)$ by the formula

$$A_h^x(t)\varphi^h(x) = \{-(a(t,x)\varphi_{\overline{x}})_{x,n} - \beta(a(t,-x)\varphi_{\overline{x}})_{x,n} + \delta\varphi_n\}_{-M+1}^{M-1} \tag{79}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_{-M}^{M}$ satisfying the conditions $\varphi_{-M} = \varphi_M = 0$. It is well-known that $A_h^x(t)$ is a self-adjoint PD operator in L_{2h}.

With the help of $A_h^x(t)$, we arrive at the initial value problem

$$\begin{cases} du^h(t,x,w_t) + A_h^x(t)u^h(t,x,w_t)dt \\ = f^h(t,x,w_t)dt + g^h(t,x,w_t)dw_t,\ 0<t<T,\ x\in[0,l]_h, \\ u^h(0,x,0)=\varphi^h(x), x\in[0,l]_h \end{cases} \tag{80}$$

for the stochastic ordinary differential equation. In the second step, we replace (80) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) = R(\tau A_{h,k}^x)\varphi_k^h, \\ \varphi_k^h = \int_{t_{k-1}}^{t_k} \big(f(s,w_s)ds + g(s,w_s)dw_s\big), \\ A_{h,k}^x = A_h^x(t_k),\ t_k = k\tau,\ 1\le k\le N,\ x\in[-l,l]_h, \\ u_0^h(x) = \varphi^h(x),\ x\in[-l,l]_h. \end{cases} \tag{81}$$

Theorem 4.2.2.2 Assume that

$$E\left\|\varphi\right\|^2_{W_2^4[-l,l]}, E\int_0^T \left\|f(t,w_t)\right\|^2_{W_2^4[-l,l]}dt, E\int_0^T \left\|g(t,w_t)\right\|^2_{W_2^4[-l,l]}dt < \infty. \tag{82}$$

Then, the solutions of DS (81) satisfy the following convergence estimate:

$$\left(\sum_{k=0}^N E\left\|v^h(t_k) - u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{1}{2}} + h\right),$$

where $C(\delta)$ do not depend on τ and h.

The proof of Theorem 4.2.2.2 is based on the abstract Theorem 4.2.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (79).

Third, the mixed problem (48) for the multidimensional stochastic parabolic equation with the Dirichlet condition is considered.

The discretization of problem (48) is carried out in two stages. In the first stage, let us define the grid sets

$$\overline{\Omega}_h = \{x = x_r = (h_1r_1, ..., h_nr_n),\ r=(r_1,...,r_n),$$

$$0 \le r_j \le N_j, \ h_j N_j = 1, \ j = 1, ..., n\},$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega, \ S_h = \overline{\Omega}_h \cap S.$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W_{2h}^r = W_2^r(\overline{\Omega}_h), r = 1, 2$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, ..., h_n r_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{n} \left|\left(\varphi^h\right)_{x_r, j_r}\right|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{n} \left|\left(\varphi^h\right)_{x_r \overline{x}_r, j_r}\right|^2 h_1 \cdots h_n\right)^{1/2},$$

respectively. To the differential operator $A(t)$ generated by problem (48), we assign the difference operator $A_h^x(t)$ by the formula

$$A_h^x(t) u^h(t, x) = -\sum_{r=1}^{n} \left(a_r(t, x) u_{\overline{x}_r}^h\right)_{x_r, j_r}, \tag{83}$$

where the difference operator $A_h^x(t)$ is defined on those grid functions $u^h(x) = 0$, for all $x \in S_h$. It is well-known that $A_h^x(t)$ is a self-adjoint PD operator in L_{2h}. Using (48) and (83), we get the following initial-value problem

$$\begin{cases} du^h(t, x, w_t) + A_h^x(t) u^h(t, x, w_t) dt = f^h(t, x, w_t) dt + g^h(t, x, w_t) dw_t, \\ 0 < t < T, \ x \in \Omega_h, \\ u^h(0, x, 0) = \varphi^h(x), \ x \in \widetilde{\Omega}_h \end{cases} \tag{84}$$

for the stochastic ordinary differential equation. In the second step, we replace

(84) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) = R(\tau A_{h,k}^x)\varphi_k^h, \\ \\ \varphi_k^h = \int_{t_{k-1}}^{t_k} \big(f(s,w_s)ds + g(s,w_s)dw_s\big), \\ \\ A_{h,k}^x = A_h^x(t_k),\ t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \\ \\ u_0^h(x) = \varphi^h(x), \quad x \in \widetilde{\Omega}_h. \end{cases} \tag{85}$$

Theorem 4.2.2.3 Assume that

$$E\Big\|\varphi\Big\|^2_{W_2^4(\Omega)}, E\int_0^T \Big\|f(t,w_t)\Big\|^2_{W_2^4(\Omega)} dt, E\int_0^T \Big\|g(t,w_t)\Big\|^2_{W_2^4(\Omega)} dt < \infty. \tag{86}$$

Then, the solution of DS (85) satisfy the following convergence estimate:

$$\left(\sum_{k=0}^{N} E\left\|v^h(t_k) - u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{1}{2}} + |h|^2\right), \tag{87}$$

where $C(\delta)$ do not depend on τ and $|h|$.

The proof of Theorem 4.2.2.3 is based on the abstract Theorem 4.2.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (83) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Fourth, the mixed problem (51) for the multidimensional stochastic parabolic equation with the Neumann condition is considered. The discretization of problem (51) is carried out in two steps. The differential operator $A^x(t)$ in (51) is replaced with

$$A_h^x(t)u^h(x) = -\sum_{r=1}^{n}\left(a_r(t,x)u^h_{\overline{x}_r}\right)_{x_r,j_r} + \delta u^h(x), \tag{88}$$

where the difference operator $A_h^x(t)$ is defined on those grid functions $D^h u^h(x) = 0$, for all $x \in S_h$, where $D^h u^h(x) = 0$ is the second order of approximation of $\frac{\partial u(t,x,w_t)}{\partial \vec{n}}$. It is well-known that $A_h^x(t)$ is a self-adjoint PD operator in L_{2h}.

Using (51) and (88), we get the following initial-value problem

$$\begin{cases} du^h(t,x,w_t) + A_h^x(t)u^h(t,x,w_t)dt = f^h(t,x,w_t)dt + g^h(t,x,w_t)dw_t, \\ 0 < t < T,\ x \in \Omega_h, \\ u^h(0,x,0) = \varphi^h(x),\ x \in \widetilde{\Omega}_h \end{cases} \tag{89}$$

for the stochastic ordinary differential equation. In the second step, we replace (89) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) = R(\tau A_{h,k}^x)\varphi_k^h, \\ \varphi_k^h = \int_{t_{k-1}}^{t_k} \big(f(s,w_s)ds + g(s,w_s)dw_s\big), \\ A_{h,k}^x = A_h^x(t_k),\ t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \\ u_0^h(x) = \varphi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{90}$$

Theorem 4.2.2.4 Assume that all assumptions of Theorem 4.2.2.3 are satisfied. Then, for the solution of (90) the estimate 4.2.2.3 holds.

The proof of Theorem 4.2.2.4 is based on the abstract Theorem 4.2.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (88) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}. (P.E. Sobolevskii, 1975)

Rothe Difference Scheme Without the Standard Wiener Process

Now, we consider problem (26), when the source term $f(t,w_t)$, $g(t,w_t)$ does not depend on stochastic noise term w_t. That means $f(t,w_t) = f(t)$, $g(t,w_t) = g(t)$ is an abstract function defined on the segment $[0,T]$ with values in H. Assume that the function $f(t)$, $g(t)$ is continuous and smooth. Replacing $R(\tau A_k)\varphi_k = R(\tau A_k)\int\limits_{t_{k-1}}^{t_k} f(s)ds + \int\limits_{t_{k-1}}^{t_k} g(s)dw_s$ by $\int\limits_{t_{k-1}}^{t_k} g(t_k)dw_s = g(t_k)(w_{t_k} - w_{t_{k-1}})$ and $\int\limits_{t_{k-1}}^{t_k} f(t_k)ds = f(t_k)(t_k - t_{k-1})$ in (59), we get RDS

$$u_k - u_{k-1} + \tau A_k u_k$$

$$= R(\tau A_k)\left[f(t_k)\tau + g(t_k)(w_{t_k} - w_{t_{k-1}})\right],\ 1 \le k \le N,\ u_0 = \varphi \tag{91}$$

for the approximate solution of the Cauchy problem

$$dv(t) = \big(-A(t)v(t) + f(t)\big)dt + g(t)dw_t, \qquad 0 < t < T,\ v(0) = \varphi \tag{92}$$

in a Hilbert space H with a self adjoint PD operator $A(t)$. It is clear that RDS (91) is uniquely solvable and the following formula holds

$$u_k = \sum_{i=1}^{k} u_\tau(k, i-1)\big(f(t_i)\tau + g(t_i)(w_{t_i} - w_{t_{i-1}})\big). \tag{93}$$

The Main Theorem on Stability and Convergence

The estimate of convergence for the solution of the DS (91) is shown in the following main theorem.

Theorem 4.3.1 Assume that

$$\|A(0)\varphi\|_H^2,\ \int_0^T \|f'(t)\|_H^2\, dt + \int_0^T \|A(t)f(t)\|_H^2\, dt,\ \int_0^T \|g'(t)\|_H^2\, dt + \int_0^T \|A(t)g(t)\|_H^2\, dt < \infty, \tag{94}$$

then the following convergence estimate is valid

$$\left(\sum_{k=0}^{N} E\,\|v(t_k) - u_k\|_H^2\right)^{\frac{1}{2}} \le \tilde{M}(\delta)\tau^{\frac{1}{2}}.$$

Here, M and $\tilde{M}_1(\delta)$ do not depend on τ.

Proof. Applying formulas (93) and (27), we get

$$v(t_k) - u_k = \sum_{i=1}^{k-1}\left[v(t_k, t_i) - u_\tau(k, i)\right]\int_{t_{i-1}}^{t_i} v(t_i, s)\big(f(s)ds + g(s)dw_s\big) \tag{95}$$

$$+\sum_{i=1}^{k} u_\tau(k, i)\Big\{\int_{t_{i-1}}^{t_i} v(t_i, s)\big(f(s)ds + g(s)dw_s\big)$$

$$-u_\tau(i, i-1)\big(f(t_i)\tau + g(t_i)(w_{t_i} - w_{t_{i-1}})\Big\}$$

$$= P_{1,k} + P_{2,k},$$

where

$$P_{1,k} = \sum_{i=1}^{k-1} \big(v(t_k,t_i) - u_\tau(k,i)\big) \int_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds + g(s)dw_s\big), \tag{96}$$

$$P_{2,k} = \sum_{i=1}^{k} u_\tau(k,i)\Big\{ \int_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds + g(s)dw_s\big)$$
$$-u_\tau(i,i-1)\big(f(t_i)\tau + g(t_i)(w_{t_i} - w_{t_{i-1}})\big)\Big\}. \tag{97}$$

We will estimate $\left(\sum\limits_{k=0}^{N} E\|P_{r,k}\|_H^2\right)^{\frac{1}{2}}$ for $r = 1$ and 2, separately. First, let us estimate $\left(\sum\limits_{k=0}^{N} E\left\|P_{1,k}\right\|_H^2\right)^{\frac{1}{2}}$. Using formula (93) , estimates (64), (35) and the triangle inequality, we obtain

$$\sum_{k=0}^{N} E\left\|P_{1,k}\right\|_H^2$$

$$\leq \sum_{k=0}^{N}\sum_{i=1}^{k-1} \left\|\big(v(t_k,t_i) - u_\tau(k,i)\big)A_i^{-1}\right\|_{H\to H}^2 \int_{t_{i-1}}^{t_i} \left\|A_i v(t_i,s)A^{-1}(s)\right\|_{H\to H}^2$$
$$\times\left\|A(s)\big(f(s)+g(s)\big)\right\|_H^2 ds$$

$$\leq M_0\tau^2 \sum_{k=0}^{N}\sum_{i=1}^{k-1} M_1^2 \int_{t_{i-1}}^{t_i} \left\|A(s)\big(f(s)+g(s)\big)\right\|_H^2 ds$$
$$\leq M_0\tau^2 M_1^2 \sum_{k=0}^{N}\sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \left\|A(s)\big(f(s)+g(s)\big)\right\|_H^2 ds$$
$$\leq M_2\tau^2 \sum_{k=0}^{N} \Big(\int_0^T \left\|A(s)f(s)\right\|_H^2 ds + \int_0^T \left\|A(s)g(s)\right\|_H^2 ds\Big) \leq M_2 M\tau.$$

Now, we estimate $\left(\sum\limits_{k=0}^{N} E\left\|P_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$. Then from formula (97) and Minkowski inequality, we obtain

$$\sum_{k=0}^{N} E\left\|P_{2,k}\right\|_H^2$$

$$\leq \sum_{k=0}^{N} E\Big\| \sum_{i=1}^{k} u_\tau(k,i) \int_{t_{i-1}}^{t_i} \big(v(t_i,s) - u_\tau(i,i-1)\big)A^{-1}(s)A(s)\big(f(s)ds + g(s)dw_s\big)\Big\|_H^2$$

$$+\sum_{k=0}^{N} E\Big\| \sum_{i=1}^{k} u_\tau(k,i-1) \int\limits_{t_{i-1}}^{t_i} \Big[\big(f(s)-f(t_{i-1})\big)ds$$

$$+\big(g(s)-g(t_{i-1})\big)dw_s\Big]\Big\|_H^2$$

$$\leq \sum_{k=0}^{N}\sum_{i=1}^{k} \Big\|u_\tau(k,i)\Big\|_{H\to H}^2 \int\limits_{t_{i-1}}^{t_i} \Big\|\big(v(t_i,s)-u_\tau(i,i-1)\big)A_s^{-\frac{1}{2}}\Big\|_{H\to H}^2$$

$$\times\Big\|A_s^{\frac{1}{2}}\big(f(s)+g(s)\big)\Big\|_H^2 ds$$

$$+\sum_{k=0}^{N}\sum_{i=1}^{k} \Big\|u_\tau(k,i-1)\Big\|_{H\to H} \int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s} \Big\|f'(z)+g'(z)\Big\|_H^2 dzds.$$

Then,

$$\left(\sum_{k=0}^{N} E\left\|P_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{N}\sum_{i=1}^{k} M_1M_2\tau^{\frac{1}{2}} \int\limits_{t_{i-1}}^{t_i} \left\|A(s)\big(f(s)+g(s)\big)\right\|_H^2 ds\right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=0}^{N}\sum_{i=1}^{k} M_3 \int\limits_{t_{i-1}}^{t_i} \Big\|f'(z)+g'(z)\Big\|_H^2 dz \int\limits_{t_{i-1}}^{t_i} ds\right)^{\frac{1}{2}}$$

$$\leq M_1M_2\tau^{\frac{1}{2}}\left(\int\limits_0^T \left\|A(s)f(s)\right\|_H^2 ds + \int\limits_0^T \left\|A(s)g(s)\right\|_H^2 ds\right)^{\frac{1}{2}}$$

$$+M_3\tau^{\frac{1}{2}}\left(\int\limits_0^T \Big\|f'(z)\Big\|_H^2 dz + \int\limits_0^T \Big\|g'(z)\Big\|_H^2 dz\right)^{\frac{1}{2}} \leq \tilde{M}\tau^{\frac{1}{2}}.$$

Then combining both estimates, we get

$$\left(\sum_{k=0}^{N} E\left\|P_{1,k}+P_{2,k}\right\|_H^2\right)^{\frac{1}{2}} \leq \tilde{M}(\delta)\tau^{\frac{1}{2}}.$$

Theorem 4.3.1 is proved.

Applications

Now, we consider applications of Theorem 4.3.1. First, let us consider the IBVP (1) for one dimensional stochastic parabolic equation with nonlocal conditions. The discretization of problem (1) is carried out in two steps. The first step is the same as in the previous case. With the help of $A_h^x(t)$, we arrive at the initial value problem

$$\begin{cases} du^h(t,x,w_t) + A_h^x(t)u^h(t,x,w_t)dt = f^h(t,x)dt + g^h(t,x)dw_t, \\ 0 < t < T,\ x \in [0,l]_h, \\ u^h(0,x,0) = \varphi^h\left(x\right), x \in [0,l]_h \end{cases} \tag{98}$$

for the stochastic ordinary differential equation. In the second step, we replace (98) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) \\ = R(\tau A_{h,k}^x)\left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1})\right], \\ f_k^h(x) = f^h(t_k,x),\ g_k^h(x) = g^h(t_k,x),\ A_{h,k}^x = A_h^x(t_k), \\ t_k = k\tau,\ 1 \le k \le N,\ x \in [0,l]_h, \\ u_0^h(x) = \varphi^h\left(x\right),\ x \in [0,l]_h. \end{cases} \tag{99}$$

Theorem 4.3.2.1 Assume that

$$\left\|\varphi\right\|_{W_2^4[0,l]}^2, \int_0^T \left\|f'(t)\right\|_{L_2[0,l]}^2 dt + \int_0^T \left\|f(t)\right\|_{W_2^4[0,l]}^2 dt < \infty, \tag{100}$$

$$\int_0^T \left\|g'(t)\right\|_{L_2[0,l]}^2 dt + \int_0^T \left\|g(t)\right\|_{W_2^4[0,l]}^2 dt < \infty. \tag{101}$$

Then, the solutions of DS (99) satisfy the following convergence estimate:

$$\max_{0\le k\le N} \left(E\left\|v^h(t_k) - u_k^h\right\|_{L_{2h}}^2\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{1}{2}} + h\right),$$

where $C(\delta)$ do not depend on τ and h.

The proof of Theorem 4.3.2.1 is based on the abstract Theorem 4.3.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (75).

Second, we consider the IBVP (99) for one dimensional stochastic parabolic equation with involution and Dirichlet conditions.

The discretization of problem (99) is carried out in two stages. The first step is the same as in the previous case. With the help of $A_h^x(t)$, we arrive at the following problem

$$\begin{cases} du_t^h(t,x,w(t)) + A_h^x(t)u^h(t,x,w(t))dt = f^h(t,x)dt + g^h(t,x)dw_t, \\ x \in [-l,l]_h, \ 0 < t < T, \\ u^h(0,x,w(0)) = \varphi^h(x), x \in [-l,l]_h. \end{cases} \tag{102}$$

In the second stage, we replace the differential equation (102) with a first order of accuracy DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_h^x(t)u_k^h(x) \\ = R(\tau A_{h,k}^x)\left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1})\right], \\ f_k^h(x) = f^h(t_k,x), \ g_k^h(x) = g^h(t_k,x), \\ x \in [-l,l]_h, \ 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in [-l,l]_h. \end{cases} \tag{103}$$

Theorem 4.3.2.2 Assume that

$$\left\|\varphi\right\|_{W_2^4[-l,l]}^2, E\int_0^T \left\|f(t,w_t)\right\|_{W_2^4[-l,l]}^2 dt, E\int_0^T \left\|g(t,w_t)\right\|_{W_2^4[-l,l]}^2 dt < \infty.$$

Let τ and h be sufficiently small numbers. For the solution of DS (103) the following convergence estimates hold

$$\left(\sum_{k=0}^N E\left\|v^h(t_k) - u_k^h\right\|_{C([0,T]_\tau, L_{2h})}^2\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{1}{2}} + |h|\right), \tag{104}$$

The proof of Theorem 4.3.2.2 is based on the abstract Theorem 4.3.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (79).

Third, the mixed problem (48) for the multidimensional stochastic parabolic equation with the Dirichlet condition is considered.

The first step is the same as in the previous case. With the help of $A_h^x(t)$, we arrive at the initial value problem

$$\begin{cases} du^h(t,x,w_t) + A_h^x(t)u^h(t,x,w_t)dt = f^h(t,x)dt + g^h(t,x)dw_t, \\ 0 < t < T,\ x \in \Omega_h, \\ u^h(0,x,0) = \varphi^h(x),\ x \in \widetilde{\Omega}_h \end{cases} \tag{105}$$

for the stochastic ordinary differential equation. In the second step, we replace (105) with the DS (91)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) \\ = R(\tau A_{h,k}^x)\left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1})\right], \\ f_k^h(x) = f^h(t_k,x),\ g_k^h(x) = g^h(t_k,x),\ A_{h,k}^x = A_h^x(t_k), \\ t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \\ u_0^h(x) = \varphi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{106}$$

Theorem 4.3.2.3 Assume that

$$\left\|\varphi\right\|_{W_2^4(\Omega)}^2, \int_0^T \left\|f'(t)\right\|_{L_2(\Omega)}^2 dt + \int_0^T \left\|f(t)\right\|_{W_2^4(\Omega)}^2 dt < \infty \tag{107}$$

and

$$\int_0^T \left\|g'(t)\right\|_{L_2(\Omega)}^2 dt + \int_0^T \left\|g(t)\right\|_{W_2^4(\Omega)}^2 dt < \infty. \tag{108}$$

Then, the solution of DS (106) satisfy the following convergence estimate:

$$\max_{0 \le k \le N}\left(E\left\|v^h(t_k) - u_k^h\right\|_{L_{2h}}^2\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{1}{2}} + |h|^2\right), \tag{109}$$

where $C(\delta)$ do not depend on τ and $|h|$.

The proof of Theorem 4.3.2.3 is based on the abstract Theorem 4.3.2.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (83) and the Theorem 4.2.2.4 on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Fourth, the mixed problem (51) for the multidimensional stochastic parabolic equation with the Neumann condition is considered. The discretization of problem (51) is carried out in two steps. The discretization of problem (51) in x is done in the same as above. Then, in the second step, we replace (105) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x u_k^h(x) \\ \\ = R(\tau A_{h,k}^x) \left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1}) \right], \\ \\ f_k^h(x) = f^h(t_k, x),\ g_k^h(x) = g^h(t_k, x), A_{h,k}^x = A_h^x(t_k), \\ \\ t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \\ \\ u_0^h(x) = \varphi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{110}$$

Theorem 4.3.2.4 Assume that the assumptions of Theorem 4.3.2.1 are satisfied. Then, for the solution of (110) the estimate (109) holds.

The proof of Theorem 4.3.2.4 is based on the abstract Theorem 4.3.2.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (88) and the Theorem 4.3.2.4 on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Crank-Nicholson Difference Scheme with the Standard Wiener Process

Let us consider again the expression $v(t_k, t_{k-1})$. Applying formula (4.2.2) and putting $t = t_k$, $s = t_{k-1}$, $p = t_{k-\frac{1}{2}} = (k - \frac{1}{2})\tau$, we get

$$v(t_k, t_{k-1}) = e^{-A_k\tau} + \int_{t_{k-1}}^{t_k} e^{-A_k(t_k - z)}\big[A(z) - A_k\big]v(z, t_{k-1})dz, \tag{111}$$

where $A_k = A(t_k - \frac{\tau}{2}), 1 \leq k \leq N$. Since

$$\int_{t_{k-1}}^{t_k} \big[t_{k-\frac{1}{2}} - z\big]dz = 0, \tag{112}$$

we have that

$$v(t_k, t_{k-1})u = e^{-A_k\tau}u + o(\tau^2)$$

for all elements $u \in D$ and

$$v(t_k, t_{k-1})u = e^{-A_k\tau}u + o(\tau^3)$$

for all elements $u \in D(A_k^2) \cap D(A_k')$.Therefore,

$$v(t_k, t_{k-1})u = B(\tau A_k)u + o(\tau^2)$$

for all elements $u \in D(A_k^2)$. Here $B(\tau A_k) = \big(I - \frac{\tau A_k}{2}\big)\big(I + \frac{\tau A_k}{2}\big)^{-1}$. We will put

$u_\tau(k, k-1) = \big(I - \frac{\tau A_k}{2}\big)\big(I + \frac{\tau A_k}{2}\big)^{-1} = B(\tau A_k)$.

Now, we consider again the expression

$f_k = \int\limits_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big)$.

We will present the approximate formula for the expression $v(t_k, s)$ for all $t_{k-1} \leq s \leq t_k$. First, we have that

$$v(t_k, s) = v(t_k, t_{k-\frac{1}{2}}) + \int\limits_{t_{k-\frac{1}{2}}}^{s} v(t_k, p)A(p)dp. \tag{113}$$

Applying the triangle inequality, estimates (34) and (36), we get

$$\Big\|\Big[v(t_k, s) - v(t_k, t_{k-\frac{1}{2}})\Big]\, A_k^{-1}\Big\|_{H\to H}$$

$$\leq \int_s^{t_k} \|v(t_k, p)\|_{H\to H}\, \big\|A(p)A_k^{-1}\big\|_{H\to H}\, dp \leq M_1\tau.$$

Therefore,

$$v(t_k,s)u = v(t_k,t_{k-\frac{1}{2}})u + o(\tau),$$

$$\int\limits_{t_{k-1}}^{t_k} v(t_k,s)dsu = \tau v(t_k,t_{k-\frac{1}{2}})u + o(\tau^2)$$

for all elements $u \in D$. Moreover, using formulas (112) and (113), we get

$$\int\limits_{t_{k-1}}^{t_k} v(t_k,s)ds = \tau v(t_k,t_{k-\frac{1}{2}}) + \int\limits_{t_{k-1}}^{t_k}\int\limits_{t_{k-\frac{1}{2}}}^{s} \left(v(t_k,p) - I\right) A(p)dpds$$

$$+ \int\limits_{t_{k-1}}^{t_k}\int\limits_{t_{k-\frac{1}{2}}}^{s} \left(A(p) - A_k\right) dpds.$$

Applying the triangle inequality, estimates (36) and condition (29), we get

$$\left\| \left[\int\limits_{t_{k-1}}^{t_k} v(t_k,s)ds - \tau v(t_k,t_{k-\frac{1}{2}}) \right] A_k^{-2} \right\|_{H\to H}$$

$$\leq \int\limits_{t_{k-1}}^{t_k} \int_s^{t_k} \left\| \left(v(t_k,p) - I\right) A^{-1}(p) \right\|_{H\to H} \left\| A^2(p) A_k^{-2} \right\|_{H\to H} dpds$$

$$+ \int\limits_{t_{k-1}}^{t_k}\int\limits_{t_{k-\frac{1}{2}}}^{s} \left\| \left(A(p) - A_k\right) A_k^{-2} \right\|_{H\to H} dpds$$

$$\leq \int\limits_{t_{k-1}}^{t_k} \int_s^{t_k} M(t_k - p)Mdpds + \int\limits_{t_{k-1}}^{t_k} \int_s^{t_k} M \left| t_{k-\frac{1}{2}} - p \right| Mdpds \leq M_1\tau^3.$$

Therefore,

$$\int\limits_{t_{k-1}}^{t_k} v(t_k,s)dsu = \tau v(t_k,t_{k-\frac{1}{2}})u + o(\tau^3) \tag{114}$$

for all elements $u \in D(A_k^2)$.

Second, using formula (**??**), we get

$$v(t_k,t_{k-\frac{1}{2}})u = e^{-A_k\frac{\tau}{2}}u + o(\tau^2) \tag{115}$$

for all elements $u \in D$. Third, we have that

$$e^{-A(s)(t_k-s)} - e^{-A_k(t_k-s)}$$

$$= \int_0^1 (t_k - s)e^{-pA(s)(t_k-s)}\big[-A(s) + A_k\big]e^{-(1-p)A_k\tau}dp.$$

Using estimate (29), (36) and (28), we get

$$\big\|\big[e^{-A(s)(t_k-s)} - e^{-A_k(t_k-s)}\big]A_k^{-1}\big\|_{H\to H}$$

$$\leq (t_k - s)\int_0^1 \big\|e^{-pA(s)(t_k-s)}\big\|_{H\to H}\big\|\big[-A(s) + A_k\big]A_k^{-1}\big\|_{H\to H}$$

$$\times \big\|e^{-(1-p)A_k(t_k-s)}\big\|_{H\to H}dp \leq M_1(t_k - s)\left|s - t_{k-\frac{1}{2}}\right| \leq M_3\tau^2.$$

From that it follows

$$e^{-A(s)(t_k-s)}u = e^{-A_k(t_k-s)}u + o(\tau^2) \tag{116}$$

for all elements $u \in D$. Fourth, we have that

$$\int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)}ds = A_k^{-1}\left(I - e^{-A_k\tau}\right),$$

$$\int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)}ds - \tau e^{-A_k\frac{\tau}{2}} = \int_{t_{k-1}}^{t_k}\big[e^{-A_k(t_k-s)} - e^{-A_k\frac{\tau}{2}}\big]ds.$$

Using estimate (116), we get

$$\left\|\int_{t_{k-1}}^{t_k} e^{-A_k(t_k-s)}ds - \tau e^{-A_k\frac{\tau}{2}}\right\|_{H\to H}$$

$$\leq \int_{t_{k-1}}^{t_k}\big\|e^{-A_k(t_k-s)} - e^{-A_k\frac{\tau}{2}}\big\|_{H\to H}ds \leq M_3\tau^2$$

for all elements $u \in D$. Therefore,

$$\int_{t_{k-1}}^{t_k} v(t_k, s)dsu = A_k^{-1}\left(I - e^{-A_k\tau}\right)u + o(\tau^3) \tag{117}$$

for all elements $u \in D(A_k^2)$.Then, applying formula (117), we get

$$\int_{t_{k-1}}^{t_k} v(t_k, s)dsu = \left(I + \frac{\tau}{2}A_k\right)^{-1}u + o(\tau^2) \tag{118}$$

for all elements $u \in D(A_k^2)$ and $t_{k-1} \le s \le t_k, 1 \le k \le N$. Thus

$$\left(I + \frac{\tau}{2} A_k\right)^{-1} \varphi_k$$

$$= \left(I + \frac{\tau}{2} A_k\right)^{-1} \int_{t_{k-1}}^{t_k} \big(f(s, w_s)ds + g(s, w_s)dw_s\big)$$

is the approximation of the expression $f_k = \int\limits_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big)$ such that

$$f_k = \left(I + \frac{\tau}{2} A_k\right)^{-1} \varphi_k + o(\tau^{\frac{5}{2}}).$$

Replacing $v(t_k, t_{k-1})$ by $B(\tau A_k) = u_\tau(k, k-1)$, $v(t_k)$ by u_k and expression $\int_{t_{k-1}}^{t_k} v(t_k, s)\big(f(s, w_s)ds + g(s, w_s)dw_s\big)$ by $\left(I + \frac{\tau}{2} A_k\right)^{-1} \varphi_k$, we get 3/2-th order of approximation CNDS

$$u_k - u_{k-1} + \big(I - u_\tau(k, k-1)\big)u_{k-1} = \left(I + \frac{\tau}{2} A_k\right)^{-1} \varphi_k,$$

$$1 \le k \le N,\ u_0 = \varphi$$

for the approximate solution of (26). From the above difference scheme it follows

$$u_k - u_{k-1} + \frac{\tau}{2} A_k (u_k + u_{k-1}) = \varphi_k,\ 1 \le k \le N,\ u_0 = \varphi. \tag{119}$$

It is clear that the DS (119) is uniquely solvable and the following formula holds

$$u_k = \sum_{i=1}^{k} u_\tau(k, i) \left(I + \frac{\tau}{2} A_i\right)^{-1} \varphi_i \tag{120}$$

$$= \sum_{i=1}^{k} u_\tau(k, i) \left(I + \frac{\tau}{2} A_i\right)^{-1} \int_{t_{i-1}}^{t_i} \big[f(s, w_s)ds + g(s, w_s)dw_s\big],$$

where

$$u_\tau(k, i) = \begin{cases} u_\tau(k, k-1) \cdots u_\tau(i+2, i+1), & k > i, \\ I, & k = i. \end{cases} \tag{121}$$

The Main Theorem on Stability

Now, we will investigate the convergence of the DS (119). Note that $u_\tau(k, i)$ is the approximation of $v(t_k, t_i)$. Therefore, we have that

Theorem 4.4.1.1 (A. Ashyralyev and P.E. Sobolevskii, 2004; P.E. Sobolevskii, 1978) For any $0 \leq t_i < t_k \leq T$ and $\alpha \in (0, \frac{1}{2}]$, the following estimates hold:

$$\begin{cases} \|u_\tau(k,i)\|_{H\to H} \leq M, \\ \|A_k^\alpha u_\tau(k,i) A_i^{-\alpha}\|_{H\to H} \leq M, \\ \|A_k^\alpha u_\tau(k,i) \left(I + \frac{\tau}{2}A_i\right)^{-1} \|_{H\to H} \leq M \frac{1}{((k-i)\tau)^\alpha} \end{cases}$$

where M does not depend on τ, k and i.

Theorem 4.4.1.2 (A. Ashyralyev and P.E. Sobolevskii, 2004; P.E. Sobolevskii, 1978)For any $0 \leq t_{i-1} \leq s \leq t_i < t_k \leq T$, the following estimates hold:

$$\left\| \left[v(t_k, t_i) - u_\tau(k,i) \right] A_i^{-2} \right\|_{H\to H} \leq M\tau^2, \tag{122}$$

$$\left\| u_\tau(k,i) \big(v(t_i, s) - \left(I + \frac{\tau}{2} A_i \right)^{-1} \big) A_i^{-2} \right\|_{H\to H} \leq M\tau^2, \tag{123}$$

where M does not depend on τ, k, s and i.

We have the following main theorem on stability of difference scheme (119).

Theorem 4.4.1.3 Suppose that

$$E\|\varphi\|_H, E\sum_{i=1}^N \left\|\varphi_i^1\right\|_H, E\sum_{i=1}^N \left\|\varphi_i^2\right\|_H^2 < \infty, \tag{124}$$

where $\varphi_i^1 = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} f(s, w_s)ds, \varphi_i^2 = \frac{1}{\sqrt{\tau}}\int_{t_{i-1}}^{t_i} g(s, w_s)dw_s$. *Then, for the* solution of DS (119) the following estimate holds

$$\max_{1\leq k\leq N} E\|u_k\|_H \leq M \left[E\|\varphi\|_H \right.$$

$$\left. + E\sum_{i=1}^N \left\|\varphi_i^1\right\|_H \tau + \left(E\sum_{i=1}^N \left\|\varphi_i^2\right\|_H^2 \tau \right)^{\frac{1}{2}} \right]. \tag{125}$$

Proof. Using formula (120) and the triangle inequality, we get

$$\max_{1\leq k\leq N} E\|u_k\|_H \leq \max_{1\leq k\leq N} E\|u_\tau(k,0)\varphi\|_H$$

$$+ \max_{1\leq k\leq N} E\sum_{i=1}^k \|u_\tau(k,i) \left(I + \frac{\tau}{2}A_i\right)^{-1} \varphi_i^1\|_H \tau + \max_{1\leq k\leq N} E\sum_{i=1}^k \|u_\tau(k,i) \left(I + \frac{\tau}{2}A_i\right)^{-1} \varphi_i^2\|_H \tau$$

$$= P_1 + P_2 + P_3.$$

Here,

$$P_1 = \max_{1\le k\le N} E\|u_\tau(k,0)\varphi\|_H,$$

$$P_2 = \max_{1\le k\le N} E\sum_{i=1}^{k} \|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^1\|_H\tau,$$

$$P_3 = \max_{1\le k\le N} E\sum_{i=1}^{k} \|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^2\|_H\tau.$$

We will estimate P_r for all $r=1,2,3$, separately. We start with P_1. Applying estimate (61), we can write

$$P_1 = \max_{1\le k\le N}\|u_\tau(k,0)\|_{H\to H}E\|\varphi\|_H \le M_3E\|\varphi\|_H.$$

Now let us estimate P_2. Using estimate (61), we get

$$P_2 \le \max_{1\le k\le N} E\sum_{i=1}^{k}\left\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\right\|_{H\to H}\|\varphi_i^1\|_H\tau \le M_4E\sum_{i=1}^{N}\|\varphi_i^1\|_H\tau.$$

Finally, let us estimate P_3. Using estimate (61), we get

$$I_3 \le \max_{1\le k\le N} E\left(\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^2\|_H\tau\right)^{\frac{1}{2}}$$

$$\le \max_{1\le k\le N} E\left(\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\|_{H\to H}^2\left\|\varphi_i^2\right\|_H\tau^2\right)^{\frac{1}{2}} \le M_5E\left(\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau\right)^{\frac{1}{2}}.$$

Combining the estimates for P_r for all $r=1,2$ and 3, we get (**??**.4.1.3. Theorem 4.4.1.3 is established.

Theorem 4.4.1.4 Suppose that

$$E\|\varphi\|_H^2, E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H^2\tau, E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau < \infty. \tag{126}$$

Then, for the solution of DS (119) the following estimate holds

$$\left(E\sum_{i=1}^{N}\|u_k\|_H\tau\right)^{\frac{2}{2}} \le M\left[(E\|\varphi\|_H)^{\frac{1}{2}}\right.$$

$$\left.+\left(E\sum_{i=1}^{N}\left\|\varphi_i^1\right\|_H^2\tau\right)^{\frac{1}{2}}+\left(E\sum_{i=1}^{N}\left\|\varphi_i^2\right\|_H^2\tau\right)^{\frac{1}{2}}\right]. \tag{127}$$

Proof. Using formula (120) and the triangle inequality, we get

$$\left(E\sum_{i=1}^{N}\|u_k\|_H\tau\right)^{\frac{1}{2}} \leq M(\delta)\left[\left(E\sum_{i=1}^{N}\|u_\tau(k,0)\varphi\|_H^2\tau\right)^{\frac{1}{2}}\right. \tag{128}$$

$$\left.+\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^1\|_H^2\tau\right)^{\frac{1}{2}}+\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^2\|_H^2\tau\right)^{\frac{1}{2}}\right]$$

$$=R_1+R_2+R_3.$$

Here,

$$R_1=\left(E\sum_{i=1}^{N}\|u_\tau(k,0)\varphi\|_H^2\tau\right)^{\frac{1}{2}},$$

$$R_2=\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^1\|_H^2\tau\right)^{\frac{1}{2}},$$

$$R_3=\left(E\sum_{i=1}^{N}\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^2\|_H^2\tau\right)^{\frac{1}{2}}.$$

We will estimate R_r for all $r=1,2,3$, separately. We start with R_1. Applying estimate (34), we can write

$$R_1\leq\left(E\sum_{i=1}^{N}\|u_\tau(k,0)\|_{H\to H}^2\|\varphi\|_H^2\tau\right)^{\frac{1}{2}}\leq M_1TE\left(\|\varphi\|_H^2\right)^{\frac{1}{2}}.$$

Now let us estimate R_2. Making the substitution $m=k-i+1$, we get

$$\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^1\|_H^2\tau=\sum_{m=1}^{k}\|u_\tau(k,k-m+1)\left(I+\frac{\tau}{2}A_m\right)^{-1}\varphi_{k-m+1}^1\|_H^2\tau$$

$$=\sum_{m=1}^{N}\|u_\tau(k,k-m+1)\left(I+\frac{\tau}{2}A_m\right)^{-1}\varphi_{k-m+1}^{*1}\|_H^2\tau.$$

Here

$$\varphi_{k-m+1}^{*1}=\begin{cases}\varphi_{k-m+1}^1, 1\leq m\leq k,\\ \\ 0, (k-m)\tau\notin[0,T]_\tau.\end{cases}$$

Using the Minkowski inequality and estimate (61), we get

$$R_2\leq E\sum_{m=1}^{N}M\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{*1}\right\|_H^2\tau\right)^{\frac{1}{2}}\tau$$

$$\leq MTE\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{1}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}.$$

Finally, let us estimate R_3. Making the substitution $m=k-i+1$, we get

$$\sum_{i=1}^{k}\|u_\tau(k,i)\left(I+\frac{\tau}{2}A_i\right)^{-1}\varphi_i^2\|_H^2\tau=\sum_{m=1}^{k}\|u_\tau(k,k-m+1)\left(I+\frac{\tau}{2}A_m\right)^{-1}\varphi_{k-m+1}^2\|_H^2\tau$$

$$=\sum_{m=1}^{N}\|u_\tau(k,k-m+1)\left(I+\frac{\tau}{2}A_m\right)^{-1}\varphi_{k-m+1}^{*2}\|_H^2\tau.$$

Here

$$\varphi_{k-m+1}^{*2}=\begin{cases}\varphi_{k-m+1}^{2},1\leq m\leq k,\\ \\ 0,(k-m)\,\tau\notin[0,T]_\tau\,.\end{cases}$$

Using the Minkowski inequality and estimate (61), we get

$$R_3\leq E\sum_{m=1}^{N}M\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{*2}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}\tau$$

$$\leq MTE\left(\sum_{m=1}^{N}\left\|\varphi_{k-m+1}^{2}\right\|_{H}^{2}\tau\right)^{\frac{1}{2}}.$$

Combining the estimates for R_r for all $r=1,2$ and 3, we get (127). Theorem 4.4.1.4 is proved.

The estimate of convergence for the solution of the DS (119) is shown in the following main theorem.

Theorem 4.4.1.5 Assume that

$$\left\|A^2(0)\varphi\right\|_H^2, E\int_0^T\left\|A^2(t)f(t,w_t)\right\|_H^2dt, E\int_0^T\left\|A^2(t)g(t,w_t)\right\|_H^2dt<\infty, \tag{129}$$

then the following convergence estimate is valid

$$\left(\sum_{k=0}^{N}E\left\|v(t_k)-u_k\right\|_H^2\right)^{\frac{1}{2}}\leq\tilde{M}(\delta)\tau^{\frac{3}{2}}. \tag{130}$$

Here, M and $\tilde{M}(\delta)$ do not depend on τ.

Proof. Using formulas (33) and (120), we get

$$v(t_k) - u_k = \sum_{i=1}^{k-1} \big[v(t_k,t_i) - u_\tau(k,i)\big] \int_{t_{i-1}}^{t_i} v(t_i,s)\big[f(s,w_s)ds + g(s,w_s)dw_s\big]$$

$$+\sum_{i=1}^{k} u_\tau(k,i) \int_{t_{i-1}}^{t_i} \Big[v(t_i,s) - \Big(I+\frac{\tau}{2}A_i\Big)^{-1}\Big]\big[f(s,w_s)ds + g(s,w_s)dw_s\big] \tag{131}$$

$$= G_{1,k} + G_{2,k},$$

where

$$G_{1,k} = \sum_{i=1}^{k-1} \big[v(t_k,t_i) - u_\tau(k,i)\big] \int_{t_{i-1}}^{t_i} v(t_i,s) \big[f(s,w_s)ds + g(s,w_s)dw_s\big], \tag{132}$$

$$G_{2,k} = \sum_{i=1}^{k} u_\tau(k,i) \int_{t_{i-1}}^{t_i} \Big[v(t_i,s) - \Big(I+\frac{\tau}{2}A_i\Big)^{-1}\Big] \times\big[f(s,w_s)ds + g(s,w_s)dw_s\big]. \tag{133}$$

We will estimate $\left(\sum\limits_{k=0}^{N} E\|G_{r,k}\|_H^2\right)^{\frac{1}{2}}$ for $r=1$ and 2, separately. First, let us estimate

$$\left(\sum_{k=0}^{N} E\left\|G_{1,k}\right\|_H^2\right)^{\frac{1}{2}}.$$

Using formula (132), estimates (122), (35) and the triangle inequality, we obtain

$$\left(\sum_{k=0}^{N} E\left\|G_{1,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$\leq \Bigg(\sum_{k=0}^{N}\sum_{i=1}^{k-1} \Big\|\big(v(t_k,t_i) - u_\tau(k,i)\big)A_i^{-2}\Big\|_{H\to H}^2 \int_{t_{i-1}}^{t_i} \Big\|A_i^2 v(t_i,s)A^{-2}(s)\Big\|_{H\to H}^2 E\Big\|A^2(s) \times\big[f(s,w_s)ds + g(s,w_s)dw_s\big]\Big\|_H^2 ds\Bigg)^{\frac{1}{2}}$$

$$\leq \; M_0\tau^2\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}M_1^2\int\limits_{t_{i-1}}^{t_i}E\Big\|A^2(s)\big[f(s,w_s)ds+g(s,w_s)dw_s\big]\Big\|_H^2 ds\Big)^{\frac{1}{2}}$$

$$\leq \; M_0\tau^2 M_1\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}\int\limits_{t_{i-1}}^{t_i}E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq \; M_2\tau^2\Big(\sum_{k=0}^{N}\int\limits_{0}^{T}E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq \; M_2M\tau^{\frac{3}{2}}.$$

Second, let us estimate $\left(\sum\limits_{k=0}^{N}E\left\|G_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$. Using the triangle inequality, formula (133), and estimates (123), we get

$$\left(\sum_{k=0}^{N}E\left\|G_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{N}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\Big\|u_\tau(k,i)\big(v(t_i,s)-\tau e^{-A_i\frac{\tau}{2}}\big)A^{-2}(s)\Big\|_{H\to H}^2\right.$$

$$\left.\times E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\right)^{\frac{1}{2}}$$

$$+\Big(\sum_{k=0}^{N}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\Big\|u_\tau(k,i)\big(e^{-A_i\frac{\tau}{2}}-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\big)A^{-2}(s)\Big\|_{H\to H}^2$$

$$\times E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{N}\sum_{i=1}^{k}M_3^2\tau^4\int_{t_{i-1}}^{t_i}E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\right)^{\frac{1}{2}}$$

$$+\Big(\sum_{k=0}^{N}\sum_{i=1}^{k}M_{31}^2\tau^4\int_{t_{i-1}}^{t_i}E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq \; M_4\tau^2\Big(\sum_{k=0}^{N}\int\limits_{0}^{T}E\Big[\Big\|A^2(s)f(s,w_s)\Big\|_H^2+\Big\|A^2(s)g(s,w_s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq \; M_4M\tau^{\frac{3}{2}}.$$

Then combining both estimates, we get

$$\left(\sum_{k=0}^{N}E\left\|G_{1,k}+G_{2,k}\right\|_H^2\right)^{\frac{1}{2}}\leq M_2M\tau^{\frac{3}{2}}+M_4M\tau^{\frac{3}{2}}\leq \tilde{M}(\delta)\tau^{\frac{3}{2}}.$$

From that it follows (130). Theorem 4.4.1.5 is proved.

Applications

Now, we consider an applications of Theorem 4.4.1.5. First, let us consider the initial-value problem for one dimensional stochastic parabolic equation. In the same manner, the discretization of problem (76) is carried out in two steps. The first step is the same as in the previous case. In the second step, we replace (76) with the DS (119)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \frac{\tau}{2} A_{h,k}^x \big(u_k^h(x) + u_{k-1}^h(x)\big) = \varphi_k^h(x), \\ \\ \varphi_k^h(x) = \int\limits_{t_{k-1}}^{t_k} f^h(s,x,w_s)ds + \int\limits_{t_{k-1}}^{t_k} g^h(s,x,w_s)dw_s, \\ \\ A_{h,k}^x = A_h^x(t_{k-\frac{1}{2}}),\ t_k = k\tau,\ 1 \le k \le N,\ x \in [0,l]_h, \\ \\ u_0^h(x) = \varphi^h(x),\ x \in [0,l]_h. \end{cases} \tag{134}$$

Theorem 4.4.2.1 Assume that

$$E\left\|\varphi\right\|^2_{W_2^4[0,l]}, E\int\limits_0^T \left\|f(t,w_t)\right\|^2_{W_2^4[0,l]}dt, E\int\limits_0^T \left\|g(t,w_t)\right\|^2_{W_2^4[0,l]}dt < \infty. \tag{135}$$

Then, the solutions of DS (134) satisfy the following convergence estimate:

$$\left(\sum_{k=0}^N E\left\|v^h(t_k) - u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{3}{2}} + h\right), \tag{136}$$

where $C(\delta)$ do not depend on τ and h.

The proof of Theorem 4.4.2.1 is based on the abstract Theorem 4.4.1.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (75).

Second, we study the one dimensional stochastic parabolic equations with involution and Dirichlet conditions. In the same manner, the discretization of problem (80) is carried out in two steps. The first step is same as previous case.

In the second step, we replace (80) with the DS (119)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \frac{\tau}{2} A_{h,k}^x \big(u_k^h(x) + u_{k-1}^h(x)\big) = \varphi_k^h, \\ \\ \varphi_k^h = \int_{t_{k-1}}^{t_k} \big(f(s,w_s)ds + g(s,w_s)dw_s\big), \\ \\ A_{h,k}^x = A_h^x(t_{k-\frac{1}{2}}),\ t_k = k\tau,\ 1 \le k \le N,\ x \in [-l,l]_h, \\ \\ u_0^h(x) = \varphi^h(x),\ x \in [-l,l]_h. \end{cases}$$

Theorem 4.4.4.2 Assume that

$$E\left\|\varphi\right\|^2_{W_2^4[-l,l]}, E\int\limits_0^T \left\|f(t,w_t)\right\|^2_{W_2^4[-l,l]} dt, E\int\limits_0^T \left\|g(t,w_t)\right\|^2_{W_2^4[-l,l]} dt < \infty. \tag{137}$$

Let τ and h be sufficiently small numbers. For the solution of DS (103) the following convergence estimates hold

$$\left(\sum_{k=0}^{N} E\left\|v^h(t_k) - u_k^h\right\|^2_{C([0,T]_\tau, L_{2h})}\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{3}{2}} + |h|\right),$$

where $C(\delta)$ do not depend on τ and h.

Proof. The proof of Theorem 4.4.2.2 is based on the Theorems 4.4.1.5 and on the self-adjointness and positivity of operator $A_h^x(t)$ defined by formula (79).

Third, let us consider the initial value problem (48) for the multidimensional parabolic equation. The discretization of problem (84) is done in the same manner as above. In the second step, we replace (84) with the DS (119)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \tau A_{h,k}^x \big(u_k^h(x) + u_{k-1}^h(x)\big) = \varphi_k^h(x), \\ \\ \varphi_k^h(x) = \int\limits_{t_{k-1}}^{t_k} \big(f^h(s,x,w_s)ds + g^h(s,x,w_s)dw_s\big), \\ \\ A_{h,k}^x = A_h^x(t_{k-\frac{1}{2}}),\ 1 \le k \le N,\ x \in \Omega_h, \\ \\ u_0^h(x) = 0,\ x \in \widetilde{\Omega}_h. \end{cases} \tag{138}$$

Theorem 4.4.4.3 Assume that

$$E\left\|\varphi\right\|^2_{W_2^4(\Omega)}, E\int\limits_0^T \left\|f(t,w_t)\right\|^2_{W_2^4(\Omega)}dt, E\int\limits_0^T \left\|g(t,w_t)\right\|^2_{W_2^4(\Omega)}dt < \infty. \tag{139}$$

Then, the solution of DS (138) satisfy the following convergence estimate:

$$\left(\sum_{k=0}^{N} E\left\|v^h(t_k) - u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \leq C(\delta)\left(\tau^{\frac{3}{2}} + |h|^2\right), \tag{140}$$

where $C(\delta)$ do not depend on τ and $|h|$.

Proof. The proof of Theorem 4.4.4.3 is based on the Theorems 4.4.1.5 and on the self-adjointness and positivity of operator $A_h^x(t)$ by formula (83) and the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Fourth, we consider the multi-dimensional parabolic equation (51) with the Neumann condition. The discretization of problem (51) is done in the same manner as above. Then, in the second step, we replace (89) with the DS (119)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \frac{\tau}{2}A_{h,k}^x\left(u_k^h(x) + u_{k-1}^h(x)\right) = \varphi_k^h(x), \\ \varphi_k^h(x) = \int\limits_{t_{k-1}}^{t_k} \left(f^h(s,x,w_s)ds + g^h(s,x,w_s)dw_s\right), \\ A_{h,k}^x = A_h^x(t_{k-\frac{1}{2}}),\ t_k = k\tau,\ 1 \leq k \leq N,\ x \in \Omega_h, \\ u_0^h(x) = \varphi^h(x),\ \ x \in \widetilde{\Omega}_h. \end{cases} \tag{141}$$

Theorem 4.4.4.4 Assume that all assumptions of Theorem 4.4.3.4 are satisfied. Then, for the solution of (141) the estimate (140) holds.

The proof of Theorem 4.4.4.4 is based on the abstract Theorem 4.4.1.5 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (88) and the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Crank-Nicolson Difference Scheme Without the Standard Wiener Process

Now, we consider problem (26), when the sourse term $f(t, w_t)$ and $g(t, w_t)$ does not dependent on stochastic noise term w_t. That means $f(t, w_t) = f(t)$ and $g(t, w_t) = g(t)$ is an abstract function defined on the segment $[0, T]$ with values in H. Assume that the function $f(t)$ and $g(t)$ is continuous and smooth. In the similary manner in section 3.2, replacing $\varphi_k = \int\limits_{t_{k-1}}^{t_k} \big(f(s)ds + g(s)dw_s\big)$ by $\int\limits_{t_{k-1}}^{t_k} f(t_{k-\frac{1}{2}})dw_s = f(t_{k-\frac{1}{2}})\tau + g(t_{k-\frac{1}{2}})(w_{t_k} - w_{t_{k-1}})$ in (119), we get the CNDS

$$u_k - u_{k-1} + \frac{\tau}{2}A_k(u_k + u_{k-1})$$

$$= f(t_{k-\frac{1}{2}})\tau + g(t_{k-\frac{1}{2}})(w_{t_k} - w_{t_{k-1}}),\ 1 \le k \le N,\ u_0 = \varphi \tag{142}$$

for the approximate solution of the Cauchy problem (92). It is clear that the CNDS (142) is uniquely solvable and the following formula holds

$$u_k = \sum_{i=1}^{k} u_\tau(k,i)\left(I + \frac{\tau}{2}A_i\right)^{-1}\big(f(t_{k-\frac{1}{2}})\tau + g(t_{i-\frac{1}{2}})(w_{t_i} - w_{t_{i-1}})\big). \tag{143}$$

The Main Theorem on Stability

The estimate of convergence for the solution of the CNDS (142) is shown in the following theorem.

Theorem 4.5.1.1 Assume that

$$\left\|A^2(0)\varphi\right\|_H, \int\limits_0^T \|f'(t)\|_H^2\,dt + \int\limits_0^T \left\|A^2(t)f(t)\right\|_H^2 dt < \infty \tag{144}$$

and

$$\int\limits_0^T \|g'(t)\|_H^2\,dt + \int\limits_0^T \left\|A^2(t)g(t)\right\|_H^2 dt < \infty, \tag{145}$$

then the following convergence estimate is valid

$$\left(\sum_{k=0}^{N} E\,\|v(t_k) - u_k\|_H^2\right)^{\frac{1}{2}} \le \tilde{M}(\delta)\tau^{\frac{3}{2}}. \tag{146}$$

Here, M and $\tilde{M_1}(\delta)$ do not depend on τ.

Proof. Using formulas (33),(143), we get

$$v(t_k)-u_k=\sum_{i=1}^{k-1}\left[v(t_k,t_i)-u_\tau(k,i)\right]\int\limits_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds+g(s)dw_s\big)$$

$$+\sum_{i=1}^{k}u_\tau(k,i)\Big\{\int\limits_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds+g(s)dw_s\big)$$

$$-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\big(f(t_{i-\frac{1}{2}})\tau+g(t_{i-\frac{1}{2}})(w_{t_i}-w_{t_{i-1}})\big)\Big\}$$

$$=K_{1,k}+K_{2,k},$$

where

$$K_{1,k} = \sum_{i=1}^{k-1}\big(v(t_k,t_i)-u_\tau(k,i)\big)\int\limits_{t_{i-1}}^{t_i} v(t_i,s) \tag{147}$$

$$\times]\int\limits_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds+g(s)dw_s\big),$$

$$K_{2,k} = \sum_{i=1}^{k}u_\tau(k,i)\Big\{\int\limits_{t_{i-1}}^{t_i} v(t_i,s)\big(f(s)ds+g(s)dw_s\big)$$

$$-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\big(f(t_{i-\frac{1}{2}})\tau+g(t_{i-\frac{1}{2}})(w_{t_i}-w_{t_{i-1}})\big)\Big\}. \tag{148}$$

We will estimate $\left(\sum\limits_{k=0}^{N}E\|K_{r,k}\|_H^2\right)^{\frac{1}{2}}$ for $r=1$ and 2, separately. First, let us estimate

$$\left(\sum_{k=0}^{N}E\left\|K_{1,k}\right\|_H^2\right)^{\frac{1}{2}}.$$

Using formula (147), estimates (122), (35) and the triangle inequality, we obtain

$$\left(\sum_{k=0}^{N}E\left\|K_{1,k}\right\|_H^2\right)^{\frac{1}{2}}$$

$$=\left(\sum_{k=0}^{N}E\Big\|\sum_{i=1}^{k-1}\big(v(t_k,t_i)-u_\tau(k,i)\big)A_i^{-2}\int\limits_{t_{i-1}}^{t_i} A_i^2v(t_i,s)A^{-2}(s)A^2(s)\big(f(s)ds+g(s)dw_s\big)\Big\|_H^2\right)^{\frac{1}{2}}$$

$$\leq\Bigg(\sum_{k=0}^{N}\sum_{i=1}^{k-1}\Big\|\big(v(t_k,t_i)-u_\tau(k,i)\big)A_i^{-2}\Big\|_{H\to H}^2\int\limits_{t_{i-1}}^{t_i}\Big\|A_i^2v(t_i,s)A^{-2}(s)\Big\|_{H\to H}^2$$

$$\times\Big[\Big\|A^2(s)f(s)\Big\|_H^2+\Big\|A^2(s)g(s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq\ M_0\tau^2\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}M_1^2\int\limits_{t_{i-1}}^{t_i}\Big[\Big\|A^2(s)f(s)\Big\|_H^2+\Big\|A^2(s)g(s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq\ M_0\tau^2M_1\Big(\sum_{k=0}^{N}\sum_{i=1}^{k-1}\int\limits_{t_{i-1}}^{t_i}\Big[\Big\|A^2(s)f(s)\Big\|_H^2+\Big\|A^2(s)g(s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}$$

$$\leq\ M_2\tau^2\Big(\sum_{k=0}^{N}\int\limits_{0}^{T}\Big[\Big\|A^2(s)f(s)\Big\|_H^2+\Big\|A^2(s)g(s)\Big\|_H^2\Big]ds\Big)^{\frac{1}{2}}\leq M_2M\tau^{\frac{3}{2}}.$$

Now, we estimate $\left(\sum\limits_{k=0}^{N}E\left\|K_{2,k}\right\|_H^2\right)^{\frac{1}{2}}$. Then from formula (148) and Minkowski inequality it follows that

$$\sum_{k=0}^{N}E\left\|K_{2,k}\right\|_H^2$$

$$=\sum_{k=0}^{N}E\Big\|\sum_{i=1}^{k}u_\tau(k,i)\Big\{\int\limits_{t_{i-1}}^{t_i}v(t_i,s)\big(f(s)ds+g(s)dw_s\big)$$

$$-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\left(f(t_{i-\frac{1}{2}})\tau+g(t_{i-\frac{1}{2}})(w_{t_i}-w_{t_{i-1}})\right)\Big\}\Big\|_H^2$$

$$\leq\tau\sum_{k=0}^{N}E\Big\|\sum_{i=1}^{k}u_\tau(k,i)\int\limits_{t_{i-1}}^{t_i}\Big(v(t_i,s)-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\Big)$$

$$\times A^{-2}(s)A^2(s)\big(f(s)ds+g(s)dw_s\big)\Big\|_H^2$$

$$+\sum_{k=0}^{N}E\Big\|\sum_{i=1}^{k}u_\tau(k,i)\Big(I+\frac{\tau}{2}A_i\Big)^{-1}$$

$$\times\Big[\int\limits_{t_{i-1}}^{t_i}\big(f(s)-f(t_{i-\frac{1}{2}})\big)ds+\int\limits_{t_{i-1}}^{t_i}\big(g(s)-g(t_{i-\frac{1}{2}})\big)dw_s\Big\|_H^2\Big]$$

$$\leq\sum_{k=0}^{N}\sum_{i=1}^{k}\Big\|u_\tau(k,i)\Big\|_{H\to H}^2\int\limits_{t_{i-1}}^{t_i}\Big\|\Big(v(t_i,s)-\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\Big)A^{-2}(s)\Big\|_{H\to H}^2$$

$$\times\Big[\Big\|A^2(s)g(s)\Big\|_H^2+\Big\|A^2(s)f(s)\Big\|_H^2\Big]ds$$

$$+\sum_{k=0}^{N}\sum_{i=1}^{k}\Big\|u_\tau(k,i)\Big(I+\frac{\tau}{2}A_i\Big)^{-1}\Big\|_{H\to H}\int\limits_{t_{i-1}}^{t_i}\int\limits_{s}^{t_i}\Big[\Big\|f'(z)\Big\|_H^2+\Big\|g'(z)\Big\|_H^2\Big]dzds.$$

Then,

$$
\begin{aligned}
&\left(\sum_{k=0}^{N} E\left\|K_{2,k}\right\|_H^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=0}^{N}\sum_{i=1}^{k} M_1 M_2 \tau^2 \int_{t_{i-1}}^{t_i}\left[\left\|A^2(s)f(s)\right\|_H^2+\left\|A^2(s)g(s)\right\|_H^2\right]ds\right)^{\frac{1}{2}} \\
&+\left(\sum_{k=0}^{N}\sum_{i=1}^{k} M_3 \int_{t_{i-1}}^{t_i}\left[\left\|f'(z)\right\|_H^2+\left\|g'(z)\right\|_H^2\right]dz\int_{t_{i-1}}^{t_i}ds\right)^{\frac{1}{2}} \\
&\leq M_1 M_2 \tau^{\frac{3}{2}}\left(\int_0^T\left[\left\|A(s)f(s)\right\|_H^2\left\|A(s)g(s)\right\|_H^2\right]ds\right)^{\frac{1}{2}} \\
&+M_3\tau^{\frac{3}{2}}\left(\int_0^T\left[\left\|f'(z)\right\|_H^2+\left\|g'(z)\right\|_H^2\right]dz\right)^{\frac{1}{2}} \leq \tilde{M}\tau^{\frac{3}{2}}.
\end{aligned}
$$

Then combining both estimates, we get

$$
\left(\sum_{k=0}^{N} E\left\|K_{1,k}+K_{2,k}\right\|_H^2\right)^{\frac{1}{2}} \leq M_2 M\tau^{\frac{3}{2}}+\tilde{M}\tau^{\frac{3}{2}} \leq \tilde{M}(\delta)\tau^{\frac{3}{2}}.
$$

Theorem 4.5.1.1 is proved.

Applications

Now, we consider applications of Theorem 4.5.1.1. First, let us consider the initial-value problem for one dimensional stochastic parabolic equation (1). In the same manner, the discretization of problem (1) is carried out in two steps. The first step is the same as in the previous case. In the second step, we replace (76) with the DS (142)

$$
\begin{cases}
u_k^h(x)-u_{k-1}^h(x)+\frac{\tau}{2}A_{h,k}^x\big(u_k^h(x)+u_{k-1}^h(x)\big)=\varphi_k^h(x), \\
\varphi_k^h(x)=f^h(t_{k-\frac{1}{2}},x)\tau+g^h(t_{k-\frac{1}{2}},x)(w_{t_k}-w_{t_{k-1}}), \\
t_k=k\tau,\ 1\leq k\leq N,\ x\in[0,l]_h, A_{h,k}^x=A_h^x(t_{k-\frac{1}{2}}), \\
u_0^h(x)=\varphi^h(x),\ x\in[0,l]_h.
\end{cases}
\tag{149}
$$

Theorem 4.5.2.1 Assume that

$$\left\|\varphi\right\|^2_{W_2^4[0,l]}, \int_0^T \left\|f'(t)\right\|^2_{L_2[0,l]} dt + \int_0^T \left\|f(t)\right\|^2_{W_2^4[0,l]} dt < \infty \tag{150}$$

and

$$\int_0^T \left\|g'(t)\right\|^2_{L_2[0,l]} dt + \int_0^T \left\|g(t)\right\|^2_{W_2^4[0,l]} dt < \infty. \tag{151}$$

Then, the solutions of DS (149) satisfy the following convergence estimate:

$$\max_{0\le k\le N} \left(E\left\|v^h(t_k) - u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \le C(\delta)\left(\tau^{\frac{3}{2}} + h\right),$$

where $C(\delta)$ do not depend on τ and h.

The proof of Theorem 4.5.2.1 is based on the abstract Theorem 4.5.1.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (75). Second, let us consider the initial value problem for one dimensional stochastic parabolic equation (99) with involution and Dirichlet conditions. The discretization of problem (99) is the same with as in the previous case. In the second step, we replace (102) with the DS (142)

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \frac{\tau}{2}A_{h,k}^x\left(u_k^h(x) + u_{k-1}^h(x)\right) = \varphi_k^h(x), \\ \varphi_k^h(x) = \left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1})\right], \\ f_k^h(x) = f^h(t_k, x),\ g_k^h(x) = g^h(t_k, x) \\ x \in [-l, l]_h,\ 1 \le k \le N, \\ u_0^h(x) = \varphi^h(x), x \in [-l, l]_h. \end{cases} \tag{152}$$

Theorem 4.5.2.2 Assume that

$$\left\|\varphi\right\|_{W_2^4[-l,l]}, \int_0^T \left\|f'(t)\right\|^2_{L_2[-l,l]} dt + \int_0^T \left\|f(t)\right\|^2_{W_2^4[-l,l]} dt < \infty \tag{153}$$

and

$$\int_0^T \left\|g'(t)\right\|^2_{L_2[-l,l]} dt + \int_0^T \left\|g(t)\right\|^2_{W_2^4[-l,l]} dt < \infty. \tag{154}$$

Let τ and h be sufficiently small numbers. For the solution of DS (152) the following convergence estimates hold

$$\left(\sum_{k=0}^{N} E\left\|v^h(t_k)-u_k^h\right\|^2_{C([0,T]_\tau,L_{2h})}\right)^{\frac{1}{2}} \leq C(\delta)\left(\tau^{\frac{3}{2}}+|h|\right),$$

Proof. The proof of Theorem 4.5.2.2 is based on the abstract Theorem 4.5.1.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (79).

Third, we the mixed problem (48) for the multidimensional stochastic parabolic equation with the Dirichlet condition is considered. The discretization of problem (105) is done in the same as above. Then, in the second step, we replace (105) with the DS (142)

$$\begin{cases} u_k^h(x)-u_{k-1}^h(x)+\frac{\tau}{2}A_{h,k}^x\left(u_k^h(x)+u_{k-1}^h(x)\right)=\varphi_k^h(x), \\ \varphi_k^h(x)=\left[f_k^h(x)\tau+g_k^h(x)(w_t-w_{t-1})\right], \\ f_k^h(x)=f^h(t_k,x),\ g_k^h(x)=g^h(t_k,x),\ A_{h,k}^x=A_h^x(t_{k-\frac{1}{2}}), \\ t_k=k\tau,\ 1\leq k\leq N,\ x\in\Omega_h, \\ u_0^h(x)=\varphi^h\left(x\right),\ x\in\widetilde{\Omega}_h. \end{cases} \tag{155}$$

Theorem 4.5.2.3 Assume that

$$\left\|\varphi\right\|_{W_2^4(\Omega)}, \int_0^T \left\|f'(t)\right\|^2_{L_2(\Omega)}dt+\int_0^T \left\|f(t)\right\|^2_{W_2^4(\Omega)}dt<\infty \tag{156}$$

and

$$\int_0^T \left\|g'(t)\right\|^2_{L_2(\Omega)}dt+\int_0^T \left\|g(t)\right\|^2_{W_2^4(\Omega)}dt<\infty. \tag{157}$$

Then, the solution of DS (155) satisfy the following convergence estimate:

$$\max_{0\leq k\leq N}\left(E\left\|v^h(t_k)-u_k^h\right\|^2_{L_{2h}}\right)^{\frac{1}{2}} \leq C(\delta)\left(\tau^{\frac{3}{2}}+|h|^2\right), \tag{158}$$

where $C(\delta)$ do not depend on τ and $|h|$.

The proof of Theorem 4.5.2.3 is based on the abstract Theorem 4.5.1.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (83) and the Theorem 3.6 on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Fourth, the mixed problem (51) for the multidimensional stochastic parabolic equation with the Neumann condition is considered. The discretization of problem (51) is carried out in two steps. The discretization of problem (51) in x is done in the same as above. Then, in the second step, we replace (105) with the DS

$$\begin{cases} u_k^h(x) - u_{k-1}^h(x) + \frac{\tau}{2} A_{h,k}^x \big(u_k^h(x) + u_{k-1}^h(x)\big) = \varphi_k^h(x), \\ \\ \varphi_k^h(x) = \left[f_k^h(x)\tau + g_k^h(x)(w_t - w_{t-1})\right], \\ \\ f_k^h(x) = f^h(t_k, x),\ g_k^h(x) = g^h(t_k, x), A_{h,k}^x = A_h^x(t_{k-\frac{1}{2}}), \\ \\ t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \\ \\ u_0^h(x) = \varphi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{159}$$

Theorem 4.5.2.3 Assume that the assumptions of Theorem 3.23 are satisfied. Then, for the solution of (159) the estimate (158) holds.

The proof of Theorem 4.5.2.3 is based on the abstract Theorem 4.5.1.1 and the symmetry properties of the difference operator $A_h^x(t)$ defined by formula (88) and the Theorem 4.5.1.1 on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

CHAPTER V

Numerical Results

Introduction

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. In this section the single-step DS's in time for the solution of one dimensional stochastic partial differential equations are presented. Numerical results are provided. We apply a procedure of modified Gauss elimination method to solve the problem. The theoretical statements for the solution of these difference schemes are supported by the result of the numerical experiment.

The Mixed Problem with Dirichlet Condition

We consider the IBVP with Dirichlet condition

$$\begin{cases} dv(t,x,w_t) - 2(1+t)v_{xx}(t,x,w_t)dt = e^{-(t+1)^2}\sin(x)dw_t, \\ \\ 0 < t < 1, 0 < x < \pi, \\ \\ v(0,x,0) = 0, 0 \le x \le \pi, \\ \\ v(t,0,w_t) = v(t,\pi,w_t) = 0, w_t = \sqrt{t}\xi, \quad \xi \in \mathcal{N}(0,1), 0 \le t \le 1 \end{cases} \tag{160}$$

for the one dimensional stochastic partial differential equations. The exact solution of problem (160) is

$$u(t,x,w_t) = e^{-(t+1)^2}\sin(x)w_t.$$

Here and in the future, we consider the uniform grid space

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k,x_n) : t_k = k\tau,\ 0 \le k \le N,\ N\tau = 1;\ x_n = nh,\ 0 \le n \le M\},$$

where $N\tau = 1, Mh = \pi$. First, we consider the DS 1/2-th order of accuracy in t and second order of accuracy in x for the approximate solution of the IBVP (160)

$$\begin{cases} u_n^k - u_n^{k-1} - \frac{\tau}{h^2} 2(1+k\tau)(u_{n+1}^k - 2u_n^k + u_{n-1}^k) \\ = f(t_k, x_n)(\sqrt{k\tau} - \sqrt{(k-1)\tau})\xi, \\ f(t_k, x_n) = e^{-(t_k+1)^2} \sin(x_n), \quad 1 \le k \le N-1,\ 1 \le n \le M-1, \\ u_n^0 = 0, 0 \le n \le M, \\ u_0^k = u_M^k = 0, \quad 0 \le k \le N. \end{cases} \tag{161}$$

Thus we have $(N+1) \times (M+1)$ system of linear equations. We will write it in the matrix form

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ u_0 = \vec{0},\ u_M = \vec{0}. \end{cases} \tag{162}$$

Here

$$\varphi_n = \begin{pmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{pmatrix}_{(N+1)\times 1},$$

$\varphi_n^0 = 0,\ \varphi_n^k = f(t_k, x_n)\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi,\ 1 \le k \le N,\ 1 \le n \le M,$

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b & c_1 & 0 & \dots & 0 & 0 \\ 0 & b & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & 0 \\ 0 & 0 & 0 & \dots & b & c_N \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$a_k = -\frac{\tau}{h^2}(2(1+t_k))$, $b = -1$, $c_k = 1 + \frac{2\tau}{h^2}(2(1+t_k))$ and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}$$

, $s = n-1, n\, n+1$. For the solution of the last matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 1,\ u_M = \vec{0}, \tag{163}$$

where α_j, are $(N+1) \times (N+1)$ square matrices and β_j, are $(N+1) \times 1$ column matrices and $(j = 1, \dots, M-1)$ defined by

$$\begin{aligned} \alpha_{n+1} &= -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} &= (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, \dots, M-1. \end{aligned} \tag{164}$$

Here $\alpha_1 = [0]_{(N+1)\times(N+1)}$, $\beta_1 = [0]_{(N+1)\times 1}$. Finally, we generate 1000 random numbers with mean 0 and variance 1, set $\xi = [y_1, y_2, \dots, y_{1000}]^T$: set $\xi_j = y_j$, $j : 1$

to 1000 result of error analysis. The errors are computed by

$$E_M^N = \left(\sum_{k=0}^{N} \frac{1}{1000} \sum_{m_\xi=1}^{1000} \sum_{n=1}^{M-1} \left| u(t_k, x_n, m_\xi) - u_n^k(m_\xi) \right|^2 h \right)^{\frac{1}{2}} \tag{165}$$

of the numerical solutions, where $u(t_k, x_n) = u(t_k, x_n, m_\xi)$ represents the exact solution at ξ and $u_n^k = u_n^k(m_\xi)$ represents the numerical solution at (t_k, x_n, ξ) and the results are given in the following table:

Table 1.

Numerical results of difference scheme (161)

DS's/N, M	$10, 10$	$20, 20$	$40, 40$
DS (161)	0.0018	0.0010	0.0004782

As it is seen in Table 1 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $1/\sqrt{2}$ for $\frac{1}{2}-$th order of accuracy difference scheme.

Second, we consider the CNDS 3/2-th order of accuracy in t and second order of accuracy in x for the approximate solution of the IBVP (160)

$$\begin{cases} u_n^k - u_n^{k-1} - \frac{\tau}{h^2}(1 + t_{k-\frac{\tau}{2}})\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) \\ -\frac{\tau}{h^2}(1 + t_{k-\frac{\tau}{2}})\left(u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right) \\ = f(t_{k-\frac{1}{2}}, x_n) \left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right) \xi, \\ f(t_{k-\frac{1}{2}}, x_n) = e^{-(t_{k-\frac{1}{2}}+1)^2} \sin(x_n), \\ t_{k-\frac{1}{2}} = \left(k - \frac{1}{2}\right)\tau, \; x_n = nh, \; 1 \le k \le N, \; 1 \le n \le M-1, \\ u_n^0 = 0, 0 \le n \le M, \\ u_0^k = u_M^k = 0, \quad 0 \le k \le N. \end{cases} \tag{166}$$

Thus, we have $(N+1)\times(M+1)$ system of linear equations. Therefore, we can transform it in matrix form (162). Here

$$\varphi_n = \begin{pmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{pmatrix}_{(N+1)\times 1},$$

$\varphi_n^0 = 0$, $\varphi_n^k = f(t_{k-\frac{\tau}{2}}, x_n)\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi$, $1 \le k \le N$, $1 \le n \le M$,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & a_N & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

,

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b_1 & c_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & 0 \\ 0 & 0 & 0 & \dots & b_N & c_N \end{bmatrix}_{(N+1)\times(N+1)}$$

,

$a_k = -\frac{\tau}{h^2}(1 + t_k - \frac{\tau}{2})$, $b_k = -1 + \frac{\tau}{h^2}(2(1 + t_{k-\frac{1}{2}}))$, $c_k = 1 + \frac{2\tau}{h^2}(1 + t_{k-\frac{\tau}{2}})$ and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

,

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}$$

, $s = n-1, n\, n+1$.

For the solution of the last matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form (163) and (164).

The following table is of the error of the CNDS 3/2-th order of accuracy in t and second order of accuracy in x for the approximate solution of the IBVP (160):

Table 2.

Numerical result of difference scheme (166)

DS's/N, M	$10, 10$	$20, 20$	$40, 40$
DS (166)	0.00077724	0.00034105	0.00015924

As it is seen in the above Table 2, we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $\sqrt{2}/4$ for $\frac{3}{2}-$th order of accuracy difference scheme.

The Mixed Problem with Neumann Condition

For the numerical experiment,we consider the IBVP with Neumann condition

$$\begin{cases} dv(t,x,w_t) - 2(1+t)v_{xx}(t,x,w_t)dt = e^{-(t+1)^2}\cos(x)dw_t, \\ 0 < t < 1, 0 < x < \pi, \\ v(0,x,0) = 0, 0 \le x \le \pi, \\ v_x(t,0,w_t) = v_x(t,\pi,w_t) = 0, w_t = \sqrt{t}\xi, \quad \xi \in \mathcal{N}(0,1), 0 \le t \le 1 \end{cases} \tag{167}$$

for the one dimensional stochastic partial differential equations. The exact solution of problem (167) is

$$u(t,x,w_t) = e^{-(t+1)^2}\cos(x)w_t.$$

For numerical solution of problem (167) we consider the same uniform grid space $[0,1]_\tau \times [0,\pi]_h$.First, we consider the DS 1/2-th order of accuracy in t and first order of accuracy in x for the approximate solution of the IBVP (167)

$$\begin{cases} u_n^k - u_n^{k-1} - \frac{\tau}{h^2}2(1+k\tau)(u_{n+1}^k - 2u_n^k + u_{n-1}^k) \\ \\ = f(t_k,x_n)(\sqrt{k\tau} - \sqrt{(k-1)\tau})\xi, \\ \\ f(t_k,x_n) = e^{-(t_k+1)^2}\cos(x_n), \quad 1 \le k \le N-1,\, 1 \le n \le M-1, \\ \\ u_n^0 = 0, 0 \le n \le M, \\ \\ u_0^k = u_1^k,\, u_M^k = u_{M-1}^k, \quad 0 \le k \le N. \end{cases} \tag{168}$$

Thus we have $(N+1)\times(M+1)$ system of linear equations. We will write it in the matrix form

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ \\ u_0 = \vec{u_1},\, u_M = \vec{u_{M-1}}. \end{cases} \tag{169}$$

Here

$$\varphi_n = \begin{pmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{pmatrix}_{(N+1)\times 1},$$

$\varphi_n^0 = 0$, $\varphi_n^k = f(t_k,x_n)\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi$, $1 \le k \le N$, $1 \le n \le M$,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b & c_1 & 0 & \dots & 0 & 0 \\ 0 & b & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & 0 \\ 0 & 0 & 0 & \dots & b & c_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$a_k = -\frac{\tau}{h^2}(2(1+t_k))$, $b = -1$, $c_k = 1 + \frac{2\tau}{h^2}(2(1+t_k))$ and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$s = n \pm 1, n$. For the solution of the last matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 1,\ u_M = (I - \alpha_u)\beta_M, \tag{170}$$

where α_j, are $(N+1)\times(N+1)$ square matrices and β_j, are $(N+1)\times 1$ column matrices and $(j=1,\ldots,M-1)$ defined by

$$\begin{aligned}\alpha_{n+1} &= (B+C\alpha_n)^{-1}A, \\ \beta_{n+1} &= -(B+C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n=1,\ldots,M-1.\end{aligned} \tag{171}$$

Here $\alpha_1 = [1]_{(N+1)\times(N+1)}, \quad \beta_1 = [0]_{(N+1)\times 1}$. Finally, we generate 1000 random numbers with mean 0 and variance 1, set $\xi = [y_1, y_2, \ldots, y_{1000}]^T$: set $\xi_j = y_j$, $j : 1$ to 1000 result of error analysis. The errors are computed by

$$E_M^N = \left(\sum_{k=0}^{N} \frac{1}{1000} \sum_{m_\xi=1}^{1000} \sum_{n=0}^{M} \left|u(t_k, x_n, m_\xi) - u_n^k(m_\xi)\right|^2 h\right)^{\frac{1}{2}} \tag{172}$$

of the numerical solutions, where $u(t_k, x_n) = u(t_k, x_n, m_\xi)$ represents the exact solution at ξ and $u_n^k = u_n^k(m_\xi)$ represents the numerical solution at (t_k, x_n, ξ) and the results are given in the following table:

Table 3.

Numerical result of difference scheme (168)

DS's/N, M	$10, 10$	$20, 20$	$40, 40$
DS (168)	0.1082	0.0770	0.0522

As it is seen in Table 3 , we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $1/\sqrt{2}$ for $\frac{1}{2}-$th order of accuracy difference scheme.

Second, we consider the CNDS 3/2-th order of accuracy in t and first order of accuracy in x for the approximate solution of the IBVP (167)

$$\begin{cases} u_n^k - u_n^{k-1} - \frac{\tau}{h^2}(1+t_{k-\frac{\tau}{2}})\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) \\ \\ -\frac{\tau}{h^2}(1+t_{k-\frac{\tau}{2}})\left(u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right) \\ \\ = f(t_{k-\frac{1}{2}},\, x_n)\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi, \\ \\ f(t_{k-\frac{1}{2}},\, x_n) = e^{-(t_{k-\frac{1}{2}}+1)^2}\cos(x_n), \\ \\ t_{k-\frac{1}{2}} = \left(k-\frac{1}{2}\right)\tau,\, x_n = nh,\, 1 \le k \le N,\, 1 \le n \le M-1, \\ \\ u_n^0 = 0, 0 \le n \le M, \\ \\ u_0^k = u_1^k,\, u_M^k = u_{M-1}^k, \quad 0 \le k \le N. \end{cases} \tag{173}$$

Thus, we have $(N+1)\times(M+1)$ system of linear equations. Therefore, we can transform it in matrix form (162). Here

$$\varphi_n = \begin{pmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{pmatrix}_{(N+1)\times 1},$$

$\varphi_n^0 = 0$, $\varphi_n^k = f(t_{k-\frac{\tau}{2}}, x_n)\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi$, $1 \le k \le N$, $1 \le n \le M$,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & a_N & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

,

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b_1 & c_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & 0 \\ 0 & 0 & 0 & \dots & b_N & c_N \end{bmatrix}_{(N+1)\times(N+1)}$$

,

$a_k = -\frac{\tau}{h^2}(1 + t_k - \frac{\tau}{2})$, $b_k = -1 + \frac{\tau}{h^2}(2(1 + t_{k-\frac{1}{2}}))$, $c_k = 1 + \frac{2\tau}{h^2}(1 + t_{k-\frac{\tau}{2}})$ and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$s = n \pm 1, n$.

For the solution of the last matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form (163) and (164). We get the following table of the error of the CNDS 3/2-th order of accuracy in t and second order of accuracy in x for the approximate solution of the IBVP (167)

Table 4.

Numerical result of difference scheme (173)

DS's/N, M	$10, 10$	$20, 20$	$40, 40$
DS (173)	0.10578	0.0706	0.0475

As it is seen in Table, we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately $\sqrt{2}/4$ for $\frac{3}{2}-$th order of accuracy difference scheme.

CHAPTER VI

Conclusion

In this study, the stability of an abstract Cauchy problem for the for the solution of SDE in a Hilbert space with the time-dependent positive operator is established. In practice, theorems on stability estimates for the solution of four types of the initial boundary value problems for the one dimensional and multidimensional stochastic parabolic equation with dependent in t and space variables are proved. Single step DS's generated by exact DS are presented. The main theorems of the convergence of these difference schemes for the approximate solutions of the time-dependent abstract Cauchy problem for the parabolic equations are established. In applications, the convergence estimates for the solution of DS's for the SPDE's are obtained. Numerical results for the $\frac{1}{2}$ and $\frac{3}{2}$ th order of accuracy difference schemes of the approximate solution of mixed problems for the stochastic parabolic equations with Dirichlet, Neumann conditions are provided. Numerical results are given.

Investigate a high order of accuracy absolute stable difference schemes for the numerical solution of stochastic parabolic equation with dependent in t and space variables.

Investigate a high order of accuracy absolute stable DS's for the numerical solution of stochastic hyperbolic equation with dependent in t and space variables

$$d\dot{v}(t) + A(t)v(t)dt = f(t, w(t))dt + g(t, w(t))dw_t,\ 0 < t < T,\ v(0) = \varphi,\ \dot{v}(0) = \psi. \tag{174}$$

in a Hilbert space H with the unbounded operators $A(t)$.

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Appendices

Appendix A

Phyton Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (160)

function (Dirichlet condition)

```
import random
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
#Input 1: N as the number of time intervals and M as the number of position intervals
N = 26
M = 26
# Input 2: Exact solution function
def u(t,x,w):
return np.exp(-(t+1)**2)*np.sin(x)*w
# Define (N+1) x (M+1) spacetime grid matrix by limits and interval
# lengths for the plot (for some reason we need this 'if conition'
# because ogrid tends produce more coordinates)
tau = 1/N
h = np.pi/M
if (N t, x = np.ogrid[0:(N+1)*tau:tau, 0:(M+1)*h:h]
else:
t, x = np.ogrid[0:(N+1)/N:1/N, 0:(M+1)*h:h]
# Value grid matrix of the exact solution exact_u = u(t,x,np.sqrt(t))
# Recover lists of coordinates
t_list = []
for line in t:
t_list.append(line[0])
x_list = []
for entry in x[0]:
x_list.append(entry)
```

```
# Here we calculate the approximate solution values
# Construct zero vector of N+1 entries
N_zero_vector = []
for n in range(N+1):
N_zero_vector.append(0)
N_one_vector = []
for n in range(N+1):
N_one_vector.append(1)
# Construct A matrix
A = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == 0:
aux_line.append(0)
elif n == n2:
aux_line.append(-2*tau/(h**2)*(1+n*tau))
else:
aux_line.append(0)
A.append(aux_line)
A = np.array(A)
# Construct B matrix
B = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == n2 and n != 0:
aux_line.append(1+(4*tau)/(h**2)*(1+n*tau))
elif n == 0 and n2 == 0:
aux_line.append(1)
elif n == n2+1:
aux_line.append(-1)
```

```
else:
aux_line.append(0)
B.append(aux_line)
B = np.array(B)
# Construct alpha matrices, "alpha" = list of alpha matrics
# and construct beta vectors, "beta" = list of beta vectors
alpha = []
aux_matrix = []
for n in range(N+1):
aux_matrix.append(N_zero_vector)
aux_matrix = np.array(aux_matrix)
alpha.append(aux_matrix)
del aux_matrix
beta = []
beta.append(np.array(N_zero_vector))
for m in range(1,M-1):
aux_matrix_1 = B + np.matmul(A, alpha[m-1])
aux_matrix_2 = inv(aux_matrix_1)
aux_matrix_3 = - np.matmul(aux_matrix_2,A)
alpha.append(aux_matrix_3)
phi_m = []
for n in range(N+1):
if n == 0:
phi_m.append(0)
else:
phi_mn = u(t_list[n],x_list[m],1)*(np.sqrt(n/N)-p.sqrt((n-1)/N))
phi_m.append(phi_mn)
phi_m = np.array(phi_m)
aux_vector_1 = phi_m - np.matmul(A,beta[m-1])
aux_vector_2 = np.matmul(aux_matrix_2, aux_vector_1)
beta.append(aux_vector_2)
# Construct array of approximate solution vectors
```

```
approx_u = []
approx_u.append(np.array(N_zero_vector))
for m in range(1,M):
aux_vector = np.matmul(alpha[M-1-m],approx_u[0]) + beta[M-1-m]
approx_u.insert(0, aux_vector)
approx_u.insert(0,np.array(N_zero_vector))
approx_u = np.array(approx_u)
# The transpose of this array is needed for the plot approx_u = approx_u.transpose()
# Calculate numerical error between exact and approximate solution
# Generate a list of 1000 uniformly random real numbers in [0,1] random_list =
[]
for i in range(0,1000):
random_list.append(random.uniform(0,1))
def ESum(xi):
quad_sum = 0
for l in range(len(exact_u)):
for e in range(len(exact_u[l])):
diff = abs(exact_u[l][e]*xi-approx_u[l]
e
*xi)**2*h
quad_sum = quad_sum + diff
return quad_sum
quad_Error_NM = 0
for xi in random_list:
quad_Error_NM = quad_Error_NM + ESum(xi)
Error_NM = np.sqrt(1/1000*quad_Error_NM)
# Output of Error value into the terminal
print("The error by 1000 uniformly random numbers with mean 0 and variance
1 is:")
print(str(Error_NM))
# Exact solution plot
# Set up ax as plot object
ax1 = plt.figure().add_subplot(projection='3d')
```

```
# create surface with on values
ax1.plot_surface(t, x, exact_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax1.set_xlabel("time t")
ax1.set_ylabel("position x")
ax1.set_zlabel("exact solution u(t,x)")
# Set function value axis limts
ax1.set_zlim(-0.02, 0.12)
# Approximate solution plot
ax2 = plt.figure().add_subplot(projection='3d')
# create surface with on values
ax2.plot_surface(t, x, approx_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax2.set_xlabel("time t")
ax2.set_ylabel("position x")
ax2.set_zlabel("approximate solution u(t,x)")
# Set function value axis limts
ax2.set_zlim(-0.02, 0.12)
# output the plot plt.show()
```

Appendix B

Phyton Implementation of second Order of Accuracy Difference Schemes of Problem (166)

function (Dirichlet condition)

```
import random
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
# Input 1: N as the number of time intervals and M as the number of position intervals
N = 20
M = 20
# Input 2: Exact solution function
def u(t,x,w):
return np.exp(-(t+1)**2)*np.sin(x)*w
# Define (N+1) x (M+1) spacetime grid matrix by limits and interval
# lengths for the plot. The data format of the t, x-arrays is chosen
# to fit the requirements of the 'plot_surface'-routine.
tau = 1/N
h = np.pi/M
t = []
for n in range(N+1):
t.append([n*tau])
t = np.array(t)
x = [[]]
for m in range(M+1):
x[0].append(m*h)
x = np.array(x)
# Value grid matrix of the exact solution
exact_u = u(t,x,np.sqrt(t))
# Interpolated time points
t2 = []
```

```
for n in range(1,N+1):
t2.append([(n - 0.5)*tau])
t2 = np.array(t2)
# Here we calculate the approximate solution values
# Construct zero vector of N+1 entries
N_zero_vector = []
for n in range(N+1):
N_zero_vector.append(0)
# Construct A matrix
A = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):

if n == 0:
aux_line.append(0)
elif n == n2:
aux_line.append(-tau/(h**2)*(1+(n-0.5)*tau))
elif n == n2+1:
aux_line.append(-tau/(h**2)*(1+(n-0.5)*tau))
else:
aux_line.append(0)
A.append(aux_line)
A = np.array(A)
# Construct B matrix
B = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == n2 and n != 0:
aux_line.append(1+(2*tau)/(h**2)*(1+(n-0.5)*tau))
elif n == 0 and n2 == 0:
```

```
aux_line.append(1)
elif n == n2+1:
aux_line.append(-1+(2*tau)/(h**2)*(1+(n-0.5)*tau))
else:
aux_line.append(0)
B.append(aux_line)
B = np.array(B)
# Construct alpha matrices, "alpha" = list of alpha matrics
# and construct beta vectors, "beta" = list of beta vectors
alpha = []
aux_matrix = []
for n in range(N+1):
aux_matrix.append(N_zero_vector)
aux_matrix = np.array(aux_matrix)
alpha.append(aux_matrix)
del aux_matrix
beta = []
beta.append(np.array(N_zero_vector))
for m in range(1,M-1):
aux_matrix_1 = B + np.matmul(A, alpha[m-1])
aux_matrix_2 = inv(aux_matrix_1)
aux_matrix_3 = - np.matmul(aux_matrix_2,A)
alpha.append(aux_matrix_3)
ph_m = []
for n in range(N+1):
if n == 0:
phi_m.append(0)
else:
phi_mn = u(t2[n-1][0],x[0][m],1)*(np.sqrt(n/N)-np.sqrt((n-1)/N))
phi_m.append(phi_mn) phi_m = np.array(phi_m)
aux_vector_1 = phi_m - np.matmul(A,beta[m-1])
aux_vector_2 = np.matmul(aux_matrix_2, aux_vector_1)
```

```
beta.append(aux_vector_2)
# Construct array of approximate solution vectors
approx_u = []
approx_u.append(np.array(N_zero_vector))
for m in range(1,M):
aux_vector = np.matmul(alpha[M-1-m],approx_u[0]) + beta[M-1-m]
approx_u.insert(0, aux_vector)
approx_u.insert(0,np.array(N_zero_vector))
approx_u = np.array(approx_u)
# The transpose of this array is needed for the plot approx_u = approx_u.transpose()
# Calculate numerical error between exact and approximate solution
# Generate a list of 1000 uniformly random real numbers in [0,1]
random_list = []
for i in range(0,1000):
random_list.append(random.uniform(0,1))
def ESum(xi):
quad_sum = 0
for l in range(len(exact_u)):
for e in range(len(exact_u[l])):
diff = abs(exact_u[l][e]*xi-approx_u[l][e]*xi)**2*h
quad_sum = quad_sum + diff
return quad_sum
quad_Error_NM = 0
for xi in random_list:
quad_Error_NM = quad_Error_NM + ESum(xi)
Error_NM = np.sqrt(1/1000*quad_Error_NM)
# Output of Error value into the terminal print("The error by 1000 uniformly random numbers with mean 0 and variance 1 is:")
print(str(Error_NM))
# Exact solution plot
# Set up ax as plot object
ax1 = plt.figure().add_subplot(projection='3d')
```

```
# create surface with on values
ax1.plot_surface(t, x, exact_u, cmap=’autumn’, cstride=1, rstride=1)
# Name axes
ax1.set_xlabel(”time t”)
ax1.set_ylabel(”position x”) ax1.set_zlabel(”exact solution u(t,x)”)
# Set function value axis limts
ax1.set_zlim(-0.02, 0.12)
# Approximate solution plot
ax2 = plt.figure().add_subplot(projection=’3d’)
# create surface with on values
ax2.plot_surface(t, x, approx_u, cmap=’autumn’, cstride=1, rstride=1)
# Name axes
ax2.set_xlabel(”time t”) ax2.set_ylabel(”position x”)
ax2.set_zlabel(”approximate solution u(t,x)”)
# Set function value axis limts
ax2.set_zlim(-0.02, 0.12)
# output the plot
plt.show()
```

Appendix C

Phyton Implementation of One Dimension First Order of Accuracy Difference Schemes of Problem (167)

function (Neumann condition)

```
import random
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
# Input 1: N as the number of time intervals and M as the number of position
intervals.
N = 50
M = 50
# Input 2: Exact solution function
def u(t,x,w):
return np.exp(-(t+1)**2)*np.cos(x)*w
# Define (N+1) x (M+1) spacetime grid matrix by limits and interval
# lengths for the plot (for some reason we need this 'if conition'
# because ogrid tends produce more coordinates)
tau = 1/N
h = np.pi/M
if (N t, x = np.ogrid[0:(N+1)*tau:tau, 0:
(M+1)*h:h]
else:
t, x = np.ogrid[0:(N+1)/N:1/N, 0:
(M+1)*h:h]
# Value grid matrix of the exact solution
exact_u = u(t,x,np.sqrt(t))
# Recover lists of coordinates
t_list = []
for line in t:
t_list.append(line[0])
x_list = []
```

```
for entry in x[0]: x_list.append(entry)
# Here we calculate the approximate solution values
# Construct zero vector of N+1 entries
N_zero_vector = []
for n in range(N+1):
N_zero_vector.append(0)
N_unit_matrix = []
for n in range(N+1):
aux_vector = []
for n2 in range(N+1):
if n == n2:
aux_vector.append(1) else:
aux_vector.append(0)
N_unit_matrix.append(aux_vector)
# Construct A matrix
A = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == 0:
aux_line.append(0)
elif n == n2:
aux_line.append(-2*tau/(h**2)*(1+n*tau))
else:
aux_line.append(0)
A.append(aux_line)
A = np.array(A)
# Construct B matrix
B = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
```

```
if n == n2 and n != 0:
aux_line.append(1+(4*tau)/(h**2)*(1+n*tau))
elif n == 0 and n2 == 0:
aux_line.append(1)
elif n == n2+1:
aux_line.append(-1)
else:
aux_line.append(0)
B.append(aux_line)
B = np.array(B)
# Construct alpha matrices, "alpha" = list of alpha matrics
# and construct beta vectors, "beta" = list of beta vectors
alpha = []
alpha.append(np.array(N_unit_matrix))
beta = []
beta.append(np.array(N_zero_vector))
for m in range(1,M-1):
aux_matrix_1 = B + np.matmul(A, alpha[m-1])
aux_matrix_2 = inv(aux_matrix_1)
aux_matrix_3 = - np.matmul(aux_matrix_2,A)
alpha.append(aux_matrix_3)
phi_m = []
for n in range(N+1):
if n == 0:
phi_m.append(0)
else:
phi_mn = u(t_list[n],x_list[m],1)*(np.sqrt(n/N)-np.sqrt((n-1)/N))
phi_m.append(phi_mn)
phi_m = np.array(phi_m)
aux_vector_1 = phi_m - np.matmul(A,beta[m-1])
aux_vector_2 = np.matmul(aux_matrix_2, aux_vector_1)
beta.append(aux_vector_2)
```

```
# Construct array of approximate solution vectors
approx_u = []
u_M = np.matmul(inv(np.array(N_unit_matrix)-alpha[M-2]),beta[M-2])
approx_u.append(u_M)
for m in range(1,M):
aux_vector = np.matmul(alpha[M-1-m],approx_u[0]) + beta[M-1-m]
approx_u.insert(0, aux_vector)
approx_u.insert(0,np.matmul(alpha[0],approx_u[1]) + beta[0])
approx_u = np.array(approx_u)
# The transpose of this array is needed for the plot
approx_u = approx_u.transpose()
print(u_M)
print(np.matmul(alpha[M-2],u_M)+beta[M-2])
# Calculate numerical error between exact and approximate solution
# Generate a list of 1000 uniformly random real numbers in [0,1]
random_list = []
for i in range(0,1000):
random_list.append(random.uniform(0,1))
def ESum(xi):
quad_sum = 0
for l in range(len(exact_u)):
for e in range(len(exact_u[l])):
diff = abs(exact_u[l][e]*xi-approx_u[l][e]*xi)**2*h
quad_sum = quad_sum + diff
return quad_sum
quad_Error_NM = 0
for xi in random_list:
quad_Error_NM = quad_Error_NM + ESum(xi)
Error_NM = np.sqrt(1/1000*quad_Error_NM)
# Output of Error value into the terminal
print("The error by 1000 uniformly random numbers with mean 0 and variance
1 is:")
```

```
print(str(Error_NM))
# Exact solution plot
# Set up ax as plot object
ax1 = plt.figure().add_subplot(projection='3d')
# create surface with on values
ax1.plot_surface(t, x, exact_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax1.set_xlabel("time t")
ax1.set_ylabel("position x")
ax1.set_zlabel("exact solution u(t,x)")
# Set function value axis limts
ax1.set_zlim(-0.02, 0.12)
# Approximate solution plot
ax2 = plt.figure().add_subplot(projection='3d')
# create surface with on values
ax2.plot_surface(t, x, approx_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax2.set_xlabel("time t")
ax2.set_ylabel("position x")
ax2.set_zlabel("approximate solution u(t,x)")
# Set function value axis limts
ax2.set_zlim(-0.02, 0.12)
# output the plot
plt.show()
```

Appendix D

Phyton Implementation of second Order of Accuracy Difference Schemes of Problem (173)

function (Neumann condition)

```
import random
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
# Input 1: N as the number of time intervals and M as the number of
# position intervals
N = 20
M = 20
# Input 2: Exact solution function
def u(t,x,w):
return np.exp(-(t+1)**2)*np.cos(x)*w
# Define (N+1) x (M+1) spacetime grid matrix by limits and interval
# lengths for the plot. The data format of the t, x-arrays is chosen
# to fit the requirements of the 'plot_surface'-routine.
tau = 1/N
h = np.pi/M
t = []
for n in range(N+1):
t.append([n*tau])
t = np.array(t)
x = [[]]
for m in range(M+1):
x[0].append(m*h)
x = np.array(x)
# Value grid matrix of the exact solution
exact_u = u(t,x,np.sqrt(t))
# Interpolated time points
t2 = []
```

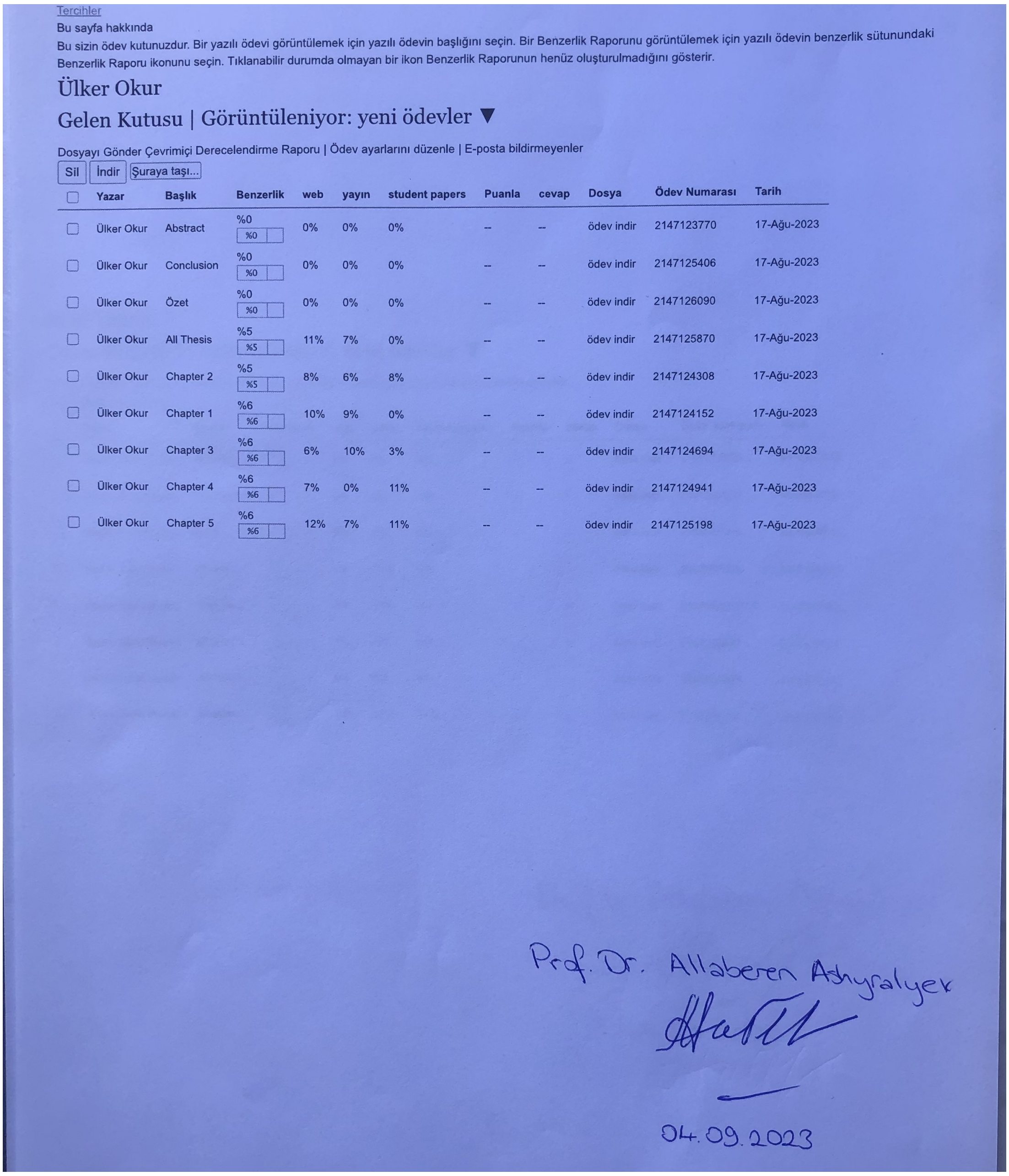
```
for n in range(1,N+1):
t2.append([(n - 0.5)*tau])
t2 = np.array(t2)
# Here we calculate the approximate solution values
# Construct zero vector of N+1 entries
N_zero_vector = []
for n in range(N+1):
N_zero_vector.append(0)
N_unit_matrix = []
for n in range(N+1):
aux_vector = []
for n2 in range(N+1):
if n == n2:
aux_vector.append(1)
else:
aux_vector.append(0)
N_unit_matrix.append(aux_vector)
# Construct A matrix
A = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == 0:
aux_line.append(0)
elif n == n2:
aux_line.append(-tau/(h**2)*(1+(n-0.5)*tau))
elif n == n2+1:
aux_line.append(-tau/(h**2)*(1+(n-0.5)*tau))
else:
aux_line.append(0)
A.append(aux_line)
A = np.array(A)
```

```
# Construct B matrix
B = []
for n in range(N+1):
aux_line = []
for n2 in range(N+1):
if n == n2 and n != 0:
aux_line.append(1+(2*tau)/(h**2)*(1+(n-0.5)*tau))
elif n == 0 and n2 == 0:
aux_line.append(1)
elif n == n2+1:
aux_line.append(-1+(2*tau)/(h**2)*(1+(n-0.5)*tau))
else:
aux_line.append(0)
B.append(aux_line)
B = np.array(B)
# Construct alpha matrices, "alpha" = list of alpha matrics
# and construct beta vectors, "beta" = list of beta vectors
alpha = []
alpha.append(np.array(N_unit_matrix))
beta = []
beta.append(np.array(N_zero_vector))
for m in range(1,M-1):
aux_matrix_1 = B + np.matmul(A, alpha[m-1])
aux_matrix_2 = inv(aux_matrix_1)
aux_matrix_3 = - np.matmul(aux_matrix_2,A)
alpha.append(aux_matrix_3)
phi_m = []
for n in range(N+1):
if n == 0:
phi_m.append(0)
else:
phi_mn = u(t2[n-1][0],x[0][m],1)*(np.sqrt(n/N)-np.sqrt((n-1)/N))
```

```
phi_m.append(phi_mn)
phi_m = np.array(phi_m)
aux_vector_1 = phi_m - np.matmul(A,beta[m-1])
aux_vector_2 = np.matmul(aux_matrix_2, aux_vector_1)
beta.append(aux_vector_2)
# Construct array of approximate solution vectors
approx_u = []
u_M = np.matmul(inv(np.array(N_unit_matrix)-alpha[M-2]),beta[M-2])
approx_u.append(u_M)
for m in range(1,M):
aux_vector = np.matmul(alpha[M-1-m],approx_u[0]) + beta[M-1-m]
approx_u.insert(0, aux_vector)
approx_u.insert(0,np.matmul(alpha[0],approx_u[1]) + beta[0])
approx_u = np.array(approx_u)
# The transpose of this array is needed for the plot
approx_u = approx_u.transpose()
# Calculate numerical error between exact and approximate solution
# Generate a list of 1000 uniformly random real numbers in [0,1]
random_list = []
for i in range(0,1000):
random_list.append(random.uniform(0,1)) def ESum(xi): quad_sum = 0 for l in
range(len(exact_u)):
for e in range(len(exact_u[l])):
diff = abs(exact_u[l][e]*xi-approx_u[l][e]*xi)**2*h
quad_sum = quad_sum + diff
return quad_sum
quad_Error_NM = 0
for xi in random_list: quad_Error_NM = quad_Error_NM + ESum(xi)
Error_NM = np.sqrt(1/1000*quad_Error_NM)
# Output of Error value into the terminal
print("The error by 1000 uniformly random numbers with mean 0 and variance
1 is:")
```

```
print(str(Error_NM))
# Exact solution plot
# Set up ax as plot object
ax1 = plt.figure().add_subplot(projection='3d')
# create surface with on values
ax1.plot_surface(t, x, exact_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax1.set_xlabel("time t")
ax1.set_ylabel("position x")
ax1.set_zlabel("exact solution u(t,x)")
# Set function value axis limts
ax1.set_zlim(-0.02, 0.12)
# Approximate solution plot
ax2 = plt.figure().add_subplot(projection='3d')
# create surface with on values
ax2.plot_surface(t, x, approx_u, cmap='autumn', cstride=1, rstride=1)
# Name axes
ax2.set_xlabel("time t")
ax2.set_ylabel("position x")
ax2.set_zlabel("approximate solution u(t,x)")
# Set function value axis limts
ax2.set_zlim(-0.02, 0.12)
# output the plot
plt.show()
```

Appendix E
Turnitin Similarity Report



Tercihler

Bu sayfa hakkında

Bu sizin ödev kutunuzdur. Bir yazılı ödevi görüntülemek için yazılı ödevin başlığını seçin. Bir Benzerlik Raporunu görüntülemek için yazılı ödevin benzerlik sütunundaki Benzerlik Raporu ikonunu seçin. Tıklanabilir durumda olmayan bir ikon Benzerlik Raporunun henüz oluşturulmadığını gösterir.

Ülker Okur

Gelen Kutusu | Görüntüleniyor: yeni ödevler ▼

Dosyayı Gönder Çevrimiçi Derecelendirme Raporu | Ödev ayarlarını düzenle | E-posta bildirmeyenler

Sil | İndir | Şuraya taşı...

Yazar	Başlık	Benzerlik	web	yayın	student papers	Puanla	cevap	Dosya	Ödev Numarası	Tarih
Ülker Okur	Abstract	%0	0%	0%	0%	--	--	ödev indir	2147123770	17-Ağu-2023
Ülker Okur	Conclusion	%0	0%	0%	0%	--	--	ödev indir	2147125406	17-Ağu-2023
Ülker Okur	Özet	%0	0%	0%	0%	--	--	ödev indir	2147126090	17-Ağu-2023
Ülker Okur	All Thesis	%5	11%	7%	0%	--	--	ödev indir	2147125870	17-Ağu-2023
Ülker Okur	Chapter 2	%5	8%	6%	8%	--	--	ödev indir	2147124308	17-Ağu-2023
Ülker Okur	Chapter 1	%6	10%	9%	0%	--	--	ödev indir	2147124152	17-Ağu-2023
Ülker Okur	Chapter 3	%6	6%	10%	3%	--	--	ödev indir	2147124694	17-Ağu-2023
Ülker Okur	Chapter 4	%6	7%	0%	11%	--	--	ödev indir	2147124941	17-Ağu-2023
Ülker Okur	Chapter 5	%6	12%	7%	11%	--	--	ödev indir	2147125198	17-Ağu-2023

Prof. Dr. Allaberen Ashyralyev

04.09.2023

Appendix F

CURRICULUM VITAE

PERSONAL DATA

Date of Birthday	20.03.1984
Place of Bith	Stuttgart-Bad Cannstatt
Nationality	German

WORK EXPERINCE

Since 01.10.2022
Württembergische-Gemeinde-Versicherung
● calculation of premiums
● Responsible for the operation management
● Optimization and further development of innovative product

01.08.2021 – 31.01.2022
Allianz-Lebensversicherungs-Aktiengesellschaft
● Business Analyst in the department of product and services
● Responsible for the operation management
● Optimization and further development of innovative product

01.09.2019 – 01.07.2021
Mercer Deutschland GmbH
● Preparation of actuarial reports on the valuation of pension liabilities according to the national law and international evaluation guidelines (IFRS, US-GAAP, etc.) as well as old-age part-time obligations
● Processing of insured events
● Information related to the pensions
● Supervision and consultancy according to the corporate-pension Business
● Contacts to maintain existing client relationships

01.07.2019 – 31.08.2019
Gassner & Partner AG in Stuttgart
● Preparation of actuarial reports on the valuation of pension liabilities according to the national law evaluation guidelines

01.06.2018 – 30.06.2019
Kern Mauch & Kollegen GmbH in Stuttgart
● Preparation of actuarial reports on the valuation of pension liabilities according to the national law and international evaluation guidelines (IFRS, US-GAAP, etc.)
● Processing of insured events
● Information related to the pensions
● Supervision and consultancy according to the corporate-pension Business
● Contacts to maintain existing client relationships

06.2011 - 09.2012
Scientific Officer at the University of Stuttgart in Germany
● Assistant in the field of statistic for economist
● Assistant in the field of higher mathematics II and III

10.2010 - 03.2011
Tutoress at the University of Stuttgart in Germany
Student care of the lecture of probability theory

01.2011. - 12.2015
Membership
Primus Bildungszentrum in Esslingen a.N.
High school graduation for preparation in Mathematics

EDUCATION

06.2011 – 06.2018	University Stuttgart
Seit 06.2020	Near East University Cyprus
Facility	**Math**
Degree expected	Automn 2022
Degree	doctor rerum naturalium
Scholarship	Algebra Engineering
Details	**Subject of the doctoral thesis:** "Oerator approach for the solution of stochastic differential equaitons" Tasks area: In the first part it treats with initial-boundary value problem with a stochastic parabolic equation. The convergence estimation for the solution of the difference schemes was established.
10.2005 - 05.2011	Universität Stuttgart
Branch of study	**Study of Mathematics**
Degree	**Diploma of Mathematics**
Details	In-depth studies: Analysis Minor field of study: business studies, computer science

EDUCATION

09.2002 - 06.2005	Robert-Franck-Schule Ludwigsburg
Degree	general qualification for university entrance (Wirtschaftsgymnasium)
09.2000 - 06.2002	Robert-Franck-Schule Ludwigsburg
Degree	secondary school certificate (Wirtschaftsschule)

Skills

Language	German (native) English (fluent) Turkish (fluent)
Computer	LaTeX, Word, Excel, Powerpoint, MATLAB
Interests	Astronomy, Sport, Theater