Partial Differential Equations

1. What is a PDE?

Def PDE is an equation that relates a function u of several variables $x_1, \dots, x_n$ and its partial derivatives. For example, if a function of two variables is denoted $u(x,y)$, then one may consider the following as examples of PDEs:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's eq.)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(the wave eq.)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad \text{(the heat eq.)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y) \qquad \text{(Poisson's eq.)}$$

We will often use subscripts:

$$u_{xx} + u_{yy} = 0, \quad u_{xx} - u_{yy} = 0, \quad u_{xx} - u_y = 0, \quad u_{xx} + u_{yy} = g$$

The order of a PDE is indicated by the highest-order derivative that appears. All of the above four examples are PDEs of second order.

In the case of a function of several variables $u(x_1, \dots, x_n)$, the most general second-order PDE can be written

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \dots, u_{x_n x_n}) = 0$$

In case $n=1$ we obtain the second-order ODE

$F(x, u, u', u'') = 0$.

Let $\mathcal{L}$ be a differential operator. The operator is said to be linear if for any two functions u, v ~~and~~ any constant c,

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v, \quad \mathcal{L}(cu) = c\mathcal{L}u.$$

A PDE is said to be linear if it can be written in the form

$$\mathcal{L}u = g \qquad (1)$$

where $\mathcal{L}$ is a linear differential operator and g is a given function.

In case $g=0$, (1) is said to be homogeneous.
For example, three of the above examples are linear homogeneous PDEs. The most general linear second-order PDE in two variables is written

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y)$$

where the f.s a,b,c,d,e,f,g are given.

2. Superposition principle for homogeneous eq.s.

Proposition 1.
If $u_1,\dots,u_N$ are solutions of the same linear homogeneous PDE $\mathcal{L}u=0$, and $c_1,\dots,c_N$ are constants (real or complex) then $c_1u_1+\dots+c_Nu_N$ is also a solution of the PDE.

Proof. We have $\mathcal{L}(u_i)=0$ for $i=1,\dots,N$. Hence

$$\mathcal{L}(c_1u_1+\dots+c_Nu_N)=c_1\mathcal{L}(u_1)+\dots+c_N\mathcal{L}(u_N)=0.$$

For example, for any constant k, the function $u(x,y)=e^{kx}\cos ky$ is a solution of Laplace's eq. $u_{xx}+u_{yy}=0$.
By the superposition principle, the function

$$u(x,y)=e^{-x}\cos y+2e^{-3x}\cos 3y-5e^{-\pi x}\cos \pi y$$

is also a solution of Laplace's equation.

The superposition principle does not apply to nonhomogeneous eq.s. For example, if u_1 and u_2 are solutions of the Poisson eq. $u_{xx}+u_{yy}=1$, then the function u_1+u_2 is the solution of a different eq., namely $u_{xx}+u_{yy}=2$.

Proposition 2 (subtraction principle for nonhomogeneous eq.s)
If u_1 and u_2 are solution of the same linear nonhomogeneous eq. $\mathcal{L}u=g$, then the function u_1-u_2 is a solution of the associated homogeneous eq. $\mathcal{L}u=0$.

Proof. We have

$$\mathcal{L}(u_1-u_2)=\mathcal{L}u_1-\mathcal{L}u_2=0.$$

For example, if u_1 and u_2 are both solutions of the Poisson eq. $u_{xx}+u_{yy}=1$, then u_1-u_2 is a solution of Laplace's eq. $u_{xx}+u_{yy}=0$.

<u>Corollary</u>. The general solution of the linear PDE $\mathcal{L}u = g$ can be written in the form

$$u = U + v$$

where U is a particular solution of the eq. $\mathcal{L}U = g$ and v is the general solution of the related homogeneous eq. $\mathcal{L}v = 0$.

<u>Example</u>. Find the general solution $u(x,y)$ of the equation $u_{xx} = 2$.

<u>Solution</u>. $u = x^2$ is a solution of the given equation. The general solution of $u_{xx} = 0$ is $u(x,y) = x\,g(y) + h(y)$

$$\left(u_{xx} = 0 \xRightarrow{\text{integrating}} u_x(x,y) = C(y) \xRightarrow{\text{second integration}} u(x,y) = x\,C(y) + D(y), \right.$$

where C and D are arbitrary functions.)

Therefore the general solution of the nonhomogeneous eq. is $u(x,y) = x^2 + x\,g(y) + h(y)$.

Classification of second-order PDEs.

$$a(x,y)\,u_{xx} + b(x,y)\,u_{xy} + c(x,y)\,u_{yy} + d(x,y)\,u_x + e(x,y)\,u_y + f(x,y)\,u = g(x,y) \tag{1}$$

If $4ac - b^2 > 0$, the PDE (1) is called elliptic,

If $4ac - b^2 = 0$, the (1) is called parabolic.

If $4ac - b^2 < 0$, the (1) is called hyperbolic.

For example, Laplace's and Poisson's eq. are both elliptic, while the wave eq. is hyperbolic. The heat eq. is parabolic.

The types of boundary conditions:

If the eq. is elliptic, we may solve the Dirichlet problem, namely, in a region D to find a solution of $\mathcal{L}u = g$ that further satisfies the boundary condition $u = \phi(x,y)$ on the boundary of D.

$$\begin{cases} u_{xx} + u_{yy} = -f(x,y), & x^2+y^2 < R^2 \\ u(x,y) = C, & x^2+y^2 = R^2 \end{cases}$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & t>0,\ 0<x<L, \\ u(x;0) = f_1(x) & 0<x<L \\ u_t(x;0) = f_2(x) & 0<x<L, \\ u(0;t) = 0,\ u(L;t) = 0, & t>0 \end{cases}$$

Separation of variables

We look for particular solutions in the form $u(x,y) = X(x)Y(y)$ and try to obtain ODEs for $X(x)$ and $Y(y)$. These eq.s will contain a parameter called the separation constant. The function $u(x,y)$ is called a separated solution. Then we can use the superposition principle to obtain more general solutions of a linear homogeneous PDE as sums of separated solutions.

· Separated solutions of Laplace's equation.

Let $u(x,y) = X(x)Y(y)$. Substitute in Laplace's eq., we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assumed to be nonzero), we obtain

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

We introduce the constant λ and obtain the system of two ordinary DEs,

$$\frac{X''(x)}{X(x)} = \lambda, \quad \frac{Y''(y)}{Y(y)} = -\lambda,$$

λ is the separation constant.

$$\Rightarrow \qquad X''(x) - \lambda X(x) = 0 \qquad (1)$$
$$Y''(y) + \lambda Y(y) = 0 \qquad (2)$$

Case 1. If $\lambda > 0$, we write $\lambda = k^2$, where $k > 0$. The general solutions to (1) and (2) are

$$X(x) = A_1 e^{kx} + A_2 e^{-kx}$$
$$Y(y) = A_3 \cos ky + A_4 \sin ky$$

where A_1, A_2, A_3, A_4 are arbitrary constants.

Case 2. If $\lambda = 0$, we have

$$X''(x) = 0, \Rightarrow X(x) = A_1 x + A_2$$
$$Y''(x) = 0 \Rightarrow Y(y) = A_3 y + A_4$$

where A_1, A_2, A_3, A_4 are arbitrary constants

$X(x) = e^{kx}$

$X' = k e^{kx}$

$X'' = k^2 e^{kx}$

$k^2 e^{kx} - \lambda e^{kx} = 0$

$k^2 = \lambda$, $k = \pm\sqrt{\lambda}$

$X(x) = A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x}$

Case 3. If $\lambda < 0$,

we write $\lambda = -l^2$, where $l > 0$; the general solutions of (1) and (2) are

$$X(x) = A_1 \cos lx + A_2 \sin lx.$$
$$Y(y) = A_3 e^{ly} + A_4 e^{-ly}.$$

We have found the following separated solutions of Laplace's eq.:

$$u(x,y) = \begin{cases} (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky), & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) \\ (A_1 \cos lx + A_2 \sin lx)(A_3 e^{ly} + A_4 e^{-ly}), & l > 0 \end{cases}$$

<u>Example</u>. Verify that the preceding separated solutions satisfy Laplace's equation.

<u>Solution</u>. In case $\lambda > 0$, we have

$$u(x,y) = (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky)$$

so that

$$u_x = \cdots$$
$$u_{xx} =$$
$$u_y =$$
$$u_{yy} = \cdot$$

$$\Rightarrow \underline{\underline{u_{xx} + u_{yy} = 0}}$$

In case $\lambda = 0$ we have

$$\left.\begin{aligned} u_x &= A_1 (A_3 y + A_4), & u_{xx} &= 0 \\ u_y &= (A_1 x + A_2) A_3, & u_{yy} &= 0 \end{aligned}\right\} \Rightarrow u_{xx} + u_{yy} = 0.$$

Similarly (at home!)

In case $\underline{\lambda < 0}$

$$\underline{\underline{u_{xx} + u_{yy} = 0}}$$

Real and complex seperated solutions

<u>Proposition</u>. Let $u(x,y) = v_1(x,y) + i v_2(x,y)$ be a complex-valued solution of the linear PDE

$$\mathcal{L}u = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

where a, b, c, d, e, f, g are real-valued functions of (x,y). Then $v_1(x,y) = \operatorname{Re} u(x,y)$ satisfies the PDE $\mathcal{L}u = g$, and $v_2(x,y) = \operatorname{Im} u(x,y)$ satisfies the associated homogeneous PDE $\mathcal{L}u = 0$.

<u>Proof</u>.

$$u_x = (v_1)_x + i(v_2)_x \qquad u_y = (v_1)_y + i(v_2)_y$$

$$u_{xx} = (v_1)_{xx} + i(v_2)_{xx} \qquad u_{yy} = (v_1)_{yy} + i(v_2)_{yy}$$

$$u_{xy} = (v_1)_{xy} + i(v_2)_{xy}$$

$$\mathcal{L}u = \mathcal{L}(v_1 + iv_2) = \mathcal{L}v_1 + i\mathcal{L}v_2$$

$$= a(v_1)_{xx} + b(v_1)_{xy} + c(v_1)_{yy} + d(v_1)_x + e(v_1)_y + fv_1$$

$$+ i\left(a(v_2)_{xx} + b(v_2)_{xy} + c(v_2)_{yy} + d(v_2)_x + e(v_2)_y + fv_2\right) = g$$

$$= \mathcal{L}v_1 + i\mathcal{L}v_2 = g \quad \text{since } \boxed{\mathcal{L}u = g}$$

$$= g + i \cdot 0 = g$$

We illustrate this technique with the example of Laplace's equation. Letting $u(x,y) = X(x) \cdot Y(y)$, consider a purely imaginary separation constant in the form $\lambda = 2ik^2$, where $k > 0$. This leads to the two ODEs

$$X''(x) - 2ik^2 X(x) = 0 \qquad (1)$$

$$Y''(y) + 2ik^2 Y(y) = 0 \qquad (2)$$

These eq.s can be solved in terms of the complex exponential function $e^{\mu x}$:

$$\xRightarrow{X''(x)-2iK^2X(x)=0} \mu^2 e^{\mu x} - 2iK^2 e^{\mu x} = 0 \Rightarrow \mu^2 = 2iK^2$$

$$\mu = \pm K\sqrt{2i}$$

$$\sqrt{2i} = R(\cos\Phi + i\sin\Phi)$$

$$\begin{cases} 2i = R^2(\cos 2\Phi + i\sin 2\Phi) \\ 2i = 2\left(\cos\left(\frac{\pi}{2}+2\pi K\right) + i\sin\left(\frac{\pi}{2}+2\pi K\right)\right) \end{cases},$$

$$R^2 = 2,\quad R = \sqrt{2},\quad \Phi = \frac{\pi}{4} + \pi K$$

$K=0$

$$\omega_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \cancel{= 1+i}$$

$K=1$:

$$\omega_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -(1+i)$$

So

$$\mu_1 = K(1+i),\quad \mu_2 = -K(1+i)$$

Thus

$$X(x) = A_1 e^{K(1+i)x} + A_2 e^{-K(1+i)x}$$

$$Y''(y) + 2iK^2 Y(y) = 0,\quad \boxed{e^{\mu y}}$$

$$\mu^2 = -2iK^2,\quad \mu = \pm K\sqrt{-2i}$$

$\sqrt{-2i} = ?$

$$\omega_1 = 1-i,\quad \omega_2 = -(1-i)$$

$$\mu_1 = K(1-i),\quad \mu_2 = -K(1-i)$$

$$Y(y) = A_3 e^{K(1-i)y} + A_4 e^{-K(1-i)y}$$

$$U(x,y) = X(x)Y(y) = \left(A_1 e^{k(1+i)x} + A_2 e^{-k(1+i)x}\right) \times \left(A_3 e^{k(1-i)y} + A_4 e^{-k(1-i)y}\right)$$

$$= \cdots = e^{k(x+y)} e^{ik(x-y)} + e^{k(x-y)} e^{ik(x+y)} + e^{k(y-x)} e^{-ik(x+y)} + e^{-k(x+y)} \cdot e^{ik(y-x)}$$

$\Rightarrow$ We obtain the complex seperated solutions

$$U(x,y) = \begin{cases} e^{k(x+y)} e^{ik(x-y)} \\ e^{k(x-y)} e^{ik(x+y)} \\ e^{k(y-x)} e^{-ik(x+y)} \\ e^{-k(x+y)} \cdot e^{ik(y-x)} \end{cases}$$

When we take the real and imaginary parts, we obtain the following real-valued solutions of Laplace's equation:

$$U(x,y) = \begin{cases} e^{k(x+y)} \cos k(x-y) \,, & e^{k(x+y)} \sin k(x-y) \\ e^{k(x-y)} \cos k(x+y) \,, & e^{k(x-y)} \sin k(x+y) \\ e^{k(y-x)} \cos k(x+y) \,, & e^{k(y-x)} \sin k(x+y) \\ e^{-k(x+y)} \cos k(y-x) \,, & e^{-k(x+y)} \sin k(y-x) \,. \end{cases}$$

Example 2. Find seperated solutions of the PDE $U_{xx}-U_t=0$ in the form $U(x,t)=e^{i\mu x}\cdot e^{\beta t}$, with μ real.

Solution. Substituting $U(x,t)=e^{i\mu x}\cdot e^{\beta t}$ in the eq.

$U_x=i\mu e^{i\mu x}\cdot e^{\beta t}$, $U_{xx}=-\mu^2 e^{i\mu x}\cdot e^{\beta t}$, $U_t=\beta e^{i\mu x}\cdot e^{\beta t}$

$$\Rightarrow \quad -\mu^2-\beta=0 \Rightarrow \beta=-\mu^2$$

$\Rightarrow$ we have the seperated solutions

$$U(x,t)=e^{i\mu x}\cdot e^{-\mu^2 t}=e^{-\mu^2 t}(\cos\mu x+i\sin\mu x)$$
$$=\cos\mu x\cdot e^{-\mu^2 t}+i\left(\sin\mu x\cdot e^{-\mu^2 t}\right)$$

Taking the real and imaginary parts, we obtain the real-valued seperated solutions

$$U(x;t)=\sin\mu x\cdot e^{-\mu^2 t},\quad U(x;t)=\cos\mu x\cdot e^{-\mu^2 t}.$$

By taking linear combinations, we may write the general real-valued seperated solution as

$$U(x;t)=\left(A_1\sin\mu x+A_2\cos\mu x\right)e^{-\mu^2 t},$$

where A_1, A_2 are arbitrary constants.

Example 3. Find separated solutions of the PDE $U_{xx}-U_t=0$ in the form $U(x,t)=e^{\alpha x}\cdot e^{i\omega t}$, where ω is real and positive.

Solution. Substituting $U(x,t)=e^{\alpha x}\cdot e^{i\omega t}$ in $U_t-U_{xx}=0$ yelds $\alpha^2-i\omega=0 \Rightarrow \alpha=\pm\sqrt{i}\cdot\sqrt{\omega}$.

$$u(x;t) = (A_1 \sin\mu x + A_2 \cos\mu x)\, e^{-\mu^2 t}$$

$u_{xx} - u_t = 0 \qquad u(x,t) = e^{\alpha x} \cdot e^{i\omega t}$, $\omega \geq 0$ is a real.

$u_t - u_{xx} = 0$

$u_t = i\omega\, e^{\alpha x} \cdot e^{i\omega t}$

$u_x = \alpha e^{\alpha x} \cdot e^{i\omega t}$

$u_{xx} = \alpha^2 e^{\alpha x} e^{i\omega t}$

$i\omega u - \alpha^2 u = 0$

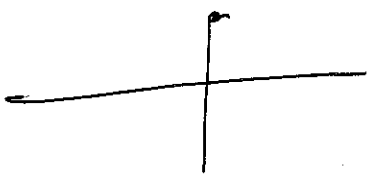
$i\omega - \alpha^2 = 0, \quad \underline{\alpha^2 = i\omega}$

$\alpha = \pm \sqrt{i} \cdot \sqrt{\omega}$

$\sqrt{i} = R(\cos\Phi + i\sin\Phi)$

$\cancel{\Phi > i}$

$i = R^2(\cos 2\Phi + i\sin 2\Phi)$



$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$R = 1, \quad \cancel{\cos 2\Phi = \cos}$

$2\Phi = \frac{\pi}{2} + 2\pi k$

$\Phi = \frac{\pi}{4} + \pi k$

$\sqrt{i} = \cos\left(\frac{\pi}{4} + \pi k\right) + i\sin\left(\frac{\pi}{4} + \pi k\right)$

$k=0$

$\omega_1 \quad \sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$k=1$

$\omega_2 = \cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right)$

$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$\sqrt{t} = \pm\frac{1}{\sqrt{2}}(1+i)$

$$\alpha = \pm (1+i)\sqrt{\frac{\omega}{2}}$$

$$\Rightarrow \quad U(x,t) = e^{x(1+i)\sqrt{\frac{\omega}{2}}} \cdot e^{i\omega t}$$

$$= e^{x\cdot\sqrt{\frac{\omega}{2}} + i\left(x\sqrt{\frac{\omega}{2}} + \omega t\right)}$$

$$= e^{x\sqrt{\frac{\omega}{2}}} \cdot e^{i\left(\omega t + x\sqrt{\frac{\omega}{2}}\right)}$$

$$U(x,t) = e^{-x(1+i)\sqrt{\frac{\omega}{2}}} \cdot e^{i\omega t} = e^{-x\sqrt{\frac{\omega}{2}} - ix\sqrt{\frac{\omega}{2}}} \cdot e^{i\omega t}$$

$$= e^{-x\sqrt{\frac{\omega}{2}}} \cdot e^{i\left(\omega t - x\sqrt{\frac{\omega}{2}}\right)}$$

$$U(x,t) = \begin{cases} e^{x\sqrt{\frac{\omega}{2}}} \cdot \cos\left(\omega t + x\sqrt{\frac{\omega}{2}}\right) \\ e^{x\sqrt{\frac{\omega}{2}}} \cdot \sin\left(\omega t + x\sqrt{\frac{\omega}{2}}\right) \\ e^{-x\sqrt{\frac{\omega}{2}}} \cos\left(\omega t - x\sqrt{\frac{\omega}{2}}\right) \\ e^{-x\sqrt{\frac{\omega}{2}}} \sin\left(\omega t - x\sqrt{\frac{\omega}{2}}\right) \end{cases}$$

These real-valued solutions are no longer in the seperated form $X(x)T(t)$. But because they arise as the real and imaginary parts of complex seperated solutions, we refer to them as <u>quasi-seperated solutions</u>.

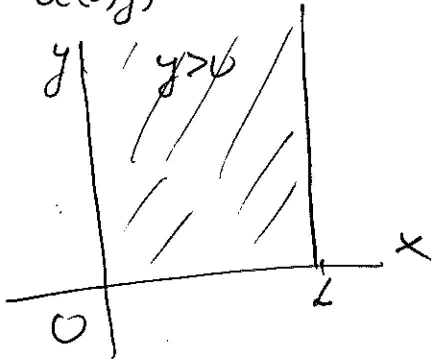
If some of the coefficients a, b, c, d, e, f are non constant, we will no longer have separated solutions in the form of exponential functions.

$$aU_{xx} + bU_{xy} + CU_{yy} + dU_x + eU_y + fu = 0$$

Separated solutions with boundary conditions.

In many problems we need separated solutions that satisfy certain additional conditions, which are suggested by the physics of the problem. They may be in the form of boundary conditions or conditions of boundedness.

Example 1. Find the separated solutions of Laplace's equation $U_{xx}+U_{yy}=0$ in the region $0<x<L$, $y>0$ that satisfy the boundary conditions $U(0,y)=0$, $U(L,y)=0$, $U(x,0)=0$



Solution

We have the separated solutions of three types, depending on the separation constant:

$$U(x,y)=\begin{cases}(A_1e^{kx}+A_2e^{-kx})(A_3\cos ky+A_4\sin ky), & k>0\\ (A_1x+A_2)(A_3y+A_4) & \\ (A_1\cos lx+A_2\sin lx)(A_3e^{ly}+A_4e^{-ly}) & l>0.\end{cases}$$

We can also write the separated solutions of Laplace's eq. in terms of hyperbolic functions. These are defined by the formulas

$$\sinh a=\frac{1}{2}(e^{a}-e^{-a}), \quad \cosh a=\frac{1}{2}(e^{a}+e^{-a})$$

$$\Rightarrow \begin{cases}e^{a}-e^{-a}=2\sinh a\\ e^{a}+e^{-a}=2\cosh a\end{cases} \Longrightarrow \boxed{\begin{array}{l}e^{a}=\cosh a+\sinh a,\\ e^{-a}=\cosh a-\sinh a\end{array}}$$

Now, we can write the separated solutions of Laplace's eq. in the equivalent form

$$\left(A_1(\cosh kx+\sinh kx)+A_2(\cosh kx-\sinh kx)=\underbrace{(A_1+A_2)}_{A_1}\cosh kx+\underbrace{(A_1-A_2)}_{A_2}\sinh kx\right)$$

$$U(x,y)=\begin{cases}(A_1\sinh kx+A_2\cosh kx)(A_3\cos ky+A_4\sin ky), & k>0\\ (A_1x+A_2)(A_3y+A_4) & \\ (A_1\cos lx+A_2\sin lx)(A_3\sinh ly+A_4\cosh ly), & l>0.\end{cases}$$

In the first case

From $U(0,y)=0$ follows

$$0 = U(0,y) = A_2(A_3 \cos ky + A_4 \sin ky), \text{ so}$$

$A_2=0$, while

$$0 = U(L,y) = A_1 \sinh kL (A_3 \cos ky + A_4 \sin ky) \text{ implies}$$

that $A_1=0$, so this case does not produce any seperated solutions that satisfy the boundary conditions.

In the second case, we must have

$$0 = U(0,y) = A_2(A_3 y + A_4), \text{ so } A_2=0, \text{ and}$$

$$0 = U(L,y) = A_1 L (A_3 y + A_4), \text{ so } A_1 = 0.$$

Therefor this case does not produce any seperated solutions that satisfy the boundary conditions.

In the third case, we must have

$$0 = U(0,y) = A_1(A_3 \sinh ly + A_4 \cosh ly), \text{ so that } A_1 = 0;$$

and $0 = U(L,y) = A_2 \sin Ll (A_3 \sinh ly + A_4 \cosh ly)$ has a nonzero solution if and only if $\sin Ll = 0 \Rightarrow Ll = n\pi$ for some $n=1,2,3,\dots$

To satisfy the boundary condition $U(x,0)=0$, we must have $A_4=0$. Writing $A = A_2 A_3$, we have obtained the following seperated solutions of Laplace's equation satisfying the boundary conditions:

$$U(x,y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2, \dots .$$

Example 2. Find the seperated solutions $U(x;t)$ of the heat equation $U_{xx} - U_t = 0$ in the region $0<x<L$, $t>0$ that satisfy the boundary conditions

$$U(0;t)=0, \quad U(L;t)=0.$$

Solution.

We found the real-valued separated solutions

$$U(x;t) = (A_1 \sin \mu x + A_2 \cos \mu x) e^{-\mu^2 t}$$

We must have $0 = U(0;t) = A_2 e^{-\mu^2 t} \Rightarrow A_2 = 0$.

$0 = u(L;t) = A_1 (\sin \mu L) e^{-\mu^2 t}$ if and only if $\Longleftrightarrow \mu L = n\pi, \ n = 1, 2, \ldots$ $\mu = \frac{n\pi}{L}$

Therefore the separated solutions satisfying the boundary conditions are of the form

$$u(x;t) = A_1 \sin \frac{n\pi x}{L} \cdot e^{-\left(\frac{n\pi}{L}\right)^2 t}, \quad n = 1, 2, \ldots$$

Example 3. Find the separated solutions of the wave equation $u_{tt} - c^2 u_{xx} = 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.

Solution

Let $u(x;t) = X(x)T(t)$. It follows that

$X(x) \cdot T''(t) - c^2 X''(x) T(t) = 0$. Thus $\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$,

$X''(x) + \lambda X(x) = 0$, $T''(t) + \underline{\lambda c^2} T(t) = 0$. The boundary conditions require $X(0) = 0$, $X(L) = 0$.

$\underline{\lambda = k^2 > 0}$ (for $\lambda < 0$, we have only zero solutions)

$\Rightarrow \ X(x) = A_3 \sin \frac{n\pi x}{L}$

$T(t) = A_1 \cos \frac{n\pi ct}{L} + A_2 \sin \frac{n\pi ct}{L}$ for constants A_1, A_2, A_3. The required separated solutions are

$u(0,t) = 0$

$X(0;t) \cdot T(t) = 0$

$\Rightarrow X(0) = 0$

and $u(L;t) = 0$

$X(L) \cdot T(t) = 0$

$\Rightarrow X(L) = 0$

$$u(x;t) = \left(A_1 \cos \frac{n\pi ct}{L} + A_2 \sin \frac{n\pi ct}{L}\right) \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

Example 4. Find the complex separated solutions $u(x;t)$ of the wave eq. $u_{tt} - c^2 u_{xx} = 0$, which are bounded in the form $|u(x;t)| \le M$ for some constant M and all t, $-\infty < t < \infty$.

Solution

Taking $u(x,t) = e^{ax+bt}$ and substituting in the wave equation, we have $b^2 - c^2 a^2 = 0$; thus $b = \pm ca$. The separated solutions are of the form $u(x;t) = e^{ax} e^{cat}$, $e^{ax} e^{-cat}$. This solutions is bounded for all t if and only if a is pure imaginary, $a = ik$ for k real.

Thus the solutions are $u(x;t) = e^{ikx} \cdot e^{cikt} = e^{ik(x+ct)}$, $e^{ik(x-ct)}$. The real (quasi-separated) solutions are $\cos k(x+ct)$, $\cos k(x-ct)$, $\sin k(x+ct)$, $\sin k(x-ct)$.

<u>Example 5</u>. Find the complex separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$, which are bounded in the form $|u(x;t)| \leq M$ for some constant M and all t, $-\infty < t < \infty$.

<u>Solution</u>. Taking $u(x;t) = e^{ax+bt}$ and substituting in the heat eq., we have $b - a^2 = 0$. In order that this solution be bounded for all t, $-\infty < t < \infty$, it is necessary that the constant b be purely imaginary; otherwise the solution would be tend to $+\infty$ for large $|t|$ if b had a nonzero real part. Hence we set $b = i\omega$, where ω is real. Assuming $\omega > 0$, the eq. $a^2 = i\omega$ has two solutions.

$$a = \sqrt{\frac{\omega}{2}}\,(1+i), \quad a = -\sqrt{\frac{\omega}{2}}\,(1+i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{\frac{\omega}{2}}(1+i)x} + A_2 e^{-\sqrt{\frac{\omega}{2}}(1+i)x}\right)$$

If $\omega < 0$, then the equation $a^2 = i\omega$ has two solutions,

$$\Rightarrow \quad a = \pm\sqrt{i} \cdot i\sqrt{|\omega|}, \quad \sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i), \quad \sqrt{i}\cdot i = \pm\frac{1}{\sqrt{2}}(i-1) = \pm\frac{1}{\sqrt{2}}(1-i)$$

$$\Rightarrow \quad a = \sqrt{\frac{|\omega|}{2}}\,(1-i), \quad a = -\sqrt{\frac{|\omega|}{2}}\,(1-i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{\frac{|\omega|}{2}}(1-i)x} + A_2 e^{-\sqrt{\frac{|\omega|}{2}}(1-i)x}\right).$$

When we consider more general linear PDEs, complex-valued separated solutions may always be found if the functions a, b, c, d, e, f are constants. These solutions may be written as exponential functions.

<u>Proposition.</u> Consider the linear homogeneous PDE

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0$$

suppose that a, b, c, d, e, f are real constants. Then there exist complex separated solutions of the form

$$u(x,y) = e^{\alpha x} \cdot e^{\beta y}$$

for appropriate choices of the complex numbers α, β.

<u>Proof.</u> Substituting $u(x,y) = e^{\alpha x} \cdot e^{\beta y}$ into the PDE, we must have ($e^{\alpha x} \cdot e^{\beta y} \neq 0$)

$$(a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f) = 0 \qquad (1)$$

For a given value of β, we may solve this eq. for α to obtain in general two roots α_1, α_2. Alternatively, we may fix α and solve for β to obtain in general two roots β_1, β_2 ●

If some of the coefficients a, b, c, d, e, f are not constant, the eq. may not admit any nonconstant separated solutions. Nevertheless, various classes of eq.s can still be solved by the separation of variables. For example, for any eq. of the form

$$a(x)u_{xx} + c(y)u_{yy} + d(x)u_x + e(y)u_y = 0$$

if we divide by $X(x)Y(y)$, we have

$$\underbrace{\left[a(x)\frac{X''(x)}{X(x)} + d(x)\frac{X'(x)}{X(x)}\right]}_{\lambda \ \text{Const.}} + \underbrace{\left[c(y)\frac{Y''(y)}{Y(y)} + e(y)\frac{Y'(y)}{Y(y)}\right]}_{-\lambda \ \text{Const}} = 0$$

$$\Rightarrow \quad a(x)X''(x) + d(x)X'(x) - \lambda X(x) = 0$$

$$c(y)Y''(y) + e(y)Y'(y) + \lambda Y(y) = 0 \qquad \text{(Euler's eq.)}$$

<u>Example</u>. Find all of the real-valued separated solutions of the PDE $u_{xx} + y^2 u_{yy} + y u_y = 0$ valid for $y > 0$.

<u>Solution.</u>

we let $u(x,y) = X(x)Y(y)$

$$\Rightarrow \quad X''(x) + \lambda X(x) = 0 \qquad (1)$$

$$y^2 Y''(y) + yY'(y) - \lambda Y(y) = 0 \qquad (2)$$

<u>If $\lambda = k^2 > 0$</u>, then the general solution of (1) is $X(x) = A_1 \cos kx + A_2 \sin kx$. The eq. (2) can be solved by a power $Y(y) = y^r$:

$$y^2 \cdot r(r-1) \cdot y^{r-2} + y \cdot r y^{r-1} - k^2 y^r = 0$$

$$\Rightarrow r(r-1) + r - k^2 = 0 \Rightarrow r^2 = k^2 \Rightarrow \underline{r = \pm k}$$

So $Y(y) = A_3 y^k + A_4 y^{-k}$ is the general solution of (2)

If $\lambda = 0$, then $X(x) = A_1 + A_2 x$, while (2) becomes

$$y^2 Y'' + yY'(y) = 0 \Rightarrow yY'' + Y'(y) = 0$$

The general sol. is $Y(y) = A_3 + A_4 \log y$, $y > 0$.

If $\lambda = -\ell^2 < 0$, then the general solution of (1) is $X(x) = A_1 e^{\ell x} + A_2 e^{-\ell x}$, while (2) becomes

$$y^2 Y'' + yY' + \ell^2 Y = 0,$$ which has the general solution

$$Y(y) = A_3 \cos(\ell \log y) + A_4 \sin(\ell \log y).$$

Putting these together, we have the real valued separated solution:

$$u(x,y) = \begin{cases} (A_1 \cos kx + A_2 \sin kx)(A_3 y^k + A_4 y^{-k}), & k>0 \\ (A_1 + A_2 x)(A_3 + A_4 \log y) \\ (A_1 e^{\ell x} + A_2 e^{-\ell x})(A_3 \cos(\ell \log y) + A_4 \sin(\ell \log y)), & \ell > 0 \end{cases}$$

Orthogonal Functions

Separated solutions of linear PDEs with suitable boundary conditions lead to systems of orthogonal functions. In order to formulate the property of orthogonality, we first introduce the general notion of inner product

1. Inner product space of functions. The notions of dot product, distance, orthogonality, and projection, which are familar for vectors in three dimensions, can also be formulated for real-valued f.s on an interval $a \le x \le b$. The inner product of two functions $\varphi(x), \psi(x)$ on the interval $a \le x \le b$ is defined by the integral

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)\,dx \qquad (1)$$

For example, on the interval $0 \le x \le 1$, ~~of two f~~ we have

$$\langle x, e^{x^2} \rangle = \int_0^1 x e^{x^2} dx = \frac{1}{2}\int_0^1 e^u du = \frac{1}{2} e^u \Big|_0^1 = \qquad \begin{matrix} u = x^2 \\ \frac{1}{2} du = x dx \end{matrix}$$

$$= \frac{1}{2}(e-1).$$

Properties of the inner product:

For any functions $\varphi_1, \varphi_2, \psi_1, \psi_2$ and any real number a,

$$\langle \varphi_1, \psi_1 + \psi_2 \rangle = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_1, \psi_2 \rangle$$

$$\langle \varphi_1 + \varphi_2, \psi_1 \rangle = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_1 \rangle$$

$$\langle a\varphi_1, \psi_1 \rangle = a \langle \varphi_1, \psi_1 \rangle$$

$$\langle \varphi_1, a\psi_1 \rangle = a \langle \varphi_1, \psi_1 \rangle \qquad \text{(Proof at Home!)}$$

Def. Two functions φ, ψ are orthogonal on the interval $a \le x \le b$ if and only if $\langle \varphi, \psi \rangle = 0$

Example 1. Show that the f.s $\Phi(x) = \sin x$, $\psi(x) = \cos x$ are orthogonal on the interval $0 \le x \le \pi$ but are not orthogonal on the interval $0 \le x \le \frac{\pi}{2}$.

Solution.

$$\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{2}(\sin x)^2 \Big|_0^{\pi} = 0 \Rightarrow \text{are orthogonal}$$

$$\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx = \frac{1}{2}(\sin x)^2 \Big|_0^{\frac{\pi}{2}} = \frac{1}{2} \Rightarrow \text{are not orthogonal}$$

Example 2. Show that the set of functions $\sin x, \sin 2x, \ldots, \sin Nx$ is orthogonal on the interval $0 \le x \le \pi$ for any $N \ge 2$.

Solution.

(For more than two functions, we say that $(\varphi_1, \ldots, \varphi_N)$ are orthogonal if $\langle \varphi_i, \varphi_j \rangle = 0$ for $i \neq j$.

The inner product is

$$\int_0^\pi \sin mx \cdot \sin nx \, dx$$

Since $\sin mx \cdot \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$,

If $m \neq n$,

$$\int_0^\pi \sin mx \cdot \sin nx \, dx = \frac{1}{2}\left[\int_0^\pi \left(\cos(m-n)x - \cos(m+n)x\right) dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{m-n} \sin(m-n)x \Big|_0^\pi - \frac{1}{m+n} \sin(m+n)x \Big|_0^\pi\right] = 0$$

$\Rightarrow$ are orthogonal.

The norm of a function is the nonnegative number $\|\varphi\|$ that satisfies

$$\|\varphi\|^2 = \langle \varphi, \varphi \rangle$$

For example, on the interval $0 \le x \le \pi$,

$$\|\sin x\|^2 = \int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1 - \cos 2x) \, dx = \frac{\pi}{2}.$$

The distance between φ and ψ is defined by

$$d(\varphi, \psi) = \|\varphi - \psi\|.$$

For example, the distance between $\sin x$ and $\cos x$ on the interval $0 \le x \le \pi$ is obtained from

$$[d(\sin x, \cos x)]^2 = \int_0^\pi (\sin x - \cos x)^2 dx = -\int_0^\pi 2 \sin x \cos x \, dx + \int_0^\pi 1 \cdot dx$$

$$\cancel{\pi} - 2\int_0^\pi \sin x \cos x \, dx = -2 \cdot (\sin x)^2 \cdot \frac{1}{2} \Big|_0^\pi + \pi = \pi,$$

so that the distance is given by $d = \sqrt{\pi}$.

In three-dimensional space we have the dot product formula $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$.

Hence $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} \Rightarrow \vec{v}\cdot\vec{w} \leq \|\vec{v}\|\cdot\|\vec{w}\|$

In order to extend this to functions on an interval, we need to know that the corresponding ~~ratios~~ ratio is not greater than 1 in absolute value.

This is known as the <u>Schwarz inequality</u>.

<u>Proposition</u>. Suppose that $\varphi(x)$, $\psi(x)$ are nonzero f.s defined on an interval $a \leq x \leq b$. Then

$$\langle\varphi,\psi\rangle^2 \leq \|\varphi\|^2\|\psi\|^2 \qquad (2)$$

<u>Proof</u>. For any real number t,

$$D(t) := \|\varphi - t\psi\|^2 = \langle\varphi - t\psi, \varphi - tw\rangle = \langle\varphi,\varphi\rangle - t\langle\varphi,w\rangle - t\langle\psi,\varphi\rangle$$

$$+ t^2\langle\psi,\psi\rangle = \|\varphi\|^2 - 2t\langle\varphi,w\rangle + t^2\|\psi\|^2 =$$

$$= \|\psi\|^2\left(t^2 - 2t\frac{\langle\varphi,\psi\rangle}{\|\psi\|^2} + \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^4}\right) + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right)$$

$$= \|\psi\|^2\left(t - \frac{\langle\varphi,\psi\rangle}{\|\psi\|^2}\right)^2 + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right)$$

The minimum value at $t = t_0$, where $t_0 = \dfrac{\langle\varphi,\psi\rangle}{\|\psi\|^2}$, is nonnegative

$$D(t_0) = \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \geq 0$$

$$\Rightarrow \langle\varphi,\psi\rangle^2 \leq \|\varphi\|^2\|\psi\|^2 \qquad \blacksquare$$

2. Projection of a function onto an orthogonal set.

Let $(\varphi_1, \dots, \varphi_N)$ be a set of orthogonal functions with $\|\varphi_i\| \neq 0$ for $1 \le i \le N$. If f is an arbitrary function, we compute the minimum of

$$D(c_1, \dots, c_N) = \|f - (c_1\varphi_1 + \dots + c_N\varphi_N)\|^2$$

where $(c_1, \dots, c_N)$ range over all real values. In other words, we are trying to find the best „mean square approximation" of the given function $f(x)$, $a \le x \le b$, by means of linear combinations of the members of the orthogonal set.

<u>Proposition</u>. The minimization problem has the following properties:

1°. The minimum is attained uniquely when

$$c_i = \hat{c}_i := \frac{\langle f, \varphi_i \rangle}{\|\varphi_i\|^2}, \quad 1 \le i \le N$$

2°. The minimum distance is given by

$$d_{min}^2 = \|f\|^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{\|\varphi_i\|^2}$$

3°. The Fourier coefficients $\hat{c}_1, \dots, \hat{c}_N$ satisfy Bessel's inequality

$$\hat{c}_1^2 \|\varphi_1\|^2 + \dots + \hat{c}_N^2 \|\varphi_N\|^2 \le \|f\|^2$$

The function $\hat{c}_1\varphi_1 + \dots + \hat{c}_N\varphi_N$ is called the <u>projection</u> of f onto the orthogonal set $(\varphi_1, \dots, \varphi_N)$; $\hat{c}_i$ is called the i-th <u>Fourier coefficient</u> of f.

Proof. $D(c_1,\dots,c_N) = \|f\|^2 - 2\sum_{i=1}^{N} c_i \langle f,\varphi_i\rangle + \sum_{i=1}^{N} c_i^2 \|\varphi_i\|^2$

$$= \sum_{i=1}^{N} \|\varphi_i\|^2 \left(c_i^2 - 2c_i \frac{\langle f,\varphi_i\rangle}{\|\varphi_i\|^2} + \frac{\langle f,\varphi_i\rangle^2}{\|\varphi_i\|^4} \right) + \|f\|^2 - \sum_{i=1}^{N} \frac{\langle f,\varphi_i\rangle^2}{\|\varphi_i\|^2}$$

$$= \sum_{i=1}^{N} \|\varphi_i\|^2 \left(c_i - \frac{\langle f,\varphi_i\rangle}{\|\varphi_i\|^2} \right)^2 + \|f\|^2 - \sum_{i=1}^{N} \frac{\langle f,\varphi_i\rangle^2}{\|\varphi_i\|^2} \geqslant$$

$$\geqslant \|f\|^2 - \sum_{i=1}^{N} \frac{\langle f,\varphi_i\rangle^2}{\|\varphi_i\|^2}$$

The minimum is achieved when $c_i = \hat{c}_i := \dfrac{\langle f,\varphi_i\rangle}{\|\varphi_i\|^2}$, as required. The value of the min. is

$$D(\hat{c}_1,\dots,\hat{c}_N) = \|f\|^2 - \sum_{i=1}^{N} \frac{\langle f,\varphi_i\rangle^2}{\|\varphi_i\|^2} = \|f\|^2 - \sum_{i=1}^{N} \hat{c}_i^2 \|\varphi_i\|^2$$

as required. Since this is nonnegative, Bessel's inequality follows from $D(\hat{c}_1,\dots,\hat{c}_N) \geqslant 0$.

Example 1. Find the projection of the function $f(x)=1$ onto the orthogonal set $\{\sin x, \sin 2x, \sin 3x\}$ on the interval $0 \le x \le \pi$ and compute d_{min}.

Solution.

$$\|\varphi_m\|^2 = \int_0^\pi \sin^2 mx\,dx = \frac{1}{2}\int_0^\pi (1-\cos 2mx)\,dx = \frac{\pi}{2}$$

The Fourier coefficients are

$$\hat{c}_1 = \frac{\int_0^\pi \sin x\,dx}{\int_0^\pi \sin^2 x\,dx} = \frac{\cos x\,\big|_\pi^0}{\frac{\pi}{2}} = \frac{2}{\frac{\pi}{2}} = \frac{4}{\pi},$$

$$\hat{c}_2 = \frac{\int_0^\pi \sin 2x\,dx}{\frac{\pi}{2}} = \frac{-\frac{1}{2}\cos 2x\,\big|_0^\pi}{\frac{\pi}{2}} = \frac{0}{\pi} = 0$$

$$\hat{c}_3 = \frac{\int_0^\pi \sin 3x\,dx}{\frac{\pi}{2}} = \frac{-\frac{1}{3}\cos 3x\,\big|_0^\pi}{\frac{\pi}{2}} = \frac{-\frac{1}{3}(-1-1)}{\frac{\pi}{2}} = \frac{\frac{2}{3}}{\frac{\pi}{2}} = \frac{4}{3\pi}$$

The projection is the function

$$S(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x.$$

$$d_{min}^2 = \|f\|^2 - \sum_{i=1}^{3} \hat{c}_i^2 \|\varphi_i\|^2 = \pi - \left(\frac{4}{\pi}\right)^2\left(\frac{\pi}{2}\right) - \left(\frac{4}{3\pi}\right)^2 \cdot \left(\frac{\pi}{2}\right) \approx 0.33$$

3. Orthonormal sets of functions.

The formulas for the Fourier coefficients and the minimum distance become simple when the functions $(\varphi_1, \dots, \varphi_N)$ are orthonormal. This means that $\langle \varphi_i, \varphi_j \rangle = 0$ for $i \neq j$ and $\langle \varphi_i, \varphi_i \rangle = 1$, $1 \le i \le N$. Thus we have for orthonormal functions

$$\hat{C}_i = \langle f, \varphi_i \rangle \qquad 1 \le i \le N \qquad (3)$$

$$d_{min}^2 = D(\hat{C}_1, \dots, \hat{C}_N) = \|f\|^2 - (\hat{C}_1^2 + \dots + \hat{C}_N^2) \qquad (4)$$

If $(\varphi_1, \dots, \varphi_N)$ is an orthogonal set of functions, we obtain an orthonormal set by replacing φ_i by $\frac{\varphi_i}{\|\varphi_i\|}$, $1 \le i \le N$.

Example. Let $\varphi_1 = 1$, $\varphi_2 = \sin x$, $\varphi_3 = \cos x$ for $-\pi < x < \pi$. Verify that this is an orthogonal set and find the corresponding orthonormal set.

Solution.
$$\langle \varphi_i, \varphi_j \rangle = 0, \quad i \neq j$$

$$\langle \varphi_1, \varphi_2 \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = -\cos\pi + \cos\pi = 0$$

$$\langle \varphi_1, \varphi_3 \rangle = \int_{-\pi}^{\pi} 1 \cdot \cos x \, dx = \sin x \Big|_{-\pi}^{\pi} = 0$$

$$\langle \varphi_2, \varphi_3 \rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 \Big|_{-\pi}^{\pi} = 0$$

$$\|\varphi_1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi, \quad \|\varphi_2\|^2 = \int_{-\pi}^{\pi} \sin^2 x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left(2\pi - \frac{1}{2} \sin 2x \Big|_{-\pi}^{\pi}\right) = \pi, \quad \|\varphi_3\|^2 = \int_{-\pi}^{\pi} \cos^2 x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left(2\pi + \frac{1}{2} \sin 2x \Big|_{-\pi}^{\pi}\right) = \pi$$

The orthonormal set is $\left(\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}\right)$.

In many problems we are given an <u>infinite</u> orthonormal set

$$(\varphi_n)_{n\geq 1} = (\varphi_1, \varphi_2, \dots)$$

The Fourier coefficients are

$$\hat{C}_i = \langle f, \varphi_i \rangle \quad , \quad 1 \leq i \leq N$$

which don't depend on N.

Bessel's inequality: for each N

$$\sum_{i=1}^{N} \hat{C}_i^2 \leq \|f\|^2 \quad , \quad N = 1, 2, \dots$$

This is valid for every $N = 1, 2, \dots$; hence the infinite series $\sum_{i=1}^{\infty} \hat{C}_i^2$ converges and we have

$$\sum_{i=1}^{\infty} \hat{C}_i^2 \leq \|f\|^2 \quad \text{— Bessel inequality.}$$

Def:

When Bessel's inequality is an equality

$$\sum_{i=1}^{\infty} \hat{C}_i^2 = \|f\|^2,$$

~~then~~ this called <u>Parseval's equality</u>.

<u>Proposition</u>. Let $(\varphi_n)_{n\geq 1}$ be an orthonormal set and f a function with $\int_a^b f^2(x)dx < \infty$. Parseval's equality is true if and only if we have mean square convergence of the series $\sum_{i=1}^{\infty} \hat{C}_i \varphi_i$. (The Def: <u>mean square convergence</u> of the series $\sum_{i=1}^{\infty} \hat{C}_i \varphi_i$ is defined as $\lim_{N\to\infty} \|f - \sum_{i=1}^{N} \hat{C}_i \varphi_i\|^2 = 0$.)

<u>Proof</u>. Let $\hat{C}_i = \langle f, \varphi_i \rangle$ be the i-th Fourier coefficient of f.

Then, we have

$$\langle f - \sum \hat{C}_i \varphi_i , f - \sum \hat{C}_j \varphi_j \rangle$$

$$\|f - \sum_{i=1}^{N} \hat{C}_i \varphi_i\|^2 = \|f\|^2 - 2\sum_{i=1}^{N} \hat{C}_i \underbrace{\langle f, \varphi_i \rangle}_{\hat{C}_i} + \sum_{i=1}^{N} \hat{C}_i^2 = \|f\|^2 - \sum_{j=1}^{N} \hat{C}_j^2$$

Letting $N \to \infty$, we see that the right side tends to zero if and only if Parseval's equality is valid. The left side tends to zero if and only if we have mean square convergence. $\blacksquare$

Remark. Parseval's equality is not true for an arbitrary function. For example the set

$$\frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \dots$$

is an orthonormal set for $-\pi \le x \le \pi$.

The function $f(x)=1$ has all Fourier coefficients zero:

$\hat{c}_i = \langle f, \varphi_i \rangle$, $i = 1, 2, \dots$

$$\langle 1, \tfrac{1}{\sqrt{\pi}}\sin x \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin x \, dx = -\frac{1}{\sqrt{\pi}} \cos x \Big|_{-\pi}^{\pi} = 0$$

$$\langle 1, \tfrac{1}{\sqrt{\pi}}\sin nx \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin nx \, dx = \frac{-1}{n\sqrt{\pi}} \cos nx \Big|_{-\pi}^{\pi} = 0$$

$$\langle 1, \tfrac{1}{\sqrt{\pi}}\cos nx \rangle = \frac{1}{n\sqrt{\pi}} \sin nx \Big|_{-\pi}^{\pi} = 0 \quad , \forall\, n \geq 1$$

But $\quad \|f\|^2 = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$

Then Bessel's inequality is true $\quad 0 = \sum_{i=1}^{\infty} \hat{c}_i^2 < \|f\|^2 = 2\pi$

and $\quad 0 \neq 2\pi \Rightarrow$ Parseval's equality is not true.

Def. If Parseval's equality holds for all functions f with $\int_a^b (f(x))^2 dx < \infty$, then we say that the orthogonal set is complete on the interval $a \le x \le b$.

(The system $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\}_{n \geq 1}$ is complete on the interval $-\pi \le x \le \pi$.)

$\hat{C}_i = \langle f, \varphi_i \rangle$, $\underline{f(x)=1}$, $\cancel{\frac{1}{\sqrt{\pi}}(\sin nx, \cos nx)_{n\geq 1}}$

$$\hat{C}_1 = \langle 1, \frac{1}{\sqrt{2\pi}} \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}\, dx = \sqrt{2\pi}$$

$$\hat{C}_2 = \langle 1, \frac{\sin x}{\sqrt{\pi}} \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin x\, dx = 0,$$

$\vdots$

$$\langle 1, \frac{\sin nx}{\sqrt{\pi}} \rangle = \langle 1, \frac{\cos nx}{\sqrt{\pi}} \rangle = 0$$

Then

$$\sum_{i=1}^{\infty} \hat{C}_i^2 = \left(\sqrt{2\pi}\right)^2 = 2\pi = \|f\|^2 = \|1\|^2 = \langle 1, 1 \rangle$$

$$= \int_{-\pi}^{\pi} 1 \cdot dx = 2\pi .$$

0.3 5. Weighted inner product

Def. 1. A weighted inner product with respect to a positive weight function $\rho(x)$, $a \leq x \leq b$ is defined by the integral

$$\langle \varphi, \psi \rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)\, dx$$

This has the same properties (of linearity and homogeneity) as the ordinary inner product.

Def. 2. We say that two functions φ, ψ are orthogonal with respect to the weight function $\rho(x)$, $a \leq x \leq b$, if $\langle \varphi, \psi \rangle_\rho = 0$.

Weighted orthogonality arises when we make a change of variables by means of an <u>increasing</u> differentiable function $x = h(y)$. The ordinary inner product is transformed as follows:

$$\int_a^b \varphi(x)\psi(x)\,dx = \int_c^d \varphi(h(y))\psi(h(y))\underbrace{h'(y)}_{\geq 0}\,dy$$

Therefore, if $\varphi(x), \psi(x)$ are orthogonal on $a \leq x \leq b$, then the f.s $\varphi(h(y)), \psi(h(y))$ are orthogonal with respect to the weight function $h'(y)$ on $c \leq y \leq d$, where

$a = h(c)$, $b = h(d)$.

<u>Example</u>. Given the orthogonal f.s :

$P_1(x) = x$, $P_2(x) = 3x^2 - 1$ on $-1 \leq x \leq 1$, find the weighted orthogonality relation on $0 \leq y \leq \pi$ under the transformation $x = -\cos y$.

<u>Sol</u>

$P_1(h(y)) = -\cos y$, $P_2(h(y)) = 3\cos^2 y - 1$, $\rho(x) = \sin y$

$x = h(y) = -\cos y$: $-1 = -\cos y \Rightarrow y = 0$
$1 = -\cos y \Rightarrow y = \pi$

$$\int_{-1}^{1} x\,(3x^2 - 1)\,dx = \int_0^{\pi} (-\cos y)(3\cos^2 y - 1)\cdot \sin y\,dy.$$

Fourier series

Many of The classical PDEs with boundary conditions have separated solutions that involve sums of trigonometric functions. This leads to the theory of Fourier series.

<u>Def.</u> A <u>trigonometric series</u> is a function of the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right) \qquad (1)$$

where $A_0, A_1, B_1, \ldots$ are constants.

We will explore the possibility of expanding a large class of functions $f(x)$, $-L<x<L$, as trigonometric series. We first prove that ~~this set~~ of functions is orthogonal on the interval $-L<x<L$.

1. <u>Orthogonality relations.</u>

Proposition 1. We have the orthogonality relations

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases} \qquad (2)$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & n \neq m \\ L & n = m \neq 0 \\ 0 & n = m = 0 \end{cases} \qquad (3)$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\, dx = 0 \quad \text{all } m, n. \qquad (4)$$

<u>Proof.</u> We use

$$\begin{aligned} \cos\alpha \cos\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \\ \sin\alpha \sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \\ \sin\alpha \cdot \cos\beta &= \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]. \end{aligned} \qquad (5)$$

$n \neq m$,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right] dx$$

$$= \frac{L}{2\pi}\left[\frac{\sin (n-m)\pi x/L}{n-m}\Big|_{-L}^{L} + \frac{\sin (n+m)\pi x/L}{n+m}\Big|_{-L}^{L}\right] = 0$$

If $n = m \neq 0$, we have

$$\int_{-L}^{L} \cos^2\frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\left(2L + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_{-L}^{L}\right) = L$$

If $n = m = 0$, the integral is $2L$. This complets the proof of (2). Prove of (3) and (4) at Home!

2. Definition of Fourier coefficients.

In order to define the Fourier series of a function, it suffices to define the Fourier coefficients A_n, B_n, which is done as follows.

Def. Let $f(x)$, $-L < x < L$, be a real-valued function. The Fourier series of f is the trigonometric series (1) where (A_n, B_n) are defined by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad (6)$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \qquad (7)$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \qquad (8)$$

(These definitions) We showed that for any orthogonal set $(\varphi_1, \dots, \varphi_N)$, the minimum of $\|f - \sum_{n=1}^{N} c_n \varphi_n\|^2$ is determined by choosing $(c_1, \dots, c_N)$ as the Fourier coefficients $\frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$, $1 \le n \le N$.

3. Even functions and odd functions.

In order to simplify the computation of Fourier series of many functions encountered in practice, we often exploit symmetry arguments. A function $f(x)$, $-L<x<L$, is <u>even</u> if $f(-x)=f(x)$, $-L<x<L$. A function $f(x)$, $-L<x<L$, is <u>odd</u> if $f(-x)=-f(x)$, $-L<x<L$. For example, $f(x)=x$, $f(x)=x^3$, and $f(x)=\sin x$ are <u>odd</u> f.s, whereas $f(x)=x^2$, $f(x)=x^4$ and $f(x)=\cos x$ are <u>even</u> functions. Many f.s are neither even nor odd. For example $f(x)=x+x^2$.

The <u>product</u> of two even f.s is an even function, the product of an odd function and an even function is an odd function, and the product of two odd f.s is an even function. $(+1)(+1)=+1$, $(-1)(+1)=-1$, $(-1)(-1)=+1$.

If $f(x)$, $-L<x<L$, is an odd function, the integral $\int_{-L}^{L} f(x)\,dx = 0$. If $f(x)$, $-L<x<L$, is an even function, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

<u>Proposition 2</u>. If $f(x)$, $-L<x<L$, is an even function, then $B_n=0$, $n=1,2,\dots$. If $f(x)$, $-L<x<L$ is an odd function, then $A_n=0$, $n=0,1,2,\dots$.

<u>Proof</u>. Since $\sin\frac{n\pi x}{L}$ is an odd and $\cos\frac{n\pi x}{L}$ is an even. Now, if $f(x)$, $-L<x<L$, is an even function, the product $f(x)\cdot\sin\frac{n\pi x}{L}$ is an odd, so $B_n=0$. If $f(x)$, $-L<x<L$ is an odd then $f(x)\cos\frac{n\pi x}{L} \Rightarrow A_n=0$. ∎

Example 1. Compute the Fourier series of $f(x)=x$, $-L<x<L$.

Solution. $f(x)$, $-L<x<L$, is an odd function; therefore $A_n=0$. To compute B_n, we note that $f(x)\sin\frac{n\pi x}{L}$ is an even; thus

$$B_n=\frac{1}{L}\int_{-L}^{L}x\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_{0}^{L}x\sin\frac{n\pi x}{L}\,dx$$

$$=\frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L+\right.$$

$u=x$, $du=dx$

$dv=\sin\frac{n\pi x}{L}dx$

$$\left.+\frac{L}{n\pi}\int_0^L\cos\frac{n\pi x}{L}dx\right]=\frac{2}{L}\left[-L\cdot\frac{L}{n\pi}\cos n\pi+\right.$$

$v=-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$

$$\left.+\frac{L}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right]=-\frac{2L}{n\pi}(-1)^n=\frac{2L}{n\pi}(-1)^{n+1}$$

Therefore the Fourier series of $f(x)=x$, $-L<x<L$, is

$$x=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\frac{n\pi x}{L}.$$

Example 2 Compute the Fourier series of $f(x)=|x|$, $-L<x<L$.

Solution. $f(x)$, $-L<x<L$, is an even function; therefore $B_n=0$. Since $f(x)\cos\frac{n\pi x}{L}$ is an even function, thus, for $n\neq 0$,

$$A_n=\frac{1}{L}\int_{-L}^{L}|x|\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx=\cdots$$

$$=-\left(\frac{2L}{n^2\pi^2}\right)\left[1-(-1)^n\right],\ \text{for } n\neq 0$$

$$A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\cdot\frac{x^2}{2}\Big|_0^L = \frac{L}{2}.$$

Therefore the Fourier series of $f(x) = |x|$, $-L < x < L$, is

$$\frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi x}{L}.$$

4. Periodic functions.

Def. A function $f(x)$, $-\infty<x<\infty$, is $2L$-periodic if
$$f(x+2L)=f(x), \quad -\infty<x<\infty.$$

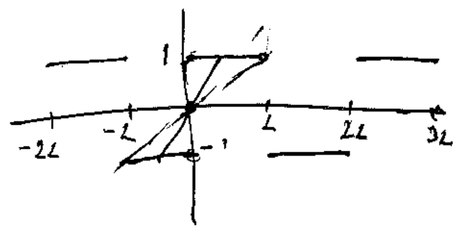
For example, $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ are $2L$-periodic for $n=1,2,\ldots$ since
$$\sin\frac{n\pi}{L}(x+2L)=\sin\left(\frac{n\pi x}{L}+2n\pi\right)=\sin\frac{n\pi x}{L}$$
$$\cos\frac{n\pi}{L}(x+2L)=\cos\left(\frac{n\pi x}{L}+2n\pi\right)=\cos\frac{n\pi x}{L}.$$

The sum, difference, or product of any two $2L$-periodic f.s is again $2L$-periodic. Therefore any convergent trigonometric series defines a $2L$-periodic function $f(x)$, $-\infty<x<\infty$.
Conversely, we can speak of the Fourier series of a $2L$-periodic function $f(x)$, $-\infty<x<\infty$, by restricting x to $-L<x<L$ and computing the Fourier series as we have just done.

Example 3. Compute the Fourier series of the $2L$-periodic function $f(x)=-1$ if $(2n-1)L<x<2nL$, $f(x)=1$ if $2nL<x<(2n+1)L$, $n=0,\pm1,\pm2,\ldots$

Solution. f is an odd function, and thus $A_n=0$,
$$B_n=\frac{2}{L}\int_0^L\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_L^0=\frac{2}{n\pi}\left[1-(-1)^n\right]$$

The Fourier series is
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L}.$$



$n=0$: $-L<x<0 \Rightarrow f(x)=-1$
$n=1$: $L<x<2L \Rightarrow f(x)=-1$
$n=-1$: $-3L<x<-2L$

$0<x<L$,
$2L<x<3L$ $\quad f(x)=1$
$-2L<x<-L$
⋮

6. Fourier sine and cosine series.

Suppose we are given a function $f(x)$, $0<x<L$, and we desire a Fourier series representation. To get this, we extend f to the interval $-L<x<L$ and then compute the Fourier coefficients. There are two natural ways of doing this, giving rise to the Fourier sine series and the Fourier cosine series.

One way of extending f is to define a new function f_0 by

$$f_0(x)=\begin{cases} f(x) & 0<x<L \\ -f(-x) & -L<x<0 \\ 0 & x=0 \end{cases} \qquad (1)$$

f_0 is called the odd extension of f to $(-L,L)$. It is an odd function, and therefore its Fourier coefficients are given as follows:

$$A_n=0 \quad , n=0,1,\dots \qquad (2)$$

$$B_n=\frac{1}{L}\int_{-L}^{L} f_0(x)\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad (3)$$

Therefore we have the Fourier sine series

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad 0\le x\le L$$

where

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Another way of extending f to the interval $(-L,L)$ is to define

$$f_E(x)\begin{cases} f(x) & 0<x<L \\ f(-x) & -L<x<0 \\ 0 & x=0 \end{cases} \qquad (4)$$

f_E is called the even extension of f to $(-L,L)$.

The Fourier coefficients of f_E are as follows:

$$B_n = 0, \quad n = 1, 2, \dots \qquad (5)$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f_E(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx \qquad (6)$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f_E(x)\cdot\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad (7)$$

Therefore, we have the Fourier cosine series

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad 0 \le x \le L$$

where
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \dots$$

Example 1. Compute the Fourier sine series of $f(x) = 1$, $0 < x < L$.

Sol. We have

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = -\frac{2\cancel{L}}{\cancel{L}n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$
$$= \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

The Fourier sine series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L}.$$

We now give an alternative method for computing the Fourier sine series of certain functions that satisfy boundary conditions. Let $f(x)$, $0 \le x \le L$, be a function with $f(0) = 0$, $f(L) = 0$, and $f''(x)$ continuous for $0 \le x \le L$. Then

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\Bigg(\overset{0}{\cancel{\frac{L}{n\pi} f(x)\cos\frac{n\pi x}{L}\Big|_L^0}} +$$
$$+ \frac{L}{n\pi}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx\Bigg) = \frac{2}{n\pi}\Bigg[\overset{0}{\cancel{\frac{L}{n\pi} f'(x)\cdot\sin\frac{n\pi x}{L}\Big|_0^L}}$$
$$- \frac{L}{n\pi}\int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx\Bigg]$$

$u = f(x)$, $du = f'(x)\,dx$
$dv = \sin\frac{n\pi x}{L}\,dx$
$v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}$

$u = f'(x)$, $du = f''(x)\,dx$
$dv = \cos\frac{n\pi x}{L}\,dx$
$v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$

So
$$B_n = -\left(\frac{L}{n\pi}\right)^2 \cdot \frac{2}{L}\int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore the Fourier sine series of $f(x)$, $0<x<L$, is obtained from the Fourier sine series of $f''(x)$, $0<x<L$, by multiplication of the n-th term of the series by $-\left(\frac{L}{n\pi}\right)^2$.

<u>Example 2</u>. Find the Fourier sine series of $f(x) = x^3 - L^2 x$, $0<x<L$.

<u>Solution</u>. The function satisfies $f(0)=0$, $f(L)=0$ with $f''(x) = 6x$. The Fourier sine series of $6x$:

$$\frac{2}{L}\int_0^L 6x\sin\frac{n\pi x}{L}\,dx = \frac{12}{L}\left[x\cdot\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_L^0 + \right.$$

$$\left. + \frac{L}{n\pi}\cancel{\int_0^L \cos\frac{n\pi x}{L}\,dx}^{\,0}\right] = -\frac{12L}{n\pi}(-1)^n = \frac{12L}{n\pi}(-1)^{n+1}$$

$\left(\begin{array}{l} u = x,\ du = dx \\ dv = \sin\frac{n\pi x}{L}\,dx \\ v = \frac{L}{n\pi}\cos\frac{n\pi x}{L}\end{array}\right.$

$$6x \sim \left(\frac{12L}{\pi}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

Therefore, the Fourier sine series of $f(x)$ is

$$-\left(\frac{L}{\pi}\right)^2 \cdot \frac{12L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin\frac{n\pi x}{L}$$

$$= \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cdot\sin\frac{n\pi x}{L}.$$

Parseval's Theorem and Mean Square Error

How Fourier series may be used in various problems?

1. Statement and proof of Parseval's theorem.

<u>Theorem 1</u> (Parseval's theorem). Let $f(x)$, $-L<x<L$, be a piecewise smooth function with Fourier series

$$A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L}+B_n\sin\frac{n\pi x}{L}\right)$$

Then

$$\frac{1}{2L}\int_{-L}^{L}f(x)^2dx=A_0^2+\frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2+B_n^2\right) \qquad (1)$$

<u>Proof</u>.

$-L \leftarrow 0 \rightarrow L$

Let $f(-L+0)=f(L-0)$. In that case we multiply the uniformly convergent Fourier series by $f(x)$ to obtain

$$f(x)^2=A_0f(x)+\sum_{n=1}^{\infty}\left[A_nf(x)\cos\frac{n\pi x}{L}+B_nf(x)\sin\frac{n\pi x}{L}\right]$$

This series is also uniformly convergent, and we may integrate term by term for $-L<x<L$;

$$\int_{-L}^{L}f(x)^2dx=A_0\int_{-L}^{L}f(x)dx+\sum_{n=1}^{\infty}\left[A_n\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}+B_n\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\right]$$

Dividing both sides by $2L$, we obtain eq. (1), the desired form of Parseval's theorem in this case

$$\frac{1}{2L}\int_{-L}^{L}f(x)^2dx=A_0^2+\frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2+B_n^2\right)$$

2. Application to mean square error. Our first application of Parseval's theorem is to the mean square error σ_N^2, defined by

$$\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}\left[f(x) - f_N(x)\right]^2 dx \qquad (2)$$

This number measures the average amount by which $f_N(x)$ differs from $f(x)$.

The Fourier series of $f(x) - f_N(x)$ is

$$\sum_{n=N+1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

and therefore, by Parseval's theorem, we have

$$\frac{1}{2L}\int_{-L}^{L}\left[f(x) - f_N(x)\right]^2 dx = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2 + B_n^2\right)$$

and the formula

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2 + B_n^2\right). \qquad (3)$$

⟹ The mean square error is half the sum of the squares of the remaining Fourier coefficients. This formula shows, in particular, that the mean square error tends to zero when N tends to infinity.

<u>Example 1.</u> Let $f(x) = |x|$, $-\pi < x < \pi$. Find the mean square error give an asymptotic estimate when $N \to \infty$.

<u>Sol.</u> We have $B_n = 0$ ($f(x) = |x|$ is <u>even</u>)

$$A_n = -\frac{2L}{n^2\pi^2}\left[1 - (-1)^n\right], \; n \neq 0, \qquad L = \pi$$

$$A_0 = \frac{L}{2}; \quad \left(A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}\right), \quad L = \pi. \qquad A_0 = \frac{\pi}{2}$$

$$A_{2m} = 0, \quad A_{2m-1} = -\frac{4\pi}{\pi^2(2m-1)^2} = -\frac{4}{\pi(2m-1)^2}$$

So

$$\sigma_{2N-1}^2 = \sigma_{2N}^2 = \frac{1}{2}\sum_{n=2N+1}^{\infty} A_n^2 = \frac{1}{2}\sum_{m=N+1}^{\infty}\left[\frac{4}{\pi(2m-1)^2}\right]^2 = \frac{8}{\pi^2}\sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4}$$

We can make a useful asymptotic estimate.

$$\frac{8}{\pi^2}\sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4} \le \frac{8}{\pi^2}\int_N^{\infty}\frac{1}{(2x-1)^4}\,dx$$

$$= \frac{8}{\pi^2}\lim_{t\to\infty}\int_N^t \frac{1}{(2x-1)^4}\,dx = \frac{8}{\pi^2}\lim_{t\to\infty}\left.\frac{(2x-1)^{-3}}{-3\cdot 2}\right|_N^t =$$

$$= \frac{8}{\pi^2}\lim_{t\to\infty}\left[\frac{1}{-3(2t-1)^3 2} - \frac{1}{-3(2N-1)^3\cdot 2}\right] = \frac{4}{3\pi^2}\cdot\frac{1}{(2N-3)^3}$$

This gives us the useful asymptotic statement

$$\sigma_N^2 = O(N^{-3}),\quad N\to\infty$$

$$\sigma_{2N-1}^2 = \frac{1}{2}\sum_{n=2N}^{\infty}(A_n^2+B_n^2)$$

$$\frac{1}{2}\sum_{n=}$$

$$= \frac{1}{2}\left(A_{2N}^2 + A_{2N+1}^2 + A_{2N+2}^2 + A_{2N+3}^2 + \dots\right)$$

$$\sigma_{2N}^2 = \frac{1}{2}\sum_{n=2N+1}^{\infty}A_n^2$$

$$= \frac{1}{2}\left(A_{2N+1}^2 + A_{2N+2}^2 + A_{2N+3}^2 + \dots\right)$$

$$= \frac{1}{2}\left(A_{2N+1}^2 + A_{2N+3}^2 + \dots\right)$$

$$\frac{1}{2}\sum_{m=N+1}^{\infty}A_{2m-1} = \frac{1}{2}\left(A^2_{\underset{2N+1}{2N+2-1}} + A^2_{2N+3} + \dots + A^2_{\underset{2N+3}{2(N+2)-1}}\right)$$

<u>Example 2</u> Let $f(x) = x$, $-\pi<x<\pi$

Find the mean square error and give an asymptotic estimate when $N\to\infty$.

<u>Solution</u> $A_m = 0$, $B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin\frac{m\pi x}{\pi}\,dx$

$$\cancel{\frac{1}{\pi}\left[-x\cdot\frac{1}{\pi}\cos\pi x\right]}$$

$u = x$, $\cancel{dv = \sin\frac{m\pi x}{\pi}dx}$, $\cancel{v = -\frac{1}{\pi}\cos nx}$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\cdot\frac{1}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$$

$u = x$

$dv = \sin nx\,dx$

$v = -\frac{1}{n}\cos nx$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right) =$$

$$= \frac{2}{n}(-1)^{n+1},\quad B_m = \frac{2}{m}(-1)^{m-1}$$

So

$$\sigma_N^2 = \frac{1}{2}\sum_{m=N+1}^{\infty}\frac{4}{m^2} = 2\sum_{m=N+1}^{\infty}\frac{1}{m^2}$$

$$2\sum_{m=N+1}^{\infty}\frac{1}{m^2} \sim 2\int_N^{\infty}\frac{dx}{x^2} = 2\lim_{t\to\infty}\int_N^t \frac{dx}{x^2}$$

$$= 2\lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_N^t = 2\lim_{t\to\infty}\left[-\frac{1}{t}+\frac{1}{N}\right] = \frac{2}{N}$$

So that

$$\sigma_N^2 = O(N^{-1}),\ N\to\infty .$$

Complex inner product. In dealing with complex-valued function it is necessary to modify of inner product and orthogonality. The guiding principle is that the norm of a function should be a nonnegative number. With this in mind, we define the complex inner product and norm on the interval as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x) \overline{\psi}(x)\, dx \qquad (1)$$

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle} \geq 0 \qquad (2)$$

where the bar denotes the complex conjugate of a function, defined by $\overline{\psi}(x) = f(x) - i g(x)$ when $\psi(x) = f(x) + i g(x)$.

Orthogonality: $\langle \varphi, \psi \rangle = 0$.

In this case

$$\langle \varphi, a\psi \rangle = 0 = \bar{a} \langle \varphi, \psi \rangle = 0 \text{ for any complex constant.}$$

$$\left(\langle \varphi, a\psi \rangle = \int_a^b \varphi(x) \cdot \overline{a\psi}\, dx = \bar{a} \int_a^b \varphi(x) \overline{\psi}(x)\, dx = \bar{a} \langle \varphi, \psi \rangle \right)$$

Also Schwarz's inequality holds

$$|\langle \varphi, \psi \rangle| \leq \|\varphi\| \cdot \|\psi\|.$$

Complex Form of Fourier Series

1. Fourier Series and Fourier coefficients.

Since
$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned} \qquad (1)$$

We apply these to a Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$
$$= A_0 + \sum_{n=1}^{\infty}\left(A_n \cdot \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + B_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i}\right)$$
$$= A_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[(A_n - iB_n)\, e^{i\cdot\frac{n\pi x}{L}} + (A_n + iB_n)\, e^{-i\frac{n\pi x}{L}}\right]$$

We let $\alpha_n = \frac{1}{2}(A_n - iB_n)$, $n = 1, 2, \dots$; $\alpha_n = \frac{1}{2}(A_{-n} + iB_{-n})$ $n = -1, -2, \dots$, and $\alpha_0 = A_0$. With this convention the Fourier series assumes the form

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \qquad (2)$$

To obtain integral formulas for the coefficients $\{\alpha_n\}$, we use $A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

$$2\alpha_n = (A_n - iB_n) = \frac{1}{L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right)dx$$
$$= \frac{1}{L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\,dx$$

when $n=0$ also true;

with a corresponding formula for the plus sign. Thus we have

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\,dx, \quad n = 0, \pm 1, \pm 2, \dots \qquad (3)$$

2. Parseval's theorem in Complex form.

If we multiply (2) by $f(x)$ and integrate on $(-L, L)$. The result is

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{n\pi x}{L}}$$

$$\Rightarrow \int_{-L}^{L} f^2(x)\,dx = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}}\,dx \Rightarrow \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = \sum_{n=-\infty}^{\infty} \alpha_n \underbrace{\frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}}\,dx}_{\alpha_n}$$

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = \sum_{n=-\infty}^{\infty} (\alpha_n)^2} \qquad (4)$$

3. Orthogonality

The functions $e^{i\frac{n\pi x}{L}}$ satisfy an orthogonality relation, which may be written in the form

$$\int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}}\,dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

These may be proved by using Euler's formula and the orthogonality of the trigonometric functions $\cos\frac{n\pi x}{L}$, $\sin\frac{n\pi x}{L}$:

$\boxed{n \neq m}$

$$\int_{-L}^{L} \left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)\left(\cos\frac{m\pi x}{L} - i\sin\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L} \Big(\cancel{\cos^2\frac{n\pi x}{L} - i\cos\frac{n\pi x}{L}} \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L} - i\cos\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}$$

$$+ i\sin\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L} + \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L} \Big)dx = 0$$

$\boxed{n = m}$ $\quad \int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{n\pi x}{L}}\,dx = \int_{-L}^{L} 1\,dx = 2L$

Example. Compute the complex Fourier series of $f(x)=e^{ax}$, $-\pi<x<\pi$, where a is a real number.

$$\alpha_n=\frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-\frac{in\pi x}{L}}dx$$

$$f(x)=\sum_{n=-\infty}^{\infty}\alpha_n e^{\frac{in\pi x}{L}}$$

Solution.

$$\alpha_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx$$

$$=\frac{1}{2\pi}\cdot\frac{1}{a-in}\,e^{(a-in)x}\Big|_{-\pi}^{\pi}=\frac{1}{2\pi}\cdot\frac{1}{a-in}\left[e^{(a-in)\pi}-e^{-(a-in)\pi}\right]$$

$$=\frac{1}{2\pi}\cdot\frac{1}{a-in}\left(e^{a\pi}\cdot e^{-in\pi}-e^{-a\pi}\cdot e^{in\pi}\right)$$

$$=\frac{1}{2\pi}\cdot\frac{1}{a-in}\left(e^{a\pi}\left(\cos n\pi-i\sin n\pi\right)-e^{-a\pi}\left(\cos n\pi+i\sin n\pi\right)\right)$$

$$=\frac{1}{2\pi}\cdot\frac{1}{a-in}\left((-1)^n e^{a\pi}-(-1)^n e^{-a\pi}\right)=$$

$$=\frac{1}{2\pi}\cdot\frac{1}{a-in}(-1)^n\underbrace{\left(e^{a\pi}-e^{-a\pi}\right)}_{2\sinh a\pi}=\frac{1}{\pi}\sinh a\pi\cdot\frac{(-1)^n}{a-in}=$$

$$=\frac{1}{\pi}\sinh a\pi\cdot\frac{(-1)^n(a+in)}{a^2+n^2}.$$

So the complex Fourier series of $f(x)=e^{ax}$, $-\pi<x<\pi$, is

$$\frac{1}{\pi}\sinh a\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}\cdot e^{inx}.$$

Sturm-Liouville Eigenvalue Problems.

Fourier series may be formulated as the orthogonal expansion in terms ~~in terms~~ of functions $\Phi(x)$ that are solutions of the differential equation

$$\Phi''(x)+\lambda\Phi(x)=0 \qquad (1)$$

on the interval $-L<x<L$ and that satisfy the periodic boundary conditions

$$\Phi(-L)=\Phi(L)\ ,\quad \Phi'(-L)=\Phi'(L).$$

Indeed, the functions $\Phi(x)=\sin\frac{n\pi x}{L}$ and $\Phi(x)=\cos\frac{n\pi x}{L}$ satisfy these conditions with the value $\lambda=\left(\frac{n\pi}{L}\right)^2$.

The general <u>two-point boundary condition</u> on the interval $a\le x\le b$ is written (arising in problems of heat conduction and wave propagation)

$$\cos\alpha\,\Phi(a)-L\sin\alpha\,\Phi'(a)=0 \qquad (2)$$

$$\cos\beta\,\Phi(b)+L\sin\beta\,\Phi'(b)=0 \qquad (3)$$

where $L=b-a$ and α,β are parameters, $0\le\alpha<\pi$, $0\le\beta<\pi$.

The λ number λ is called an <u>eigenvalue</u> and $\Phi(x)$ is called an <u>eigenfunction</u> of the Sturm-Liouville (S-L) eigenvalue <u>problem</u> defined by (1), (2), and (3).

1. <u>Examples of Sturm-Liouville</u> eigenvalue problems.

Fourier sine series and Fourier cosine series both arise from (S-L) problems with a two-point b.c. on the interval $0<x<L$.
In the first case we use $\alpha=0,\beta=0 \Rightarrow \Phi(0)=0,\Phi(L)=0$;
in the second case we use $\alpha=\frac{\pi}{2},\beta=\frac{\pi}{2}\Rightarrow \Phi'(0)=0,\Phi'(L)=0$.

<u>Example 1.</u> ($\alpha=0,\beta=0$). Find all nontrivial solutions of (1) on the interval $0<x<L$ satisfying the b.c.s $\Phi(0)=0,\Phi(L)=0$.

<u>Solution</u>

Solution

1) $\lambda = 0 \Rightarrow \Phi(x) = Ax + B$

$$0 = \Phi(0) = B, \quad 0 = \Phi(L) = AL + B \iff (A,B) = (0,0)$$

2) $\lambda = -\mu^2 < 0 \Rightarrow \Phi(x) = Ae^{\mu x} + Be^{-\mu x}$

$$0 = A + B$$
$$0 = Ae^{\mu L} + Be^{-\mu L} \iff (A,B) = (0,0).$$

3) $\lambda > 0 \Rightarrow \Phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$

$$0 = A, \quad 0 = A\cos L\sqrt{\lambda} + B\sin(L\sqrt{\lambda})$$

$$B \neq 0, \quad L\sqrt{\lambda} = n\pi, \quad n = 1, 2, \ldots$$

$$\Rightarrow \underbrace{\lambda_n = \left(\frac{n\pi}{L}\right)^2}_{\text{Eigenvalues}}, \quad \underbrace{\Phi_n(x) = \sin\frac{n\pi x}{L}}_{\text{Eigenfunctions}}, \quad n = 1, 2, \ldots$$

Example 2. $\left(\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}\right)$. Find all nontrivial solutions of (1) on the interval $0 < x < L$ satisfying the b.cs $\Phi'(0) = 0, \Phi'(L) = 0$.

Solution.

1) $\lambda = 0 \Rightarrow \Phi(x) = Ax + B$

$$0 = \Phi'(0) = A, \quad 0 = \Phi'(L) = A$$

which gives a nontrivial solution if and only if $A = 0$ and B is nonzero.

2) $\lambda = -\mu^2 < 0 \Rightarrow \Phi(x) = Ae^{\mu x} + Be^{-\mu x}$.

$$0 = \mu A - \mu B, \quad 0 = A\mu e^{\mu L} - B\mu e^{-\mu L}$$
$$\iff (A,B) = (0,0)$$

3) $\lambda > 0 \Rightarrow \Phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$.

$$0 = \Phi'(0) = B\sqrt{\lambda}$$
$$0 = -A\sqrt{\lambda}\sin L\sqrt{\lambda} + B\sqrt{\lambda}\cos(L\sqrt{\lambda})$$

$$\Rightarrow B = 0, \quad L\sqrt{\lambda} = n\pi, \quad n = 1, 2, \ldots$$

$$\Rightarrow \lambda_0 = 0, \; \Phi_0(x) = 1, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; \Phi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \; n = 1, 2, \ldots$$

Some general properties of S-L eigenvalue problems.

Theorem 1 Consider the Sturm-Liouville eigenvalue problem

$$\Phi''(x) + \lambda\Phi(x) = 0 \quad , \quad a<x<b \qquad (1)$$

$$\cos\alpha\,\Phi(a) - L\sin\alpha\,\Phi'(a) = 0 \qquad (2)$$

$$\cos\beta\,\Phi(b) + L\sin\beta\,\Phi'(b) = 0 , \qquad (3)$$

where $L = b-a$, $0 \le \alpha < \pi$, $0 \le \beta < \pi$.

(1). Suppose that $\Phi(x), \Psi(x)$ are nontrivial solutions of (1)–(3) ~~with different~~ the same eigenvalues λ ~~$\lambda_1 \neq \lambda_2$~~. Then there is a constant $C \neq 0$ such that

$$\Phi(x) = C\,\Psi(x)$$

(2). Suppose that $\Phi_1(x), \Phi_2(x)$ are nontrivial solutions of (1)–(3) with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal:

$$\int_a^b \Phi_1(x)\cdot\Phi_2(x)\,dx = 0$$

Proof Let $\alpha = 0 \Rightarrow \Phi(a), \Psi(a) = 0$. Both $\Phi(x)$ and $\Psi(x)$ satisfy the same second-order linear homogeneous d.e., and so does any linear combination.

We set
$$f(x) = \Psi'(a)\Phi(x) - \Phi'(a)\Psi(x)$$

The function $f(x)$, $a<x<b$, also satisfies (1) and the initial condition $f(a) = 0$, $f'(a) = 0$. This requires that $f(x) \equiv 0$. But if $\Psi'(a) = 0$ (resp. $\Phi'(a) = 0$), then $\Psi(x) \equiv 0$ (resp. $\Phi(x) \equiv 0$),
($\Psi(x)$ or $\Phi(x)$ are eigenfunctions $\neq 0$)
a contradiction. We have proved the theorem with the value $C = \frac{\Phi'(a)}{\Psi'(a)}$
$f(x) \equiv 0 \Rightarrow$

In the general case $\alpha \neq 0$, we set

$$f(x) = \Psi(a)\,\Phi(x) - \Phi(a)\,\Psi(x)$$

$$\left(\begin{aligned} f'(a) &= \Psi(a)\Phi'(a) - \Phi(a)\Psi'(a) \\ &= \Psi(a)\cdot\frac{\cos\alpha}{L\sin\alpha}\Phi(a) - \Phi(a)\Psi'(a) = 0 \end{aligned}\right.$$

The function $f(x)$, $a<x<b$, also satisfies (1) and $f(a) = 0$, $f'(a) = 0$
$\Rightarrow f(x) \equiv 0$. But if $\Psi(a) = 0$ (resp. $\Phi(a) = 0$) then (2) it follows that $\Psi'(a) = 0$ (resp. $\Phi'(a) = 0$), so $\Phi(x) \equiv 0$ (resp. $\Psi(x) \equiv 0$), a contradiction. Theorem is proved with the value $C = \frac{\Phi(a)}{\Psi(a)}$.

(2). we write (1) for $\varphi_1(x)$:

$$\varphi_1''(x) + \lambda_1 \varphi_1(x) = 0 \qquad (4)$$

Multiply (4) by $\varphi_2(x)$ and integrate on the interval $a<x<b$:

$$\int_a^b \varphi_2(x)\varphi_1''(x)\,dx + \lambda_1 \int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

$$\hookrightarrow \varphi_2(x)\varphi_1'(x)\Big|_{x=a}^{x=b} - \int_a^b \varphi_1'(x)\varphi_2'(x)\,dx + \lambda_1\int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

$u = \varphi_2(x)$
$du = \varphi_2'(x)dx$
$dv = \varphi_1''(x)dx$
$v = \varphi_1'(x)$

Now we interchange the roles of (φ_1, λ_1) and (φ_2, λ_2)

$$\varphi_1(x)\,\varphi_2'(x)\Big|_{x=a}^{x=b} - \int_a^b \varphi_2'(x)\varphi_1'(x)\,dx + \lambda_2\int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

We subtract these two eq.s:

$$\left(\varphi_2(x)\varphi_1'(x) - \varphi_1(x)\varphi_2'(x)\right)\Big|_{x=a}^{x=b} + (\lambda_1 - \lambda_2)\int_a^b \varphi_1(x)\,\varphi_2(x)\,dx = 0$$

From the boundary conditions we conclude that the endpoint terms contribute zero, so

$$\underbrace{\varphi_2(b)\varphi_1'(b) - \varphi_1(b)\cdot\varphi_2'(b) - \left(\varphi_2(a)\,\varphi_1'(a) - \varphi_1(a)\,\varphi_2'(a)\right)}_{=0} + (\lambda_1-\lambda_2)\int_a^b \varphi_1(x)\varphi_2(x)\,dx$$

$$\underbrace{(\lambda_1-\lambda_2)}_{\neq 0}\underbrace{\int_a^b \varphi_1(x)\varphi_2(x)\,dx}_{=0} = 0 \Rightarrow \lambda_1 \neq \lambda_2 \quad \int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

$$\varphi_1'(b) = -\frac{\cos\beta}{L\sin\beta}\varphi_1(b) \Rightarrow -\varphi_2(b)\frac{\cos\beta}{L\sin\beta}\varphi_1(b) - \varphi_1(b)\varphi_2'(b)$$

$$= -\varphi_1(b)\frac{\left[\cos\beta\,\varphi_2(b) + L\sin\beta\,\varphi_2'(b)\right]}{L\sin\beta} = 0$$

Also $$\varphi_1'(a) = \frac{\cos\alpha}{L\sin\alpha}\varphi_1(a) \Rightarrow \varphi_2(a)\cdot\frac{\cos\alpha}{L\sin\alpha}\varphi_1(a) - \varphi_1(a)\varphi_2'(a) =$$

$$= \varphi_1(a)\frac{\overbrace{\left[\cos\alpha\,\varphi_2(a) - L\sin\alpha\,\varphi_2'(a)\right]}^{=0}}{L\sin\alpha} = 0$$

Example of transcendental eigenvalues.

Example ($\alpha = 0$, $0 < \beta < \frac{\pi}{2}$). Find all nontrivial solution of (1) on the interval $0 < x < L$ satisfying the b.cs
$\Phi(0) = 0$, $h\Phi(L) + \Phi'(L) = 0$, where $h > 0$.

Solution. 1) $\lambda = 0 \Rightarrow \Phi(x) = Ax + B$.

$$0 = \Phi(0) = B, \quad 0 = h\Phi(L) + \Phi'(L) = h(AL + B) + A$$
$$= hAL + A = A(1 + hL)$$
$$\Rightarrow A = 0,\ B = 0 \Rightarrow \text{Trivial solution } \Phi(x) \equiv 0.$$

2) $\lambda = -\mu^2 < 0 \Rightarrow \Phi(x) = Ae^{\mu x} + Be^{-\mu x}$

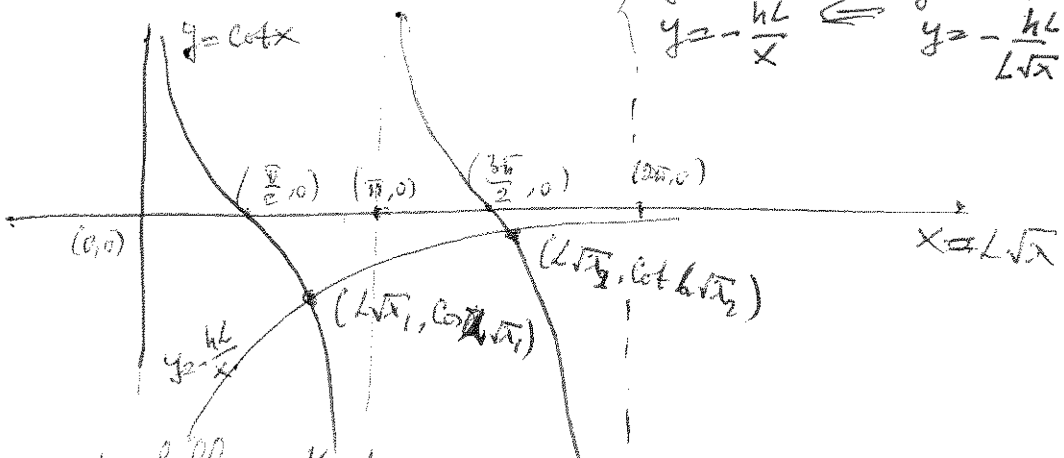
$$0 = A + B, \quad 0 = h(Ae^{\mu L} + Be^{-\mu L}) + A\mu e^{\mu L} - B\mu e^{-\mu L}$$
$$= hA(e^{\mu L} - e^{-\mu L}) + \mu A(e^{\mu L} + e^{-\mu L})$$
$$\Rightarrow A = 0 \Rightarrow B = 0.$$

3) $\lambda > 0 \Rightarrow \Phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$.

$$0 = \Phi(0) = A, \quad 0 = h\Phi(L) + \Phi'(L) =$$
$$= hB\sin(L\sqrt{\lambda}) + B\sqrt{\lambda}\cos(L\sqrt{\lambda}) \Rightarrow \cancel{\cot(L\sqrt{\lambda})}$$
$$\Longleftrightarrow \cot(L\sqrt{\lambda}) = -\frac{h}{\sqrt{\lambda}} = -\frac{hL}{L\sqrt{\lambda}} \tag{5}$$

Therefore we have found all eigenfunctions in the form
$$\Phi_n(x) = \sin(x\sqrt{\lambda_n}), \quad n = 1, 2, \dots$$
where the eigenvalues λ_n are determined by solving (5).

$$\begin{cases} y = \cot x \\ y = -\frac{hL}{x} \end{cases} \Longleftarrow \begin{array}{l} y = \cot(L\sqrt{\lambda}) \\ y = -\frac{hL}{L\sqrt{\lambda}} \end{array}$$



From the graph follows that
$$\frac{\pi}{2} < L\sqrt{\lambda_1} < \pi, \quad \frac{3\pi}{2} < L\sqrt{\lambda_2} < 2\pi, \quad L\sqrt{\lambda_n} - \left(n - \frac{1}{2}\right)\pi \to 0, \ n \to \infty.$$

Completeness and positivity of eigenfunctions.

By analogy with Fourier series, we may expect to be able to expand a piecewise smooth function in a series of (S-L) eigenfunctions in the form

$$f(x) \sim \sum_{n=1}^{\infty} A_n \Phi_n(x) \quad , \qquad (6)$$

where

$$A_n = \frac{\int_a^b f(x)\,\Phi_n(x)\,dx}{\int_a^b \Phi_n(x)^2\,dx} \qquad n = 1, 2, \ldots \qquad (7)$$

The following theorem shows that we may always expect a complete set of eigenfunctions for the S-L eigenvalue problem.

Theorem 1. There exist an infinite sequence of solutions $\lambda_n, \Phi_n(x)$ of the S-L eigenvalue problem defined by (1)-(3) that possess the following properties

(i). $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \to \frac{\pi}{L}$, $n \to \infty$

(ii). If $f(x)$, $a<x<b$, is a piecewise smooth function, the series (6) converges to $\frac{f(x+0)+f(x-0)}{2}$, $a<x<b$.

(iii) Parseval's relation holds, in the form

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b \Phi_n(x)^2\,dx = \int_a^b f(x)^2\,dx$$

Theorem 2. Suppose that the parameters α, β satisfy the inequalities $0 \le \alpha < \frac{\pi}{2}$, $0 \le \beta < \frac{\pi}{2}$. Then all eigenvalues of the Sturm-Liouville eigenvalue problem (1) with the boundary conditions (2), (3) satisfy $\lambda_n > 0$.

<u>Proof</u>. Let $\Phi(x)$ be a $\not\equiv 0$ solution of the (S-L) problem (1)-(3). Then

$$\Phi''(x)+\lambda\Phi(x)=0 \Longrightarrow \int_a^b \Phi(x)\Phi''(x)\,dx + \lambda\int_a^b \Phi^2(x)\,dx = 0$$

$$\Rightarrow \quad \lambda\int_a^b \Phi^2(x)\,dx = \int_a^b (\Phi'(x))^2 dx + \Phi(a)\Phi'(a) - \Phi(b)\Phi'(b) \quad \begin{cases} u=\Phi(x) \\ du=\Phi'(x)dx \\ dv=\Phi''(x)dx \\ v=\Phi'(x) \end{cases}$$

$\int_a^b (\Phi')^2 dx > 0$ otherwise $\int_a^b (\Phi')^2 dx = 0$

$\Rightarrow \Phi'(x)=0 \Rightarrow \Phi(x)\equiv$ Constant, which is possible if and only if $\alpha=\frac{\pi}{2}$, $\beta=\frac{\pi}{2}$, which is excluded.

On the other hand, we can rewrite the b.c.s in the form

$\Phi(a) = L\tan\alpha\,\Phi'(a)$,
$\Phi(b) = -L\tan\beta\,\Phi'(b)$, which leads to

$$\lambda\int_a^b \Phi^2(x)\,dx > L\tan\alpha(\Phi'(a))^2 + L(\Phi'(b))^2\tan\beta \geq 0$$

since α and β both $0\leq\alpha,\beta<\frac{\pi}{2}$.

General Sturm-Liouville problems

Many of the properties of the eigenfunctions of the simple differential equation $\Phi''(x)+\lambda\Phi(x)=0$ are shared by the eigenfunctions of the more general equation

$$[S(x)\Phi'(x)]'+[\lambda\rho(x)-q(x)]\Phi(x)=0,\quad a<x<b \qquad (1')$$

where $S(x)$, $\rho(x)$, $q(x)$ are given functions on the interval $a<x<b$ with $\rho(x)>0$. We have already studied the special case $S(x)\equiv 1$, $\rho(x)\equiv 1$, $q(x)=0$. The eigenfunctions will satisfy a property of weighted orthogonality with respect to the weight function $\rho(x)$, $a<x<b$.

Def 1. Weighted inner product with respect to a positive weight function $\rho(x)$, $a\le x\le b$ is defined by the integral

$$\langle\varphi,\psi\rangle_\rho=\int_a^b\varphi(x)\psi(x)\rho(x)dx.$$

Def. 2 We say that two functions φ,ψ are orthogonal with respect to the weight function $\rho(x)$, $a\le x\le b$, if $\langle\varphi,\psi\rangle_\rho=0$.

Now we state and prove the corresponding orthogonality properties of the Sturm-Liouvill eigenfunctions.

Theorem 1. Consider the (S-L) problem (1'), (2), (3). Suppose that $\Phi_1(x)$, $\Phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1\neq\lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$, $a<x<b$:

$$\int_a^b\Phi_1(x)\Phi_2(x)\rho(x)\,dx=0.$$

If the two eigenfunctions belong to the same eigenvalue $\lambda_1=\lambda_2$, then the eigenfunctions must be proportional: $\Phi_2(x)=C\Phi_1(x)$ for some constant C.

Proof. Write the (S–R) equation satisfied by Φ_1:

$$[S\Phi_1']' + (\lambda_1\rho - q)\Phi_1 = 0$$

Multiply this eq. by Φ_2 and integrate on $a<x<b$

$$\int_a^b \Phi_2(x)\left(S\Phi_1'(x)\right)' dx + \int_a^b \Phi_2(x)\left(\lambda_1\rho(x) - q(x)\right)\Phi_1(x)\,dx = 0$$

$$\Rightarrow \quad \Phi_2(x)\,S(x)\Phi_1'(x)\Big|_a^b - \int_a^b \Phi_2'(x)\,S(x)\Phi_1'(x)\,dx + \int_a^b \Phi_2(x)\left(\lambda_1\rho(x) - q(x)\right)\Phi_1(x)\,dx = 0 \qquad (8)$$

Now we interchange the roles of $\Phi_1(x)$ and $\Phi_2(x)$ to yield

$$\Phi_1(x)\,S(x)\Phi_2'(x)\Big|_a^b - \int_a^b \Phi_1'(x)\,S(x)\Phi_2'(x)\,dx + \int_a^b \Phi_1(x)\left(\lambda_2\rho(x) - q(x)\right)\Phi_2(x)\,dx = 0 \qquad (9)$$

We subtract (8) and (9) and apply the b.c.s, ~~yet~~ we have

$$(\lambda_1 - \lambda_2)\int_a^b \Phi_1(x)\Phi_2(x)\rho(x)\,dx = 0, \quad \text{if } \lambda_1 - \lambda_2 \neq 0,$$

it follows that Φ_1 and Φ_2 must be orthogonal with respect to the weight function ρ. ▣

Example 1. Find the orthogonality relation for eigenfunctions of the Bessel equation of order zero:

$$(x\Phi')' + \lambda x \Phi = 0.$$

Solution. We have $S(x) = x$, $\rho(x) = x$, $q(x) = 0$.
If $\Phi_1(x)$ and $\Phi_2(x)$ both satisfy the same two-point b.c.s with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form

$$\int_a^b \Phi_1(x)\Phi_2(x)\,x\,dx = 0 \qquad ▣$$

Example 2. Find the orthogonality relation for eigenfunctions of the Bessel equation of order m:

$$(x\Phi')' + \left(\lambda x - \frac{m^2}{x}\right)\Phi = 0.$$

Solution. We have

$$S(x) = x, \quad \rho(x) = x, \quad q(x) = \frac{m^2}{x}.$$

If $\Phi_1(x)$ and $\Phi_2(x)$ both satisfy the same two-point b.c.s with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form

$$\int_a^b \Phi_1(x)\Phi_2(x)\, x\, dx = 0 \quad \boxed{}$$

Boundery-value problems in rectangular coordinates

1. Steady-state solutions in a slab. (ПЛИТА) (plita) $\left(\frac{\partial u}{\partial t} = 0\right.$, heat independent of t)

Proposition 1. Steady-state solutions of of the heat equation, with no internal heat sources, are solutions of Laplace's equation.

Example 1. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$ satisfying the boundary conditions $u(x,y,0) = T_1$, $\left(\frac{\partial u}{\partial z} + hu\right)(x,y,L) = 0$ where T_1 and h are positive constants.

Solution. Steady-state solutions of the heat equation are solutions of Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$. Since the boundery conditions are independent of (x,y), we look for the solution in the form $u(x,y,z) = U(z)$, independent of (x,y). Thus U must satisfy $U''(z) = 0$, whose general solution is $U(z) = A + Bz$. The b.c. at $z=0$ requires $A = T_1$, while the b.c. at $z = L$ requires $B + h(A + BL) = 0$. Thus $B + hA + hBL = 0$

$\Rightarrow B(1+hL) = -hA = -hT_1$, and the solution is

$$U(z) = T_1 - \frac{hT_1}{1+hL} z.$$

Example 2. Find the steady-state solution of the heat eq. $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$ satisfying the b.c.s $u(x,y,0) = T_1$, $\left(\frac{\partial u}{\partial z} + hu\right)(x,y,L) = 0$, where r, K, h and T_1 are positive constants.

Solution. The b.c.s are independent of (x,y); hence we look for the solution in the form $u(x,y,z) = U(z)$, independent of (x,y). Thus $U(z)$ must satisfy $KU''(z) + r = 0$, whose general solution is

$$U''(z) = -\frac{r}{K}, \quad U'(z) = -\frac{r}{K}z + B \Rightarrow U(z) = -\frac{r}{2K}z^2 + Bz + A$$

B.C. at $z = 0$: $T_1 = A$

$z = L$: $-\frac{r}{K}L + B + h\left(-\frac{r}{2K}L^2 + BL + A\right) = 0$

$$B(1+hL) = \frac{rL}{K} + \frac{hrL^2}{2K} - hT_1$$

$$B = \frac{1}{1+hL}\left(\frac{rL}{K} + \frac{hrL^2}{2K} - hT_1\right)$$

The solution is

$$U(z) = -\frac{r}{2K}z^2 + \frac{1}{1+hL}\left(\frac{rL}{K} + \frac{hrL^2}{2K} - hT_1\right)z + T_1 .$$

2. Time-periodic solutions.

Let us solve the problem

$$u_t = K u_{zz} \quad , z>0, \ -\infty < t < \infty$$

$$u(0;t) = u_0(t), \quad -\infty < t < \infty.$$

where $u_0(t)$ is periodic with period τ, and

$$|u(z;t)| \le M.$$

To solve this problem, we first look for complex separated solutions, of the form

$$u(z;t) = Z(z)T(t).$$

Since the heat equations has real coefficients, the real and imaginary parts of a complex-valued solution are again solutions. Thus we may allow $Z(t)$, $T(t)$ to be complex-valued.

Substituting into the heat equation, we have

$$Z(t)T'(t) = K Z''(z)T(t)$$

$$\Rightarrow \quad \frac{KZ''(z)}{Z(z)} = \frac{T'(t)}{T(t)} = -\lambda$$

$$\Rightarrow \quad T'(t) + \lambda T(t) = 0$$

$$Z''(t) + \frac{\lambda}{K} Z(t) = 0$$

$\Rightarrow T(t) = e^{-\lambda t}$

Since we require bounded solutions for $-\infty < t < \infty$, λ must be pure imaginary $\lambda = i\beta$ with β real.

We try $Z(z) = e^{\gamma z}$. Thus we must have

$$\Rightarrow \gamma^2 e^{\gamma z} + \frac{\lambda}{K}e^{\gamma z} = 0 \Rightarrow \gamma^2 + \frac{i\beta}{K} = 0$$

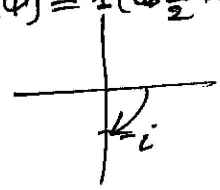
In the case where $\beta > 0$, this has two solutions:

$$\gamma = \pm\sqrt{\frac{\beta}{2K}}\cdot\sqrt{-i}$$

$\sqrt{-i} = w = R(\cos\Phi + i\sin\Phi)$

$w^2 = -i$

$R^2(\cos 2\Phi + i\sin 2\Phi) = 1\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$



$\underline{R=1}$

$2\Phi = -\left(\frac{\pi}{2}\right) + 2k\pi$

$\Phi = -\frac{\pi}{4} + k\pi$

$k=0$: $w_1 = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(-1+i)$

$w_2 = \cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-1+i)$

$$\Rightarrow \gamma = \pm(-1+i)\sqrt{\frac{\beta}{2K}}.$$

Since we require bounded solutions for $z>0$, we must take the solution with $\mathrm{Re}\,\gamma<0$, that is, the plus sign. Therefore we have the complex seperated solutions

$$e^{-i\beta t}\cdot e^{(-1+i)\sqrt{\frac{\beta}{2K}}\cdot z}$$

Taking the real and imaginary parts, we have the real solutions

$$\left(e^{-i\left(\beta t - z\sqrt{\frac{\beta}{2K}}\right)}\cdot e^{-\sqrt{\frac{\beta}{2K}}z} = e^{-cz}\cdot e^{-i(\beta t - cz)}, \quad c = \sqrt{\frac{\beta}{2K}}\right)$$

$$e^{-cz}\cos(\beta t - cz), \quad e^{-cz}\sin(\beta t - cz), \quad c = \sqrt{\frac{\beta}{2K}}$$

We refer to these as the quasi-seperated solutions

To solve the original problem, we suppose that the boundary temperature has been expanded as a Fourier series.

$$u_0(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{2n\pi t}{\tau} + B_n\sin\frac{2n\pi t}{\tau}\right)$$

Then, we obtain the solution in the form

$$u(z,t) = A_0 + \sum_{n=1}^{\infty} e^{-c_n z}\left[A_n\cos(\beta_n t - c_n z) + B_n\sin(\beta_n t - c_n z)\right],$$

where $\beta_n = \frac{2n\pi}{\tau}$, $c_n = \sqrt{\frac{\beta_n}{2K}} = \sqrt{\frac{\frac{2n\pi}{\tau}}{2K}} = \sqrt{\frac{n\pi}{K\tau}}$.

<u>Example</u> 3. Solve the heat equation $u_t = K u_{zz}$ for $z>0$, $-\infty < t < \infty$, with the boundary condition

$$u(0;t) = A_0 + A_1 \cos\frac{2\pi t}{\tau},$$

where A_0, A_1, and τ are positive constants.

<u>Solution</u>. Referring to the general solution just obtained, we let $B_n = 0$ for $n \geq 1$ and $A_n = 0$ for $n \geq 2$. The solution is

$$u(z;t) = A_0 + A_1 e^{-c_1 z} \cos\left(\frac{2\pi t}{\tau} - z\sqrt{\frac{\pi}{K\tau}}\right).$$

Homogeneous BCs on a Slab

A homogeneous b.c. at $z=0$ has one of the following forms:

$$u(0;t)=0 \quad \text{or} \quad u_z(0;t)=0 \quad \text{or} \quad u_z(0;t)=h\,u(0;t)$$

where h is a nonzero constant. All three of these may be included in the following succinct form:

$$\cos\alpha\, u(0;t) - L\sin\alpha\, u_z(0;t)=0, \qquad (1)$$

where $0\le\alpha<\pi$. When $\alpha=0 \Rightarrow u(0;t)=0$ (first b.c.)
when $\alpha=\frac{\pi}{2} \Rightarrow u_z(0;t)=0$ (second b.c.)
and when $\cot\alpha=hL \Rightarrow u_z(0;t)=h\,u(0;t)$ (the third b.c.)

$$\left(u_z(0;t)=\underbrace{\frac{1}{L}\cot\alpha}_{h}\, u(0;t)\right)$$

Similarly, the general homogeneous b.c. at $z=L$ is written in the form

$$\cos\beta\, u(L;t) + L\sin\beta\, u_x(L;t)=0 \qquad (2)$$

where $0\le\beta<\pi$. The constant β is not related to α, in general.

1. Separated solutions with b.c.s.

We now discuss separated solutions of the heat eq. $u_t = K u_{zz}$ with the homogeneous b.c.s (1), (2).

A separated solution of the heat eq. is written

$$u(z;t)=\Phi(z)T(t).$$

From $u_t=Ku_{zz}$, we obtain

$$\Phi(z)T'(t)=K\Phi''(z)T(t)$$

Dividing by $K\Phi(z)T(t)$, we obtain

$$\underbrace{\frac{T'(t)}{KT(t)}}_{\sim t} = \underbrace{\frac{\Phi''(z)}{\Phi(z)}}_{\sim z} = -\lambda$$

$$T'(t) + \lambda K T(t) = 0 \qquad (3)$$

$$\Phi''(z) + \lambda \Phi(z) = 0 \qquad (4)$$

$$\Rightarrow T(t) = e^{-\lambda K t} \neq 0$$

To the second eq. (4), we must add the b.c.s (1) and (2). The product

$$u(z,t) = \Phi(z) T(t)$$

satisfies (1) <u>iff</u> $\Phi(z)$ satisfies the b.c.

$$\cos\alpha\, \Phi(0) - L \sin\alpha\, \Phi'(0) = 0.$$

Similarly, $u(z,t)$ satisfies (2) <u>iff</u> ~~it~~ $\Phi(z)$ satisfies the b.c.

$$\cos\beta\, \Phi(L) + L \sin\beta\, \Phi'(L) = 0$$

This leads us to the following

<u>Proposition 1</u>. The separated solutions of the heat eq. $u_t = K u_{zz}$ with the b.c.s (1) and (2) are of the form $u_n(z,t) = e^{-\lambda_n K t} \Phi_n(z)$ where λ_n is an eigenvalue and $\Phi_n(z)$ is an eigenfunction of the (S-L) eigenvalue problem

$$\Phi''(z) + \lambda \Phi(z) = 0$$

with the b.c.s

$$\cos\alpha\, \Phi(0) - L \sin\alpha\, \Phi'(0) = 0$$

$$\cos\beta\, \Phi(L) + L \sin\beta\, \Phi'(L) = 0.$$

These eigenfunctions satisfy the orthogonality relation

$$\int_0^L \Phi_n(z) \Phi_m(z)\, dz = 0 \quad \text{for } m \neq n.$$

Example 1. Find all the separated solutions of the heat equation $U_t = KU_{zz}$ for $0<z<L$ satisfying the boundary conditions $U(0;t)=0$, $U(L;t)=0$. (corresponds to a slab with both faces maintained at temperature zero)

Solution. The associated (S-L) problem:

$$\Phi''(z)+\lambda\Phi(z)=0,$$
$$\Phi(0)=0, \quad \Phi(L)=0.$$

$$\Rightarrow \Phi_n(z) = \sin\frac{n\pi z}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Thus we have the separated solutions

$$U_n(z;t) = \sin\frac{n\pi z}{L}\cdot e^{-\left(\frac{n\pi}{L}\right)^2 Kt}, \quad n=1,2,\ldots$$

Example 2 (Corresponds to a slab with one face insulated and the other face maintained at temperature zero).
Find all the separated solutions of the heat equation $U_t = KU_{zz}$ for $0<z<L$ satisfying the b.c.s $U(0;t)=0$, $U_z(L;t)=0$.

Solution. (S-L) problem:

$$\Phi''(z)+\lambda\Phi(z)=0$$
$$\Phi(0)=0, \quad \Phi'(L)=0.$$

a) $\lambda=0$:

$$\Phi(z)=Az+B$$

$$\left.\begin{array}{l}\Phi(0)=0 \Rightarrow B=0\\ \Phi'(L)=0\Rightarrow A=0\end{array}\right\} \Rightarrow \lambda=0 \text{ is not an eigenvalue.}$$

b) $\lambda=-\mu^2<0$

$$\Phi(z)=A\sinh(\mu z)+B\cosh(\mu z)$$

$$\left(\sinh a = \frac{e^a-e^{-a}}{2}, \quad \cosh a = \frac{e^a+e^{-a}}{2}\right.$$

$$\Phi(0)=0 \Rightarrow B=0:$$
$$\Phi(z)=A\sinh(\mu z)$$
$$\Phi'(L)=0 \Rightarrow 0=A\mu\cosh(\mu L) \Rightarrow A=0$$

Hence $\lambda<0$ is not a possible eigenvalue.

c) $\lambda>0 \Rightarrow \Phi(z)=A\sin z\sqrt{\lambda}+B\cos z\sqrt{\lambda}$

$\Phi(0)=0\Rightarrow B=0$, $\Phi'(0)=0 \Rightarrow A\sqrt{\lambda}\cos \cancel{z}\sqrt{\lambda}=0$

$$\Rightarrow L\sqrt{\lambda}=\left(n-\tfrac{1}{2}\right)\pi, \quad n=1,2,\ldots$$

$$\Rightarrow \Phi_n(z)=\sin\left(n-\tfrac{1}{2}\right)\pi z/L, \quad \lambda=\frac{\left(n-\frac{1}{2}\right)^2\pi^2}{L^2}.$$

The separated solutions of the heat equation are

$$U_n(z;t) = \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left[-\left(n-\frac{1}{2}\right)^2\pi^2\frac{Kt}{L^2}\right], \quad n=1,2,\ldots$$

2. Solution of the initial-value problem in a slab.

Having obtained the separated solutions of the heat equation with homogeneous boundary conditions, we can solve the following initial-value problem:

$$U_t = KU_{zz} \quad , t>0,\ 0<z<L$$

$$\cos\alpha\, U(0;t) - L\sin\alpha\, U_z(0;t) = 0 \ , \ t>0$$

$$\cos\beta\, U(L;t) + L\sin\beta\, U_z(L;t) = 0 \ , \ t>0$$

$$U(z;0) = f(z), \quad 0<z<L,$$

where $f(z)$, $0<z<L$, is a piecewise smooth function.

To solve this initial-value problem, we first expand $f(z)$ in a series of eigenfunctions of the (S–L) problem, in the form

$$f(z) = \sum_{n=1}^{\infty} A_n \Phi_n(z) \quad , \ 0<z<L.$$

[If f is discontinuous at z, the series converges to $\frac{1}{2}f(z+0) + \frac{1}{2}f(z-0)$.] The formal solution of the initial-value problem is given by the series

$$U(z;t) = \sum_{n=1}^{\infty} A_n \Phi_n(z)\, e^{-\lambda_n K t} \qquad (5)$$

The solution has been written as a superposition of separated solutions of the heat equation satisfying the indicated homogeneous boundary conditions. The Fourier coefficients A_n are obtained from the orthogonality relations by the formulas

$$\int_0^L f(z)\Phi_n(z)\,dz = A_n \int_0^L (\Phi_n(z))^2\,dz \quad , n=1,2,\ldots$$

Example 3. Solve the initial-value problem $u_t = K u_{zz}$ for $t>0$, $0<z<L$, with the b.c.s $u(0;t)=0$, $u(L;t)=0$ and the initial condition $u(z;0)=1$.

Solution. The seperated solutions of the heat equation satisfying the b.c.s are

$$\sin\left(\frac{n\pi z}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Kt}, \quad n=1,2,\dots$$

To satisfy the initial condition, we must expand the function $f(z)=1$ is a Fourier sine series. The Fourier coefficients are given by

$$A_n \int_0^L \sin^2 \frac{n\pi z}{L} dz = \int_0^L \sin \frac{n\pi z}{L} dz = -\frac{L}{n\pi} \cos \frac{n\pi z}{L} \Big|_0^L =$$

$$= \frac{L}{n\pi}\left[1-(-1)^n\right] \Rightarrow A_n = \frac{\frac{L}{n\pi}\left[1-(-1)^n\right]}{\int_0^L \sin^2 \frac{n\pi z}{L} dz}$$

$$\int_0^L \sin^2 \frac{n\pi z}{L} dz = \frac{1}{2}\int_0^L \left(1-\cos\frac{2n\pi z}{L}\right) dz =$$

$$= \frac{L}{2}; \quad \text{So} \quad A_n = \frac{\frac{L}{n\pi}\left[1-(-1)^n\right]}{\frac{L}{2}} = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$u(z;t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi z}{L} \cdot e^{-\left(\frac{n\pi}{L}\right)^2 Kt}.$$

~~For $t>0$ and $0<z<L$, this series~~

Uniqueness of solutions.

We now discuss the uniqueness of the solution of the initial-value problem. We have found a solution as a series of separated solutions, but it is conceivable that by by another method we might produce a distinct solution of the heat equation with the same initial conditions and boundary conditions. We shall prove that this is impossible. To be specific, we take the b.c.s

$$u(0;t) = 0, \quad u(L;t)=0.$$

Suppose that u_1 and u_2 are two solutions with the same initial and b.c.s, and set $u = u_1 - u_2$. Then u satisfies the heat equation with zero b.c.s and zero initial conditions.

Let

$$W(t) = \frac{1}{2}\int_0^L (u(z;t))^2 dz$$

Then

$$W'(t) = \int_0^L u(z;t)\, u_t(z;t)\, dz \qquad (1)$$

$$= K\int_0^L u(z;t)\, u_{zz}(z;t)\, dz \qquad (2)$$

$$= K\, u(z;t)\, u_z(z;t)\Big|_0^L - K\int_0^L (u_z(z;t))^2 dz \qquad (3)$$

0, b.c.s

Therefore

$$W'(t) = -K\int_0^L (u_z(z;t))^2 dz$$

Since $K>0$ (constant), $(u_z(z;t))^2 \geq 0$, we have

$$W'(t) \leq 0 \quad \text{and} \quad W(t) \geq 0.$$

But $u(z;0)=0$, which means that $W(0)=0$. Now, we use the fundamental theorem of calculus:

$$W(t) = W(0) + \int_0^t W'(s)ds \leq 0.$$

Since $W(t) \geq 0$, we have $W(t) \equiv 0$, which means that $u(z;t)=0$ for each t, that is, $u_1(z;t) = u_2(z;t)$.

Hence we have proved uniqueness of the solution.

Let us note that we have used the b.c.s only to show that $uu_z\Big|_0^L = 0$. So our proof applies also to other b.c.s, for example, $u_z(0)=0$, $u_z(L)=0$.

Nonhomogeneous Boundary Conditions

(1)
$$u_t = K u_{zz} + r \qquad (1)$$
$$\cos\alpha\, u(0;t) - L \sin\alpha\, u_z(0;t) = T_1 \qquad (2)$$
$$\cos\beta\, u(L;t) + L \sin\beta\, u_z(L;t) = T_2 \qquad (3)$$
$$u(z;0) = f(z), \qquad (4)$$

where $f(z)$, $0<z<L$, is a piecewise smooth function and $\alpha, \beta, r, T_1, T_2$, and K are constants.
We seek the solution $u(z;t)$ for all $t>0$, $0<z<L$.

(2) Five-stage method of solution.

Stage 1: Steady-state solution. ~~We first~~ ignore the initial conditions and look for a function $U(z)$ that satisfies the heat eq. (1) and b.c.s (2) and (3). Thus we must have
$$K U''(z) + r = 0,$$
whose general solution is
$$U(z) = -\frac{r}{2K} z^2 + A + Bz$$
where A and B are determined from the b.c.s.

Stage 2. Transformation of the problem.

We define a new unknown function,
$$v(z;t) = u(z;t) - U(z).$$

We have $v_t(z;t) = u_t(z;t)$, $v_z(z;t) = u_z(z;t) - U'(z)$,
$$v_{zz}(z;t) = u_{zz}(z;t) - U''(z).$$
Thus
$$v_t - K v_{zz} = u_t - K\left[u_{zz} - U''(z)\right] = u_t - K u_{zz} + K U''(z)$$
$$= r - r = 0.$$
Likewise, v satisfies the b.c.s (2) and (3) with $T_1=0$, $T_2=0$.
Thus, we have

$$v_t = K v_{zz} \qquad (5)$$

$$\cos\alpha\, v(0;t) - L \sin\alpha\, v_z(0;t) = 0 \qquad (6)$$

$$\cos\beta\, v(L;t) + L \sin\beta\, v_z(L;t) = 0 \qquad (7)$$

$$v(z;0) = f(z) - U(z) \qquad (8)$$

Thus $v(z;t)$ satisfies a homogeneous eq. with homogeneous b.c.s and a new initial condition. This type of problem was treated.

Stage 3: <u>seperation of variables</u>.

To determine $v(z;t)$, we use a superposition of separated solutions with homogeneous b.c.s (6) and (7).

$$v(z;t) = \sum_{n=1}^{\infty} A_n \Phi_n(z) e^{-\lambda_n K t}.$$

The coefficients A_n are determined by expanding the initial condition $f(z) - U(z)$ in a series eigenfunctions $\sum_{n=1}^{\infty} A_n \Phi_n(z)$. They may be computed from the integrals

$$\int_0^L [f(z) - U(z)]\, \Phi_n(z)\, dz = A_n \int_0^L \Phi_n(z)^2\, dz, \quad n = 1, 2, \ldots$$

The formal solution of the initial-value problem is

$$u(z;t) = U(z) + \sum_{n=1}^{\infty} A_n \Phi_n(z) e^{-\lambda_n K t} \qquad (9)$$

Stage 4: Verification of the solution.

Stage 5: Asymptotic behavior.

Stage 4: Verification of the solution.

We have to verify that the formal solution (9) is unique and satisfies the initial-value problem. To illustrate the proof, we assume that $\alpha = 0$, $\beta = 0$ where the eigenfunctions are $\Phi_n(z) = \sin\frac{n\pi z}{L}$ and the eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Then

$$u: \quad U(z) + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L}\, e^{-\lambda_n K t}$$

$$u_z: \quad U'(z) + \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \cos\frac{n\pi z}{L} \cdot e^{-\lambda_n K t}$$

$$u_{zz}: \quad U''(z) \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi z}{L} \cdot e^{-\lambda_n K t}$$

$$u_t: \quad -\sum_{n=1}^{\infty} A_n K \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi z}{L} \cdot e^{-\lambda_n K t}$$

We have $|A_n| \le 2M$, where M is the maximum of $|f(z) - U(z)|$. Therefore, for each $t > 0$, each of these series is uniformly convergent for $0 \le z \le L$ and u satisfies the heat equation $u_t = K u_{zz} + r$, together with the boundary conditions.

If u_1 and u_2 are two solutions of the problem. then as we proved that $u = u_1 - u_2 \equiv 0$ (see uniqueness ...)

Stage 5. Asymptotic behavior.

Asymptotic behavior and relaxation time

Let b.cs $u(0;t)=0$, $u(L;t)=0$

and the initial conditions is

$u(z;0)=f(z)$, a piecewise smooth function.

The solution is

$$u(z;t)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 Kt}$$

where A_n are the Fourier sine coefficients of the piecewise smooth function $f(z)$, $0<z<L$. Thus

$$A_n=\frac{2}{L}\int_0^L f(z)\sin\frac{n\pi z}{L}\,dz \quad \text{and} \quad |A_n|\le 2M,$$

where M is the maximum of $|f(z)|$, $0<z<L$.

Writing $a=\frac{\pi^2 K}{L^2}$ and noting that $\left|\sin\frac{n\pi z}{L}\right|\le 1$, we have

$$|u(z;t)|\le 2M\sum_{n=1}^{\infty} e^{-n^2 at}$$

But $n^2\ge n$ for $n\ge 1$, and thus $e^{-n^2at}\le e^{-nat}=(e^{-at})^n$. Hence

$$|u(z;t)|\le 2M\sum_{n=1}^{\infty}(e^{-at})^n=2M\frac{e^{-at}}{1-e^{-at}}.$$

When $t\to\infty$, $e^{-at}\to 0$, and we have shown that

$$u(z;t)=O(e^{-at}),\quad t\to\infty$$

In particular $u(z;t)\to 0$ when $t\to\infty$, which means that $u(z;t)$ is a <u>transient solution</u>.

We define the <u>relaxation time</u> τ by the formula

$$\frac{1}{\tau}=-\lim_{t\to\infty}\frac{1}{t}\ln|u(z;t)|$$

provided that the limit exists and is independent of z, $0<z<L$. For transient solutions of the heat equation, the relaxation time can be computed explicitly from the first nonzero term of the series solution.

In general:

Theorem 1. For the heat equation $u_t = K u_{zz}$ with the boundary conditions (1) and (2), suppose that all eigenvalues λ_n are positive. Then

$$u(z;t) = \sum_{n=1}^{\infty} A_n \Phi_n(z) e^{-\lambda_n K t}$$

is a transient solution of the heat equation, and the relaxation time is given by

$$\tau = \frac{1}{\lambda_1 K} \quad \text{if } A_1 \neq 0.$$

Example. Compute the relaxation time for the solution

$$u(z;t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L} \cdot e^{-\left(\frac{n\pi}{L}\right)^2 K t}$$

Solution. We write

$$u(z;t) = A_1 \sin\frac{\pi z}{L} \cdot e^{-\left(\frac{\pi}{L}\right)^2 K t} + \underbrace{\sum_{n=2}^{\infty} A_n \sin\frac{n\pi z}{L} e^{-\left(\frac{n\pi}{L}\right)^2 K t}}_{O\left(e^{-\frac{4\pi^2 K t}{L^2}}\right),\ t\to\infty}$$

If $A_1 \neq 0$, we may write

$$u(z;t) = A_1 \sin\frac{\pi z}{L} \cdot e^{-\left(\frac{\pi}{L}\right)^2 K t} \left[1 + O\left(e^{-\frac{3\pi^2 K t}{L^2}}\right) \right]$$

$$\ln|u(z;t)| = \ln|A_1| + \ln \sin\frac{\pi z}{L} - \left(\frac{\pi}{L}\right)^2 K t + O\left(e^{-\frac{3\pi^2 K t}{L^2}}\right)$$

Thus $\displaystyle\lim_{t\to\infty} \frac{1}{t} \ln|u(z;t)| = -\frac{\pi^2 K}{L^2}$.

We have proved that $\tau = \frac{L^2}{\pi^2 K}$ provided that $A_1 \neq 0$.

From this analysis follows that, for large t, the solution $u(z;t)$ is well approximated by the first term of the series.

Since $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \lambda_1 = \frac{\pi^2}{L^2}$

$$\Rightarrow \tau = \frac{1}{K} \cdot \frac{1}{\frac{\pi^2}{L^2}} = \frac{1}{K\lambda_1} \Rightarrow \boxed{\tau = \frac{1}{\lambda_1 K}}$$

To illustrate the five-stage method we consider the following

Example 1. Solve the initial-value problem for the heat equation $u_t = K u_{zz}$ with the boundary conditions $u(0;t) = T_1$, $u_z(L;t) = 0$ and the initial condition $u(z;0) = T_3$, where T_1 and T_3 are positive constants.

Solution.

Stage 1. Steady-state solution.

$$U_{zz} = 0,\ U(0) = T_1,\ U_z(L) = 0 \Longrightarrow U(z) = Az + B$$

$$B = T_1,\ A = 0 \Longrightarrow U(z) = T_1.$$

Stage 2. Transformation of the problem. We set $v(z;t) = u(z;t) - U(z)$

Then $v(z;t)$ satisfies:

$$v_t = K v_{zz} \qquad (1')$$

$$v(0;t) = 0,\ t > 0 \qquad (2')$$

$$v_z(L;t) = 0,\ t > 0 \qquad (3')$$

$$v(z;0) = T_3 - T_1 \qquad (4')$$

Stage 3. Separation of variables.

$v(z;t) = \Phi(z)T(t)$:

$$T'(t) + K\lambda T(t) = 0 \qquad (11) \qquad \to T(t) = C e^{-\lambda K t}$$

$$\Phi''(z) + \lambda \Phi(z) = 0 \qquad (12)$$

we have

$$\Phi(z) = A \cos z\sqrt{\lambda} + B \sin z\sqrt{\lambda} \qquad (\lambda > 0)$$

$$0 \equiv \Phi(z) = A + Bz \qquad (\lambda = 0)$$

$$0 \equiv \Phi(z) = A \cosh z\sqrt{-\lambda} + B \sinh z\sqrt{-\lambda} \qquad (\lambda < 0)$$

The separated solutions of the heat equation that satisfy the homogeneous boundary are therefore of the form

$$\sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right) \exp\left\{-\left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 K t\right\},\quad n = 1, 2, \ldots$$

The superposition principle shows that any function of the form

$$V(z;t)=\sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left\{-\left[\left(n-\frac{1}{2}\right)\frac{\pi}{L}\right]^2 Kt\right\}$$

is a solution of the heat equation with homogeneous b.c.s.
To satisfy the new initial conditions, we set $t=0$ and obtain

$$T_3-T_1=\sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)=\sum_{n=1} A_n \sin\left((2n-1)\frac{\pi z}{2L}\right).$$

$$\Rightarrow \quad A_n=\frac{2}{L}\int_0^L (T_3-T_1)\sin\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)dz$$

$$=-\frac{2}{L}\cdot\frac{T_3-T_1}{(n-\frac{1}{2})\frac{\pi}{L}}\cos\left(\left(n-\frac{1}{2}\right)\frac{\pi z}{L}\right)\Big|_0^L$$

$$=\frac{2}{\pi}\cdot\frac{T_3-T_1}{n-\frac{1}{2}}, \quad n=1,2,\dots$$

Therefore the solution to the original problem is

$$U(z;t)=T_1+\frac{2}{\pi}(T_3-T_1)\sum_{n=1}^{\infty}\frac{\sin(n-\frac{1}{2})\frac{\pi z}{L}}{n-\frac{1}{2}}\exp\left\{-\left[\left(n-\frac{1}{2}\right)\frac{\pi}{L}\right]^2 Kt\right\}$$

Stage 4. As before...

Stage 5. <u>Asymptotic behavior.</u>

we note that $\{A_n\} \leq \frac{2}{\pi}\cdot\frac{T_3-T_1}{\frac{2n-1}{2}}=\frac{4}{\pi}\cdot\frac{T_3-T_1}{2n-1}$

$$\Rightarrow \quad |A_n|\leq\frac{4}{\pi}[T_3-T_1]$$

and $(n-\frac{1}{2})^2\geq\frac{1}{2}(n-\frac{1}{2})$, and therefore

$$|u(z;t)-U(z)|\leq\frac{4}{\pi}(T_3-T_1)\sum_{n=1}^{\infty}\exp\left[-\frac{1}{2}\left(n-\frac{1}{2}\right)\frac{\pi^2}{L^2}Kt\right]$$

$$=\frac{4}{\pi}(T_3-T_1)\,\cancel{\exp\left(-\frac{\pi^2}{}\right)}\sum_{n=1}^{\infty}\left(e^{-\frac{\pi^2}{4L^2}\cdot Kt}\right)^{(2n-1)}$$

$$=\frac{4}{\pi}(T_3-T_1)\frac{\exp\left(-\frac{\pi^2 Kt}{4L^2}\right)}{1-\exp\left(-\frac{\pi^2 Kt}{4L^2}\right)}$$

$\Rightarrow \quad u(z;t) \to U(z)$ when $t \to \infty$.

$$\{u(z;t) - T_1\} = \frac{2}{\pi}(T_3 - T_1)\frac{\sin\frac{\pi z}{2L}}{\frac{1}{2}} \cdot e^{-\frac{\pi^2}{4L^2}Kt} + \frac{2}{\pi}(T_3 - T_1)\sum_{n=2}^{\infty}$$

$$\ln|u(z;t) - T_1| = \ln\left|\frac{2}{\pi}(T_3 - T_1)\, 2\sin\frac{\pi z}{2L}\right| - \frac{\pi^2}{4L^2}Kt + \ln\left[1 + O(e^{-t})\right]$$

$$\frac{1}{\tau} = -\lim_{t\to\infty}\frac{1}{t}\ln|u(z;t) - T_1| = +\frac{\pi^2 K}{4L^2}$$

The relaxation time is given by

$\tau = \frac{4L^2}{\pi^2 K}$, provided that $T_1 \neq T_3$

Temporally nonhomogeneous problems.

Let

$$u_t - K u_{zz} = r(z;t)$$

$$\cos\alpha\, u(0;t) - L\sin\alpha\, u_z(0;t) = T_1(t)$$

$$\cos\beta\, u(L;t) + L\sin\beta\, u_z(L;t) = T_2(t)$$

$$u(z;0) = f(z)$$

Here $r(z;t)$, $f(z)$, $T_1(t)$, and $T_2(t)$ are given functions, assumed to be piecewise smooth in each variable.

To solve a problem of this type, we first consider the case of homogeneous b.c.s, that is $T_1(t) \equiv 0$, $T_2(t) \equiv 0$. The solution is sought in the form of a series of eigenfunctions of the homogeneous problem. Thus

$$\cancel{r(z;t) = \sum_{n=1}^{\infty} r_n(t)\,\Phi_n(z)}, \quad f(z) = \sum_{n=1}^{\infty} f_n^{v_n} \Phi_n(z), \quad u(z;t) = \sum_{n=1}^{\infty} u_n(t)\Phi_n(z)$$

where $\Phi_n(z)$ are normalized eigenfunctions of the (S-L) eigenvalue problem with the associated b.c.s. $r_n(t)$, v_n, and $u_n(t)$ are obtained from the orthogonality of the eigenfunctions as the generalized Fourier coefficients:

$$r_n(t) = \int_0^L r(z;t)\Phi_n(z)\,dz, \quad f_n^{v_n} = \int_0^L f(z)\Phi_n(z)\,dz,$$

$$u_n(t) = \int_0^L u(z;t)\,\Phi_n(z)\,dz.$$

Substituting the series for $u(z;t)$ into the nonhomogeneous heat equation, we have

$$u_t - K u_{zz} = \sum_{n=1}^{\infty} \left[u_n'(t)\,\Phi_n(z) - K u_n(t)\Phi_n''(z) \right]$$

$$= \sum_{n=1}^{\infty} \left[u_n'(t)\,\Phi_n(z) + K\lambda_n \Phi_n(z) \cdot u_n(t) \right]$$

$$= \sum_{n=1}^{\infty} \left[u_n'(t) + K\lambda_n u_n(t) \right] \Phi_n(z) = \sum_{n=1}^{\infty} r_n(t)\,\Phi_n(z)$$

$$u_n'(t) + \underline{k\lambda_n} u_n(t) = r_n(t)$$

$$e^{\int k\lambda_n dt} = e^{k\lambda_n t}$$

$$e^{k\lambda_n t} u_n'(t) + k\lambda_n e^{k\lambda_n t} u_n(t) = e^{k\lambda_n t} r_n(t)$$

$$\frac{d}{dt}\left[e^{k\lambda_n t} \cdot u_n(t)\right] = r_n(t) \cdot e^{k\lambda_n t}$$

$$\underbrace{k\lambda_n e^{k\lambda_n t} \cdot u_n(t)} + \underbrace{e^{k\lambda_n t} \cdot u_n'(t)} = r_n(t) e^{k\lambda_n t}$$

$$e^{k\lambda_n t} u_n(t) = \int_0^t r_n(t) e^{k\lambda_n t} dt + C$$

$$u_n(t) = C \cdot e^{-k\lambda_n t} + e^{-k\lambda_n t} \int r_n(s) e^{k\lambda_n s} dt$$

$$u_n(t) = C_n e^{-k\lambda_n t} + \int_0^t r_n(s) \cdot e^{-\lambda_n k(t-s)} ds$$

$$\frac{dy}{dx} + P(x) y = Q(x)$$

integrating factor: $e^{\int P(x) dx}$

$$e^{\int P(x)dx} \frac{dy}{dx} + e^{\int P(x)dx} P(x) y = Q(x) e^{\int P(x) dx}$$

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = Q(x) e^{\int P(x)dx}$$

$$e^{\int P(x)dx} \cdot y = \int e^{\int P(x)dx} \cdot Q(x) dx + C$$

Therefore we choose $u_n(t)$ as the solution of the ODE: $u_n'(t)+K\lambda_n u_n(t) = r_n(t)$, with the initial conditions $u_n(0)=v_n$. This first-order ODE is easily solved in the form

$$u_n(t) = v_n e^{-\lambda_n K t} + \int_0^t r_n(s) e^{-\lambda_n K(t-s)} ds$$

It can be shown that the above series obtained converges uniformly for $0<z<L$ together with the differentiated series for u_z, u_{zz}, u_t and that the function obtained satisfies the eq. $u_t = K u_{zz} + r(z;t)$.

To solve the original problem with $T_1(t)$ and $T_2(t)$ nonzero, we reduce to homogeneous b.c.s by defining a new function,

$$v(z;t) = u(z;t) - [A(t) + zB(t)]$$

$$\Rightarrow u(z;t) = v(z;t) + [A(t) + zB(t)]$$

$$\Rightarrow v_t + A'(t) + zB'(t) - Kv_{zz} = r(z;t)$$

$$\begin{cases} v_t - Kv_{zz} = r(z;t) - [A'(t) + zB'(t)] \\ v(z;0) = f(z) - [A(0) + zB(0)] \end{cases}$$

$u(z;0) = v(z;0) + [A(0)+zB(0)] = f(z)$

The f.s $A(t)$, $B(t)$ are chosen so that the linear function $A(t)+zB(t)$ satisfies the nonhomogeneous b.c.s of the given problem. This requires that

$$\cos\alpha\, A(t) - L\sin\alpha\, B(t) = T_1(t)$$

$$\cos\beta[A(t)+LB(t)] + L\sin\beta\, B(t) = T_2(t)$$

$$\cos\beta\, A(t) + [L\cos\beta + L\sin\beta]B(t) = T_2(t)$$

If $\det(\;) \neq 0$ $\quad \cos\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta \neq 0$

which is equivalent to the statement that $\lambda=0$ is not an eigenvalue of the (S-L) problem with the homogeneous b.c.s.

$$\cos\alpha[v(0;t)+A(t)] - L\sin\alpha[v_z(0;t)+B(t)] = T_1(t)$$

$$\cos\beta[v(L;t)+A(t)+LB(t)] + L\sin\beta[v_z(L;t)+B(t)] = T_2(t)$$

$$\cos\alpha\, v(0;t) - L\sin\alpha\, v_z(0;t) = \underbrace{-\cos\alpha\, A(t) + L\sin\alpha\, B(t) + T_1(t)}_{=0}$$

$\Rightarrow$ $v(z;t)$ satisfies the homogeneous b.c.s. This leads to the trial solution in the form

$$v(z;t) = \sum_{n=1}^{\infty} v_n(t)\,\Phi_n(z)$$

We must have

$$v_t - K v_{zz} = r(z;t) - [A'(t) + z B'(t)]$$
$$v(z;0) = f(z) - [A(0) + z B(0)]$$

To solve the problem, we must find the generalized Fourier series of the linear function $a+bz$,

$$a+bz = \sum_{n=1}^{\infty} \left(a\langle 1,\Phi_n\rangle + b\langle z,\Phi_n\rangle\right)\Phi_n(z)$$

where we have used the inner product notation for the generalized Fourier coefficients. Replacing $r_n(t)$ and V_n suitably, we are led to the solution

$$\Rightarrow \quad v_n(t) = e^{-\lambda_n K t}\left[V_n - A(0)\langle 1,\Phi_n\rangle - B(0)\langle z,\Phi_n\rangle\right]$$

$$+ \int_0^t e^{-\lambda_n K(t-s)}\left[r_n(s) - A'(s)\langle 1,\Phi_n\rangle - B'(s)\langle z,\Phi_n\rangle\right]ds$$

<u>Example</u>. Find the solution of the heat equation $U_t - K U_{zz} = 0$ with the b.c.s $U(0;t) = a_0 + b_0 t$, $U(L;t) = a_1 + b_1 t$ and the initial conditions $U(z;0) = 0$.

<u>Solution</u>

$$\Rightarrow U(z;t) = a_0 + b_0 t + \frac{z}{L}\left[a_1 - a_0 + (b_1 - b_0)t\right]$$

The new function $v(z;t)$ satisfies the nonhomogeneous heat equation

$$v_t - K v_{zz} = -\left[b_0 + (z/L)(b_1 - b_0)\right]$$

with

$$v(z;0) = -a_0 - \frac{z}{L}(a_1 - a_0).$$

$$\begin{cases} v(0;t) + A(t) = a_0 + b_0 t \\ v(L;t) + A(t) + L B(t) = a_1 + b_1 t \end{cases}$$

$$\Rightarrow \quad A(t) = a_0 + b_0 t$$

$$a_0 + b_0 t + L B(t) = a_1 + b_1 t$$

$$B(t) = \frac{1}{L}\left(a_1 + b_1 t - a_0 - b_0 t\right)$$

$$\Rightarrow B(t) = \frac{1}{L}\left(a_1 - a_0 + (b_1 - b_0)t\right)$$

The appropriate Fourier series is

$$b_0 + (b_1 - b_0)\left(\frac{z}{L}\right) = \sum_{n=1}^{\infty}\left(b_0\int_0^L \sin\frac{n\pi z}{L}\,dz + \frac{(b_1-b_0)}{L}\int_0^L z\sin\frac{n\pi z}{L}\,dz\right)\sin\frac{n\pi z}{L}$$

$$\int_0^L \sin\frac{n\pi z}{L}\,dz = -\frac{L}{n\pi}\cos\frac{n\pi z}{L}\Big|_0^L = \frac{L}{n\pi}\left(1-(-1)^n\right)$$

$$\int_0^L z\sin\frac{n\pi z}{L}\,dz = -\frac{zL}{n\pi}\cos\frac{n\pi z}{L}\Big|_0^L +$$

$$+\frac{L}{n\pi}\int_0^L \cos\frac{n\pi z}{L}\,dz = -\frac{L^2}{n\pi}\left((-1)^n \cancel{-1}\right)$$

$$= \frac{L^2}{n\pi}(-1)^{n+1}$$

$u = z$
$du = dz$
$dv = \sin\frac{n\pi z}{L}dz$
$v = -\frac{L}{n\pi}\cos\frac{n\pi z}{L}$

$$b_0 + (b_1 - b_0)\left(\frac{z}{L}\right) = \sum_{n=1}^{\infty}\left(b_0\frac{L}{n\pi}\left(1-(-1)^n\right) + \frac{(b_1-b_0)L}{n\pi}(-1)^{n+1}\right)\sin\frac{n\pi z}{L}$$

$$= \frac{L}{\pi}\sum_{n=1}^{\infty}\left(\frac{b_0}{n}\left[1-(-1)^n\right] + \frac{b_1-b_0}{n}(-1)^{n+1}\right)\sin\frac{n\pi z}{L}$$

The coefficients $v_n(t)$ are given by

$$v_n(t) = \frac{L}{\pi}e^{-\lambda_n K t}\left(-\frac{a_0}{n}\left[1-(-1)^n\right] - \frac{a_1-a_0}{n}(-1)^{n+1}\right)$$

$$-\frac{L}{\pi}\left(\frac{b_0}{n}\left[1-(-1)^n\right] + \frac{b_1-b_0}{n}(-1)^{n+1}\right)\underbrace{\int_0^t e^{-\lambda_n K(t-s)}\,ds}$$

$$e^{-\lambda_n K t}\int_0^t e^{\lambda_n K s}\,ds$$

$$e^{-\lambda_n K t}\cdot\frac{1}{\lambda_n K}e^{\lambda_n K s}\Big|_0^t$$

$$= \frac{1}{\lambda_n K}\left(1 - e^{-\lambda_n K t}\right)$$

The solution of the given is

$$u(z;t) = U(z;t) + \sum_{n=1}^{\infty} v_n(t)\Phi_n(z)$$

when we make the appropriate substitutions for $U(z;t)$ and $v_n(t)$ from above.

Problem for the hyperpolic equation

$$\left.\begin{aligned} &y_{tt}(s;t) = c^2 y_{ss} \\ &y(0;t) = 0 = y(L;t) \\ &y(s;0) = f_1(s) \\ &y_t(s;0) = 0 \end{aligned}\right\} \quad (1)$$

Separated solutions:

Let $y(s;t) = S(s)T(t)$, it follows that

$$y_{tt} = c^2 y_{ss} \Rightarrow S(s)T''(t) - c^2 S''(s)\,T(t) = 0$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{S''(s)}{S(s)} = -\lambda$$

$$S''(s) + \lambda S(s) = 0 \quad , \quad S(0) = 0,\ S(L) = 0$$

$$T''(t) + \lambda c^2 T(t) = 0$$

$$S(s) = A\cos s\sqrt{\lambda} + A_3 \sin s\sqrt{\lambda}$$

$$\underline{0 = S(0) = A}$$

$$0 = A_3 \sin L\sqrt{\lambda}$$

$$L\sqrt{\lambda} = n\pi$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$S(s) = A_3 \sin\frac{n\pi s}{L},$$

$$T(t) = A_1 \cos\frac{n\pi ct}{L} + A_2 \sin\frac{n\pi ct}{L},$$

$$T_t(0) = 0 \Rightarrow \left(-A_1\frac{n\pi c}{L}\sin\frac{n\pi ct}{L} + A_2\frac{n\pi c}{L}\cos\frac{n\pi ct}{L}\right)\Big|_{t=0} = 0 \quad (A_2 = 0)$$

where A_1, A_2, A_3 are constants.

The separated solutions are

$$y(s;t) = B_n \cos\frac{n\pi ct}{L}\cdot\sin\frac{n\pi s}{L}$$

By superposition principle the formal solution of problem (1) is

$$\boxed{y(s;t) = \sum_{n=1}^{\infty} B_n \cos\frac{n\pi ct}{L}\cdot\sin\frac{n\pi s}{L}} \quad (2)$$

To find the coefficients B_n, we set $t=0$. This requires that

$$f_1(s) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L} \qquad (3)$$

Therefore B_n is the n-th Fourier sine coefficient of f_1

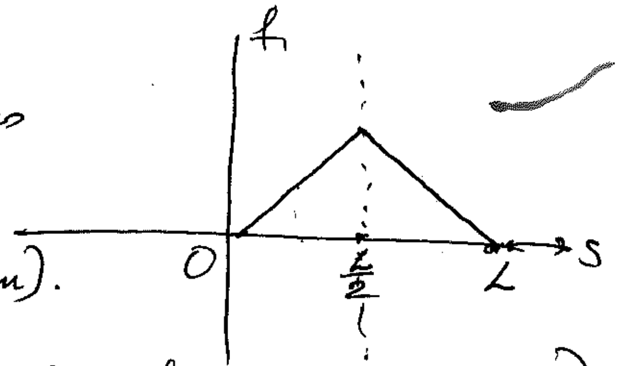
$$B_n = \frac{2}{L}\int_0^L f_1(s) \sin\frac{n\pi s}{L}\, ds \qquad (4)$$

Thus to solve the problem (1) we have a simple rule: Given f_1, compute B_n from (4) and substitute into (1). This is called the Fourier representation of the solution.

<u>Example 1</u>. Solve the problem (1) in the case where

$$f_1(s) = \begin{cases} s, & 0 < s < \frac{L}{2} \\ L-s, & \frac{L}{2} < s < L. \end{cases}$$

<u>Solution</u>.



f_1 is even about $s = \frac{L}{2}$, whereas $\sin\frac{n\pi s}{L}$ is even (resp. odd) about $s=\frac{L}{2}$ if n is odd (resp. even).

$$\left(\sin\frac{n\pi(L-s)}{L} = \sin\left(n\pi - \frac{n\pi s}{L}\right) = -\cos n\pi \cdot \sin\frac{n\pi s}{L} = -(-1)^n \sin\frac{n\pi s}{L} \right)$$

Therefore $B_n = 0$ if n is even. If <u>n is odd</u>, we have

$$B_n = \frac{2}{L}\int_0^L f_1(s)\sin\frac{n\pi s}{L}\,ds = \frac{4}{L}\int_0^{\frac{L}{2}} f_1(s)\sin\frac{n\pi s}{L}\,ds = \frac{4}{L}\int_0^{L/2} s\sin\frac{n\pi s}{L}\,ds$$

$$= \frac{4}{L}\frac{Ls}{n\pi}\cdot\cos\frac{n\pi s}{L}\Big|_{L/2}^{0} + \frac{4\cdot L}{L\cdot n\pi}\int_0^{L/2}\cos\frac{n\pi s}{L}\,ds = \frac{4}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi}{2} = \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Let $n = 2m-1$, $\sin\frac{n\pi}{2} = \sin\frac{(2m-1)\pi}{2} = \sin\left(m\pi - \frac{\pi}{2}\right) = (-1)^{m+1}$

Therefore, we have solved the problem

$$y(s;t) = \frac{4L}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \cos \frac{(2m-1)\pi ct}{L} \sin \frac{(2m-1)\pi s}{L}.$$

Now, we will rewrite the solution

$$y(s;t) = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi s}{L} \qquad (2)$$

another form. To do this, we use the trigonometric indentity $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$:

$$y(s;t) = \frac{1}{2} \sum_{n=1}^{\infty} B_n \left[\sin \frac{n\pi}{L}(s+ct) + \sin \frac{n\pi}{L}(s-ct)\right]$$

Let $\bar{f}_1$ is the odd, $2L$-periodic extension of $f_1(s)$, $0<s<L$.

Then
$$\bar{f}_1(s+ct) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(s+ct),$$
$$\bar{f}_1(s-ct) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(s-ct)$$

and from (2), we have

$$y(s;t) = \frac{1}{2}\left[\bar{f}_1(s+ct) + \bar{f}_1(s-ct)\right] \qquad (5)$$

$y(s;t)$ in (5) satisfies the wave equation

$$y_t = \frac{1}{2} c \bar{f}_1'(s+ct) - \frac{1}{2} c \bar{f}_1'(s-ct)$$
$$y_{tt} = \frac{1}{2} c^2 \bar{f}_1''(s+ct) + \frac{1}{2} c^2 \bar{f}_1''(s-ct)$$
$$y_s = \frac{1}{2} \bar{f}_1'(s+ct) + \frac{1}{2} \bar{f}_1'(s-ct)$$
$$y_{ss} = \frac{1}{2} \bar{f}_1''(s+ct) + \frac{1}{2} \bar{f}_1''(s-ct)$$

Clearly, $y_{tt} = c^2 y_{ss}$

To check the b.c.s, we have

$$y(0;t) = \frac{1}{2}[\bar{f}_1(ct) + \bar{f}_1(-ct)] = 0 \qquad (\bar{f}_1(-ct) = -\bar{f}_1(ct))$$

Since $\bar{f}$ is odd, and

$$y(L;t) = \frac{1}{2}[\bar{f}(L+ct) + \bar{f}(L-ct)]$$
$$= \frac{1}{2}[\bar{f}(L+ct) - \bar{f}(-L+ct)]$$
$$= \frac{1}{2}[\bar{f}(L+ct) - \bar{f}(L+ct)] = 0$$

$\bar{f}(-L+ct) =$
$= \bar{f}(2L-L+ct)$
$= \bar{f}(L+ct)$
Since $\bar{f}$ is $2L$-periodicity

The initial condition

$$y(s;0) = \bar{f}(s),$$
$$y_t(s;0) = 0 \quad \text{, wherever } \bar{f}' \text{ is defined.}$$

We now solve the problem with nonzero initial condition $y_t(s;0)$ (velocity). Thus we must solve

$$y_{tt}(s;t) = c^2 y_{ss}, \quad t>0,\ 0<s<L$$
$$y(0;t) = 0 = y(L;t), \quad t>0,$$
$$y(s;0) = 0, \quad 0<s<L,$$
$$y_t(s;0) = f_2(s), \quad 0<s<L.$$

To solve this problem, ~~$f_2(s)$~~ we begin with the separated solutions that satisfy the wave equation, the boundary conditions, and the first initial condition. These are of the form

$$\sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}$$

To satisfy the second initial condition, we try a superposition of these,

$$y(s;t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L}\cdot\sin\frac{n\pi ct}{L} \qquad (6)$$

$$\Rightarrow \quad y_t(s;t) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi s}{L}\cdot\cos\frac{n\pi ct}{L}$$

$$\Rightarrow \quad y_t(s;0) = f_2(s) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi s}{L} \qquad (7)$$

In other words, $\frac{n\pi c}{L} B_n$ is the n-th Fourier sine coefficient of $f_2(s)$, $0<s<L$:

$$\frac{n\pi c}{L} B_n = \frac{2}{L}\int_0^L f_2(s)\sin\frac{n\pi s}{L}\,ds\ , \quad n=1,2,\dots$$

To obtain an explicit representation, we apply

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$ to the Fourier

representation. Thus

$$y(s;t)=\frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\cos\frac{n\pi}{L}(s-ct)-\cos\frac{n\pi}{L}(s+ct)\right]$$

$$\Rightarrow\quad y(s;t)=\frac{1}{2}\sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)\int_{s-ct}^{s+ct}\sin\frac{n\pi z}{L}\,dz$$

$$=\frac{1}{2}\int_{s-ct}^{s+ct}\Big\{\underbrace{\sum_{n=1}^{\infty} B_n\frac{n\pi}{L}\cdot\sin\frac{n\pi z}{L}}\Big\}\,dz .$$

$$\bar{f}_2(z) = \frac{1}{c}\sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\sin\frac{n\pi z}{L}$$

~~where~~ If $f_2(s)$, $0<s<L$, is continuous and piecewise smooth, the Fourier series (7) converges for all s to the odd periodic extension of f_2, denoted $\bar{f}_2$. This completes the Fourier representation of the solution.

$\frac{1}{c}\sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\sin\frac{n\pi z}{L}$ is the Fourier sine series for $\bar{f}_2$, the odd, $2L$-periodic extension of $f_2(s)$, $0<s<L$. Thus we have the explicit representation

$$y(s;t)=\frac{1}{2c}\int_{s-ct}^{s+ct}\bar{f}_2(z)\,dz \qquad (8)$$

d'Alember's general solution.

The explicit solutions just obtained can be combined to obtain a solution of the wave equation for the general initial conditions $y(s;0)=f_1(s)$, $y_t(s,0)=f_2(s)$. For this purpose, consider the function

$$y(s;t)=\frac{1}{2}\left[f_1(s+ct)+f_1(s-ct)\right]+\frac{1}{2c}\int_{s-ct}^{s+ct} f_2(z)\,dz \qquad (9)$$

Suppose that f_1, f_2', f_1'', f_2, and f_2' are continuous functions. Then

$$y_t=\frac{c}{2}\left[f_1'(s+ct)-f_1'(s-ct)\right]+\frac{1}{2}\left[f_2(s+ct)+f_2(s-ct)\right]$$

$$y_{tt}=\frac{c^2}{2}\left[f_1''(s+ct)+f_1''(s-ct)\right]+\frac{c}{2}\left[f_2(s+ct)-f_2(s-ct)\right]$$

$$y_s=\frac{1}{2}\left[f_1'(s+ct)+f_1'(s-ct)\right]+\frac{1}{2c}\left[f_2(s+ct)-f_2(s-ct)\right]$$

$$y_{ss}=\frac{1}{2}\left[f_1''(s+ct)+f_1''(s-ct)\right]+\frac{1}{2c}\left[f_2'(s+ct)-f_2'(s-ct)\right]$$

We observe that $y(s;0)=f_1(s)$, $y_t(s;0)=f_2(s)$. $y_{tt}=c^2y_{ss}$ for all (s,t).

This general solution is called d'Alembert's solution of the wave equation.

Thus d'Alembert's formula (9) gives a solution of the wave equation $y_{tt}=c^2y_{ss}$ valid for all $t>0$, $-\infty<s<\infty$, and satisfies the initial conditions $y(s;0)=f_1(s)$, $y_t(s;0)=f_2(s)$.