Rotational Motion

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Learning Objectives:

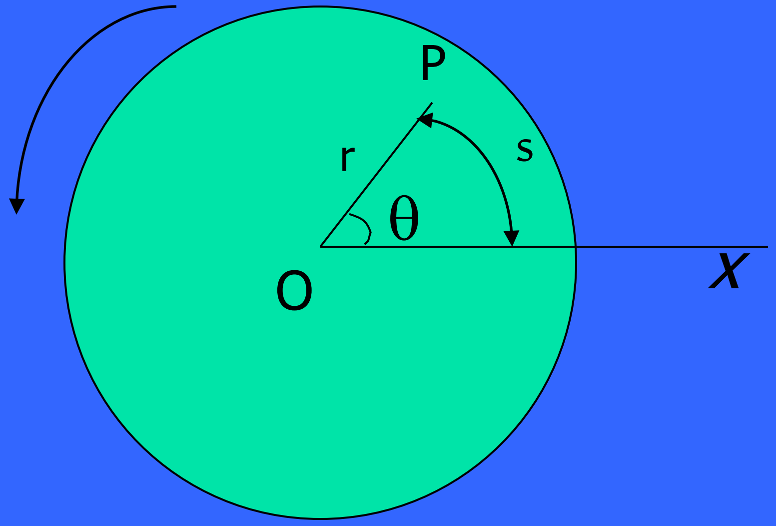
- to learn the concept of rigid body
- to understand the motion of rigid bodies
- to define angular velocity and angular acceleration
- to learn the relation between angular acceleration, centripetal and tangential acceleration
- to understand the causes of rotational motion: torque
- application of those term to the muscles and bones of the human body.

- The motion of real-world bodies can be very complex. They can have rotational as well as translational motion and they can deform.
- Real life object are NOT point-like. To describe a real-life object we need a body, which has a perfectly definite and unchanging shape and size.
- This idealized model is called the rigid body.

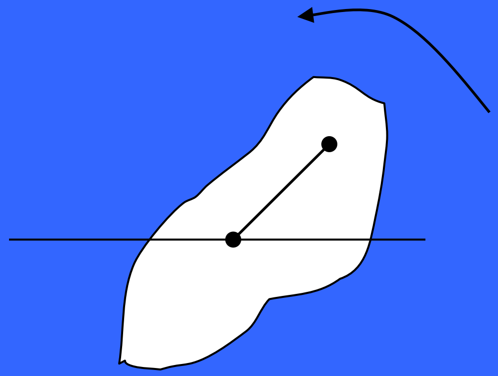
Rigid Bodies

- A body with a definite shape that doesn’t change, so that the particles composing it stay in fixed positions relative to one another.
- Translational + rotational motion about its center of mass
- Translational motion: only changes inposition is considered, changes of orientation are ignored.
- Rotational motion: all points in the body move in circles and centers of these circles lie on a line called axis of rotation

- Axis of rotation:
- Perpendicular to the page



Looking down on a wheel that is rotating counterclockwise about an axis through the wheel's center at point O.



Angular Quantities

- Every point in a body rotating about a fixed axis moves in a circle whose center is on the axis and whose radius is r, the perpendicular distance of that point from the axis of rotation.
- A perpendicular line drawn from the axis to any point sweeps out the same angle θ in the same time.
- Position of the body is specified with angle θ. w.r.t reference line, x-axis.

- Point P moves through an angle θ when it travel distance s measured along the circumference of circular path.
- In circular motion, *radian* is the angular measure.
- *One radian (rad) is defined as the angle subtended by an arc whose length is equal to the radius*.
- If point P moved a distance s and if $s=r$, then θ is exactly equal to 1 rad.

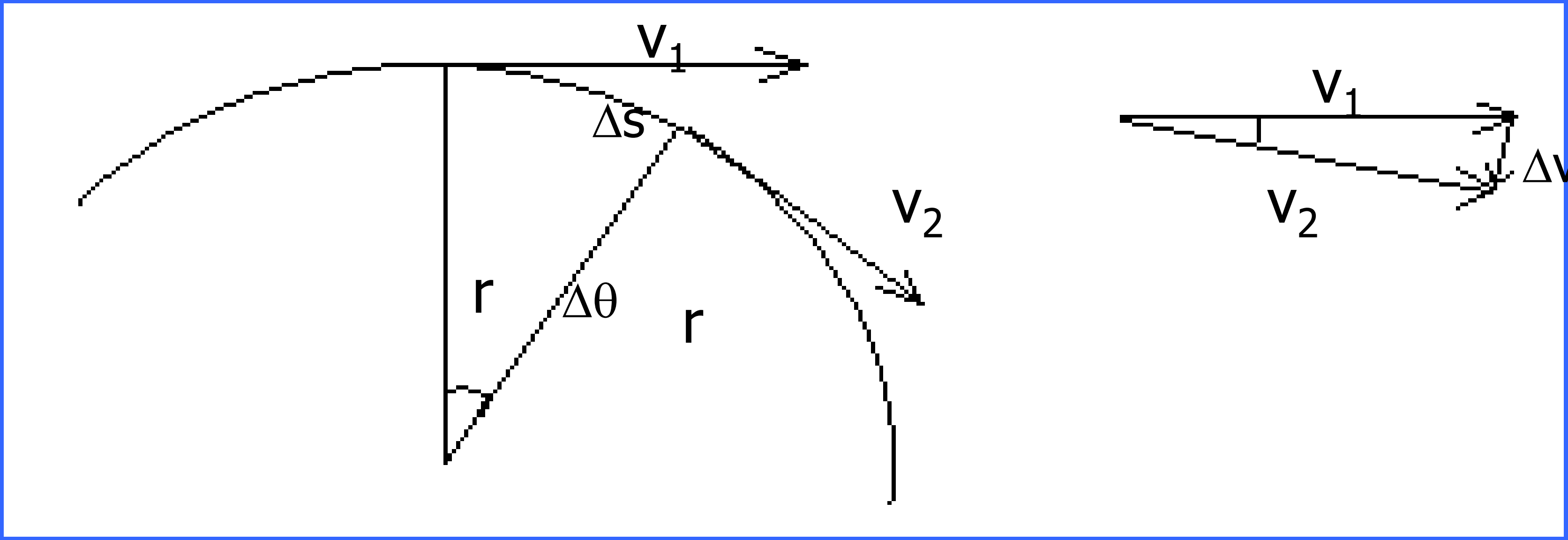
- $\theta = s / r$
- 360 ° = 2 π rad; therefore:
- 360 / 6.28 ≈ 57.3 °
- Radian is dimensionless
- *Angular velocity*:
  - Angular distance (angle of rotation) per unit time
  - Average angular velocity:
  - $w_{av} = \theta / t$

- *Instantenous angular velocity*:
- $w = \Delta \theta / \Delta t$
- (Δt very small, approaching zero)
- Radians/second
- !!All points in the body rotate with same angular velocity
- *Angular acceleration*:
- Change in angular velocity divided by time required to make this change
- $$\alpha_{av} = \frac{w - w_0}{t}$$

- *Instantenous angular acceleration*:
- $\alpha = \Delta w / \Delta t$; rad /s^2
- (Δ t very small, approaching zero)
- **α is also same for all points. α and w are properties of the rotating body as a whole.**
- For a rigid body is rotating around a fixed axis, every part of the body has the same angular velocity [omega] and the same angular acceleration a, but points that are located at different distances from the rotation axis have different linear velocities and different linear accelerations.

- *Relation between linear and angular quantities*:
- Linear velocity is tangent to its circular path:
- $v = \Delta s / \Delta t = r \, \Delta\theta / \Delta t = r w$
- !!Linear velocity is greater for points farther from the axis.

- *Tangential linear acceleration*:
- $a_T = \Delta v / \Delta t = r \, \Delta w / \Delta t = r \, \alpha$
- *The total linear acceleration:*
- $a = a_T + a_c$ where a_c is the radial component which is also known as "centripetal acceleration"



- An object moves in a circle at constant speed v is said to experience uniform circular motion
- $\Delta v/v \approx \Delta s / r \ (v = v_1 = v_2)$
- $\Delta v \approx v/r \ \Delta s$
- $a_c = \Delta v / \Delta t = (v/r)\ (\Delta s/\Delta t)$
- $a_c = v^2 / r$
- An object moving in a circle of radius r with constant speed v has an acceleration whose direction is toward the center of the circle and whose magnitude v^2 / r.
- $a_c = v^2 / r = (w\ r)^2/r = w^2\ r$

- Relation between angular velocity and frequency:
- Frequency: number of complete revolutions per second.
- One revolution corresponds to an angle 2π radians, and thus 1 rev/s = 2π radians/s.
- $f = w / 2\pi$; $w = 2\pi f$
- The time required for one complete revolution is called the period and
- $T = 1 / f$

Example 1: What is the linear speed of a point 1.2 m from the center of a steadily rotating merry-go-round that rotates one complete revolution in 4.0 s?

- First we find angular velocity by the help of period:
- $f=1/T = 0.25\ s^{-1}$
- $w = 2\pi f = 6.28 . 0.25 = 1.6$ rad/s
- $v= r\ w = 1.2 . 1.6 = 1.9$ m/s

Example 2: What is the magnitude of the acceleration of a child placed at the point on the merry-go-round described in the previous example?

- w = 1.6 rad/s; v = 1.9 m/s
- Since w is constant, then $a_T = r\,\alpha = 0$
- $a_c = w^2\, r = (1.6)^2 \,.\, (1.2) = 3\ m/s^2$
- or $(1.9)^2 / (1.2) = 3\ m/s^2$

- Example 3: A centrifuge rotor is accelerated from rest to 20.000 rpm in 5 min. What is its average angular acceleration?

- Initially w=0. The final angular velocity is
- w= (20.000 rev/min) . 2π (rad/rev)
- 60 (min/s)
- w= 2100 rad/s
- since $\alpha_{av} = \Delta w / \Delta t$; $\alpha_{av} = 2100 / 300 = 7$ rad/s^2

- Example 4: A wheel turns with an angular acceleration of $\alpha = 50$ rad/s^2. Find tangential and perpendicular components of acceleration when w=10 rad/s. Radius of the wheel is 0.8 m long
- $a_T = w^2 . r = 80 \text{ m} / \text{s}^2$
- $a_{//} = r\,\alpha = 40 \text{ m} / \text{s}^2$
- $a = \sqrt{\;} a_T{}^2 + a_{//}{}^2 = 89 \text{ m/s}^2$

Kinematics of Uniformly Accelerated Rotational Motion

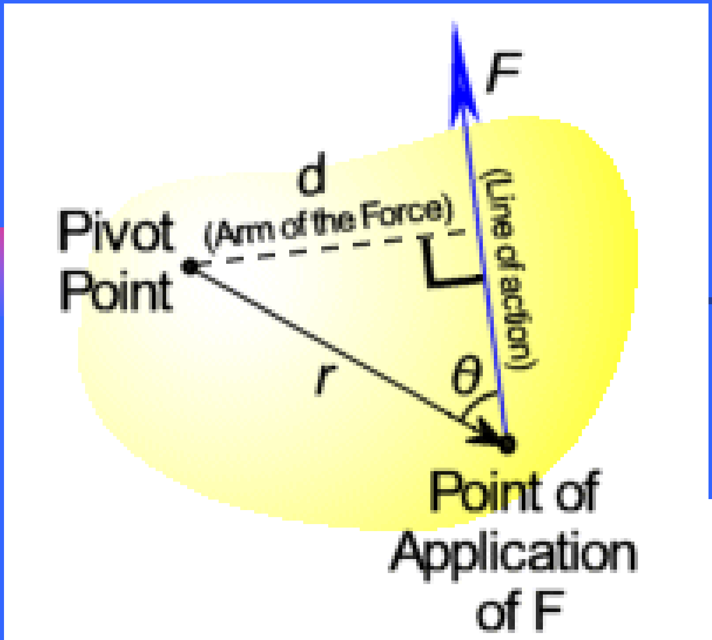
- Angular equations for constant angular acceleration will be analogous to equations of motion for uniform linear acceleration except that:
- x is replaced with θ;
- v is replaced by w
- a is replaced by α

Example 5: Through how many turns has the centrifuge rotor of example 3 turned during its acceleration period? Assume constant angular acceleration.

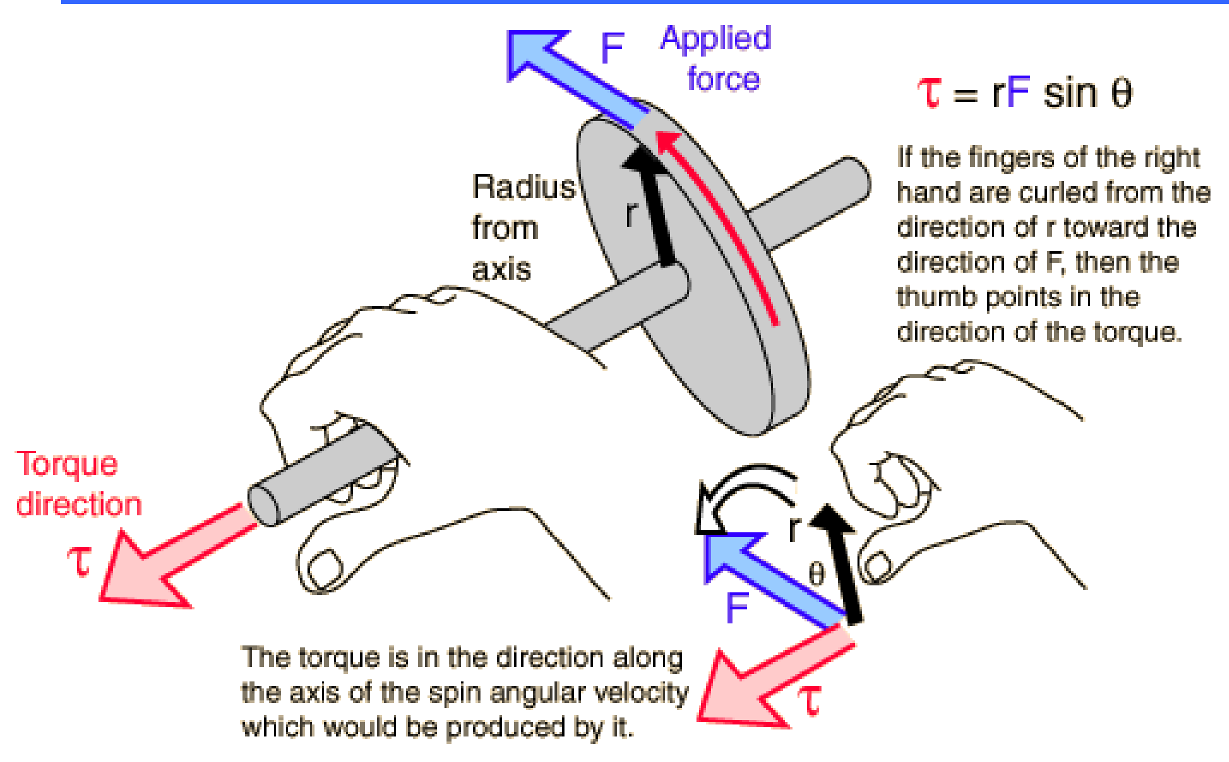
- $w_0 = 0$; $w = 2100$ rad/s; $\alpha = 7$ rad/s^2
- and $t = 300$ s
- $\theta = \theta_0 t + ½ \alpha t^2 = 0 + ½ (7) (300)^2$
- $= 3.2 \times 10^5$ rad
- divided by 2π will give number of revolutions:
- 5×10^4 revolutions

Torque: Rotational Dynamics

- causes of rotational motion
- if you apply a force closer to the hinge, you will need greater force to open it than you apply the force to the end. The effect of the force is less.
- Angular acceleration of the door is proportional not only to the magnitude of force, but is also proportional to the *perpendicular distance from the axis of rotation to the line along which the force acts.*



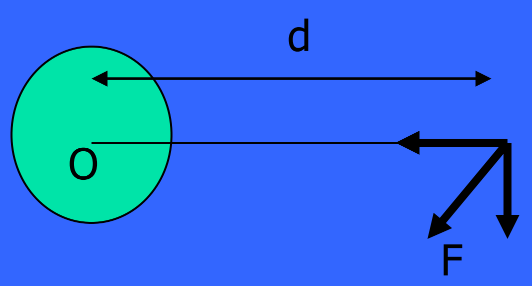
$\tau = rF \sin\theta = Fd$

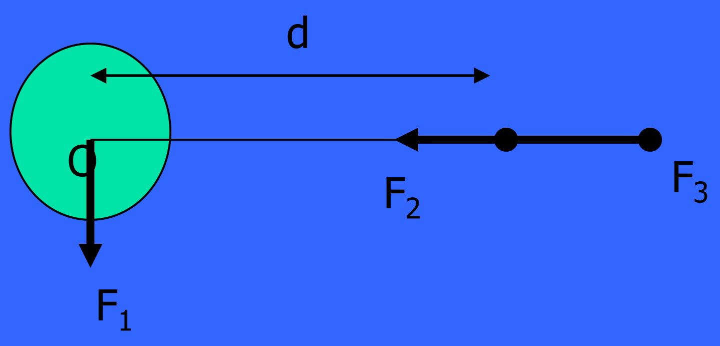


- This distance is called *the lever arm or moment arm* .
- Torque (τ) is a measure of the tendency of a force to rotate a body
- This force should be perpendicular or at least should have a perpendicular component to the rotation line

$$\tau = F \cdot d \cdot \sin\theta$$

Forces with zero torque

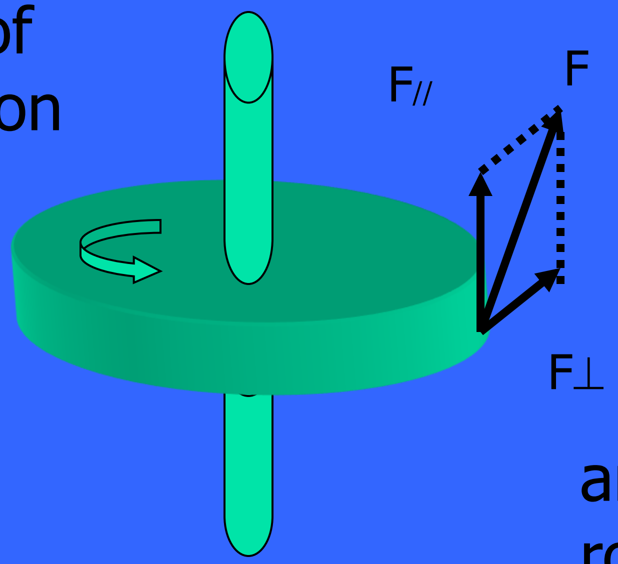




- $\tau \propto \alpha$ (torque gives rise to angular acceleration)
- Force applied with an angle will also be less effective than the force applied straight on.
- $\tau = r_{\perp} F$ or $\tau = F_{\perp} r$
- Unit of torque is N.m

- If all torques rotate the body in same direction, take summation; if one turns in one direction and the other turns it reverse, take difference.

Axis of rotation



only a force (or a component of force) in a plane perpendicular to the axis will give rise to rotation about the axis

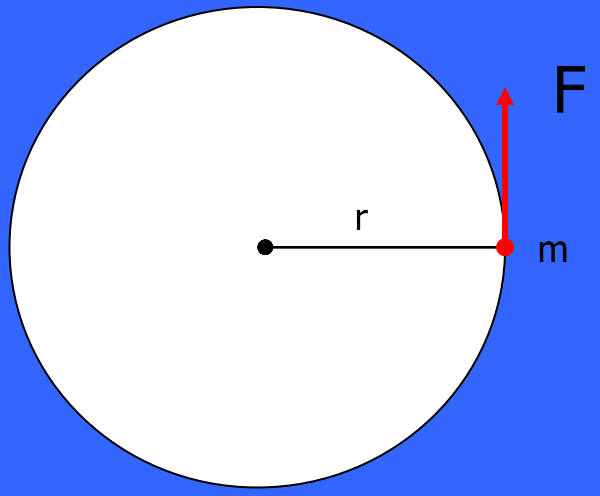
angular acceleration α of a rotating body is proportional to the net torque applied to it

Angular Momentum:

- Linear momentum of a motion is given with $P = m v$
- Angular momentum is closely analogous to that .
- We define the angular momentum as the product of the magnitude of its momentum and the perpendicular distance from the axis to its instantenous line of motion.
- $\mathbf{L = m\ v\ r = m\ w\ r^2}$
- $\mathbf{L = I\ w}$

Torque and Rotational Inertia

- $\alpha \propto \Sigma\tau$: this corresponds to the Newton's second law $a \propto \Sigma F$
- a is also inversely proportional to m
- what m corresponds in rotation?



$$F = m\,a\;;\; a_T = r\,\alpha\;;$$

$$F = m\,r\,\alpha$$

$$\tau = r\,F = m\,r^2\,\alpha$$

[single particle]

- The quantity $m r^2$ represents the rotational inertia of the particle or moment of inertia.
- Now let us consider a rigid body, such as a wheel rotating about an axis through its center.
- We can think of the wheel as consisting of many particles located at various distances from the axis of rotation.
- To find the total torque, we have to take the sum over all the particles. Since location of each particle (r) from the origin will be different, the sum of the various torques for each point should be calculated separately:
- $\Sigma\tau = (\Sigma mr^2)\,\alpha$ [α is same for all particles]

- $I = \Sigma mr^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots.$

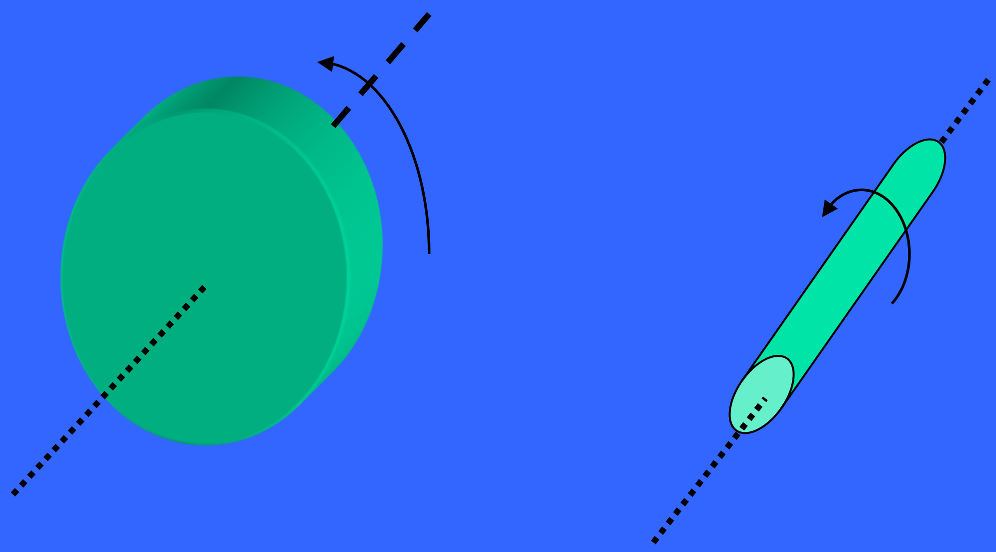
$\Sigma\tau = (\Sigma mr^2)\,\alpha = I\,\alpha$

THIS IS THE ROTATIONAL EQUIVALENT OF NEWTON'S SECOND LAW.

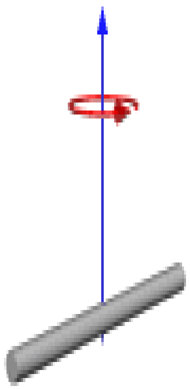
IT IS VALID FOR THE ROTATION OF A RIGID BODY ABOUT A FIXED AXIS.

Rotational inertia of an object depends not only on its mass, but also on how that mass is distributed with respect to the axis.

- !!!mass can not be considered concentrated at CM for rotational motion.
- a large diameter wheel has greater rotational inertia than one of smaller diameter but equal mass.



- **Thin rod** of mass m and length ℓ, rotating around the axis which passes through its center and is perpendicular to the rod.



Let Oz be the axis of rotation, and Ox the axis along the rod. If ρ is the density, and s the cross-section of the rod (so that $m = \rho\ell s$), then the volume element for the integral formula will be equal to $\mathrm{d}V = s \cdot \mathrm{d}x$, where x changes from $-\frac{1}{2}\ell$ to $\frac{1}{2}\ell$. The moment of inertia can be found by computing the integral:

$$I = \int_{-\ell/2}^{\ell/2} \rho x^2 s \mathrm{d}x = \rho s \left. \frac{x^3}{3} \right|_{-\ell/2}^{\ell/2} = \frac{m}{s\ell} \cdot s \cdot 2 \frac{\ell^3/8}{3} = \frac{1}{12} m\ell^2.$$

Center of Mass

- General motion (rotational and translational) of extended bodies –like human body- can be considered as the sum of their trans. and rot. motion of center of mass (CM).
- we can consider any extended body as consisted of many tiny particles
- first consider two particles m_1 and m_2 located at x_1 and x_2 on x-axis respectively
- $x_{CM} = \underline{m_1 x_1 + m_2 x_2}$
- m_1+m_2
- If two masses are equal, x_{CM} is midway between them

- we can extend this for more than two particles or more than one dimension

- Note that CM can sometimes lie outside the body-like e.g. doughnut, whose center of mass is at the center of hole.

- knowing the CM of the body is of great use in studying body mechanics

Center of Gravity

- In many equilibrium problems, one of the forces acting on the body is its weight.
-
- To calculate torque of this force with respect to any axis is not a simple problem, because the weight does not act at a single point but is ditributed over the entire body.
-
- However we can calculate the torque due to the body's weight by assuming that entire force of gravity (weight) is concentrated at the center of mass of the body, which is called as **center of gravity.**

- Each particle of weight contributes to the total torque depending on their distance from the axis of rotation.
- $w_1 x_1 + w_2 x_2 + w_3 x_3 + ... = \Sigma w x$
- $W X = w_1 x_1 + w_2 x_2 + w_3 x_3 + ...;$
- then $X = \Sigma w x / W$

If we apply the same thought for the vertical axis:

-

$W Y = w_1 y_1 + w_2 y_2 + w_3 y_3 + ...;$

then $Y = \Sigma w y / W$

If we divide both equation to g (gravity constant), we will see that the center of gravity of any body is identical to its center of mass.

- Total torque due to weight:

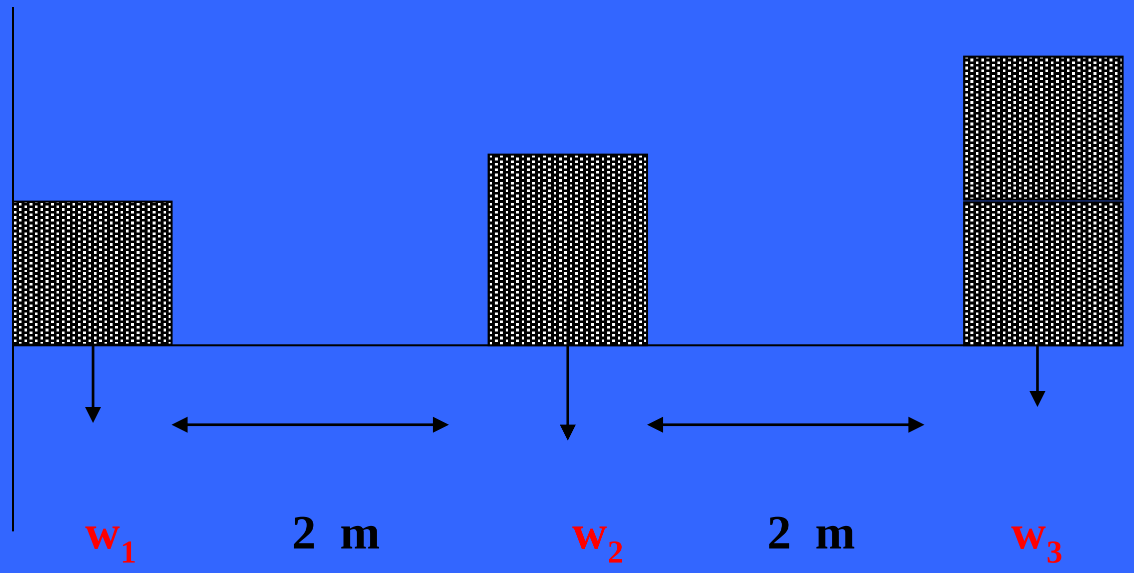
$$\boldsymbol{\Gamma} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \ldots..}{M} \times M \mathbf{g} =$$

- $\mathbf{R}_{CM} \times \mathbf{W}$ center of mass

- The total torque is the same as though the total weight were acting at the position **R** of the center of mass. (which gives us center of gravity).

A weightless plastic rod 4 m long has three blocks as shown in the figure:

Find the center of gravity of the system



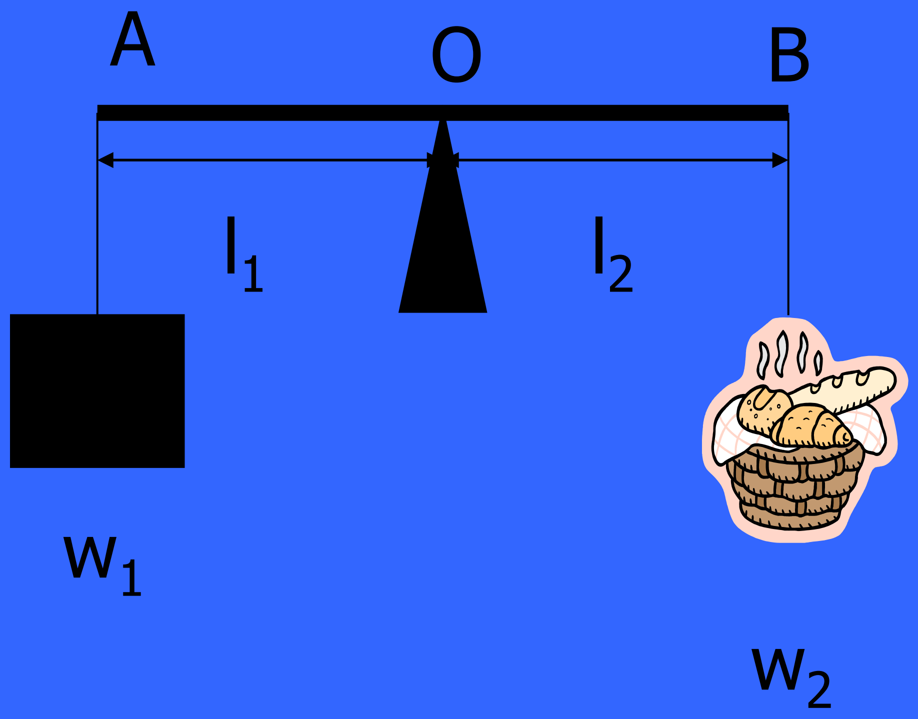
- $\mathbf{w_1 = w_0}$
- $\mathbf{w_2 = w_0}$
- $\mathbf{w_3 = 2\ w_0}$

- $\Sigma\ w = W = 4\ w_0$

$$X = \frac{\Sigma\, w\, x}{W} = \frac{0 + 2\ w_0 + 8\ w_0}{4\ w_0}$$

$= 2.5\ m$

- A hanging bread basket B having weight w_2 , is hung out over the edge of a balcony on a horizantal beam.
- Basket is counterbalanced by weight w_1 .
- Find the weight w_1 needed to balance the basket, and the total upward force exerted on beam at point O.



- $\Sigma F_y = F_o - w_1 - w_2 = 0$
-
- $\Sigma \Gamma_o = w_1 l_1 - w_2 l_2 = 0$
-
- $l_1 = 1.2$ m ; $l_2 = 1.6$ m and $w_2 = 15$ N
-
- Then from the equations above $w_1 = 20$ N and
- $F_o = 35$ N
-
- torque w.r.t. point A
- $\Sigma \Gamma_A = F_o l_1 - w_2 (l_1 + l_2) =$
- $(35)(1.2) - (15)(2.8) = 0$

- **Equilibrium of a Rigid Body :**
-
- Bodies are in equilibrium whenever the sum of forces acting on them is zero.
- If sum of the torques about an axis is zero, body won't have a tendency to rotate and we say that it is in rotational equilibrium.
- Equilibrium means that the object either remains at rest or continues to move with a constant vector quantity.
-
-

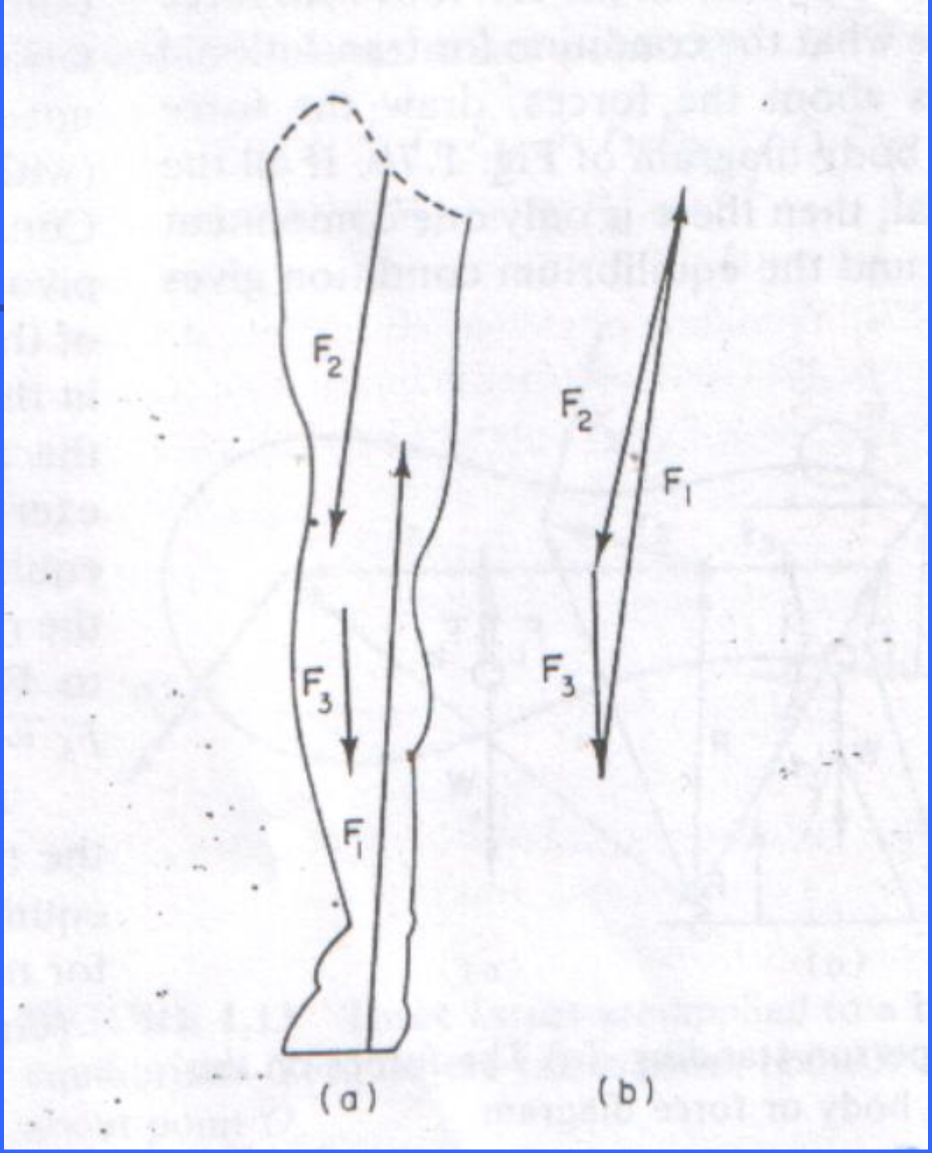
- There are two conditions of equilibrium
- 1st Condition: Translational Equilibrium
- $\Sigma F_x = 0$; $\Sigma F_y = 0$
- Vector sum of all forces acting on the body will be zero.
- The forces need NOT act at one point on the object

- 2nd Condition: Rotational Equilibrium
- $\Sigma \Gamma = 0$ (NET TORQUE IS ZERO) about any axis
-

- Translational
- Equilibrium

- $\sum_i F_i = 0$

- Rotational equilibrium
- $\sum_i \tau_i = \sum_i r_i F_i = 0$

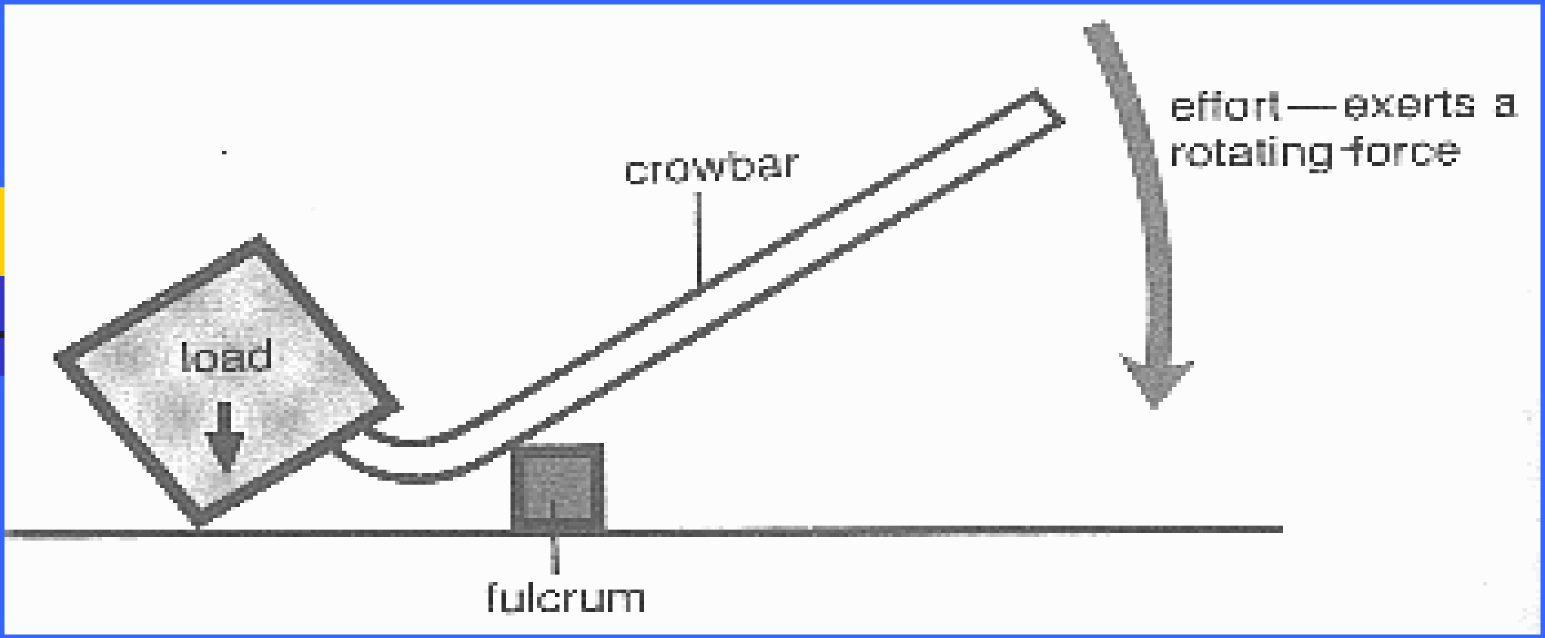


Body Statics

- Bones - moved by the **alternate contraction and relaxation of the skeletal muscles**.

- Skeletal muscles act on the bones as a system of levers.

- For every muscle or group of muscles, there is another muscle or group of muscles which bring about an **opposite movement** called **antagonistic muscles** (e.g.biceps and triceps).



- **Types of Levers in the Body**
- **A lever is an inflexible or rigid rod that is able to rotate about a fixed point called the fulcrum.**

Parts of Levers

- The force to be moved or overcome is called the load (point of resistance). In body this is the weight to be moved

- The force arm is the distance between the fulcrum and the point of applied force (effort). In the body, the bone acts as the lever arm, and the joint is fulcrum.

- The moment of the force (torque) is
- the rotating force. Applied force: contraction of muscles

- **The moment of a force depends not only on the size of the force but also on the distance from the fulcrum that the force is applied:**

  - **Moment of force = Effort x Effort Arm.**

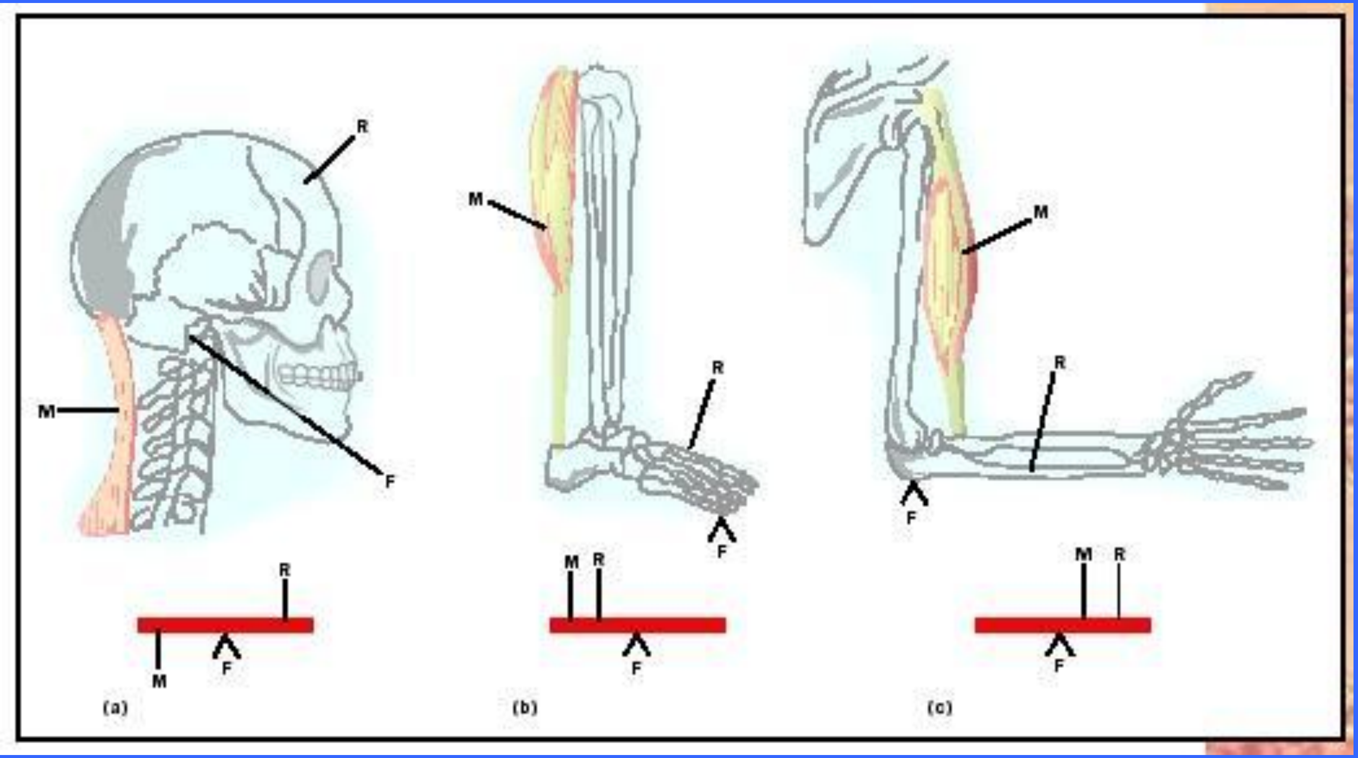
    or

    = Load x Load Arm.

- **The force (i.e. the effort or resistance) is multiplied by the perpendicular distance between the fulcrum and the direction in which the force is applied (i.e. the force arm or the arm of the load).**

Types of Levers Levers are subdivided into three classes on the basis of the arrangement of the fulcrum in relation to the point of effort and point of resistance (load point).

- 1st class: fulcrum between load and force
- E.g. Crowbar
- 2nd class: weight is between force and fulcrum. E.g. Wheelbarrow
- 3rd class: force is between weight and fulcrum and close to the fulcrum.
- E.g.levers in the body



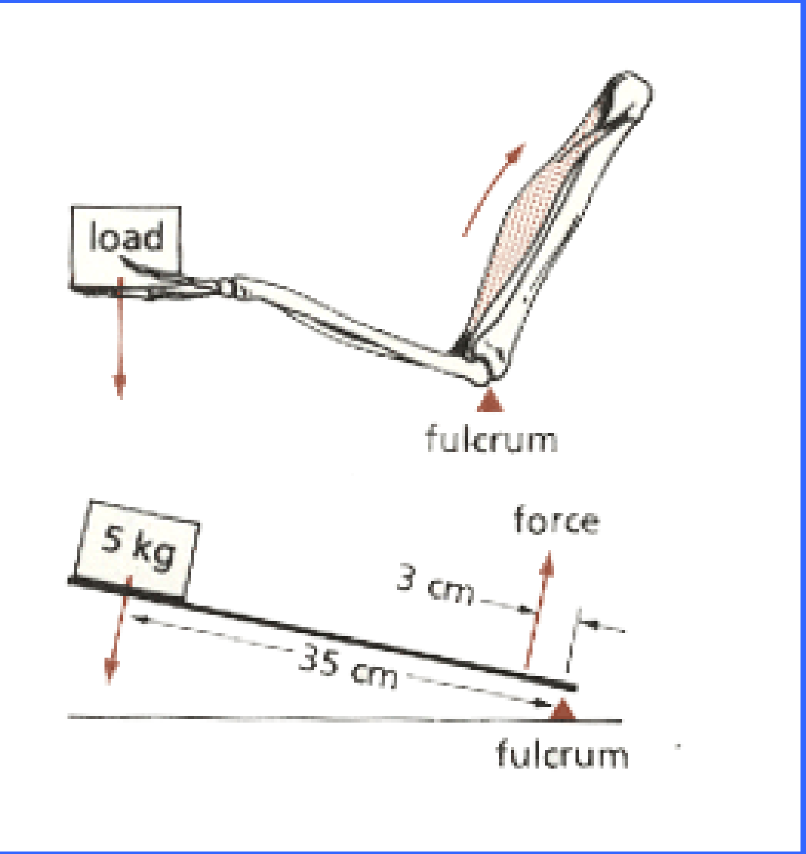
- Classes of levers. (a) In a first-class lever, the fulcrum (F) is set up between the resistance (R) and the effort (M). (b) In a second-class lever, the resistance is between the fulcrum and the effort. (c) In a third-class lever, the effort is between the fulcrum and the resistance.

- In this figure the elbow acts as a fulcrum. A load of 5 kg is placed in the hand, the center of which is 35 cm from the fulcrum (elbow) The biceps muscle is attached at a point 3 cm from the elbow.

- The force required to lift 5 kg is 5 X 10 N.

$$3x = 35 \text{ cm} \times 50 \text{ N}$$
$$= 1750 \text{cm-N}$$
$$x = 583 \text{ N}$$

- The biceps muscle then exerts a force of 583 N (58.3 kg) to raise a load 5 kg.



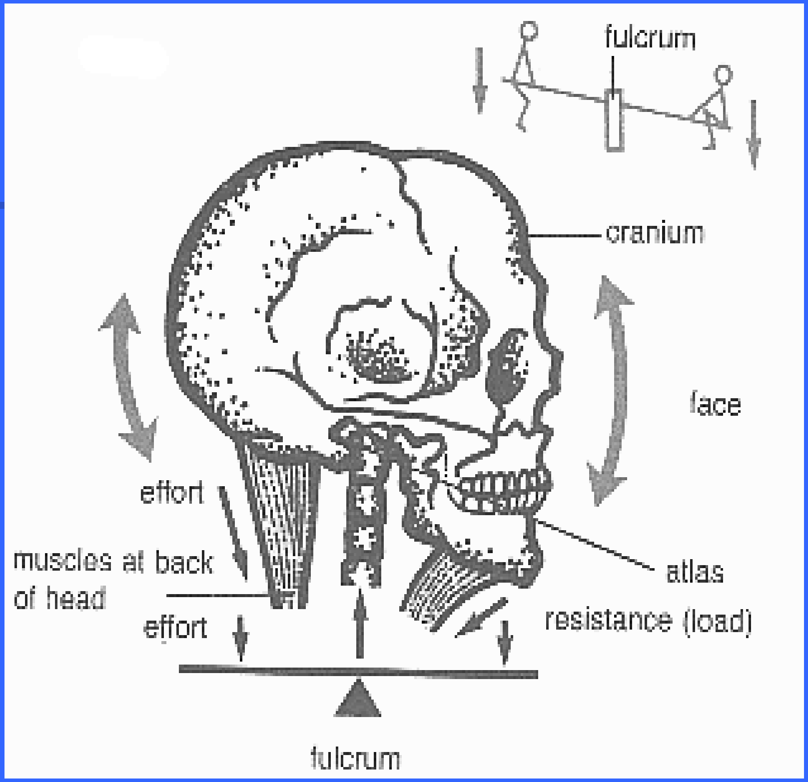
Mechanical Advantage

- M.A. = F_L / F_a ; L:load; a: applied force

- $\dfrac{F_L}{F_a} = \dfrac{X_a}{X_L}$

- Short limbs able to exert large forces however rapid movement requires long limbs

- **Levers of the First Class**
- Here the fulcrum lies between the effort and the load. In our bodies, a lever of the first class can be found when the head undergoes nodding movements. The weight of the face and the head are the resistance. The contraction of the neck muscles is the effort to lift the weight.



Levers of the Second Class

Here the **load lies between the fulcrum and the effort**. A lever of the second class operates on the same principle as a **wheelbarrow**.

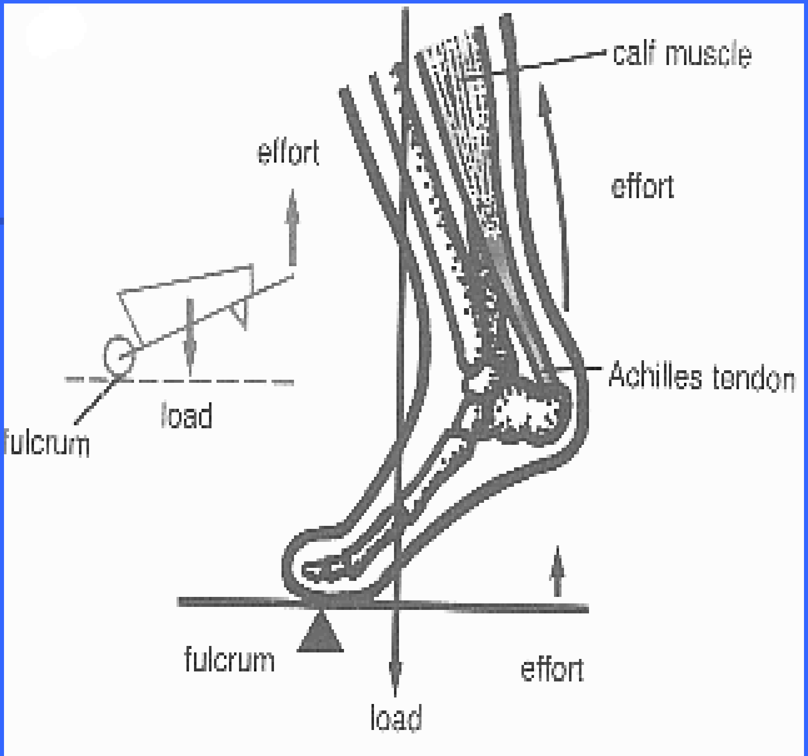
A **small upward force** applied to the handles can overcome a much **larger force (weight)** acting downwards in the barrow.

Similarly a relatively small **muscular effort is required** to raise the body weight.

In our bodies, a lever of the second class can be found in our **feet** when we stand on our toes and lift our heels of the ground.

The **resistance (load) is the weight of our body resting on the arch of the foot**.

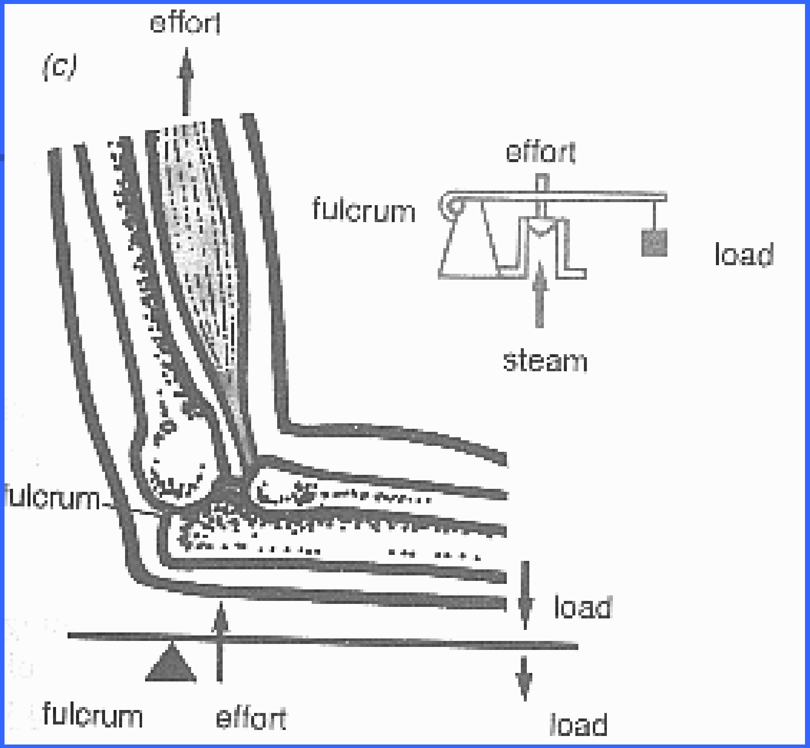
The **effort is brought about by the contraction of the calf muscle attached to the heel**. This **leverage allows us to walk**. The main purpose of a lever of the second class is to **overcome the resistance**.



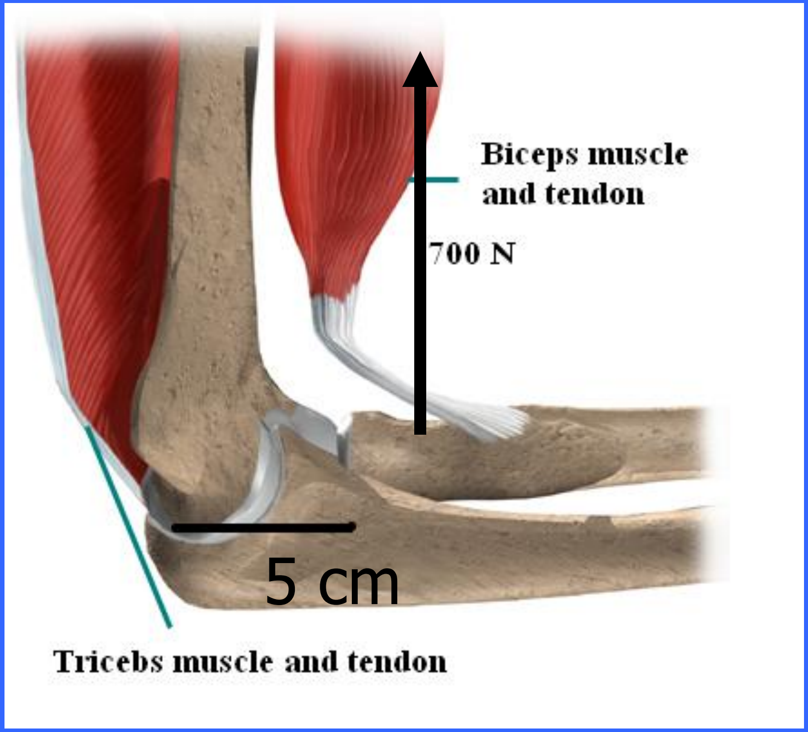
Levers of the Third Class

- Here the **effort lies between the fulcrum and the load**. In our bodies, an example of a lever of the third class is when the **biceps contracts, allowing us to lift something in our hand**.
- The **elbow is the fulcrum**, the **hand and its contents are the resistance (or load)** and the **biceps muscles creates the effort**.

- **The load can be moved rapidly over a large distance, while the point of application moves over a relatively short distance**. The main purpose of this type of lever is to obtain **rapid movement**.



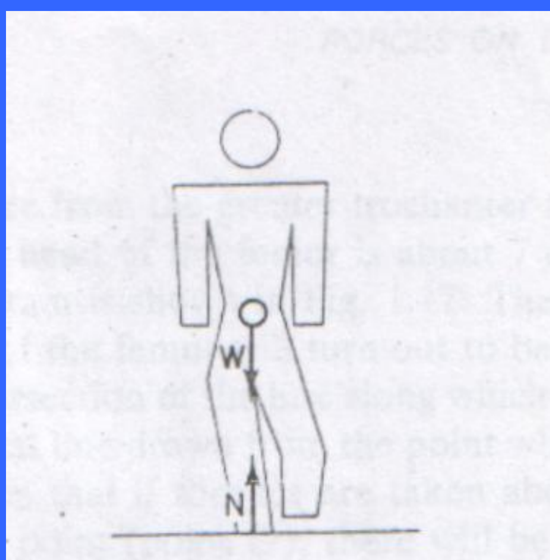
The biceps muscle exerts a vertical force on the lower arm as shown. Calculate the torque about the axis of rotation through elbow joint.



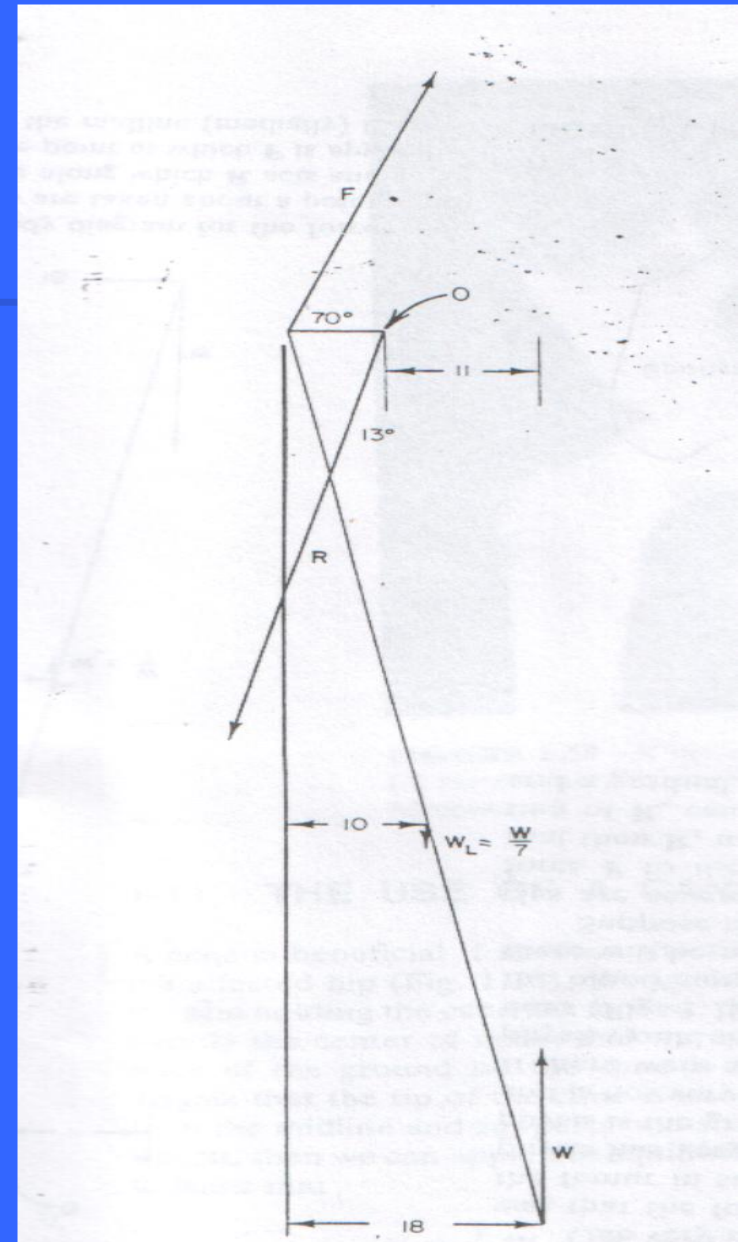
- F = 700 N
- $r \perp = 0.050$ m
- So $\tau = 35$ N.m
- If lower arm is at 45 °
- Lever arm will be shorter:
- $r \perp = (0.050)$ (sin 45)
- $\tau = 25$ N.m

Forces on a hip

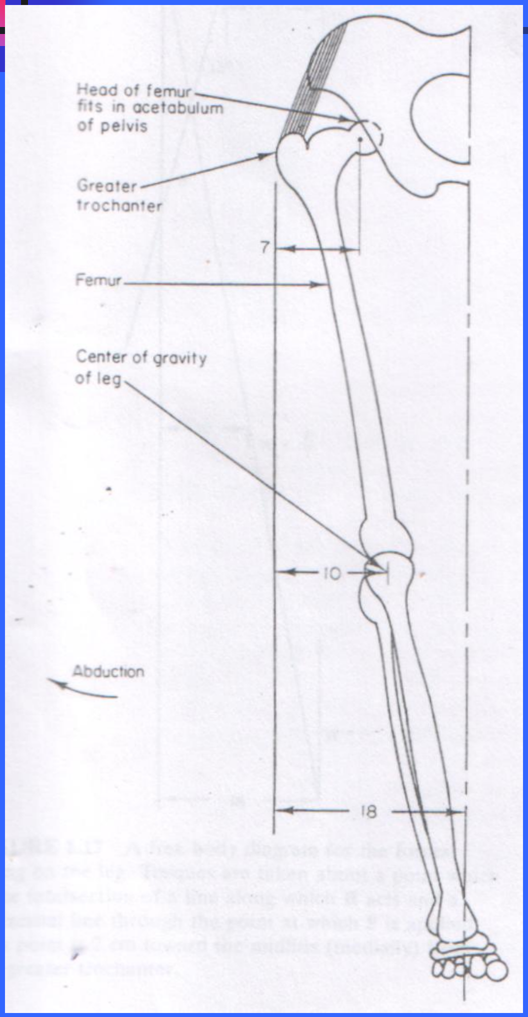
- F: net force of the abductor muscles, acting on great trochanter
- R: The force of the acetabulum (the socket of the pelvis) on the head of femur



- N: upward force of the floor on the bottom of foot



- W_L: weight of the leg



$$\sum F_y = F \sin(70°) - R_y - W/7 + W = 0$$

$$\sum F_x = F \cos(70°) - R_x = 0$$

$$\sum \tau = -(F)(\sin 70°)(7) - (W/7)(10 - 7) + W(18 - 7) = 0$$

The last of these equations can be written as

$$(11)W - \frac{3}{7}W - (6.6)F = 0$$

from which

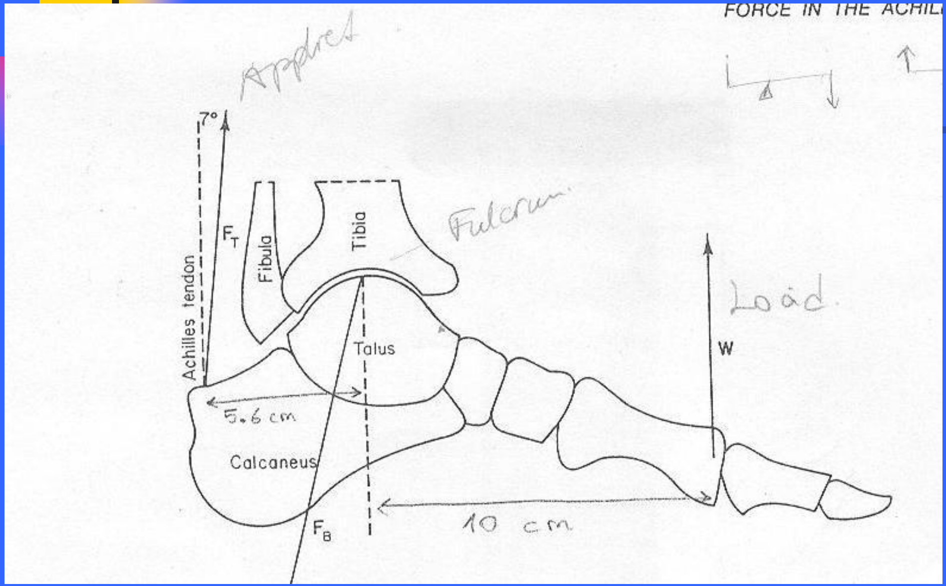
$$F = 1.6W$$

$$R_x = F\cos(70°) = (1.6)(W)(0.342) = 0.55W$$

$$R_y = F\sin(70°) + (6/7)W = (1.6)(W)(0.94) + (0.86)W = 2.4W$$

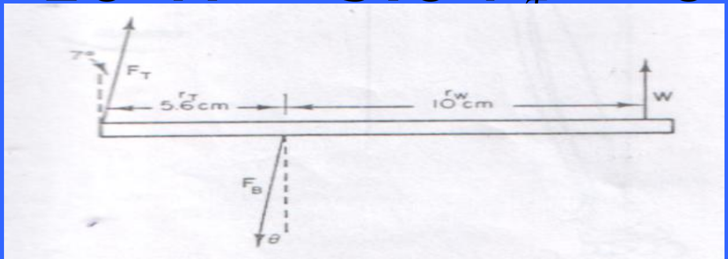
- The magnitude of the force in the abductor muscles is about 1.6 times the body weight.
- If patient had not had to put the foot under CG, F will be smaller. This can be done by using a cane.

FORCE IN THE ACHILLES TENDON



$$F_T \cos(7°) + W - F_B \cos\theta = 0$$
$$F_T \sin(7°) - F_B \sin\theta = 0$$

- 10 W – 5.6 F_T = 0



This equation can be solved to give the tension in the tendon:

$$F_T = \frac{10W}{5.6} = 1.8W \tag{1.20}$$

This result can now be used in Eq. 1.18 to find $F_{By} = F_B \cos\theta$:

$$\begin{aligned}(1.8)(W)(0.993) + W &= F_B \cos\theta \\ 2.8W &= F_B \cos\theta\end{aligned} \tag{1.21}$$

From Eqs. 1.19 and 1.20, we get

$$\begin{aligned}(1.8)(W)(0.122) &= F_B \sin\theta \\ 0.22W &= F_B \sin\theta\end{aligned} \tag{1.22}$$

Equations 1.21 and 1.22 are squared and summed to give

$$2.8W = F_B$$

while they can be divided to give

$$\tan\theta = \frac{0.22}{2.8} = 0.079$$

$$\theta = 4.5°$$

The tension in the tendon is nearly twice the weight, while the force exerted by the leg on the talus is nearly three times the body weight.

- **Spinal Column:**
- The human spinal column is made up of 24 vertebrae separated by fluid-filled disks.
-
- When a person bends, the spine is effectively a lever with a small M.A.
-
- Hence bending over to pick up even a light object produces a very large force on the lumbosacral disk; which separates the last vertebra from sacrum, the bone supporting the spine.
-
- If weakened, this disk can rupture or be deformed, causing pressure on nearby nerves and severe pain.

- If we treat spine as lever :
-
- sacrum ----> fulcrum (pivot or support point) exerts force R
-
- muscles of the back --------> T , which has an angle 12° with horizantal
-
- w (weight of the torso, head and arms, presenting 65% of total body weight)
-
- Because α is small, lever arm of T is small.
- However, the weight acts at right angles to the spine and its lever arm is much longer.
-

- to lift a child 40 N, the forces at T and R should be around 750 N. Such force in the muscles and on the disks are quite hazardous.
-
- An often abused part of the body is the lumbar (lower back) region.
-
- The calculated force at the fifth lumbar vertebra (L5) with the body tipped forward at 60° to the vertical and with a weight of 225 N in the hands, can approach to 3800 N.

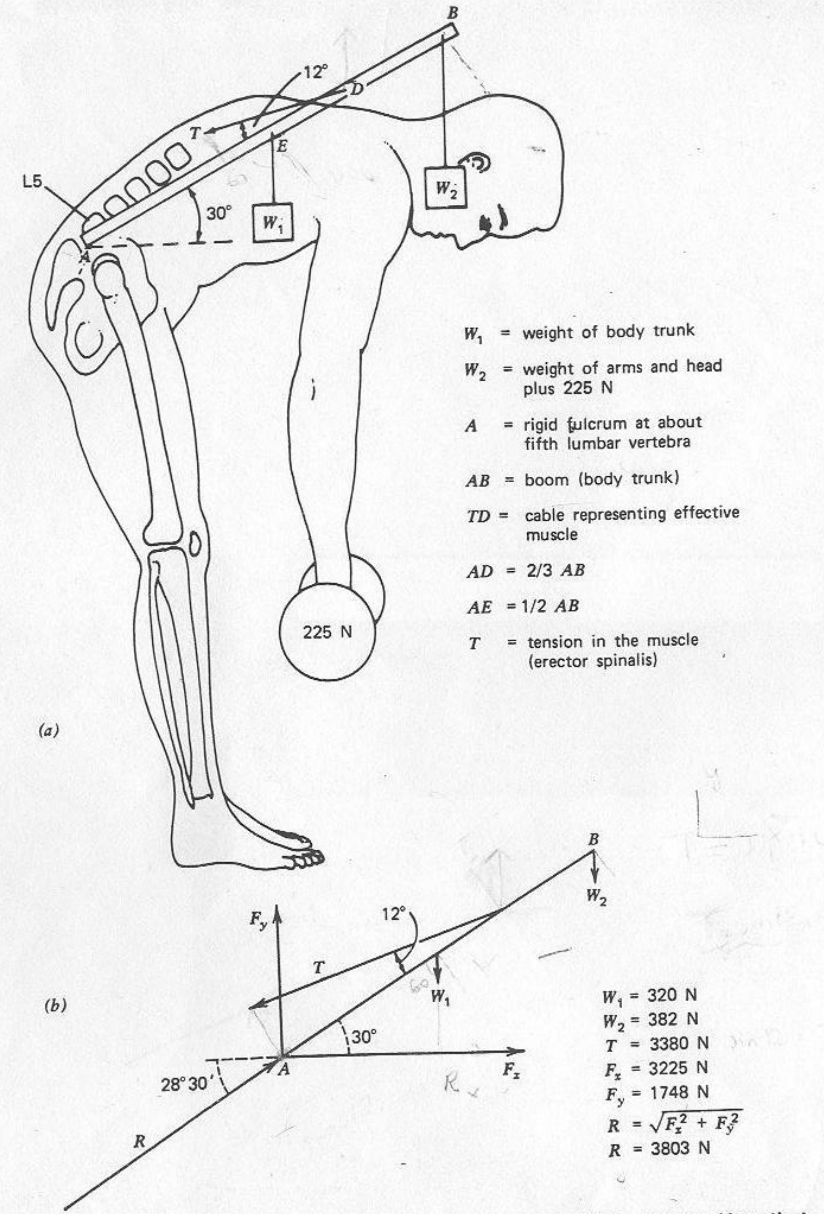


Figure 2.6. Lifting a weight. (*a*) Schematic of forces used. (*b*) The forces. Note that the reaction force *R* at the fifth lumbar vertebra is quite substantial. (Adapted from L.A. Strait, V.T. Inman, and H.J. Ralston, *Amer. J. Phys.*, 15, 1947, pp. 377–378.)

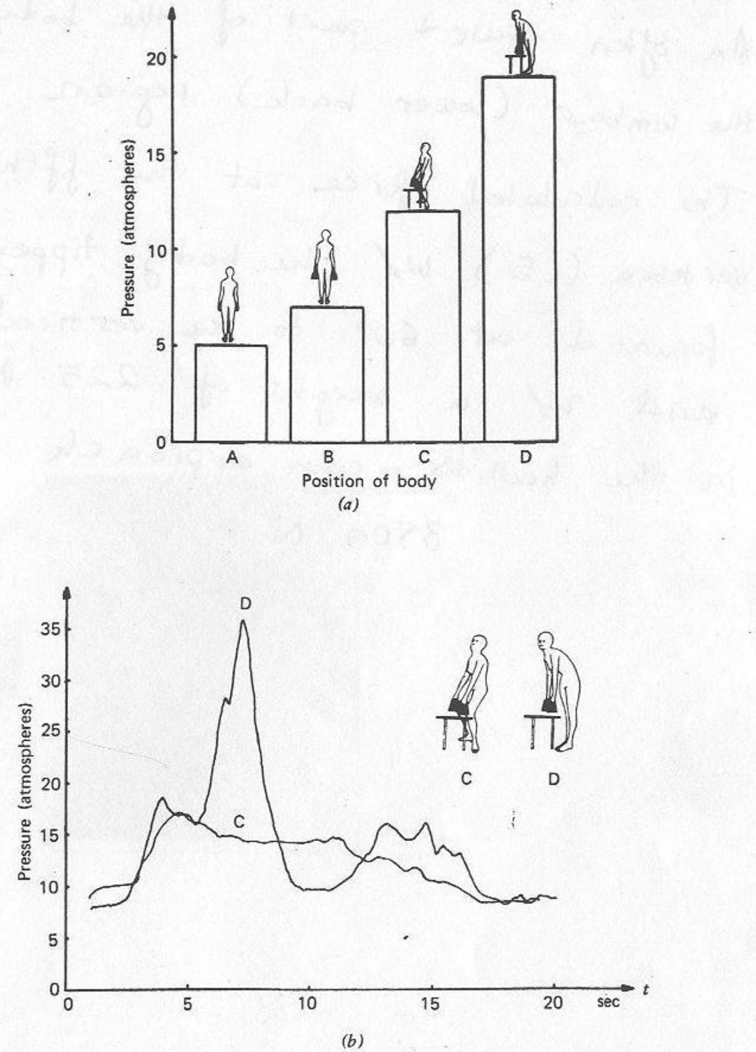


Figure 2.7. Pressure on the spinal column. (*a*) The pressure on the third lumbar disc for a subject (A) standing, (B) standing and holding 20 kg, (C) picking up 20 kg correctly by bending the knees, and (D) picking up 20 kg incorrectly without bending the knees. (*b*) The instantaneous pressure in the third lumbar disc while picking up and replacing 20 kg correctly and incorrectly. Note the much larger peak pressure during incorrect lifting. (Adapted from A. Nachemson and G. Elfstrom, *Scand. J. Rehab. Med.*, Suppl. 1, 1970, pp. 21–22.)