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Some properties of q-biorthogonal polynomials ☆

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Abstract

Almost four decades ago, Konhauser introduced and studied a pair of biorthogonal polynomials

$$Y_n^\alpha(x;k) \quad \text{and} \quad Z_n^\alpha(x;k) \quad \big(\alpha > -1;\ k \in \mathbb{N} := \{1, 2, 3, \ldots\}\big),$$

which are suggested by the classical Laguerre polynomials. The so-called Konhauser biorthogonal polynomials $Z_n^\alpha(x;k)$ of the second kind were indeed considered earlier by Toscano without their biorthogonality property which was emphasized upon in Konhauser's investigation. Many properties and results for each of these biorthogonal polynomials (such as generating functions, Rodrigues formulas, recurrence relations, and so on) have since been obtained in several works by others. The main object of this paper is to present a systematic investigation of the general family of q-biorthogonal polynomials. Several interesting properties and results for the q-Konhauser polynomials are also derived.

Keywords: Biorthogonal polynomials; q-Laguerre polynomials; q-Biorthogonal polynomials; q-Konhauser polynomials; Rodrigues formulas; Raising operators

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1. Introduction and definitions

Motivated essentially by an earlier study on biorthogonal polynomials by Preiser [9], Joseph D.E. Konhauser (1924–1992) [5] investigated two sets of polynomials $\{R_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$, which satisfy the following extension of the usual orthogonality condition:

$$\int_a^b \rho(x) R_m(x) S_n(x)\,dx = \begin{cases} 0 & (m \neq n) \\ \neq 0 & (m = n) \end{cases} \quad \bigl(m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\bigr), \tag{1.1}$$

where $\rho(x)$ is an admissible weight function over an interval (a, b), and $R_m(x)$ and $S_n(x)$ are polynomials of degrees m and n, respectively, in the basic polynomials $r(x)$ and $s(x)$, both of which are polynomials in x. The polynomials $R_m(x)$ and $S_n(x)$ are called *biorthogonal* with respect to the weight function $\rho(x)$ over the interval (a, b). In fact, Konhauser [5] showed that the condition (1.1) is equivalent to the following conditions:

$$\int_a^b \rho(x)\bigl[r(x)\bigr]^i S_n(x)\,dx = \begin{cases} 0 & (i = 0, 1, \ldots, n-1), \\ \neq 0 & (i = n) \end{cases} \tag{1.2}$$

and

$$\int_a^b \rho(x)\bigl[s(x)\bigr]^i R_m(x)\,dx = \begin{cases} 0 & (i = 0, 1, \ldots, m-1), \\ \neq 0 & (i = m). \end{cases} \tag{1.3}$$

Konhauser [5] also obtained many properties and results for these biorthogonal polynomials.

In the year 1967, using his basic results of the general theory of biorthogonal polynomials presented in [5], Konhauser [6] introduced the following pair of biorthogonal polynomials:

$$Y_n^{\alpha}(x;k) \quad \text{and} \quad Z_n^{\alpha}(x;k) \quad \bigl(\alpha > -1;\ k \in \mathbb{N} := \{1, 2, 3, \ldots\}\bigr),$$

which are suggested by the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ given by

$$L_n^{(\alpha)}(x) = Y_n^{\alpha}(x;1) = Z_n^{\alpha}(x;1). \tag{1.4}$$

These polynomial sets are biorthogonal with respect to the weight function $x^{\alpha}e^{-x}$ $(\alpha > -1)$ over the interval $(0, \infty)$ and were subsequently studied rather extensively by (for example) Carlitz [3], Prabhakar [8], Srivastava [11,12], and Rassias and Srivastava [10]. We remark in passing that the so-called Konhauser *biorthogonal* polynomials $Z_n^{\alpha}(x;k)$ of the *second* kind were indeed considered earlier by Letterio Toscano (1905–1977) [15], but *without* their biorthogonality property which was emphasized upon in Konhauser's investigation [5,6].

In 1983, Al-Salam and Verma [1] constructed some q-extensions of the polynomials $Y_n^{\alpha}(x;k)$ and $Z_n^{\alpha}(x;k)$, which they called the q-Konhauser polynomials (see also [2]). More recently, Jain and Srivastava [4] derived linear and bilinear generating functions for one of these q-Konhauser polynomials and also suggested an *alternative* pair of q-Konhauser biorthogonal polynomials (see, for details, [4, pp. 342–343]). Further information concerning some of these q-biorthogonal polynomials will be presented in Sections 2 and 3.

With a view to presenting our proposed investigation of the general family of q-biorthogonal polynomials, we first recall the following notations and definitions.

For a real or complex number q ($|q| < 1$), $(a; q)_n$ is given by (see, for example, [13, pp. 346 *et seq.*])

$$(a; q)_n := \begin{cases} 1 & (n = 0), \\ \prod_{j=0}^{n-1}(1 - aq^j) & (n \in \mathbb{N}) \end{cases} \tag{1.5}$$

and

$$(a; q)_\infty = \prod_{j=0}^{\infty}\big(1 - aq^j\big). \tag{1.6}$$

Let $q \in \mathbb{R} \setminus \{1\}$. Then the q-analogue of a number a is given by

$$[a]_q := \frac{1 - q^a}{1 - q} \tag{1.7}$$

and the q-Pochhammer symbol is defined by

$$[a]_{n,q} := \prod_{m=0}^{n-1} [a + m]_q \tag{1.8}$$

for a real parameter a (see, for instance, [14]). Furthermore, the q-derivative operator D_q is defined by

$$D_q\big(f(x)\big) = \frac{f(qx) - f(x)}{(q - 1)x}, \tag{1.9}$$

so that, clearly, we have

$$D_q\big(x^a\big) = [a]_q x^{q-1} \quad (a \in \mathbb{R}).$$

Let $f(x)$ and $g(x)$ be two piecewise continuous functions. Then we have

$$\begin{aligned} D_q\big(f(x)g(x)\big) &= f(x)D_q\big(g(x)\big) + g(x)D_q\big(f(x)\big) \\ &\quad + (q - 1)xD_q\big(f(x)\big)D_q\big(g(x)\big). \end{aligned} \tag{1.10}$$

For $q \to 1-$, these definitions would reduce to the corresponding relatively more familiar definitions.

The q-integral of a piecewise continuous function $f(x)$ is defined as follows:

$$\int_a^b f(x)\,d_q x = \sum_{n=0}^{\infty}\big(bq^n - bq^{n+1}\big)f\big(bq^n\big) - \sum_{n=0}^{\infty}\big(aq^n - aq^{n+1}\big)f\big(aq^n\big)$$

and

$$\int_0^\infty f(x)\,d_q x = (1 - q) \sum_{k=-\infty}^{\infty} q^k f\big(q^k\big). \tag{1.11}$$

The q-partial integration is defined by

$$\int_0^\infty f(x) D_q\big(g(x)\big)\, d_q x$$

$$= \lim_{n\to\infty}\big\{f\big(q^{-n}\big)g\big(q^{-n}\big) - f\big(q^{n+1}\big)g\big(q^{n+1}\big)\big\} - \int_0^\infty g(x) D_q\big(f(x)\big)\, d_q x$$

$$- (q-1)\int_0^\infty x D_q\big(f(x)\big) D_q\big(g(x)\big)\, d_q x \tag{1.12}$$

for two piecewise continuous functions $f(x)$ and $g(x)$.

The q-exponential function $e_q(x)$ is defined by

$$e_q(x) = \sum_{k=0}^\infty \frac{((1-q)x)^k}{(q;q)_k} = \frac{1}{((1-q)x;q)_\infty}, \tag{1.13}$$

which, in conjunction with the definition (1.9), yields

$$D_q\big(e_q(ax)\big) = a e_q(ax).$$

For $|q| < 1$, let $w(x;q)$ be a positive weight function which is defined on the set $\{aq^n, bq^n;\ n \in \mathbb{N}_0\}$. If the polynomials $\{P_n(x;q)\}_{n\in\mathbb{N}_0}$ satisfy the following property:

$$\int_a^b P_m(x;q) P_n(x;q) w(x;q)\, d_q x = \begin{cases} 0 & (m \neq n) \\ \neq 0 & (m = n) \end{cases} \quad (m, n \in \mathbb{N}_0), \tag{1.14}$$

then the polynomials $P_n(x;q)$ are called q-orthogonal polynomials with respect to the weight function $w(x;q)$ over the interval (a,b). Using this definition, Moak [7] introduced the q-Laguerre polynomials $L_n^\alpha(x;q)$ given explicitly by

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^n \frac{(q^{-n};q)_k q^{\binom{k}{2}} (1-q)^k (q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k (q;q)_k} \quad (\alpha > -1). \tag{1.15}$$

These polynomials are monic polynomials in the sense that the leading coefficient of the polynomials is 1. The polynomials $L_n^\alpha(x;q)$ are q-orthogonal polynomials with respect to the weight function $x^\alpha e_q(-x)$ over the interval $(0,\infty)$, and we have

$$L_n^\alpha(x;q) \to L_n^{(\alpha)}(x) \quad \text{as } q \to 1-,$$

where $L_n^{(\alpha)}(x)$ are the classical Laguerre polynomials occurring in (1.4).

For $\alpha > 0$, we denote by $\mathcal{R}$ the *raising operator* for the q-Laguerre polynomials, which is given by

$$\mathcal{R}(\cdots) := D_q\big(x^\alpha e_q(-x)(\cdots)\big), \tag{1.16}$$

so that

$$\begin{aligned}\mathcal{R}\big(L_n^\alpha(x;q)\big) &= D_q\big(x^\alpha e_q(-x) L_n^\alpha(x;q)\big) \\ &= -\big[1 + [\alpha]_q(q-1) + [n]_q(q-1)\big(1 + (q-1)[\alpha]_q\big)\big] \\ &\quad \cdot x^{\alpha-1} e_q(-x) L_{n+1}^{\alpha-1}(x;q)\end{aligned} \tag{1.17}$$

and the Rodrigues formula for the q-Laguerre polynomials is given by

$$\begin{aligned} D_q^n\bigl(x^{\alpha+n}e_q(-x)\bigr) = (-1)^n \prod_{k=1}^{n}\bigl\{&1+[\alpha+n+1-k]_q(q-1) \\ &+[k-1]_q(q-1)\bigl(1+(q-1)[\alpha+n+1-k]_q\bigr)\bigr\} \\ &\cdot x^\alpha e_q(-x)L_n^\alpha(x;q). \end{aligned} \tag{1.18}$$

2. Definition of the q-biorthogonal polynomials

In this section, we first give some further definitions and notations, which would help us in our construction of the definition of the q-biorthogonal polynomials.

Definition 1. For $|q| < 1$, let $r(x;q)$ and $s(x;q)$ be polynomials in x of degrees h and k, respectively ($h, k \in \mathbb{N}$). Also let $R_m(x;q)$ and $S_n(x;q)$ denote polynomials of degrees m and n in $r(x;q)$ and $s(x;q)$, respectively. Then $R_m(x;q)$ and $S_n(x;q)$ are polynomials of degrees mh and nk in x. The polynomials $r(x;q)$ and $s(x;q)$ are called the q-basic polynomials.

For $|q| < 1$, let $\{R_n(x;q)\}_{n=0}^\infty$ denote the set of polynomials

$$R_0(x;q), R_1(x;q), \ldots, R_n(x;q), \ldots$$

of degrees

$$0, 1, \ldots, n, \ldots \quad \text{in } r(x;q).$$

Similarly, let $\{S_n(x;q)\}_{n=0}^\infty$ denote the set of polynomials

$$S_0(x;q), S_1(x;q), \ldots, S_n(x;q), \ldots$$

of degrees

$$0, 1, \ldots, n, \ldots \quad \text{in } s(x;q).$$

Definition 2. For $|q| < 1$, let $w(x;q)$ be an admissible weight function which is defined on the set

$$\bigl\{aq^n, bq^n;\ n \in \mathbb{N}_0\bigr\}.$$

If the polynomial sets

$$\bigl\{R_n(x;q)\bigr\}_{n=0}^\infty \quad \text{and} \quad \bigl\{S_n(x;q)\bigr\}_{n=0}^\infty$$

satisfy the following condition:

$$\int_a^b R_m(x;q)S_n(x;q)w(x;q)\,d_qx = \begin{cases} 0 & (m \neq n) \\ \neq 0 & (m = n) \end{cases} \quad (m, n \in \mathbb{N}_0), \tag{2.1}$$

then the polynomial sets

$$\bigl\{R_n(x;q)\bigr\}_{n=0}^\infty \quad \text{and} \quad \bigl\{S_n(x;q)\bigr\}_{n=0}^\infty$$

are said to be q-biorthogonal over the interval (a, b) with respect to the weight function $w(x;q)$ and the q-basic polynomials $r(x;q)$ and $s(x;q)$.

The q-biorthogonality condition (2.1) is analogous to the q-orthogonality condition (1.14). We also note that, when $q \to 1-$, the q-biorthogonality condition (2.1) gives us the usual biorthogonality condition (1.1).

Remark 1. If we take the weight function

$$w(x;q)=x^{\alpha}e_q(-x)$$

over the interval $(0,\infty)$, we obtain the following q-Konhauser polynomials:

$$Z_n^{(\alpha)}(x,k;q)=\frac{(q^{1+\alpha};q)_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j q^{\frac{1}{2}kj(kj-1)+kj(n+\alpha+1)}}{(q^k;q^k)_j(q^{1+\alpha};q)_{jk}}x^{kj} \tag{2.2}$$

and

$$Y_n^{(\alpha)}(x,k;q)=\frac{1}{(q;q)_n}\sum_{r=0}^{n}\frac{x^r q^{\frac{1}{2}r(r-1)}}{(q;q)_r}\sum_{j=0}^{r}\frac{(q^{-r};q)_j(q^{1+\alpha+j};q^k)_n}{(q;q)_j}q^j, \tag{2.3}$$

which were considered by Al-Salam and Verma [1], who proved that

$$\int_0^{\infty} Z_n^{(\alpha)}(x,k;q)Y_m^{(\alpha)}(x,k;q)x^{\alpha}e_q(-x)\,d_qx=\begin{cases}0 & (n\neq m),\\ \neq 0 & (n=m).\end{cases} \tag{2.4}$$

Equation (2.4) does indeed exhibit the fact that the polynomials $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$ are q-biorthogonal polynomials with respect to the weight function $x^{\alpha}e_q(-x)$ over the interval $(0,\infty)$.

Remark 2. For $k=1$, the q-Konhauser polynomials in (2.2) and (2.3) reduce to the q-Laguerre polynomials given by (1.15).

Remark 3. Just as we indicated in the preceding section, Jain and Srivastava [4] gave another pair of q-Konhauser polynomials which are defined by

$$z_n^{(\alpha)}(x,k\mid q)=\frac{(\alpha q;q)_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j}{(\alpha q;q)_{kj}}\frac{(xq)^{kj}}{(q^k;q^k)_j} \tag{2.5}$$

and

$$y_n^{(\alpha)}(x,k\mid q)=\frac{1}{(q;q)_n}\sum_{j=0}^{n}\frac{(xq)^j}{(q;q)_j}\sum_{l=0}^{j}\frac{(q^{-j};q)_l(\alpha q^{l+1};q^k)_n}{(q;q)_l}q^{(j-n)l}. \tag{2.6}$$

3. General properties of the q-biorthogonal polynomials

3.1. Equivalent conditions for q-biorthogonality

Theorem 1 provides equivalent conditions for q-biorthogonality.

Theorem 1. *For $|q|<1$, let $w(x;q)$ be a weight function which is defined on the set $\{aq^n,bq^n;\ n\in\mathbb{N}_0\}$. Suppose also that $r(x;q)$ and $s(x;q)$ are q-basic polynomials. If*

$$\int_a^b w(x;q)\big[r(x;q)\big]^j S_n(x;q)\,d_qx=\begin{cases}0 & (j=0,1,\dots,n-1),\\ \neq 0 & (j=n)\end{cases} \tag{3.1}$$

and

$$\int_a^b w(x;q)\big[s(x;q)\big]^j R_m(x;q)\,d_qx = \begin{cases} 0 & (j=0,1,\ldots,m-1), \\ \neq 0 & (j=m), \end{cases} \tag{3.2}$$

then

$$\int_a^b w(x;q)R_m(x;q)S_n(x;q)\,d_qx = \begin{cases} 0 & (m \neq n), \\ \neq 0 & (m=n) \end{cases} \tag{3.3}$$

for $n \in \mathbb{N}_0$. Conversely, if the condition (3.3) *holds true, then both* (3.1) *and* (3.2) *also hold true.*

Proof. Suppose that the conditions (3.1) and (3.2) hold true. Then, clearly, constants

$${}_qc_{m,j} \quad (j=0,1,\ldots,m) \quad \text{and} \quad {}_qc_{m,m} \neq 0$$

exist such that

$$R_m(x;q) = \sum_{j=0}^{m} {}_qc_{m,j}\big[r(x;q)\big]^j. \tag{3.4}$$

For $m \leqq n$, we find that

$$\begin{aligned} &\int_a^b w(x;q)R_m(x;q)S_n(x;q)\,d_qx \\ &\quad = \int_a^b w(x;q)\left(\sum_{j=0}^{m} {}_qc_{m,j}\big[r(x;q)\big]^j\right)S_n(x;q)\,d_qx \\ &\quad = \sum_{j=0}^{m} {}_qc_{m,j}\int_a^b w(x;q)\big[r(x;q)\big]^j S_n(x;q)\,d_qx. \end{aligned}$$

By virtue of (3.1), the following q-integral:

$$\int_a^b w(x;q)\big[r(x;q)\big]^j S_n(x;q)\,d_qx$$

vanishes except when $j=n=m$.

If $m>n$, then constants

$${}_qd_{n,j} \quad (j=0,1,\ldots,m) \quad \text{and} \quad {}_qd_{n,n} \neq 0$$

exist such that

$$S_n(x;q) = \sum_{j=0}^{n} {}_qd_{n,j}\big[s(x;q)\big]^j, \tag{3.5}$$

and the proof is completed as in the case when $m \leqq n$.

We now assume that (3.3) holds true. Then constants

$$ {}_q e_{m,i} \quad \text{and} \quad {}_q f_{n,i} $$

exist such that

$$ \big[r(x;q)\big]^j = \sum_{i=0}^{j} {}_q e_{m,i} R_i(x;q) \tag{3.6} $$

and

$$ \big[s(x;q)\big]^j = \sum_{i=0}^{j} {}_q f_{n,i} S_i(x;q). \tag{3.7} $$

Thus, if $0 \leqq j \leqq n$, we obtain

$$ \begin{aligned} &\int_a^b w(x;q)\big[r(x;q)\big]^j S_n(x;q)\,d_q x \\ &\quad = \int_a^b w(x;q)\Bigg(\sum_{i=0}^{j} {}_q e_{m,i} R_i(x;q)\Bigg) S_n(x;q)\,d_q x \\ &\quad = \sum_{i=0}^{j} {}_q e_{m,i} \int_a^b w(x;q) R_i(x;q) S_n(x;q)\,d_q x. \end{aligned} $$

If $j < n$, then each integral on the extreme right-hand side vanishes, since (3.3) is assumed to hold true. If $j = n$, then each of these integrals is different from zero. Therefore, we conclude that (3.1) holds true.

In a similar manner, we can establish (3.2). This evidently completes our proof of Theorem 1. □

Corollary 1. *If the conditions* (3.1) *and* (3.2) *hold true, then*

$$ \int_a^b w(x;q) S_n(x;q) F_{n-1}(x;q)\,d_q x = 0 \tag{3.8} $$

and

$$ \int_a^b w(x;q) R_m(x;q) G_{m-1}(x;q)\,d_q x = 0, \tag{3.9} $$

where $F_{n-1}(x;q)$ and $G_{m-1}(x;q)$ are arbitrary polynomials of degrees not exceeding $n-1$ and $m-1$ in the polynomials $r(x;q)$ and $s(x;q)$, respectively.

In Section 2, we pointed out that the q-Konhauser polynomials $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$ are q-biorthogonal with respect to the weight function $x^\alpha e_q(-x)$ over the interval $(0,\infty)$. From (2.2) and (2.3), we can easily see that $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$ are polynomials in x^k and x of degree n, respectively. Consequently, it follows from Theorem 1 that the polynomials $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$ satisfy the assertion of Corollary 2 below.

Corollary 2. (Al-Salam and Verma [1]) *For the q-Konhauser polynomials $Z_n^{(\alpha)}(x,k;q)$ and $Y_n^{(\alpha)}(x,k;q)$,*

$$\int_0^\infty x^\alpha e_q(-x)x^j Z_n^{(\alpha)}(x,k;q)\,d_qx = \begin{cases} 0 & (j=0,1,\ldots,n-1), \\ \neq 0 & (j=n) \end{cases} \tag{3.10}$$

and

$$\int_0^\infty x^\alpha e_q(-x)x^{kj} Y_n^{(\alpha)}(x,k;q)\,d_qx = \begin{cases} 0 & (j=0,1,\ldots,n-1), \\ \neq 0 & (j=n), \end{cases} \tag{3.11}$$

respectively.

3.2. Pure recurrence relation for q-biorthogonal polynomials

In the case of q-biorthogonal polynomials, if we choose one of the q-basic polynomials with respect to the other one, we can derive several pure recurrence relations, each of which would connect $m+2$ successive polynomials.

Theorem 2. *For $|q|<1$, if the q-basic polynomials $r(x;q)$ and $s(x;q)$ are such that $s(x;q)$ is a polynomial $p(x;q)$ of degree m in $r(x;q)$, and if the q-biorthogonal polynomial sets*

$$\{R_n(x;q)\} \quad \textit{and} \quad \{S_n(x;q)\}$$

are known to exist for an admissible weight function $w(x;q)$ over the interval (a,b), then there exist pure recurrence relations of the following forms:

$$p(x;q)R_n(x;q) = \sum_{i=n-1}^{n+m} {}_qa_{n,i}R_i(x;q) \tag{3.12}$$

and

$$p(x;q)S_n(x;q) = \sum_{i=n-m}^{n+1} {}_qb_{n,i}S_i(x;q), \tag{3.13}$$

each connecting $m+2$ successive polynomials. The coefficients ${}_qa_{n,i}$ and ${}_qb_{n,i}$ depend on n, but not on x.

Proof. The polynomial $R_n(x;q)$ is of degree n in the q-basic polynomial $r(x;q)$ and the polynomial $p(x;q)$ is of degree m in $r(x;q)$. Therefore, the product $p(x;q)R_n(x;q)$ is of degree $n+m$ in $r(x;q)$ and constants ${}_qa_{n,i}$ exist such that

$$p(x;q)R_n(x;q) = \sum_{i=0}^{n+m} {}_qa_{n,i}R_i(x;q). \tag{3.14}$$

Here we use $R_i(x;q)$ for $r(x;q)$, because the polynomial $R_i(x;q)$ $(i\in\mathbb{N}_0)$ is of degree $i\in\mathbb{N}_0$ in the q-basic polynomial $r(x;q)$.

Now, multiplying both sides of (3.14) by $w(x;q)S_j(x;q)$ and integrating over (a,b), we obtain

$$\int_a^b w(x;q)p(x;q)R_n(x;q)S_j(x;q)\,d_qx$$
$$=\sum_{i=0}^{m+n} {}_qa_{n,i}\int_a^b w(x;q)S_j(x;q)R_i(x;q)\,d_qx$$
$$= {}_qa_{n,j}\int_a^b w(x;q)S_j(x;q)R_j(x;q)\,d_qx, \tag{3.15}$$

where we have made use of the q-biorthogonality conditions.

The product $p(x;q)S_j(x;q)$ is a linear combination of

$$S_{j+1}(x;q), S_j(x;q), \ldots, S_0(x;q)$$

and $R_n(x;q)$ is q-biorthogonal to $p(x;q)S_j(x;q)$ for $j+1<n$. It follows that

$${}_qa_{n,j}=0 \quad \text{for } j=0,1,\ldots,n-2,$$

and the sum in (3.14) runs only from $n-1$ to $n+m$. Therefore, a recurrence relation of the form (3.12) exists, which would connect $m+2$ successive polynomials $R_n(x;q)$.

In order to establish (3.13), we consider the product $p(x;q)S_n(x;q)$, which is a polynomial of degree $n+1$ in the q-basic polynomial $s(x;q)$. Therefore, constants ${}_qb_{n,i}$ exist such that

$$p(x;q)S_n(x;q)=\sum_{i=0}^{n+1} {}_qb_{n,i}S_i(x;q). \tag{3.16}$$

Thus, upon multiplying both sides of (3.16) by $w(x;q)R_j(x;q)$ and integrating over (a,b), we obtain

$$\int_a^b w(x;q)p(x;q)R_j(x;q)S_n(x;q)\,d_qx$$
$$= {}_qb_{n,j}\int_a^b w(x;q)S_j(x;q)R_j(x;q)\,d_qx. \tag{3.17}$$

The product $p(x;q)R_j(x;q)$ is a linear combination of

$$R_{j+m}(x;q), R_{j+m-1}(x;q), \ldots, R_j(x;q).$$

By q-biorthogonality conditions, $p(x;q)R_j(x;q)$ is q-biorthogonal to $S_n(x;q)$ for $j+m<n$. It follows that ${}_qb_{n,j}=0$ for $j=0,1,\ldots,n-m$, and the sum in (3.16) runs only from $n-m$ to $n+1$; that is, a recurrence relation of the form (3.13) exists, which would connect $m+2$ successive polynomials $S_n(x;q)$. Our proof of Theorem 2 is thus completed. □

For $m=1$, the q-basic polynomials are the same and they are found to be simply q-orthogonal. Therefore, the corresponding recurrence relations will connect three successive polynomials. When $q\to 1-$, we obtain the familiar three-term recurrence relations for classical orthogonal polynomials.

4. Further properties of the q-Konhauser polynomials

Jain and Srivastava [4] obtained linear and bilinear generating functions for the polynomials $Z_n^{(\alpha)}(x,k;q)$, which were defined in Section 2 (see also [1, Section 4]). In this section, we obtain some further properties of the q-Konhauser polynomials $Z_n^{(\alpha)}(x,k;q)$. We first obtain a raising operator and then derive a Rodrigues formula for $Z_n^{(\alpha)}(x,k;q)$. Here we choose the polynomial $Z_n^{(\alpha)}(x,k;q)$ to be monic.

Lemma. *For $k \in \mathbb{N}$ and* [*cf. Eq.* (1.16)]

$$\mathcal{R}(\cdots) := D_q\bigl(x^\alpha e_q(-x)(\cdots)\bigr) \quad (\alpha > 0), \tag{4.1}$$

the raising operator for the q-Konhauser polynomials $Z_n^{(\alpha)}(x,k;q)$ is given by

$$\begin{aligned}\mathcal{R}\bigl(Z_n^{(\alpha)}(x,k;q)\bigr) &= D_q\bigl(x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q)\bigr)\\ &= -\bigl[1+[\alpha]_q(q-1)+[nk]_q(q-1)\bigl(1+(q-1)[\alpha]_q\bigr)\bigr]\\ &\quad\cdot x^{\alpha-k}e_q(-x)Z_{n+1}^{(\alpha-k)}(x,k;q).\end{aligned} \tag{4.2}$$

Proof. Let $Q_{n+1}(x,k;q)$ be a monic polynomial of degree $n+1$. By using (1.10), we find that

$$\begin{aligned}&D_q\bigl(x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q)\bigr)\\ &\quad= -\bigl[1+[\alpha]_q(q-1)+[nk]_q(q-1)\bigl(1+(q-1)[\alpha]_q\bigr)\bigr]\\ &\qquad\cdot x^{\alpha-k}e_q(-x)Q_{n+1}(x,k;q),\end{aligned} \tag{4.3}$$

which, by means of (1.12), yields

$$\begin{aligned}&-\bigl[1+[\alpha]_q(q-1)+[nk]_q(q-1)\bigl(1+(q-1)[\alpha]_q\bigr)\bigr]\\ &\qquad\cdot\int_0^\infty x^{i+\alpha-k}e_q(-x)Q_{n+1}(x,k;q)\,d_qx\\ &\quad=\int_0^\infty x^i D_q\bigl(x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q)\bigr)\,d_qx\\ &\quad=-[i]_q\int_0^\infty x^{i-1}x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q)\,d_qx\\ &\qquad-[i]_q(q-1)\int_0^\infty x^i D_q\bigl(x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q)\bigr)\,d_qx.\end{aligned} \tag{4.4}$$

For $i=1,\dots,n$, the first term on the right-hand side of the last equality in (4.4) vanishes by virtue of the q-orthogonality relation (3.10) for the polynomials $Z_n^{(\alpha)}(x,k;q)$. Therefore, we obtain

$$\int_0^\infty x^{i+\alpha-k}e_q(-x)Q_{n+1}(x,k;q)\,d_qx = 0 \quad (i=0,1,\dots,n). \tag{4.5}$$

Equation (4.5) shows that the monic polynomials $Q_{n+1}(x,k;q)$ are q-orthogonal with respect to the weight function $x^{\alpha-k}e_q(-x)$ over the interval $(0,\infty)$. Because of the observation that a monic polynomial, which is q-orthogonal with respect to a given weight function $x^{\alpha-k}e_q(-x)$ over a given interval $(0,\infty)$, is unique, we can replace

$$Q_{n+1}(x,k;q) \quad \text{by} \quad Z_{n+1}^{(\alpha-k)}(x,k;q),$$

and the proof of the lemma is thus completed. □

For $k=1$, (4.2) reduces to the raising operator for the q-Laguerre polynomials just as asserted by (1.17).

Finally, we obtain a Rodrigues formula for the q-Konhauser polynomials $Z_n^{(\alpha)}(x,k;q)$ by applying the raising operator $\mathcal{R}$ to the polynomial

$$Z_0^{(\alpha+nk)}(x,k;q)=1$$

successively. We thus find that

$$\begin{aligned}
&D_q^n\big(x^{\alpha+nk}e_q(-x)\big)\\
&\quad=(-1)^n\prod_{i=1}^{n}\big\{1+\big[\alpha+nk+(1-i)k\big]_q(q-1)\\
&\qquad+[i-1]_q(q-1)\big(1+(q-1)\big[\alpha+nk+(1-i)k\big]_q\big)\big\}\\
&\qquad\cdot x^\alpha e_q(-x)Z_n^{(\alpha)}(x,k;q).
\end{aligned} \tag{4.6}$$

In its special case when $k=1$, (4.6) reduces to the Rodrigues formula (1.18) for the q-Laguerre polynomials.

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