

(b-) if $\exists (a,b)$ containing x_0 s.t $f'(x) < 0$ on (a,x_0) and $f'(x) > 0$ on (x_0,b)
 $\Rightarrow f$ has a local min at x_0 .

Part II Testing endpoints of domain.

Suppose x_0 is a left endpoint of the dom f and f is right cont at x_0

(c-) If $f'(x) > 0$ on $(x_0,b) \Rightarrow f$ has a local min at x_0

(d-) if $f'(x) < 0$ " $\Rightarrow f$ " local max at x_0

Suppose x_0 is the right end-point of the domain of f and f is left cont at x_0 .

(e-) If $f'(x) > 0$ on $(a,x_0) \Rightarrow f$ has a local max at x_0

(f-) if $f'(x) < 0$ " $(a,x_0) \Rightarrow f$ has a local min at x_0

Remark: If f' is post or negative for $\forall x \in D_f$
 $\Rightarrow f$ has neither max nor a minimal at any point

Ex: Find the local and absolute extreme values of $f(x) = x^4 - 2x^2 - 3$ on $[-2, 2]$. Sketch the graph of f .

Soln: $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

Critical points: $f'(x) = 4x(x-1)(x+1) = 0 \Rightarrow x=1, x=0, x=-1$

$$f(0) = -3$$

$$f(1) = -4$$

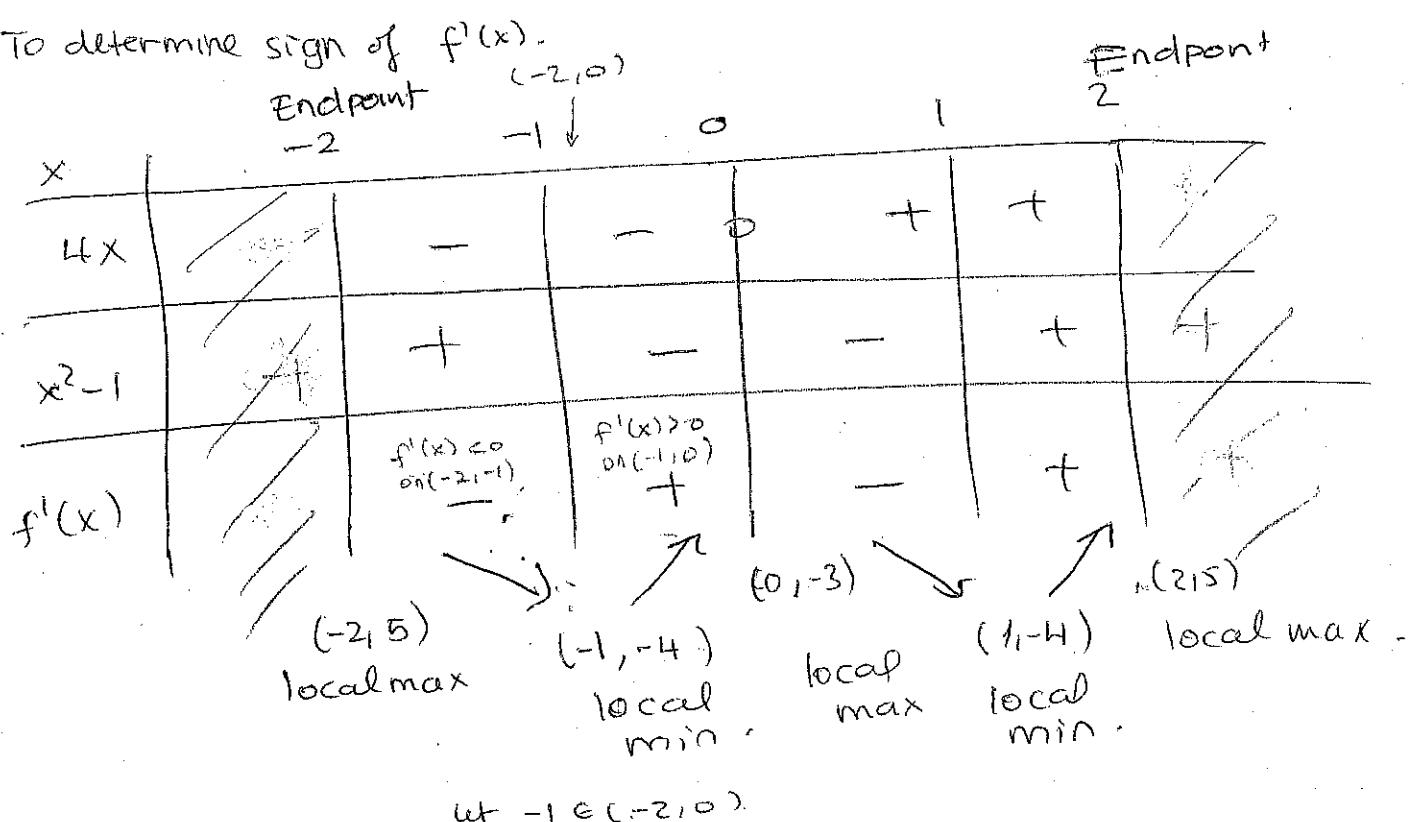
$$f(-1) = -4$$

Singular point: it has no since f is poly it is diff.

end points: $f(2) = 5$

$$f(-2) = 5$$

To determine sign of $f'(x)$.

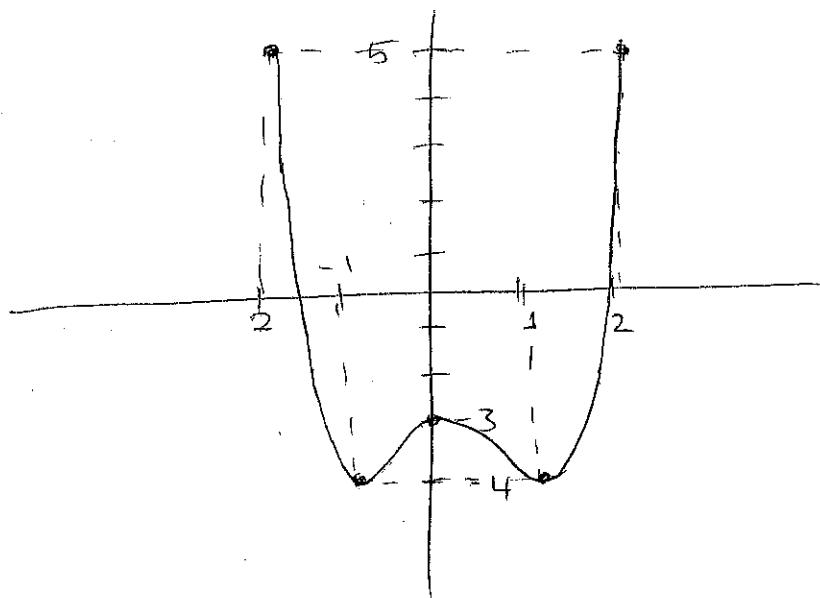


$D_f = [-2, 2] \rightarrow$ closed interval

$\Rightarrow f$ has abs max and min values.

f has abs max ~~val~~ 5 at $x = \pm 2$

" min val -4 at $x = \pm 1$



Ex: Find and classify the local and abs. extreme

values of ~~func~~

$$f(x) = x - x^{2/3} \text{ on } [-1, 2]$$

Sketch the graph of f .

Soln: $f'(x) = 1 - \frac{2}{3}x^{-1/3} = \frac{x^{1/3} - \frac{2}{3}}{x^{1/3}}$ $x \neq 0$

singular point - $x = 0$

critical point: $f'(x) = 0 \Rightarrow x^{1/3} = \frac{2}{3} \Rightarrow x = \underline{\underline{\frac{8}{27}}}$

Endpoints

$$f(-1) = (-1) - (-1)^{\frac{2}{3}} = -2$$

$$f(2) = 2 - 2^{\frac{2}{3}} \approx 0.4126$$

$$\frac{2^3}{3^3} = \left(\frac{2}{3}\right)^{\frac{3}{3}} = \frac{4}{9}$$

$$f(0) = 0$$

$$\frac{1}{2} - \left(\frac{1}{2}\right)^{\frac{2}{3}}$$

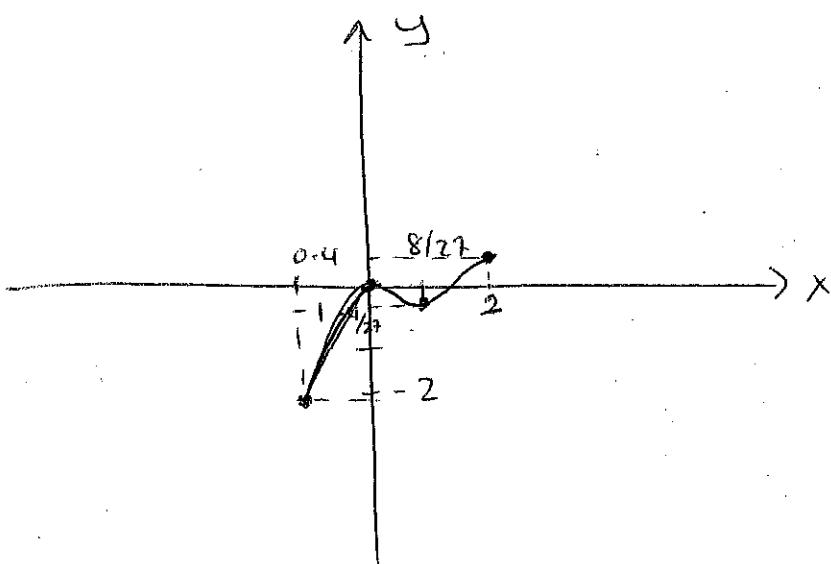
$$f\left(\frac{8}{27}\right) = \frac{8}{27} - \left(\frac{8}{27}\right)^{\frac{2}{3}} = -\frac{4}{27}$$

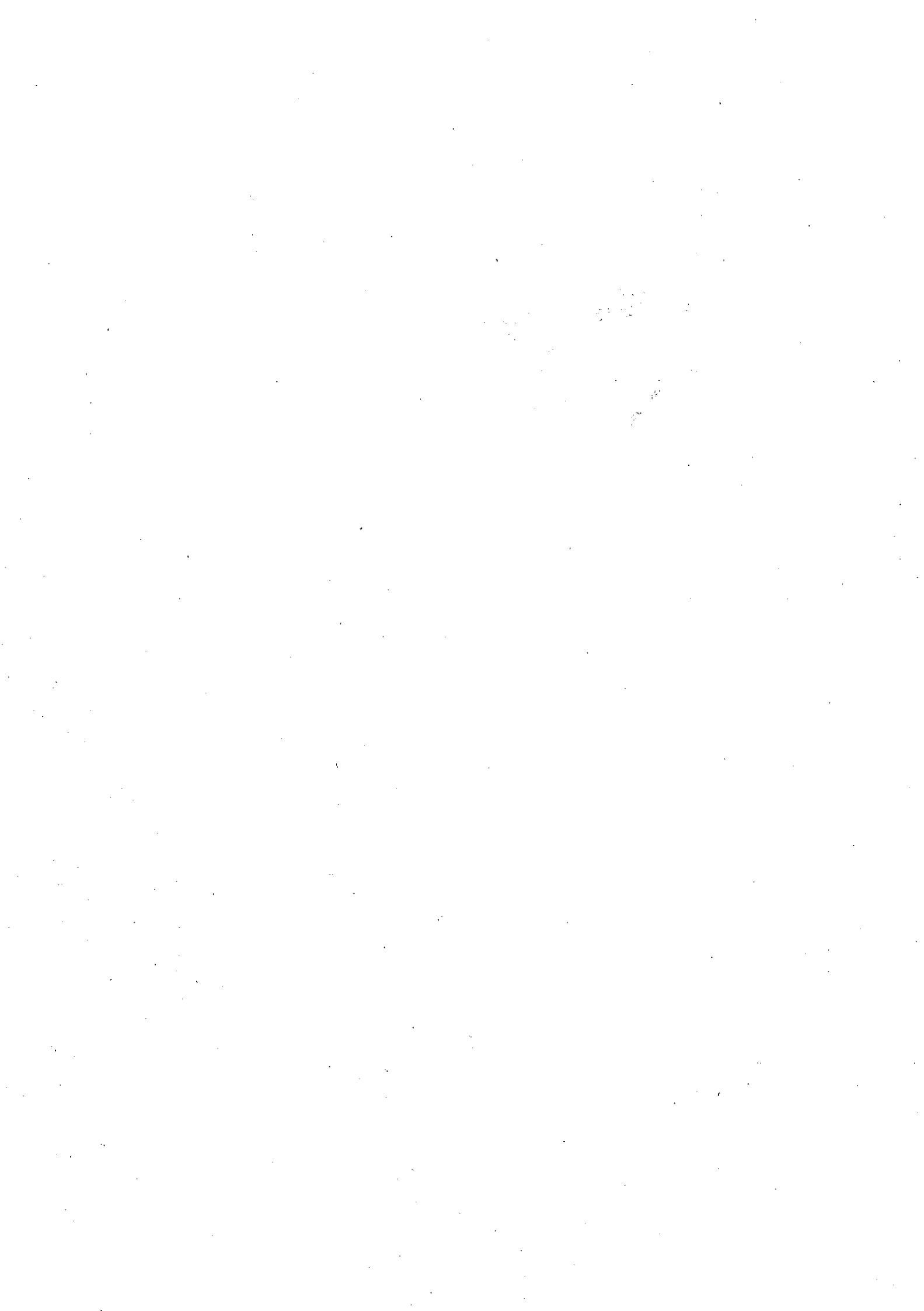
x	-1	0	$\frac{8}{27}$	2	Endpt
f'	-	+	-	+	+ (2, 0.4126)
(-1, -2)	(0, 0)		$\left(\frac{8}{27}, -\frac{4}{27}\right)$		local max.
local min.	local max.		local min.		

There are 2 local max & 2 local min.

abs max is 0.4126 at $x=2$

abs min is -2 at $x=-1$





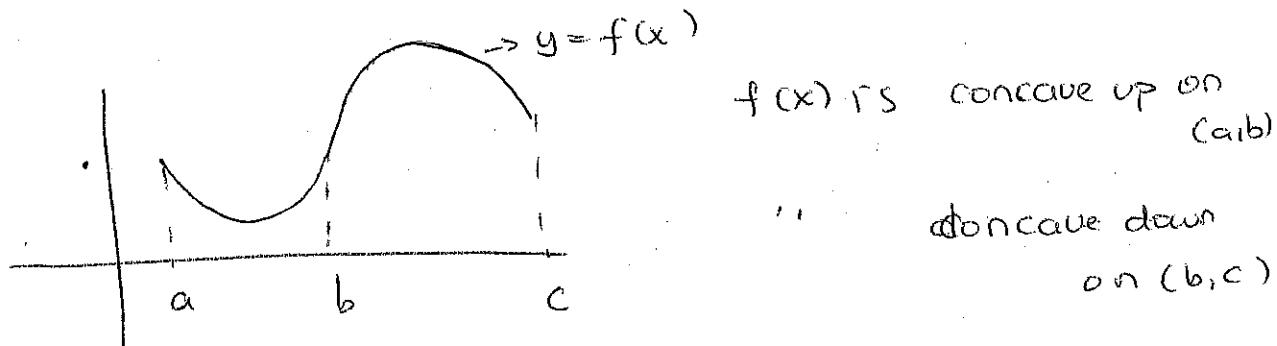
Sec 4-3 Concavity and Inflections:

2nd derivative \rightarrow inf about shape of graph.

whether it is bending upward or downward.

Defn: f is concave up on open interval I if it is differentiable there and f' is increasing on I .
 f is concave down on I if f' exists and is decreasing on I .

Ex



Defn: Inflection Points

$(x_0, f(x_0))$ is an inflection point of the curve $y = f(x)$

(or $f(x)$ has an inflection point at $x=x_0$) if

1-) graph of $y = f(x)$ has a tangent line at x_0

\Rightarrow either f is diff at x_0 or its graph has a vertical tangent line there

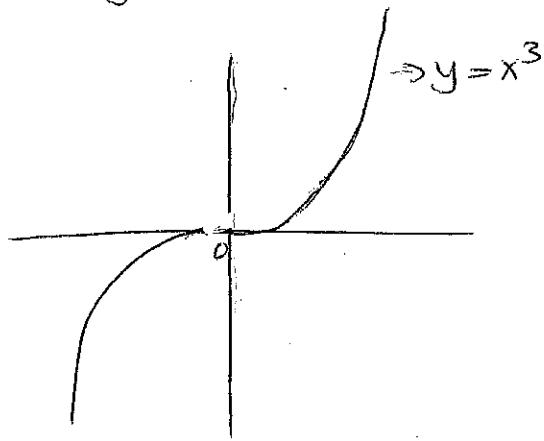
2-) the concavity of f is opposite on opposite sides of x_0

\Rightarrow graph crosses its tangent line at x_0 .

Inflection point is a point on graph of f .

critical, singular points are points in dom f .

Ex: $y = x^3$

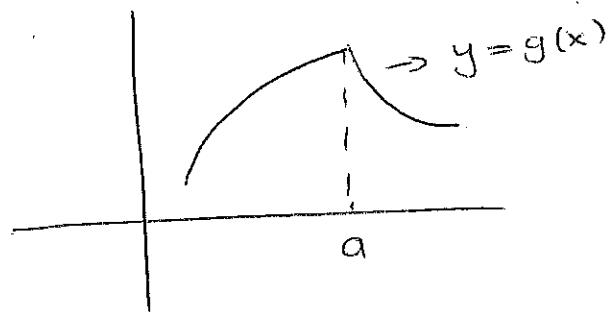


$x=0$ is inflection point:
 $y=x^3$ has horizontal tangent at $x=0$

$y = x^3$ concave down on $(-\infty, 0)$
 concave up on $(0, \infty)$

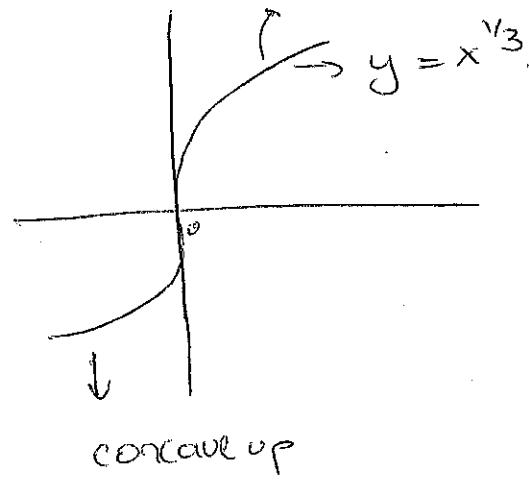
$x=0$ not inflection pt, no tangent line at $x=a$.

Ex:



g is concave down
 on left of $x=a$ and
 concave up on right of $x=a$
 but g has no inflection
 point at $x=a$
 because no tangent line there

Ex:



there is vertical asymptote
 at $x=0$.

so $x=0$ is inflection point

Thm Concavity and second derivative

(a-) If $f''(x) > 0$ on interval I $\Rightarrow f$ is concave up on I

(b) if $f''(x) < 0$ " $\Rightarrow f$ is concave down on

(c-) if f has inflect pt at x_0 and $f''(x_0)$ exists

Ex: Determine the intervals of concavity of

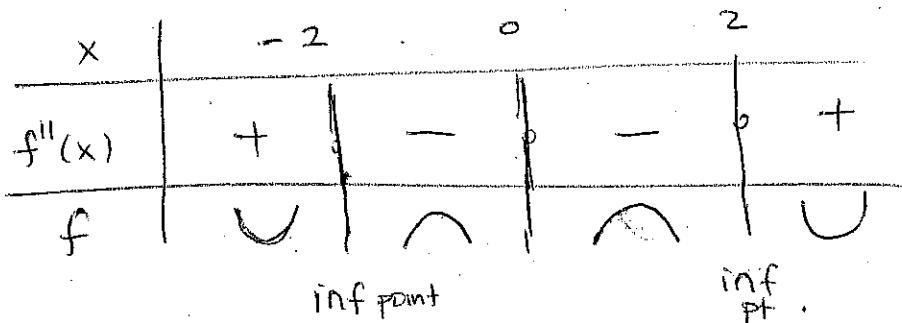
$$f(x) = x^6 - 10x^4$$

and the inflection points of its graph.

Sln: $f'(x) = 6x^5 - 40x^3$

$$f''(x) = 30x^4 - 120x^2 = 30x^2(x^2 - 4) = 30x^2(x-2)(x+2)$$

$$f''(x) = 0 \text{ when } x=0 \quad x=2 \quad x=-2$$



f is concave up on $(-\infty, -2)$ and $(2, \infty)$.

f is concave down on $(-2, 0)$ and $(0, 2)$.

graph of f has inf. points at $(-2, 96)$ and $(2, 96)$.

Ex: Determine the intervals of increase and decrease

the local extreme values and the concavity of

$$f(x) = x^4 - 2x^3 + 1$$

Use the information to sketch the graph of f .

Soln:

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x = 12x(x-1)$$

$$f'(x) = 0 \Rightarrow 2x^2(2x^2 - 3) = 0 \quad x = 0$$

$$\bullet x = \frac{3}{2}$$

$$f''(x) = 0 \quad 12x(x-1) = 0 \quad x = 0$$

$$x = 1$$

x	0	$\frac{3}{2}$
f'	-	-
f	↓	↓

local min

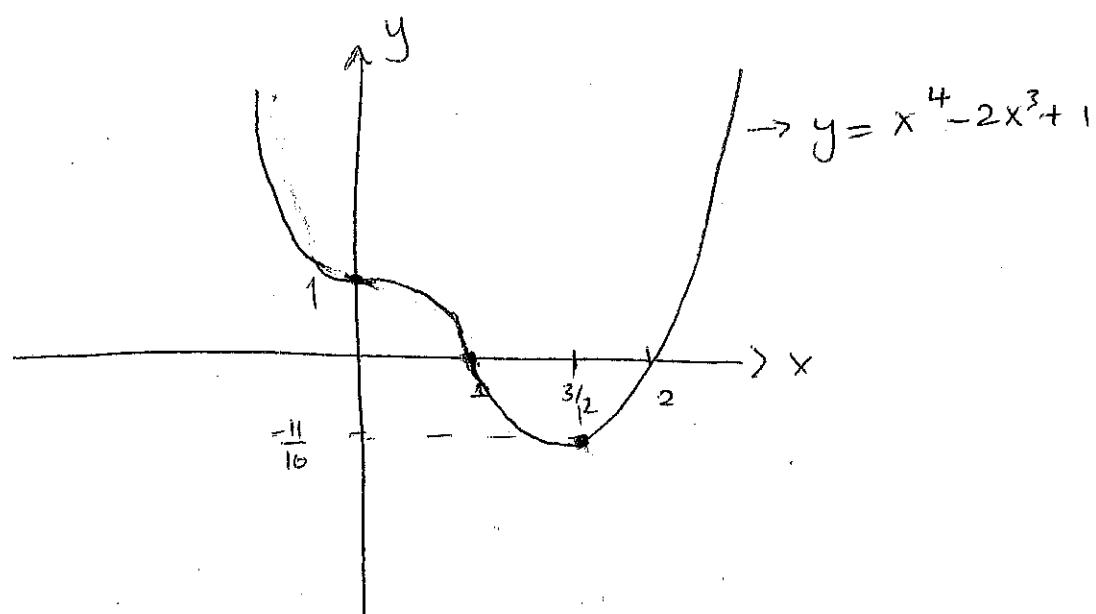
x	0	1
f''	+	-
f	↑	↑

inf pt inf

$$f\left(\frac{3}{2}\right) = -\frac{11}{16} \quad \left(\frac{3}{2}, -\frac{11}{16}\right) \text{ local min}$$

$$f(0) = 0 \quad (0, 0) \text{ inf pt}$$

$$f(1) = 0 \quad (1, 0) \text{ inf pt}$$



The Second Derivative Test:

x_0 critical point

f is concave down

Thm: a) if $f'(x_0) = 0$ and $f''(x_0) < 0 \Rightarrow f$ has a local max value at x_0 .

f is concave up

b) if $f'(x_0) = 0$ and $f''(x_0) > 0 \Rightarrow f$ has a local min value at x_0 .

c) If $f'(x_0) = 0$ and $f''(x_0) = 0 \Rightarrow$ no conclusion can be drawn

f may have a local max at x_0 or a local min or it may have an inflection point instead.

Ex: Find and classify the critical points of

$$f(x) = x^2 e^{-x}$$

Soln: $f'(x) = 2x e^{-x} - x^2 e^{-x} = e^{-x}(2-x) = 0$

$$x=2 \quad x=0$$

Critical points

$$f''(x) = 2e^{-x} - 2x e^{-x} - 2x e^{-x} + x^2 e^{-x}$$

$$= e^{-x}(2-4x+x^2)$$

$$f''(0) = 2 > 0 \quad f''(2) = -2e^{-2} < 0$$

$\Rightarrow f$ has local min at $x=0$

f has local max at $x=2$.

Q13

Ex: Determine the intervals of constant concavity of function and locate any inflection pts.

$$f(x) = x + \sin 2x$$

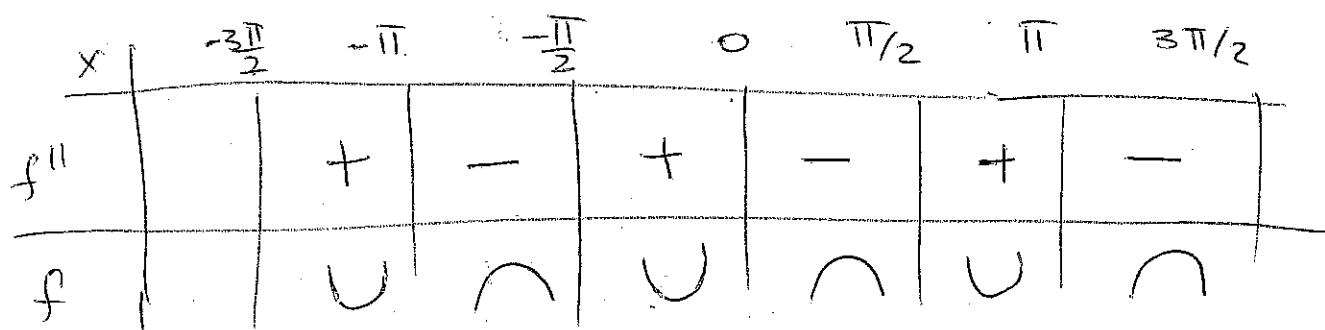
Soln: $f'(x) = 1 + 2\cos 2x$

$$f''(x) = -4 \sin 2x = 0$$

$$\sin 2x = 0$$

$$2x = n\pi$$

$$x = \frac{n\pi}{2} \quad n=0, \pm 1, \pm 2, \dots$$



f is concave down

f is concave up

$$n=-1: \left(-\pi, -\frac{\pi}{2}\right)$$

$$\left(-\frac{3\pi}{2}, -\pi\right) \quad n=-1$$

$$n=0: \left(0, \frac{\pi}{2}\right)$$

$$\left(-\frac{\pi}{2}, 0\right) \quad n=0$$

$$n=1: \left(\pi, \frac{3\pi}{2}\right)$$

$$\left(\frac{\pi}{2}, \pi\right) \quad n=1$$

$$\boxed{\left(n\pi, (n+\frac{1}{2})\pi\right)}$$

$$\boxed{\left((n-\frac{1}{2})\pi, n\pi\right)}$$

$$n=0, \pm 1, \pm 2, \dots$$

$$n=0, \pm 1, \pm 2, \dots$$

f has inflection at $x = \frac{n\pi}{2}$

Section 4 Sketching the Graph of a Function

When sketching $y = f(x)$ we have 3 sources of info:

- 1- f itself → determine some points on graph
→ symmetry of graph
→ asymptotes
- 2- f' → intervals of increase and decrease
location of local extreme values
- 3- f'' → determine concavity and inflection points
and some extreme values

Asymptotes:

There are 3 kinds: vertical, horizontal and oblique.

Defn:

The graph $y = f(x)$ has a vertical asymptote at $x=a$

If either

$$\lim_{x \rightarrow a^-} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \pm \infty$$

or both.

Note: we have this situation when $f(x)$ is a quotient of two expressions and the denominator is zero at $x=a$.

Ex: Find the vertical asymptotes of

$$f(x) = \frac{1}{x^2 - x}$$

How does the graph approach these asymptotes?

Soln:

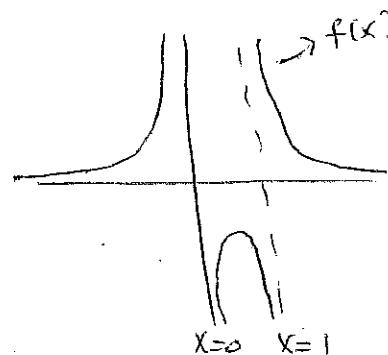
$$x^2 - x = x(x-1)$$

denominator $\rightarrow 0$ when $x \rightarrow 0$ and $x \rightarrow 1$

f has vertical asymptotes at $x=0$ and $x=1$.

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2 - x} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x^2 - x} = -\infty$$



$$\lim_{x \rightarrow 0^+} \frac{1}{x^2 - x} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x^2 - x} = \infty$$

Defn: The graph of $y = f(x)$ has a horizontal asymptote $y = L$ if

either $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$ or both.

Ex: Find the horizontal asymptotes of

a.) $f(x) = \frac{1}{x^2 - x}$

b.) $g(x) = \frac{x^4 + x^2}{x^4 + 1}$

Ex: $y = \frac{xe^x}{1+e^x}$

$$\lim_{x \rightarrow -\infty} \frac{xe^x}{1+e^x} = \frac{x \cdot \frac{1}{e^\infty}}{1 + \frac{1}{e^\infty}} = \frac{0}{1} = 0.$$

horizontal asymptote $y=0$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{xe^x}{1+e^x} - x \right) &= \lim_{x \rightarrow \infty} \frac{xe^x - x - xe^x}{1+e^x} \\ &= \lim_{x \rightarrow \infty} \frac{x(e^x - 1 - e^x)}{1+e^x} = \lim_{x \rightarrow \infty} \frac{-x}{1+e^x} = 0 \end{aligned}$$

↓
apply L'Hopital

Asymptotes of a Rational Function :

Suppose $f(x) = \frac{P_m(x)}{Q_n(x)}$, P_m and Q_n are polynomials

of degree m and n respectively. Suppose that P_m and Q_n have no common linear factors.

then,

(1-) Graph of f has a vertical asymptote at every x s.t $\boxed{Q_n(x) = 0}$.

(2-) The graph of f has a two-sided horizontal asymptote $y=0$ if $m < n$.

Soln:

$$a) \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - x} = \lim_{x \rightarrow \mp\infty} \frac{1}{x^2(1 - \frac{1}{x})} = \lim_{x \rightarrow \mp\infty} \frac{\frac{1}{x^2}}{1 - \frac{1}{x}} = 0$$

f has horizontal asymptote $y = 0$

$$b) \lim_{x \rightarrow \mp\infty} \frac{x^4 + x^2}{x^4 + 1} = \lim_{x \rightarrow \mp\infty} \frac{x^4(1 + \frac{1}{x^2})}{x^4(1 + \frac{1}{x^2})} = \frac{1}{1} = 1$$

g has horizontal asymptote $y = 1$

Note: ~~f, g have two~~

Here the horizontal asymptotes are two-sided because graph \rightarrow asym as $x \rightarrow \infty$ and $x \rightarrow -\infty$.

Defn: The straight line $y = ax + b$ ($a \neq 0$) is an oblique asymptote of $y = f(x)$ if

either $\lim_{x \rightarrow -\infty} (f(x) - (ax + b)) = 0$ or

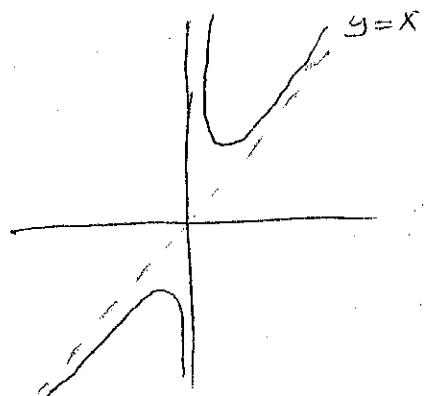
$\lim_{x \rightarrow \infty} (f(x) - (ax + b)) = 0$ or both.

Ex:

$$f(x) = \frac{x^2 + 1}{x}$$

$$f(x) = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

$$\lim_{x \rightarrow \mp\infty} (f(x) - x) = \lim_{x \rightarrow \mp\infty} \frac{1}{x} = 0$$



So $y = x$ is a two-sided oblique asymptote.

(3-) The graph of f has a two-sided horizontal asymptote $y=L$ ($L \neq 0$) if $m=n$. L is the quotient of the coefficients of the highest degree terms in P_m and Q_n .

(4-) The graph of f has a two-sided oblique asymptote if $m=n+1$. This is found by dividing Q_n into P_m to obtain linear quotient $ax+b$, and remainder R is a polynomial of degree at most $n-1$.

$$f(x) = ax+b + \frac{R(x)}{Q_n(x)}$$

$y = ax+b$ is oblique asymptote.

(5-) Graph of f has no horizontal or oblique asymptotes if $m > n+1$.

Ex: Find the oblique asymptote of $y = \frac{x^3}{x^2+x+1}$

Sln:

$$\begin{array}{r} x-1 \\ \hline x^2+x+1 \end{array} \overline{) x^3} \\ -x^3-x^2-x \\ \hline -x^2-x \\ \hline \pm x^2 \pm x-1 \\ \hline 1$$

$$\frac{x^3}{x^2+x+1} = x-1 + \frac{1}{x^2+x+1}$$

Formal Curve Sketching:

Checklist For Curve Sketching:

- (1-) Calculate $f'(x)$ and $f''(x)$. and express them in factored form.
- (2-) Examine $f(x)$ to determine its domain and the following
 - a-) vertical asymptotes. (zeros of denominator)
 - b-) horizontal or oblique asymp. (consider $\lim_{x \rightarrow \pm\infty} f(x)$)
 - c-) any obvious symmetry (is f even or odd?)
 - d-) any easily calculated intercepts, $(x_1, 0), (y_1, 0)$
- (3-) Examine $f'(x)$ for:
 - (a-) any critical points
 - (b-) any points where f' is not defined
(singular points, endpoints of dom f , vertical asymp)
 - (c-) intervals on which f' is post or negative.
Make a chart ~~for~~ intervals where f is increasing and decreasing and classify singular and critical points as local max or local minima.
- (4-) Examine f'' for:
 - (a-) points where $f''(x) = 0$
 - (b-) " " " $f''(x)$ is undefined.

use a chart to find

(c) intervals where f'' is post and negative

\Rightarrow where f is concave up or down.

(d) any inflection pts.

Ex: sketch the graph of $y = \frac{x^2+2x+4}{2x}$ $\rightarrow P_n$ $\rightarrow Q_m$.

Sdn:

1.)

$$y' = \frac{(2x+2)(2x) - 2(x^2+2x+4)}{(2x)^2}$$

$$= \frac{4x^2 + 4x - 2x^2 - 4x - 8}{4x^2}$$

$$\boxed{y' = \frac{2x^2 - 8}{4x^2} = \frac{x^2 - 4}{2x^2}}$$

$$y'' = \frac{2x(2x^2) - (4x)(x^2 - 4)}{4x^4} = \frac{4x^3 - 4x^3 + 16x}{4x^4}$$

$$\boxed{y'' = \frac{4}{x^3}}$$

2.) From y :

(a) domain $y : \mathbb{R} - \{0\}$.

(b) vertical asymptote: $\lim_{x \rightarrow 0^+} \frac{x^2+2x+4}{2x} = \infty$

so $x=0$ is vertical asymptote.

$\deg P_n(x) = 1 + \deg \text{ of } Q_m(x)$.

(c) oblique asymptote:

$$\begin{array}{r} \frac{1}{2}x + 1 \\ \hline 2x \left[\begin{array}{r} x^2 + 2x + 4 \\ - x^2 \\ \hline 2x + 4 \\ - 2x \\ \hline 4 \end{array} \right] \end{array}$$

oblique asymptote: $y = \frac{1}{2}x + 1$

$$\lim_{x \rightarrow \pm\infty} y - \left(\frac{x}{2} + 1\right) = \lim_{x \rightarrow \pm\infty} \frac{x^2 + 2x + 4 - \frac{1}{2}x - 1}{2x}$$

$$= \lim_{x \rightarrow \pm\infty} \frac{4}{2x} = 0$$

Symmetry:

$$y = f(-x) = \frac{(-x)^2 + 2(-x) + 4}{2(-x)} = \frac{x^2 - 2x + 4}{-2x}$$

y is neither odd nor even.

Intercepts:

$x = 0 \Rightarrow y$ is not defined no y intercept

$$y = 0 \Rightarrow 0 = \frac{x^2 + 2x + 4}{2x} \Rightarrow 0 = x^2 + 2x + 4$$

No x -intercept -

$$x = \frac{-2 \mp \sqrt{4 - 4(1)(4)}}{2}$$

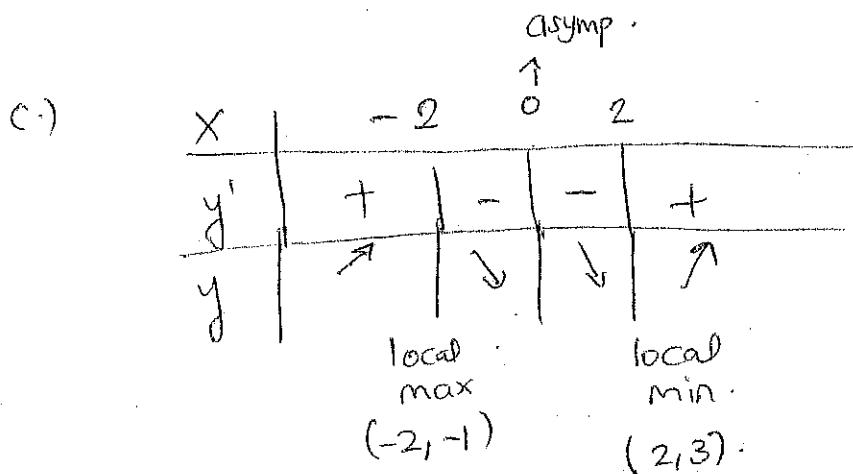
No
real
roots

From y' :

a) critical points: $y' = \frac{x^2 - 4}{2x^2} = 0 \Rightarrow x^2 - 4 = 0$
 $x=2, x=-2$

$(2, 3), (-2, -1)$

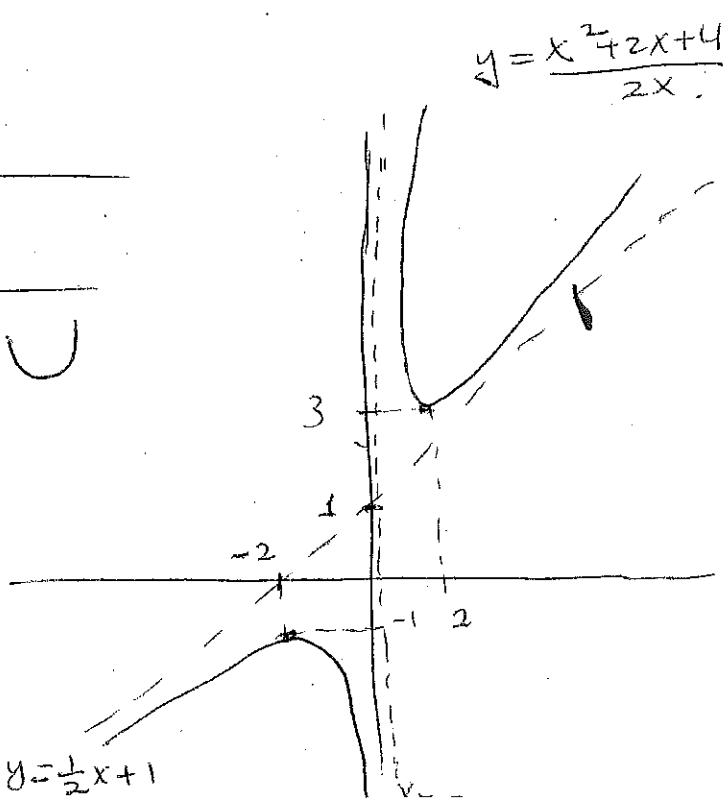
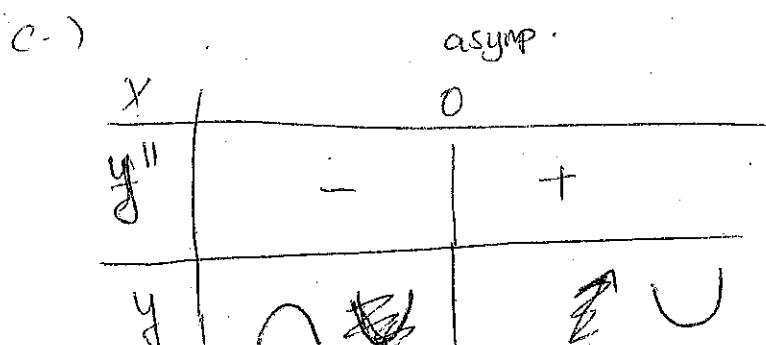
b.) y' is not defined at $x=0$ (vertical asymptote).



From y'' :

a.) $y''=0 \Rightarrow 0=\frac{4}{x^2}$ not possible

b.) y'' is not defined as $x=0$.



Ex: Sketch the graph of

$$f(x) = \frac{x^2 - 1}{x^2 - 4}$$

Soln:

$$f'(x) = \frac{(2x)(x^2 - 4) - (2x)(x^2 - 1)}{(x^2 - 4)^2} = \frac{-6x}{(x^2 - 4)^2}$$

$$f''(x) = \frac{-6(x^2 - 4)^2 - 2(x^2 - 4)(2x)(-6x)}{(x^2 - 4)^4}$$

$$f''(x) = \frac{6(3x^2 + 4)}{(x^2 - 4)^3}$$

From f:

Domain: $\mathbb{R} - \{2, -2\}$

Vertical asymptote $\lim_{x \rightarrow 2} f(x) = \infty$ $\lim_{x \rightarrow -2} f(x) = \infty$

$x=2, x=-2$ are vertical asymptotes.

Horizontal asymptote: $\deg P_n(x) = \deg P_m(x)$.

$$\lim_{x \rightarrow \pm\infty} \frac{x^2 - 1}{x^2 - 4} = \lim_{x \rightarrow \pm\infty} \frac{x^2(1 - \frac{1}{x^2})}{x^2(1 - \frac{4}{x^2})} = 1$$

$y=1$ is horizontal asymptote.

Symmetry:

$$f(-x) = \frac{(-x)^2 - 1}{(-x)^2 - 4} = \frac{x^2 - 1}{x^2 - 4} = f(x) \rightarrow f \text{ is even funct.}$$

f is symm about y -axis

Intercepts: $x=0 \Rightarrow y = \frac{1}{4}$ $(0, \frac{1}{4}) \rightarrow y\text{-intercept}$

$y=0 \Rightarrow x=\mp 1$ $(1,0) (-1,0) \rightarrow x\text{-intercepts}$

Other points: $x=-2$ and $x=2$ two vertical asymptotes
divide graph into 3 components we need points
on each of them.

$$\left(-3, \frac{8}{5}\right), \left(3, \frac{8}{5}\right)$$

From f' :

Critical points: $\frac{-6x}{(x^2-4)^2} = 0 \Rightarrow x=0$

f' is not defined at $x=2$ or $x=-2$.

x	asyp	critp.	asyp
	-2	0	2
f'	+	+	-

f

local max.

$(0, \frac{1}{4})$ local max pt.

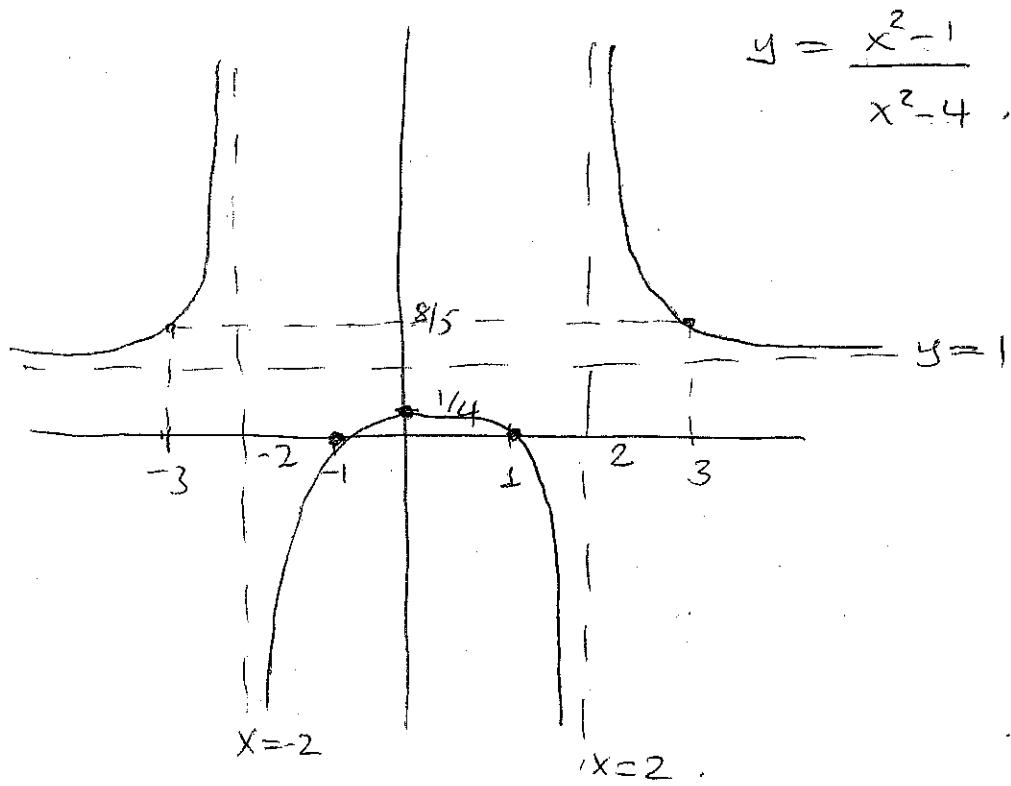
From f'' :

$$f''(x)=0 \Rightarrow \frac{6(3x^2+4)}{(x^2-4)^3}=0 \Rightarrow 18x^2 = -24$$

not possible.

f'' is not defined at $x=2$, $x=-2$.

f''	-2	2
	+	-
f	U	↗



(Read Ex 8)

Ex: Sketch the graph of $y = x e^{-x^2/2}$

Soln:

$$y' = 1 \cdot e^{-x^2/2} + (-1)x^2 e^{-x^2/2} = \boxed{e^{-x^2/2} (1-x^2)}$$

$$y'' = -x e^{-x^2/2} (1-x^2) + (-2x) e^{-x^2/2}.$$

$$y'' = \boxed{e^{-x^2/2} (x^2 - 3)}.$$

From y:

$$\underline{\text{domain}} = \mathbb{R}.$$

$$\underline{\text{Symmetry}}: y = f(-x) = -x e^{-x^2/2} = -f(x)$$

y is odd function \Rightarrow symmetric about origin

Intercepts: $x=0 \Rightarrow y=0$. (0,0) intercept.

$y = L$ is horizontal asymptote. 16

Horizontal asymptote

$\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$.

Let $t = \frac{x^2}{2} \Rightarrow |x e^{-x^2/2}| = \sqrt{2t} e^{-t}$

? $\lim_{t \rightarrow \infty} \frac{\sqrt{2t}}{e^t} = \ln \frac{\frac{1}{2}(2t)^{-1/2} \cdot 2}{e^t} = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{2t} e^t} = 0$.

When $t \rightarrow \infty \Rightarrow x \rightarrow \pm\infty$ so $y = 0$ is horizontal asymptote.

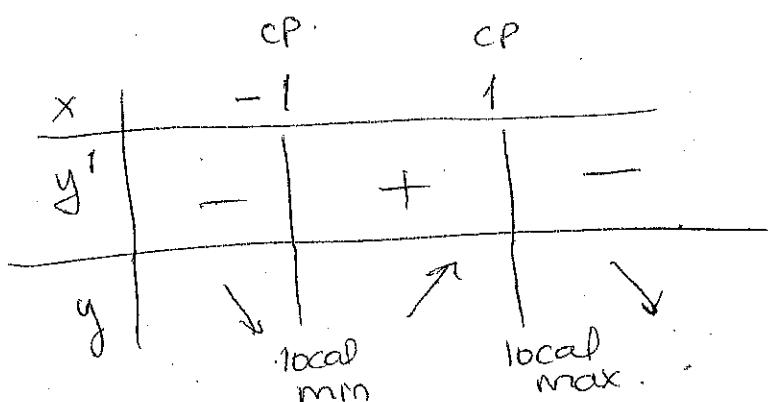
From y' :

critical pts

$$* e^{-x^2/2} f'(x) = 0$$

$$1-x^2=0 \Rightarrow x = \pm 1$$

$$(\pm 1, \pm \frac{1}{\sqrt{e}}) = (\pm 1, \pm 0.61)$$



local min $(-1, -0.61)$

local max $(1, 0.61)$

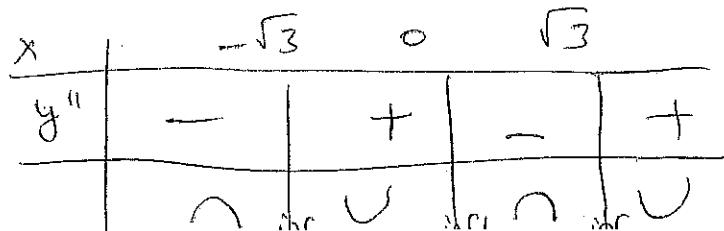
From y'' :

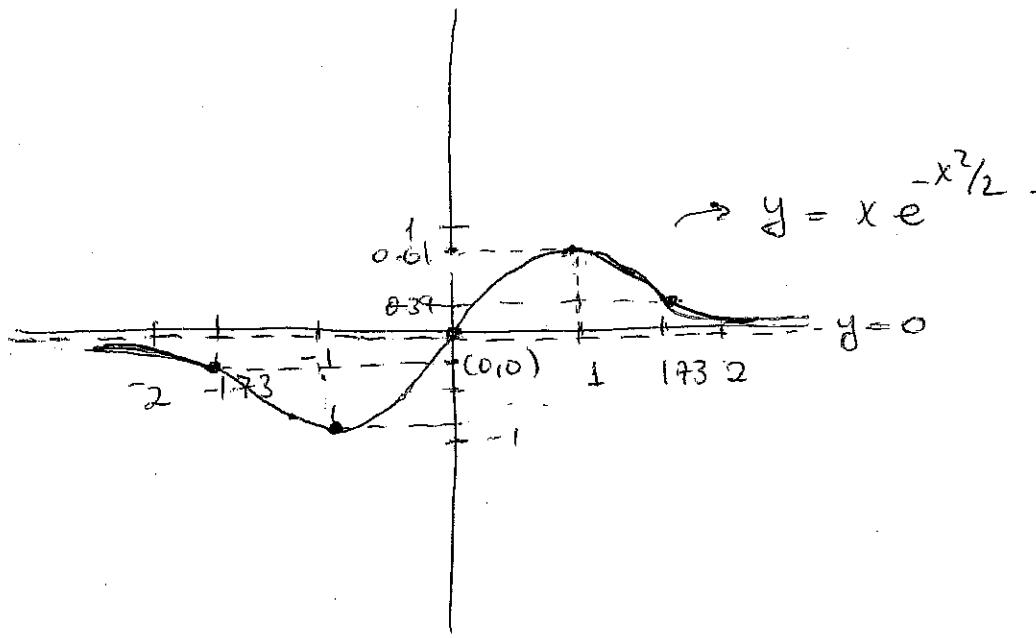
$$y'' = 0 \Rightarrow e^{-x^2/2} (x^2 - 3)x = 0$$

$$x = 0, x = \pm \sqrt{3}$$

inflection points

$$(0, 0), (\pm \sqrt{3}, \pm \sqrt{3} e^{3/2}) = (\pm 1.73, \pm 0.39)$$





(Read Ex 9)

Ex: Sketch the graph of $f(x) = (x^2 - 1)^{2/3}$

Soln: $f'(x) = \frac{2}{3} (x^2 - 1)^{-1/3} \cdot 2x = \boxed{\frac{4}{3} x (x^2 - 1)^{-1/3}}$

$$\begin{aligned} f''(x) &= \frac{4}{3} (x^2 - 1)^{-1/3} + \frac{-1}{3} (x^2 - 1)^{-4/3} \cdot 2x \cdot \frac{4}{3} x \\ &= \frac{4}{3} (x^2 - 1)^{-1/3} - \frac{8}{9} x^2 (x^2 - 1)^{-4/3} \\ &= \frac{\frac{4}{3}}{(x^2 - 1)^{4/3}} - \frac{\frac{8}{9} x^2}{(x^2 - 1)^{4/3}} = \frac{\frac{4}{3} x^2 - \frac{4}{3} - \frac{8}{9} x^2}{(x^2 - 1)^{4/3}} \end{aligned}$$

From f:

a.) $\text{Dom } f = \mathbb{R}$.

$$\boxed{f''(x) = \frac{4}{9} \frac{(x^2 - 3)}{(x^2 - 1)^{4/3}}}$$

b.) Asymptotes: none. $x \rightarrow \pm\infty$ $f(x)$ grow like $x^{4/3}$ no horizontal

c.) Symmetry: $f(-x) = ((-x)^2 - 1)^{2/3} = (x^2 - 1)^{2/3} = f(x)$ f is even funct.

d.) Symmetric about y-axis.

d.) Intercepts: $f(x) = (x^2 - 1)^{2/3}$

$$x=0 \Rightarrow f(x)=1.$$

$$y=f(x)=0 \Rightarrow [0]^{2/3} = [(x^2 - 1)^{2/3}]^{3/2} \Rightarrow 0 = x^2 - 1$$

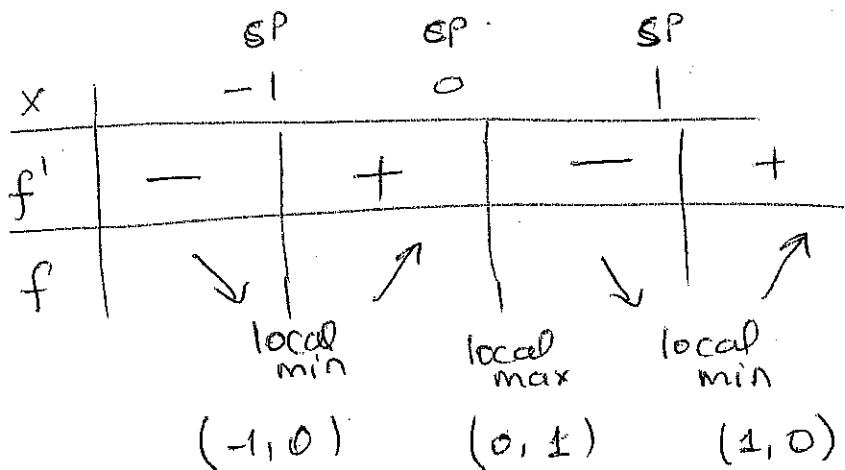
$$\Rightarrow x = \mp 1$$

$$\boxed{(-1, 0) \text{ & } (0, 1)}$$

From f' : $f'(x) = \frac{4x(x^2 - 1)^{-1/3}}{(x^2 - 1)^{4/3}} = \frac{\frac{4}{3}x}{(x^2 - 1)^{1/3}}$

$$f'(x)=0 \Rightarrow \frac{4}{3}x=0 \Rightarrow \boxed{x=0} \text{ critical point}$$

f' is undefined when $x=\mp 1 \Rightarrow \boxed{x=\mp 1 \text{ are singular pts}}$



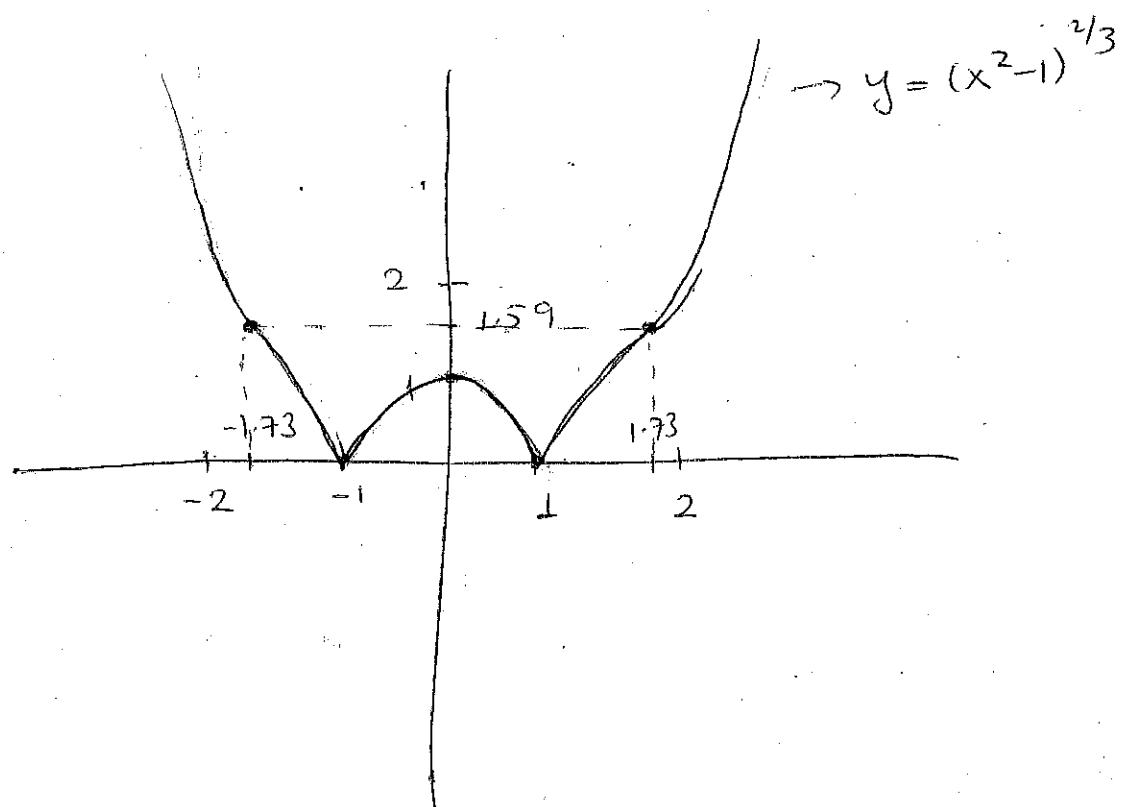
From f'' : $f''(x) = \frac{4}{9} \frac{(x^2 - 3)}{(x^2 - 1)^{4/3}}$

$$f''(x)=0 \Rightarrow \frac{4}{9}(x^2 - 3)=0 \Rightarrow x^2 - 3=0 \Rightarrow x = \mp \sqrt{3}$$

$$(\mp \sqrt{3}, 2^{2/3}) \approx (\mp \sqrt{3}, 1.59)$$

x	$-\sqrt{3} \approx -1.73$	-1	1	$\sqrt{3} \approx 1.73$
$f''(x)$	+	-	-	-
f	U	⌞	⌞	⌞

infl.
 $(-1.73, 1.59)$ $(1.73, 1.59)$



Sec 4.5 Extreme Value - Problems :

Ex 1 Find the largest possible product of two nonnegative numbers whose sum is 7.

Soln: Let the numbers be

$$x, 7-x \quad \text{so} \quad x+7-x=7$$

both numbers are nonnegative. $0 \leq x \leq 7$.

$$P(x) = x(7-x) = 7x - x^2$$

$$P(0) = 0 \quad \text{but} \quad P(1) = 7-1 = 6 > 0$$

$$P(7) = 49-49 = 0$$

So $P(x)$ doesn't take max value at end points so it must have the max value at a critical point in $(0, 7)$.

To find critical point:

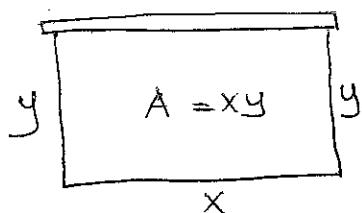
$$P'(x) = 7-2x \quad P'(x)=0 \Rightarrow 7-2x=0$$

$$x = \frac{7}{2} = \boxed{3\frac{1}{2}} \quad \text{critical point}$$

Largest possible product: $P\left(\frac{7}{2}\right) = 7 \cdot \frac{7}{2} - \frac{49}{4} = \frac{49}{4} - \frac{49}{4} = \frac{49}{4}$

Ex:2 A rectangular animal enclosure is to be constructed having one side along an existing long wall and the other 3 sides fenced. If 100m of fence are available, what is the largest possible area for the enclosure?

Soln:



$$x + 2y = 100$$

$$x = 100 - 2y.$$

$$A = (100 - 2y)y = 100y - 2y^2$$

in order to have $A \geq 0$, $y > 0$ and $x \geq 0$ (if $y \leq 50$).

So we want to maximize

$$A = A(y) = 100y - 2y^2 \text{ on } [0, 50].$$

A is continuous on $[0, 50]$ \Rightarrow it must have max value by them

$$A(0) = 100(0) - 2(0)^2 = 0 \quad \text{and } A'(y) > 0 \text{ when } 0 < y < 50$$

$$A(50) = 100(50) - 2(50)^2 = 0$$

Since A has no singl^{po}

So A cannot have max at an endpoint \uparrow Max must occur at critical point

$$A'(y) = 100 - 4y \quad 0 = 100 - 4y \Rightarrow \boxed{y = 25} \\ \text{critical point}$$

Largest
Greatest possible area is $A(25) = 100(25) - 2(25)^2 = \underline{\underline{1250 \text{ m}^2}}$

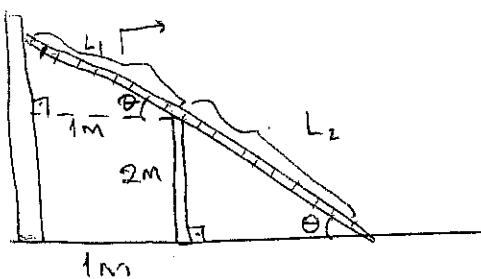
Procedure For Solving Extreme-Value Problems : (No.)

Solving extreme-value problems:

- 1.- Read the problem very carefully.
- 2.- Make a diagram if appropriate
- 3.) Define any symbols you wish to use.
- 4.) Express quantity Q to be maximized or minimized as a funct of one or more variables.
- 5.) if Q depends on n variables, find $n-1$ eqns (constraints) linking these variables.
- 6.) Use the constraints to eliminate variables and hence express Q as a function of only one variable.
Determine the interval(s) in which this variable must lie for the problem to make sense.
- 7.) Find the required extreme value of Q :
Remember to consider critical pts, singular pts and endpt!
- 8.) Make a conclusion statement answering the question asked.

(No)

Ex4: Find the length of the shortest ladder that can extend from a vertical wall over a fence 2m high located 1m away from the wall, to a point on the ground outside the fence.



$$\cos \theta = \frac{1}{L_1} \Rightarrow L_1 = \frac{1}{\cos \theta}$$

$$\sin \theta = \frac{2}{L_2} \Rightarrow L_2 = \frac{2}{\sin \theta}$$

$$L = L(\theta) = L_1 + L_2 = \frac{1}{\cos \theta} + \frac{2}{\sin \theta} \quad 0 < \theta < \frac{\pi}{2}$$

we want to minimize L .

$$\lim_{\theta \rightarrow 0^+} L(\theta) = \infty \quad (\frac{2}{\sin \theta} \rightarrow \infty) \quad \lim_{\theta \rightarrow \left(\frac{\pi}{2}\right)^-} L(\theta) = \infty \quad (\frac{1}{\cos \theta} \rightarrow \infty)$$

so $L(\theta)$ must have min between $(0, \frac{\pi}{2})$ ^{but not at endpoints}.

critical pt.

$$L'(\theta) = \frac{\sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta} = \frac{\sin^3 \theta - 2 \cos^3 \theta}{\cos^2 \theta \sin^2 \theta} = 0.$$

so any critical pts satisfies $\sin^3 \theta = 2 \cos^3 \theta$

$$\text{or } \tan^3 \theta = 2.$$

$$\theta = \tan^{-1}(2^{1/3})$$

$$\tan \theta = 2^{1/3}.$$

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + 2^{2/3}$$

$$\cos \theta = \frac{1}{\sec \theta} = \frac{1}{(1+2^{2/3})^{1/2}}$$

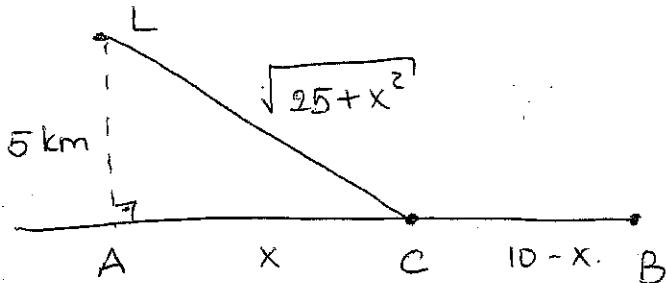
$$\sin \theta = \tan \theta \cos \theta = \frac{2^{1/3}}{(1+2^{2/3})^{1/2}}$$

Ex3 A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B and from there to B along the shoreline. The part of the cable lying in the water costs \$ 5000/km and the part along the shoreline costs \$ 3000/km.

(a.) Where should C be chosen to minimize the total cost of the cable?

(b.) Where should C be chosen if B is only 3 km from A?

Soln:



Then, total cost function.

$$T = T(x) = 5000(\sqrt{25+x^2}) + 3000(10-x), \quad 0 \leq x \leq 10$$

T is continuous funct on $[0, 10]$ so it has min value either at endpoints $x=0, x=10$, or critical point or at singular

$$T(0) = 25000 + 30000 = \boxed{55000}$$

$$T(10) \approx \boxed{55,902.70}$$

No singular point. $\frac{dT}{dx}$ defined for all x.

Critical points :

$$\frac{dT}{dx} = 5000 \cdot \frac{1}{x} (25+x^2)^{-\frac{1}{2}} \cdot 2x + 3000$$

$$\frac{dT}{dx} = \frac{5000x}{\sqrt{25+x^2}} - 3000$$

$$\frac{dT}{dx} = 0 \Rightarrow 5000x = 3000 \sqrt{25+x^2}$$

$$25x^2 = 9(25+x^2)$$

$$25x^2 - 9x^2 - 225 = 0$$

$$16x^2 = 225$$

$$x^2 = \frac{225}{16} = \frac{15^2}{4^2}$$

$$x = \pm \frac{15}{4}$$

only $x = \frac{15}{4} = 3\frac{3}{4} \in (0, 10)$. $x = -\frac{15}{4} \notin (0, 10)$.

$$\boxed{T\left(\frac{15}{4}\right) = 50000}$$

minimum value of T is 50000 when $x = \frac{15}{4} = 3.75$.

So for minimal cost C should be 3.75 km from A.

(b-) $T(x) = 5000\sqrt{25+x^2} + 3000(3-x) \quad 0 \leq x \leq 3$.

$$T(0) = 26000 + 9000 = 34000$$

$$T(3) \approx 29,155$$

$$\frac{dT}{dx} = \frac{5000x}{\sqrt{25+x^2}} - 3000$$

when $\frac{dT}{dx} > 0$ we have same critical points $x = \mp 3.75$

but $x = \mp 3.75 \notin (0, 3)$. So T has min val when $x = 3$.

So to minimize cost cable should go straight from L to B.

min value of $L(\theta)$ is :

$$\frac{1}{\cos \theta} + \frac{2}{\sin \theta} = (1+2^{2/3})^{1/2} + 2 \frac{(1+2^{2/3})^{1/2}}{2^{1/3}} =$$

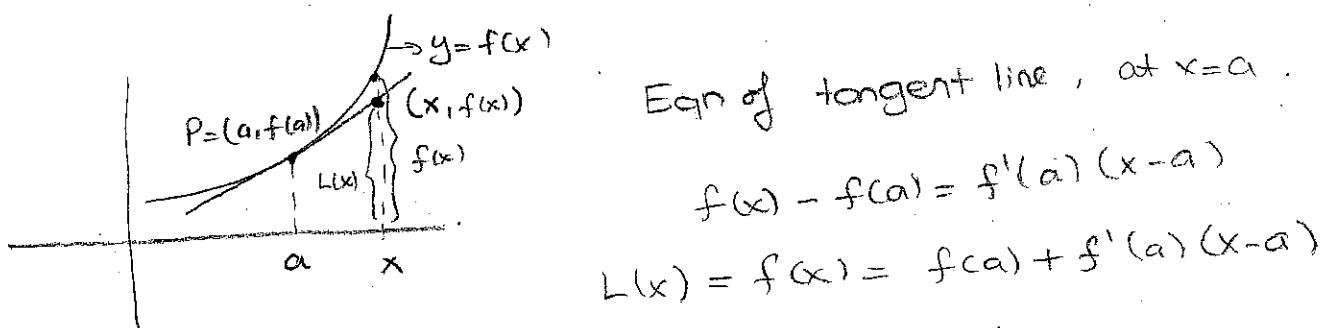
$$= \frac{(1+2^{2/3})^{1/2} + 2^{1/3} + 2(1+2^{2/3})^{1/2}}{2^{1/3}} = (1+2^{2/3})^{3/2} \approx 4.16$$

So the shortest ladder can be 4.16 m long.

Sec 4.7 Linear Approximations:

Defn: The linearization or linear approximation of the function f about $\underline{x=a}$ is function $L(x)$ defined as

$$L(x) = f(a) + f'(a)(x-a)$$



Eqn of tangent line, at $x=a$.

$$f(x) - f(a) = f'(a)(x-a)$$

$$L(x) = f(x) = f(a) + f'(a)(x-a)$$

The tangent to graph

$y=f(x)$ at $x=a$ describes

the behaviour of that graph near $P=(a, f(a))$

the behaviour of that graph near P , because it goes better than any other straight line through P , through P in the same direction as the curve $y=f(x)$.

Ex: Find the linearization for

(a-) $f(x) = \sqrt{1+x}$ about $x=0$

(b-) $g(x) = \frac{1}{x}$ about $x=\frac{1}{2}$

Sln: a-) $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$

$$L(x) = f(0) + f'(0)(x-0)$$

$$L(x) = 1 + \frac{1}{2}x \rightarrow \text{linear approximation of } f \text{ about } x=0.$$

b-) $g'(x) = -\frac{1}{x^2}$

$$L(x) = g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$$

$$= 2 - 4\left(x-\frac{1}{2}\right)$$

$$L(x) = 4 - 4x \rightarrow \text{linear approx of } f \text{ about } x=\frac{1}{2}.$$

Approximating Values of Functions

We have already made use of linearization in sec 2.7

where,

$$\boxed{\Delta y \approx \frac{dy}{dx} \Delta x}$$

is used to approximate small change in Δy when ~~Δx~~ when there is small change a to $a + \Delta x$.

$$\Delta y = f(a + \Delta x) - f(a)$$

$$f(a + \Delta x) \approx f(a) \approx f'(a) \Delta x$$

$$f(a + \Delta x) \approx f(a) + f'(a) \Delta x.$$

Ex: A ball of ice melts so that its radius decreases from 5 cm to 4.92 cm. By approximately how much does the volume of the ball decrease?

Soln: $V = \frac{4}{3} \pi r^3$

$$\Delta V \approx \frac{dV}{dr} \Delta r$$

$$\Delta V \approx 4\pi r^2, \Delta r. \quad \Delta r = 5 - 4.92 = 0.$$

$$\Delta r = 4.92 - 5 = -0.08$$

$$\Delta V \approx 4\pi(5)^2(-0.08) = -8\pi$$

$$\Delta V \approx -25.13$$

So volume of ball decreases by about 25 cm^3 .

Ex: Use the linearization for \sqrt{x} about $x=25$ to find an approximate value of $\sqrt{26}$.

Soln: $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(25) + \frac{1}{10}(x-25) = 5 + \frac{1}{10}(x-25)$$

linearization
of $f(x)$ abo
 $x=25$.

$$f(26) = \sqrt{26} \approx L(26) = 5 + \frac{1}{10}(26-25) = 5.1$$

Error Analysis

In any approximation,

$$\boxed{\text{Error} = \text{true value} - \text{approximate value.}}$$

Ex: $\sqrt{26} = 5.0990195.$

approximate value $\sqrt{26} = 5.1$

$$\begin{array}{r} 5.1 \\ 5.0990195 \\ \hline 5.1 \end{array}$$

$$5.0990195 - 5.1 = -0.0009805 \quad 0.0009805$$

$$\frac{5.0990195}{5.1} \times 100\% =$$

Ex: For the function $y = f(x) = \sqrt{3+x}$

a) Find the linear approximation at $x=6$.
 b) Use this linear approximation to estimate $\sqrt{9.3}$.

$$a) f(1) = \sqrt{3+1} \Rightarrow f'(1) = \frac{1}{2\sqrt{3+1}} \Rightarrow f'(6) = \frac{1}{6}$$

$$f(6) \approx \sqrt{3+6} = 3.$$

$$f(x) \approx \text{Lin } 3 + \frac{1}{6}(x-6)$$

$$b) \sqrt{3+3} = f(3) \approx L(3) = 3 + \frac{1}{6}(3-6) = 3 - \frac{1}{6}(-3) \approx 3.167$$

$$(\sqrt{3+0.3}) - (\sqrt{3+0.9}) \approx L(3.3) - L(3.9) = 3 + \frac{1}{6}(3.3-6) - 3 + \frac{1}{6}(3.9-6) \approx 3 - 0.1 \approx 2.98$$

$$c) \sqrt{3+6.3} = f(6.3) \approx L(6.3) = 3 + \frac{1}{6}(6.3-6) = 3 + \frac{1}{6}(0.3) \approx 3.05$$

Sec 4.9 Indeterminate Forms:

Types of indeterminate forms

$$\frac{0}{0} \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\frac{\infty}{\infty} \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow 0} \frac{\ln(1/x^2)}{\cot(x^2)}$$

$$0 \cdot \infty \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow 0^+} x \ln \frac{1}{x}$$

$$\infty - \infty \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow (\frac{\pi}{2})^-} \left(\tan x - \frac{1}{\pi - 2x} \right)$$

$$0^\circ \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow 0^+} x^x$$

$$0^\circ \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow (\frac{\pi}{2})^-} (\tan x)^{\cos x}$$

$$1^\infty \quad \underline{\text{Ex:}} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

L' Hospital's Rule:

We can evaluate many indeterminate forms of type $\frac{0}{0}$ by cancelling common factors from numerator and denominator.

L' Hospital's Rule is also used for evaluating limits of indeterminate forms of the types $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

The other types of indeterminate forms can usually be reduced to one of these two by algebraic manipulation and taking of logarithms.

Thm: The First L'Hopital Rule:

Suppose f and g are differentiable on interval (a, b) and $g'(x) \neq 0$, then

Suppose also

$$\text{i.) } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0 \text{ and}$$

$$\text{ii.) } \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \quad (L \text{ is finite or } \infty \text{ or } -\infty)$$

$$\Rightarrow \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$$

We get similar results if $\lim_{x \rightarrow a^+}$ is replaced by $\lim_{x \rightarrow b^-}$

or $\lim_{x \rightarrow c}$ where $a < c < b$

The cases $a = -\infty$ and $b = \infty$ are also allowed.

Ex: Evaluate $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$

$$\text{Sln: } \lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} = \frac{0}{0}$$

Apply L'Hopital,

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$$

2c

Ex: Evaluate $\lim_{x \rightarrow 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$

Sln:

$$\lim_{x \rightarrow 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2} \in \left(\frac{0}{0}\right)$$

Apply L'Hopital,

$$= \lim_{x \rightarrow 0} \frac{2\cos x - 2\cos(2x)}{2e^x - 2 - 2x} = \lim_{x \rightarrow 0} \frac{\cancel{2}(\cos x - \cos(2x))}{\cancel{2}(e^x - 1 - 2x)} \in \left(\frac{0}{0}\right)$$

Apply L'Hopital

$$= \lim_{x \rightarrow 0} \frac{-\sin x + 2\sin(2x)}{e^x - 1} \in \left(\frac{0}{0}\right)$$

Apply L'Hopital

$$= \lim_{x \rightarrow 0} \frac{-\cos x + 4\cos(2x)}{e^x} = \frac{-1 + 4}{1} = 3 //$$

Ex: Evaluate,

a-) $\lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{2x - \pi}{\cos^2 x}$

b-) $\lim_{x \rightarrow 1^+} \frac{x}{\ln x}$.

Sln: $\lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{2x - \pi}{\cos^2 x} \in \left(\frac{0}{0}\right)$

$$= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{2}{-\frac{1}{2} \sin x \cos x} = -\frac{2}{0} = -\infty$$

b.) $\lim_{x \rightarrow 1^+} \frac{x}{\ln x} = \infty$ not indeterminate form.
 when $x \rightarrow 1^+$
 $\ln x \rightarrow 0$

Ex: Evaluate $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

Soln: $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \infty - \infty$

at this step we can't apply L'Hopital Rule.

$$= \lim_{x \rightarrow 0^+} \left(\frac{\sin x - x}{x \sin x} \right) \not\in \left(\frac{0}{0} \right)$$

Apply L'Hopital

$$= \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x + x \cos x} \not\in \left(\frac{0}{0} \right)$$

Apply L'Hopital

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \frac{-0}{2} = 0$$

Thm The Second L'Hopital Rule

Suppose that f and g are differentiable on interval (a, b)
 and that $g'(x) \neq 0$ there. Suppose that

(i) $\lim_{x \rightarrow a^+} g(x) = \pm \infty$ and

(ii) $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L$ (L is finite or ∞ or $-\infty$)

then,

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$$

Similar results hold for $\lim_{x \rightarrow b^-}$, $\lim_{x \rightarrow c}$ and cases

$a = -\infty$ and $b = \infty$

Ex Evaluate,

$$(a) \lim_{x \rightarrow \infty} \frac{x^2}{e^x}$$

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \underset{\infty}{\cancel{\left(\frac{\infty}{\infty} \right)}}$$

$$\text{apply L'Hopital} \\ = \lim_{x \rightarrow \infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty} \right)$$

$$\text{apply L'Hopital} \\ = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0 //$$

$$(b) \lim_{x \rightarrow 0^+} x^\alpha \ln x \quad \alpha > 0$$

$$= \lim_{x \rightarrow 0^+} \frac{x^\alpha \ln x}{x^\alpha} \quad (0, (-\infty))$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-\alpha}} \quad \left(\frac{\infty}{\infty} \right)$$

Apply L'Hopital

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\alpha x^{-\alpha-1}} = \lim_{x \rightarrow 0^+} \frac{1}{-\alpha x^{-\alpha-1} \cdot x^1} = \lim_{x \rightarrow 0^+} \frac{x^\alpha}{-\alpha} = 0$$

Ex/

Note: To deal with indeterminate forms of types $\frac{0}{0}$, ∞^0

∞^∞ , and 1^∞ , we take logarithms of the expressions involved.

Ex: Evaluate $\lim_{x \rightarrow 0^+} x^x$

Soln: $\lim_{x \rightarrow 0^+} x^x = (0^\circ)$.

Let $y = x^x \Rightarrow \ln y = x \ln x$.

$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \ln x = 0$. (by prev ex (b))

So $\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} \ln y = e^0 = 1$.

Ex: Evaluate $\lim_{x \rightarrow (\frac{\pi}{2})^-} (\tan x)^{\cos x}$

$= \lim_{x \rightarrow (\frac{\pi}{2})^-} (\tan x)^{\cos x} = (\infty^\infty)$

$y = (\tan x)^{\cos x} \Rightarrow \ln y = \cos x \ln \tan x$.

$\lim_{x \rightarrow (\frac{\pi}{2})^-} \ln y = \lim_{x \rightarrow (\frac{\pi}{2})^-} \cos x \ln \tan x (0 \cdot \infty)$

$= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\ln \tan x}{\sec x} \underset{\infty}{\underset{\infty}{\approx}}$

Apply L'Hopital

$$= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\frac{1}{\tan x} \sec^2 x}{\sec x + \tan x} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x}{\sec x + \tan x} \cdot \frac{1}{\sec x \tan x}$$

$$\frac{1}{\cos 90^\circ} = \infty = \sec \frac{\pi}{2} \quad \tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \infty$$

$$= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x}{\tan^2 x} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

$$\lim_{x \rightarrow (\frac{\pi}{2})^-} (\tan x)^{\cos x} = e^0 = 1.$$

Ex: Evaluate $\lim_{x \rightarrow \infty} \left(1 + \sin \frac{3}{x} \right)^x$

Soln: $\lim_{x \rightarrow \infty} \left(1 + \sin \frac{3}{x} \right)^x \rightarrow [1^\infty]$

Let $y = \left(1 + \sin \frac{3}{x} \right)^x \Rightarrow \ln y = x \ln \left(1 + \sin \frac{3}{x} \right)$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln \left(1 + \sin \frac{3}{x} \right) \quad [\infty \cdot 0]$$

$$= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \sin \frac{3}{x} \right)}{\frac{1}{x}} \quad \left[\frac{0}{0} \right].$$

Apply L'Hopital

$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{1+\sin\frac{3}{x}}\right) \left(\cos\frac{3}{x}\right) \left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} =$$

$$= \lim_{x \rightarrow \infty} \frac{3 \cos \frac{3}{x}}{1 + \sin \frac{3}{x}} = \frac{3}{1} = 3.$$

$$\lim_{x \rightarrow \infty} \left(1 + \sin \frac{3}{x}\right)^x = e^3$$

Chapter 5 INTEGRATION

Sec 5.1 Sums and Sigma Notation

Defn: If m, n are integers with $m \leq n$, and if f is defined at the integers $m, m+1, m+2, \dots, n$.

$$\sum_{i=m}^n f(i) = f(m) + f(m+1) + \dots + f(n).$$

Ex: $\sum_{i=1}^5 i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

Ex: a) $\sum_{j=1}^{10} j = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10.$

b) $\sum_{m=1}^n 1 = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$

c) $\sum_{i=0}^n x^i = x^0 + x^1 + x^2 + \dots + x^n$

Infinite Series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

$$\sum_{i=m}^n (A f(i) + B g(i)) = A \sum_{i=m}^n f(i) + B \sum_{i=m}^n g(i)$$

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^n f(i+m)$$

Ex: Express $\sum_{j=0}^{99} \sin(j)$ in the form $\sum_{i=A}^n f(i)$.

Soln: $\sum_{j=0}^{99} \sin(j) = \sum_{i=1}^{100} \sin(i-1)$

Ex: Express $\sum_{j=3}^{17} \sqrt{1+j^2}$ in the form $\sum_{i=1}^n f(i)$

Soln: $\sum_{j=3}^{17} \sqrt{1+j^2} = \sum_{i=1}^{15} \sqrt{1+(i+2)^2}$

Thm: Summation Formulas:

(a-) $\sum_{i=1}^n 1 = \underbrace{1+1+\dots+1}_{n \text{ terms}} = n$

(b-) $\sum_{i=1}^n i = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

(c-) $\sum_{i=1}^n i^2 = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(d-) $\sum_{i=1}^n r^{i-1} = 1+r+r^2+\dots+r^{n-1} = \frac{r^n - 1}{r - 1} \quad \text{if } r \neq 1$

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Ex: Evaluate $\sum_{k=m+1}^n (6k^2 - 4k + 3)$ where $1 \leq m < n$.

Soln:

$$\begin{aligned}\sum_{k=1}^n (6k^2 - 4k + 3) &= 6 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + 3 \sum_{k=1}^n 1 \\ &= 6 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + 3n \\ &= 2n^3 + n^2 + 2n.\end{aligned}$$

$$\begin{aligned}\sum_{k=m+1}^n (6k^2 - 4k + 3) &= \sum_{k=1}^n (6k^2 - 4k + 3) - \sum_{k=1}^m (6k^2 - 4k + 3) \\ &= 2n^3 + n^2 + 2n - 2m^3 - m^2 - 2m\end{aligned}$$

Q27

Ex: $\sum_{k=1}^n (\pi^k - 3)$

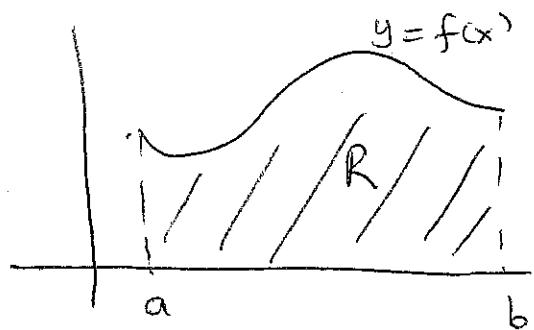
Soln:

$$\begin{aligned}&= \sum_{k=1}^n \pi^k - 3 \sum_{k=1}^n 1 \\ &= \sum_{k=1}^n \pi \cdot \pi^{k-1} - 3 \sum_{k=1}^n 1 = \pi \sum_{k=1}^n \pi^{k-1} - 3 \sum_{k=1}^n 1 \\ &= \pi \left(\frac{\pi^n - 1}{\pi - 1} \right) - 3n\end{aligned}$$

Sec 5.2Areas as Limits of Sums :

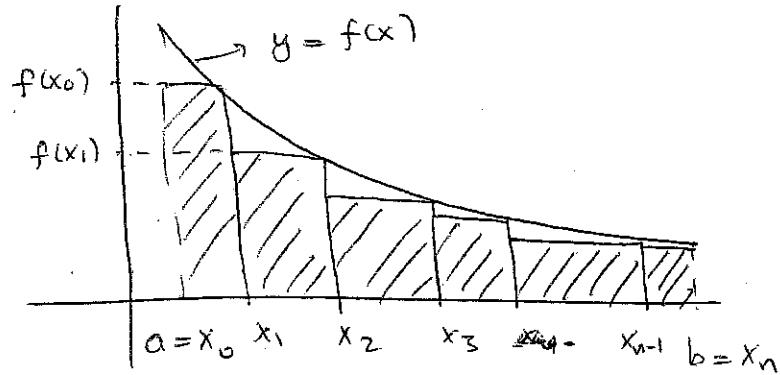
If a region has a curved boundary its area can only be approximated by using rectangles or triangles and calculating the exact area requires the evaluation of limit.

The Basic Area Problem:



To find area of region R
lying under graph of $y = f(x)$
(f is nonnegative, continuous)
above x -axis and between the
vertical lines $x=a$ and $x=b$

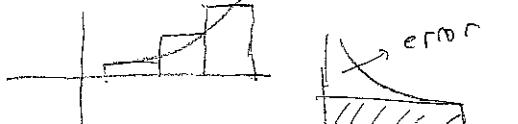
For a decreasing function $y = f(x)$



Divide $[a, b]$ into n subintervals

$$\Delta x_i := x_i - x_{i-1} \quad (i=1, \dots, n)$$

For increasing function,



Area of rectangle $(\Delta x_i) f(x_i)$

$$S_n = f(x_1) \Delta x_1 + f(x_2) \Delta x_2 + \dots + f(x_n) \Delta x_n = \sum_{i=1}^n f(x_i) \Delta x_i$$



approximation to the area of region R .

Δx_i

* Approximation gets better as n increases and the width of the widest rectangle approaches to zero.

$\text{Area of } R = \lim_{n \rightarrow \infty} S_n$ $\max \Delta x_i \rightarrow 0$
--

If all the subintervals Δx_i will have equal length

then,

$$\boxed{\Delta x_i = \frac{b-a}{n} \quad x_i = a + i \cdot \Delta x \quad i=0, 1, \dots, n}$$

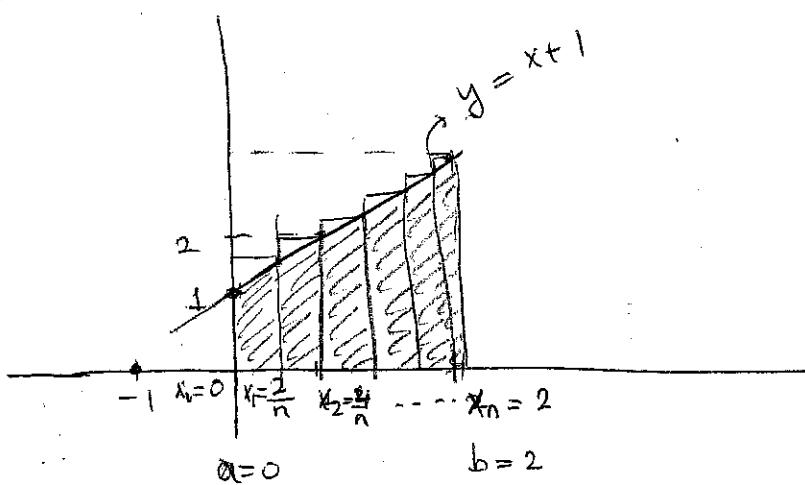
Some Area Calculations:

Ex: Find the area of the region lying under the straight line $y = x+1$, above the x -axis and between the lines $x=0$ and $x=2$.

Soln: $y = x+1$

Divide the interval $[0, 2]$

into n subintervals of equal length.



$$\Delta x_i = \frac{2-0}{n} = \frac{2}{n} \quad x_0 = 0 + 0 \cdot \frac{2}{n} = 0$$

$$x_1 = 0 + 1 \cdot \frac{2}{n} = \frac{2}{n}$$

$$x_2 = 0 + 2 \cdot \frac{2}{n} = \frac{4}{n}$$

$$x_n = 0 + n \cdot \frac{2}{n} = 2$$

$$x_i = 0 + \Delta x_i = 0 + \frac{2}{n} \cdot i$$

$$f(x_i) = x_i + 1 = \frac{2i}{n} + 1$$

$i^{\text{th}} \text{ subinterval } \left[\frac{2(i-1)}{n}, \frac{2i}{n} \right] \Rightarrow \Delta x_i = \frac{2i}{n} - \frac{2(i-1)}{n} = \frac{2}{n}$

$\downarrow \quad \downarrow$

$x_{i-1} \quad x_i$

$\Delta x_i = \frac{2}{n}$

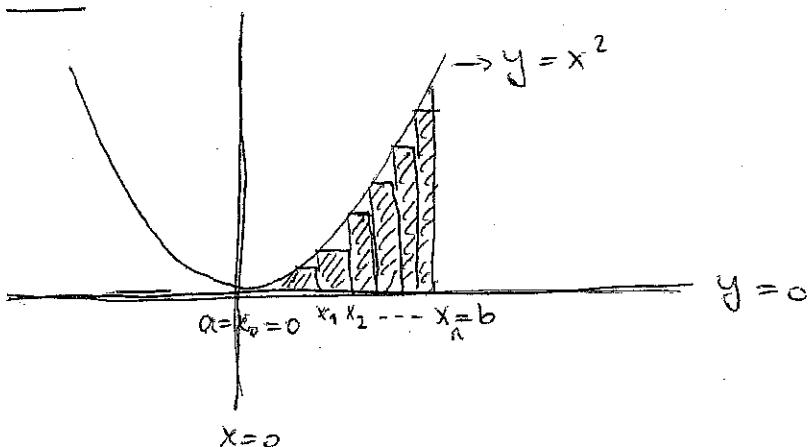
$$n \rightarrow \infty \quad \Delta x_i \rightarrow 0$$

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i) \Delta x_i \\ &= \sum_{i=1}^n \left(\frac{2i}{n} + 1 \right) \frac{2}{n} \\ &= \frac{2}{n} \left[\frac{2}{n} \sum_{i=1}^n i + \sum_{i=1}^n 1 \right] \\ &= \frac{2}{n} \left[\frac{2}{n} \left(\frac{n(n+1)}{2} \right) + n \right] = \frac{2}{n} \left[\frac{n(n+1)}{2} + n \right] = 4 + \frac{2}{n}. \end{aligned}$$

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(4 + \frac{2}{n} \right) = 4$$

Ex: Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y=0$, $x=0$ and $x=b$, $b > 0$

Soln:



$$y = f(x) = x^2 \quad [0, b]$$

$$\Delta x_i = \frac{b - 0}{n} = \frac{b}{n}$$

$$x_0 = 0 + 0 \cdot \frac{b}{n} = 0$$

$$x_1 = 0 + 1 \cdot \frac{b}{n} = \frac{b}{n}$$

$$\vdots$$

$$x_n = 0 + n \cdot \frac{b}{n} = b$$