CHAPTER 1

BASIC DEFINITIONS

1.1 Introduction

At the begining of the 19. century, the function ${}_2F_1(a, b; c; x)$ is the first Hypergeometric function to be studied by Gauss and so is frequently known as Gauss's hypergeometric function. Subsequently, most mathematicians had studied about hypergeometric functions and several deifferent forms and applications of them had been obtained.

Almost all of the elementary functions of mathematics are either hypergeometric or ratios of hypergeometric functions. This is the importance of hypergeometric functions.

The aim of this thesis is investigation of basic and elementary properties of hypergeometric functions. Anyone, who want to study further and harder properties of htpergeometric functions, this thesis can be a first step of them.

At the first chapter, some basic definitions and theorems that will be need at the other chapters are given.

At the second chapter, first convegency of the Hypergeometric Series for $|x| \leq 1$ is shown and then the sum of Hypergeometric series which are called Hypergeometric functions is obtained.

At the third chapter, it will be shown that the solutions of Hypergeometric Differential Equations are given by Hypergeometric Function $F(a, b, c; x)$.

Additional properties of the hypergeometric fumctions are going to be obtained at the fourth chapter.

Finally, definitions and properties of generalized hypergeometric functions that they can de reduced to the hypergeometric functions by special choice of parameters are given.

In this Chapter, the basic definitions and theorems about the topics that it will be needed while obtaining the properties of Hypergeometric functions will be given.

1.2 Series and Power Series

First, start with the basic idea of series. (Adams, R. A. & Essex, C. , 2010)

Definition 1.1

In mathematics, given an infinite sequence of numbers { an }, a series is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + ...$ These can be written more compactly using the summation symbol $\sum\limits_{n=1}^{\infty} a_n$.

Definition 1.2

A series $\sum a_n$ is convergent if its sequence of partial sums $\{S_n\}$ converges that is, if $\lim\limits_{n\to\infty} S_n = S$ for some real number S. The limit S is the sum of the series $\sum a_n$ and we write

$$\sum_{k=1}^{\infty} a_k = S = a_1 + a_2 + ... + a_n \tag{1.1}$$

Thus, if the sequence of partial sums of a series converges then the series converges. The converse of this implication is true.

Another definition about the convergence of a seris by using the absolute value of series can be given.

Definition 1.3

A series $\sum\limits_{n=0}^{\infty} a_n$ is said to converge absolutely if the series $\sum\limits_{n=0}^{\infty} |a_n|$converges, where $|a_n|$ denotes the absolute value.

It is known that, if a series ia absolutely convergent then it is convergent.

Some infinite series have positive general terms. These type of series are called positive series. There is a usefull test for the convergence of positive series. Now, let define this test which is called ratio test.

Definition 1.4

Suppose that $a_n > 0$ and that $\rho = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is $+\infty$.

a) If $0 \leq \rho < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

b) If $1 < \rho \leq \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum\limits_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $\rho = 1$, this test gives no information; the series may either converge or diverge to infinity. This test is called the ratio test.

There is an alternative test for the convergence and absolute convergence of the series which is called Gauss Test. (Kosmala, 2004)

Definition 1.5

If $\left|\frac{a_n}{a_{n+1}}\right| = 1+\frac{h}{n}+\frac{C_n}{n^r}$ where $r > 1$ and C_n is bounded, then the series $\sum a_n$ convergent if $h > 1$ and diverges if $h \leq 1$. It is same with $\left|\frac{a_{n+1}}{a_n}\right| = 1-\frac{p}{n}+\frac{B(n)}{n^r}$ the series converges absolutely if and only if $p > 1$. This test is called Gauss Test.

Some infinite series consist variable x and so the series can be convergent or divergent with respect to the values of x. Now, the definition of these special kind of infinite series which are called power series is given.

Definition 1.6

A series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + ... \qquad (1.2)$$

is called a power series in powers of $x - c$ or a power series about the point $x = c$. The constants a_0, a_1, a_2, a_3...are called the coefficients of the power series. The point c is the centre of covergence of the power series.

The convergence of a power series depends to the values of x. It means there can be an interval that the power series is converge.

Theorem 1.1

For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ one of the following alternatives must hold:

(i) the series may converge only at $x = c$,

(ii) the series may converge at every real number x, or

(iii) there may exist a positive real number R such that the series converges at every x satisfying $|x-c| < R$ and diverges at every x satisfying $|x-c| > R$. In this case the series may or may not converge at either of the two endpoints $x = c - R$ and $x = c + R$.

By Theoem 1.1, the set of values x for which the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges is an interval centered at $x = c$. This interval is called the interval of convergence of the power series. It must be of one of the following forms:

(i) the isolated point $x = c$

(ii) the entire line $(-\infty, \infty)$

(iii) a finite interval centered at c, $[c-R, c+R]$, or $[c-R, c+R)$, or $(c-R, c+r]$, or $(c-R, c+r)$.

The number R in (iii) is called the radius of convergence of the power series.

Definition 1.7

Suppose that $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exist or is ∞. Then the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R = \frac{1}{L}$.

For later works, some rules about elementary series manupilations will be needed. Here are two basic lemmas about rearrangement of terms in iterated series with their proofs. (Rainville, E. D. (1965))

Lemma 1.1

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{1.3}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k) \tag{1.4}$$

Proof

Let consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\ t^{n+k} \tag{1.5}$$

If introduce new indices of summation j and m by

$$k = j \quad , \quad n = m - j \tag{1.6}$$

then the exponent $(n+k)$ in (1.3) becomes m. Because of (1.6), the inequalities

$$n \geq 0 \quad , \quad k \geq 0$$

become

$$m - j \geq 0 \quad , \quad j \geq 0$$

or $0 \leq j \leq m$. Thus arrive at

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\ t^{n+k} = \sum_{m=0}^{\infty}\sum_{j=0}^{m} A(j,m-j)\ t^{m}$$

and by putting $t = 1$ and replacing dummy indices j and m on the right by dummy indices k and n, (1.3) is obtained. (1.4) can be obtained by writing the (1.3) in reverse.

Lemma 1.2

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n-2k) \tag{1.7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+2k) \tag{1.8}$$

Proof

Consider the seires

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\ t^{n+2k} \tag{1.9}$$

and introduce new indices as

$$k=j \quad , \quad n=m-2j \tag{1.10}$$

so that n+2k=m. Since

$$n \geq 0 \quad , \quad k \geq 0 \tag{1.11}$$

it is concluded that

$$m-2j \geq 0 \quad , \quad j \geq 0 \tag{1.12}$$

from which $0 \leq 2j \leq m$ and $m \geq 0$. Since $0 \leq j \leq \frac{1}{2}m$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\ t^{n+2k} = \sum_{m=0}^{\infty}\sum_{j=0}^{\left[\frac{1}{2}m\right]} A(j,m-2j)\ t^{m} \tag{1.13}$$

is obtained. By taking $t=1$ and making the proper change of letters for the dummy indices on the right side in (1.13), (1.7) is obtained. Equation (1.8) is (1,7) in reverse order.

Note also that a combination of Lemmas 1.1 and Lemma 1.2 gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} C(k,n-k).$$

Now, definition of another important series as binomial series is given.

Theorem 1.2

If $|x| < 1$, then

$$\begin{aligned}(1+x)^r &= 1 + r\ x + \frac{r\ (r-1)}{2!}x^2 + \frac{r\ (r-1)\ (r-2)}{3!}x^3 + ... \\ &= 1 + \sum_{n=0}^{\infty} \frac{r\ (r-1)\ (r-2)\ ...(r-n+1)}{n!}\ x^n \quad , \quad (-1 < x < 1)\end{aligned}$$

If take $x = -z$ and $r = -a$, particılarly it can be obtain

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n\ z^n}{n!} \tag{1.14}$$

1.2 Gamma Function

Now, the definition of a special function which is defined by an improper integral is going to be given. This function is called Gamma Function and has several applications in Mathematics and Mathematical Physics.(Marcellan, F. & Van Assche, W. (2006))

Definition 1.8

The improper integral

$$\int_0^{\infty} t^{x-1}e^{-t}dt \tag{1.15}$$

converges for $x > 0$. The function which gives the value of (1.15) respect to x is called "Gamma Function" and denoted by Γ:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt \tag{1.16}$$

Several properties of the Gamma function can be obtained easily. Now we will give some basic properties of Gamma function without their proofs.

1.

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t} dt = n! \tag{1.17}$$

where n is a positive integer.

2.

$$\Gamma(x+1) = x\ \Gamma(x) \tag{1.18}$$

where $-1 < x$.

3.

$$\Gamma(b).\Gamma\left(b+\frac{1}{2}\right) = 2^{1-2b}.\sqrt{\pi}.\Gamma(2b) \tag{1.19}$$

where b is a non-negative integer.

4.

$$\Gamma\left(b+\frac{1}{2}n\right)\ \Gamma\left(b+\frac{1}{2}n+\frac{1}{2}\right) = 2^{1-2b-n}\ \sqrt{\pi}\ \Gamma(2b+n) \tag{1.20}$$

where $Re(b) > 0$ and n is a non-negative integer.

5.

$$\Gamma(a) = \frac{(n-1)!\ n^a}{(a)_n} \tag{1.21}$$

where $Re(a) > 0$ and n is a non-negative integer.

In the theory of special functions and applied mathematics, there is a usefull symbol which is called Pochammer symbol. (Andrews, G. E. & Askey, R. & Roy, R. (1999))

Definition 1.9

Let x is a real or complex number and n is a positive number or zero.

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1).....(x+n-1) \tag{1.22}$$

$$(x)_0 = 1$$

is known as "Pochammer Symbol".

Let give some properties of this symbol.

1.

$$\frac{(c)_{n+k}}{(c)_n} = (c+n)_k$$

where c is a real or complex number and n and k are natural numbers.

2.

$$\frac{n!}{(n-k)!} = \frac{(-n)_k}{(-1)^k}$$

where n and k are natural numbers.

3.

$$\frac{(c)_{2k}}{2^{2k}} = \left(\frac{c}{2}\right)_k \left(\frac{c}{2}+\frac{1}{2}\right)_k$$

where c is a complex number and k is a natural number.

4.

$$\frac{(2k)!}{2^{2k}\ k!} = \left(\frac{1}{2}\right)_k$$

where k is a natural number.

Lemma 1.3

For a non-negative number n

$$(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n \tag{1.23}$$

Proof

$$\begin{aligned}
(\alpha)_{2n} &= \alpha\ (\alpha+1)\ (\alpha+2)\ ...\ (\alpha+2n-1) \\
&= 2^{2n}\ \left(\frac{\alpha}{2}\right)\ \left(\frac{\alpha+1}{2}\right)\ \left(\frac{\alpha}{2}+1\right)\ \left(\frac{\alpha+1}{2}+1\right) ... \left(\frac{\alpha}{2}+n-1\right)\ \left(\frac{\alpha+1}{2}+n-1\right) \\
&= 2^{2n}\ \left(\frac{\alpha}{2}\right) \left(\frac{\alpha}{2}+1\right) ... \left(\frac{\alpha}{2}+n-1\right)\ \left(\frac{\alpha+1}{2}\right) \left(\frac{\alpha+1}{2}+1\right)\ ... \left(\frac{\alpha+1}{2}+n-1\right) \\
&= 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n
\end{aligned}$$

Lemma 1.4

For $0 \leq k \leq n$,

$$(\alpha)_{n-k} = \frac{(-1)^k\ (\alpha)_n}{(1-\alpha-n)_k} \tag{1.24}$$

Proof

By definition (1.20)

$$(\alpha)_{n-k} = (\alpha)\ (\alpha+1)\ ...\ (\alpha+n-k-1)$$

If multiply and divide the right hand side of above equality with

$$(\alpha+n-k)\ (\alpha+n-k+1)\ ...\ (\alpha+n-1)$$

then

$$
\begin{aligned}
(\alpha)_{n-k} &= \frac{(\alpha)\ (\alpha+1)\ ...\ (\alpha+n-k-1)\ (\alpha+n-k)\ (\alpha+n-k+1)\ ...\ (\alpha+n-1)}{(\alpha+n-k)\ (\alpha+n-k+1)\ ...\ (\alpha+n-1)} \\
&= \frac{(\alpha)_n}{(-1)^k\ (1-n-\alpha)\ \ ...\ (1-n-\alpha+k)\ (1-n-\alpha+k-1)} \\
&= \frac{(-1)^k\ (\alpha)_n}{(1-\alpha-n)_k}
\end{aligned}
$$

is obtained.

Particularly, from Lemma 1.4, for $\alpha = 1$,

$$(1)_{n-k} = \frac{(-1)^k\ (1)_n}{(-n)_k}$$

and

$$(n-k)! = \frac{(-1)^k\ \ n!}{(-n)_k} \quad , \quad 0 \le k \le n \tag{1.25}$$

1.3 Differential Equations

In the theory of second order linear differential equation with variable coefficients, series method solutions are important. For these solutions, let give the definitions of critical points of an second order differential equation with variable coefficients.(Agarwa, R. P. & O'Regan, D. (2009))

Definition 1.10

Consider a second-order ordinary differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1.26}$$

It will be assumed that x_o is a regular singular point for this equation. The definition of regular singular point implies the following three conditions.

1) $P(x_o) = 0$

2) $\lim\limits_{x \to x_0} \frac{(x-x_0).Q(x)}{P(x)}$ exist

3) $\lim\limits_{x \to x_0} \frac{(x-x_0)^2.R(x)}{P(x)}$

A point x_0 is an ordinary point for $P(x)y'' + Q(x)y' + R(x)y = 0$ if $P(x) \neq 0$.

For solving differential equations by series method, there are two types of series..

1. If x_0 is an ordinary point for (1.26), then we can find a solution in the neighborhood of x_0 for the differential equation (1.26) by using the Taylor Series Method where $y = \sum\limits_{n=0}^{\infty} a_n (x - x_0)^n$.

2. If x_0 is a regular singular point for the differential equation (1.26), then we can find a solution in the neighborhood of x_0 for the differential equation (1.26) by using the Frobenius Series Method where $y = \sum\limits_{n=0}^{\infty} a_n (x - x_0)^n$.

CHAPTER 2

HYPERGEOMETRIC SERIES

Before obtaining the Hypergeometric functions, it needs to be shown that the series which give Hypergeometric functions convergent and have finite sum. In this chapter, the convergency of hypergeometric series and then find the sum of them will be investigated. (Kummer, E. E. (1836); Rainville, E. D. (1965))

2.1 Definition of Hypergeometric Series

Definition 2.1

A series of the form

$$1 + \frac{ab}{c.1!}x + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}x^3 + ...$$

$$= \left(\sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)m!} x^m \right) \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \tag{2.1}$$

is called a hypergeometric series.

This sum can be obtained by using properties of the Gamma function . The series from the right side of the equation (2.1) is in the form of

$$\sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)m!} x^m$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} + \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(c+1)1!}x + \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(c+2)2!}x^2 + ... \tag{2.2}$$

If multipling both sides of (2.2) by $\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}$ and using the property (1.17), then

$$\begin{aligned}
\left(\sum_{m=0}^{\infty} \tfrac{\Gamma(a+m)\ \Gamma(b+m)}{\Gamma(c+m)\ m!} x^m\right) \tfrac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} &= \left(\tfrac{\Gamma(a)\ \Gamma(b)}{\Gamma(c)} + \tfrac{\Gamma(a+1)\ \Gamma(b+1)}{\Gamma(c+1)\ 1!} x + ...\right) \cdot \tfrac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \\
&= \left(\tfrac{\Gamma(a)\ \Gamma(b)}{\Gamma(c)} \tfrac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)}\right) + \left(\tfrac{a!\ b!}{c!} x\right) \left(\tfrac{(c-1)!}{(a-1)!\ (b-1)!}\right) \\
&+ \left(\tfrac{(a+1)!\ (b+1)!}{(c+1)!\ 2!}\ x^2\right) \left(\tfrac{(c-1)!}{(a-1)!\ (b-1)!}\right) + ... \\
&= 1 + \tfrac{a\ b}{c} x + \tfrac{a\ (a+1)\ b\ (b+1)}{c\ (c+1)\ 2!} x^2 + ...
\end{aligned}$$

is obtained which gives (2.1).

2.2 Sum of Hypergeometric Series

Now, first let give the definition of Hypergeometric function and then show that the Hypergeometric series converges to this function.

Definition 2.2

The sum of the hypergeometric series denoted by $F(a, b; c; x)$ is called Hypergeometric Function

$$F(a, b, c; z) = \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m\ m!} x^m \tag{2.3}$$

Now, investigate where the series (2.3) converge. For this, let we use the ratio test. The ratio of the coefficients of $x^{m+1}$and x^m in the series (2.3) is

$$\begin{aligned}
&\frac{(a)_{m+1}\,(b)_{m+1}}{(c)_{m+1}\ (m+1)!} \frac{(c)_m\ \ m!}{(a)_m\ \ (b)_m} \\
= &\frac{(a+m)\,(b+m)}{(c+m)\,(m+1)}
\end{aligned} \tag{2.4}$$

which tends to 1 uniformly as $m \to \infty$, regardless of the values of a, b and c. So by ratio test, for

$$u_m = \frac{(a)_m(b)_m}{(c)_m m!} x^m$$

$$\lim_{m\to\infty}\left|\frac{u_{m+1}}{u_m}\right| = \lim_{m\to\infty}\left|\frac{(a+m)(b+m)}{(c+m)(m+1)}.x\right| = |x|$$

can be obtained and for $|x| < 1$ the series (2.3) converges.

Also, (2.4) can be written as in the form of

$$1 - \frac{1+c-a-b}{m} + O(\frac{1}{m^2}) \tag{2.5}$$

Really,

$$\begin{aligned} \frac{(m+a)\ (m+b)}{(m+1)\ (m+c)} &= \left(\left(1+\tfrac{a}{m}\right)\left(1+\tfrac{b}{m}\right)\right)\left(\left(1+\tfrac{1}{m}\right)\left(1+\tfrac{c}{m}\right)\right)^{-1} \\ &= \left(1+\tfrac{a+b}{m}+O\left(\tfrac{1}{m^2}\right)\right)\left(1-\tfrac{1+c}{m}+O\left(\tfrac{1}{m^2}\right)\right) \\ &= 1+\tfrac{a+b-c-1}{m}+O\left(\tfrac{1}{m^2}\right) \end{aligned}$$

Thus, the series (2.3) converges absolutely at $x = \pm 1$ by the Gauss test if $c > a + b$. Therefore, the series (2.3) is convergent for $|x| \leq 1$.

CHAPTER 3

THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

It is known that, a second order linear differential equation can be solved by series method. Thus, a convergent series could be a solution of any second order linear differential equation. In this Chapter, the solutions of the Hypergeometric differential equations will be obtain as Hypergeometric series.(Rainville, E. D. (1965))

Now, let start to solve the hypergeometric differential equation

$$x(1-x)y'' + \left[c - (a+b+1)\,x\right] y' - aby = 0 \tag{3.1}$$

where a, b, c are parameters. It is clear that $x = 0$ and $x = 1$ are regular singular points of the equation (2.6) whereas all other points are ordinary points.

$$P(0) = 0\ (1-0) = 0$$

$$\lim_{x \to 0} \frac{(x-0)\ \left[c - (a+b+1)\,x\right]}{x\ (1-x)} = c$$

and

$$\lim_{x \to 0} \frac{(x-0)^2\ (-ab)}{x\ (1-x)} = 0$$

gives that $x = 0$ is a regular singular point. Similarly, it can be shown that $x = 1$ is a regular singular point, too.

The equation (2.6) can be solved by using the Frobenius series method at a neigborhood of regular singular point $x = 0$. By using the series

$$y(x) = x^r \sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{\infty} c_m x^{r+m} \tag{3.2}$$

then the derivatives

$$y' = \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} \tag{3.3}$$

and

$$y'' = \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m-2} \tag{3.4}$$

can be obtained. If (3.2), (3.3) and (3.4) are substituted into the equation (3.1), it will be obtained that

$$\begin{aligned} & x \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m-2} - x^2 \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m-2} \\ & + c \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} - (a+b+1)\, x \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} - ab \sum_{m=0}^{\infty} c_m x^{r+m} \\ & = 0 \end{aligned} \tag{3.5}$$

In the equation (3.5), if the x terms inside the summations are rewritten, then

$$\begin{aligned} & \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m-1} - \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m} \\ & + c \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} - (a+b+1) \sum_{m=0}^{\infty} c_m (r+m) x^{r+m} - ab \sum_{m=0}^{\infty} c_m x^{r+m} \\ & = 0 \end{aligned} \tag{3.6}$$

Now, in the equation (3.6), equalize the powers of x to the smallest power $r+m-1$, and obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} c_m (r+m)(r+m-1) x^{r+m-1} - \sum_{m=1}^{\infty} c_m (r+m-1)(r+m-2) x^{r+m-1} \\ & + c \sum_{m=0}^{\infty} c_m (r+m) x^{r+m-1} - (a+b+1) \sum_{m=1}^{\infty} c_{m-1} (r+m-1) x^{r+m-1} \\ & - ab \sum_{m=1}^{\infty} c_{m-1} x^{r+m-1} \\ & = 0 \end{aligned} \tag{3.7}$$

If the series start from $m = 1$, then (3.7) can be written as

$$
\begin{aligned}
&c_0(r)(r-1)x^{r-1} + \sum_{m=1}^{\infty} c_m(r+m)(r+m-1)x^{r+m-1} \\
&- \sum_{m=1}^{\infty} c_{m-1}(r+m-1)(r+m-2)x^{r+m-1} + c(c_0 r)x^{r-1} + c\sum_{m=1}^{\infty} c_m(r+m)x^{r+m-1} \\
&- (a+b+1)\sum_{m=1}^{\infty} c_{m-1}(r+m-1)x^{r+m-1} - ab\sum_{m=1}^{\infty} c_{m-1}x^{r+m-1} \\
&= c_0(r(r-1)+rc)x^{r-1} + \sum_{m=1}^{\infty} c_m(r+m)(r+m-1)x^{r+m-1} \\
&- \sum_{m=1}^{\infty} c_{m-1}(r+m-1)(r+m-2)x^{r+m-1} + c\sum_{m=1}^{\infty} c_m(r+m)x^{r+m-1} \\
&-(1+a+c)\sum_{m=1}^{\infty} c_{m-1}(r+m-1)x^{r+m-1} - ab\sum_{m=1}^{\infty} c_{m-1}x^{r+m-1} = 0
\end{aligned}
\tag{3.8}
$$

So the indical equation and its roots are obtain as

$$r(r-1) + cr = 0 \tag{3.9}$$

and

$$r_1 = 0 \qquad r_2 = 1 - c \tag{3.10}$$

Also from the rest of terms in the equation (3.8), we can obtain the recurrence relation

$$
\begin{aligned}
&[(r+m)(r+m-1) + c(r+m)]\, c_m \\
&+ [-(r+m-1)(r+m-2) - (a+b+1)(r+m-1) - ab]\, c_{m-1} \\
&= 0
\end{aligned}
$$

or

$$
\begin{aligned}
&[(r+m+1)(r+m) + c(r+m+1)]\, c_{m+1} \\
&= [(r+m)(r+m-1) + (a+b+1)(r+m) + ab]\, c_m
\end{aligned}
$$

and so

$$(r+m+1)\,(r+m+c)\,c_{m+1} = (r+a+m)(r+b+m)c_m \quad (for \;\; m = 0, 1, ...) \tag{3.11}$$

Therefore, by taking $m = 0, 1, 2, ...,$

$$c_{m+1} = \frac{(a+m)\;(b+m)}{(c+m)\;(m+1)}\;c_m$$

and

$$c_1 = \frac{a\;b}{c}\;c_0$$

$$c_2 = \frac{(a+1)\;(b+1)}{(c+1)\;2}\;c_1$$

$$.....$$

So, for the exponent $r_1 = 0$, the recurrence relation (3.11) leads to the solution $c_0F(a, b, c, x)$. Taking $c_0 = 1$, first solution of the differential equation (3.1) can be found as

$$y(x) = F(a, b, c, x) \tag{3.12}$$

The second solution with the exponent $r_2 = 1 - c$, when c is neither zero nor a negative integer, can be obtained as follows: In the differential equation (3.1), by using the substitution

$$y = x^{1-c}w \tag{3.13}$$

with

$$y' = (1-c)x^{-c}.w + x^{1-c}.w$$

and

$$\begin{aligned} y'' &= (1-c)(-c)x^{-c-1}w + (1-c)\ x^{-c}\ w' + (1-c)\ x^{-c}\ w' + x^{1-c}\ w'' \\ &= c(c-1)x^{-c-1}\ w + 2(1-c)\ w' + x^{1-c}\ w'' \end{aligned}$$

then

$$\begin{aligned} &x\ (1-x)\ c\ (c-1)\ x^{-c-1}\ w + 2\ (1-c)\ x^{-c}\ x\ (1-x)\ w' + x\ (1-x)\ x^{1-c}\ w'' \\ &+ [c-(a+b+1)\ x]\ (1-c)\ x^{-c}\ w + [c-(a+b+1)x]\ \ (1-c)\ x^{1-c}\ w' \\ &-a\ b\ x^{1-c}\ w = 0 \end{aligned}$$

is obtained. After making some arrangements, we obtain

$$\begin{aligned} &x\ (1-x)\ x^{1-c}\ w'' + [2\ (1-c)\ x^{-c}\ x\ (1-x) + [c-(a+b+1)\ x]\ \ x^{1-c}]\ \ w' \\ &+ [x\ (1-x)\ c\ (c-1)\ x^{-c-1} + [c-(a+b+1)\ x](1-c)\ x^{-c} - a\ b\ x^{1-c}]\ \ w \\ &= 0 \end{aligned} \tag{3.14}$$

If divide both sides of the equation (3.14) by x^{1-c}, second order differential equation can be obtained as

$$\begin{aligned} &x(1-x).w'' + [1-c-[(a-c+1)+(b-c+1)+1]\,x]\,w' \\ &-(a-c+1)(b-c+1)w = 0 \end{aligned} \tag{3.15}$$

If pick

$$a_1 = a-c+1 \qquad b_1 = b-c+1 \qquad c_1 = 2-c$$

then the equation (3.15) turns in the form of

$$x(1-x)w'' + [c_1-(a_1+b_1+1)x]\,w' - a_1 b_1 w = 0 \tag{3.16}$$

It is known that the differential equation (3.16) is the Hypergeometric Differential Equation and has the solution

$$w(x) = F(a_1, b_1, c_1, x) \tag{3.17}$$

If the values of a_1, b_1, c_1 and the substitution (3.13) are written, then the solution of the Hypergeometric differential equation can be obtained as

$$y(x) = x^{1-c}F(a-c+1, b-c+1, 2-c, x) \tag{3.18}$$

The solutions (3.12) and (3.18) are linearly independent. So, the general solution of the Hypergeometric Differential Equations about $x = 0$ is written as

$$y(x) = c_1F(a, b, c; x) + c_2x^{1-c}F(a-c+1, b-c+1, 2-c, x) \tag{3.19}$$

For the solution at the singular point $x = 1$, it is deduced from the preceding solutions by a change of independent variable $t = 1-x$.Let try to express the solutions of this case in terms of the solutions for the point $x = 0$. Let $t = 1 - x$ Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{dy}{dt}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}(-\frac{dy}{dt}) = \frac{d}{dt}\frac{dt}{dx}(-\frac{dy}{dt}) = \frac{d^2y}{dt^2}$$

So, the equation (3.1) can be written as

$$t\,(1-t)\,\frac{d^2y}{dt^2} - (c_1 - (1+a+b)(1-t))\frac{dy}{dt} - aby = 0 \tag{3.20}$$

If $c_1 = a + b - c + 1$ is taken, it turns the equation (3.1) and has the solutions

$$y(x) = F(a, b, a+b-c+1, 1-x) \tag{3.21}$$

and

$$y(x) = (1-x)^{c-a-b}F(c-b, c-a, c-a-b+1, 1-x) \tag{3.22}$$

where $c-a-b$ is not a positive integer. So, the general solution of the Hypergeometric Differential Equations near the regular singular point $x = 1$ is

$$y(x) = c_3F(a, b, a+b-c+1, 1-x)+c_4(1-x)^{c-a-b}F(c-b, c-a, c-a-b+1, 1-x) \quad (3.23)$$

CHAPTER 4

ADDITIONAL PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

In the theory of special functions and applied mathematics, some properties of Hypergeometric function make important roles. Sometimes, alternative definitions or new explicit formulas of the Hypergeometric function are usefull. In this chapter, several properties of the hypergeometric functions are going to be obtained.

For our later works of this chapter, an important terminated value of hypergeometric functions which can be obtained by changing the values of the variables will be needed. (Rainville, 1965)

Theorem 4.1

$$F(-n,\ c-b;\ c;\ 1) = \frac{\Gamma(c)\ \Gamma(b+n)}{\Gamma(c+n)\ \Gamma(b)} \tag{4.1}$$

4.1 A Simple Transformation for Hypergeometric Function

In this section, we are going to give a theorem about a transformation between Hypergeometric functions with different variables. Then, we are going to obtain an equality for Hypergeometric functions. (Rainville, 1965)

Theorem 4.2

If $|z| < 1$ and $\left|\frac{z}{1-z}\right| < 1$, then

$$F\left[\begin{matrix} a,\ b; & \\ c; & \end{matrix} z\right] = (1-z)^{-a}\ F\left[\begin{matrix} a,\ c-b; & \\ c; & \end{matrix} \frac{-z}{1-z}\right] \tag{4.2}$$

Proof

It is known that

$$F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a)_k \, (c-b)_k \ \ (-1)^k \ z^k}{(c)_k \ \ k! \ (1-z)^k}$$

If both sides multiplied by $(1-z)^{-a}$, then

$$(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a)_k \, (c-b)_k \ \ (-1)^k \ z^k}{(c)_k \ \ k! \ (1-z)^{k+a}} \tag{4.3}$$

is obtained. By (1.14),

$$(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_k \, (c-b)_k \ \ (a+k)_n \ (-1)^k \ z^{n+k}}{(c)_k \ \ k! \ n!}$$

and from the first properties of Poachammer symbol, $(a)_k \ (a+k)_n = (a)_{n+k}$,

$$(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c-b)_k \ \ (a)_{n+k} \ (-1)^k \ z^{n+k}}{(c)_k \ \ k! \ n!}$$

can be written. By using (1.3) from Lemma 1.1, and the equality (1.24)

$$\begin{aligned}(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c-b)_k \ \ (a)_n \ (-1)^k \ z^n}{(c)_k \ \ k! \ (n-k)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c-b)_k \ \ (-n)_k}{(c)_k \ \ k!} \frac{(a)_n \ z^n}{n!}\end{aligned}$$

can be obtained. The inner sum on the right side of above equality gives a terminating hypergeometric series. Hence

$$(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} F\left[\begin{matrix} -n, \ c-b; & 1 \\ c; & \end{matrix}\right] \frac{(a)_n \ z^n}{n!}$$

By Theorem 4.1,

$$
\begin{aligned}
(1-z)^{-a} \ F\left[\begin{matrix} a, \ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] &= \sum_{n=0}^{\infty} \frac{\Gamma(c) \ \Gamma(b+n) \ (a)_n \ z^n}{\Gamma(c+n) \ \Gamma(b) \ n!} \\
&= \sum_{n=0}^{\infty} \frac{(b)_n \ (a)_n}{(c)_n \ n!} z^n \\
&= F(a, b; c; z)
\end{aligned}
$$

which gives the proof.

The roles of a and b may be interchanged in Theorem 4.2.

$$
F\left[\begin{matrix} a, \ b; & z \\ c; & \end{matrix}\right] = (1-z)^{-b} \ F\left[\begin{matrix} c-a, \ b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] \tag{4.4}
$$

A result of Theorem 4.2 can be given as a new theorem.

Theorem 4.3

If $|z| < 1$, then

$$
F(a, \ b; \ c; \ z) = (1-z)^{c-b} \ F\left(c-a, \ c-b; \ c; \ z\right) \tag{4.5}
$$

Proof

Let apply the new form of Theorem 4.2 which was given by (4.4) to the Hypergeometric function on the right in (4.2). If put

$$
y = \frac{-z}{1-z}
$$

then

$$
F\left[\begin{matrix} a, \ c-b; & y \\ c; & \end{matrix}\right] = (1-y)^{-c+b} \ F\left[\begin{matrix} c-a, \ c-b; & \dfrac{-y}{1-y} \\ c; & \end{matrix}\right]
$$

Because of $1-y=(1-z)^{-1}$ and $-y/(1-y)=z$,

$$F\left[\begin{matrix} a,\ c-b; & \dfrac{-z}{1-z} \\ c; & \end{matrix}\right] = (1-z)^{c-b}\ F\left[\begin{matrix} c-a,\ c-b; & z \\ c; & \end{matrix}\right]$$

can be written. If combine this result with Theorem 4.2, then the desired result is obtained.

4.2 A Quadratic Transformation for Hypergeometric Function

By using a linear fractional transformation on the independent variable, it can be studied the transformations of equation (3.1) into itself. By using quadratic transformations and relations among a, b and c, several properties of Hypergeometric functions can be obtained as one of them is given by the following theorem.

Theorem 4.5

If $2b$ is neither zero nor a negative integer, and if both $|x|<1$ and $\left|4x\,(1+x)^{-2}\right|<1$, then

$$(1+x)^{-2a}\ F\left[\begin{matrix} a,\ b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = F\left[\begin{matrix} a,\ a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}; & \end{matrix}\right] \tag{4.6}$$

Proof

If put $c=2b$ in the equation (4.1), then

$$z(1-z)w'' + [2b-(a+b+1)\,z]\,w' - abw = 0 \tag{4.7}$$

obtained. It is known that one solution of the equation (4.2) is

$$w = F(a,b;2b;z) \tag{4.8}$$

Now, let take

$$z = \frac{4x}{(1+x)^2} \tag{4.9}$$

and apply this subtitution to the equation (4.2).

$$w' = \frac{dw}{dz} = \frac{(1+x)^3}{4-4x}.\frac{dw}{dx} \tag{4.10}$$

and

$$\begin{aligned} w'' &= \left[\frac{8x-16}{(1+x)^4}.\frac{dw}{dx} + \frac{4-4x}{(1+x)^3}.\frac{d^2w}{dx^2}\right]\left[\frac{(1+x)^3}{4-4x}\right] \\ &= \frac{8x-16}{(1+x)(4-4x)}\frac{dw}{dx} + \frac{d^2w}{dx^2} \end{aligned} \tag{4.11}$$

are obtained. When put (4.5) and (4.6) in (4.2),

$$\begin{aligned} & \frac{4x}{(1+x)^2}\left(1-\frac{4x}{(1+x)^2}\right)\left[\frac{8x-16}{(1+x)(4-4x)}\frac{dw}{dx} + \frac{d^2w}{dx^2}\right] \\ & + \left[2b-(a+b+1)\frac{4x}{(1+x)^2}\right]\left[\frac{(1+x)^3}{4-4x}.\frac{dw}{dx}\right] - abw \\ = {} & 0 \end{aligned}$$

After some simplifications

$$\begin{aligned} & \frac{x}{(1+x)^2}\left[\frac{(x-1)^2}{(1+x)^2}.\frac{8x-16}{(1+x)(1-x)}\frac{dw}{dx}\right] + \frac{4x\,(x-1)^2}{(1+4)^4}\frac{d^2w}{dx^2} \\ & + \left[2b-(a+b+1)x.\frac{(1+x)}{(1-x)}\frac{dw}{dx}\right] - 4abw \\ = {} & 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{8x(x-2)}{(1+x)^5}\frac{dw}{dx} + \frac{4x(1-x)}{(1+x)^4}\frac{d^2w}{dx^2} \\ & + \left[2b-(a+b+1)x\frac{(1+x)}{(1-x)^2}\frac{dw}{dx}\right] - \frac{4ab}{(1-x)}w \\ = {} & 0 \end{aligned} \tag{4.12}$$

is obtained. By multiplying both sides of the equation (4.12) with

$$\frac{(1+x)^6}{4}$$

then the differential equation is obtained as

$$x(1-x)(1+x)^2\frac{d^2w}{dx^2}+2(1+x)\left(b-2ax+bx^2-x^2\right)\frac{dw}{dx}-4ab(1-x)w=0 \quad (4.13)$$

which has a solution in the form of

$$w=F\left[\begin{matrix} a,b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] \quad (4.14)$$

If making substitution as

$$w=(1+x)^{2a}y$$

in the equation (4.8), by the derivatives

$$w'=2a(1+x)^{2a-1}y+(1+x)^{2a}y'$$

and

$$\begin{aligned} w'' &= (2a-1)(2a)(1+x)^{2a-1}y+2a(1+x)^{2a-1}y'+2a(1+x)^{2a-1}y'+(1+x)^{2a}y'' \\ &= (1+x)^{2a}y''+4a(1+x)^{2a-1}y'+(2a-1)(2a)(1+x)^{2a-1}y \end{aligned}$$

then

$$\begin{aligned} & x(1-x)(1+x)^2\left[(1+x)^{2a}y''+4a(1+x)^{2a-1}y'+(2a-1)(2a)(1+x)^{2a-1}y\right] \\ & +2(1+x)\left(b-2ax+bx^2-x^2\right)\left[2a(1+x)^{2a-1}y+(1+x)^{2a}y'\right] \\ & -4ab(1-x)w \\ = & \ 0 \end{aligned}$$

is obtained which gives

$$\begin{aligned} & x(1-x)(1+x)^{2a+2}y''+\left[4ax(1-x)+2\left(b-2ax+bx^2-x^2\right)\right](1+x)^{2a+1}y' \\ & +\left[x(1-x)(2a-1)(2a)+4a\left(b-2ax+bx^2-x^2\right)\right](1+x)^{2a}y \\ = & \ 0 \end{aligned}$$

If multipling the last equation by

$$\frac{1}{(1+x)^{2a+1}}$$

then

$$\begin{aligned} & x\,(1-x)\,(1+x)y'' + \left[4ax - 4ax^2 + 2b - 4ax + 2bx^2 - 2x^2\right] y' \\ & + \left[4a^2x - 2ax - 4a^2x^2 + 2ax^2 + 4ab - 8a^2x + 4abx^2 - 4ax^2\right] (1+x)^{-1}\, y \\ = \;& 0 \end{aligned}$$

and

$$x(1-x^2)y'' + 2\left[b - (2a-b+1)x^2\right] y' - 2ax(1+2a-2b)y = 0 \tag{4.15}$$

Hypergeometric equation is obtained. (4.5) has a solution

$$y = (1+x)^{-2a} F \left[\begin{matrix} a, b\;; & \dfrac{4a}{(1+x)^2} \\ 2b; & \end{matrix} \right] \tag{4.16}$$

If x changed by $-x$ and introduce a new independet variable $v = x^2$ in equation (4.5), it is easily obtained the equation

$$v(1-v)\frac{d^2y}{dv^2} + \left[b + \frac{1}{2} - \left(2a - b + \frac{3}{2}\right) v\right] \frac{dy}{dv} - a(a - b + \frac{1}{2})y = 0 \tag{4.17}$$

which has the general solution

$$y = AF \left[\begin{matrix} a, a-b+\frac{1}{2}; & \\ b+\frac{1}{2}\,; & v \end{matrix} \right] + B\; v^{\frac{1}{2}-b}\; F \left[\begin{matrix} a-b+\frac{1}{2}, a-2b+1; & \\ \frac{3}{2}-b\,; & v \end{matrix} \right] \tag{4.18}$$

with $|v| < 1$.

The differential equation (4.2) has a solution in (4.4) as $2b$ is neither zero nor a negative integer. At same time the differential equation (4.7) has the general solution in $|v| < 1$ with $v = x^2$.

Therefore, if both

$$|v| < 1 \; and \; \left|\frac{4x}{(1+x)^2}\right| < 1$$

and if $2b$ is neither zero nor a negative integer, there exist constants A and B such that

$$(1+x)^{-2a}\ F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right]$$

$$= A\ F\left[\begin{matrix} a, a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}\,; & \end{matrix}\right] + B\ x^{1-2b}\ F\left[\begin{matrix} a-b+\frac{1}{2}, a-2b+1; & x^2 \\ \frac{3}{2}-b; & \end{matrix}\right] \tag{4.19}$$

In this equation, the term

$$(1+x)^{-2a}\ F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right]$$

and

$$A\ F\left[\begin{matrix} a, a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}\,; & \end{matrix}\right]$$

are analytic at $x = 0$ but the term

$$B\ x^{1-2b}\ F\left[\begin{matrix} a-b+\frac{1}{2}, a-2b+1; & x^2 \\ \frac{3}{2}-b; & \end{matrix}\right]$$

is not analytic at $x = 0$ because it has the factor x^{1-2b}.

Hence $B = 0$ should be choosen and A is easily determined by using the resultant identity

$$(1+x)^{-2a}\ F\left[\begin{matrix} a, b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = A\ F\left[\begin{matrix} a, a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}\,; & \end{matrix}\right]$$

By putting $x = 0$,

$$A = 1$$

is obtained and the desired result

$$(1+x)^{-2a}\ F\left[\begin{matrix} a,\ b; & \dfrac{4x}{(1+x)^2} \\ 2b; & \end{matrix}\right] = F\left[\begin{matrix} a,\ a-b+\frac{1}{2}; & x^2 \\ b+\frac{1}{2}; & \end{matrix}\right]$$

is found.

4.3 Other Quadratic Transformation for Hypergeometric Function

Now, let give another properties of quadratic transformation.

Theorem 4.6

If $2b$ is a non-negative integer and if $|y| < \frac{1}{2}$ and $\left|\frac{y}{(1-y)}\right| < 1$,

$$(1-y)^{-a}\ F\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}; & \dfrac{y^2}{(1-y)^2} \\ & b+\frac{1}{2}; & \end{matrix}\right] = F\left[\begin{matrix} a, b; & 2y \\ 2b; & \end{matrix}\right] \tag{4.20}$$

Proof

Let Ψ denote the left member of (4.20). Then, with the aim of Lemma 1.3,

$$\begin{aligned} \Psi &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_k \left(\frac{1}{2}a+\frac{1}{2}\right)_k .y^{2k}}{\left(b+\frac{1}{2}\right)_k (1-y)^{2k+a}\ k!} \\ &= \sum_{k=0}^{\infty} \frac{(a)_{2k}\ .y^{2k}}{2^{2k}\ \left(b+\frac{1}{2}\right)_k (1-y)^{2k+a}\ \ k!} \end{aligned} \tag{4.21}$$

can be written. Also

$$(1-y)^{-2k-a} = \sum_{n=0}^{\infty} \frac{(a+2k)_n\, y^n}{n!}$$

and

$$(a)_{2k}\,(a+2k)_n = (a)_{n+2k}$$

Hence

$$\Psi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{n+2k}\quad y^{n+2k}}{2^{2k}\left(b+\frac{1}{2}\right)_k n!\ k!}$$

By Lemma 3.2,

$$\begin{aligned}\Psi &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(a)_{n+2k-2k}\, .y^{n+2k-2k}}{2^{2k}\left(b+\frac{1}{2}\right)_k k!\,(n-2k)!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(a)_n \;\; y^n}{2^{2k}\left(b+\frac{1}{2}\right)_k \; k! \;\; (n-2k)!}\end{aligned}$$

It is know that

$$(n-2k)! = \frac{n!}{(-n)_{2k}}$$

and that gives

$$(-n)_{2k} = 2^{2k}\, \left(-\frac{n}{2}\right)_k \, \left(-\frac{n}{2}+\frac{1}{2}\right)_k$$

Therefore,

$$\begin{aligned}\Psi &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left(-\frac{n}{2}\right)_k\left(-\frac{n}{2}+\frac{1}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k k!} . \frac{(a)_n\, y^n}{n!} \\ &= \sum_{n=0}^{\infty} F\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2}; & \\ & b+\frac{1}{2}; & \end{matrix}\; 1\right] \frac{(a)_n\, y^n}{n!}\end{aligned}$$

is written. The value of Hypergeometric function

$$\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2}; & \\ & b+\frac{1}{2}; & \end{matrix}\; 1\right] = \frac{2^n\;(b)_n}{(2b)_n}$$

and the desired result is obtained as

$$\Psi = \sum_{n=0}^{\infty} \frac{2^n\;(b)_n\;(a)_n\;y^n}{(2b)_n\;n!} = F\left[\begin{matrix} a, b; & 2y \\ 2b; & \end{matrix}\right]$$

(4.15) can be written as in different form. Let put

$$y = \frac{2x}{(1+x)^2} \tag{4.22}$$

in (4.15). Then

$$1 - y = \frac{1+x^2}{(1+x)^2} \tag{4.23}$$

and

$$\frac{y}{1-y} = \frac{2x}{1+x^2} \tag{4.24}$$

are obtained. So

$$\left(\frac{1+x^2}{(1+x)^2}\right)^{-a} F\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}; \\ & b+\frac{1}{2}; \end{matrix} \left(\frac{2x}{1+x^2}\right)^2\right] = F\left[\begin{matrix} a, \ b; \\ 2b; \end{matrix} \left(\frac{4x}{(1+x)^2}\right)\right] \tag{4.25}$$

can be written. In the view of Theorem 4.1, right side of the equality below can be written as

$$F\left[\begin{matrix} a, \ b; \\ 2b; \end{matrix} \left(\frac{4x}{(1+x)^2}\right)\right] = (1+x)^{2a} \ F\left[\begin{matrix} a, a-b+\frac{1}{2}; \\ b+\frac{1}{2}\,; \end{matrix} \ x^2\right] \tag{4.26}$$

and (4.20) can be obtained as

$$\left(1+x^2\right)^{-a} \ (1+x)^{2a} \ F\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}; \\ & b+\frac{1}{2}; \end{matrix} \ \frac{4x^2}{(1+x^2)^2}\right] \tag{4.27}$$

$$= \ (1+x)^{2a} \ F\left[\begin{matrix} a, a-b+\frac{1}{2}; \\ b+\frac{1}{2}\,; \end{matrix} \ x^2\right] \tag{1}$$

By putting $x^2 = z$ and replace b by $\left(\frac{1}{2}+a-b\right)$,

$$(1+z)^{-a} \ F\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}; \\ & a-b+1; \end{matrix} \ \frac{4z}{(1+z)^2}\right] = F\left[\begin{matrix} a, b; \\ 1+a-b\,; \end{matrix} \ z\right] \tag{4.28}$$

can be written.

CHAPTER 5

GENERALIZED HYPERGEOMETRIC FUNCTIONS

In this chapter, generalized hypergeometric functions which are more general types of hypergeometric functions will be defined. Then some main properties of them will be given.

5.1 Definition of Generalized Hypergeometric Functions

It is known that, the Hypergeometric function

$$F\left[\begin{matrix} a, \quad b; \\ c; \end{matrix} \qquad z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}$$

which has two numerator parameters, a and b, and one denominator parameter, c, and given by (2.3). The generalized Hypergeometric functions are a natural generalization of definition (2.3) to a similar function with any number of numerator and denominator paramters.

Definition 5.1

The generalized Hypergeometric function is defined by

$${}_pF_q\left[\begin{matrix} \alpha_1, \alpha_2, ..., \alpha_p; \\ \beta_1, \beta_2, ..., \beta_q; \end{matrix} \qquad z\right] = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{q} (\beta_j)_n} . \frac{z^n}{n!} \tag{5.1}$$

where α_i $(1 \leq i \leq p)$ are numerator parameters and β_j $(1 \leq j \leq q)$ are denominator parameters. Here no denominator parameter β, is allowed to be zero or a negative integer and if any numerator parameter α; is zero or a negative integer, the series terminates.

5.2 Convergency of Generalized Hypergeometric Functions

The convergency of the series (5.1) can be shown by using an application of the elementary ratio test to the power series. By ratio test,

$$1+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}.\frac{z^n}{n!}$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\prod_{i=1}^{p}(\alpha_i)_{n+1}\ \ z^{n+1}}{\prod_{j=1}^{q}(\beta_j)_{n+1}\ \ (n+1)!}.\frac{\prod_{j=1}^{q}(\beta_j)_n\ \ n!}{\prod_{i=1}^{p}(\alpha_i)_n\ \ z^n} \\ &= \frac{\prod_{j=1}^{q}(\beta_j+n)\ .\ (n+1)}{\prod_{i=1}^{p}(\alpha_i+n)}\end{aligned}$$

and the limit

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\prod_{i=1}^{p}(\alpha_i+n)\ .\ (n+1)}{\prod_{j=1}^{q}(\beta_j+n)}\right| \\ &= \lim_{n\to\infty}\left|\frac{(\alpha_1+n)\ (\alpha_2+n)\ ...(\alpha_p+n).}{(\beta_1+n)\ (\beta_2+n)\ ...\ (\beta_q+n)\ .(n+1)}\right|\end{aligned}$$

is obtained. Therefore, the convergency of generalized hypergeometric funcitions can be explained by

a) convergent for all finite z, if $p \leq q$;

b) converges for $|z| < 1$ and diverges for $|z| > 1$, if $p = q+1$;

c) diverges for $z \neq 0$, if $p = q + 1$.

5.3 Some Special Generalized Hypergeometric Functions

When indicate the number of numerator parameters and of denominator parameters is wanted but not to specify them, the notation ${}_pF_q$ used. For example, the ordinary Hypergeometric function is a ${}_2F_1$. There are several examples for specific Hypergeometric function as well-known elementary functions.

If put 0 in p and 1 in q,

$$ {}_0F_1(-; b; z) = \sum_{n=0}^{\infty} \frac{z^n}{(b)_n n!} \tag{5.2} $$

is obtained and it is called Bessel function.

Another example is the Exponential function. If no numerator or denominator parameters are present, the result is

$$ {}_0F_0\,(-; -; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) \tag{5.3} $$

Let give one more example. For one numerator parameter and no denominator parameter,

$$ {}_1F_0\,(a; -; z) = \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{n!} \tag{5.4} $$

is obtained. It is known that

$$ \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{n!} = (1 - z)^{-a} \tag{5.5} $$

and by using (5.5),

$$ {}_1F_0\,(a; -; z) = (1 - z)^{-a} \tag{5.6} $$

is obtained, which gives the binomial function.

5.4 Differential Equations of Generalized Hypergeometric Functions

Now, let define the differential equation of generalized Hypergeometric functions. It is known that $F(a; b; c; z)$ satisfies the differential equation (3.1)

$$z(1-z)\frac{d^2w}{dw^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0$$

By changing the differential operator $\theta = z\frac{d}{dz}$, the differential equation can be obtained in the form of

$$[\theta\,(\theta + c - 1) - z(\theta + a)\,(\theta + b)]\,w = 0 \tag{5.7}$$

First of all, let write

$$w =_p F_q = \sum_{k=0}^{\infty} \frac{(a_1)_k\,(a_2)_k \,\ldots\ldots\ldots\ldots\ldots\ldots\, (a_p)_k}{(b_1)_k\,(b_2)_k \,\ldots\ldots\ldots\ldots\ldots\ldots\, (b_q)_k} . \frac{z^k}{k!} \tag{5.8}$$

Since $\theta z^k = kz^k$, it follows that

$$\begin{aligned} \theta \prod_{j=1}^{q} (\theta + b_j - 1)w &= \sum_{k=1}^{\infty} \frac{k.\Pi_{i=1}^{q}(k+b_j-1).\Pi_{i=1}^{p}(a_i)_k}{\Pi_{j=1}^{q}(b_j)_k} \\ &= \sum_{k=1}^{\infty} \frac{\Pi_{i=1}^{p}(a_i)_k}{\Pi_{j=1}^{q}(b_j)_{k-1}} \cdot \frac{z^k}{(k-1)!} \end{aligned} \tag{5.9}$$

Now replace k by $(k+1)$ at (5.9) then

$$\begin{aligned}
\theta\prod_{j=1}^{q}(\theta+b_j-1)w &= \sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(a_i)_{k+1}}{\prod\limits_{i=1}^{q}(b_j)_k}\ \frac{z^{k+1}}{(k)!} \\
&= \sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(a_i+k)\ \prod\limits_{i=1}^{p}(a_i)_k}{\prod\limits_{i=1}^{q}(b_j)_k} \\
&= z\prod_{i=1}^{p}\ (\theta+a_i)\,w
\end{aligned}$$

Thus, it was shown that $w = {}_pF_q$ is a solution of differential equation

$$\left[\theta\prod_{j=1}^{q}(\theta+b_j-1) - z\prod_{i=1}^{p}(\theta+a_i)\right] w = 0$$

which is differential equation of generalized hypergeometric function.

5.5 Conclusion

The aim of this thesis is giving a general information about Hypergeometric functions and make a usefull and suffucient base for later works about Hypergeometric functions.

Hypergeometric functions have several applications for differential equations and mathematical analysis. Moreover, several generalizations of Hypergeometric functions were obtained. Thus, for later works, by adding some parameters for Hypergeometric functions, specially generalized Hypergeometric functions, new generalizations can be obtained and several properties of these generalizations can be investigated.

Additionally, if there any differential equation exists such that it can be reduced to the Hypergeometric differential equation, then solutions of these type equations can be given by these new generalized Hypergeometric functions.

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