SOLVING TWO-PERSON ZERO-SUM GAME BY LINEAR PROGRAMMING USING EXCEL

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Dilzar Abdulrahman Hussein: SOLVING TWO-PERSON ZERO-SUM GAME BY LINEAR PROGRAMMING USING EXCEL

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To those who believed in me...

ABSTRACT

This research concentrates on two-person zero-sum games in which one person's net gain is exactly equal to the other person's total loss, the solution of such games requires to obtain the value of the game and each players' optimal strategy, this research utilized linear programming method to achieve this goal, all games' payoff matrices are converted to Linear programming, since solving linear programming problems by simplex method manually needs tedious algebraic calculations so this thesis prefers to use Microsoft Office Excel to obviate miscalculations thereby to obtain the solution faster. The study discussed strategic games that consist of both pure and mixed strategies. Typically, game containing pure strategy has special methods to be solved. And 2×2 payoff matrices have special formulas for solving. However, the research demonstrates the more complex problems, other than these of pure strategy and 2×2 matrix games.

Keywords: Two-person games; zero-sum games; matrix games; payoff matrix; games of strategy; optimal strategy; saddle point; constant-sum games; mixed strategy; linear programming; simplex method

ÖZET

Bu çalışma, bir insanın net kazancının diğer insanların toplam kayıplarına eşit olan iki kişilik sıfır toplamlı oyunlar üzerine yoğunlaşmaktadır. Bu tür oyunların çözümü her bir oyuncunun optimal stratejisi ve oyunun değerini kazanmayı gerektirir. Simpleks yöntemi ile doğrusal programlama probleminin çözümünde Microsoft Office programlarından Excel kullanılmıştır, bu sayede elle sıkıcı cebirsel hesaplamalar yerine yanlış hesaplamaları ortadan kaldırıp daha hızlı çözümler elde etmiş oldu. Bu çalışma hep saf hem de karışık stratejilerin oluşturduğu stratejik oyunları tartışır. Tipik olarak, saf stratejide çözülmesi gereken özel yöntemler vardır. 2x2 sonuç matrisi çözüm için özel formüllere sahiptir. Fakat bu çalışma daha karmaşık problemler içermektedir.

Anahtar Kelimeler: Iki kişilik oyunlar; sıfır toplamlı oyunlar; matris oyunlar; sonuç matrisi; strateji oyunları; optimal strateji; eyer noktası; sabit toplamlar oyunları; karma strateji; doğrusal programlama; simpleks yöntemi

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LIST OF ABBRIVIATIONS

BV:	Basic Variable
GT:	Game Theory
H:	Head
LP:	Linear Programming
LPP:	Linear Programming Problem
MOE:	Microsoft Office Excel
NE:	Nash Equilibrium
OR:	Objective Row
PC:	Pivotal column
PR:	Pivotal row
RHS:	Right Hand Side
SLLP:	Standard Linear Programming Problem
SM:	Simplex Method
SV:	Slack Variable
T:	Tail

LIST OF SYMBOLS

S^*	Profile strategy
S^{NE}	Nash equilibrium profile strategy
S_i	Set of actions or strategies
s_{-i}^{NE}	Nash equilibrium profile strategy accept s_i^{NE}
π_i	Utility function for player i
$\mathbf{A}^T$	Vector transposes
$\mathbf{Max_q}$	A strategy having maximum payoff
$\mathbf{Min_p}$	A strategy having least payoff
$\mathbf{P_D}$	Dual problem
$\mathbf{P_P}$	Primal problem
$\aleph$	Optimal solution in objective function

CHAPTER 1
INTRODUCTION

Two years ago in February 2014 the Russian military intervention in Ukraine and the subsequent annexation of Crimea by the Russian Federation, prompted a number of countries to force sanctions against officials, businesses, and individuals from Russia. Sanctions were approved by some countries, international organizations, and the United States. In response, Russia has imposed sanctions on a number of countries, including a total ban on food imports from the United States, European Union, Australia, Norway, and Canada.

Has Russia made the right decision for military intervention in Ukraine? Similarly, one can ask, has the United States and others who involved, made the right decision to apply sanctions against Russia? Was there any other choice available to them to reduce the size of damage or not? Because the sanctions have contributed to the collapse of the Russian Ruble currency and consequently, the 2014-15 Russian financial crisis (Walker and Nardelli, 2015), they have also caused economics damage to a number of countries in European Union with total losses estimated at €100 billion (Sharkov, 2015). Before even thinking to answer these questions; are you qualified to form an opinion on this serious situation? Have you ever studied the topic of game theory (GT)? Because the above topic is about politics and decision making! Do not be confused, your opinion is fine, since the game theory is the study of strategic decision making (Neumann and Morgenstern, 1944). The term game is not to be confused with games like soccer or other games played by kids, etc. Rather, game theory means strategy theory, is a branch of applied mathematics which is used in a wide range of areas as the social sciences, most notably in economics, as well as in political science, biology (specially ecology), engineering, computer science, philosophy, and international relations (Crider, 2012). So game, is to make a successful choice in response to the choices of others' (Myerson, 1991), whereas initially created to break down rivalries in such a way, one individual improves his gain to another individual's detriment (Crider, 2012). Nowadays, game theory is a kind of umbrella covering the rational side of social sciences, while the word social is deciphered comprehensively to incorporate human and non-human players (plants, computers, animals) (Aumann, 1987).

Conventional uses of GT endeavor to achieve Equilibria in these games. In equilibrium every player has a strategy that they are unlikely to change. The most well-known one among other available equilibriums is that of John Nash called Nash Equilibrium (NE). Of course any game needs at least two players/parties to play it. They maybe two persons or more, or maybe they are markets, organization unions, politicians competing each other, or maybe countries. So consider players that are competing for market shares such as competitors are selling their product in the same market, they are launching some new product. So they have the common goal to maximize profit and minimize loss. The point here is the gain of one player is exactly the loss of the other, and for that they have different strategies; for example, the strategies could be, one player offers a discount of 10% on his products while the other player's strategy is to give 1+1 that means if you buy one product the second product will be free; similarly one competitor is giving advertisement to a radio channel while the other competitor is giving advertisement in the TV channel, or one player is using magazines for advertisement for his product the other is using some other means of advertising such as distributing leaflets. So different players have different strategies and these strategies are known to the players before starting the game. These strategies are used carefully so that the players' position in the game is optimized, his gain is maximized, his loss is minimized, and this is the ultimate aim of any player while playing the game. Game theory covers a very broad spectrum of situations, it is unrealistic for a thesis of this size and specification to investigate and present the entire subject, therefore this dissertation emphasizes on a small part of game theory, namely, two-person zero-sum game.

The games that are played between two persons or two players called Two Person Game whereas games among three or more players are called n-person game, there is a reward at the end of each game, that is, when a game is finished there are gains and losses for each player, this gain/loss is called payoff or an outcome, the details are explained in chapter 2, consequently, the two-person games are classified as zero-sum game and non zero-sum game; zero-sum game means adding payoffs of each player at the end of the game will always equal to zero, i.e. what gained by one player is exactly equals to the amount that was lost by the other player.

So this thesis sheds light on two-person zero-sum games attempts to apply some powerful tools for solving these types of games. Here, solving a game problem means finding the strategies for both players as well as finding the value of the game. The meaning and the definition of the strategies of the players and the value of the game will be presented later.

1.1 Aims of the Study

As it was mentioned earlier, solving any game problem in game theory means evaluating the value of the game as well as the optimal strategies that each player can adopt throughout the game; there is more than one method to solve a game problem. However, this research uses the most powerful method for this purpose; furthermore, the technique is impressive in a way that is fairly easy to learn and to use, and also leads to the correct solution very fast. The linear programming (LP) is a powerful tool to be used after arranging the problem in a suitable format. Moreover, the simplex method (SM) is to be used to solve the linear programming problem representing the game problem. The other objective is to perform the complex computation mechanism involved in the solution process by using Microsoft Office Excel (MOE). Since Excel has some unexplored powers for performing complex calculations, it is employed in this research for solving game problems. So in summary, the research attempts to answer the question; are all cases of Two-Person Zero-Sum games solvable by linear programming? Also seeks to search for weaknesses in the execution process. It is also compares other methods with linear programming to identify the most practical one.

1.2 Thesis Outline

The thesis is divided into five chapters; Chapter one is devoted to introduction and the aim of the study.

Chapter two contains a background and literature review; in literature review a short history of game theory has been given and also some important terminologies and concepts have been introduced. Some well-known game problems in the real life have been presented with illustrative example such as prisoners' dilemma. How to construct the payoff matrix for a game problem is also presented in Chapter 2.

Chapter three consists of methods and methodology for solving game problems; discussed three methods for solving game problems and methodology for the fourth method is also given with step by step procedures of the solution. Further, how to use Excel for this purpose is explained in detail with illustrations.

Chapter four discusses solution of two-person zero-sum game, this is done by setting up the linear programming problem (LPP) representing the game problem and proceed with solution using simplex method.

In Chapter 5, the conclusion of this work is presented; it summarizes and analyses the entire work conducted in this thesis.

CHAPTER 2
LITERATURE REVIEW

When you are playing a game with someone and you know little about that person's background skills or the complete information of that game's rules, it is certain that you will fail to win or you will end up with bad results.

Likewise for reading this topic, the theory of games, if you do not have background knowledge of the game theory, it will be hard for you to understand all aspects of the game and consequently this thesis. Thus the main aim of this chapter is to give the reader an easy to comprehend background and history of game theory, from where it began? How did it start from the beginning? Who was the first who talked about the game? By whom it was developed? In which field it has been used and for what purpose? Etc... Also to illustrate some concepts and definitions of games on one hand, and on the other hand to classify game models and rules. Since this thesis is concerned mainly with its strong relation with linear programming so the illustration of its concepts and applications will also be our main purpose.

2.1 History of the Theory of Games

At the beginning game theory was used in Economics area because of human needs and economic revolution at 20th century (Shapiro, 1989), however, this does not mean that there was no any movements on game theory prior to this date; earlier of that time the solution of two-person zero-sum game was appeared in a letter from James waldegrave to Pierre-Remond around year 1713 (Bellhouse, 2007).

In 1913 the German mathematician Ernest Zermelo talked about possible choices for players at every decision point, in a statement (Screpanti et al., 2005), after then four notes were published by the mathematician Emile Borel on strategic games between years 1921 and 1927 but his notes have not been at an interest point until Von Neumann appeared on the stage (Tucker and Luce, 1959). The Hungarian mathematician John Von Neumann published a paper in 1928, we can say that this was a birthday and sunrise for the dark days of game theory, his paper followed by his book *theory of Games and Economic Behavior* in 1944 with the help of the economist Oskar Morgenstern who joined Von Neumann at Princeton

(Weintraub, 1992), in that book many things illustrated and provided theory of utility which renewed Bernoulli's old theory of utility.

Till that point in time the work on game theory was focused on cooperative game theory, a fantastic alteration on this static mentality originated from a beautiful mind of a great mathematician John Nash, when he developed a players' strategies known as 'Nash equilibrium', this equilibrium lead them to a non-cooperative games which is wider and vaster field.

Finally in 1994 Nobel Prize was awarded to John Nash and two other mathematicians called John Harsanyi and Rienhard Selten for their work on non-cooperative games (Goluch, 2012).

2.2 Classifying Games

In this section we introduce some terminology and definitions that will be required from the reader in order to get the best out of this topic. Some concepts may slightly differ from one author to another within the wide range of books and publications that have been written on game theory. However, in this text we tried our best towards bringing the differences closer to each other as much as possible. For example, putting your money on one square and rounding the roulette ball, is a game which requires no skill on your part to gain additional money, rather, the outcome is depending on probability rules, and such games are called Game of Chance that we do not talk about it anymore, in this work. The other type of games is called game of strategy like the Chess, Checkers and Poker that require some kind of skill on the part of the player. Moreover there are many games in business, economics, warfare, etc. (Crider, 2012; Kelly,2003), for example, "the theory of Nash equilibrium has been used to study oligopolistic and political competition, the theory of mixed equilibrium has been used to explain the distributions of tongue length in bees and tube length in flowers" (Osborne and Rubinstein, 1994). Some books refer to the third type of games known as Game of Skill (Kelly, 2003); our goal here is to discuss the second type of games (games of strategy).

The main idea behind game theory is to maximize gains or profits and minimize losses, in other words to optimize situation in particular decision for a problem. So game theory is abstract model of decision making (Neumann and Morgenstern, 1944). In the diagram below games and their branches are classified.

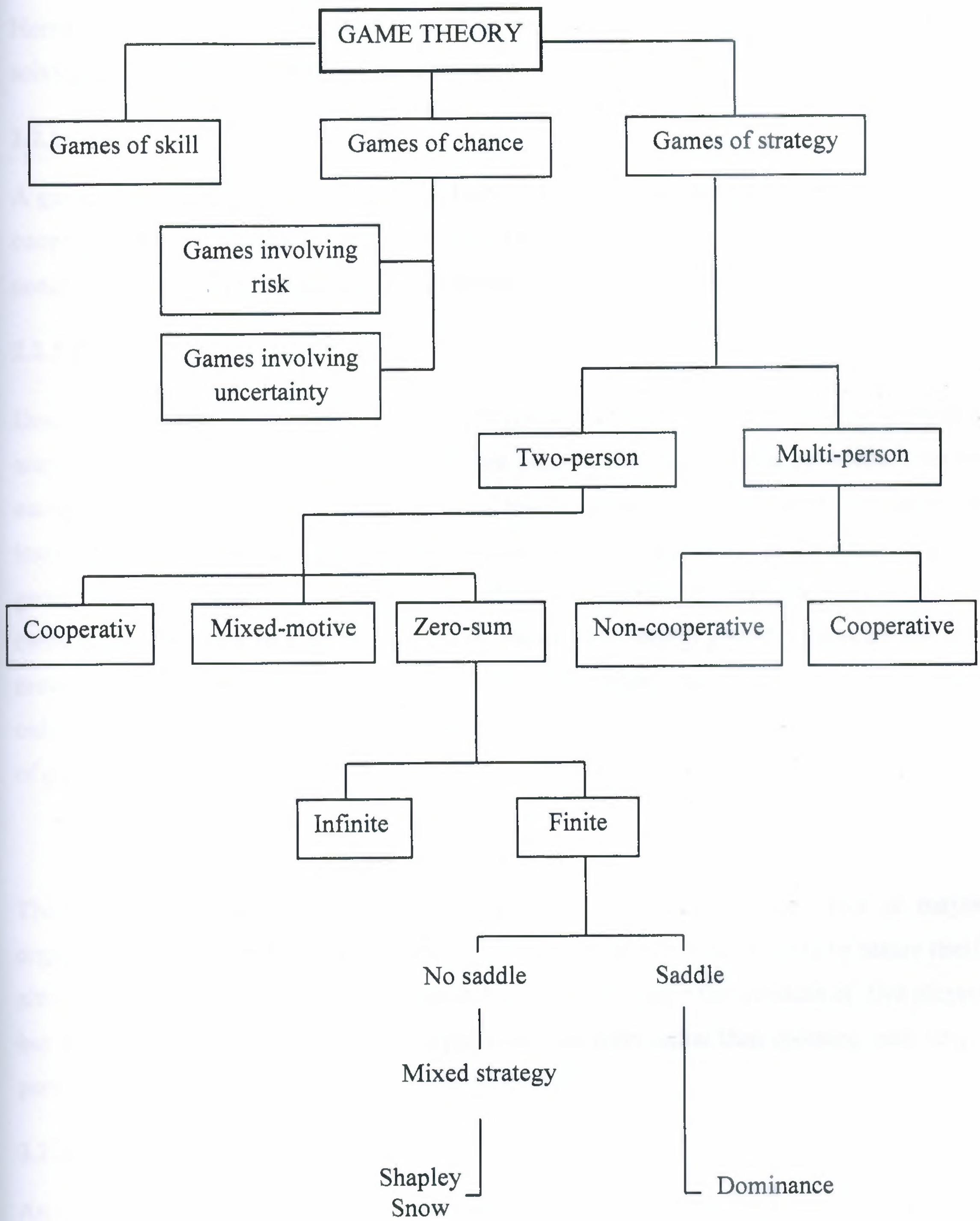

Figure 2.1: Classifying games

Hereinafter some underlying concepts of game theory will be shown, they are important for solving games and for the study of game theory.

2.2.1 Game

A game simply can be anything that two players or competitors conflict or compete or maybe cooperate about it, which can be used to describe simple games like chess, poker or can be complex as political campaign situations, international wars, marketing and so on.

2.2.2 Player

Describing competitors or players rely on the type of the game, for instance, if the game is a war between two countries then these two are players, also players can be workers versus management, supplier versus supplier, political leader versus political leader etc. Any game at least needs two players to play it, whereas one of them maybe nature, "in the terminology of game theory, nature is not 'counted' as one of the players. So, for example, when a deck of cards is shuffled prior to a game of solitaire, nature –the second player- is making the first move in what is a one-player game" (Kelly, 2003). This means that there is no game contains only one player because the nature will get its role, thus we can classify games by the number of players involved:

- Two-person game.
- N-person game, more than two players.

These players maybe individuals as in Chess. One person against the other or maybe organizations such as volleyball team against another volleyball team or it can be nature itself, although teams consist of many members such as basketball team that consists of five players but it can be described as an individual player as one team rather than counting each single person in that team.

2.2.3 Outcome

An outcome or the payoff is a final result reached through an activity or a set of activities that are taken by players or happens under a process of logical decisions made by the participants.

This payoff maybe gains, losses or both, the negative sign indicates a loss while the positive sign indicates a gain, the type of game effects on the outcome, for example if the game is strictly conflictive the gain by one player equals the loss by the other, this type of outcome is called the zero-sum, payoff is not always the largest number possible, in some cases it is a minimum number such as in the case where a company wants to design a special product at a minimum cost. Payoffs can be money or percent market share, etc. Thus we can classify games based on the sum of gains and losses as zero-sum game.
Zero-sum game, example: consider two players (say player A and player B) throw a coin for choosing a Head (H) or Tail (T), if the coin shows Head, player B will pay $1 to player A and vice versa, thus what player A gains (+1$, the sign + indicates gain) is exactly equal to what player B loses (-1$, the – sign indicates loss), therefore, the sum of +1 and -1 is zero hence the name is zero-sum game.

On the other hand there are games where the players are not in direct conflict with each other. In fact they may communicate an outcome that is to the benefit of both players, a simple example is when two drivers approach a junction they both use strategies to avoid collision. This type of game is known as non Zero-Sum Game.

2.2.4 Payoff matrix

This table shows the payoff matrix for player A.

		Player B			
		B_1	B_2	B_j	B_n
Player A	A_1	a_{11}	a_{12}	a_{1j}	a_{1n}
	A_2	a_{21}	a_{22}	a_{2j}	a_{1n}
	A_i	a_{i1}	a_{i2}	a_{ij}	a_{in}
	A_m	a_{m1}	a_{m2}	a_{mj}	a_{mn}

Figure 2.2: Payoff matrix

With A_1, A_2 ..., A_m representing strategies of player A and B_1, B_2..., B_n representing strategies for player B. The a_{ij} entry of the matrix represents the payoff to the player A for player A choosing strategy A_i and player B choosing strategy B_j. So as it is this matrix represents the payoff matrix for player A. The payoff matrix for player B can be constructed similarly if the entries of the matrix say b_{ij} were to represent the payoff for player B. However, often authors choose to insert both a_{ij} and b_{ij} in the same square using different colors or writing them in different corners of each square. See Figure 2.3

Firm B

		Advertise	Don't advertise
Firm A	Advertise	5 5	3 25
	Don't advertise	25 3	10 10

Figure 2.3: Payoff matrix

The example below illustrates payoff matrix in numbers between two markets competing; market I and market II.

Market II

		N_1	N_2
Market I	M_1	4, -4	5, -5
	M_2	1, -1	3, -3

Payoff matrix 2.1: A game between two markets

In the above payoff matrix player A is market I, player B is market II. Market I has two strategies M_1 and M_2. Market II has again two strategies N_1 and N_2. Now if market I plays M_1 strategy and market II plays N_1 strategy then gain for the market I is 4 and the loss to the market II is -4. Consider that market I plays M_2 strategy and market II is playing N_2 strategy then 3 is the payoff and this payoff is the gain for market I and loss to the market II. Similarly the same procedure can be applied on the rest of strategies.

2.2.5 Strategy

The strategy is designing a plan through choosing a set of actions available to the player/participant in advance which affects the outcome directly. There is a difference between a strategy and a move that the player takes during the game steps. A strategy is a decision making that taken once and for all but the move is an option that can be taken at some point throughout the game period. A strategy can be divided into two subcategories they are pure and mixed strategy both of them will be explained at the next pages of the current section, moreover this strategy spread along with types of games, which exists strategic games or sometimes called *'normal games'* as named like that by Von Neumann, and there exists game of chance that involves risk and involving uncertainty and the last one is a game of skill. Strategies can be anything like discounting in shopping, or it can be advertisement of companies, etc. if we have a payoff matrix A of the size $m \times n$ and suppose p_i, $1 \leq i \leq m$ is the probability for player D that he selects his ith move. Let the probability of player E be q_j when $1 \leq j \leq n$ that E selects jth column of the payoff matrix A. Hence this row vector $p = [p_1\ p_2 \dots p_m]$ is the strategy for the player D; and the column vector

$$q=\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

is said to be the strategy for the player E.

2.2.5.1 *Pure strategy*: When the player chooses a specific move or the same strategy with certainty in every situation that he could face and follows this strategy during the entire time of game then it is said to be a pure strategy. Such moves that he takes through this strategy may not be random. We also can say that if the matrix game has a saddle point (saddle point will be explained at a moment) for its payoff matrix then the pure strategy for both players will have 1 as one component and zero for all the other components.

the following example is for illustrating pure strategy: suppose a game with payoff matrix

$$\begin{array}{cc} & E \\ D & \begin{bmatrix} 1 & -2 & -4 & 4 \\ 3 & 2 & 2 & 4 \\ 2 & 5 & 0 & 6 \end{bmatrix} \end{array}$$

The pure strategy for the player D is p = $[0 \quad 1 \quad 0]$ and q = $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ indicates the pure strategy for player E; player D chooses his second row each time because it is the best result for him to maximize the gains whatever player E intends to do, likewise player E will choose his 3rd column every time because it is the least payoff that he losses since he wants to minimize the loss as much as possible, as we said before, 1 as one component and zero for the rest in the vector of strategy.

2.2.5.2 *Mixed strategy*: when the player uses a combination of different actions/choices within his strategy then the strategy is known as mixed strategy. Thus there is a probabilistic situation, and the main aim for the player is to optimize outcomes, to maximize gains and to minimize expected losses. Mixed strategy is probabilistic in nature. For example if the strategy for a player is p_1, p_2, p_3, p_4, in a pure strategy only one of the p_i say p_2 will be 1 and all the others are zero. This means that the player will choose action p_2 all the time with certain probability $p_2 = 1$, and chooses the other actions p_1, p_3, $p_4$with probabilities p1 = p3 = p4 = 0. While in mixed strategy p1, p2, p3 and p4 they have probabilistic values not necessarily zero, as long as their sum is equal to 1.

Mathematically a mixed strategy for a player with at least two possible choices of action; is denoted by the set N, while N consists of $\{x_1, x_2, \ldots, x_n\}$ such that $x_1 + x_2 + \cdots + x_n = 1$ and for all $x_i \geq 0$. One more important note should not be forgotten is that the mixed strategy can apply on those games that do not have a saddle point; the saddle point is the major difference between those games which can be solved by applying the concept of mixed strategy and those games that can be solved by pure strategies. Consider the following payoff matrix for explanation of mixed strategy:

	B_1	B_2
A_1	0	-2
A_2	-3	2

Payoff matrix 2.2: Explains mixed strategy between two players

Since there is no entry which is the minimum of its row, at the same time maximum of its column so there is no saddle point in this game therefore none of participants are ready to choose a single strategy with certainty, there exist a risk to give the other competitor an advantage of selecting such a strategy for that reason each player will keep changing his choice of strategy for optimizing his situation and get better position in the game.

2.2.5.3 *Optimal strategy*: Is a perfect game plan that makes a player in the most preferred position regardless of what his competitor intends to do. It means any deviation from the preferred position would make a payoff to be decreased.

2.3 Rules of the Game

As other games rules are fixed before starting the game so there are certain rules to be followed, these are described below:

1. The players act intelligently and rationally, it means they know the different strategies available to them and to the other participants, also they have knowledge about the results of playing different strategies by the other players and by them, so they are rational means they are logical and the purpose of their playing is winning they are not playing to lose the game.
2. There is a finite set of actions available to each player and the impact of those courses of action is known to him.
3. All relevant information is known to each player, here again it is the rational behavior feature; the player is playing to maximize the gains not the opposite. The player cannot say this was not known to me and I played a wrong strategy, everything is familiar to them.
4. For zero-sum game players take individual without direct communication.
5. In the strategic form (non-extensive game) the player chooses his respective courses of action simultaneously, it means that if one takes a strategy the other also takes another strategy based on the strategy played by the competitor.
6. The payoffs of playing different strategies is known to them in advance based on that they take the action, the payoff determined in advanced and fixed.

2.4 Saddle Point

A saddle point is an entry in a payoff matrix that is smallest number in its row, at the same time is the largest number of its column. This implies that 'the row player' guarantees the least winning among all payoffs, although it is a small gain but better than a large gain under a risk without warranty, likewise for the column player he guarantees the minimum loss by choosing this point and avoid the larger loss, at the same time he does not let his competitor to exceed the minimum win, this is the best act for both of them. Moreover, saddle point is called the value of the game. A game may have more than one saddle points and necessarily all these saddle points must be equal. On the other hand a game may not have a saddle point at all. However if the matrix game has a saddle point then this game is said to be strictly determined, moreover if the game is a zero sum game and the value of the game is zero then the game is called a Fair Game (Kolman and Hill, 2005).

The 3D graph of a saddle point and the other payoffs can be sketched out then the corresponding part to a saddle point will look like a saddle therefore it is called 'saddle point' (Straffin, 1993).

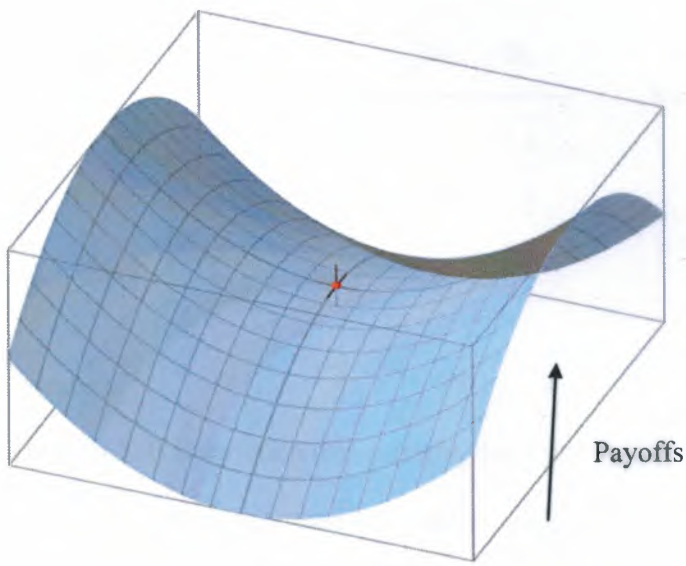

Figure 2.4: Describes saddle point (in red)

Here are two examples, the first one for a payoff matrix having more than one saddle point the other one has no saddle point.

	B_1	B_2	B_3	B_4
A_1	6	4	7	4
A_2	7	-2	3	2
A_3	7	4	6	4

Payoff matrix 2.3: A game having more than one saddle point

To determine the saddle point/s of the above payoff matrix, first check all entries in the first row (row A_1) and identify the minimum entry of the row which is 4. Repeat the procedure on the second and the third row. For the columns search for maximum values not minimum values. If a number is the minimum of its row and at the same time is the maximum of its column, then that number is a saddle point. These points are circled below.

	B_1	B_2	B_3	B_4	Row minima
A_1	6	**4**	7	**4**	4
A_2	7	-2	3	2	-2
A_3	7	**4**	6	**4**	4
Column maxima	7	4	7	4	

Payoff matrix 2.4: Illustration of a game having more than one saddle point

It is clear that this payoff matrix has more than one saddle point, since entry 4 in the first row is the minimum of all corresponding entries in its row which are 6 and 7 at the same time it is greater than all the entries in its column (column 2 or column B_2) hence entry 4 is a saddle point of the matrix game, repeat the same mechanism to find all other saddle points. As one can see there are four saddle points and they are all equal. Hence 4 is the value of the game.

The following example shows that the payoff matrix does not have a saddle point. Consider the matching pennies between two players and the result of the game or the payoffs are written directly without any explanation of the game mechanism because at least, now it is adequate for our purpose of saddle point to be clear; at the next sections it will be explained with details.

	H	T
H	1	-1
T	-1	1

Payoff matrix 2.5: A game having no saddle point

It is obvious that the game does not have any saddle points.

2.5 Game Models

Game theory can be described in two different ways by which players can use their strategies, they are normal form and extensive form; in the first type (normal) players must choose the moves at the same time or simultaneously that is they do not have knowledge about the other players' plans, while the game in the second type is sequential; both types are presented below in more detail:

2.5 Normal form

Before starting the game in normal form each player selects his strategy and he is unaware of the strategy that is chosen by the other participant, since it is simultaneous. However, the strategy must work for every possible situation for the duration of the game; the normal form can be represents by a payoff matrix and must fulfill the conditions below:

- A finite set of players I= 1, … N
- A none empty set of actions for each player i, that is S_i.
- For each player a payoff (profile strategy S^*). (Fudenberg and Tirole, 1991)

The best example of this sort of game is the well-known dilemma known as "The prisoner's dilemma" it is presented as follows:

Two suspects (say A is girl, B is guy) have been arrested for some minor crime, the police think they committed a more serious crime but they do not have enough evidence to convict them. They need a confession; both of the suspects are held in separate rooms for investigation so that they cannot talk to each other. The police offer them a plan for punishment: confess that your partner committed the crime, and you will go free we will pardon you for the minor offense but your partner will receive a sentence of 5 years. If you stay silent and your partner tells us that you were the one who committed the crime then you are going to get the same sentence which is 5 years in jail. They know that the police do not have any evidence and if they both stay silent or do not confess they will only go to prison for 1 year each for the minor crime. If both of them confess each other then they will go to prison for 2 years each. Here each partner has two strategies available to them either stay silent or confess, staying silent would be cooperating and confessing would be defecting, if they both stay silent they each receives a year in jail, if one confesses and the other stay silent then the confessor goes free and silent spends 5 years in prison. If they both confess then it is 2 years each. So what is the best thing they have to do? Well, one may think that they should cooperate, is the best thing for them, actually this is not a case completely because if one partner cooperate there is no warranty the other will also do so, even if they can talk to each other. However If she think player B is going to stay silent then she should confess so she can go free. Going free is better than a year in prison, if she thinks he is going to confess her then she should definitely confess, two years in jail is better than 5. The other suspect is in the same situation and will think the same thing; he should confess if she stays silent and he should confess if she confesses. They should have both cooperated but from an individual stand point they noticed they could always gain by defecting. If they have no control over what the other person is going to do, so they will both defect to get prefer position of their own situation. The Figure below illustrates this dilemma more clearly.

		B	
		Stay silent	Confess
A	Stay Silent	1, 1	5, 0
	Confess	0, 5	2, 2

Payoff matrix 2.6: Shows prisoner's dilemma

2.5.2 Extensive form

Extensive form goes under a dynamic branch of game theory. Where the strategies that the players employ is sequential. It is important that the strategy will be characterized at different points in time not only at the beginning .That is the decisions are simultaneous. Game trees can be used for describing the extensive form of Game. This form consists of a set of players and a set of actions that are taken sequentially, the actions may be finite or infinite. Moreover there exists extensive game with perfect and imperfect information, Depending on how much information about the other player/s is available prior to the action.

This is an example about extensive form: there are two players, player 1 and 2. Player 1 has two strategies x and y, player 2 again has two strategies A and B. The strategy profiles are: (x to A, x to B, y to A and y to B) and the payoffs are: (**6**, 11), (**9**, 2), (**-6**, 3), (**10**, 7) as described in the Figure 2.5. The first components of the payoffs (6, 9, -6, and 10) which are written in bold belong to player 1. The second components of the payoffs are for player 2. So player 1 moves first and has two choices x and y then player 2 comes and he has two choices A and B, out of this player 2 will make a choice between the payoffs of A and B. If player 1 is playing x, then player 2 will choose A, because 11 is greater than 2. Suppose player 1 chooses y, then again player 2 has two choices A and B he will choose 7 since 7 is greater than 3. Hence there are two equilibriums (A and B) in this game. Equilibrium will be explained at a moment.

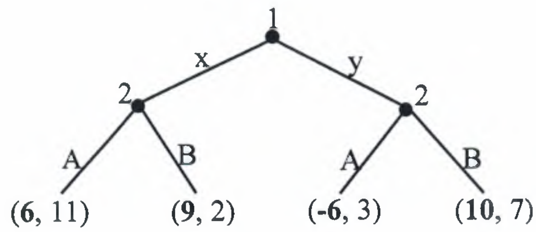

Figure 2.5: A game tree between two players

Hence there are two equilibriums (A and B) in this game. Equilibrium will be explained at a moment.

This game can also be putted in a normal form that is in a table or a matrix form see payoff matrix 2.7. There are two players 1 and 2, player 1 has two choices or strategies x and y. player 2 has two strategies A and B. While player 1 plays strategy x the best strategy for player 2 to choose between 11 and 2 is A, (means payoff 11) and so on for the rest of strategies.

		Player 2	
		A	B
Player 1	X	6, 11	9, 2
	Y	-6, 3	10, 7

Payoff matrix 2.7: Converting game tree to the payoff matrix

2.6 Nash Equilibrium

It is not reasonable to study a course on game theory without talking about Nash Equilibrium since it has a broad sense for solving many games in economic that could not be solved by other concepts of game theory. The definition of Nash Equilibrium can be stated as follows: Is a solution concept of a game containing more than one player and a collection of strategies such that the strategies available to each player makes his situation best preferred position in a game and optimize his strategic response to others. It means that if the players have Nash equilibrium then they do not have any other strategy that is better than this equilibrium strategy, even if one of participant changes his plan he cannot benefit unless the other player keeps his own unchanged.

Mathematically Nash equilibrium can be defined as *"Given a game G in strategic form, the strategy profile* $S^{NE} \equiv (s_1^{NE}, s_2^{NE}, \ldots, s_N^{NE})$ *is called Nash equilibrium if*

$$\forall\, i = 1, 2, \ldots, N, \;\; \forall\, s_i \in S_i, \;\; \pi_i(s_i^{NE},\; s_{-i}^{NE}) \geq \pi_i(s_i,\; s_{-i}^{NE}).$$

There are too many situations in real life being played by people in every day which are solved by Nash equilibrium; an example of NE is stoplight games that face most people in every normal day. Assume that two cars are travelling from perpendicular directions towards an intersection; the stoplight is red for one and green for the other. Suppose that the police do not exist, the normal law is that the driver who has red light stops and who has green light goes, would anyone want to break the law or not! By looking at a matrix that represents this situation and the answer is being 'no' corresponding to the payoffs in the table then the case is a Nash equilibrium replaced instead of a normal law. Verifying payoff matrix in the payoff matrix 2.8 it can be seen two options available for each player they could choose Go or Stop.

		E	
		Go	Stop
D	Go	-6, -6	1, 0
	Stop	0, 1	-1, -1

Payoff matrix 2.8: Stop light game explains Nash equilibrium

If they both go then they crash into each other that is a bad outcome for them also in matrix this accident indicates payoff of -6, if they both decide to stop, again it is bad for them because they have to wait there a long time and the payoff for them is -1 according to the matrix above. If one goes and the other stops, say stoplight is telling to player D go and player E stop, then she gets to her destination in time that is good for her, at the same time it is good option for player E to stop since if he switches his strategy and decides to go then he will crash the car and get payoff of -1 that is bad outcome, likewise, the situation for player D is the same case for player E also.

2.7 Cooperative Games

In cooperative game, participants, rather, they have a common gain/benefit. There are two types of cooperative games: pure and minimal social situation games (Sidowski, 1957). Moreover, purely cooperative game can be divided into two branches: perfect and imperfect information. However purely game even not counted as a game by some authors in game theory (Luce and Raiffa, 1989). An example on cooperative games is that if someone wants to sell his house while another one needs to buy it but for less than that price which the seller asks for. If each of them keeps his decision unchanged, means they are not cooperating but if the seller accepts an offer that is lower than the original asking price and also if the buyer increases his offer to a level that the seller accepts, then there is a deal and both parties are happy. This process is called cooperative game (Hogarth, 2006).

2.8 Non-Cooperative Games

In this type of games there is no permission players communicate with each other, they attempt not to reveal their strategies. This type is completely opposite to cooperative games, they always try to minimize losses.

2.9 Constant-Sum Games

In constant sum game players pay and receive to each other from a constant amount directly, that is; adding gains and losses always equal to the same value, therefore it is called constant sum game, also it is known as pure competitive since there is no cooperation between players at all. Maybe total of these gains and losses add up to zero, however, if the case is so, then it is called a zero sum game which is a class of constant sum game. One example of this game; consider two players competing on $60 that they earned in a specific game together but they do not want to divide this amount between themselves equally, they wish to play on $60 again so they decided to throw a die under condition that when someone gets number 4 in die they will take $40 and the other one has to accept the remaining $20, in case the die shows the number 6 for one of them then he takes the total $60 and his partner gets nothing and so on.

Since the total gain and loss in every situation add up to same constant $60, hence it is called constant sum game.

2.10 Zero-Sum Game

A special kind of constant sum game is a zero-sum game as illustrated before. The total profit got by one player in zero-sum game is equal to the loss of the other participant at the end of the game; there is no chance for cooperating at all, only transferring payoffs from one competitor to the other (Kelly, 2003). One example on zero-sum is matching pennies, suppose there are two players competing as follows; the symbols H and T imply Head and Tail side of coin respectively, the coins will be flipping, if both coins appear same side (H-H or T-T) player A gets $1 from player B, otherwise player B wins $1, player A loses $1. There is no other possibility; that is why it is called zero-sum. Payoff matrix 2.9 shows the problem into payoff matrix.

		B: H	B: T
A	H	1	-1
	T	-1	1

Payoff matrix 2.9: Matching pennies, a zero-sum game

Negative sign indicates gain of B or loss of A.

2.11 Two-Person Zero-Sum Game

The term zero-sum game has been explained before and it is still valid for two-person zero-sum game also, the only change is two-person added here, the convenient way for representing two-person zero sum game is using a payoff matrix, in this class of games there maybe finite number of pure strategies exist for both of them, maybe infinite pure strategies exist for one of them or two of them (Kelly, 2003), thus depending on these two features two person zero sum game can be classified respectively as:

- Finite zero-sum game.

- Infinite zero-sum game.

The payoff matrix 2.10 represents two-person zero sum-game of player 1 and 2.

		2	
		Strategy D	Strategy E
1	Strategy D	D, D	D, E
	Strategy E	E, D	E, E

Payoff matrix 2.10: Shows two-person zero-sum game

The matrix entries are the payoffs for player 1; they also belong to player 2 if they represent as negative sign.

2.12 Minimax Theorem

The cornerstone of minimax theorem was laid by John Von Neumann since the theorem reveals the most important secret goal that players attempt to reach it in finite two-person zero-sum game. The theorem states that two players in a game trying to optimize their situations, player D wants to choose his best strategy which maximizes gains available to him, while player E wants to minimize player D's gains as much as possible, therefore both trying to get to a suitable value v. Each player has mixed strategy available to him, this mixed strategy makes player D gains a payoff not less than the value v, also player E's strategy makes him to lose a payoff not greater than v what ever his competitor does (Neumann and Morgenstern, 1944).

Suppose player D has x_i strategy and player E has y_j strategy in a payoff matrix labeled as m rows and n columns with payoff z_{ij}. During the use of two players' mixed strategies there exist probability p available to player D and probability q for player E such that $\sum p_i = 1$ for $i = 1, \dots, m$, $\sum q_j = 1$ for $j = 1, \dots, n$.

When player D chooses strategy x_i with probability p_i and strategy y_j is chosen by player E with probability q_j, so the chance for getting a payoff u_{ij} is $p_i q_j$ and the expected payoff game will be then: $\sum u_{ij} p_i q_j$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ (Kelly, 2003).

Player D tries to maximize $\sum U_{ij} p_i q_j$ and player E unlikely wants to minimize it, therefore if player E going to choose a specific column say column b then player D will attempt to choose a strategy which maximizes this expected payoff:

$$\text{Max}_\text{p} \sum u_{ib} p_i q_b$$

Likewise if player D decides to select say strategy d then the other player should select that strategy which makes this payoff minimum, that is:

$$\text{Min}_\text{q} \sum u_{dj} p_d q_j$$

Therefore player $D\&E$ can assure that the expected payoff for them will not be less than:

$$\text{Max}_\text{p}\ \text{Min}_\text{q} \sum U_{ij} p_i q_j$$

And not greater than:

$$\text{Min}_\text{q}\ \text{Max}_\text{p} \sum U_{ij} p_i q_j$$

The minimax theorem states that $\text{Min}_\text{q}\ \text{Max}_\text{p} \sum U_{ij} p_i q_j = \text{Max}_\text{p}\ \text{Min}_\text{q} \sum U_{ij} p_i q_j$ for the proof of the theorem the reader is invited to (Neumann and Morgenstern, 1944) the book name is "theory of games and economic behavior".

2.13 Linear Programming

The aim of linear programming is to evaluate the optimal solution of a linear function (objective function) subject to some linear constraints. The optimal solution here can be a maximization or minimization depending on the problem in hand, as we try to maximize profit and minimize loss.

The historical backdrop of linear programming can be traced back to the 1930s; the early part of the history is portrayed by McClosky (1987), while the later authoritative record is provided by (Gass and Assad, 2005).

More serious work on linear programming started in the mid-1930s. (Motzkin, 1936) for solution of systems of linear inequalities, also work was carried out on output-input models (Leontief, 1936).

In the year 1939 a monograph was published by the mathematician L.V. Kantorovich entitled "Mathematical Methods in the Organization and Planning of production" Kantorovich (1939). He realized that there were many production issues that are solved by the same numerical methods; however the work of this Russian mathematician went unrecognized.

Later, in 1941the transportation problem was formulated by Frank Hitchcock (Hitchcock, 1941), after that George Stigler in '1945' considered the diet problem which discusses the minimal cost (Stigler, 1945), the next chapter will discuss the diet problem, thus because of these problems and others related to the World War II the need of optimal methods for solving linear programming problems became so necessity. Moreover, in the year 1951 "George Dantzig" found the Simplex method (Dantzig, 1951), also John Von Neumann realized that how significant the concept of 'Duality' is for serving LPP, and first theorem was proved and then published by Gale, Tucker, and Kuhn (Gale, et al., 1951).

The real techniques and strategies of linear programming have been effectively connected to petroleum industry, steel and iron industry, food processing industry, and some more (Thie and Keough, 2008). Linear programming is also used in the fields of business engineering, banking, oil refining, agriculture, etc.

The following is an example adopted from (Kolman and Beck, 1995) on linear programming with detailed explanation on how to adjust it in mathematical model without figuring out complete solution.

A timber mill received a log that was asked to be sawed into two types of board; first-grade and second-grade boards. Assume that, for the first grade, the mill takes an hour to saw and three hours to plane every 500 feet board. For the second-grade, the timber mill takes an hour to saw and 2 hours to plane every 500 feet board. However, the timber saw is available 4 hours per day while the planer is available 10 hours daily. If the profit on each 500 board feet of first-grade and each 500 board feet of second-grade is \$100 and \$80 respectively, then, how many board feet of timber of each kind must be sawed to gain maximum profit?

Now, to translate this linear programming problem to a mathematical model; Let x indicates the number of units of first-grade timber per day, and y denotes the number of units of second-grade timber can be made ready per day, then the required hours daily for the saw is

$$x + y$$

Since only 4 hours the saw is available daily, x and y should satisfy the below inequality

$$x + y \leq 4$$

By the similar way, the number of hours required for the planer is

$$3x + 2y$$

Thus, x and y must satisfy the below inequality

$$3x + 2y \leq 10$$

Of course, the amount of both types is nonnegative so

$$x \geq 0 \text{ and } y \geq 0$$

The objective of this problem is to maximize the profit, so the function of this objective is defined as

$$z = 100x + 80y$$

This function z is called the *objective function.*

Therefore the complete problem is stated as: Find values of x and y to

$$\text{Maximize } z=100x+80y$$

Subject to the constraints

$$x+y \leq 4$$

$$3x+2y \leq 10$$

$$x \geq 0$$

$$y \geq 0.$$

This was an example on mathematical formulation of linear programming problem, the next chapter; section 3.2 is devoted with details for this type of problems and methods for solving them.

CHAPTER 3

METHODS AND METHODOLOGY FOR SOLVING GAMES

In this chapter methods for solving games in game theory will be discussed with examples; there are four ways for solving games: Algebraic, Calculus, Graphical and linear programming method; the last mentioned one (LP) will be discussed in the next chapter. Here, the other three types will be demonstrated, the other goal of chapter 3 is to explain procedures and methods for solving games by the last mentioned type, simplex method can be used softly and accurately for LP to organize linear systems and achieve goals precisely for finding game values. Furthermore, applying Microsoft office Excel format is a great thing to excel in using game theory for evaluating value of the game to find optimal strategies, etc. Excel is a smooth, swift, and light program available in most computers, simple and easy to use not such a complicated program whereas learning excel is not difficult, does not require too many skills. Learning how to use excel for game theory purpose with pictures' explanation is one of this chapter's mission.

3.1 Methods for Solving Games

As it was mentioned earlier games having pure strategies do not require much effort to solve, as there is only one strategy available to each player, it is sufficient to determine the saddle point (value of the game) and hence both strategies are determined. However, games containing mixed strategies require some procedures for evaluating the value of the game and the set of strategies.

3.1.1 Algebraic method

Looking at payoff matrix 3.1 below, company *A* having strategies A_1 and A_2 Company *B* is having strategies B_1 and B_2.

		Co. B	
		B_1 (q)	B_2 (1-q)
Co. A	A_1 (p)	1	7
	A_2 (1-p)	6	2

Payoff matrix 3.1: A game between company A and company B

Assume that player A plays option A_1 (p) fraction of times then A_2 is played $(1-p)$ fraction of times. In fact p represents the probability of A_1 being selected and $(1 - p)$ represents the probability of A_2 being selected since the sum of all probabilities is equal to one. Similarly, for company B having strategies B_1 and B_2.

B_1 is played (q) fraction of times and B_2 is played $(1-q)$ fraction of the times, payoffs are 1, 7, 6, and 2 it means that when A_1 is played by company A gives him the profit of 1 unit when company B plays B_1 strategy and a profit of 7 units when company B plays B_2 strategy, 1 is the gain for company A and loss for company B similarly 7 is the gain for company A and loss for company B. Now to proceed using algebraic method we have the following, p being the probability of A_1 strategy and $(1-p)$ the probability of A_2 strategy, so, multiplying p by 1 and $(1-p)$ by 6 when company B plays B_1 strategy. So gains to company A will be $p+6(1-p)$ and the gain will be equal to $7p+2(1-p)$ when company B playing B_2 strategy, in both cases the gains will be same so equating gains $p+6(1-p)=7p+2(1-p)$ and then solving for p :

$$p+6-6p=7p+2-2p$$

$$10p=4$$

Hence

$$p=0.4 \quad \text{and} \quad 1-p=0.6$$

It means that company A will play A_1 strategy 40% of the time along the game time and A_2 strategy 60% of the time. Similarly, the company B will have the loss of $q + 7(1 - q)$ and this will be equal to $6q + 2(1 - q)$ while company A is playing strategy A_2 so solving for q getting:

$$q + 7 - 7q = 6q + 2 - 2q$$

$$10q = 5$$

$$q = 0.5 \quad \text{And} \quad 1 - q = 0.5$$

It means that company B will play 50% of the time the strategy B_1 and 50% of the time strategy B_2 to minimize his loss.

So the value of the game can be found by substituting the value of $p = 0.4$ into $p + 6(1 - p)$ to get

$$0.4 + 6 - 2.4$$

$$0.4+3.6=4$$

Therefore, four is the gain for the company A and the corresponding loss for the company B.

3.1.2 Analytical or calculus method

Using the same payoff matrix 3.1 to demonstrate the calculus method for finding value of the game and its strategies, let v be the value of the game while p and q are strategies for the two companies A and B.

In the first left cell the payoff is one, one is get by the probability of p and q for player A and B respectively, thus it should be written pq, similarly in the second right cell the payoff is 7 and 7 has probability p and $1 - q$ so we write $7p(1 - q)$, the same procedure will be done for the other remaining payoffs.

Hence

$$v = pq + 7p(1 - q) + 6q(1 - p) + 2(1 - p)(1 - q)$$

Simplifying

$$v = pq + 7p - 7pq + 6p - 6pq + 2 - 2q - 2p + 2pq$$

$$v = -10pq + 5p + 4q + 2$$

Now to use calculus method, using partial differentiation with respect to q and p

$$\frac{\partial v}{\partial q} = -10p + 4 = 0$$

Equating this to zero we get

$$p = 0.4 \;, \qquad 1 - p = 0.6$$

And similarly

$$\frac{\partial v}{\partial p} = -10q + 5 = 0$$

$$q = 0.5 \;, \qquad 1 - q = 0.5$$

$$v = -10 * 0.4 * 0.5 + 5 * 0.4 + 4 * 0.5 + 2$$

Or

$$-2 + 2 + 2 + 2 = 4$$

Thus, the game is biased to the company A by four units.

3.1.3 Graphical method

To illustrate the graphical method for the same problem of company A and company B above: A graph has to be made for solving this problem by graphical method, first, x-axis *'horizontal line'* will be drawn then a *'vertical line'* y-axis has to be drawn and a parallel vertical line is also drawn as shown in Figure 3.1

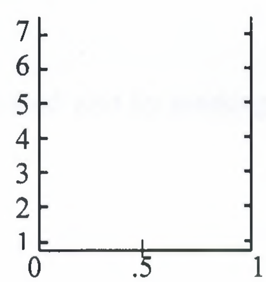

Figure 3.1: Making graph for graphical method

Distance between these two vertical lines is equal to one representing the sum of all probabilities, so starting from zero probability and ending up to one and middle point between them is 0.5.

Now, mark the divisions like Figure 3.1 on left hand side as well on the right hand side, numbers of divisions are given by the payoff matrix, so maximum payoff value in the matrix is seven; will be the maximum number of divisions, in the payoff matrix 3.1 maximum payoff is 7, so minimum 7 divisions have to be taken on the vertical lines. So the number of divisions must be equal to the maximum value in the payoff matrix. Now strategies have to be marked, for the company A the value of p and $1 - p$ will be found, drawing straight line from 1 to 6 the payoff of A_1strategy is one, the payoff of A_2 strategy is six. Remember 1 should to be marked in the right hand side vertical line; so that whatever percentage will get these percentage will be for the company A and the value of p only. So mark one on the right hand side and six in the left hand side of the vertical line and join them by a straight line as it shown in Figure 3.2.

Similarly 7 and 2 must be marked; 7 is with respect to A_1 strategy 2 is with respect to A_2 strategy, 7 will mark in right hand side vertical line and 2 will mark in left hand side vertical line and join them.

Note that starting from left hand side vertical line is also acceptable but in that case the value of $(1 - p)$ will be determined and not value of p.

Hence, the point of intersection between two lines will give the value of p on the x-axis, and it is 0.4 which we determined the same 0.4 in both previous methods; so 40% of the time company A will play A_1 strategy.

Similarly the value of q can be determined also by marking points 1 and 7, 6 and 2 and the point of intersection will give the value of q.

Now, the value of the game will be given on the vertical line; the particular point of intersection 0.4 and corresponding value on the vertical axis (y-axis) will give the value of the game; so this value on the y-axis is 4 corresponds to intersection point.

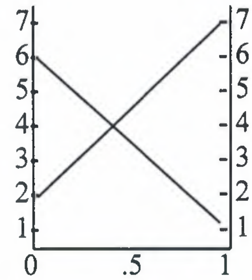

Figure 3.2: Completing the graph of graphical method

3.2 Linear Programming

The aim in solving linear programming problems as discussed in chapter 2 is to evaluate the optimal solution of a linear function subject to a set of linear constraints, in fact linear programming is a branch of mathematical programming that is sometimes called linear optimization (Eiselt and sandblem, 2007) therefore linear programming problems are optimization problems, thus there are three main issues that linear programming must go through them, they are normally organized into three phases. They are feasibility, optimality, and sensitivity (Eiselt and sandblem, 2007). Feasibility means to check whether the constraints can be satisfied or not. If the answer is no! Then the given model must be reformulated. Once the feasibility has been satisfied, that is we have a feasible region that contains the optimum, then the second phase starts, that is to find the optimal solution. There is at least one optimal solution because in the first phase at least one feasible solution was found. The last phase (third phase) tests what happens, if some values of parameters in the model have changed. Therefore sensitivity can be called a post optimality analysis and includes the "what-if" terms.

To illustrate these concepts with applying step by step procedures needs one real life problem. So consider the diet problem which has been adopted from (Thie and Keough, 2008) with slight changes:

A nutritionist is about to minimize the cost of meeting the daily requirements of vitamin C, proteins, and iron. With a diet restricted to bananas, apples, carrots, eggs, and dates. Below is the table of all five ingredients with their cost of a unit and nutrient values.

Table 3.1: The diet problem

Food	Measure of a unit	Protein (g/unit)	Vitamin C (mg/unit)	Iron (mg/unit)	Cost (cents/unit)
Apples	1 med.	0.5	6	0.5	9
Bananas	1 med.	1.3	10	0.7	11
Carrots	1 med.	0.7	4	0.5	4
Dates	$\frac{1}{2}$ cup	0.7	2	0.3	20
Eggs	2 med.	12.3	0	2.7	15

The requirement of this daily diet is at least 55 mg of vitamin C, 75 g of protein, and 12 mg of iron. The problem is the combination of these ingredients to meet the requirement at the minimum cost .Translating this example to a mathematical form we need to introduce five variables as:

Let A indicates number of units of apples in the daily diet, B indicates the number of units of bananas, C indicates the number of units of carrots, D indicates the number of units of dates, and E indicates the number of units for eggs. The cost is measured by the function $f(A,B,C,D,E) = 9A + 11B + 4C + 20D + 15E$ so the aim is to minimize the objective function f. Since the amount of three requirements are chosen at least, then the below inequalities should be satisfied:

$$0.5A + 1.3B + 0.7C + 0.7D + 12.3E \geq 75$$

$$6A + 10B + 4C + 2D \qquad \geq 55$$

$$0.5A + 0.7B + 0.5C + 0.3D + 2.7E \geq 12$$

Hence the complete LP problem is in this form:

Determine the minimum value of

$$f(A,B,C,D,E) = 9A + 11B + 4C + 20D + 15E$$

Subject to the constraints

$$0.5A + 1.3B + 0.7C + 0.7D + 12.3E \geq 75$$

$$6A + 10B + 4C + 2D \qquad \geq 55$$

$$0.5A + 0.7B + 0.5C + 0.3D + 2.7E \geq 12$$

The above problem is a typical example of linear programming. In general, the mathematical form will look like:

Find values of $x_1, x_2, x_3, \ldots, x_n$ that will maximize or minimize

$$z = k_1x_1 + k_2x_2 + \cdots + k_nx_n \tag{3.1}$$

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n (\geq)(=)(\leq) b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n (\geq)(=)(\leq) b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n (\geq)(=)(\leq) b_m \end{array}\right\} \tag{3.2}$$

$$x_i \geq 0 \qquad (1 \leq i \leq n), \tag{3.3}$$

While one and only one of signs $\leq, \geq, =$ occurs in every statement in 3.2. The function in 3.1 is called the objective function which is linear. The linear inequalities or equalities in 3.2 are called constraints. The term linear in linear programming refers to linearity of all constraint as well as the objective function. The term programming means the applications in planning, it is not the same meaning as it is used in computer programming; rather '*programming*' here is a synonym of the term optimization (Eiselt and sandblem, 2007).

Consider a problem that a rancher in order to feed her stock she can buy two sorts of feed. The rancher has established that the herd requires on daily basis, 72, 84, and 60 units of the nutritional C, B, and A, respectively. The following table gives the information about these two feeds and their costs.

Table 3.2: Nutritional elements of feed 1 and feed 2

Nutritional elements (unit/lb)				
	A	B	C	Cost (units/lb)
Feed 1	3	7	3	10
Feed 2	2	2	6	4

It is clear that the rancher has more than one option to choose, for example she can use only one feed to meet daily requirements or to use both feeds. However, the rancher seeks to find least costly way to combine these two feeds for providing a suitable diet (an adequate one). Thus the rancher should examine all possibilities that satisfy requirements and choose such a suitable diet which consists of minimal cost. Now, changing this problem to mathematical language, let x indicates the number of pounds of feed 1 and y indicates the number of pounds of feed 2 for using in the diet, by this way the units of nutritional element A in the diet can be represented by the inequality $3x + 2y \geq 60$ here 60 chosen because at least 60 units are required of element A. By the same way, units of the remaining nutritionals must be $7x + 2y \geq 84$ and $3x + 6y \geq 72$. In order to measure the minimum cost of diet the function $f(x, y) = 10x + 4y$ will be composed corresponding to the cost column in the above table.

Now, the linear programming problem is composed, the second step is to solve the problem for optimality, there exist more than one way to solve the problem, one of them is called *geometrical* solution and can be illustrated graphically.

The above problem can be reset to this context:

Find the values of x and y that minimize $z = 10x + 4y$

Subject to the constraints

$$3x + 2y \geq 60$$

$$7x + 2y \geq 84$$

$$3x + 6y \geq 72$$

$$x, y \geq 0$$

First, search for the points satisfying the inequality $3x + 2y \geq 60$, to do so, see Figure 3.3

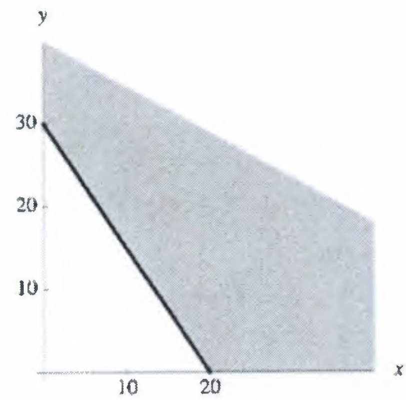

Figure 3.3: The shaded area represents all points (feasible) satisfying inequality

The straight line separates the points that satisfy the inequality, and the points that do not satisfy the inequality. To determine the region that contains all satisfied points choose any point not on the line $3x + 2y = 60$ to test on which side of the line this test point lies. Hence the shaded area is the set of all points satisfying the inequality. By the same way the other inequalities could be checked.

Finally, all the points satisfying the system of inequalities is the intersection of all the sets satisfying every inequality individually. See Figure 3.4

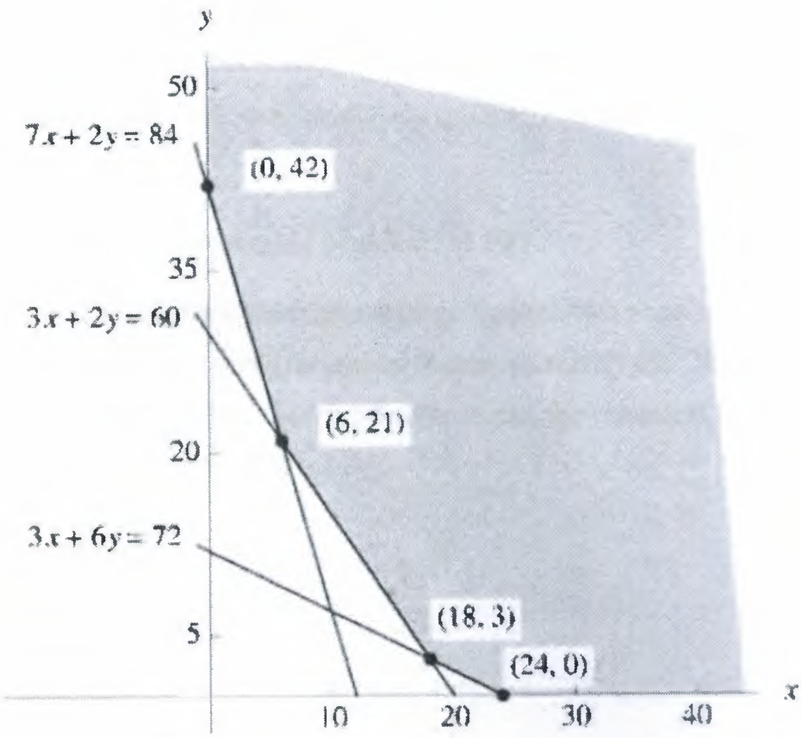

Figure 3.4: Set of all points satisfying all inequalities

The optimum solution will be among the several corner/extreme points where the border lines intersect. Evaluating the function at each of these corner points and then selecting the point with minimum value of the objective function.

Moreover, there are situations where the constraints involve equalities and inequalities in both directions (($\geq$ and $\leq$) such as the following LP problem:

$$\text{Minimize } z = 2x + 3y$$

Subject to

$$x + y \geq 3$$

$$2x + 5y \leq 10$$

However, there is a way for dealing with all of these complex and difficult situations and that is to transform the *Standard Form,* and the *Simplex Method* can be employed to solve the LP problem (Kolman and Hill, 2005; Thie and Keough, 2008).

Since Simplex Method is more effective more flexible also produce fast results and it can be also used to solve game problems once they have written in LP format. Moreover it is useful for practical applications and can handle exceptionally large cases therefore this research focuses on this method.

3.2.1 Standard linear programming problem (SLPP)

Formulating problem into standard form needs go through two phases; the first phase is to set the problem into what is called *Canonical Form*, (Dantzig and Thapa, 2003; Eiselt and sandlom, 2007) and others. The second phase is to put the canonical form into the Standard Form by adding some *Slack Variables* (SV).

3.2.1.1. Canonical form

The canonical form is given by:

$$\text{Max } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}\right\} \tag{3.4}$$

$$x_i \ge 0 \quad , \quad 1 \le i \le n$$

The function z which is to be optimized called objective function, inequalities are called constraints including nonnegative variables (x_i).

There are three conditions to formulate problem into a canonical form

1. The objective function should be maximized.
2. All inequalities other than nonnegative variables are on the form less than or equal to some constants.
3. Non negativities are on the form greater than or equal to zero.

Consider a problem that maximizes the objective function $z = 10x + 8y$

Subject to

$$2x + y \leq 40$$

$$x + 3y \leq 60$$

$$x, y \geq 0$$

is on the canonical form, while this linear programming problem:

$$\text{Minimize } z = 4x - 3y$$

Subject to

$$3x - 2y \leq 5$$

$$x + y \leq 7$$

$$x \geq 0, \quad y \geq 0$$

is not a canonical form, since the objective function is minimized and not maximized

The linear programming problem Max $z = 10x - 16y$

Subject to

$$4x - 2y \geq 3$$

$$x + 2y \leq 4$$

$$x \geq 0, \quad y \geq 0$$

is not a canonical form, since one of the constraints is of the form greater than or equal to right hand side (RHS).

The linear programming problem Max $z = 6x + 11y$

Subject to

$$2x + 2y = 3$$

$$x + 3y \leq 1$$

$$x \geq 0$$

$$y \geq 0$$

is not a canonical form, since the first constraint is an equation and not inequality.

However, there is good news that every LP problems can be transformed to canonical form as well as to standard form (Kolman and Hill, 2005). There are some procedures to follow for writing each problem in a standard linear programming problem.

3.2.1.2 Minimization problem as a maximization problem

Every minimization problem can be viewed as a maximization problem, and vice versa, by multiplying the objective function by -1 and also reversing each inequality with the symbol $\geq$ to the symbol $\leq$ by multiplying both sides of the inequality by -1. Mathematically this becomes:

Minimum of $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ = maximum of $\{-(c_1x_1 + c_2x_2 + \cdots + c_nx_n)\}$

Given the inequality $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq -b$, multiplying both sides by -1 to obtain:

$$-a_1x_1 - a_2x_2 - \cdots = a_nx_n \leq b$$

Consider this minimization LPP Min $w = 3x - 2y$

Subject to

$$2x - 2y \geq -6$$

$$x + 3y \leq 11$$

$$x \geq 0$$

$$y \geq 0$$

Multiplying both sides of each of the objective function and first inequality yielding:

$$\text{Max } z = -3x + 2y$$

$$-2x + 2y \leq 6$$

$$x + 3y \leq 11$$

$$x \geq 0$$

$$y \geq 0$$

Then, the minimum value of the first function (w) can be found by taking negative of the maximum value of the function z.

3.2.1.3 Slack variables

This research will restrict attention to problems in canonical form with nonnegative right-hand sides, in other words:

$$\text{Max} \;\; z = \sum_{i=1}^{n} c_i x_i$$

Subject to

$$\left.\begin{aligned} &\textstyle\sum_{i=1}^{n} a_{ij} x_i \leq b_j \quad , \quad j = 1, \ldots m. \\ &x_i \geq 0 \quad , \qquad i = 1, \ldots, n. \\ &\text{Where,} \\ &b_j \geq 0 \quad , \quad \forall\, j = 1, \ldots, m. \end{aligned}\right\} \qquad (3.5)$$

To change this form to a standard form with all constraints having equalities, slack variables should be added to the left side of each inequality (≤).

Consider the constraint $c_1x_1 + \cdots + c_nx_n \leq b$ this inequality can be converted to equality by adding some non negative quantities (say u) to the left-hand side of the inequality to become $c_1x_1 + \cdots + c_nx_n + u = b$ the added quantity is called slack variable. Now, adding slack variables x_{n+i} where $1 \leq i \leq m$ to a complete system like the system in 3.1 will look like this:

$$\text{Max} \;\; z = c_1x_1 + \cdots + c_nx_n$$

Subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} &= b_m \end{aligned}\right\} \qquad (3.6)$$

$$x_i \geq 0, \qquad \forall\, i = 1, \dots, n+m$$

Thus, solving this new problem with m equations and $m + n$ variables is equivalent to solving the original problem, i.e. adding slack variables $s_1 + \cdots + s_n$ to 3.5 to obtain a standard form:

$$\text{Max } z = \sum_{i=1}^{n} c_i x_i$$

Subject to

$$\sum_{i=1}^{n} a_{ij} x_i + s_j = b_j, \qquad j = 1, \dots, m$$

$$x_i \geq 0, \quad i = 1, \dots, n$$

$$s_j \geq 0, \quad j = 1, \dots, m$$

Consider the problem below:

$$Max\ z = x + 2y$$

Subject to

$$2x + 2y \leq 40$$

$$3x + y \leq 70$$

$$x \geq 0, \quad y \geq 0.$$

This is a correct canonical form, to convert into standard form add nonnegative slack variables u and v to obtain:

$$2x + 2y + u = 40$$

$$3x + y + v = 70$$

$$x \geq 0, \quad y \geq 0, \quad u \geq 0, \quad v \geq 0.$$

3.2.2 Applying Simplex method

The simplex method, in a very efficient way, starts from an extreme point and evaluates the function for optimality, and then moves to the neighboring point and so on until it reaches the optimum it also insures that the given problem has no finite optimal solution.

The original (non slack) variables can be taken as non basic variables, and slack variables can be taken as basic variables (BV) and then solving for these basic variables. Applying simplex method to a system goes through a sequence of iterations. In such a way, at each iteration the set of variables that are organized in a Table will be updated until the optimum is achieved. Herein, initial Table will be presented. The system 3.6 was given above is in a standard form. Preparing the problem for simplex method, we take all the variables to one side in the objective function and make the right hand side to be zero;

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + z = 0 \tag{3.7}$$

Making z a new variable with slack variables were introduced in the same system 3.6 as

$$x_{n+1}, x_{x+2}, \ldots, x_{n+m}$$

Then the initial Table is:

Table 3.3: An initial Table for simplex method

BV	x_1	x_2	...	x_n	x_{n+1}	x_{n+2}	...	x_{n+m}	z	RHS	Ratio
x_{n+1}	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	0	b_1	
x_{n+2}	a_{21}	a_{22}	...	a_{2n}	0	1	...	0	0	b_2	
$\vdots$	$\vdots$						$\vdots$			$\vdots$	
x_{n+m}	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	0	b_m	
OR	$-c_1$	$-c_2$	...	$-c_n$	0	0	...	0	1	0	

The Table consists of m rows; m is the number of slack variables introduced; Plus two extra rows (the top and the bottom row), in the top row, all variables (basic and non basic) are written including variable z. Thus the number of columns is equivalent to the number of all variables with two additional columns; they are labeled as (RHS) and ratio column, while bottom row is called Objective Row (OR).

Now, how to fill the Table, the coefficients of the variables in equation 3.7 are written in the objective row including coefficient of variable z which is +1 and RHS of the equation which is equal to zero. The other constraints in 3.6 are interred in the top rows; each constraint corresponds to its containing slack variable. In the first left-hand side column of the above Table slack (basic) variables are written, finding their values is the mission of this powerful method, the other non basic variables do not appear in the mentioned column, so their values are considered to equal to zero. Applying simplex method to solve this initial Table is explained by the following problem.

Evaluate the value of two variables x and y such that

$$\text{Maximize } z = 8x + 10y$$

Subject to

$$2x + y \leq 50$$

$$x + 2y \leq 70$$

$$x \geq 0\,,\ \ y \geq 0$$

After adding slack variables s and w the problem becomes: find the value of x, y, s, w such that

$$Max\ \ z = 8x + 10y$$

Subject to

$$2x + y + s = 50$$

$$x + 2y + w = 70$$

$$x, y, s, w \geq 0.$$

First of all, deriving objective function $z - 8x - 10y = 0$ then the initial Table will be

Table 3.4: The initial simplex method Table for an example

BV	x	y	s	w	z	RHS	ratio
s	2	1	1	0	0	50	
w	1	2	0	1	0	70	
OR	-8	-10	0	0	1	0	

Simplex method seeks to increase the objective function, to do so; all numbers in the objective row must be nonnegative, if there is no negative entry in the objective row at the first (initial) Table it means the optimal solution is already exists, however, there are operations to change each negative to positive number.

In the above initial Table basic variables are s and w, while x and y are not in the set of basic variables, so they are non basic variables and their values equal to zero; $x = 0\ , y = 0$. Taking a look to objective row, obviously it is not an optimal solution, since it can be rewritten as follows

$$z = 8x + 10y - 0s - 0w \tag{3.8}$$

$$z = 0$$

To increase the value of z, the value of x or y must be increased, since both x and y have positive coefficients in equation 3.8 so they appear as negative coefficient in the objective row, then most negative would be chosen to be increased, as well as the value of equation 3.8 increases. So:

Rule number one; take the most negative entry (if any) in the objective row, if there are equal entries then choose any one. So in Table 3.4 the most negative element is -10 belongs to the non basic variable y, selecting the column that containing -10 because non basic variable y will enter to the set of basic variables and hence become basic variable. The selected column is called *pivotal column* (PC). So variable y will enter and one slack variable will get out. One of them is entering variable and the other is departing variable. To decide which slack variable will be replaced by variable y *pivotal row* (PR) must be determined.

For determination of PR the *ratio* should be formed. Row containing smallest ratio is a pivotal row; for evaluating ratio; divide each entry in the column labeled RHS (except of the intersection cell between RHS and OR) by its corresponding entry in the pivotal column and then choose smallest one. In the above problem, smaller ratio between $\frac{50}{1}, \frac{70}{2}$ obviously is $\frac{70}{2}$ so pivotal row is the second row and departing variable is s. Therefore replace variable s by variable y. Note that while evaluating ratio, dividing by negative entries and zero is not allowable. So if there exist zero or negative entries in the pivotal column they should be neglected. If all entries in PC are negative or zero then the given problem has infinite optimal solutions. The intersection cell between pivotal row and pivotal column is called pivot entry or pivot element, thus pivot element in the given problem is $a_{22} = 2$. Table 3.5 describes each of pivotal row, pivotal column, and pivot element.

Table 3.5: Illustration of pivotal elimination in simplex method

↓ Entering variable

BV	x	y	s	w	z	RHS	ratio
s	2	1	1	0	0	50	50/1
w	1	2	0	1	0	70	70/2
OR	-8	-10	0	0	1	0	

Departing variable ←

This is the smallest ratio

The most negative entry in row

Pivot element

The shaded row and column by green color are pivotal row, column respectively, while the yellow cell is pivot element, the arrows ← and ↓ indicate departing and entering variables respectively.

Hence, increasing variable y implies increasing objective function, to obtain a new Table for increasing value of y and objective function there are several steps to do.

These procedures will be done by row operation to get the desired results. First of all select and circle pivot element thereby pivotal row and column will be decided and chosen, so mark them by a different color, as well as the entering and departing variables are automatically determined.

It is necessary to make pivot element equal to one. And the other elements in its column above and below the pivot element have to be zero, so suppose pivot element is k then multiply the pivotal row by $\frac{1}{k}$, that way; now pivot element is equal to one.

For making all entries zero in pivotal column above and below the pivot element, an appropriate multiple of pivot row should be added to OR and all other rows. That way a new initial Table is obtained. Hence label pivotal row by entered variable in new Table. These steps are called pivotal elimination. So solving above problem, pivot element must be divided by 2, as well as the other elements in the same row so the Table becomes as follows:

Table 3.6: Shows how to make pivot element equal to 1

BV	x	y	s	w	z	RHS	ratio
s	2	1	1	0	0	50	
w	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35	
OR	-8	-10	0	0	1	0	

To make the entry that locates above the pivot element equal to zero, the pivot row should be subtracted one time from its row (in current example, the first row), see below Table:

Table 3.7: Illustrates how to make all entries in pivot column equal to zero

BV	x	y	s	w	z	RHS	ratio
s	$^3/_2$	0	1	$^{-1}/_2$	0	15	
w	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35	
OR	-8	-10	0	0	1	0	

By the same way, in order to make the element that locates below pivot element equal to zero, so pivot row should be added to objective row ten times to obtain the following Table

Table 3.8: Explains how to take operations in pivotal elimination

BV	x	y	s	w	z	RHS	ratio
s	$^3/_2$	0	1	$^{-1}/_2$	0	15	
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35	
OR	-3	0	0	5	1	350	

Table 3.8 is a new initial Table, the pivotal elimination again should apply if steal there exist nonnegative elements in objective row, however, if all entries in OR are positive so optimal solution has reached. In given problem, since only negative element in OR is -3 then the pivotal column is the first one labeled with variable x. The two ratios are $\frac{15}{\frac{3}{2}}$ and $\frac{35}{\frac{1}{2}}$, since 10

is less than 70 so the pivot row is the first one and variable s is leaving, hence circle the first entry $\frac{3}{2}$ as a new pivot. Again performing pivot elimination, Table 3.8 yields Table 3.9.

Table 3.9: The optimal solution has reached

BV	x	y	s	w	z	RHS	ratio
x	1	0	$\frac{2}{3}$	$-1/3$	0	10	
y	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	30	
OR	0	0	2	4	1	380	

The objective row of this last Table is free of negative elements so the solution is optimal and the computation stops here, these optimal solutions are:

$$x = 10 \ , \ y = 30, \quad s = 0\,, \quad w = 0.$$

These values make the objective function $z = 380$ that is the maximum value which z can reaches to it.

The flow chart below in Figure 3.5 describes simplex method in general

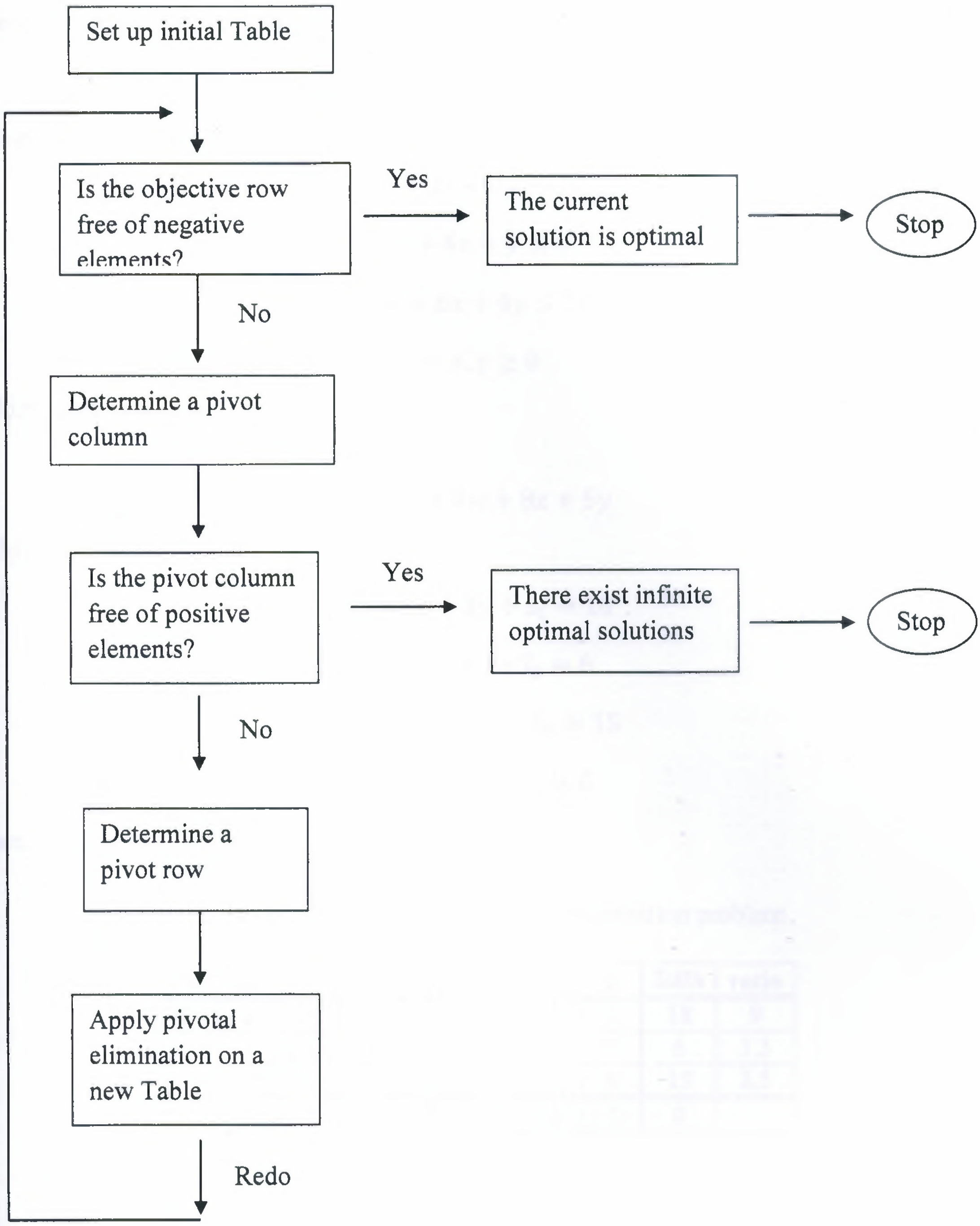

Figure 3.5: Flowchart of pivotal elimination

Consider another maximization problem

$$\text{Max } z = 4w + 8x + 5y$$

Subject to

$$w + 2x + 3y \leq 18$$

$$w + 4x + y \leq 6$$

$$2w + 6x + 4y \leq 15$$

$$w, x, y \geq 0$$

Adding SVs to obtain

$$\text{Max } z = 4w + 8x + 5y$$

Subject to

$$w + 2x + 3y + S_1 = 18$$

$$w + 4x + y + S_2 = 6$$

$$2w + 6x + 4y + S_3 = 15$$

$$w, x, y, S_1, S_2, S_3 \geq 0$$

Then, all tables are stated below

Table 3.10: Initial table for a maximization problem

BV	w	x ↓	y	S_1	S_2	S_3	z	RHS	ratio
S_1	1	2	3	1	0	0	0	18	9
← S_2	1	(4)	1	0	1	0	0	6	1.5
S_3	2	6	4	0	0	1	0	15	2.5
OR	-4	-8	-5	0	0	0	1	0	

Table 3.11: Continuous of pivotal elimination of a maximization problem

BV	w	x	y	S_1	S_2	S_3	z	RHS	ratio
S_1	$\frac{1}{2}$	0	$\frac{5}{2}$	1	$\frac{-1}{2}$	0	0	15	6
x	$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{3}{2}$	6
S_3	$\frac{1}{2}$	0	$\frac{5}{2}$	0	$\frac{-3}{2}$	1	0	6	2.4
OR	-2	0	-3	0	2	0	1	12	

Table 3.12: Continuous of pivotal elimination of a maximization problem

BV	w	x	y	S_1	S_2	S_3	z	RHS	ratio
S_1	0	0	0	1	1	-1	0	9	
x	$\frac{1}{5}$	1	0	0	$\frac{2}{5}$	$-\frac{1}{10}$	0	$\frac{9}{10}$	4.5
y	$\frac{1}{5}$	0	1	0	$\frac{-3}{5}$	$\frac{2}{5}$	0	$\frac{12}{5}$	12
OR	$-\frac{7}{5}$	0	0	0	$\frac{1}{5}$	$\frac{6}{5}$	1	$\frac{96}{5}$	

Table 3.13: Optimal solution of a maximization problem has reached

BV	w	x	y	S_1	S_2	S_3	z	RHS	ratio
S_1	0	0	0	1	1	-1	0	9	
w	1	5	0	0	2	$-\frac{1}{2}$	0	$\frac{9}{2}$	
y	0	-1	1	0	-1	$\frac{1}{2}$	0	$\frac{3}{2}$	
OR	0	7	0	0	3	$\frac{1}{2}$	1	$\frac{51}{2}$	

Hence, optimal solution is

$$w = \frac{9}{2}, \qquad x = 0, \qquad y = \frac{3}{2}.$$

The slack variables are

$$S_1 = 9, \qquad S_2 = 0, \qquad S_3 = 0.$$

And the optimal value of z is $\frac{51}{2}$.

It should be noted that simplex method can apply onto all cases of linear programming problem; however, the dissertation is focusing on standard linear programming problems which contains only nonnegative elements in the right hand side to fulfill the needs of game theory.

3.2.3 Duality

All previous problems dealt with maximization problems with restrictions such that all constraints must be less than or equal to the RHS, one question herein! What about minimization problems? Is standardizing LPPs stops here without solution! The answer of course no; there is a mechanism called duality which converts minimization to maximization problem as well to standard form, so, there is a link between Min and Max problems which is duality. So suppose the below pair of LPP:

$$\text{Max } z = b^T X$$

Subject to

$$AX \leq C \tag{3.9}$$

$$X \geq 0$$

And

$$\text{Min } V = C^T y$$

Subject to

$$A^T y \geq b \tag{3.10}$$

$$y \geq 0$$

Where A is $m \times n$, C is $m \times 1$, b is $n \times 1$, X is $n \times 1$, y is $m \times 1$

Problem 3.9 is called original or primal problem (P_p), the other one 3.10 is called the dual problem (P_D).

Note that in performing dual problem, coefficients of primal constraints become transpose coefficients of dual constraints and vice versa. Moreover, coefficients of objective function of P_p become the RHS of the constraints of P_D and vice versa.

Below is one example to clarify the above speech. Consider the primal problem is

$$\text{Max} \quad z = 3x_1 + 4x_2$$

Subject to

$$2x_1 + 3x_2 \leq 3$$

$$3x_1 - x_2 \leq 4$$

$$5x_1 + 4x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Thus, the dual problem is Min $V = 3y_1 + 4y_2 + 2y_3$

Subject to

$$2y_1 + 3y_2 + 5y_3 \geq 3$$

$$3y_1 - y_2 + 4y_3 \geq 4$$

$$y_1, y_2, y_3 \geq 0$$

Theorem: Dual of the dual problem is original primal problem.

Proof: rewriting the dual problem 3.10 in standard form to obtain

$$\text{Max} \quad V' = -C^T y$$

Subject to

$$-A^T y \leq -b \qquad (3.11)$$

$$y \geq 0$$

Now the dual of 3.11 is

$$\text{Min } V'' = (-b)^T w$$

Subject to

$$(-A^T)^T w \geq (-C^T)^T$$

$$w \geq 0$$

Or

$$\text{Max } z = b^T w$$

Subject to

$$Aw \leq C$$

$$w \geq 0 \tag{3.12}$$

Putting $w = X$ it is obvious problem 3.12 is primal problem.

Below is one example of linear programming problem to find its dual problem

$$\text{Min } z = 2x_1 + 3x_2$$

Subject to

$$3x_1 + 4x_2 \geq 5$$

$$x_1 + 2x_2 \geq 2$$

$$5x_1 + 3x_2 \geq 7$$

$$x_1 \geq 0 \text{ , } x_2 \geq 0$$

Hence dual problem is going to be like the following system

$$\text{Max } V = 5y_1 + 2y_2 + 7y_3$$

Subject to

$$3y_1 + y_2 + 5y_3 \leq 2$$

$$4y_1 + 2y_2 + 3y_3 \leq 3$$

$$y_1 \geq 0 \text{ , } y_2 \geq 0 \text{ , } y_3 \geq 0$$

Theorem: if the primal problem has an optimal solution with finite objective value, then the dual problem is also has an optimal solution and conversely. Moreover, the objective values of the dual problem and primal problem are equal.

Observe that, the objective row in the final Table of primal problem contains optimal solution to the dual problem while solving problem by simplex method. These optimal solutions can be found exactly under the columns of slack variables.

Hereinafter, one example to show that optimal solution for both problems (primal and dual) consider the minimization problem

$$\text{Min } v = 30x_1 + 40x_2$$

Subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 12 \\ x_1 + x_2 &\geq 9 \\ x_1 + 3x_2 &\geq 15 \\ x_1, x_2 &\geq 0 \end{aligned} \tag{3.13}$$

Then the dual for the problem is

$$\text{Max } z = 12y_1 + 9y_2 + 15y_3$$

Subject to

$$\begin{aligned} 2y_1 + y_2 + y_3 &\leq 30 \\ y_1 + y_2 + 3y_3 &\leq 40 \\ y_i \geq 0 \quad \text{For all } i &= 1,2,3. \end{aligned} \tag{3.14}$$

So this is a standard linear programming problem, introducing slack variables S_1 and S_2 to obtain

$$\text{Max } z = 12y_1 + 9y_2 + 15y_3$$

Subject to

$$2y_1 + y_2 + y_3 + S_1 = 30$$

$$y_1 + y_2 + 3y_3 + S_2 = 40$$

$$y_i \geq 0\,,\ S_i \geq 0 \qquad \forall\, i = 1,2,3.$$

Then solving the problem by simplex method obtaining the below tables.

Table 3.14: Initial table of dual problem

BV	y_1	y_2	y_3 ↓	S_1	S_2	Z	RHS	ratio
S_1	2	1	1	1	0	0	30	30
← S_2	1	1	(3)	0	1	0	40	40/3
OR	-12	-9	-15	0	0	1	0	

Table 3.15: Continuous of pivotal elimination of dual problem

BV	y_1 ↓	y_2	y_3	S_1	S_2	Z	RHS	ratio
← S_1	($\frac{5}{3}$)	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{50}{3}$	10
y_3	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{40}{3}$	40
OR	-7	-4	0	0	5	1	200	

Table 3.16: Continuous of pivotal elimination of dual problem

BV	y_1	y_2 ↓	y_3	S_1	S_2	Z	RHS	ratio
← y_1	1	($\frac{2}{5}$)	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	10	25
y_3	0	$\frac{1}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	10	50
OR	0	$-\frac{6}{5}$	0	$\frac{21}{5}$	$\frac{18}{5}$	1	270	

Table 3.17: Shows the optimal solution of the dual problem reached

BV	y_1	y_2	y_3	S_1	S_2	Z	RHS	ratio
y_2	$\frac{2}{5}$	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	25	
y_3	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	5	
OR	3	0	0	6	3	1	300	

Thus, the optimal solution for problem 3.14 is

$$y_1 = 0, \quad y_2 = 25, \quad y_3 = 5,$$

And the value of objective function is 300.

The optimal solution for P_p is found in the OR under S_1 and S_2 columns

$$x_1 = 6 \ , \quad x_2 = 3$$

Indeed, this mechanism is very useful for preparing games in game theory.

3.3 Microsoft Office Excel

Simplex method has been shown that is a great powerful method for solving linear programming problems, however, performing pivotal elimination and row operation techniques somewhat difficult to calculate manually like adding and multiplying elements (arithmetic calculations) so there is a possibility of falling into mathematical traps. Using Microsoft offices excel is the best way to avoid miscalculating and tedious algebraic calculations. The mysterious powers of MOE appears here since it is faster than calculator for solving formulas, light and free program available in almost computer devices, doing simple and complex computations, formulas, charts, schedules, and many more. Moreover, gives precise results step by step in such a way keeping the user in touch with how each step is going on.

3.3.1 How to use Excel

3.3.1.1 Entering data, formula, function, and some other tips

Figure 3.6 is a one sample worksheet of Microsoft Office Excel. A worksheet also called spreadsheet contains organized series of cells identified by rows and columns; is a store for data and documents that are used in Microsoft Excel spreadsheet. A collection of worksheets is called workbook.

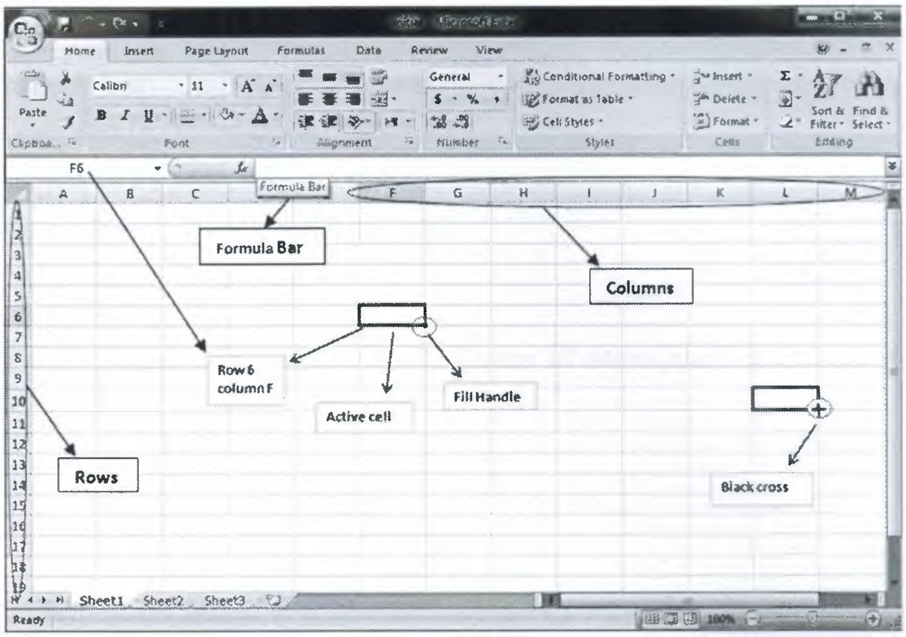

Figure 3.6: A sample of Excel worksheet with illustration

In Figure 3.6 above, there are letters A, B, C ...etc. these letters indicate columns; across the side there are numbers 1, 2, 3 ...etc. they are rows. When a cell has been selected at any given time, that is sign to a row and column, this selected cell is called the *active cell*; is that cell when you are going to type in. Only one cell is active at a time and is bounded by a heavy border. In Figure 3.6 column F and row 6 has been highlighted to notify you that you are in cell F6 the Microsoft workbook is consist of multiple sheets; sheet can be renamed to desired topic's name, to rename a sheet just right-click on a sheet bar then choose '*rename*' and then type the name of topic. For writing a text in any particular cell user can just click on the cell or moving by arrows in keyboard or hit the button *Tap* and then simply type text in the selected cell.

3.3.1.2 Simple arithmetic in Excel

First of all, there is a *formula Bar* to start writing anything in the spreadsheet, see Figure 3.6, for typing any formula or function, first the equal symbol (=) must be typed meaning that you are going to write an equation.

Arithmetic operators are:

- The plus sign (+) for addition
- The minus sign (-) for subtraction
- The cross sign (*) for multiplication
- The slash sign (/) for division

These symbols are available on the keyboard. In Figure 3.7 below, there is addition of two numbers

3.3.1 How to use Excel

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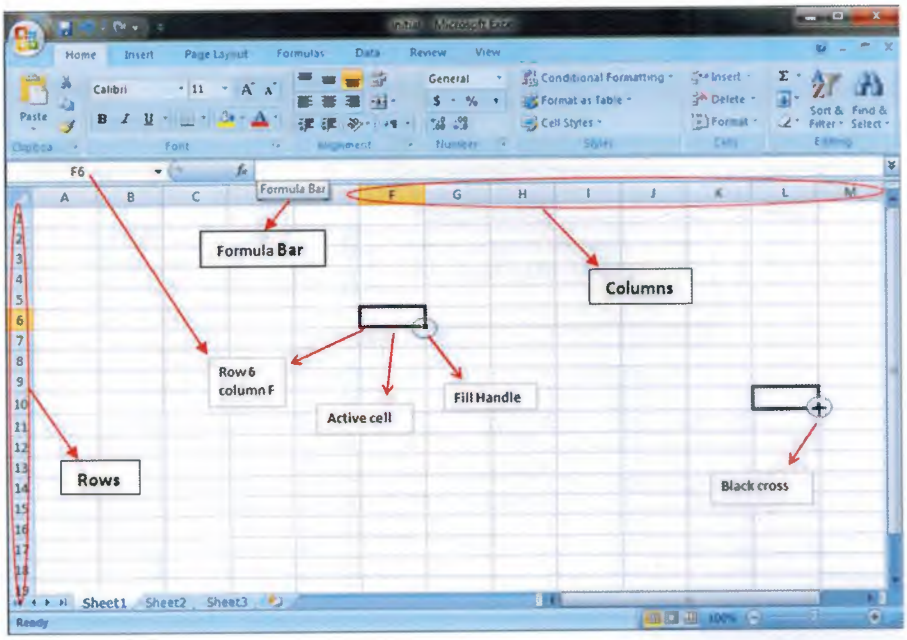

Figure 3.6: A sample of Excel worksheet with illustration

In Figure 3.6 above, there are letters A, B, C …etc. these letters indicate columns; across the side there are numbers 1, 2, 3 …etc. they are rows. When a cell has been selected at any given time, that is sign to a row and column, this selected cell is called the *active cell*; is that cell when you are going to type in. Only one cell is active at a time and is bounded by a heavy border. In Figure 3.6 column F and row 6 has been highlighted to notify you that you are in cell F6 the Microsoft workbook is consist of multiple sheets; sheet can be renamed to desired topic's name, to rename a sheet just right-click on a sheet bar then choose '*rename*' and then type the name of topic. For writing a text in any particular cell user can just click on the cell or

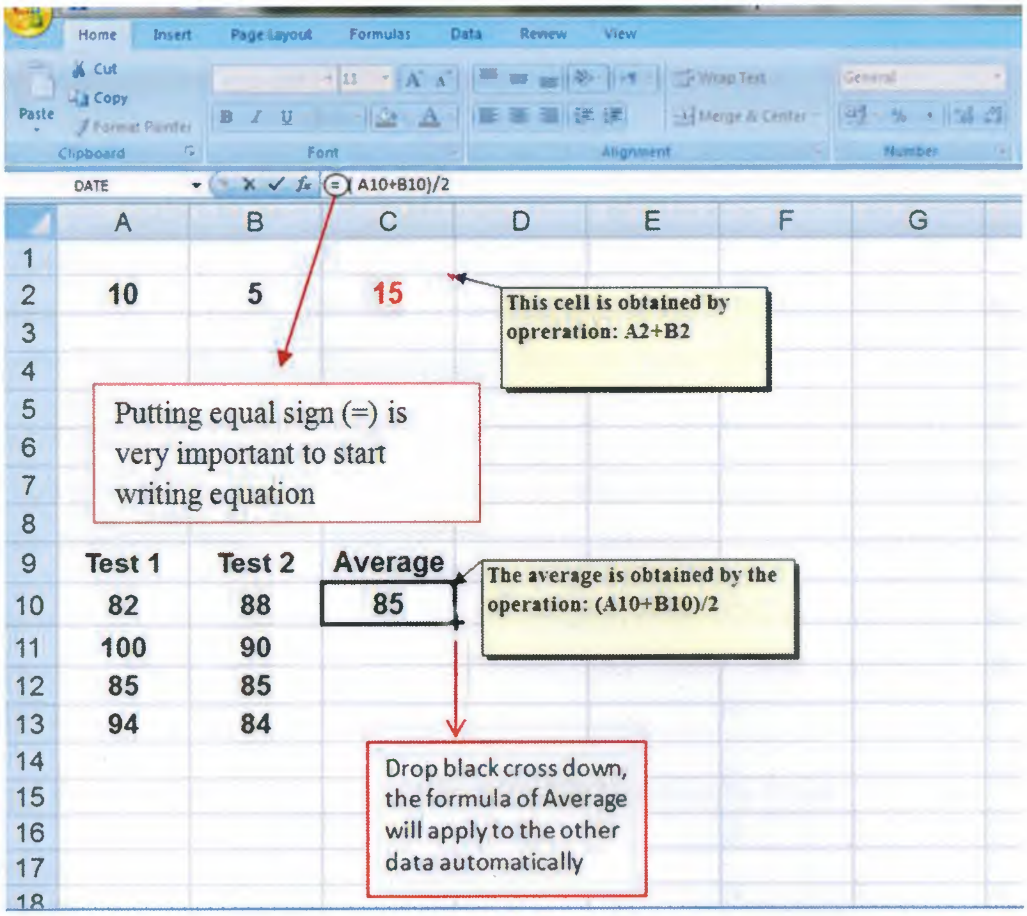

Figure 3.7: Illustration of some arithmetic operations in spreadsheet

In the box C2 has been written (=A_2+B_2) that means column A in row 2 is added to column B in row 2. To see the result of this equation just press *enter* button. At the same table there is two tests with grades of students, also the average of these two tests, in the box just below the average cell has been written the equation: (A10+B10)/2 this means the average of two tests. A10 indicates test number 1 which lies in column A and row 10, the same meaning for (B10). Slash symbol (/) means these two tests dividing by 2. The formula (A10+B10) should be put in parenthesis in order to Excel do the addition operation before division. To show the result of average just press the *enter* tap, herein, one thing to be noted; for finding the average of remaining grades below the test 1's and test 2's column does not need to rewrite the same equation in the corresponding opposite cell to each row each time; it is enough to do just one simple technique and you get all results at once. Here is the technique:

Click on the evaluated average in cell C10, so, it becomes an active cell, and then a small black square appears in the lower-right corner of the selected cell. This small square is called *Fill Handle*, see Figure 3.6, the pointer changes from White Square to a black cross while pointing to the Fill Handle, then select a range of cells and drag down fill handle to copy and apply the formula to all other data and you are done, see Figure 3.7.

Thus, whenever needed to apply some procedures in a particular cell to the below cells just pull down the fill handle, the same technique is valid for pulling fill handle to the left/right side for changing a row.

There is some other functions such as SUM, EXP, SIN, COS, LOG, and much more, for more information reader invited to visit *Microsoft Office Excel Help Center* for complete explanation.

3.3.1.3 Formulas, equations, and solving complete problems by Excel

Consider the SLPP

$$\text{Max } z = 2x_1 + 3x_2$$

Subject to

$$3x_1 + 5x_2 + S_1 = 6$$

$$2x_1 + 3x_2 + S_2 = 7$$

$$x_1, x_2 \geq 0, \quad S_1, S_2 \geq 0$$

Illustration of how to use Excel for solving this problem is demonstrated in Figure 3.8 up to 3.13.

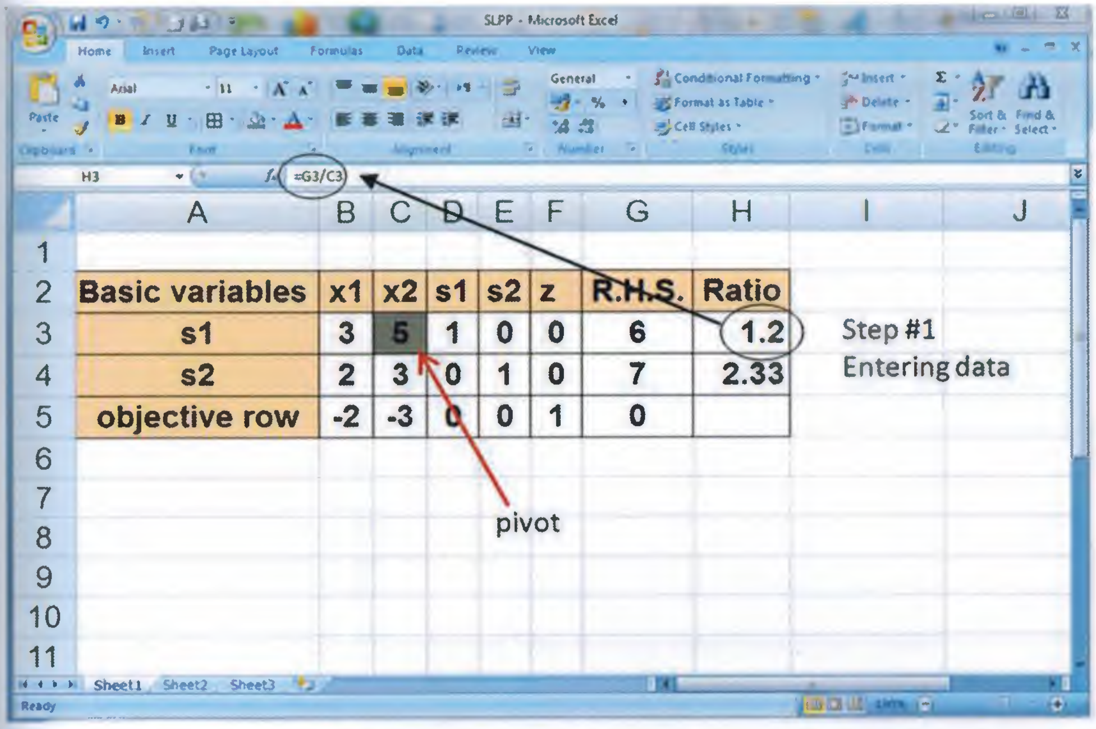

Figure 3.8: Illustration of how to type formulas in Excel

Figure 3.8 shows entering data into worksheet, the given table in the mentioned Figure is obtained from the above standard linear programming problem; the shaded cell in gray color is pivot element as pointed in Figure above, here, ratio is obtained by dividing entries in column labeled RHS by corresponding elements in pivot column (except of last element which is not allowable to be divided by the corresponding element in OR). The operation of evaluating value of ratio column is also featured in formula bar. Since minimum ratio between 1.2 and 2.3 is the first one so pivot row is determined.

The next following Figure shows that how to copy data for the second Table without entering all previous data one by one again, so it is useful for saving a lot of time.

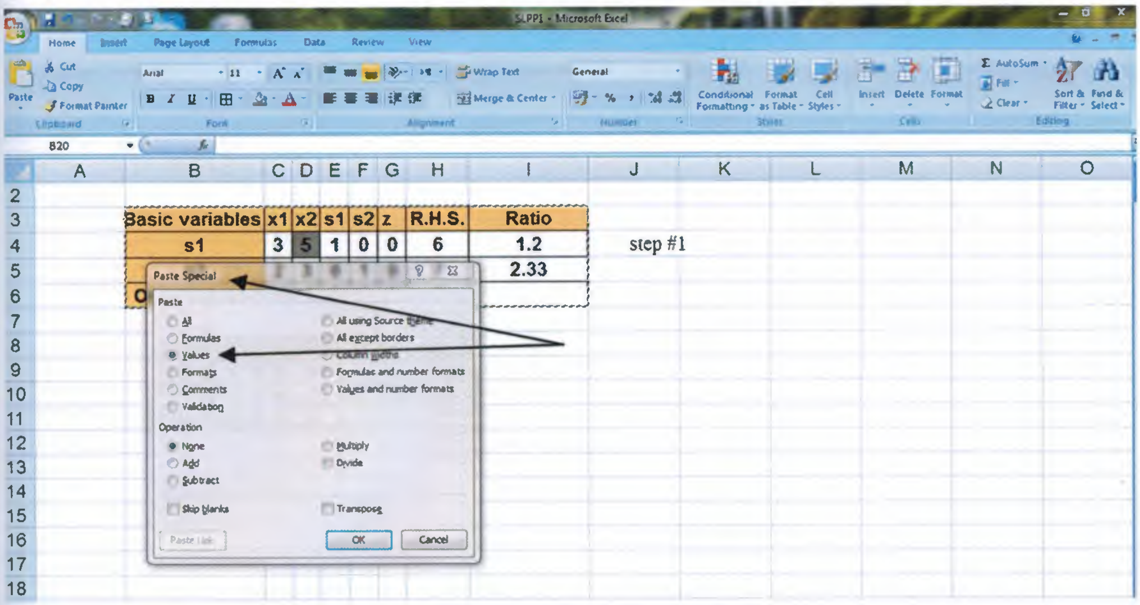

Figure 3.9: Explanation of how to copy data from initial Table

To do so, select entire Table then right click on the Table, after that choose *paste special* will appear the above dialog box in Figure 3.9, herein you have to select *values* button it means you want to copy all data as values not as formulas or any other not desired information.

When the Table has been copied you will attempt to make the pivot element equal to one, the procedure of making this element as one is described in the below Figure 3.10

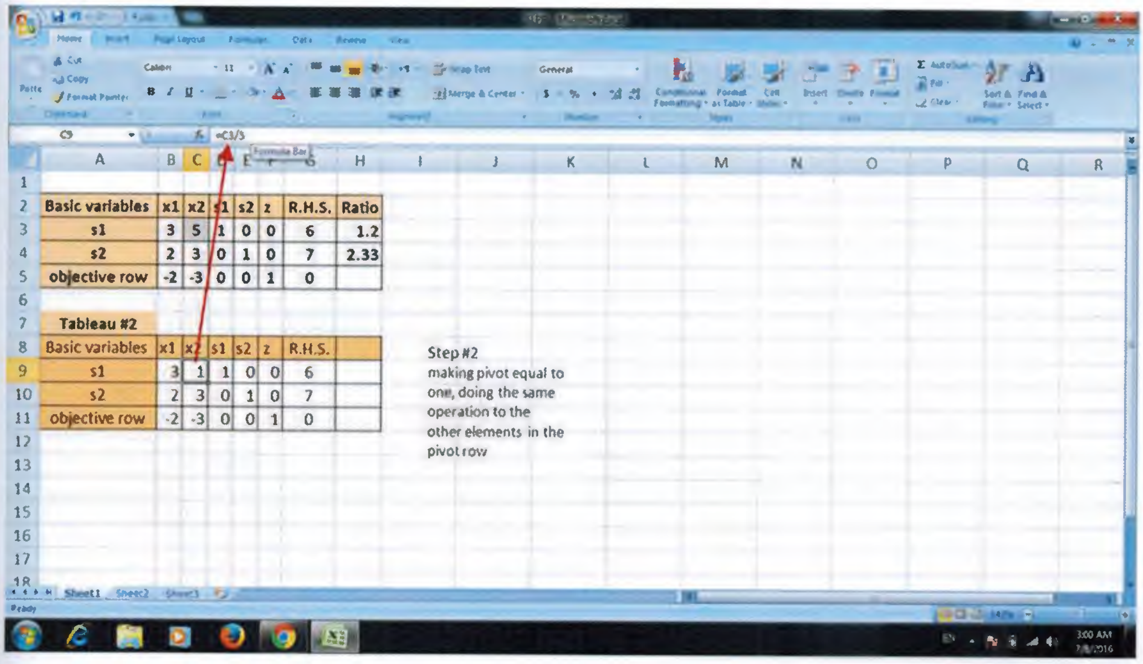

Figure 3.10: Explanation of making pivot element equal to one

It is obvious in formula bar the operation is dividing pivot by 5, however pivot is given from cell C3 to be changed in new Table, thus changing pivot in new Table depends on the old (previous) Table.

The following Figure 3.11 demonstrates how to change all other elements in pivot row with the same operation.

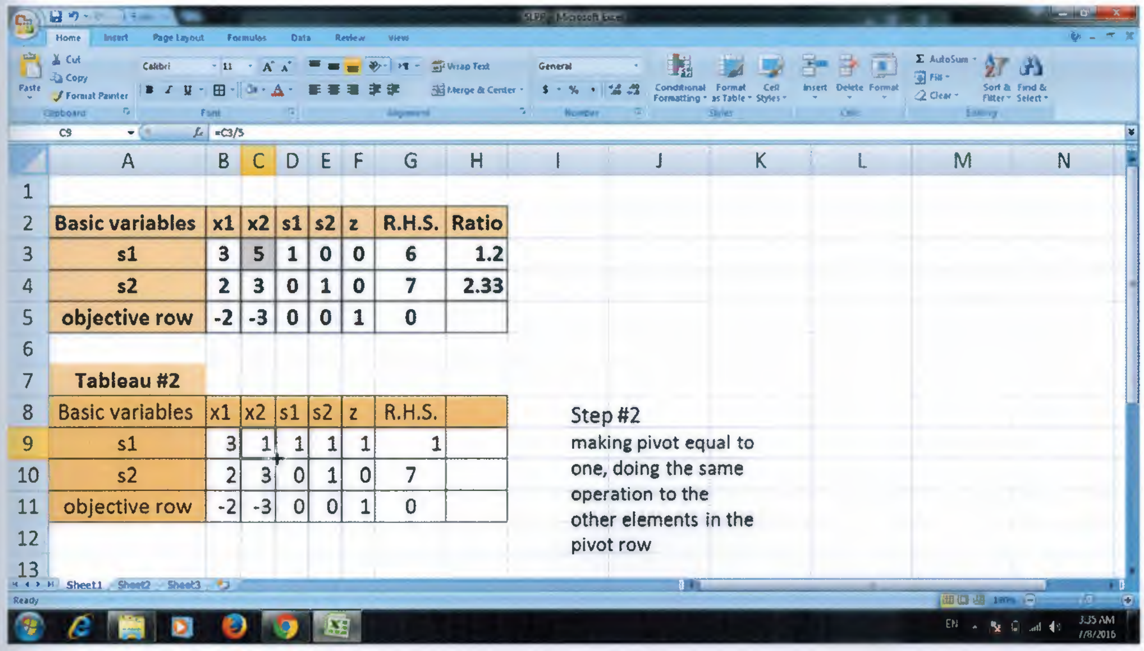

Figure 3.11: Explanation of changing all entries in pivot row

When the pointer point to the lower-right corner of pivot's box there appears a black cross or plus sign, this notifies that the pointer is ready to apply the operation onto the other entries of pivot row.

Figure 3.12 shows that how to apply operation to pivotal row's entries.

While black cross appears, drag it to the right hand side as well to the opposite (left hand) side elements (if any). In Figure 3.12 all entries at right hand side have been covered by a dashed border. Now by doing this technique, the same operation that did on pivot is also done onto other corresponding row elements, which is; in the current case, (D3 divided by 5, E3 divided by 5 as well B3 divided by 5).

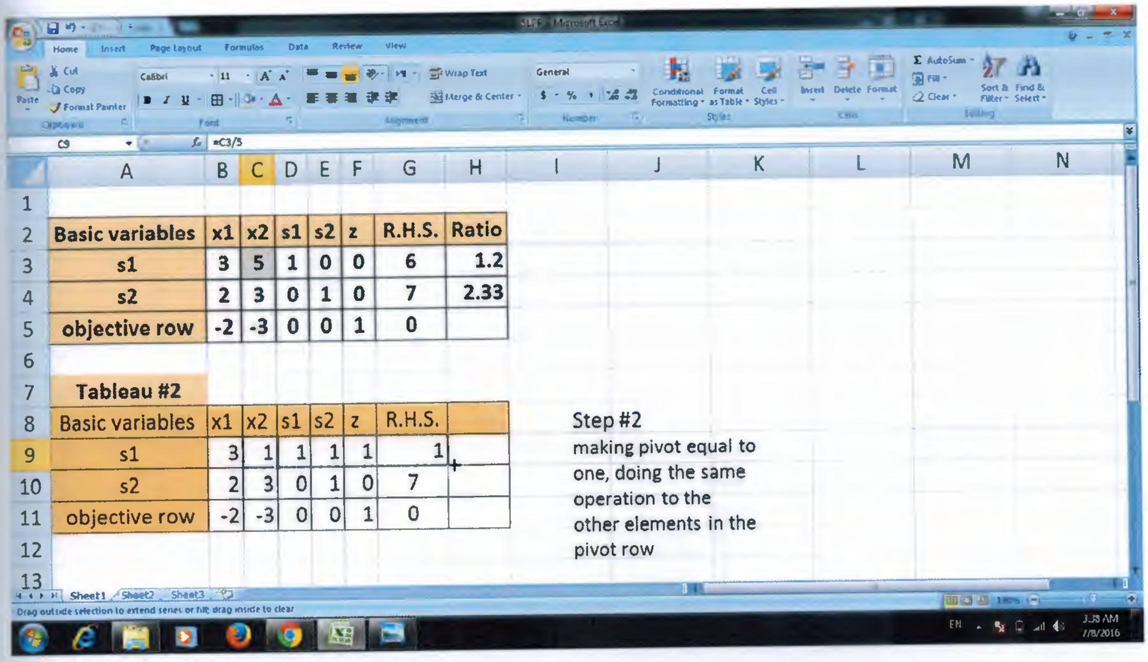

Figure 3.12: Illustration of applying row operation on pivot row

The next step is to make all entries of pivot column equal to zero except of pivot element, the answer of how to do this, is given in Figure 3.13 below

The element in the cell C10 has got a small red color triangle in the upper-right corner which notifies that there exists a comment, this comment says the operation that has been applied to this box is (-3*C9+C4); mathematically it means pivot row has been subtracted from the second row three times and hence element C10 became zero, see Figure 3.13

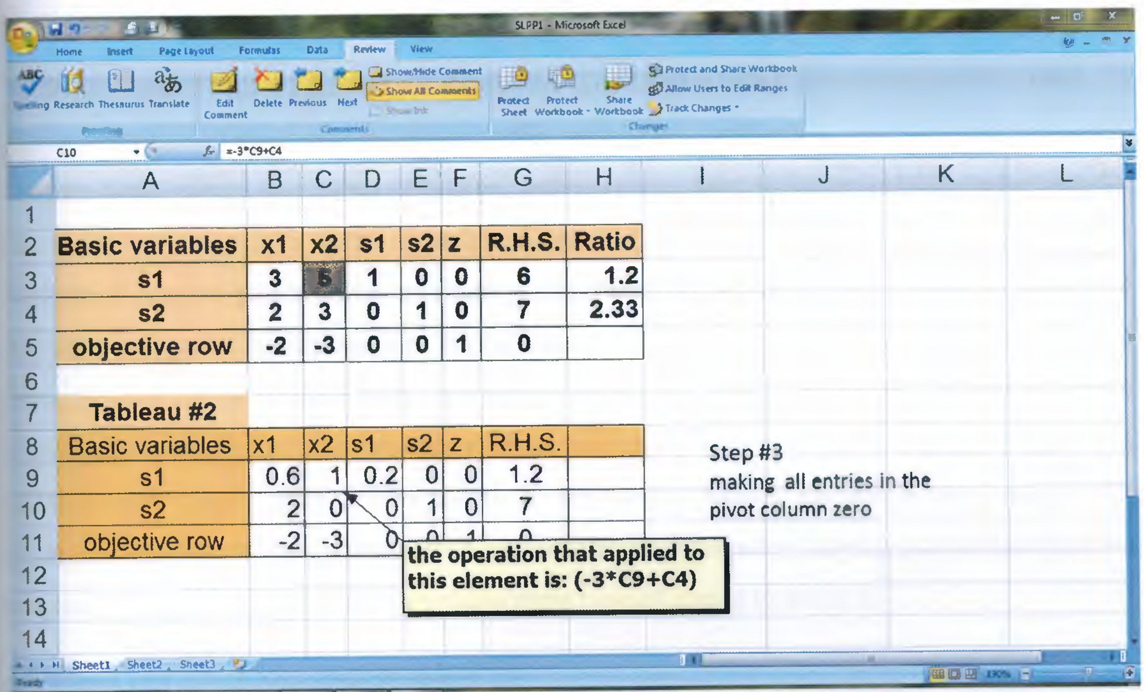

Figure 3.13: Explanation of changing entries in pivotal column

It should be noted that Table #2 again depended on the previous Table, therefore C4 brought to be subtracted by pivot. Note that, however, in the normal cases the red triangle belonging to comment does not appear automatically; the researcher did that to be clearer to the reader.

These techniques and procedures by a quite same manner and same steps must continue for the other none zero elements until the problem gets an optimal solution i.e. to get all entries in objective row as zero.

This was an illustration example, remaining steps is left to the reader.

However, the mentioned information about using Excel is a little tiny bit; there are many more magical powers.

CHAPTER 4
SOLVING TWO-PERSON ZERO-SUM GAME BY LINEAR PROGRAMMING

Chapter 4 will merge two forms of games, extensive and normal forms; solving game trees by LP, converting extensive form to matrix form using excel and then solving it by simplex method; solving a game means evaluating the value of the game and finding at least one optimal strategy for both players or more than one if it is possible. Linear programming is a powerful tool for finding value of games, but occasionally there exist some types of games that require special rules and formulas without needing LP, there exist other ways to reach to a solution more simply, they will be shown in the current chapter, but this is not a case for all situations these methods are inadequate to cover each instance for solving games therefore the magic of LP appears here since it contains most type of games to solve it.

4.1 Types of Games Solvable Without LP

4.1.1 Games with Saddle point

As it was discussed earlier, there are game problems which require no lengthy mathematical procedures for evaluating their values except that you have to check whether there is a saddle point or not, if the saddle point exists then it is the value of the game and the strategies are pure for the players. Reminding ourselves with some more examples, consider the game problem with the payoff matrix below

		Player Q	
		q_1	q_2
	p_1	3	0
Player P	p_2	-2	3
	p_3	7	5

Payoff matrix 4.1: A game between two persons for searching a saddle point

Since the entry in third row and the second column namely entry 5 is the minimum of its row and at the same time the maximum of its column so it represents the saddle point. Hence it is the value of the game; the sign of the value is positive means that row player (P) gains 5 units while column player (Q) loses 5 units. The strategy for each player is pure, and is represented by the vector: [0 0 1]. This means the row player should always choose the third row, this way he guaranties to win at least 5 units. The strategy for the column player is given by $[0 \;\; 1]^T$ which means the column player should always play the second column, this way he guaranties not lose more than 5 units. And this is the best act for them.

4.1.2 Matrix game of the size 2×2

A special case of matrix games while saddle point does not exist is; when the payoff matrix is of the size 2×2, there are some formulas to evaluate the value and optimal strategies of such a game. The formulas can be stated at as follows:

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a general form of 2×2 game. First of all check for the saddle point if does not exist then it can be shown that $a_{11} + a_{22} - a_{12} - a_{21} \neq 0$.

If the game is not strictly determined then one of these two cases will arise; either:

- $a_{11} > a_{12}\, , a_{12} > a_{22}\, , a_{22} > a_{21}$ *and* $a_{21} > a_{11}$ or;
- $a_{11} < a_{12}\, , a_{12} > a_{22}\, , a_{22} < a_{21}$ *and* $a_{21} > a_{11}$

Then formulas for optimal strategies and value v of the game could be developed as follows:

Consider row player chooses his first row with certain probability p_1, column player chooses column 1 and 2 then

$$a_{11}p_1 + a_{21}(1 - p_1) = a_{12}p_1 + a_{22}(1 - p_1)$$

Solving for p

$$a_{11}p_1 + a_{21} - a_{21}p_1 = a_{12}p_1 + a_{22} - a_{22}p_1$$

$$a_{11}p_1 - a_{21}p_1 - a_{12}p_1 + a_{22}p_1 = a_{22} - a_{21}$$

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

This is the strategy one, if he chooses probability p_2 then

$$a_{11}(1 - p_2) + a_{21}p_2 = a_{12}(1 - p_2) + a_{22}p_2$$

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

By the similar way player 2's optimal strategy can be found and they are:

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

And the value of the game is given by

$$v = a_{11}p + a_{21}(1 - p)$$

$$= \frac{a_{11}a_{22} - a_{11}a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} + a_{21} - a_{21}p$$

$$= \frac{a_{11}a_{22} - a_{11}a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} + a_{21} - \frac{a_{21}a_{22} - a_{21}a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

Example: A=$\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ optimal strategies and the value of the matrix game are as follows:

$$p_1 = \frac{1 + 2}{1 + 1 + 2} = \frac{3}{4}$$

$$p_2 = \frac{0+1}{1+1+2} = \frac{1}{4}$$

$$q_1 = \frac{1+1}{1+1+2} = \frac{1}{2},$$

$$q_2 = \frac{0+2}{1+1+2} = \frac{1}{2},$$

$$v = \frac{0-2}{1+1+2} = \frac{-1}{2}.$$

Thus the set of optimal strategies for row player is $\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}$, for column player is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

So the row player should choose the first row 75% of the time and the second row 25% the time. While the column player should choose each of his options 50% of the time

4.1.3 Dominated strategy

In this type of game problems, as mentioned earlier, the goal of row player is to maximize his gain and the goal of column player to minimize his loss. Therefore, row player aims at choosing large rows (row containing large entries) he does not need row with small entries he will not use it thus he will eliminate the smallest row among the others, attempting to get large payoff, likewise column player will eliminate the largest column among the other to obviate large losses he has to delete weak strategy. Occasionally this situation is available for some payoff matrices and has two advantages for solving game problems: sometimes leads to get the value of the game directly other times reduces the size of the large payoff matrices to a 2×2 matrix and is then solved by the previous method discussed in 4.1.2. Suppose A=(a_{ij}) is a matrix if every element of row r in A is less than or equal to the corresponding elements of row k then r row is said recessive row, and the k row dominates row r i.e. if $a_{rj} \leq a_{kj}$ $\forall j$ then row k is said to dominate the rth row. Moreover column k is said to dominate the

jth column if $a_{ik} \leq a_{ij}$ $\forall i$. It should be noted that deleting the recessive row and recessive column do not change the value of the game.

To give an example, consider a game between two players A and B each player has 3 choices in their strategies, look at payoff matrix 4.2. Player A will never choose the third option/row because it is a recessive row, thus will be dropped to avoid small gains, player B will observe that the 3[ed] option/column in his strategy has large entries so it is a recessive column he will avoid selecting it therefore, option/column 3 in his strategy will be dominated by strategy 1 because all its values greater than or equal to the respective values in the column 1.

		B		
		1	2	3
A	1	1	3	5
	2	2	4	8
	3	-1	0	2

Payoff matrix 4.2: Payoff matrix consisting of dominated strategies

After deleting the recessive Row3 and the recessive column3, the payoff matrix becomes:

	1	2
1	1	3
2	2	4

Payoff matrix 4.3: Removing dominated strategies by both players

Again player A deletes row 1 to obtain:

	1	2
2	2	4

Payoff matrix 4.4: Deleting dominated strategy by player

While player B should eliminate 2[nd] column and here the value of the game becomes 2 biased towards player A.

In some instances the value of the game cannot be found like the above case, as the payoff matrix can be only reduced to a 2×2 matrix and not a 1×1 matrix. To illustrate such a case by an example;

Given the below payoff matrix 4.5

		B: 1	2	3
	1	2	-1	3
A	2	-2	2	4
	3	3	0	4

Payoff matrix 4.5: Payoff matrix containing dominated strategy

The first row will be dominated by the third row, so row1 is recessive it can be deleted and the matrix becomes:

-2	2	4
3	0	4

Payoff matrix 4.6: First row of payoff matrix is dominated by the third row

Here the 2nd column dominates the 3ed column, so the later is recessive it can be removed to obtain

-2	2
3	0

Payoff matrix 4.7: The reduced 2x2 matrix

It is clear that there are no more recessive rows or columns; also no saddle point, therefore, the 2×2 matrix can be solved by 2×2 matrix method formulas presented earlier:

$$p_1 = \frac{0-3}{-2+0-3-2} = \frac{3}{7},$$

$$p_2 = \frac{-2-2}{-2+0-2-3} = \frac{4}{7},$$

$$q_1 = \frac{0-2}{-2+0-2-3} = \frac{2}{7},$$

$$q_2 = \frac{-2-3}{-2+0-2-3} = \frac{5}{7},$$

$$v = \frac{0-6}{-2+0-2-3} = \frac{6}{7}.$$

The set of optimal strategies are $p = \begin{pmatrix} 0 & \frac{3}{7} & \frac{4}{7} \end{pmatrix}$ for player A and $q = \begin{pmatrix} \frac{2}{7} \\ \frac{5}{7} \\ 0 \end{pmatrix}$ for the player B,

note that the zero entry in the strategies indicates rows/columns and this means these options should not be selected at all.

4.2 Linear Programming for Solving Matrix Games

Linear programming is a powerful tool for solving most game problems in game theory; it is a sufficient application to contain every situation in games without using any other methods which mentioned earlier in this dissertation. On the other hand, the previous techniques are simple and inadequate to solve large and complex problems. However, using these methods are for special cases, maybe using them are more convenient for that cases instead of using LP, anywise, it is not impossible to apply LP method even on those cases also, for the sake of comprehensiveness of LP to appear its power and magic so these cases will also be analyzed in LP format. The second objective of this chapter is to convert extensive to normal form it means solving game tree, evaluating the value of tree by using LP format.

Now the dissertation will focus on the essential subject; that is converting game problems to an LP format, there are number of steps to follow before using the method they can be summarized as:

1. Check whether the payoff matrix has a saddle point or not; if saddle point exists then the saddle point is the value of the game as it was mentioned in earlier sections.
2. Check for dominant/recessive rows/columns, if any exists then delete the recessive row/column from the strategies and follow the solution method that was illustrated earlier.
3. If the game has 2×2 payoff matrix or it can be reduced to one that has 2x2 payoff matrix, then it is solvable by special algebraic formulas introduced earlier.

Of course the possibility of reducing the problem to a simpler one with smaller payoff matrix should be explored first and then the procedures for solving the problem should be followed. However, what is important here is that, even, if no possible reduction procedures are followed, the LP will still solve the game problem producing the correct answer.

The idea behind LP method depends on two theorems one of them Fundamental Theorem of Matrix Games which states: "*every matrix game has a solution. Moreover* $v = v'$". (Kolman and Hill, 2005) the second one is the theory of duality (section 3.2.3) this is so because the value of the game could be found from row players' standpoint who aims at maximizing his gain or by column player who aims at minimizing his loss. Either way we can reach the same solution, .In fact, in the linear programming, the constraints of one player is the dual of the other and vice versa.

In order for this method, to work properly, the entries of the payoff matrix must be positive. To achieve that, a suitable constant k could be added to each entry of the payoff matrix to obtain all positive entries. It should be noted that this constant does not change the optimal strategies for the players, and the value of the new game problem is $k + v$ (v is the value of the old game problem, while the new game is the game after adding a constant to its entries). Recessive row/column if any, are to be deleted to reduce the size of the problem, however, the results will be the same, without removing the recessive columns/rows.

Let G be an $m \times n$ matrix game, and let D and E be the row and column players respectively, player D is trying to find p_i , $i = 1, \ldots, m$ where p_i is player D's strategy, such that

$$v \text{ is maximum}$$

Subject to

$$a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m - v \geq 0$$

$$a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m - v \geq 0 \qquad (4.1)$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \quad \vdots$$

$$a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m - v \geq 0$$

$$p_1 + p_2 + \cdots + p_m = 1$$

$$p_i \geq 0\,,\; i = 1, \ldots, m. \qquad v \geq 0$$

Let

$$y_i = \frac{p_i}{v}\;,$$

note that $v \neq 0$ because every entry of the matrix are positives

Since

$$\sum_{i=1}^{m} p_i = 1$$

then

$$\sum_{i=1}^{m} y_i = \frac{1}{v}$$

therefore the value of the game v is a maximum if and only if

$$\sum_{i=1}^{m} y_i$$

is minimum.

The row player problem becomes:

$$\text{Minimize } y_1 + y_2 + \cdots + y_m$$

Subject to

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq 1$$

$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq 1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad (4.2)$$

$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq 1$$

$$y_i \geq 0 \;, \qquad \forall\, i = 1, \ldots, m$$

This is a linear programming problem and can be solved by simplex method as explained in section 3.2

Now, player E (column player) needs to find minimum v' and q_i, (where q_i is player E's strategy) such that:

$$v' \text{ is minimum}$$

Subject to

$$a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n - v' \leq 0$$

$$a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n - v' \leq 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad (4.3)$$

$$a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n - v' \leq 0$$

$$\sum_{i=1}^{n} q_i = 1$$

$$q_i \geq 0 \,, \qquad \forall\, i = 1, \ldots, n \qquad\qquad v' \geq 0$$

Let

$$x_i = \frac{q_i}{v'}$$

since $v^{'} = v$ (as fundamental theorem states that)

So we can restate

$$x_i = \frac{q_i}{v}$$

Thus:

$$\sum_{i=1}^{n} x_i = \frac{1}{v}$$

Since

$$\sum_{i=1}^{n} q_i = 1$$

So v is minimum if and only if $\sum_{i=1}^{n} x_i$ is maximum.

The problem of player E could be rewritten as follows:

$$\text{Maximize } x_1 + x_2 + \cdots + x_n$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq 1 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq 1 \\ x_i \geq 0, \qquad \forall\, i = 1, \ldots, n & \end{aligned} \tag{4.4}$$

This is a linear programming problem in its standard form. The problem 4.4 is the dual of problem 4.2. Applying simplex method to 4.4 until all entries in the objective row becomes positive or zero. Then the optimal strategies for player D will be in the objective row under the slack variables' columns, while the optimal strategies for player E will appear in the RHS column. It should be noted that v can be found by summation of x_i , $i = 1, \ldots, n$. When summation x_i is maximum, v is minimum.

Recall the matching pennies game whose value is already known, it is equal to zero, we demonstrate that LP and simplex method will lead to the right answer.

	H	T
H	1	-1
T	-1	1

Payoff matrix 4.8: Applying LP on matching pennies game

A suitable constant k is to be added to all entries to make the entries positive. For that let $k = 2$, the new payoff matrix will be:

	H	T
H	3	1
T	1	3

Payoff matrix 4.9: With positive entries

Then LP problem 4.4 for player E is

$$\text{Max } x_1 + x_2$$

Subject to

$$3x_1 + x_2 \leq 1$$

$$x_1 + 3x_2 \leq 1$$

$$x_1 \geq 0\,,\; x_2 \geq 0$$

Adding slack variables x_3 and x_4 to constraints to obtain the equalities;

$$3x_1 + x_2 + x_3 = 1$$

$$x_1 + 3x_2 + x_4 = 1$$

$$x_i \geq 0\,, \quad \forall\, i = 1, \dots ,5$$

Using Excel work sheet, the simplex method can be applied as follows:

The circled number is pivot element, hence x_1 is entering variable and x_3 is leaving variable

Table 4.1: Iteration 0 of simplex method

BV	x1	x2	x3	x4	Z	RHS	Ratio
x3	(3)	1	1	0	0	1	1/3
x4	1	3	0	1	0	1	1
OR	-1	-1	0	0	1	0	

The circled number is pivot element, hence x_2 is entering variable and x_4 is leaving variable

Table 4.2: Iteration 1 of simplex method

	x1	x2	x3	x4	Z	RHS	Ratio
x1	1	0.33	0.33	0	0	0.33	1
x4	0	(2.67)	-0.33	1	0	0.67	0.25
Objective row	0	-0.67	0.33	0	1	0.33	

Table 4.3: Iteration 2 of simplex method

	x1	x2	x3	x4	z	RHS
x1	1	0	3/8	- 1/8	0	1/4
x2	0	1	- 1/8	3/8	0	1/4
Objective row	0	0	1/4	1/4	1	(1/2)

Thus $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{4}$

Entry 1/2 in the objective row under the RHS column represents the optimal solution ($\aleph$) and it is the maximum of $x_1 + x_2$. So the minimum value of v is

$$v = \frac{1}{\frac{1}{2}} = 2$$

and it is the value of the new game, so the inverse of entry $\aleph$ is always gives v, that is

$$\frac{1}{\aleph} = v$$

Since $x_i = \frac{q_i}{v}$ hence optimal strategies for player E could be found by $q_i = vx_i$ therefore

$q_1 = vx_1 \rightarrow q_1 = 2 \times \frac{1}{4} = \frac{1}{2}$.

$q_2 = 2 \times \frac{1}{4} = \frac{1}{2}$, So the set of optimal strategies for column player is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

The same technique can be applied for row player. The values under the columns of slack variables x_3 and x_4 in objective row is y_i belong to player D, $y_1 = \frac{1}{4}$, $y_2 = \frac{1}{4}$,

$$p_i = vy_i$$

$p_1 = 2 \times \frac{1}{4} = \frac{1}{2}$, $p_2 = 2 \times \frac{1}{4} = \frac{1}{2}$. Thus the set of optimal strategies for player D is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

The value of the original game is $v - k \rightarrow 2 - 2 = 0$.

At final step, the added constant (k) must be subtracted from the new value to get the value of the original game.

Recall the payoff matrix 4.1 which has a saddle point equal to 5 and pure strategies (0 0 1); $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We apply LP method also demonstrates that the same result can be obtained. Adding $k = 3$ to every entry the payoff matrix becomes:

	q_1	q_2
p_1	6	3
p_2	1	6
p_3	10	8

Payoff matrix 4.10: Applying LP method on payoff matrix 4.1

The problem of column player becomes

$$\text{Max } x_1 + x_2$$

Subject to

$$6x_1 + 3x_2 \leq 1$$

$$x_1 + 6x_2 \leq 1$$

$$10x_1 + 8x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Adding slack variables

$$6x_1 + 3x_2 + x_3 = 1$$

$$x_1 + 6x_2 + x_4 = 1$$

$$10x_1 + 8x_2 + x_5 = 1$$

$$x_i \geq 0\ ,\ \ \forall i = 1, \ldots, 5$$

Apply simplex method using Excel

The circled number (8) is pivot element, hence x_2 is entering variable and x_5 is leaving variable

Table 4.4: Iteration 0 of simplex method

slack variables	x1	x2	x3	x4	x5	R.H.S.
x3	6	3	1	0	0	1
x4	1	6	0	1	0	1
x5	10	(8)	0	0	1	1
objective row	-1	-1	0	0	0	0

Table 4.5: Iteration 1 of simplex method

slack variables	x1	x2	x3	x4	x5	R.H.S.
x3	2 1/4	0	1	0	- 3/8	5/8
x4	-6 1/2	0	0	1	- 3/4	1/4
x2	1 1/4	1	0	0	1/8	1/8
Objective row	1/4	0	0	0	1/8	1/8

Here entry ℵ is equal to 1/8 thus $\frac{1}{\aleph} = v = 8$.

Since x_1 does not exist in the column labeled as slack variables then it is equal to zero

So $q_1 = 0$ and $q_2 = vx_2$, $\rightarrow$ $q_2 = 8 \times \frac{1}{8} = 1$

The set of strategies for column player is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Returning to the other player, y_1 and y_2 are zero under columns labeled x_3, x_4 respectively so p_1 and p_2 are zero, $y_3 = \frac{1}{8}$ so $p_3 = vy_3$ $\rightarrow$ $p_3 = 8 \times \frac{1}{8} = 1$.

Hence the set of optimal strategies for the row player is [0 0 1] it means that the strategies for both players are pure. The value of the old game is given by $v - k$ so 8-3=5 which is a saddle point indeed.

Recall the payoff matrix 4.4 that has dominated strategies of the first row and last column, resolving the game problem with LP method as its original payoff matrix without deleting any recessive row/column. Add a constant 3 to each entry to obtain all entries as positive numbers;

		B 1	2	3
A	1	5	2	6
	2	1	5	7
	3	6	3	7

Payoff matrix 4.11: Applying SM to the payoff matrix containing dominated strategies

The linear programming problem will be

$$\text{Max } x_1 + x_2 + x_3$$

Such that

$$5x_1 + 2x_2 + 6x_3 \leq 1$$

$$x_1 + 5x_2 + 7x_3 \leq 1$$

$$6x_1 + 3x_2 + 7x_3 \leq 1$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Add slack variables x_4, x_5, x_6 to obtain

$$5x_1 + 2x_2 + 6x_3 + x_4 = 1$$

$$x_1 + 5x_2 + 7x_3 + x_5 = 1$$

$$6x_1 + 3x_2 + 7x_3 + x_6 = 1$$

$$x_i \geq 0\ , \text{ For all } \quad i = 1, \ldots, 6$$

Apply simplex method to become as follows:

The circled number (6) is pivot element, hence x_1 is entering variable and x_6 is leaving variable

Table 4.6: Iteration 0 of simplex method

slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x4	5	2	6	1	0	0	1
x5	1	5	7	0	1	0	1
x6	(6)	3	7	0	0	1	1
objective row	-1	-1	-1	0	0	0	0

The circled number is pivot element, hence x_2 is entering variable and x_5 is leaving variable

Table 4.7: Iteration 1 of simplex method

slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x4	0	-0.5	0.2	1	0	-0.8	0.2
x5	0	(4.5)	5.8	0	1	-0.2	0.8
x1	1	0.5	1.2	0	0	0.2	0.2
objective row	0	-0.5	0.2	0	0	0.2	0.2

Table 4.8: Iteration 2 of simplex method

slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x4	0	0	0.815	1	0.1	-0.85	0.25
x2	0	1	1.296	0	0.2	-0.04	0.18
x1	1	0	0.519	0	-0.1	0.185	0.07
objective row	0	0	0.815	0	0.1	0.148	0.25

So $\aleph = 0.25 \quad \frac{1}{\aleph} = v = 3.85$

Since x_3 does not appear in the column that labeled 'slack variables' hence it is equal to zero

$x_1 = 0.07 \quad \rightarrow \quad q_1 = vx_1 \quad \rightarrow q_1 = 3.85 \times 0.07 = 0.28$

$x_2 = 0.18 \quad \rightarrow \quad q_2 = vx_2 \quad \rightarrow q_2 = 3.85 \times 0.18 = 0.71$

Then the set of optimal strategies is $(0.28 \quad 0.71 \quad 0)^T$. Likewise for the row player, since y_1 under column x_4 is zero then p_1 is zero.

$y_2 = 0.1, \ y_3 = 0.14$ Then the strategies are as follows

$p_2 = 3.85 \times 0.1 = 0.42$, $\qquad p_3 = 3.85 \times 0.14 = 0.57$

The set of optimal strategies is $(0 \quad 0.42 \quad 0.14)$

The value of the original game is $3.85 - 3 = 0.85 = \frac{6}{7}$

Now, suppose a game with the below payoff matrix

2	-3	0
3	1	-2

Payoff matrix 4.12: Game having no saddle point, no dominated strategies

This payoff matrix neither has a saddle point nor dominated strategies. Also it is not a 2×2 payoff matrix, then the only way to solve this problem is by linear programming method, add a suitable constant (4) to every entry then the payoff matrix will be

6	1	4
7	5	2

Payoff matrix 4.13: Adding suitable constant to matrix 4.12

Now, the LP form is:

$$\text{Max } x_1 + x_2 + x_3$$

Such that

$$6x_1 + x_2 + 4x_3 \leq 1$$

$$7x_1 + 5x_2 + 2x_3 \leq 1$$

$$x_1 \geq 0\,, \quad x_2 \geq 0\,, \quad x_3 \geq 0$$

Add slack variables to obtain constraints equal to one

$$6x_1 + x_2 + 4x_3 + x_4 = 1$$

$$7x_1 + 5x_2 + 2x_3 + x_5 = 1$$

$$x_i \geq 0 \quad \forall\, i = 1, \dots, 5$$

Now, simplex method shows that the circled number is a pivot element, x_2 is entering variable and x_5 is leaving variable as follows:

Table 4.9: Iteration 0 of simplex method

slack variables	x1	x2	x3	x4	x5	R.H.S.
x4	6	1	4	1	0	1
x5	7	5	2	0	1	1
objective row	-1	-1	-1	0	0	0

Also, the circled number is a pivot element, x_3 is entering variable and x_4 is leaving variable

Table 4.10: Iteration 1 of simplex method

slack variables	x1	x2	x3	x4	x5	R.H.S.
x4	4.6	0	3.6	1	-0.2	0.8
x2	1.4	1	0.4	0	0.2	0.2
objective row	0.4	0	-0.6	0	0.2	0.2

Now, since there are no negative elements in the objective row, so the optimal solution has been reached; see the Table 4.11

Table 4.11: Iteration 2 of simplex method

slack variables	x1	x2	x3	x4	x5	R.H.S.
x3	1 2/7	0	1	2/7	- 1/18	2/9
x2	8/9	1	0	- 1/9	2/9	1/9
objective row	1 1/6	0	0	1/6	1/6	1/3

Hence $x_1 = 0,\ x_2 = \frac{1}{9},\quad x_3 = \frac{2}{9}.$

$\aleph = \frac{1}{3},\ v = \frac{1}{\aleph} = 3.$

Thus $q_1 = 0,\quad q_2 = vx_2 = 3 \times \frac{1}{9} = \frac{1}{3},\quad q_3 = vx_3 = 3 \times \frac{2}{9} = \frac{2}{3}.$

The optimal strategy set for the column player is $\begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

y_1 and y_2 can be found in the objective row under the column of slack variables x_4 and x_5 so $y_1 = \frac{1}{6}$ and $y_2 = \frac{1}{6}$

$p_1 = vy_1 = 3 \times \frac{1}{6} = \frac{1}{2}$ and $p_2 = vy_2 = 3 \times \frac{1}{6} = \frac{1}{2}$

So the optimal strategy for the row player is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

The value of the old game is $3 - 4 = -1$.

4.3 Converting Game Trees to a Payoff Matrix

The value of the game trees can be evaluated by LP, game trees can be transformed to payoff matrices by converting extensive to a normal form. Doing this procedure does not require too many skills only you have to organize every payoff at each terminal node into a correct cell available to each players' strategy in the payoff matrix.

Suppose there is a game called *'Rock- Paper – scissors'* between two players, assume that the winner gets the payoff of +1 unit and the loser -1 unit. In the case of tie, they will get nothing. Suppose that player one (p) plays first after that player two (q) plays. Player 1 shows the symbol for rock, paper or scissor, then player 2 shows up the symbol for either rock, paper or scissor; payoffs are written in the game tree, Figure 4.1 below: [this example adopted from (Griffin, 2012)]

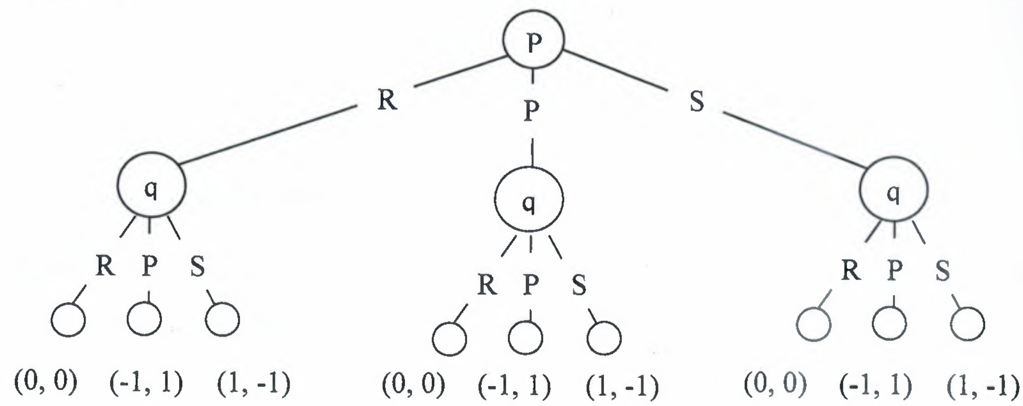

Figure 4.1: Game tree of Rock, Paper, and scissor

R= Rock P= Paper S= Scissor

Suppose that player p chooses strategy R, then player q has 3 options (either to choose R or P, or to choose S). Consider player q chooses strategy P; in this case the payoff is -1 for player p and +1 for player q.

Steps for converting tree to payoff matrix:

- Each player has 3 strategies. Label these strategies in the payoff matrix from the top to bottom for player P, and from the left to the right for player q.
- Suppose that player p chooses the first (row) strategy. The other player chooses the second (column) strategy. Then select the intersection cell between these two chosen strategies.
- Insert in the selected cell that payoff, which belongs to player one. In this tree, the payoff is (-1, 1) according to the first strategy that has been chosen by player p and the second strategy that has been chosen by player q. So the left hand side number into the parentheses which is -1 belongs to player one (p).

The payoff matrix will look like below:

		q		
		R	P	S
	R		-1	
P	P			
	S			

Payoff matrix 4.14: Converting game tree to the payoff matrix

This is the payoff matrix for the Rock-Paper-scissor game problem, the same technique should continuo for remaining payoffs, and the complete converted payoff matrix is:

		q		
		R	P	S
	R	0	-1	1
p	P	1	0	-1
	S	-1	1	0

Payoff matrix 4.15: Converting game tree to the payoff matrix

Now this game is solvable by LP format. Add constant 2 to each entry the problem becomes:

$$\text{Max} \quad x_1 + x_2 + x_3$$

Such that

$$2x_1 + x_2 + 3x_3 \leq 1$$

$$3x_1 + 2x_2 + 2x_3 \leq 1$$

$$x_1 + 3x_2 + 2x_3 \leq 1$$

$$x_1, x_2, x_3 \geq 0$$

Add slack variables x_4, x_5, x_6 to obtain:

$$2x_1 + x_2 + 3x_3 + x_4 = 1$$

$$3x_1 + 2x_2 + 2x_3 + x_5 = 1$$

$$x_1 + 3x_2 + 2x_3 + x_6 = 1$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Using simplex method in Excel spreadsheet for the above problem shows that the circled element (3) is the pivot element, and x_3 is the entering variable, while x_4 is leaving variable:

Table 4.12: Iteration 0 of simplex method

Slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x4	2	1	3	1	0	0	1
x5	3	2	1	0	1	0	1
x6	1	3	2	0	0	1	1
Objective row	-1	-1	-1	0	0	0	0

Again, the circled entry is the pivot element, therefore x_2 is entering variable and x_6 is leaving variable

Table 4.13: iteration 1 of simplex method

Slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x3	0.6667	0.3333	1	0.3333	0	0	0.3333
x5	2.3333	1.6667	0	-0.333	1	0	0.6667
x6	-0.333	2.3333	0	-0.667	0	1	0.3333
Objective row	-0.333	-0.667	0	0.3333	0	0	0.3333

Also, the circled entry is the pivot element, therefore x_1 is entering variable and x_5 is leaving

Table 4.14: Iteration 2 of simplex method

slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x3	0.7143	0	1	0.4286	0	-0.143	0.2857
x5	2.5714	-6E-15	0	0.1429	1	-0.714	0.4286
x2	-0.143	1	0	-0.286	0	0.4286	0.1429
objective row	-0.429	1E-15[1]	0	0.1429	0	0.2857	0.4286

[1] The symbol (E-No.) in Excel means $1 \times 10^{-NO.}$, for example 1E-15=1×10^{-15} which is almost zero.

Table 4.15: Iteration 3 of simplex method

Slack variables	x1	x2	x3	x4	x5	x6	R.H.S.
x3	0	2E-15	1	0.3889	-0.278	0.0556	0.1667
x1	1	-2E-15	0	0.0556	0.3889	-0.278	0.1667
x2	0	1	0	-0.278	0.0556	0.3889	0.1667
Objective row	7E-16	2E-16	0	0.1667	0.1667	0.1667	0.5

Hence

$$\aleph = \frac{1}{2}$$

So, $v = 2$

$x_1, x_2, x_3 = 0.1667$ So $q_i = vx_i$

$$q_1 = q_2 = q_3 = \frac{1}{3}$$

Also $y_1 = y_2 = y_3 = 0.1667$ then

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

Set of strategies are $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ for the row and column players respectively.

The Battle of the Bismarck Sea (pacific war) this example is adopted from (Mendelson, 2004): games can be used to clear up the importance of intelligence in combat. In World War II; in February 1943, the battle for New Guinea had reached a critical point of contact. The New Guinea was divided into two parts, the southern half was controlled by the Allies (United States, Australia) and the other half was controlled by Japanese. Leakage information said Japanese were assembling troops to reinforce their military on New Guinea trying to control the complete island. These troops were expected to deliver by naval convoy. The Japanese had two choices, sailing north or south of New Britain; the first route (north) was expected to be rain and poor visibility, the weather of the second route (south of New Britain) was expected to be fine. Both two ways required equal time of sailing to reach to the destination. The Allies Forces Commander "General Kenny" was received instruction to damage Japanese fleet by aircraft as much as possible. But he needs to make a right decision for searching northern or southern route to do his mission successfully. Game tree in Figure 4.2 summarizes the strategies for Japanese (J) and Allies (A) players (commanders). The payoffs are available number of days for bombing Japanese fleet. The strategies for Japanese are either to sail north or south while for Allies are to search south or north.

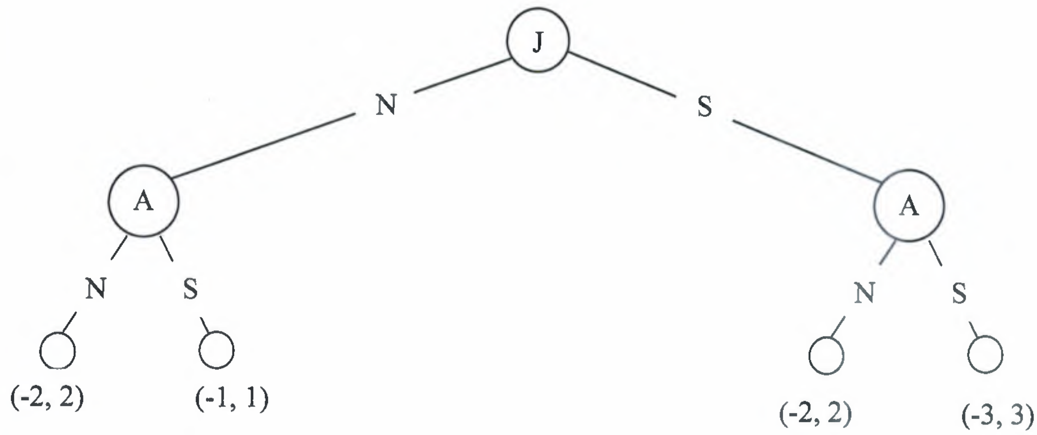

Figure 4.2: Game tree shows the battle of Bismarck Sea

In this tree players are assumed to be rational, such that, if Japanese intend to sail North then Americans have to search north to be able of bombing Japanese fleet for 2 days, if Japanese intend to sail south then the Americans should search south to bomb Japanese fleet for 3 days. However, Japanese also plays intelligently, therefore they know the best move for them is to take north to receive least damages of only 2 days which is better than 3 days; thus the Nash equilibrium for both of them is: taking north-north.

Now, the payoff matrix for this tree will look like payoff matrix 4.16 below. Row player is Japanese, since Japanese cannot benefit any way so all payoffs will be negative numbers.

		A	
		North	South
J	North	-2	-1
	South	-2	-3

Payoff matrix 4.16: The battle of the Bismarck Sea

Adding a suitable constant to all entries of the payoff matrix in order to be positives:

		A: North	A: South
J	North	2	3
	South	2	1

Payoff matrix 4.17: The battle of the Bismarck Sea

The problem for Allies is to find Max $x_1 + x_2$

Such that

$$2x_1 + 3x_2 \leq 1$$

$$2x_1 + x_2 \leq 1$$

$$x_1 \geq 0 , \quad x_2 \geq 0$$

Adding slack variables x_3 , x_4 to obtain:

$$2x_1 + 3x_2 + x_3 = 1$$

$$2x_1 + x_2 + x_4 = 1$$

$$x_1 \geq 0 , \quad x_2 \geq 0 , \; x_3 \geq 0 , \; x_4 \geq 0$$

Then, the circled entry in the Table 4.16 is pivot element, so x_2 is entering variable and x_3 is leaving variable

Table 4.16: Iteration 0 of simplex method

Slack variables	x1	x2	x3	x4	R.H.S.
x3	2	(3)	1	0	1
x4	2	1	0	1	1
Objective row	-1	-1	0	0	0

Now, the circled entry (0.6667) is the pivot element, x_1 is entering variable, and x_2 is leaving variable

Table 4.17: Iteration 1 of simplex method

Slack variables	x1	x2	x3	x4	R.H.S.
x2	0.6667	1	0.3333	0	0.3333
x4	1.3333	0	-0.333	1	0.6667
Objective row	-0.333	0	0.3333	0	0.3333

Table 4.18: Iteration 2 of simplex method

Slack variables	x1	x2	x3	x4	R.H.S.
x1	1	1.5	0.5	0	0.5
x4	4.21885E-15	-2	-1	1	2.10942E-15
objective row	-5.55112E-16	0.5	0.5	0	0.5

Thus $\aleph = \frac{1}{2}$ implies $v = \frac{1}{\frac{1}{2}} = 2$

$x_1 = 0.5 \quad \rightarrow \quad q_1 = vx_1 \quad \rightarrow \quad 2 \times 0.5 = 1$

$x_2 = 0$ because it is not in the list of the slack variables, so $q_2 = 0$.

$y_1 = 0.5$, $y_2 = 0$, Implies $p_1 = 1$, $p_2 = 0$

The value of the old game is $2 - 4 = -2$ biased to Allies.

The set of strategy for Japanese is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and for Allies is $[1 \quad 0]$

Taking look to this problem from another standpoint, it is clear that this game has a saddle point at the first entry. That is, taking the strategy north - north of the payoff matrix 4.16 is equal to -2, means Japanese loses 2 and Allies wins 2, and the strategies for them are pure.

4.4 Applications on Game Theory

Consider a game between two players that they are politicians. Each of them candidates for mayor of a specific city; they have to plan the last two days of the campaign. Politicians intend to campaign in two different neighborhoods (B and F). Each candidate has 3 strategies available to him; they can campaign two days in B, or campaign two days in F, or campaign one day in each place, see payoff matrix 4.18. Each politician wants to make a correct decision there for they can use linear programming system to select the best choice for campaigning. [Adopted from (Aguiar et al., 2014)]

		Politician 2		
		One day in each place	Two days in B	Two days in F
Politician 1	One day in each place	0	-2	2
	Two days in B	5	4	-3
	Two days in F	2	3	-4

Payoff matrix 4.18: Two politicians competing for mayor position

Now, to solve this game problem, a suitable constant is to be added to each entry in the payoff matrix 4.18 for making all negatives as positive numbers; therefore, constant k=5 will be added, then the payoff matrix will look like the below one:

		Politician 2		
		One day in each place	Two days in B	Two days in F
Politician 1	One day in each place	5	3	7
	Two days in B	10	9	2
	Two days in F	7	8	1

Payoff matrix 4.19: Two politicians competing for mayor position

The problem for column player is going to be Max $x_1 + x_2 + x_3$

Subject to

$$5x_1 + 3x_2 + 7x_3 \leq 1$$

$$10x_1 + 9x_2 + 2x_3 \leq 1$$

$$7x_1 + 8x_2 + x_3 \leq 1$$

Adding slack variables to obtain

$$5x_1 + 3x_2 + 7x_3 + x_4 = 1$$

$$10x_1 + 9x_2 + 2x_3 + x_5 = 1$$

$$7x_1 + 8x_2 + x_3 + x_6 = 1$$

$$x_i \geq 0, \quad \forall i = 1 \ldots 6$$

Then the tables of solution are going to be like below Tablex:

Simplex method shows that the circled number (10) is pivot element, and x_1 is entering variable, while x_5 is leaving variable

Table 4.19: Iteration 0 of Simplex method

SV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	5	3	7	1	0	0	1
x_5	(10)	9	2	0	1	0	1
x_6	7	8	1	0	0	1	1
OR	-1	-1	-1	0	0	0	0

Now, number 6 is the pivot element, the variables x_3 and x_4 are entering and leaving variables, respectively

Table 4.20: Iteration 1 of Simplex method

SV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	-1.5	(6)	1	-0.5	0	0.5
x_1	1	0.9	0.2	0	0.1	0	0.1
x_6	0	1.7	-0.4	0	-0.7	1	0.3
OR	0	-0.1	-0.8	0	0.1	0	0.1

Similarly, the number 0.95 is the pivot element, and the variables x_2 and x_1 are entering and leaving variables, respectively

Table 4.21: Iteration 2 of Simplex method

SV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	-0.25	1	0.166667	-0.08333	0	0.083333
x_1	1	0.95	0	-0.03333	0.116667	0	0.083333
x_6	0	1.6	0	0.066667	-0.73333	1	0.333333
OR	0	-0.3	0	0.133333	0.033333	0	0.166667

Table 4.22: Iteration 3 of Simplex method

SV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0.263158	0	1	0.157895	-0.05263	0	0.105263
x_2	1.052632	1	0	-0.03509	0.122807	0	0.087719
x_6	-1.68421	0	0	0.122807	-0.92982	1	0.192982
OR	0.315789	0	0	0.122807	0.070175	0	0.192982

Hence the value of $\aleph = 0.192982$ implies the value of

$$v = 1/\aleph = 5.181818$$

And the value of the old game is equal to

$$5 - 5.181818 = 0.181818 \approx \frac{2}{11}$$

And the set of strategy is $[0.636363 \quad 0.363636 \quad 0]$, $\begin{bmatrix} 0 \\ 0.545454 \\ 0.454545 \end{bmatrix}$

Example: Consider a game between two players A and B where each roll a die once, then the player with higher outcome (winner) gets from the other player (loser) a payment in number of units which is equal to the difference of the two outcomes, so if one player's outcome is 5 and the other player's outcome is 2, then the player whose outcome is 5 gets 5-2 = 3$ from the player whose outcome is 2. This game problem is relatively large with a 6x6 pay off matrix below:

Player B

		1	2	3	4	5	
	1	0	-1	-2	-3	-4	-5
Player A	2	1	0	-1	-2	-3	-4
	3	2	1	0	-1	-2	-3
	4	3	2	1	0	-1	-2
	5	4	3	2	1	0	-1
	6	5	4	3	2	1	0

Payoff matrix 4.20: Two throwing dice game

To solve this problem by LP method a suitable constant, in this example constant 6, is to be added to all entries of the payoff matrix.

Player B

		1	2	3	4	5	
	1	6	5	4	3	2	1
Player A	2	7	6	5	4	3	2
	3	8	7	6	5	4	3
	4	9	8	7	6	5	4
	5	10	9	8	7	6	5
	6	11	10	9	8	7	6

Payoff matrix 4.21: Two throwing dice game

Then the problem for the column player is going to be:

$$\text{Max} \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Subject to

$$6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 \leq 1$$

$$7x_1 + 6x_2 + 5x_3 + 4x_4 + 3x_5 + 2x_6 \leq 1$$

$$8x_1 + 7x_2 + 6x_3 + 5x_4 + 4x_5 + 3x_6 \leq 1$$

$$9x_1 + 8x_2 + 7x_3 + 6x_4 + 5x_5 + 4x_6 \leq 1$$

$$10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 \leq 1$$

$$11x_1 + 10x_2 + 9x_3 + 8x_4 + 7x_5 + 6x_6 \leq 1$$

$$x_i \geq 0 \text{ For all } \; i = 1 \ldots 6$$

Now, adding slack variables to each constraint to obtain equality

$$6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 + y_1 = 1$$

$$7x_1 + 6x_2 + 5x_3 + 4x_4 + 3x_5 + 2x_6 + y_2 = 1$$

$$8x_1 + 7x_2 + 6x_3 + 5x_4 + 4x_5 + 3x_6 + y_3 = 1$$

$$9x_1 + 8x_2 + 7x_3 + 6x_4 + 5x_5 + 4x_6 + y_4 = 1$$

$$10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + y_5 = 1$$

$$11x_1 + 10x_2 + 9x_3 + 8x_4 + 7x_5 + 6x_6 + y_6 = 1$$

$$x_i, y_i \geq 0 \text{ For all } \; i = 1 \ldots 6$$

Then in the initial Table the circled number is pivot element. So x_1 is entering variable, and y_6 is leaving variable

Table 4.23: Iteration 0 of Simplex method

SV	x1	x2	x3	x4	x5	x6	y1	y2	y3	y4	y5	y6	RHS
y1	6	5	4	3	2	1	1	0	0	0	0	0	1
y2	7	6	5	4	3	2	0	1	0	0	0	0	1
y3	8	7	6	5	4	3	0	0	1	0	0	0	1
y4	9	8	7	6	5	4	0	0	0	1	0	0	1
y5	10	9	8	7	6	5	0	0	0	0	1	0	1
y6	11	10	9	8	7	6	0	0	0	0	0	1	1
OR	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0

Applying the procedures of simplex method to this LP problem, then the number 0.545 is the pivot element. Hence x_6 is entering variable and x_1 is leaving variable

Table 4.24: Iteration 1 of Simplex method

SV	x1	x2	x3	x4	x5	x6	y1	y2	y3	y4	y5	y6	RHS
y1	0	-0.45	-0.91	-1.36	-1.82	-2.27	1	0	0	0	0	-0.55	0.455
y2	0	-0.36	-0.73	-1.09	-1.45	-1.82	0	1	0	0	0	-0.64	0.364
y3	0	-0.27	-0.55	-0.82	-1.09	-1.36	0	0	1	0	0	-0.73	0.273
y4	0	-0.18	-0.36	-0.55	-0.73	-0.91	0	0	0	1	0	-0.82	0.182
y5	0	-0.09	-0.18	-0.27	-0.36	-0.45	0	0	0	0	1	-0.91	0.091
x1	1	0.909	0.818	0.727	0.636	0.545	0	0	0	0	0	0.091	0.091
OR	0	-0.09	-0.18	-0.27	-0.36	-0.45	0	0	0	0	0	0.091	0.091

Now, since all entries in the objective row are positives, hence the optimal solution has been reached

Table 4.25: Iteration 2 of Simplex method

SV	x1	x2	x3	x4	x5	x6	y1	y2	y3	y4	y5	y6	RHS
y1	4.167	3.333	2.5	1.667	0.833	0	1	0	0	0	0	-0.17	0.833
y2	3.333	2.667	2	1.333	0.667	4E-15	0	1	0	0	0	-0.33	0.667
y3	2.5	2	1.5	1	0.5	0	0	0	1	0	0	-0.5	0.5
y4	1.667	1.333	1	0.667	0.333	0	0	0	0	1	0	-0.67	0.333
y5	0.833	0.667	0.5	0.333	0.167	0	0	0	0	0	1	-0.83	0.167
x6	1.833	1.667	1.5	1.333	1.167	1	0	0	0	0	0	0.167	0.167
OR	0.833	0.667	0.5	0.333	0.167	8E-16	0	0	0	0	0	0.167	0.167

Hence the number $\aleph = 0.167$ implies that $v = \frac{1}{\aleph} = 6$

The value of the old game (before adding constant 6) is $v - 6 = 0$ which is already known since the game has a saddle point of zero.

The optimal strategies for both players are pure, thus for player A is:

$$P = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \text{ and for player B is } q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This simple example, demonstrates how well the LP and simplex method can handle the game problem to produce a perfect result.

It is very obvious that given the choice of selecting a number, both players would choose to select the highest number which is six, and never choose any other number. This was very well identified by simplex method, giving the strategies of both players with all entries as zero except for last location which gives the number six the certain probability of 1.

Hence, simply simplex method showed that the game has a saddle point and both players have pure strategy.

Example of attendance problem (adopted from (Kelly, 2003)): A high school has a serious problem with student attendance in the run up to the public examination. This is also the case even among students who are not applicant. Teachers must choose whether to teach on, actively revise coursework already done during the year, or passively supervise students studying in the reduced class groups. Students must choose whether to go to class or study independently at home. Research has demonstrated that, if teachers teach on, and students attend, year-on-year results enhance by 12%, but fall by 8% if they do not. If teachers passively supervise group study and students stay at home, results enhance by 2%, but unchanged if they attend. If teacher actively revise coursework, and students go to school, results improve 5% and 1% if student do not attend.

Payoff matrix 4.22 demonstrates this serious problem.

		Teachers		
	Strategy	**Teach on**	**Passively supervise group study**	**Actively give revision workshops**
Students	Attend lessons	12	0	5
	Do not attend lessons	-8	2	1

Payoff matrix 4.22: Attendance problem

To apply LP method to this problem, a suitable constant is to be added to all the entries, so 9 is a constant that fulfils our objective. The payoff matrix would be:

		Teachers		
	Strategy	**Teach on**	**Passively supervise group study**	**Actively give revision workshops**
Students	Attend lessons	21	9	14
	Do not attend lessons	1	11	10

Payoff matrix 4.23: Attendance problem

The teachers' problem is Max $x_1 + x_2 + x_3$

Subject to

$$21x_1 + 9x_2 + 14x_3 \leq 1$$

$$x_1 + 11x_2 + 10x_3 \leq 1$$

$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$$

Introducing slack variables S_1 and S_2 to add to each of inequality constraints to obtain

$$x_1 + x_2 + x_3 + S_1 = 1$$

$$x_1 + x_2 + x_3 + S_2 = 1$$

$$x_1 \geq 0,\ x_2 \geq 0,\ \ x_3 \geq 0,\ S_1 \geq 0,\ S_2 \geq 0$$

Now, the LP form shown in Excel spreadsheet is:

Table 4.26: Iteration 0 of simplex method

SV	x_1	x_2	x_3	S_1	S_2	RHS
S_1	21	9	14	1	0	1
S_2	1	11	10	0	1	1
OR	-1	-1	-1	0	0	0

Table 4.27: Iteration 1 of simplex method

SV	x_1	x_2	x_3	S_1	S_2	RHS
x_1	1	0.428571429	0.666667	0.047619	0	0.047619
S_2	0	10.57142857	9.333333	-0.04762	1	0.952381
OR	0	-0.571428571	-0.33333	0.047619	0	0.047619

Table 4.28: Iteration 2 of simplex method

SV	x_1	x_2	x_3	S_1	S_2	RHS
x_1	1	7.21645E-16	0.288288	0.04955	-0.04054	0.009009
x_2	0	1	0.882883	-0.0045	0.094595	0.09009
OR	0	-1.9984E-15	0.171171	0.045045	0.054054	0.099099

Now, the solution has been found, $\aleph = 0.099099$ and v =10.09091.

So, the optimal strategies for the teachers are given by $\begin{bmatrix} 0.090909 \\ 0.909091 \\ 0 \end{bmatrix}$, while for students are $[0.454545 \quad 0.545455]$.

This implies that teachers should omit the strategy (Actively give revision workshops) and they have to adopt the strategies 'Teach on' and 'Passively supervise group study' with the probabilities $^{1}/_{11}$ and $^{10}/_{11}$ respectively. While students have to attend or not attend with the probabilities $^{5}/_{11}$ and $^{6}/_{11}$ respectively

In other words, the teachers choose option " teach on" 1 out of 11 times with probability 0.90909 and choose option "Passively supervise group study" 10 times out of 11 times. On the other hand the student should choose option "attend" 5 times out of 11 times with probability 0.454545 and choose not attend 6 times out of 11 times with probability 0.545455.

The value of the game is 10.09091-9 = 1.09091

This gives enhance of 1.09091% of results.

CHAPTER 5
CONCLUSIONS

The work done in this thesis studied the two-person zero-sum game which is a simple form of general game problems. The basic definitions of the concept are presented and discussed. When and how the problem arises in real life is well covered. The formulation of the problem and its payoff matrix is also explained. The conversion of the payoff matrix in to the linear programming standard form is also dealt with. The use of simplex method to solve the LP problem presented and its efficiency in handling these types of problems cannot be over looked or under estimated.

It is noted that the linear programming method is the most powerful tool in handling these problems. It has the upper hand in detecting the saddle point by producing pure strategies at the end of solution procedure. It solves problems regardless of the payoff matrix size being reducible or not. It should be emphasized here, that the way Excel sheet is used in the process of applying simplex method, has never been used before by anybody, at least to the best of our knowledge. Having said that there is a routine in Excel called "Solver" that can handle LP problems, but this work as a box where user enters information as input from one end and receives the answer from another end, without knowing what happened in between. For this reason we preferred to use the Excel in a way that the calculation is done manually and the user is aware and involved in every single step of calculation procedure.

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