UNIFICATION OF q-EXPONENTIAL FUNCTIONS AND RELATED q-POLYNOMIALSAND q-NUMBERS

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NEU 2018

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By

DYA MAHMOOD AHMED

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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Approval of Director of Graduate School of Applied Sciences

Prof. Dr. Nadire ÇAVUŞ

We certify that, this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics

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CHAPTER 1

INTRODUCTION

1.1 q- Calculus History

q-Calculus is one of classical branch of mathematics. The quantum calculus (q- calculus) is an old, since it goes back to eighteen century, which can be traced back to Euler and Gauss (Ernst, 2000; Kupershmidt, 2000). With important contributions of Jackson a century ago (Jackson, 1903; Jackson, 1910). The original object as limits when q tends to 1, is q-analogues of mathematical objects, we are searching in q-calculus. The Eigen function of q- derivative in combinatorial mathematics is a q- analogue of exponential function. They are many q-derivatives, for example, the classical q- derivative, the Askey Wilson operator, etc. (Exton, 1983). In eighteenth and nineteenth the q-Taylor formula encompasses many results of exponential function, Gauss's q- binomial formula, and Heine's formula for a q- hypergeometric function, Euler's identities for q-exponential function, Jacobi triple-product identity celebrated by Euler's identities. He had recurrent formula for the classical partition function and Hine's formula leads to the remarkable Ramanujan product formula, Gauss's formula for the number of sums of two squares while Jacobi's formula for the number of sums of four squares. At beginning of the twentieth century q-calculus was developed by F. H. Jackson who was first introducing lot of theorems and definitions in this field.

Definition 1.1: The q- analogue of m as a complex number is defined by (Kac and Cheung, 2001)

$$[m]_q = \frac{q^m - 1}{q - 1}, \qquad |q| < 1 \;\&\; m \in ¢. \tag{1.1}$$

Definition 1.2: The q- analogue of m-exponent of $(x{-}\propto)$, for any $x, a, q \in ¢$, $|q| < 1$ is defined by (Kac and Cheung, 2001)

$$(x{-}\propto)_q^m = \begin{cases} 1 & \text{if} \quad m = 0, \\ (x{-}\propto)(x - q \propto) \dots (x - q^{m-1} \propto) & \text{if} \quad m \geq 1. \end{cases} \tag{1.2}$$

Definition 1.3: The q- shifted factorial defined by (Ernst, 2000)

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1 - q^j a), \qquad n \in N \tag{1.3}$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j), \qquad |q| < 1. \tag{1.4}$$

Definition 1.4: For any real or complex valued function we may define the q- differential as (Kac and Cheung, 2001)

$$d_q\big(g(x)\big) = g(q\,x) - g(x). \tag{1.5}$$

Where g is a function and d_q is a q- derivative operator.

1.2 q-Products for two Functions

The q- product rule for g and h as a complex valued function, can be expressed as (Ernst, 2000)

$$\begin{aligned} d_q\big(g(x)h(x)\big) &= g(qx)h(qx) - g(x)h(x) \\ &= g(qx)h(qx) - g(qx)h(x) + g(qx)h(x) - g(x)h(x), \end{aligned}$$

We get

$$d_q\big(g(x)h(x)\big) = g(qx)d_q h(x) + h(x)d_q g(x). \tag{1.6}$$

Definition 1.5: We can define the q- derivative of an arbitrary complex valued function as (Kac and Cheung, 2001)

$$D_q g(x) = \frac{d_q g(x)}{d_q x} = \frac{g(qx) - g(x)}{(q-1)x}. \tag{1.7}$$

Note 1.1: We notice that

$$\lim_{q \to 1} D_q g(x) = \frac{dg(x)}{dx}.$$

Property 1.1: The q- derivative is linear by the means of (Ernst, 2000)

$$D_q\big(\alpha g(x) + \beta h(x)\big) = \alpha D_q g(x) + \beta D_q h(x).$$

Example 1.1: In this example we will consider the simple function $g(x) = x^n$, we have.

$$D_q x^n = \frac{(qx)^n - x^n}{(q-1)x} = [n]_q x^{n-1}. \tag{1.8}$$

Definition 1.6: We have two forms of quotient rules for q- derivatives both of them are true and useful

(Kac and Cheung, 2001)

$$D_q\left(\frac{g(x)}{h(x)}\right) = \frac{h(x)D_q g(x) - g(x)D_q h(x)}{h(x)h(qx)}, \tag{1.9}$$

$$D_q\left(\frac{g(x)}{h(x)}\right) = \frac{h(qx)D_q g(x) - g(qx)D_q h(x)}{h(x)h(qx)}. \tag{1.10}$$

1.3 Generalization Taylor Expansion

Assume that $Q_0(x)$, $Q_1(x)$, ... $Q_m(x)$ is sequence of polynomials and α be any scalar numbers, D is a linear operator, such that satisfy (Kac and Cheung, 2001)

1) $Q_0(\alpha) = 1$ and $Q_m(\alpha) = 0$ for $m \geq 1$;
2) We assume that the degree of all polynomials equal to m;
3) $D(Q_m(x)) = Q_{m-1}(x)$ for any $m \geq 1$, and $D(1) = 0$.

Then, for any polynomial F(x) of degree m, we have the generalized Taylor formula:

$$F(x) = \sum_{n=0}^{m} (D^n F)(\propto) Q_n(x). \tag{1.11}$$

Example 1.2: Let

$$D = \frac{d}{dx}, \qquad Q_m(x) = \frac{(x-1)^m}{m!},$$

Then all conditions of theorem 1.12 are satisfied and the theorem provides the Taylor expansion about ∝ of a polynomial. D is a linear operator of polynomial degree m onto the space polynomials of degree m - 1.

1.4 Taylor Formula

Assume g is a real function on closed interval [a, b], m is a positive integer $g \in ¢^{m}[a,b]$. Let θ, μ be distinct points of [a, b], and define (Rudin, 1976)

$$Q(x) = \sum_{j=0}^{m-1} \frac{g^{(j)}(\theta)}{j!}(x-\theta)^{j}. \qquad (1.12)$$

Then there exist a point t between θ and μ such that

$$f(\mu) = Q(\mu) + \frac{f^{n}(t)}{n!}(\mu-\theta)^{n}.$$

Definition 1.7: We have the following q- analogue of m!:

$$[m]_q! = \begin{cases} 1 & \text{if} \quad m = 0, \\ [m]_q[m-1]_q \ldots [1]_q. & \text{if} \quad m = 1,2,\ldots \end{cases} \qquad (1.13)$$

1.5 q-Derivative of Binomial Expression

The following formula is q-derivative of binomial expression (Kac and Cheung, 2001)

$$D_q(x-\propto)_q^n = [n](x-\propto)_q^{n-1}. \qquad (1.14)$$

1.6 q- Taylor Expansion

If Q(x) will be any polynomials such that degree (Q(x)) = m, and $\propto$ be any scalar, the following q- Taylor expansion can be expressed.

$$Q(x) = \sum_{j=0}^{m} (D_q^j\, Q)(\propto)\frac{(X-\propto)_q^j}{[j]!}. \qquad (1.15)$$

Proof: we assume that $D \equiv D_q$ and $Q_m(x) = \frac{(x-\propto)_q^m}{[m]!}$ at theorem 1.4.

Note 1.2: There are several q- Taylor formulae that arise for the different aspect .The classical q- Taylor formula involves many results, Euler's identities for q-exponential function and Gauss's q- binomial formula and Heine's formula for a q- hypergeometric function (Kac and Cheung, 2001). But the new q- Taylor formula is presented generalized determinates, symmetric function and representation theory of symmetric group are treated (Ernst, 2000).

Definition 1.8: The q- analogue of combination of two numbers $\binom{m}{j}_q$ can be expressed as

$$\binom{m}{m-j}_q = \frac{[m]_q!}{[j]_q!\,[m-j]_q!} = \binom{m}{j}_q . \tag{1.16}$$

Example 1.3: Consider Q(x) = x^m and $\propto = 1$, where m is a positive integer. For $j \le m$, we can evaluate $D_q^j\,(Q(x))$ by using induction. The first and second q-derivative will be $[m]_q x^{m-1}$ and $[m]_q[m-1]_q x^{m-2}$ respectively and

$$\left(D_q^j\, Q\right)(x) = [m]_q[m-1]_q \ldots [m-j+1]_q x^{m-j}, \tag{1.17}$$

Thus,

$$\left(D_q^j\, Q\right)(1) = [m]_q[m-1]_q \ldots [m-j+1]_q \tag{1.18}$$

The q- Taylor formula for x^m about x = 1 then gives

$$x^m = \sum_{j=0}^{m} \frac{[m]_q \ldots [m-j+1]_q}{[j]_q!} (x-1)_q^j = \sum_{j=0}^{m} \binom{m}{j}_q (x-1)_j^q, \tag{1.19}$$

Example 1.4: Let $g(x) = (x+\propto)_q^m$, x=0 and m be a nonnegative integer, $\propto$ be a number, using q- Taylor's formula to evaluate the expansion for g(x). For $j \le m$

We have

$$\left(D_q^j g\right)(x) = [m]_q[m-1]_q \ldots [m-j+1]_q (x+\propto)_q^{m-j}. \tag{1.20}$$

So, with x=0, $(x+\propto)_q^{m-j}$ will be

$$(\propto)(q\propto)\ldots(q^{m-1}\propto) = q^{\frac{m(m-1)}{2}} \propto^m .$$

Put this to (1.20) to get for $j \le m$,

$$(D_q^j g)(0) = [m]_q[m-1]_q \ldots [m-j+1]_q q^{\frac{(m-j)(m-j-1)}{2}} \propto^{m-j}.$$

Hence, the q- Taylor formula gives

$$(x+\propto)_q^m = \sum_{j=0}^{m} \binom{m}{j}_q q^{\frac{(m-j)(m-j-1)}{2}} \propto^{m-j} x^j. \qquad (1.21)$$

1.7 q- Analogue of Combination

The q- analogue of $\binom{m}{k}$ has the following properties, where k<m are belongs to N.

a) $\binom{m}{k}_q = \frac{[m]_q!}{[k]_q![m-k]_q!} = \binom{m}{m-k}_q$

b) $\binom{m}{k}_q = \binom{m-1}{k-1}_q + \binom{n-1}{j}_q, \qquad 1 \le j \le n-1.$

c) $\binom{m}{k}_q = \binom{m-1}{k-1}_q + q^j \binom{m-1}{k}_q$

d) $\binom{m}{0}_q = \binom{m}{m}_q = 1.$

Proof: All proofs are straight forward and can be reached directly from the definition. The detail of proof can found at (Kac and Cheung, 2001; Ernst, 2000).

1.8 Hine's Binomial Formula

Consider the function g(x) = $\frac{1}{(1-x)_q^m}$. It is production of some algebraic expression. Let us expand g(x) by q- Taylor's formula about x=0.

We have

$$D_q g(x) = D_q \frac{1}{(1-x)_q^m} = \frac{[m]_q}{(1-x)_q^{m+1}},$$

And, by induction,

$$D_q^j g(x) = \frac{[m]_q [m-1]_q \dots [m+1-j]_q}{(1-x)_q^{m+j}},$$

$$\text{Therefore,} \quad D_q^j g(0) = [m]_q [m+1]_q \dots [m+j-1]_q \ \text{ for any } j \ \geq 1,$$

This expansion of $\frac{1}{(1-x)_q^m}$ is called Hine's binomial formula which is

$$\frac{1}{(1-x)_q^m} = 1 + \sum_{j=1}^{\infty} \frac{[m][m+1]\dots[m+j-1]}{[j]!} x^j. \quad m, j \in ¢ \ \& \ x \in N \qquad (1.22)$$

Definition 1.9: The classical q- exponentials function can be described by (Ernst, 2000)

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \prod_{k=0}^{\infty} \frac{1}{1-(1-q)q^k x}, \quad 0 < |q| < 1, \quad |x| < \frac{1}{|1-q|}, \ (1.23)$$

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{[k]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q)q^k x\right). \quad 0 < |q| < 1, \ x \in ¢ \quad (1.24)$$

1.9 The Product of the Heine's Binomial Formula

The product of the expressions are evaluated by using the Heine's binomial theorem when $m \to \infty$ (Ernst, 2000)

$$\lim_{m\to\infty} \frac{1}{(1-x)_q^m} = \prod_{j=1}^{\infty} \frac{1}{(1-xq^j)}.$$

The above formula is the left side of the Heine's binomial formula when $m \to \infty$, but to proof right side we need (1.25) and (1.26)

$$\lim_{n\to\infty} [n] = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q} \qquad (1.25)$$

And

$$\lim_{n\to\infty} \binom{n}{j} = \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-j+1})}{(1-q)(1-q^2)\dots(1-q^j)}$$
$$= \frac{1}{(1-q)(1-q^2)\dots(1-q^j)} \qquad (1.26)$$

Hence

$$\sum_{j=1}^{\infty} \frac{[m][m+1]\dots[m+j-1]}{[j]!} x^j = \prod_{j=1}^{\infty} \frac{1}{(1-q^j x)}. \qquad (1.27)$$

1.10 q-Exponential Functions Properties

The q- exponential function has the following properties:

1) $e_q^x E_q^{-x} = 1,$ (1.28)

2) $e_{1/q}^x = E_q^x,$ (1.29)

3) $e_q^x e_q^y = e_q^{x+y} \quad \text{if} \quad yx = qxy,$ (1.30)

4) $D_q e_q^x = e_y^x \quad \text{and} \quad D_q E_q^x = E_q^{qx}.$ (1.31)

Proof: the first and second properties are the direct corollary of definition of q-exponential function by products. The third one can be reached where we write the summation of q-exponential function. In this case, we should care about the non-commutative condition. The q-exponential functions have not the same properties of exponential function in general. Last one can be calculated by taking q-derivatives from the 1.23 and 1.24. The detail of proof is available at (Kac and Cheung, 2001; Ernst, 2000).

Definition 1.10: The production of two gives series $\sum a_m$and $\sum b_m$ is defined by

$$c_m = \sum_{k=o}^{m} a_k b_{m-k} \qquad (m = 0, 1, 2, \dots) \qquad (1.32)$$

This production sometimes called Cauchy product. (Rudin, 1976)

Remark 1.1: The motivation of the definition (1.32) is coming from the following product

$$\left(\sum_{m=0}^{\infty} a_m z^m\right)\left(\sum_{m=0}^{\infty} b_m z^n\right) = (a_o + a_1 z + a_2 z^2 \dots)(b_0 + b_1 z + b_2 z^2 \dots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 \dots$$

$$= c_0 + c_1 z + c_2 z^2 \dots$$

Putting z=1 we get the definition (1.32).

1.11 Abel Theorem

If the series $\sum a_m$, $\sum b_m$, $\sum c_m$ converges to α, β, γ and $c_n = a_0 b_n + \cdots + a_n b_0$, then $\gamma = \alpha\beta$.

Proof: details of proof are available at (Rudin, 1976). Actually, this theorem is not strong enough to guarantee that the production of two convergent series will be convergent. There is a discussion at (Rudin, 1976) about absolute convergent of at least one of these series. Also the counterexample of two convergent series that their product is not convergent is discussed (Rudin, 1976).

CHAPTER 2

SOME q- EXPONENTIAL FUNCTIONS AND ITS PROPERTIES

Introduction

The aim of this chapter is to extend the new q- exponential function such a way that we reach to new properties. The definition is the main part of these generalizations of q-analogue of the exponential function. It should be mentioned that the new q- exponential function in our definition is based on the two q- exponential function which has been discussed in the previous chapter. In addition, we will discuss symmetric properties and the conditions for making the q-exponential convertible in the means of multiplication. First, we introduce some another q-exponential function and we will study their properties in this chapter.

2.1 Improved q- Exponential Function

Definition 2.1.1: The improved q- exponential function ε_q^z is define as (Cieśliński, 2011)

$$\varepsilon_q^z = e_q^{\frac{z}{2}} E_q^{\frac{z}{2}} = \prod_{j=0}^{\infty} \frac{1 + q^j(1-q)\frac{z}{2}}{1 - q^j(1-q)\frac{z}{2}}, \tag{2.1}$$

Where e_q^z, E_q^z are standard q- exponential functions (and the finite product representation is valid for $|q| < 1$).

2.1.1 Representation of improved q-exponential function by summation

The introduced q-exponential function (improved one) can be expressed by the summation as follow

$$\varepsilon_q^z = E_q\left(\frac{z}{2}\right) e_q\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \frac{(-1;q)_n}{2^n}, \tag{2.2}$$

In addition the interval of convergence can be found as

$$\delta = R_q = \begin{cases} \frac{2}{1-q} & \text{for } 0 < q < 1, \\ \frac{2q}{q-1} & \text{for } \quad q > 1, \\ \infty & \text{for } \ q = 1, \end{cases} \tag{2.3}$$

Therefore $\varepsilon_q(z)$ is absolutely convergent where $|z| < \delta$. (Cieśliński, 2011)

Proof: We apply (1.32) Cauchy product of two summations for (1.23) and (1.24). In addition, according to the definition (1.24) we have

$$\varepsilon_q(z) = E_q(\frac{z}{2})e_q\left(\frac{z}{2}\right).$$

But if we apply Cauchy product rule (1.32) and use (1.16) we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}}\binom{n}{k}_q \frac{(z)^n}{2^n[n]_q!} \tag{2.4}$$

Put $\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}}\binom{n}{k}_q$ is equal to $(-1,q)_n$ or $(1+1)_q^n$ by (1.21). Thus

$$\varepsilon_q(z) = \sum_{n=0}^{\infty} \frac{(-1,q)_n}{2^n}\frac{(z)^n}{[n]_q!}. \tag{2.5}$$

For the interval of convergent we will apply the ratio test then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1,q)_{n+1}z^{n+1}}{[n+1]_q!\,2^{n+1}}\right|\left|\frac{[n]_q!\,2^n}{(-1,q)_n\,z^n}\right| \tag{2.6}$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{(1+q^n)(1-q)|z|}{1-q^{n+1}}$$

Let $0 < q < 1$ then $\lim_{n\to\infty} q^n = 0$ and we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|1-q||z|}{2} < 1$$

Or equally $\quad R = \frac{2}{|1-q|}$, $\quad |z| < \frac{2}{|1-q|}$ (2.7)

Let $q > 1$ then we have

$$= \frac{1}{2}\lim_{n\to\infty}\frac{(1+q^n)(1-q)|z|}{1-q^{n+1}}$$

$$= \frac{1}{2}\ \frac{|1-q||z|}{|q|} < 1$$

Then

$$|z| < \left|\frac{2q}{1-q}\right| = \frac{2q}{q-1} \tag{2.8}$$

If q=1 then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \tag{2.9}$$

So the interval convergent in this case is infinity. ■

2.1.2 Relation between different factorial with different factors

For $n \in N$ we have the following formula

$$[n]_{\frac{1}{q}}! = [n]_q!\, q^{-\binom{n}{2}}. \tag{2.10}$$

Proof: Use definition (1.2) to get the following result

$$[n]_{1/q} = \frac{1-(1/q)^n}{1-1/q} = \frac{q^n-1}{q^{n-1}(q-1)} = q^{1-n}[n]_q. \tag{2.11}$$

Then by (2.11) we get

$$[n]_{\frac{1}{q}}! = q^{1-1}[1]_q q^{1-2}[2]_q \dots = [n]_q!\, q^{-\frac{n(1-n)}{2}},$$

Hence

$$[n]_{\frac{1}{q}}! = [n]_q!\, q^{-\binom{n}{2}}. \quad ■$$

2.1.3 Improved q-exponential functions properties

The improved q-exponential function has the following properties:

1. $\varepsilon_q^{-z} = \left(\varepsilon_q^z\right)^{-1},$ (2.12)
2. $\varepsilon_q^z = \varepsilon_{1/q}^z$, (2.13)
3. $D_q\varepsilon_q^z = \langle\varepsilon_q^z\rangle$, (2.14)

Where $z \in ¢$ and $x \in R$ and we use the notation $\langle f(z)\rangle =: \frac{f(z)+f(qz)}{2}$.

Proof: 1. from the definition (2.1) we can proof it

$$\varepsilon_q^{-z} = e_q^{-\frac{z}{2}} E_q^{-\frac{z}{2}} = \frac{1}{E_q^{\frac{z}{2}}} \frac{1}{e_q^{\frac{z}{2}}} = \frac{1}{\varepsilon_q^z} = (\varepsilon_q^z)^{-1}$$

Thus,

$$\varepsilon_q^{-z} = (\varepsilon_q^z)^{-1}.$$

2. We can use the formula (2.2) as follow; in the aim of lemma (2.3) it's easy to say

$$\varepsilon_{\frac{1}{q}}^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{\frac{1}{q}}!} \frac{(-1; q^{-1})_n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{[n]!\, q^{-\binom{n}{2}}} \frac{(1+1)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{q^2}\right)\ldots\left(1+\frac{1}{q^{n-1}}\right)}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{z^n\, q^{\binom{n}{2}}}{[n]!} . \frac{(1+1)(q+1)(q^2+1)\ldots(q^{n-1}+1)}{2^n\ \ q.q^2q^3\ldots q^{n-1}}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \frac{(1;q)_n}{2^n} = \varepsilon_q^z.$$

3. We can proof q-derivative of q-exponential function by using (1.7) as follow

$$D_q \varepsilon_q^z = \frac{\varepsilon_q^{qz} - \varepsilon_q^z}{qz - z} = \frac{\varepsilon_q^z}{(q-1)z}\left(\frac{1-(1-q)\frac{z}{2}}{1+(1-q)\frac{z}{2}} - 1\right) = \frac{\varepsilon_q^{qz}}{1+(1-q)\frac{z}{2}},$$

$$\langle \varepsilon_q^z \rangle = \frac{1}{2}\left(\varepsilon_q^{qz} + \varepsilon_q^z\right) = \frac{1}{2}\left(\frac{1-(1-q)\frac{z}{2}}{1+(1-q)\frac{z}{2}} + 1\right)\varepsilon_q^z = \frac{\varepsilon_q^{qz}}{1+(1-q)\frac{z}{2}}. \quad \blacksquare$$

2.2 Exton q-Exponential Function

In this section we will study the properties of another q-exponential function. This q-exponential function is symmetric by the means of invariant by changing q to $\frac{1}{q}$. We start our study by definition of q- exponential function.

Definition 2.2.1: The Exton q- exponential function is defined by

$$E_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} q^{\frac{\binom{n}{2}}{2}} \ . \qquad (2.15)$$

2.2.1 Exton q-exponential functions properties

For Exton q- exponential function the following relation holds true:

$$E_q^z = E_{q^{-1}}^z \ . \qquad (2.16)$$

By another words, Exton q- exponential function is invariant for changing q to q^{-1}.

Proof: we can use the definition (2.1) to prove this relation

$$E_{q^{-1}}^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} q^{\frac{-\binom{n}{2}}{2}}$$

And by (2.3) we get

$$\sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} q^{\binom{n}{2}} q^{\frac{-\binom{n}{2}}{2}}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} q^{\frac{\binom{n}{2}}{2}} = E_q^z. \quad \blacksquare$$

In this chapter we introduced two different q- exponential functions. Actually, there are many q- exponential functions. For making a generating function, we will use this q-exponential. We could reach to a lot of forms of q- exponential function form different aspects. For instance, solving any suitable q- difference equation makes a new q- exponential function. At chapter four, we will unify all of them by using one extra parameter.

CHAPTER 3

SOME q-NUMBER AND q-POLYNOMIALS AND THEIR PROPERTIES

3.1 q- Bernoulli Number and Polynomials

Bernoulli numbers were raised from the Archimedes times. They try to find the summation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n + 1)(2n+1)$$

The main question was about making a formula for $\sum_{n=1}^{m} n^k$, to achieve this, The Bernoulli made a general case. For this reason let us assume that

$$s_n(r) = \sum_{k=1}^{n-1} k^r = 1^r + 2^r + \cdots + (n - 1)^r$$

We make the generator for this summation means

$$\sum_{m=0}^{\infty} s_n(m)\frac{t^m}{m!} = 1 + \sum_{m=1}^{\infty}(1^m + 2^m + \cdots + (n - 1)^m)\frac{t^m}{m!}$$

$$= 1 + \sum_{m=0}^{\infty}\frac{t^m}{m!} + \sum_{m=0}^{\infty}\frac{(2t)^m}{m!} + \cdots + \sum_{m=0}^{\infty}\frac{((n - 1)t)^m}{m!}$$

$$= 1 + e^t + e^{2t} + \cdots + e^{(n-1)t}$$

So by geometric series we get

$$\frac{1 - e^{nt}}{1 - e^t} = \frac{1 - e^{nt}}{t} \cdot \frac{t}{1 - e^t} \qquad (3.1)$$

Therefore let us define the generating function for Bernoulli numbers as follow:

Definition 3.1.1: The Bernoulli numbers are defined by the means of generating function as follow:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \tag{3.2}$$

This definition is made by the means of generating functions and we will apply this to evaluate $S_n(m)$.

According to (3.1) we have:

$$\sum_{m=0}^{\infty} S_n\,(m) \frac{t^m}{m!} = \left(\sum_{k=0}^{\infty} \frac{n^{k+1} t^k}{(k+1)!}\right)\left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)$$

Now, we apply the Cauchy product for two series,

Then

$$\sum_{m=0}^{\infty} S_n(m) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} B_k \frac{t^k}{k!} . \frac{n^{m-k+1} t^{m-k}}{(m-k+1)!}$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \left(\binom{m+1}{k} B_k . n^{m-k+1}\right) \frac{t^m}{m!} . \frac{1}{m+1}$$

Therefore, if we equal the coefficient then we have;

$$s_n(m) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m-k+1}$$

Thus, this summation can be written in terms of Bernoulli numbers. Let us evaluate some Bernoulli numbers by following lemma:

3.1.1 Recurrence formula for Bernoulli numbers

Bernoulli numbers can be expressed by recurrence formula as

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0 \quad \text{and} \quad B_0 = 1 \tag{3.3}$$

Proof: According to the definition of Bernoulli numbers we have

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$$

Now, we write the Taylor expansion of e^t and applying Cauchy product

$$\left(\sum_{k=0}^{\infty} B_k \frac{t^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{t^n}{n!}\right) = t$$

$$\sum_{k=0}^{\infty}\sum_{k=0}^{n} B_k \frac{t^k}{k!}\frac{t^{n-k+1}}{(n-k+1)!} = t$$

$$\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\binom{n+1}{k}B_k)\frac{t^{n+1}}{n!} = t$$

Therefore by making the coefficient of both sides equal, we reach to (3.3), clearly

$$\begin{cases} \sum_{k=0}^{n}\binom{n+1}{k}B_k = 0 & n \neq 0 \\ \sum_{k=0}^{n}\binom{n+1}{k}B_k = 1 & n = 0 \end{cases}$$

$B_0 = 1$ since $n = 0$, But if n=1

$$\sum_{k=0}^{1} \binom{2}{k} B_k = 0$$

$$\binom{2}{0} B_0 + \binom{2}{1} B_1 = 0 \qquad \rightarrow \qquad B_1 = -\frac{1}{2}$$

If n=2

$$\sum_{k=0}^{2} \binom{3}{k} B_k = 0$$

$$\binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 = 0 \qquad \rightarrow \qquad B_2 = \frac{1}{6}$$

So, by the same way we get, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $\quad B_5 = 0$ ...

3.1.2 Determining odd coefficient of B_n

All the odd coefficient of B_k are zero except B_1. Means that

$$B_{2k+1} = 0 \qquad\qquad k = 1,2, \ldots$$

Proof: Let define g (t) as follow

$$g(t) = \sum_{k=0}^{\infty} B_K \frac{t^k}{k!} - B_1 t = \frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1}\right) \qquad (3.4)$$

Then g (t) is even function, because

$$g(-t) = \frac{-t}{2}\left(\frac{e^{-t} + 1}{e^{-t} - 1}\right) = -\frac{t}{2}\left(\frac{1 + e^t}{1 - e^t}\right) = \frac{t}{2}\left(\frac{1 + e^t}{e^t - t}\right) = g(t)$$

Therefore, the odd coefficient of Taylor expansion of even function is zero.■

3.1.3 Bernoulli numbers properties

The classical Bernoulli numbers has some properties which is (Elias and Dennis, 1990)

1. $\frac{t}{2}\coth\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} B_{2n}\frac{t^{2n}}{(2n)!}$ $\quad t \in [-t, t],$ (3.5)
2. $t\cot(t) = \sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2t)^{2n}}{(2n)!}$ $\quad t \in [-t, t],$ (3.6)
3. $\tan(t) = \sum_{n=1}^{\infty}(-1)^n\frac{2(4^n-1)B_{2n}(2t)^{2n-1}}{(2n)!}$ $\quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$ (3.7)
4. $\tanh(t) = \sum_{n=1}^{\infty}\frac{2(4^n-1)B_{2n}(2t)^{2n-1}}{(2n)!}$ $\quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$ (3.8)

Proof: the relation between trigonometric function and Bernoulli numbers can be found by using the following identity

1. $$\frac{t}{e^t-1} + \frac{t}{2} = \frac{2t + t(e^t-1)}{2(e^t-1)} = \frac{t}{2}\left(\frac{2+e^t-1}{e^t-1}\right)$$

$$= \frac{t}{2}\left(\frac{e^t+1}{e^t-1}\right) = \frac{t}{2}\coth\left(\frac{t}{2}\right)$$

Now let us definition (3.1) to reach (1)

$$\left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right) + (-B_1 t) = \frac{t}{2}\coth\left(\frac{t}{2}\right).$$

Thus (1) is proved and for (2) we should replace t by 2it in (1) as follow

2. $$\frac{2it}{e^{2it}-1} + \frac{2it}{2} = \frac{4it + 2it\left(e^{2it}-1\right)}{2(e^{2it}-1)} = \frac{2it}{2}\left(\frac{2+e^{2it}-1}{e^{2it}-1}\right)$$

$$= it\left(\frac{e^{2it}+1}{e^{2it}-1}\right) = t\coth(t).$$

3. We can use the following identity

$$\tan(t) = \cot(t) - 2\cot(2t)$$

Then we have

$$\tan(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} t^{2n-1} - 2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} (2t)^{2n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} t^{2n-1} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n} B_{2n}}{(2n)!} (t)^{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} (1 - 4^n)}{(2n)!} t^{2n-1}$$

Thus

$$\tan(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2(1 - 4^n) B_{2n}}{(2n)!} (2t)^{2n-1} \qquad \text{for all} |t| < \frac{\pi}{2}$$

Thus (3) is proved and for (4) we should replace t by it in (3) as follow

$$\tanh(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} (it)^{2n-1} - 2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} (2it)^{2n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} (it)^{2n-1} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n} B_{2n}}{(2n)!} (it)^{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} (4^n - 1)}{(2n)!} t^{2n-1}$$

Thus

$$\tanh(t) = \sum_{n=0}^{\infty} \frac{2(4^n - 1) B_{2n}}{(2n)!} (2t)^{2n-1} \qquad \text{for all} |t| < \frac{\pi}{2} \qquad \blacksquare$$

Definition 3.1.4: The Bernoulli polynomial is defined by the following generating function. (Ernst, 2000)

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (3.9)$$

Note 3.1.1: The Bernoulli numbers can be evaluated when we put x=0 at (3.9).

3.1.4 Bernoulli polynomials properties

Bernoulli polynomial has the following properties

a) $B'_n(x) = nB_{n-1}(x),$ (3.10)

b) $B_n(x+1) - B_n(x) = nx^{n-1},$ (3.11)

c) $B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j},$ (3.12)

d) $\sum_{j=0}^{n-1} \binom{n}{j} B_J(x) n x^{n-1}$ (3.13)

Proof: a) by taking derivative from generating function (3.9) we can easily reach it

$$\frac{t^2 e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B'_n\,(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n\,(x)\frac{t^{n+1}}{n!}$$

In this result we get

$$\sum_{n=0}^{\infty} B'_n\,(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n\,(x)\frac{t^{n+1}}{n!}$$

So

$$\sum_{n=0}^{\infty} B'_n\,(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} B_{n-1}\,(x)\frac{t^n}{(n-1)!}$$

Equating the coefficient of t^n

$$\frac{B'_n(x)}{n!} = \frac{B_{n-1}(x)}{(n-1)!}$$

Hence

$$B'_n(x) = nB_{n-1}(x).$$

b) To proof this we start left hand sides follow

$$B_n(x+1) - B_n(x) = \sum_{n=0}^{\infty} \frac{B_n(x+1)}{n!} t^n - \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$$

$$\frac{te^{x+1} - te^{tx}}{e^t - 1} = \frac{te^{tx}(e^t - 1)}{(e^t - 1)} = te^{tx}$$

Which is a derivative of e^{tx}respect to x, where

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$$

We have

$$B_n(x+1) - B_n(x) = \frac{d}{dx} x^n = nx^{n-1}$$

c) Since the Bernoulli functions uniquely characterize by two properties, we can show that $F_n(x) = \sum_{j=0}^{\infty} \binom{n}{j} B_j x^{n-j}$ has these properties, then $F_n(x)$ should be as the same as $B_n(x)$. these properties are $F_n(0) = B_n$ and $F_n'(x) = F_{n-1}(x)$.

d) We prove that by induction. If n=1 then it is trivial. If we assume that (3.13) is true for $k \geq 1$, we take the left side by (3.10) to get the right side as follow:

$$\frac{d}{dx} \sum_{j=0}^{k} \binom{k+1}{j} B_j(x) = \sum_{j=1}^{k} j \binom{k+1}{j} B_{j-1}(x) = (k+1) \sum_{j=1}^{k} \binom{k}{j-1} B_{j-1}(x)$$

$$= (k+1) \sum_{j=0}^{k-1} \binom{k}{j} B_j(x) = (k+1)kx^{k-1} = (k+1) \frac{d}{dx} x^k,$$

Thus by mathematical induction, (3.13) is true for any positive integer number. ■

Remark 3.1.1: Some Bernoulli polynomials listed below, (hint: we can get evaluates B_0, B_1, B_2 ... at (3.2))

If n=0

$$B_0(x) = \sum_{j=0}^{0} \binom{0}{0} B_0 x^{0-0}$$

$$=\binom{1}{0}\mathrm{B}_0\mathrm{x}^{0-0}=1$$

If n=1

$$\mathrm{B}_1(\mathrm{x})=\sum_{\mathrm{j}=0}^{1}\binom{1}{0}\mathrm{B}_1\mathrm{x}^{1-0}$$

$$=\binom{1}{0}\mathrm{B}_0\mathrm{x}^{1-0}+\binom{1}{1}\mathrm{B}_1\mathrm{x}^{1-1}=\mathrm{x}-\frac{1}{2}$$

If n=2

$$\mathrm{B}_2(\mathrm{x})=\sum_{\mathrm{j}=0}^{2}\binom{2}{0}\mathrm{B}_2\mathrm{x}^{2-0}$$

$$=\binom{2}{0}\mathrm{B}_0\mathrm{x}^{2-0}+\binom{2}{1}\mathrm{B}_1\mathrm{x}^{2-1}+\binom{2}{2}\mathrm{B}_2\mathrm{x}^{2-2}$$

$$\mathrm{B}_0\mathrm{x}^2+\mathrm{B}_1\mathrm{x}+\mathrm{B}_2=\mathrm{x}^2-\mathrm{x}+\frac{1}{6}$$

If n=3

$$\mathrm{B}_3(\mathrm{x})=\sum_{\mathrm{j}=0}^{3}\binom{3}{0}\mathrm{B}_3\mathrm{x}^{3-0}$$

$$=\binom{3}{0}\mathrm{B}_0\mathrm{x}^{3-0}+\binom{3}{1}\mathrm{B}_1\mathrm{x}^{3-1}+\binom{3}{2}\mathrm{B}_2\mathrm{x}^{3-2}+\binom{3}{3}\mathrm{B}_3\mathrm{x}^{3-3}$$

$$\mathrm{B}_0\mathrm{x}^3+\mathrm{B}_1\mathrm{x}^2+\mathrm{B}_2\mathrm{x}+\mathrm{B}_3=\mathrm{x}^3-\frac{3}{2}\mathrm{x}^2+\frac{1}{2}\mathrm{x}.$$

Definition 3.1.2: We can define Bernoulli polynomials with more than one variable. For this, let us define 2D –Bernoulli polynomial as follow

$$\frac{te^{tx}e^{ty}}{e^t - 1} = \sum_{n=0}^{\infty} B_n\,(x, y)\,\frac{t^n}{n!}. \qquad (3.14)$$

Note 3.1.2: The Bernoulli polynomial of one variable can be found by putting y=0, $B_n\,(x, 0) = B_n(x)$.

3.2. q-Bernoulli Number and Polynomials and their Properties

The q-Bernoulli polynomials $B_{n,q}(x)$ are introduced and studied from different approaches. These polynomials arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several others mathematical majority. These classes of q- class of q- Bernoulli numbers were introduced by Carlitz at the beginning of 19 century (Carlitz, 1948). If we use the different forms of q- exponential functions which are introduced at the last chapter, then we can reach to the several forms of q- Bernoulli numbers and polynomials.

If we modify the form of equation (3.3) then we can rewrite it as follow:

$$\sum_{k=0}^{m} \binom{m}{k} B_k - B_m = \delta_{1,m} \qquad (3.15)$$

Where δ_{ij} is delta kroneker function.by using (3.15) equation and adding q parameter, we may reach to the q-analogue of Bernoulli numbers. This is done by Carlitz at (Mahmudov and Momenzadeh, 2014). He introduced the following formula

$$\sum_{k=0}^{m} \binom{m}{k} B_{k,q}q^{k+1} - B_{m,q} = \delta_{1,m} \qquad (3.16)$$

But in this study, we apply the generating function with different q- exponentials and we reach to the more natural forms of q- Bernoulli polynomials and numbers.

Our study is based on making q- Bernoulli numbers and polynomials by using the generating function. The properties of q- Bernoulli, q-Euler and q-Genocchi numbers and polynomials are investigated at (Mahmudov and Momenzadeh, 2014) by using q-improved exponential function. In the next chapter, we will unify the q- exponential function and we will reach to q- analogue of the properties that were discussed in this chapter.

CHAPTER 4

ON A CLASS OF UNIFICATION OF q-EXPONENTIAL FUNCTIONS

4.1 Unification of q- Exponential Function and Related q- Numbers and Polynomials

In this chapter we introduce the general form of q-exponential function by using an extra parameter. We investigate the related properties of this function and the relation between this parameter and related properties of q- numbers. First of all we define this q-exponential function as follow.

Definition 4.1.1: The unification of q- exponential function is define as

$$\varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\ \alpha(q,n). \tag{4.1}$$

When z is any complex number and $\alpha(q,n)$ is a function of q and n. In addition $\alpha(q,n)$ approaches to 1, where q tends one from the left side.

4.1.1 Ratio test

The q-exponential function $\varepsilon_{q,\alpha}(z)$ is analytic in the disc $|z| < (k)^{-1}$. if

$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{[n+1]_q\ \alpha_n}\right|$, does exist and is equal to k.

Proof: we can use d'Alembert's test to obtain the radius of convergence as follow

$$\lim_{n\to\infty}\left|\frac{z^{n+1}\alpha_{n+1}}{[n+1]_q!}\right|\left|\frac{[n]_q!}{z^n\alpha_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{[n+1]\alpha_n}\right| |z|$$

Then, we get for $(q \neq 1)$ the radius of convergence.

Example 4.1.1: The special case of this q-exponential function can be found as follow

If $\alpha(q,n) = 1$, then by definition (1.23)

$$\varepsilon_{q,\alpha}^{z} = e_q^z \tag{4.2}$$

If $\alpha(q,n) = q^{\binom{n}{2}}$, then by definition (1.24)

$$\varepsilon_{q,\alpha}^{z} = E_q^z \tag{4.3}$$

If $\alpha(q,n) = \frac{(-1,q)_n}{2^n}$, then by definition (2.2)

$$\varepsilon_{q,\alpha}^{z} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \quad \frac{(-1;q)_n}{2^n} \tag{4.4}$$

If $\alpha(q,n) = q^{\frac{\binom{n}{2}}{2}}$, then by definition (2.15)

$$\varepsilon_{q,\alpha}^{z} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \, q^{\frac{\binom{n}{2}}{2}} \; . \tag{4.5}$$

Then the interval of convergent is as the some as the discussion of each q- exponential function at chapter 1 and 2.

Definition 4.1.2: Supposes q belongs to the complex number and the magnitude of q which is between zero and one. The q- Genocchi numbers and polynomials and q- Euler numbers and polynomials and q- Bernoulli number and polynomials in two variables x, y respectively are defined as follow:

$$\frac{2t}{\varepsilon_{q,\alpha}(t)+1} = \sum_{n=0}^{\infty} G_{n,q,\alpha} \frac{t^n}{[n]_q!}, \qquad |t| < \pi \tag{4.6}$$

$$\frac{2t}{\varepsilon_{q,\alpha}(t)+1} \varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(ty) = \sum_{n=0}^{\infty} G_{n,q,\alpha}(xy) \frac{t^n}{[n]_q!}, \qquad |t| < \pi \tag{4.7}$$

$$\frac{2}{\varepsilon_{q,\alpha}(t)+1} = \sum_{n=0}^{\infty} E_{n,q,\alpha} \frac{t^n}{[n]_q!}, \qquad |t| < \pi \tag{4.8}$$

$$\frac{2}{\varepsilon_{q,\alpha}(t)+1} \varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(ty) = \sum_{n=0}^{\infty} E_{n,q,\alpha}(x,y) \frac{t^n}{[n]_q!}, \qquad |t| < \pi \tag{4.9}$$

$$\frac{t}{\varepsilon_{q,\alpha}(t)-1} = \sum_{n=0}^{\infty} B_{n,q,\alpha} \frac{t^n}{[n]_q!}, \qquad |t| < 2\pi \tag{4.10}$$

$$\frac{t}{\varepsilon_{q,\alpha}(t)-1}\varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(ty)=\sum_{n=0}^{\infty}B_{n,q,\alpha}(x,y)\frac{t^n}{[n]_q!}. \quad |t|<2\pi \quad (4.11)$$

4.2 Unification of q- Exponential Functions Properties

In this section, we investigate the classical properties of Bernoulli, Euler and Genocchi numbers and polynomial for new generating function. Let us started by the following lemma shows the condition for inversing q to q^{-1}.

4.2.1 Some relation for $\varepsilon_{q,\propto}$

For unification of q- exponential function we have

$$\varepsilon_{q,\alpha}(z)=\varepsilon_{q^{-1},\alpha}(z)\leftrightarrow\ q^{\binom{n}{2}}\alpha(q^{-1},n)=\alpha(q,n) \qquad (4.12)$$

Proof: It's straight forward and by using (2.15) we can achieve it.

In the remind part, we try to investigate the classic properties of Bernoulli numbers and polynomials by using q- exponential unification. For this reason, let us start it by the following proposition:

4.2.2 Difference equation

For the new q-Bernoulli numbers and polynomials, the following relation holds true:

$$B_{n,q,\alpha}(x+1)-B_{n,q,\alpha}(x)=[n]_q\alpha_{n-1}x^{n-1} \qquad (4.13)$$

Proof: we will use the following identities for generating function

$$\sum_{n=0}^{\infty}B_{n,q,\alpha}(x,1)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}B_{n,q,\alpha}(x)\frac{t^n}{[n]_q!}$$

$$=\frac{t\varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(t)}{\varepsilon_{q,\alpha}(t)-1}-\frac{t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)-1}$$

$$=\frac{t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)-1}(\varepsilon_{q,\alpha}(t)-1)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha_n t^{n+1} x^n}{[n]_q!}$$

Comparing the coefficients leads to (4.13).

In a similar way, we can reach to the formula for q- Euler and q- Genocchi polynomial as well:

$$E_{n,q,\alpha}(x,1) + E_{n,q,\alpha}(x,0) = 2\alpha_n x^n \tag{4.14}$$

$$G_{n,q,\alpha}(x,1) + G_{n,q,\alpha}(x,0) = 2[n]_q \alpha_{n-1} x^{n-1} \tag{4.15}$$

These relations hold true, Because of the following identities:

$$\sum_{n=0}^{\infty} E_{n,q,\alpha}(x,1) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} E_{n,q,\alpha}(x) \frac{t^n}{[n]_q!}$$

$$= \frac{2\varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(t)}{\varepsilon_{q,\alpha}(t)+1} + \frac{2\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)+1}$$

$$= \frac{2\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)+1}(\varepsilon_{q,\alpha}(t)+1)$$

$$= 2\sum_{n=0}^{\infty} \frac{t^n x^n}{[n]_q!} \alpha_n$$

In addition for q- Genocchi polynomials we have:

$$\sum_{n=0}^{\infty} G_{n,q,\alpha}(x,1) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} G_{n,q,\alpha}(x) \frac{t^n}{[n]_q!}$$

$$= \frac{2t\varepsilon_{q,\alpha}(tx)\varepsilon_{q,\alpha}(t)}{\varepsilon_{q,\alpha}(t)+1} + \frac{2t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)+1}$$

$$= \frac{2t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t)+1}(\varepsilon_{q,\alpha}(t)+1)$$

$$= 2\sum_{n=0}^{\infty} \frac{t^{n+1} x^n}{[n]_q!} \alpha_{n+1}.$$

Most of the properties are satisfying by a condition that q- exponential function should be invertible in a means of multiplication. Means that, these properties are true if $\varepsilon_{q,\alpha}(-t) = \frac{1}{\varepsilon_{q,\alpha}(t)}$. One of these examples is the next proposition.

4.2.3 Recurrence formula of q- Bernoulli polynomial

The q- Bernoulli polynomials can be presented as following

$$B_{n,q,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q,\alpha}(-1)^k x^{n-k} \alpha_{n-k} \qquad (4.16)$$

This statement is true if $\alpha_n$is chosen such that $\varepsilon_{q,\alpha}(-t) = \frac{1}{\varepsilon_{q,\alpha}(t)}$. We will discuss about this condition later on.

Proof: write the generating function for q- Bernoulli polynomials then we have

$$\sum_{n=0}^{\infty} B_{n,q,\alpha}(x) \frac{t^n}{[n]_q!} = \frac{t}{\varepsilon_{q,\alpha}(t) - 1} \varepsilon_{q,\alpha}(-tx)$$

Multiplying by $\frac{\varepsilon_{q,\alpha}(-t)}{\varepsilon_{q,\alpha}(-t)}$ we get

$$\frac{-t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(-t) - 1} \varepsilon_{q,\alpha}(-t)$$

$$= \left(\sum_{n=0}^{\infty} B_{n,q,\alpha} \frac{(-t)^n}{[n]_q!} \right) \left(\sum_{m=0}^{\infty} \frac{(tx)^m}{[m]_q!} \alpha_{m,q} \right)$$

If we apply the Cauchy product rule then

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q B_{k,q,\alpha} (-1)^k x^{n-k} \alpha_{n-k} \right) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} B_{n,q,\alpha} (x) \frac{t^n}{[n]_q!}$$

Now, equating the coefficient to find (4.16)

In the next lemma we will show that how much the condition $\varepsilon_q(-z) = \frac{1}{\varepsilon_q(z)}$ is important and this condition is strong as well.

4.2.4 Relation between normal derivatives of q-Bernoulli polynomials

The q-analogue of the property which is given at (3.10) can be written as follow

$$B'_{n,q,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k}_q k x^k \alpha_k \beta_{n-k,q,\alpha} \tag{4.17}$$

Proof: we can write the generating function for q- Bernoulli polynomial $B_{n,q,\alpha}(x)$ as follow.

$$\sum_{n=0}^{\infty} B_{n,q,\alpha}(x) \frac{t^n}{[n]_q!} = \frac{t\varepsilon_{q,\alpha}(tx)}{\varepsilon_{q,\alpha}(t) - 1} \tag{4.18}$$

Now take the derivative respect to x, from both sides of (4.18) then we have

$$\sum_{n=0}^{\infty} B'_{n,q,\alpha}(x) \frac{t^n}{[n]_q!} = \frac{t}{\varepsilon_{q,\alpha}(t) - 1} [\varepsilon_{q,\alpha}(tx)]'$$

But $\quad B_{q,\alpha}(tx) = \sum_{n=0}^{\infty} \frac{t^n x^n \alpha_{n,q}}{[n]_q!} \quad$ and $\frac{d}{dx} \ \varepsilon_{q,\alpha}\,(tx) = \sum_{n=0}^{\infty} \frac{n t^n x^{n-1} \alpha_{n,q}}{[n]_q!}$

In the lead of that derivative, we have

$$\sum_{n=0}^{\infty} B'_{n,q,\alpha}(x) \frac{t^n}{[n]_q!} = \left(\sum_{n=0}^{\infty} B_{n,q,\alpha} \frac{t^n}{[n]_q!}\right)\left(\sum_{m=0}^{\infty} \frac{m t^m x^{m-1} \alpha_{m,q}}{[m]_q!}\right)$$

Use the Cauchy product of the right hand side and making the coefficients equal reach to (4.17).

Definition 4.2.5: Anew q- addition and q- subtraction similar to Daehee formula is defined by

$$(x \oplus_q y)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha_{k,q} x^k y^{n-k} \alpha_{n-k,q} \tag{4.19}$$

$$(x \ominus_q y)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha_{k,q} x^k (-y)^{n-k} \alpha_{n-k,q} \tag{4.20}$$

4.2.5 Addition formula

We can evaluate $\varepsilon_{q,\alpha}(x)\varepsilon_{q,\alpha}(y)$ by using the last notation. By the another words, we have

$$\varepsilon_{q,\alpha}(x)\varepsilon_{q,\alpha}(y) = \varepsilon_{q,\alpha}\left(x \oplus_q y\right) \tag{4.21}$$

Proof: we can write the definition of $\varepsilon_{q,\alpha}(x)$, the reminds are straight forward

$$\varepsilon_{q,\alpha}(x)\varepsilon_{q,\alpha}(y) = \left(\sum_{n=0}^{\infty} \frac{x^n \alpha^n}{[n]_q!}\right)\left(\sum_{m=0}^{\infty} \frac{y^m \alpha_m}{[m]_q!}\right)$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k} \alpha_k \alpha_{n-k}\right)\frac{1}{[n]_q!}$$

$$\sum_{n=0}^{\infty} \frac{(x \oplus_q y)^n}{[n]_q!} = \varepsilon_{q,\alpha}\left(x \oplus_q y\right).$$

4.2.6 Addition theorem

For all $x, y \in ¢$, the following relation hold true (assume that $\varepsilon_q(z^{-1}) = \frac{1}{\varepsilon_q(z)}$.)

$$B_{n,q,\alpha}(x,y) = \sum_{k=0}^{n} \binom{n}{k}_q \beta_{k,q,\alpha}(x \oplus_q y)^{n-k} \tag{4.22}$$

$$E_{n,q,\alpha}(x,y) = \sum_{k=0}^{n} \binom{n}{k}_q \epsilon_{k,q,\alpha}(x \oplus_q y)^{n-k} \tag{4.23}$$

$$G_{n,q,\alpha}(x,y) = \sum_{k=0}^{n} \binom{n}{k}_q g_{k,q,\alpha}(x \oplus_q y)^{n-k} \tag{4.24}$$

Proof: Since the proofs are similar, we just prove the relation for q- Bernoulli polynomials.

According to the generating function of $B_{n,q,\alpha}$ we have:

$$\sum_{n=0}^{\infty} \beta_{k,q,\alpha}(x,y)\frac{t^n}{[n]_q!} = \frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx)\varepsilon_q(ty)$$

$$= \frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx) \oplus_q \varepsilon_q(ty)$$

$$= \left(\sum_{n=0}^{\infty} \beta_{n,q,\alpha} \frac{t^n}{[n]_q!}\right)\left(\sum_{m=0}^{\infty} \frac{(tx \oplus_q ty)^m}{[m]_q!}\right)$$

We can prove that $\left(tx \oplus_q ty\right) = t^m\left(x \oplus_q y\right)$. For proving this we only need to substitute it at definition (4.23). Therefore

$$\sum_{n=0}^{\infty} B_{n,q,\alpha}(x,y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \beta_{k,q,\alpha}(x \oplus_q y)^{n-k} \binom{n}{k}_q\right)\frac{t^n}{[n]_q!} . \quad \blacksquare$$

Corollary 4.2.1: In particular, setting y=0 at addition theorem give the following relations

$$B_{n,q,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k}_q \beta_{k,q,\alpha} x^{n-k} \alpha_{n-k,q}\, \alpha_{k,q} \qquad (4.25)$$

$$E_{n,q,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k}_q \epsilon_{k,q,\alpha} x^{n-k} \alpha_{n-k,q}\, \alpha_{k,q} \qquad (4.26)$$

$$G_{n,q,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k}_q g_{k,q,\alpha} x^{n-k} \alpha_{n-k,q}\, \alpha_{k,q} \qquad (4.27)$$

Comparing the result with proposition (4.14) shows the rule for inversing $\varepsilon_q(-z)$.

CHAPTER 5

DISCUSSION FOR THE CASE THAT $\varepsilon_{q,\alpha}(z)$ IS INVERTIBLE

In this section we study the condition that make the unification of q- exponential function, invertible. Since this property leads to a lot of classical theorems, we study this in a separated section. In addition, we will discuss about q-derivative of $\varepsilon_{q,\alpha}(z)$.

5.1 Formula for Nonlinear System of Equations

$\varepsilon_{q,\alpha}(z)$ Satisfies $\varepsilon_{q,\alpha}(-z) = \varepsilon_{q,\alpha}(z)^{-1}$ if and only if $\alpha(q,0) = 1$ and $2\sum_{k=0}^{s-1}\binom{n}{k}_q(-1)^k$ $\alpha_k\alpha_{n-k} = \binom{n}{s}_q(-1)^{s+1}\alpha_s^2$ where n=2s & s=1, 2...

Proof: Because $\varepsilon_{q,\alpha}(-z)\ \varepsilon_{q,\alpha}(z) = 1$ must be hold, we compose the extension for this condition.

$$\varepsilon_{q,\alpha}(-z)\ \varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k\alpha_k\alpha_{n-k}\right)\frac{t^n}{[n]_q!} = 1$$

Assume that the expression on a bracket as $\beta_{q,k}$. If n is an odd number then

$\beta_{n-k,q} = \binom{n-k}{k}_q(-1)^{n-1}\alpha(q,k)\alpha(q,n-k) = -\binom{n}{k}_q(-1)^k\alpha(q,k)\alpha(q,n-k) = -\beta_{k,q}$

Where k = 0, 1 …

For n is an odd number is trivial because $\binom{n-k}{k}_q = \binom{n}{k}_q$, the same discussion for even and equating z^n-coefficient together lead us to the proof.

Remark 5.1: We notice that $\alpha_k \to 1$ where $q \to 1^{-1}$ and $\alpha_0 = 1$ and we can write a condition for a_k in the following system so the proposition (5.1) can be written as following system of nonlinear equation

$$\begin{cases} 2\alpha_2\alpha_1 - \binom{2}{1}_q\alpha_0\alpha_0 = 0 \\ 2\alpha_4\alpha_1 - 2\binom{4}{1}_q\alpha_3\alpha_{2+}\binom{4}{2}_q\alpha_2\alpha_2 = 0 \\ 2\alpha_6\alpha_1 - 2\binom{6}{1}_q\alpha_5\alpha_2 + 2\binom{6}{2}_q\alpha_4\alpha_3 - 2\binom{6}{3}_q\alpha_3\alpha_3 = 0 \end{cases}$$

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$$2\alpha_n\alpha_1 - 2\binom{n}{1}_q \alpha_{n-1}\alpha_2 + 2\binom{n}{1}_q \alpha_{n-2}\alpha_3 - \cdots + (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}_q \alpha_{\frac{n}{2}}\alpha_{\frac{n}{2}} = 0$$

For even n we can found n unknown variable and $\frac{n}{2}$ equation, by recurrence equation we can find α_k respect to $\frac{n}{2}$ parameters by the following case

$$\begin{cases} \alpha_0 = 1 \\ \alpha_2 = \dfrac{1+q}{2}\dfrac{1}{\alpha_1} \\ \alpha_4 = \dfrac{[4]_q}{2\alpha_1^2}\left([2]_q\alpha_3 - \dfrac{[3]_q!}{4\alpha_1}\right) \\ \alpha_6 = \binom{6}{1}_q + \binom{6}{3}_q - \dfrac{1}{2}\left(\binom{6}{2}_q\left(\dfrac{1+q}{2}\dfrac{1}{\alpha_1}\right)\left(\dfrac{[4]_q}{2\alpha_1^2}\left([2]_q\alpha_3 - \dfrac{[3]_q!}{4\alpha_1}\right)\right)\right) \end{cases}$$

This $\alpha(q,k)$ leads us to the improved exponential function. The familiar solution of this system is $\alpha(q,k) = \frac{(-1,q)_k}{2^k}$. On the other hand, we can assume that all k for odd k are 1. . Then by solving the system for these parameters, we reach another exponential function that satisfies $\varepsilon_{q,\alpha}(-z) = (\varepsilon_{q,\alpha}(z))^{-1}$.

5.2 q- Derivative of $\boldsymbol{\varepsilon_{q,\propto}(z)}$

If $\frac{\alpha(q,n+1)}{\alpha(q,n)}$ can be represented as a polynomial of q. that means $\frac{\alpha(q,n+1)}{\alpha(q,n)} = \sum_{j=0}^{m} a_j q^j$, then

$$\begin{aligned} D_q\big(\varepsilon_{q,\alpha}(z)\big) &= \sum_{n=1}^{\infty} \frac{z^{n-1}}{[n-1]_q!}\alpha(q,n) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\left(\alpha(q,n)\sum_{j=0}^{m} a_j q^j\right) \\ &= \sum_{j=0}^{m} a_j \frac{(zq^{\frac{j}{n}})}{[n]_q!}\alpha(q,n) = \sum_{j=0}^{m} a_k \varepsilon_{q,\alpha}\left(zq^{\frac{j}{n}}\right). \end{aligned}$$ ■

Example 5.1: The ratio $\frac{\alpha(q,n+1)}{\alpha(q,n)}$ becomes 1, q^n and $\frac{1+q^n}{2}$ respectively for $\alpha(q,n) = 1, q^{\binom{n}{2}}$ and $\frac{(-1,,q)_n}{2^n}$, then the following derivatives holds true

- $D_q\left(e_q(z)\right) = e_q(z)$
- $D_q\left(E_q(z)\right) = E_q(qz)$
- $D_q\left(\varepsilon_q(z)\right) = \frac{\varepsilon_q(z)+\varepsilon_q(zq)}{2}$.

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CHAPTER 6

CONCLUSION AND FUTURE WORKS

There is a large class of q-Numbers that gathered all the forms. The q-Appell polynomials $An, q(x)$ are defined as following by means of generating function:

$$A_{\mathrm{q},\alpha}(t)\varepsilon_{\mathrm{q},\alpha}(\mathrm{xt}) = \sum_{n=0}^{\infty} A_{\mathrm{n,q},\alpha}(x)\frac{t^n}{[n]_q!} \quad ,0 < q < 1, \qquad (6.1)$$

In this case

$$A_{\mathrm{q},\alpha}(t) := \sum_{n=0}^{\infty} A_{\mathrm{n,q},\alpha}\frac{t^n}{[n]_q!}, \qquad A_{0,\mathrm{q},\alpha} = 1,\ A_{\mathrm{q},\alpha}(t) \neq 0$$

It is clear that $A_{\mathrm{q},\alpha}(t)$ is analytic function at $t = 0$ and $A_{\mathrm{n,q},\alpha}(0) = A_{\mathrm{n,q},\alpha}$ and these numbers are called the q-Appell numbers. Based on appropriate selection for the function $A_{\mathrm{q},\alpha}(t)$, we can obtain different members belonging to the family of q-Appell polynomials. The q-Bernoulli, q-Euler and q-Genocchi numbers can be obtained from this class of function in general case.

The classical Appell polynomials are generalized as q-Appell polynomials. The classic forms can be determined by $A_n(x)$ (Appell, 1880) which can be expressed as a power series expansion

$$A(x,t) \coloneqq A(t)\mathrm{e}^{\mathrm{xt}} = \sum_{n=0}^{\infty} A_{\mathrm{n}}(x)\frac{t^n}{[n]_q!}$$

Sequence of polynomials $A_n(x)$ has a generating function in the form of $A(t)\mathrm{e}^{\mathrm{xt}}$. The properties of these classes of functions are studied at (Appell, 1880; Douak, 1996; Khan and Raza, 2013; Raza, 2013). Recently, certain mixed special polynomial families corresponded to the Appell number sequences are studied in a systematic way, see for example, (Özarslan, 2013). This field is interested due to their applications in various fields of mathematics, physics and engineering. There are several approaches for finding this class on polynomials. One of these approaches is using the generating function and since we unify q-exponential function so we can lead to this class of functions in a general case. A lot of classical properties and their q-analogues are investigated. The unification of q-exponential functions leads to the general class of q-Appell functions. The form of the definition and important rule of exponential function in generating function show the importance of unifying the q-exponential functions. Several forms of these polynomials with more than one variable are introduced before, the degenerated and different order of these polynomials and their properties were investigated. This new approach of this numbers encourages us to investigate the corresponding properties and finding the q-analogues of them.

There are numerous formulae and relations that can be rewrite by unification of q-exponential function. In this case, the implicit relations by q-trigonometric function and some classical properties should be discovered. We can focus on the properties of q-exponential function which help us to reach to these equations. In addition, we introduced two conditions on unification of q-exponential function that can be helpful to achieve q-analogue of classical relations. There exist more conditions and properties which can be studied.

The matrix presentation and algebraic properties of these classes of polynomials were investigated (Momenzadeh and Kakangi, 2017) and can be rewrite in the terms of an extra parameter α and we can use this unification to find the reason of these properties.

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ACKNOWLEDGMENTS

First and foremost, my utmost gratitude goes to my parents who are always there for me. Your prayers and affection always give me courage in all that I do; my appreciation can never be overemphasized. Thank you.

To my supervisor Assistant Professor Dr. Mohammad Momenzadeh your contributions are enormous. I thank you for your valuable guidance and corrections.

To my parents …

ABSTRACT

The several q-Analogues of exponential function were introduced since the last century. In this thesis, we study most of important q-exponential functions and we unify them by using an extra parameter. We apply the unification of q-exponential function as a generator of different q-Euler, q-Genocchi and q-Bernoulli numbers and polynomials. Some conditions at this unification will be studied. The addition theorem and q-analogue of some classical results are studied as well.

Keywords: q-calculus; q-Bernoulli; q-Euler; q-Exponential function; q-Gennochi; Cauchy product; generating function; unification; q-Appell

ÖZET

Üstel fonksiyonun birkaç q-Analogları geçen yüzyıldan beri tanıtılmıştır. Bu tezde, önemli q-üstel fonksiyonların çoğunu inceliyoruz ve bunları ek bir parametre kullanarak birleştiriyoruz. Farklı q-Euler, q-Genocchi ve q-Bernoulli sayıları ve polinomlarının bir jeneratörü olarak q-üstel fonksiyonun birleşmesini uygularız. Bu birleşmede bazı koşullar incelenecektir. Bazı klasik sonuçların ek teoremi ve q-analoğu da incelenmiştir.

Anahtar Kelimeler: q-calculus; q-Bernoulli; q-Euler; q-Üstel fonksiyon; q-Gennochi; Cauchy ürünü; üretim fonksiyonu; birleşme; q-Appell

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