NUMERICAL SOLUTIONS OF THE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS FOR OBSERVING EPIDEMIC MODELS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL OF APPLIED SCIENCES OF NEAR EAST UNIVERSITY

By
LAWIN DHAHIR HAYDER HAYDER

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

NICOSIA, 2019

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Approval of Director of Graduate School of Applied Sciences

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We certify this thesis is satisfactory for the award of the degree of Masters of Science in Mathematics Department

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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To my family…

ABSTRACT

In this thesis, the system of fractional differential equations for observing epidemic models problems are investigated. Applying Fourier series, Laplace transform and Fourier transform methods, the solutions of six problems are obtained. First and second order of accuracy difference schemes are presented for the solution of the one-dimensional epidemic models problem and the numerical procedure for implementation of these schemes is discussed.

Keywords: Epidemic models; fractional differential equations; Fourier series method; Laplace transform solution; difference scheme

ÖZET

Bu tez çalışmasında, epidemik model problemleri için kesirli türevli diferansiyel denklem sistemleri incelenmiştir. Fourier serileri, Laplace dönüşümü ve Fourier dönüşümü yöntemlerini uygulama, ile altı problemlerin çözümleri bülümüştür. Birinci ve ikinci dereceden doğruluk farkı şemaları tek boyutlu epidemik model probleminin çözümü için sunulmuş ve bu şemaların uygulanmasına yönelik sayısal prosedür ele alınmıştır.

Anahtar Kelimeler: Epidemik modeller; kesirli türevli diferansiyel denklemler; Fourier serisi yöntemi; Laplace dönüşümü çözümü; fark şeması

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CHAPTER 1
INTRODUCTION

Fractional differential equations take an important role in applied mathematics (A. Guner & S. Bekir, 2017), engineering (Y. Xiao-Jun, 2011), physics (M. Bayram, A. Seçer, & A. Adıgüzel, 2017), biology (V. Srivastava, K. Kumar, M. K. Awasthi, & B. K. Singh, 2014) and other fields of science. The system of a fractional differential equation is used in various fields in applied science.

Fractional differential equations are formed from fractional calculus which is a branch of mathematics that deals with the properties of integrals and derivatives of non-integer orders (W. F. Ames, 1999). The concept was first mentioned in a letter sent to L'Hopital by Leibniz in 1695, where the idea of semi-derivatives was suggested. Other famous mathematicians, Liouville, Grunwald, Riemann, etc. have proposed original approaches to improve fractional calculus over time (K. Shukla & P. Sapra, 2019). So that fractional differential equations can be applied in the various fields listed above, it is important to develop methods of solutions. Several methods already employed are perturbation techniques (O. Abdulaziz, I. Hashim, & S. Momani, 2008), variational iterative method (V. Gejji & S. Bhalekar, 2007), decomposition methods (H. Jafari & V. Gejji, 2006), integral transform methods (B. Sontakke, G. Kamble, & S. Acharya, 2017), and numerical methods (Y. Yan, K. Pal, & N. Ford, 2014).

An epidemiological study is important to help understand the impact of infectious diseases in a community. The mathematical model is used to analyze data and study the spread and transmission of infectious diseases. With new ideas in epidemiology, we can investigate models by model building, perform estimation of parameters, check sensitivity of models by varying parameters, and compute their numerical simulations. Over the course of history, we have seen examples of epidemic outbreaks infecting large numbers of people. Examples of such are the 1918 Spanish flu outbreak which killed millions and more recently, we have had cases such as HIV/AIDs, SARS, and Ebola outbreaks. World Health Organization (WHO) reports indicating that an estimated 13 million people worldwide die from infectious diseases (WHO, 2012). In light of this, the issue of developing realistic epidemic spreading models and controlling the outbreak and spread of infectious diseases should be considered paramount. The research of this kind helps to understand the ratio of disease spread in the population and to control their parameters (I. Abubakar et al., 2012; B. T. Grenfell, 1992).

Various classical epidemic models have been proposed and studied such as SIR, SIS, SEIR, and SIRS. Kermack and McKendrick developed the first known mathematical and population-level model applicable in studying influenza outbreaks (N. Bacaër, 2011). The model contains three groups: Susceptible (containing those individuals who have a high tendency of contracting the disease), infectious (who currently have the disease who can transmit it to the susceptible individuals), and recovered (this contains individuals that have previously contracted the infection and have now been removed from the epidemic either by recovery or by death). This model is known as the SIR model.

In the SIR model, an infected individual is brought right into a population in which all the individuals are all susceptible. Vertical transmission (transmission of a disease by an infected parent to their children) may be integrated into the SIR model if we consider that a portion of the children of the infected individuals is infected at birth (H. W. Hethcote, 2000; S. Waziri, S. Massawe, & D. Makinde, 2012). This will come in handy when considering a case such as the HIV mother-to-child transmission (MTCT) epidemic. The dynamics of diseases like measles (A. Ahmad, 2018) and influenza (G. H. Li & Y. X. Zhang, 2017) have also been explained using the modified SIR model. This model may be extended to add a state of temporary immunity where individuals who have been removed are returned to the susceptible class after they've missed out on their immunity. This extension is called the SIRS model as has been stated earlier.

In the SIS model, no perennial immunity from the infection exists, individuals can be infected again and can return to the susceptible class. El-Saka studied the stability of equilibrium points for a fractional-order SIS epidemic model (H. A. A. El-Saka, 2014). According to him using fractional differential equations can aid in reducing errors that arise from the neglected parameters in modeling real-life phenomena. The numerical solutions of the models were given and he was able to verify the theoretical analysis using numerical simulations. In the paper, Prakash et al. (B. Prakash, A. Setia, & D. Alapatt, 2017) employed a fractional-order nonlinear SEIR model with a non-constant population mathematical model to model infectious diseases. They proposed a faster and simpler numerical methods based on Harr wavelets to solve the SEIR model deriving and validating the error bounds.

Jun-Jie Wang et al. (J. J. Wang, K. H. Reilly, H. Han, Z. H. Peng, & N. Wang, 2010) employed a deterministic transmission model for the Chinese HIV MTCT epidemic to demonstrate how it is affected by some key parameters. They presented a system of ordinary differential equations and their solutions were derived using this model. HIV positive children delivered by the infected mothers were taken as the susceptible group (S), the transmission rate for HIV positive mothers was (β), and the screening proportion (α) was defined as the percentage of pregnant women who have tested HIV positive to the proportion of HIV positive pregnant cases. They found out that in China, these three factors have the biggest influence on the epidemic. This led them to conclude that proper testing for pregnant women, strengthening prevention of mother-to-child transmission (PMTCT) interventions, and reducing the amount of HIV positive occurrences in women of reproductive ages are steps that will aid in curbing the HIV MTCT epidemic in China.

Ashyralyev et al. (A. Ashyralyev, E. Hincal, & B. Kaymakamzade, 2018) studied the stability of initial-boundary value problem for the system of partial differential equations for observing HIV mother to child transmission epidemic models. The study was aimed at helping to understand the estimation of the transmission rate from mathematical models representing the dynamics of the population of infectious diseases using numerical methods. In their paper, various initial-boundary-value problems for the system of partial differential equations they presented as the initial-value problem for the system of ordinary differential equations

$$\begin{cases} \frac{du^1(t)}{dt} + \alpha u^1(t) + Au^1(t) = f^1(t), \\ \frac{du^2(t)}{dt} + \beta u^2(t) - \beta_1 u^1(t) + cAu^2(t) = f^2(t), \\ \frac{du^3(t)}{dt} + \gamma u^3(t) - \gamma_1 u^1(t) + eAu^3(t) = f^3(t), \\ \frac{du^4(t)}{dt} + du^4(t) - d_1 u^3(t) - d_2 u^2(t) + lAu^4(t) = f^4(t), \\ 0 < t < T, u^m(0) = \varphi^m, m = 1, 2, 3, 4 \end{cases} \tag{1.1}$$

in a Hilbert space H with a self-adjoint positive definite operator A. They proved theorems on stability by applying the operator approach. Moreover, difference schemes for approximate solution of system (1.1) were presented and theorems on stability of these difference schemes were proved. Numerical result was given.

Ameera Masour (2018) in her master's thesis obtained the solution of a system of partial differential equations by solving analytically using Fourier series, Laplace transform and Fourier transform methods. The first order of accuracy difference scheme for the numerical solution of the initial-boundary value problem for one-dimensional partial differential equations was presented. Numerical results were given.

In the present study, systems of fractional differential equations which an extension of partial differential equations are used to modify the system (1.1). We considered stable solution of the initial value problem for solving of fractional differential equations for observing epidemic models and we used classical methods to solve the initial value problem for the system of one dimensional partial differential equation

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} = f^1(t,x), \\ \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - -\frac{\partial^2 u^2(t,x)}{\partial x^2} = f^2(t,x), \\ \\ \frac{\partial u^3(t,x)}{\partial t} + \gamma D_t^{\frac{1}{2}} u^3(t,x) - \gamma_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} = f^3(t,x), \\ \\ \frac{\partial u^4(t,x)}{\partial t} + d D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) - \frac{\partial^2 u^4(t,x)}{\partial x^2} = f^4(t,x), \\ \\ 0 < t < T,\ 0 < x < \pi, \\ \\ u^m(t,0) = u^m(t,\pi) = 0,\ 0 \leqslant t \leqslant T, \\ \\ u^m(0,x) = \varphi^m(x),\ 0 \leqslant x \leqslant \pi,\ m = 1,2,3,4. \end{cases} \tag{1.2}$$

Here,

$$D_t^\alpha = D_{0+}^\alpha$$

is the standard Riemann-Liouville's derivative of order $\alpha \in (0,1)$. This system of fractional differential equations corresponding to the Basset problem (A. Ashyralyev, 2011). The present work aims to study numerical solutions of the initial value problem for the system of fractional differential equations observing the HIV mother-to-child transmission epidemic. The first and second-order of accuracy difference schemes for the numerical solution of the system of one-dimensional fractional partial differential equations are presented and the illustrative numerical results are provided.

The thesis organization is as follows. Chapter 1 is an introduction. The history of epidemiology problems with the system of fractional partial differential equations is presented. In Chapter 2, the methods of solution of the system of fractional partial differential equations by solving analytically using Fourier series, Laplace transform, and Fourier transform methods are presented. In chapter 3, the first and second-order of accuracy single-step difference schemes for the approximate solutions of one-dimensional epidemiology problem for the system of fractional partial differential equations are presented. Numerical results are provided by the Gauss elimination method. Chapter 4 is a conclusion.

CHAPTER 2

METHODS OF SOLUTION OF SYSTEM OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

It is known that the system of fractional partial differential equations can be solved analytically by Fourier series, Laplace transform and Fourier transform methods. In this section, three different analytical methods by examples are illustrated.

2.1 Fourier Series Method

First, we consider the Fourier series method for the solution of the mixed problems for the system of fractional partial differential equations.

Example 2.1. Consider the mixed problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} = (2t + t^2 + \alpha \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}}) \sin x, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2(t,x)}{\partial x^2} \\ = (2t + \beta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2) \sin x, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} \\ = (2t + \delta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2) \sin x, \\ \frac{\partial u^4(t,x)}{\partial t} + d D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) - \frac{\partial^2 u^4(t,x)}{\partial x^2} \\ = (2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2) \sin x, \\ 0 < t < 1, \ 0 < x < \pi, \\ u^1(t,0) = u^2(t,0) = u^3(t,0) = u^4(t,0) = 0, 0 \le t \le 1, \\ u^1(t,\pi) = u^2(t,\pi) = u^3(t,\pi) = u^4(t,\pi) = 0, 0 \le t \le 1, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = 0, 0 \le x \le \pi. \end{cases} \tag{2.1}$$

Solution. To solve this problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0,\ 0 < x < \pi,\ u(0) = u(\pi) = 0, \tag{2.2}$$

generated by the space operator of problem (2.1). It is clear to see that the solution of the Sturm-Liouville problem (2.2) is

$$\lambda_k = -k^2,\ u_k(x) = \sin kx,\ k = 1, 2, ...$$

Then, with using Fourier series solution of problem (2.1) by formula

$$\begin{cases} u^1(t,x) = \sum\limits_{k=1}^{\infty} A_k(t)\sin kx, \\ u^2(t,x) = \sum\limits_{k=1}^{\infty} B_k(t)\sin kx, \\ u^3(t,x) = \sum\limits_{k=1}^{\infty} C_k(t)\sin kx, \\ u^4(t,x) = \sum\limits_{k=1}^{\infty} D_k(t)\sin kx, \end{cases} \tag{2.3}$$

where $A_k(t), B_k(t), C_k(t)$ and $D_k(t)$ are unknown functions. Putting system (2.3) to the system (2.1), we obtain

$$\begin{cases} \sum_{k=1}^{\infty} A_k^{'}(t)\sin kx + \alpha\sum_{k=1}^{\infty} D_t^{\frac{1}{2}} A_k(t)\sin kx + \sum_{k=1}^{\infty} k^2 A_k(t)\sin kx \\ = (2t + t^2 + \alpha\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}})\sin x, \\ \sum_{k=1}^{\infty} B_k^{'}(t)\sin kx + \beta\sum_{k=1}^{\infty} D_t^{\frac{1}{2}} B_k(t)\sin kx - \beta_1\sum_{k=1}^{\infty} A_k(t)\sin kx + \sum_{k=1}^{\infty} k^2 B_k(t)\sin kx \\ = (2t + \beta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2)\sin x, \\ \sum_{k=1}^{\infty} C_k^{'}(t)\sin kx + \delta\sum_{k=1}^{\infty} D_t^{\frac{1}{2}} C_k(t)\sin kx - \delta_1\sum_{k=1}^{\infty} A_k(t)\sin kx + \sum_{k=1}^{\infty} k^2 C_k(t)\sin kx \\ = (2t + \delta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2)\sin x, \\ \sum_{k=1}^{\infty} D_k^{'}(t)\sin kx + d\sum_{k=1}^{\infty} D_t^{\frac{1}{2}} D_k(t)\sin kx - d_1\sum_{k=1}^{\infty} C_k(t)\sin kx - d_2\sum_{k=1}^{\infty} B_k(t)\sin kx \\ +\sum_{k=1}^{\infty} k^2 D_k(t)\sin kx = (2t + d\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2)\sin x, \\ 0 < t < 1,\ 0 < x < \pi. \end{cases}$$

Applying the initial conditions to the system (2.3), we can write

$$\begin{cases} u^1(0,x) = \sum_{k=1}^{\infty} A_k(0)\sin kx = 0, \\ u^2(0,x) = \sum_{k=1}^{\infty} B_k(0)\sin kx = 0, \\ u^3(0,x) = \sum_{k=1}^{\infty} C_k(0)\sin kx = 0, \\ u^4(0,x) = \sum_{k=1}^{\infty} D_k(0)\sin kx = 0, \\ 0 \le x \le \pi. \end{cases}$$

Equating coefficients $\sin kx$, $k = 1, 2, ...$, we get

$$\begin{cases} A_1'(t) + \alpha D_t^{\frac{1}{2}} A_1(t) + A_1(t) = 2t + t^2 + \alpha \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}}, \\ B_1'(t) + \beta D_t^{\frac{1}{2}} B_1(t) - \beta_1 A_1(t) + B_1(t) \\ = 2t + \beta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2, \\ C_1'(t) + \delta D_t^{\frac{1}{2}} C_1(t) - \delta_1 A_1(t) + C_1(t) \\ = 2t + \delta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2, \\ D_1'(t) + d D_t^{\frac{1}{2}} D_1(t) - d_1 C_1(t) - d_2 B_1(t) + D_1(t) \\ = 2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2, \\ 0 < t < 1, \ A_1(0) = B_1(0) = C_1(0) = D_1(0) = 0 \end{cases} \tag{2.4}$$

and for $k \neq 1$

$$\begin{cases} A_k'(t) + \alpha D_t^{\frac{1}{2}} A_k(t) + k^2 A_k(t) = 0, \\ B_k'(t) + \beta D_t^{\frac{1}{2}} B_k(t) - \beta_1 A_k(t) + k^2 B_k(t) = 0, \\ C_k'(t) + \delta D_t^{\frac{1}{2}} C_k(t) - \delta_1 A_k(t) + k^2 C_k(t) = 0, \\ D_k'(t) + d D_t^{\frac{1}{2}} D_k(t) - d_1 C_k(t) - d_2 B_k(t) + k^2 D_k(t) = 0, \\ 0 < t < 1, \ A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0. \end{cases} \tag{2.5}$$

So, we have initial value problems for the system of ordinary differential equations. For solving the systems the Laplace transform method is applied.

Here and in future we assume that

$$\begin{cases} \mathcal{L}\{A_k(t)\} = A_k(s), \\ \mathcal{L}\{B_k(t)\} = B_k(s), \\ \mathcal{L}\{C_k(t)\} = C_k(s), \\ \mathcal{L}\{D_k(t)\} = D_k(s). \end{cases}$$

Taking Laplace transform of both sides of system of fractional partial differential equations in the systems (2.4), (2.5) and using the following conditions $A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0$, $k \geq 1$, we obtain the following systems of algebraic equations

$$\begin{cases} sA_1(s) + \alpha s^{\frac{1}{2}} A_1(s) + A_1(s) = \frac{2}{s^2} + \frac{2}{s^3} + \alpha \frac{2}{s^{\frac{5}{2}}}, \\ sB_1(s) + \beta s^{\frac{1}{2}} B_1(s) - \beta_1 A_1(s) + B_1(s) = \frac{2}{s^2} + \beta \frac{2}{s^{\frac{5}{2}}} - \beta_1 \frac{2}{s^3} + \frac{2}{s^3}, \\ sC_1(s) + \delta s^{\frac{1}{2}} C_1(s) - \delta_1 A_1(s) + C_1(s) = \frac{2}{s^2} + \delta \frac{2}{s^{\frac{5}{2}}} - \delta_1 \frac{2}{s^3} + \frac{2}{s^3}, \\ sD_1(s) + ds^{\frac{1}{2}} D_1(s) - d_1 C_1(s) - d_2 B_1(s) + D_1(s) \\ = \frac{2}{s^2} + d \frac{2}{s^{\frac{5}{2}}} - d_1 \frac{2}{s^3} - d_2 \frac{2}{s^3} + \frac{2}{s^3}, \end{cases} \tag{2.6}$$

and for $k \neq 1$

$$\begin{cases} (s+\alpha s^{\frac{1}{2}}+k^2)A_k(s)=0, \\ (s+\beta s^{\frac{1}{2}}+k^2)B_k(s)-\beta_1 A_k(s)=0, \\ (s+\delta s^{\frac{1}{2}}+k^2)C_k(s)-\delta_1 A_k(s)=0, \\ (s+d s^{\frac{1}{2}}+k^2)D_k(s)-d_1 C_k(s)-d_2 B_k(s)=0. \end{cases} \tag{2.7}$$

For $k \neq 1$ from system (2.7) it follows $A_k(s) = B_k(s) = C_k(s) = D_k(s) = 0$. Taking the inverse Laplace transform with respect to t, we get

$$A_k(t) = B_k(t) = C_k(t) = D_k(t) = 0.$$

For finding $A_1(t), B_1(t), C_1(t)$ and $D_1(t)$, we use the system (2.6). First, we obtain $A_1(s)$. We have that

$(s+\alpha s^{\frac{1}{2}}+1)A_1(s) = \frac{2}{s^2} + \frac{2}{s^3} + \alpha\frac{2}{s^{\frac{5}{2}}}$.

Therefore,

$$A_1(s) = \frac{2!}{s^3}. \tag{2.8}$$

Second, we obtain $B_1(s)$. Using formula (2.8) in the second equation, we get

$$(s+\beta s^{\frac{1}{2}}+1)B_1(s) - \beta_1\frac{2!}{s^3} = \frac{2!}{s^3}(s+\beta s^{\frac{1}{2}}-\beta_1+1).$$

Therefore,

$$B_1(s) = \frac{2!}{s^3}. \tag{2.9}$$

Third, we obtain $C_1(s)$. Applying formula (2.8) in the third equation, we obtain

$$(s+\delta s^{\frac{1}{2}}+1)C_1(s) - \delta_1\frac{2!}{s^3} = \frac{2!}{s^3}(s+\delta s^{\frac{1}{2}}-\delta_1+1)$$

or

$$C_1(s) = \frac{2!}{s^3}. \tag{2.10}$$

Fourth, we obtain $D_1(s)$. Applying formula (2.9) and (2.10) in the last equation, we get

$$(s + ds^{\frac{1}{2}} + 1)D_1(s) - d_1\frac{2}{s^3} - d_2\frac{2}{s^3} = \frac{2}{s^3}(s + ds^{\frac{1}{2}} - d_1 - d_2 + 1)$$

or

$$D_1(s) = \frac{2!}{s^3}. \tag{2.11}$$

Finally, applying formulas (2.8), (2.9), (2.10) and (2.11) and taking the inverse Laplace transform with respect to t, we get

$$A_1(t) = B_1(t) = C_1(t) = D_1(t) = t^2.$$

Therefore, the exact solution of the problem (2.1) is

$$\begin{cases} u^1(t,x) = A_1(t)\sin x = t^2 \sin x, \\ u^2(t,x) = B_1(t)\sin x = t^2 \sin x, \\ u^3(t,x) = C_1(t)\sin x = t^2 \sin x, \\ u^4(t,x) = D_1(t)\sin x = t^2 \sin x. \end{cases}$$

Using similar procedure we can get the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u^1(t,x)}{\partial t} + \alpha \, D_t^{\frac{1}{2}} u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^1(t,x)}{\partial x_r^2} = f_1(t,x), \\
\frac{\partial u^2(t,x)}{\partial t} + \beta \, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^2(t,x)}{\partial x_r^2} = f_2(t,x), \\
\frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} \, u^3(t,x) - \delta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^3(t,x)}{\partial x_r^2} = f_3(t,x), \\
\frac{\partial u^4(t,x)}{\partial t} + d D_t^{\frac{1}{2}} u^4(t,x) - d_1 \, u^3(t,x) - d_2 \, u^2(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^4(t,x)}{\partial x_r^2} \\
= f_4(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega}, \; 0 < t < T, \\
u^1(0,x) = \varphi(x), \; u^2(0,x) = \psi(x), \; u^3(0,x) = \xi(x), \; u^4(0,x) = \lambda(x), \\
x = (x_1, ..., x_n) \in \overline{\Omega}, \\
u^1(t,x) = u^2(t,x) = u^3(t,x) = u^4(t,x) = 0, \; x \in S, \; 0 \le t \le T
\end{cases}
\tag{2.12}
$$

for the system of multidimensional fractional partial differential equations. Note that $a_r > a_0 > 0$ and $f_k(t,x), \; k = 1,2,3,4 \; \left(t \in (0,T), \; x \in \overline{\Omega}\right)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\lambda(x)$, $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future Ω is the unit open cube in the $n-$dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with the boundary
S, $\overline{\Omega} = \Omega \cup S$.

Note that the Fourier series method described in solving (2.12) can be used only in the case when (2.12) has constant coefficients.

Example 2.2. Consider the mixed problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} = (2t + t^2 + \alpha \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}}) \cos x, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2(t,x)}{\partial x^2} \\ = (2t + \beta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2) \cos x, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} \\ = (2t + \delta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2) \cos x, \\ \frac{\partial u^4(t,x)}{\partial t} + d D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) - \frac{\partial^2 u^4(t,x)}{\partial x^2} \\ = (2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2) \cos x, \\ 0 < t < 1, \ 0 < x < \pi, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = \cos x, \ 0 \le x \le \pi, \\ u_x^1(t,0) = u_x^2(t,0) = u_x^3(t,0) = u_x^4(t,0) = 0, \ 0 \le t \le 1, \\ u_x^1(t,\pi) = u_x^2(t,\pi) = u_x^3(t,\pi) = u_x^4(t,\pi) = 0, \ 0 \le t \le 1. \end{cases} \tag{2.13}$$

Solution. To solve the mixed problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, \ 0 < x < \pi, \ u_x(0) = u_x(\pi) = 0, \tag{2.14}$$

generated by the space operator of problem (2.13). It is clear to see that the solution of this Sturm-Liouville problem (2.14) is

$$\lambda_k = -k^2, \ u_k(x) = \cos kx, \ k = 0, 1, ...$$

Then, with using Fourier series solution of problem (2.13) by formula

$$\begin{cases} u^1(t,x) = \sum\limits_{k=0}^{\infty} A_k(t)\cos kx, \\ u^2(t,x) = \sum\limits_{k=0}^{\infty} B_k(t)\cos kx, \\ u^3(t,x) = \sum\limits_{k=0}^{\infty} C_k(t)\cos kx, \\ u^4(t,x) = \sum\limits_{k=0}^{\infty} D_k(t)\cos kx, \end{cases} \tag{2.15}$$

where $A_k(t)$, $B_k(t), C_k(t)$ and $D_k(t)$ are unknown functions. Putting system (2.15) to the system (2.13), we obtain

$$\begin{cases} \sum\limits_{k=0}^{\infty} A_k^{'}(t)\cos kx + \alpha \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} A_k(t)\cos kx + \sum\limits_{k=0}^{\infty} k^2 A_k(t)\cos kx = (2t + t^2 + \alpha\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}})\cos x, \\ \sum\limits_{k=0}^{\infty} B_k^{'}(t)\cos kx + \beta \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} B_k(t)\cos kx - \beta_1 \sum\limits_{k=0}^{\infty} A_k(t)\cos kx + \sum\limits_{k=0}^{\infty} k^2 B_k(t)\cos kx \\ = (2t + \beta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2)\cos x, \\ \sum\limits_{k=0}^{\infty} C_k^{'}(t)\cos kx + \delta \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} C_k(t)\cos kx - \delta_1 \sum\limits_{k=0}^{\infty} A_k(t)\cos kx + \sum\limits_{k=0}^{\infty} k^2 C_k(t)\cos kx \\ = (2t + \delta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2)\cos x, \\ \sum\limits_{k=0}^{\infty} D_k^{'}(t)\cos kx + d \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} D_k(t)\cos kx - d_1 \sum\limits_{k=0}^{\infty} C_k(t)\cos kx - d_2 \sum\limits_{k=0}^{\infty} B_k(t)\cos kx \\ + \sum\limits_{k=0}^{\infty} k^2 D_k(t)\cos kx = (2t + d\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2)\cos x, \\ 0 < t < 1,\ 0 < x < \pi. \end{cases}$$

Applying the initial conditions to the system (2.15), we can write

$$\begin{cases} u^1(0,x) = \sum\limits_{k=0}^{\infty} A_k(0)\cos kx = \cos x, \\ \\ u^2(0,x) = \sum\limits_{k=0}^{\infty} B_k(0)\cos kx = \cos x, \\ \\ u^3(0,x) = \sum\limits_{k=0}^{\infty} C_k(0)\cos kx = \cos x, \\ \\ u^4(0,x) = \sum\limits_{k=0}^{\infty} D_k(0)\cos kx = \cos x, \\ \\ 0 \le x \le \pi. \end{cases}$$

Equating coefficients $\cos kx, k = 0, 1, ...,$ we get

$$\begin{cases} A_1^{'}(t) + \alpha D_t^{\frac{1}{2}} A_1(t) + A_1(t) = 2t + t^2 + \alpha \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}}, \\ \\ B_1^{'}(t) + \beta D_t^{\frac{1}{2}} B_1(t) - \beta_1 A_1(t) + B_1(t) \\ \\ = 2t + \beta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + t^2, \\ \\ C_1^{'}(t) + \delta D_t^{\frac{1}{2}} C_1(t) - \delta_1 A_1(t) + C_1(t) \\ \\ = 2t + \delta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + t^2, \\ \\ D_1^{'}(t) + d D_t^{\frac{1}{2}} D_1(t) - d_1 C_1(t) - d_2 B_1(t) + D_1(t) \\ \\ = 2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + t^2, \\ \\ 0 < t < 1,\ A_1(0) = B_1(0) = C_1(0) = D_1(0) = 0 \end{cases} \tag{2.16}$$

for $k \neq 1$

$$\begin{cases} A_k'(t) + \alpha D_t^{\frac{1}{2}} A_k(t) + k^2 A_k(t) = 0, \\ B_k'(t) + \beta D_t^{\frac{1}{2}} B_k(t) - \beta_1 A_k(t) + k^2 B_k(t) = 0, \\ C_k'(t) + \delta D_t^{\frac{1}{2}} C_k(t) - \delta_1 A_k(t) + k^2 C_k(t) = 0, \\ D_k'(t) + d D_t^{\frac{1}{2}} D_k(t) - d_1 C_k(t) - d_2 B_k(t) + k^2 D_k(t) = 0, \\ 0 < t < 1,\ A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0. \end{cases} \tag{2.17}$$

Taking Laplace transform of both sides of system fractional partial differential equations in the systems (2.16) and (2.17) and using the following conditions $A_k(0) = B_k(0) = C_k(0) = D_k(0) = 0$, $k \geq 1$, we obtain the following systems of algebraic equations

$$\begin{cases} sA_1(s) + \alpha s^{\frac{1}{2}} A_1(s) + A_1(s) = \frac{2}{s^2} + \frac{2}{s^3} + \alpha \frac{2}{s^{\frac{5}{2}}}, \\ sB_1(s) + \beta s^{\frac{1}{2}} B_1(s) - \beta_1 A_1(s) + B_1(s) = \frac{2}{s^2} + \beta \frac{2}{s^{\frac{5}{2}}} - \beta_1 \frac{2}{s^3} + \frac{2}{s^3}, \\ sC_1(s) + \delta s^{\frac{1}{2}} C_1(s) - \delta_1 A_1(s) + C_1(s) = \frac{2}{s^2} + \delta \frac{2}{s^{\frac{5}{2}}} - \delta_1 \frac{2}{s^3} + \frac{2}{s^3}, \\ sD_1(s) + d s^{\frac{1}{2}} D_1(s) - d_1 C_1(s) - d_2 B_1(s) + D_1(s) \\ = \frac{2}{s^2} + d \frac{2}{s^{\frac{5}{2}}} - d_1 \frac{2}{s^3} - d_2 \frac{2}{s^3} + \frac{2}{s^3} \end{cases} \tag{2.18}$$

and for $k \neq 1$

$$\begin{cases} (s+\alpha s^{\frac{1}{2}}+k^2)A_k(s)=0, \\ \\ (s+\beta s^{\frac{1}{2}}+k^2)B_k(s)-\beta_1 A_k(s)=0, \\ \\ (s+\delta s^{\frac{1}{2}}+k^2)C_k(s)-\delta_1 A_k(s)=0, \\ \\ (s+d s^{\frac{1}{2}}+k^2)D_k(s)-d_1 C_k(s)-d_2 B_k(s)=0. \end{cases} \tag{2.19}$$

For $k \neq 1$ from system (2.19) it follows $A_k(s) = B_k(s) = C_k(s) = D_k(s) = 0$. Taking the inverse Laplace transform with respect to t, we get $A_k(t) = B_k(t) = C_k(t) = D_k(t) = 0$.

For finding $A_1(t)$, $B_1(t)$, $C_1(t)$ and $D_1(t)$, we use the system (2.18). First, we obtain $A_1(s)$. We have that

$$(s+\alpha s^{\frac{1}{2}}+1)A_1(s)=\tfrac{2}{s^2}+\tfrac{2}{s^3}+\alpha\tfrac{2}{s^{\frac{5}{2}}}.$$

Therefore,

$$A_1(s)=\frac{2!}{s^3}. \tag{2.20}$$

Second, we obtain $B_1(s)$. Using formula (2.20) in the second equation, we get

$$(s+\beta s^{\frac{1}{2}}+1)B_1(s)-\beta_1\frac{2}{s^3}=\frac{2}{s^3}(s+\beta s^{\frac{1}{2}}-\beta_1+1).$$

Therefore,

$$B_1(s)=\frac{2!}{s^3}. \tag{2.21}$$

Third, we obtain $C_1(s)$. Applying formula (2.20) in the third equation, we obtain

$$(s+\delta s^{\frac{1}{2}}+1)C_1(s)-\delta_1\frac{2!}{s^3}=\frac{2!}{s^3}(s+\delta s^{\frac{1}{2}}-\delta_1+1)$$

or

$$C_1(s)=\frac{2!}{s^3}\;. \tag{2.22}$$

Fourth, we obtain $D_1(s)$. Applying formula (2.21) and (2.22) in the last equation, we get

$$(s + ds^{\frac{1}{2}} + 1)D_1(s) - d_1\frac{2}{s^3} - d_2\frac{2}{s^3} = \frac{2}{s^3}(s + ds^{\frac{1}{2}} - d_1 - d_2 + 1)$$

or

$$D_1(s) = \frac{2!}{s^3}\ . \tag{2.23}$$

Finally, applying formulas (2.20), (2.21), (2.22), (2.23) and taking the inverse Laplace transform with respect to t, we get $A_1(t) = B_1(t) = C_1(t) = D_1(t) = t^2$.
Therefore, the exact solution of the problem (2.13) is

$$\begin{cases} u^1(t,x) = A_1(t)\cos x = t^2\cos x, \\ u^2(t,x) = B_1(t)\cos x = t^2\cos x, \\ u^3(t,x) = C_1(t)\cos x = t^2\cos x, \\ u^4(t,x) = D_1(t)\cos x = t^2\cos x. \end{cases}$$

Using similar procedure we can get the solution of the following initial boundary value problem

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^1(t,x)}{\partial x_r^2} = f_1(t,x), \\ \frac{\partial u^2(t,x)}{\partial t} + \beta\, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^2(t,x)}{\partial x_r^2} \\ = f_2(t,x), \\ \frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^3(t,x)}{\partial x_r^2} \\ = f_3(t,x), \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1\, u^3(t,x) - d_2\, u^2(t,x) \\ - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^4(t,x)}{\partial x_r^2} = f_4(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\ u^1(0,x) = \varphi(x),\ u^2(0,x) = \psi(x),\ u^3(0,x) = \xi(x), \\ u^4(0,x) = \lambda(x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, \\ \frac{\partial u^1(t,x)}{\partial \overline{m}} = \frac{\partial u^2(t,x)}{\partial \overline{m}} = \frac{\partial u^3(t,x)}{\partial \overline{m}} = \frac{\partial u^4(t,x)}{\partial \overline{m}} = 0, \\ x \in S,\ 0 \le t \le T \end{cases} \tag{2.24}$$

for the system of multidimensional fractional partial differential equations. Note that $a_r > a_0 > 0$ and $f_k(t,x)$, $k = 1,2,3,4$ $\left(t \in (0,T),\ x \in \overline{\Omega}\right)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\lambda(x)$, $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here and in future $\overline{m}$ is the normal vector to S.

Note that the Fourier series method described in solving (2.24) can be used only in the case when (2.24) has constant coefficients.

Example 2.3. Consider the mixed problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1}{\partial x^2} = (2t \\ +\alpha \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} + 4t^2) \cos 2x, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2}{\partial x^2} \\ = (2t + \beta \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + 4t^2) \cos 2x, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3}{\partial x^2} \\ = (2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + 4t^2) \cos 2x, \\ \frac{\partial u^4(t,x)}{\partial t} + d D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) \\ -\frac{\partial^2 u^4}{\partial x^2} = (2t + d \frac{8}{3} \frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + 4t^2) \cos 2x, \\ 0 < t < 1, \ 0 < x < \pi, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = 1, \\ 0 \le x \le \pi, \\ u^1(t,0) = u^1(t,\pi), \ u^1_x(t,0) = u^1_x(t,\pi), \ 0 \le t \le 1, \\ u^2(t,0) = u^2(t,\pi), \ u^2_x(t,0) = u^2_x(t,\pi), \ 0 \le t \le 1, \\ u^3(t,0) = u^3(t,\pi), \ u^3_x(t,0) = u^3_x(t,\pi), \ 0 \le t \le 1, \\ u^4(t,0) = u^4(t,\pi), \ u^4_x(t,0) = u^4_x(t,\pi), \ 0 \le t \le 1. \end{cases} \tag{2.25}$$

Solution. To solve the mixed problem, we consider the Sturm-Liouville problem

$$-u''(x) - \lambda u(x) = 0, 0 < x < \pi,\ u(0) = u(\pi),\ u_x(0) = u_x(\pi), \tag{2.26}$$

generated by the space operator of problem (2.25). It is clear to see that the solution of the Sturm-Liouville problem (2.26) is

$$\lambda_k = -4k^2,\ u_k(x) = \sin 2kx,\ k = 1, ...,\ u_k(x) = \cos 2kx,\ k = 0, 1, ...$$

Then, with using Fourier series solution of problem (2.25) by formula

$$\begin{cases} u^1(t,x) = \sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx, \\ \\ u^2(t,x) = \sum\limits_{k=1}^{\infty} C_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} D_k(t)\cos 2kx, \\ \\ u^3(t,x) = \sum\limits_{k=1}^{\infty} E_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} F_k(t)\cos 2kx, \\ \\ u^4(t,x) = \sum\limits_{k=1}^{\infty} M_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} N_k(t)\cos 2kx, \end{cases} \tag{2.27}$$

where $A_k(t)$, $B_k(t)$, $C_k(t)$, $D_k(t)$, $E_k(t)$, $F_k(t)$, $M_k(t)$ and $N_k(t)$ are unknown functions. Putting system (2.27) to the system (2.25), we obtain

$$
\begin{cases}
\sum\limits_{k=1}^{\infty} A_k'(t)\sin 2kx + \sum\limits_{k=0}^{\infty} B_k'(t)\cos 2kx + \alpha \sum\limits_{k=1}^{\infty} D_t^{\frac{1}{2}} A_k(t)\sin 2kx + \alpha \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} B_k(t)\cos 2kx \\
+ \sum\limits_{k=1}^{\infty} 4k^2 A_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} 4k^2 B_k(t)\cos 2kx = (2t + \alpha \frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} + 4t^2)\cos 2x, \\
\sum\limits_{k=1}^{\infty} C_k'(t)\sin 2kx + \sum\limits_{k=0}^{\infty} D_k'(t)\cos 2kx + \beta \sum\limits_{k=1}^{\infty} D_t^{\frac{1}{2}} C_k(t)\sin 2kx + \beta \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} D_k(t)\cos 2kx \\
-\beta_1 \sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx - \beta_1 \sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx + \sum\limits_{k=1}^{\infty} 4k^2 C_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} 4k^2 D_k(t)\cos 2kx \\
= (2t + \beta \frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + 4t^2)\cos 2x, \\
\sum\limits_{k=1}^{\infty} E_k'(t)\sin 2kx + \sum\limits_{k=0}^{\infty} F_k'(t)\cos 2kx + \delta \sum\limits_{k=1}^{\infty} D_t^{\frac{1}{2}} E_k(t)\sin 2kx + \delta \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} F_k(t)\cos 2kx \\
-\delta_1 \sum\limits_{k=1}^{\infty} A_k(t)\sin 2kx - \delta_1 \sum\limits_{k=0}^{\infty} B_k(t)\cos 2kx + \sum\limits_{k=1}^{\infty} 4k^2 E_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} 4k^2 F_k(t)\cos 2kx \\
= (2t + \delta \frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + 4t^2)\cos 2x, \\
\sum\limits_{k=1}^{\infty} M_k'(t)\sin 2kx + \sum\limits_{k=0}^{\infty} N_k'(t)\cos 2kx + d \sum\limits_{k=1}^{\infty} D_t^{\frac{1}{2}} M_k(t)\sin 2kx + d \sum\limits_{k=0}^{\infty} D_t^{\frac{1}{2}} N_k(t)\cos 2kx \\
-d_1 \sum\limits_{k=1}^{\infty} E_k(t)\sin 2kx - d_1 \sum\limits_{k=0}^{\infty} F_k(t)\cos 2kx - d_2 \sum\limits_{k=1}^{\infty} C_k(t)\sin 2kx - d_2 \sum\limits_{k=0}^{\infty} D_k(t)\cos 2kx \\
+ \sum\limits_{k=1}^{\infty} 4k^2 M_k(t)\sin 2kx + \sum\limits_{k=0}^{\infty} 4k^2 N_k(t)\cos 2kx = (2t + d \frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + 4t^2)\cos 2x, \\
0 < t < 1,\ 0 < x < \pi.
\end{cases}
$$

Applying initial conditions to the system (2.27), we can write

$$
\begin{cases}
u^1(0,x) = \sum\limits_{k=1}^{\infty} A_k(0)\sin 2kx + \sum\limits_{k=0}^{\infty} B_k(0)\cos 2kx = 1, \\
u^2(0,x) = \sum\limits_{k=1}^{\infty} C_k(0)\sin 2kx + \sum\limits_{k=0}^{\infty} D_k(0)\cos 2kx = 1, \\
u^3(0,x) = \sum\limits_{k=1}^{\infty} E_k(0)\sin 2kx + \sum\limits_{k=0}^{\infty} F_k(0)\cos 2kx = 1, \\
u^4(0,x) = \sum\limits_{k=1}^{\infty} M_k(0)\sin 2kx + \sum\limits_{k=0}^{\infty} N_k(0)\cos 2kx = 1, \\
0 \le x \le \pi.
\end{cases}
$$

Equating coefficients $\sin 2kx$, $k = 1, 2, ...$, we get

$$
\begin{cases}
A_k'(t) + \alpha D_t^{\frac{1}{2}} A_k(t) + 4k^2 A_k(t) = 0, \\
C_k'(t) + \beta D_t^{\frac{1}{2}} C_k(t) - \beta_1 A_k(t) + 4k^2 C_k(t) = 0, \\
E_k'(t) + \delta D_t^{\frac{1}{2}} E_k(t) - \delta_1 A_k(t) + 4k^2 E_k(t) = 0, \\
M_k'(t) + d D_t^{\frac{1}{2}} M_k(t) - d_1 E_k(t) - d_2 C_k(t) + 4k^2 M_k(t) = 0, \\
0 < t < 1,\ A_k(0) = C_k(0) = E_k(0) = M_k(0) = 0.
\end{cases}
\tag{2.28}
$$

Equating coefficients $\cos 2kx$, $k = 0, 1, ...$, we get

$$\begin{cases} B_1'(t) + \alpha D_t^{\frac{1}{2}} B_1(t) + 4B_1(t) = 2t + \alpha\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} + 4t^2, \\ D_1'(t) + \beta D_t^{\frac{1}{2}} D_1(t) - \beta_1 D_1(t) + 4D_1(t) \\ = 2t + \beta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \beta_1 t^2 + 4t^2, \\ F_1'(t) + \delta D_t^{\frac{1}{2}} F_1(t) - \delta_1 B_1(t) + 4F_1(t) \\ = 2t + \delta\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - \delta_1 t^2 + 4t^2, \\ N_1'(t) + dD_t^{\frac{1}{2}} N_1(t) - d_1 F_1(t) - d_2 D_1(t) + 4N_1(t) \\ = 2t + d\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + 4t^2, \\ 0 < t < 1,\ B_1(0) = D_1(0) = F_1(0) = N_1(0) = 1 \end{cases} \tag{2.29}$$

and for $k \neq 1$, we get

$$\begin{cases} B_k'(t) + \alpha D^{\frac{1}{2}} B_k(t) + 4k^2 B_k(t) = 0, \\ D_k'(t) + \beta D_t^{\frac{1}{2}} D_k(t) - \beta_1 B_k(t) + 4k^2 D_k(t) = 0, \\ F_k'(t) + \delta D_t^{\frac{1}{2}} F_k(t) - \delta_1 B_k(t) + 4k^2 F_k(t) = 0, \\ N_k'(t) + dD_t^{\frac{1}{2}} N_k(t) - d_1 F_k(t) - d_2 D_k(t) + 4k^2 N_k(t) = 0, \\ 0 < t < 1,\ B_k(0) = D_k(0) = F_k(0) = N_k(0) = 0. \end{cases} \tag{2.30}$$

Here, we assume that

$$\begin{cases} \mathcal{L}\{A_k(t)\} = A_k(s), \\ \mathcal{L}\{B_k(t)\} = B_k(s), \\ \mathcal{L}\{C_k(t)\} = C_k(s), \\ \mathcal{L}\{D_k(t)\} = D_k(s), \\ \mathcal{L}\{E_k(t)\} = E_k(s), \\ \mathcal{L}\{F_k(t)\} = F_k(s), \\ \mathcal{L}\{M_k(t)\} = M_k(s), \\ \mathcal{L}\{N_k(t)\} = N_k(s). \end{cases}$$

Taking Laplace transform of both sides of system of fractional partial diffrential equations in the systems (2.28) and (2.30) and using the following conditions
$A_k(0) = B_k(0) = C_k(0) = D_k(0) = E_k(0) = F_k(0) = M_k(0) = N_k(0) = 0$, $k \geq 1$, we obtain the following system of algebraic equations

$$\begin{cases} sA_k(s) + \alpha s^{\frac{1}{2}} A_k(s) + 4k^2 A_k(s) = 0, \\ sC_k(s) + \beta s^{\frac{1}{2}} C_k(s) - \beta_1 A_k(s) + 4k^2 C_k(s) = 0, \\ sE_k(s) + \delta s^{\frac{1}{2}} E_k(s) - \delta_1 A_k(s) + 4k^2 E_k(s) = 0, \\ sM_k(s) + d s^{\frac{1}{2}} M_k(s) - d_1 E_k(s) - d_2 C_k(s) + 4k^2 M_k(s) = 0, \end{cases} \tag{2.31}$$

and

$$\begin{cases} sB_k(s) + \alpha s^{\frac{1}{2}} B_k(s) + 4k^2 B_k(s) = 0, \\ \\ sD_k(s) + \beta s^{\frac{1}{2}} D_k(s) + 4k^2 D_k(s) - \beta_1 B_k(s) = 0, \\ \\ sF_k(s) + \delta s^{\frac{1}{2}} F_k(s) + 4k^2 F_k(s) - \delta_1 B_k(s) = 0, \\ \\ sN_k(s) + d s^{\frac{1}{2}} N_k(s) + 4k^2 N_k(s) - d_1 F_k(s) - d_2 D_k(s) = 0. \end{cases} \tag{2.32}$$

From the system (2.29), we get

$$\begin{cases} sB_1(s) + \alpha s^{\frac{1}{2}} B_1(s) + 4B_1(s) = \frac{2}{s^3} + \alpha \frac{2}{s^{\frac{5}{2}}} + \frac{8}{s^3}, \\ \\ sD_1(s) + \beta s^{\frac{1}{2}} D_1(s) - \beta_1 B_1(s) + 4D_1(s) \\ \\ = \frac{2}{s^2} + \beta \frac{2}{s^{\frac{5}{2}}} - \beta_1 \frac{2}{s^3} + \frac{8}{s^3}, \\ \\ sF_1(s) + \delta s^{\frac{1}{2}} F_1(s) - \delta_1 B_1(s) + 4F_1(s) \\ \\ = \frac{2}{s^2} + \delta \frac{2}{s^{\frac{5}{2}}} - \delta_1 \frac{2}{s^3} + \frac{8}{s^3}, \\ \\ sN_1(s) + d s^{\frac{1}{2}} N_1(s) - d_1 F_1(s) - d_2 D_1(s) + 4N_1(s) \\ \\ = \frac{2}{s^2} + d \frac{2}{s^{\frac{5}{2}}} - d_1 \frac{2}{s^3} - d_2 \frac{2}{s^3} + \frac{8}{s^3}. \end{cases} \tag{2.33}$$

For $k \neq 1$ from the system (2.31) and (2.32) it follows $A_k(s) = B_k(s) = C_k(s) = D_k(s) = E_k(s) = F_k(s) = M_k(s) = N_k(s) = 0$. Taking the inverse Laplace transform, we get
$A_k(t) = B_k(t) = C_k(t) = D_k(t) = E_k(t) = F_k(t) = M_k(t) = N_k(t) = 0$.
For finding $B_1(t), D_1(t), F_1(t)$ and $N_1(t)$, we use the system (2.33). First, we obtain $B_1(s)$.

We have that

$$sB_1(s) + \alpha s^{\frac{1}{2}}B_1(s) + 4B_1(s) = \frac{2}{s^3} + \alpha\frac{2}{s^{\frac{5}{2}}} + \frac{8}{s^3}.$$

Therefore,

$$B_1(s) = \frac{2!}{s^3}. \tag{2.34}$$

Second, we obtain $D_1(s)$. Using formula (2.34) in the second equation, we get

$$(s + \beta s^{\frac{1}{2}} + 4)D_1(s) - \beta_1\frac{2!}{s^3} = \frac{2!}{s^3}(s + \beta s^{\frac{1}{2}} - \beta_1 + 4).$$

Therefore,

$$D_1(s) = \frac{2!}{s^3}. \tag{2.35}$$

Third, we obtain $F_1(s)$. Applying formula (2.34) in the third equation, we obtain

$$(s + \delta s^{\frac{1}{2}} + 4)F_1(s) - \delta_1\frac{2!}{s^3} = \frac{2!}{s^3}(s + \delta s^{\frac{1}{2}} - \delta_1 + 4)$$

or

$$F_1(s) = \frac{2!}{s^3}. \tag{2.36}$$

Fourth, we obtain $N_1(s)$. Applying formulas (2.35) and (2.36) in the last equation, we get

$$(s + ds^{\frac{1}{2}} + 4)N_1(s) - d_1\frac{2}{s^3} - d_2\frac{2}{s^3} = \frac{2}{s^3}(s + ds^{\frac{1}{2}} - d_1 - d_2 + 4)$$

or

$$N_1(s) = \frac{2!}{s^3}. \tag{2.37}$$

Finally, appling formulas (2.34), (2.35), (2.36) and (2.37) and taking the inverse Laplace transform with respect to t, we get

$B_1(t) = D_1(t) = F_1(t) = N_1(t) = t^2$.

Therefore, the exact solution of problem (2.25) is

$$\begin{cases} u^1(t,x) = B_1(t)\cos 2x = t^2 \cos 2x, \\ u^2(t,x) = D_1(t)\cos 2x = t^2 \cos 2x, \\ u^3(t,x) = F_1(t)\cos 2x = t^2 \cos 2x, \\ u^4(t,x) = N_1(t)\cos 2x = t^2 \cos 2x. \end{cases}$$

Using similar procedure we can get the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^1(t,x)}{\partial x_r^2} = f_1(t,x), \\
\frac{\partial u^2(t,x)}{\partial t} + \beta\, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^2(t,x)}{\partial x_r^2} = f_2(t,x), \\
\frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^3(t,x)}{\partial x_r^2} = f_3(t,x), \\
\frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1\, u^3(t,x) - d_2\, u^2(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^4(t,x)}{\partial x_r^2} \\
= f_4(t,x), \\
x = (x_1, ..., x_n) \in \overline{\Omega},\ 0 < t < T, \\
u^1(0,x) = \varphi(x),\ u^2(0,x) = \psi(x),\ u^3(0,x) = \xi(x),\ u^4(0,x) = \lambda(x), \\
x = (x_1, ..., x_n) \in \overline{\Omega}, \\
u^1(t,x)\big|_{S_1} = u^1(t,x)\big|_{S_2},\ \frac{\partial u^1(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u^1(t,x)}{\partial \overline{m}}\Big|_{S_2},\ 0 \le t \le T, \\
u^2(t,x)\big|_{S_1} = u^2(t,x)\big|_{S_2},\ \frac{\partial u^2(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u^2(t,x)}{\partial \overline{m}}\Big|_{S_2},\ 0 \le t \le T, \\
u^3(t,x)\big|_{S_1} = u^3(t,x)\big|_{S_2},\ \frac{\partial u^3(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u^3(t,x)}{\partial \overline{m}}\Big|_{S_2},\ 0 \le t \le T, \\
u^4(t,x)\big|_{S_1} = u^4(t,x)\big|_{S_2},\ \frac{\partial u^4(t,x)}{\partial \overline{m}}\Big|_{S_1} = \frac{\partial u^4(t,x)}{\partial \overline{m}}\Big|_{S_2},\ 0 \le t \le T,
\end{cases}
\tag{2.38}
$$

for the system of multidimensional fractional patrial differential equations. Note that $a_r > a_0 > 0$ and $f_k(t,x)$, $k = 1,2,3,4$ $\left(t \in (0,T),\ x \in \overline{\Omega}\right)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\lambda(x)$, $\left(x \in \overline{\Omega}\right)$ are given smooth functions. Here $S = S_1 \cup S_2, \varnothing = S_1 \cap S_2$.

Note that the Fourier series method described in solving (2.38) can be used only in the case when (2.38) has constant coefficients

2.2 Laplace Transform Method

Now, we consider Laplace transform solution of problems for the system fractional partial differential equations.

Example 2.4. Consider the initial-boundary-value problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} = (2t - t^2 + \alpha \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta\, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2(t,x)}{\partial x^2} \\ = (2t - \beta_1 t^2 - t^2 + \beta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} \\ = (2t - \delta_1 t^2 - t^2 + \delta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) \\ - \frac{\partial^2 u^4(t,x)}{\partial x^2} = (2t - d_1 t^2 - d_2 t^2 - t^2 + d \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ 0 < t < 1,\ 0 < x < \infty, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = 0,\ 0 \le x < \infty, \\ u^1(t,0) = u^2(t,0) = u^3(t,0) = u^4(t,0) = t^2,\ 0 \le t \le 1, \\ u_x^1(t,0) = u_x^2(t,0) = u_x^3(t,0) = u_x^4(t,0) = -t^2,\ 0 \le t \le 1. \end{cases} \tag{2.39}$$

Solution. We will use Laplace transform solution of problem (2.39). Here and in future we assume that

$$\begin{cases} \mathcal{L}\left\{u^1(t,x)\right\} = u^1(t,s), \\ \\ \mathcal{L}\left\{u^2(t,x)\right\} = u^2(t,s), \\ \\ \mathcal{L}\left\{u^3(t,x)\right\} = u^3(t,s), \\ \\ \mathcal{L}\left\{u^4(t,x)\right\} = u^4(t,s). \end{cases}$$

Using formula

$$\mathcal{L}\{e^{-x}\} = \frac{1}{s+1}, \tag{2.40}$$

taking Laplace transform of both sides of the system of fractional partial differential equations (2.39) and using the following conditions
$u^1(t,0) = u^2(t,0) = u^3(t,0) = u^4(t,0) = t^2$, $u^1_x(t,0) = u^2_x(t,0) = u^3_x(t,0) = u^4_x(t,0) = -t^2$,

we get

$$\begin{cases} u_t^1(t,s) + \alpha D_t^{\frac{1}{2}} u^1(t,s) - s^2 u^1(t,s) + st^2 + \gamma_1(t) \\ = (2t - t^2 + \alpha \frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}}) \frac{1}{s+1}, \\ u_t^2(t,s) + \beta D_t^{\frac{1}{2}} u^2(t,s) - \beta_1 u^1(t,s) - s^2 u^2(t,s) + st^2 \\ + \gamma_2(t) = (2t - \beta_1 t^2 - t^2 + \beta \frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}}) \frac{1}{s+1}, \\ u_t^3(t,s) + \delta D_t^{\frac{1}{2}} u^3(t,s) - \delta_1 u^1(t,s) - s^2 u^3(t,s) + st^2 \\ + \gamma_3(t) = (2t - \delta_1 t^2 - t^2 + \delta \frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}}) \frac{1}{s+1}, \\ u_t^4(t,s) + d D_t^{\frac{1}{2}} u^4(t,s) - d_1 u^3(t,s) - d_2 u^2(t,s) - s^2 u^4(t,s) \\ + st^2 + \gamma_4(t) = (2t - d_1 t^2 - d_2 t^2 - t^2 + d \frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}}) \frac{1}{s+1}, \\ u^1(0,s) = u^2(0,s) = u^3(0,s) = u^4(0,s) = 0. \end{cases} \tag{2.41}$$

Applying the Laplace transform with respect to t of the system (2.41), we get

$$\begin{cases} \mu u^1(\mu,s)+\alpha\mu^{\frac{1}{2}}u^1(\mu,s)-s^2u^1(\mu,s)+s\frac{2}{\mu^3}-\frac{2}{\mu^3} \\ =(\frac{2}{\mu^2}-\frac{2}{\mu^3}+\alpha\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}, \\ \mu u^2(\mu,s)+\beta\mu^{\frac{1}{2}}u^2(\mu,s)-\beta_1u^1(\mu,s)-s^2u^2(\mu,s) \\ +s\frac{2}{\mu^3}-\frac{2}{\mu^3}=(\frac{2}{\mu^2}-\beta_1\frac{2}{\mu^3}-\frac{2}{\mu^3}+\beta\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}, \\ \mu u^3(\mu,s)+\delta\mu^{\frac{1}{2}}u^3(\mu,s)-\delta_1u^1(\mu,s)-s^2u^3(\mu,s) \\ +s\frac{2}{\mu^3}-\frac{2}{\mu^3}=(\frac{2}{\mu^2}-\delta_1\frac{2}{\mu^3}-\frac{2}{\mu^3}+\delta\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}, \\ \mu u^4(\mu,s)+d\mu^{\frac{1}{2}}u^4(\mu,s)-d_1u^3(\mu,s)-d_2u^2(\mu,s) \\ -s^2u^4(\mu,s)+s\frac{2}{\mu^3}-\frac{2}{\mu^3}=(\frac{2}{\mu^2}-d_1\frac{2}{\mu^3} \\ -d_2\frac{2}{\mu^3}+d\frac{2}{\mu^{\frac{5}{2}}}-\frac{2}{\mu^3})\frac{1}{(s+1)}. \end{cases} \tag{2.42}$$

For finding $u^1(\mu,s)$, $u^2(\mu,s)$, $u^3(\mu,s)$ and $u^4(\mu,s)$ we use the system (2.42). First, we obtain $u^1(\mu,s)$. We have that

$$(\mu+\alpha\mu^{\frac{1}{2}}-s^2)u^1(\mu,s)+s\frac{2}{\mu^3}-\frac{2}{\mu^3}=(\frac{2}{\mu^2}-\frac{2}{\mu^3}+\alpha\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}.$$

Therefore,

$$u^1(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}. \tag{2.43}$$

Second, we obtain $u^2(\mu,s)$. Using formula (2.43) in the second equation, we get

$$(\mu+\beta\mu^{\frac{1}{2}}-s^2)u^2(\mu,s)-\beta_1\frac{2}{\mu^3}\frac{1}{(s+1)}+s\frac{2}{\mu^3}-\frac{2}{\mu^3}$$

$$= (\frac{2}{\mu^2} - \beta_1 \frac{2}{\mu^3} - \frac{2}{\mu^3} + \beta \frac{2}{\mu^{\frac{5}{2}}}) \frac{1}{(s+1)}.$$

Therefore,

$$u^2(\mu, s) = \frac{2}{\mu^3} \frac{1}{(s+1)}. \tag{2.44}$$

Third, we obtain $u^3(\mu, s)$. Applying formula (2.43) in the third equation, we obtain

$$(\mu + \delta\mu^{\frac{1}{2}} - s^2) u^3(\mu, s) = \frac{2}{\mu^3} \frac{1}{(s+1)} ((1-s)(s+1) + \delta_1 + \mu - \delta_1 - 1 + \delta\mu^{\frac{1}{2}})$$

or

$$u^3(\mu, s) = \frac{2}{\mu^3} \frac{1}{(s+1)}. \tag{2.45}$$

Fourth, we obtain $u^4(\mu, s)$. Applying formulas (2.44) and (2.45) in the last equation, we get

$$(\mu + d\mu^{\frac{1}{2}} - s^2) u^4(\mu, s) - d_1 \frac{2}{\mu^3} \frac{1}{(s+1)} - d_2 \frac{2}{\mu^3} \frac{1}{(s+1)} + s\frac{2}{\mu^3} - \frac{2}{\mu^3}$$

$$= (\frac{2}{\mu^2} - d_1 \frac{2}{\mu^3} - d_2 \frac{2}{\mu^3} + d \frac{2}{\mu^{\frac{5}{2}}} - \frac{2}{\mu^3}) \frac{1}{(s+1)}.$$

Therefore,

$$u^4(\mu, s) = \frac{2}{\mu^3} \frac{1}{(s+1)}. \tag{2.46}$$

Finally, applying formulas (2.43), (2.44), (2.45), (2.46) and taking the inverse Laplace transforms with respest to t and x, we obtain

$$u^1(t, x) = u^2(t, x) = u^3(t, x) = u^4(t, x) = t^2 e^{-x}.$$

Example 2.5. Consider the initial-boundary-value problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} = (2t - t^2 + \alpha \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta\, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2(t,x)}{\partial x^2} \\ = (2t - \beta_1 t^2 - t^2 + \beta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} \\ = (2t - \delta_1 t^2 - t^2 + \delta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1\, u^3(t,x) - d_2\, u^2(t,x) \\ -\frac{\partial^2 u^4(t,x)}{\partial x^2} = (2t - d_1 t^2 - d_2 t^2 - t^2 + d \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}})e^{-x}, \\ 0 < t < 1,\ 0 < x < \infty, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = 0,\ 0 \le x < \infty, \\ u^1(t,0) = u^2(t,0) = u^3(t,0) = u^4(t,0) = t^2,\ 0 \le t \le 1, \\ u^1(t,\infty) = u^2(t,\infty) = u^3(t,\infty) = u^4(t,\infty) = 0,\ 0 \le t \le 1. \end{cases} \tag{2.47}$$

Solution. Applying formula (2.40) and taking the Laplace transform of both sides of the system of fractional partial differential equations in the system (2.47) and using the following conditions

$u^1(t,0) = u^2(t,0) = u^3(t,0) = u^4(t,0) = t^2$, we can write

$$\begin{cases} \mathcal{L}\left\{\frac{\partial u^1(t,x)}{\partial t}\right\} + \alpha\, \mathcal{L}\left\{D_t^{\frac{1}{2}} u^1(t,x)\right\} - \mathcal{L}\left\{\frac{\partial^2 u^1(t,x)}{\partial x^2}\right\} = \mathcal{L}\left\{(2t - t^2 + \alpha\frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}})e^{-x}\right\}, \\ \mathcal{L}\left\{\frac{\partial u^2(t,x)}{\partial t}\right\} + \beta\, \mathcal{L}\left\{D_t^{\frac{1}{2}} u^2(t,x)\right\} - \beta_1 \mathcal{L}\left\{u^1(t,x)\right\} - \mathcal{L}\left\{\frac{\partial^2 u^2(t,x)}{\partial x^2}\right\} \\ = \mathcal{L}\left\{(2t - \beta_1 t^2 - t^2 + \beta\frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}})e^{-x}\right\}, \\ \mathcal{L}\left\{\frac{\partial u^3(t,x)}{\partial t}\right\} + \delta\mathcal{L}\ \left\{D_t^{\frac{1}{2}} u^3(t,x)\right\} - \delta_1 \mathcal{L}\left\{u^1(t,x)\right\} - \mathcal{L}\left\{\frac{\partial^2 u^3(t,x)}{\partial x^2}\right\} \\ = \mathcal{L}\left\{(2t - \delta_1 t^2 - t^2 + \delta\frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}})e^{-x}\right\}, \\ \mathcal{L}\left\{\frac{\partial u^4(t,x)}{\partial t}\right\} + d\mathcal{L}\ \left\{D_t^{\frac{1}{2}} u^4(t,x)\right\} - d_1 \mathcal{L}\left\{\ u^3(t,x)\right\} - d_2\ \mathcal{L}\left\{u^2(t,x)\right\} - \mathcal{L}\left\{\frac{\partial^2 u^4(t,x)}{\partial x^2}\right\} \\ = \mathcal{L}\left\{(2t - d_1 t^2 - d_2 t^2 - t^2 + d\frac{8t^{\frac{3}{3}}}{3\sqrt{\pi}})e^{-x}\right\}, \\ 0 < t < 1, \\ \mathcal{L}\left\{u^1(0,x)\right\} = \mathcal{L}\left\{u^2(0,x)\right\} = \mathcal{L}\left\{u^3(0,x)\right\} = \mathcal{L}\left\{u^4(0,x)\right\} = \mathcal{L}\left\{0\right\}. \end{cases}$$

Here, we assume that

$$\begin{cases} \gamma_1(t) = u_x^1(t,0), \\ \gamma_2(t) = u_x^2(t,0), \\ \gamma_3(t) = u_x^3(t,0), \\ \gamma_4(t) = u_x^4(t,0). \end{cases}$$

Now, we using the Laplace transform with respect to t, we get

$$\begin{cases} (\mu+\alpha\mu^{\frac{1}{2}}-s^2)u^1(\mu,s)=(\frac{2}{\mu^2}-\frac{2}{\mu^3}+\alpha\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)} \\ -s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_1(\mu), \\ (\mu+\beta\mu^{\frac{1}{2}}-s^2)u^2(\mu,s)-\beta_1u^1(\mu,s)=(\frac{2}{\mu^2}-\beta_1\frac{2}{\mu^3} \\ -\frac{2}{\mu^3}+\beta\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_2(\mu), \\ (\mu+\delta\mu^{\frac{1}{2}}-s^2)u^3(\mu,s)-\delta_1u^1(\mu,s)=(\frac{2}{\mu^2}-\delta_1\frac{2}{\mu^3} \\ -\frac{2}{\mu^3}+\delta\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_3(\mu), \\ (\mu+d\mu^{\frac{1}{2}}-s^2)u^4(\mu,s)-d_1u^3(\mu,s)-d_2u^2(\mu,s) \\ =(\frac{2}{\mu^2}-d_1\frac{2}{\mu^3}-d_2\frac{2}{\mu^3}+d\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3} \\ -\gamma_4(\mu). \end{cases} \tag{2.48}$$

Taking the Laplace transform for the following conditions

$$u^1(t,\infty)=u^2(t,\infty)=u^3(t,\infty)=u^4(t,\infty)=0,$$

we get

$$u^1(\mu,\infty)=0,\ u^2(\mu,\infty)=0,\ u^3(\mu,\infty)=0,\ u^4(\mu,\infty)=0. \tag{2.49}$$

For finding $u^1(\mu,s)$, $u^2(\mu,s)$, $u^3(\mu,s)$ and $u^4(\mu,s)$, we use the system (2.48). First, we obtain $u^1(\mu,s)$. We have that

$$(\mu+\alpha\mu^{\frac{1}{2}}-s^2)u^1(\mu,s)=(\frac{2}{\mu^2}-\frac{2}{\mu^3}+\alpha\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_1(\mu).$$

Then

$$u^1(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}-\gamma_1(\mu)\frac{1}{(\mu+\alpha\mu^{\frac{1}{2}}-s^2)}.$$

Using the formula

$$\frac{1}{\mu+\alpha\mu^{\frac{1}{2}}-s^2}=\left(\frac{1}{\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}-s}+\frac{1}{\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}+s}\right)\frac{1}{2\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}},$$

we obtain

$$u^1(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}-\gamma_1(\mu)\frac{1}{2\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}}\left(\frac{1}{s+\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}}+\frac{1}{s-\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}}\right). \quad (2.50)$$

Taking the inverse Laplace transform with respect to x, we get

$$u^1(\mu,x)=\frac{2}{\mu^3}e^{-x}+\gamma_1(\mu)\frac{1}{2\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}}\left(e^{-\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}x}-e^{\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}x}\right). \quad (2.51)$$

Passing to limit in (2.51) when $x\to\infty$ and using (2.49), we get

$$u^1(\mu,\infty)=\gamma_1(\mu)\frac{1}{2\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}}\lim_{x\to\infty}e^{\sqrt{\mu+\alpha\mu^{\frac{1}{2}}}x}=0.$$

From that it follows

$$\gamma_1(\mu)=0. \quad (2.52)$$

Appling (2.50), (2.51) and (2.52), we get

$$\left(\mu+\alpha\mu^{\frac{1}{2}}-s^2\right)u^1(\mu,s)=\frac{2}{\mu^3}\frac{\mu+\alpha\mu^{\frac{1}{2}}-s^2}{(s+1)},$$

then

$$u^1(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}. \quad (2.53)$$

Second, we obtain $u^2(\mu,s)$. Using formula (2.53) in the second equation, we obtain

$$(\mu+\beta\mu^{\frac{1}{2}}-s^2)u^2(\mu,s)-\beta_1u^1(\mu,s)=(\frac{2}{\mu^2}-\beta_1\frac{2}{\mu^3}-\frac{2}{\mu^3}+\beta\frac{2}{\mu^{\frac{5}{2}}})\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_2(\mu).$$

Then

$$(\mu+\beta\mu^{\frac{1}{2}}-s^2)u^2(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}(\mu-\beta_1-1+\beta\mu^{\frac{1}{2}}+(1-s)(s+1)+\beta_1)-\gamma_2(\mu)$$

or

$$u^2(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)} - \gamma_2(\mu)\frac{1}{(\mu + \beta\mu^{\frac{1}{2}} - s^2)}.$$

Using the formula

$$\frac{1}{(\mu + \beta\mu^{\frac{1}{2}} - s^2)} = \left(\frac{1}{\sqrt{\mu + \beta\mu^{\frac{1}{2}}} - s} + \frac{1}{\sqrt{\mu + \beta\mu^{\frac{1}{2}}} + s}\right)\frac{1}{2\sqrt{\mu + \beta\mu^{\frac{1}{2}}}},$$

we get

$$u^2(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)} - \gamma_2(\mu)\frac{1}{2\sqrt{\mu + \beta\mu^{\frac{1}{2}}}}\left(\frac{1}{s + \sqrt{\mu + \beta\mu^{\frac{1}{2}}}} + \frac{1}{s - \sqrt{\mu + \beta\mu^{\frac{1}{2}}}}\right). \quad (2.54)$$

Taking the inverse Laplace transform with respect to x, we get

$$u^2(\mu, x) = \frac{2}{\mu^3}e^{-x} + \gamma_2(\mu)\frac{1}{2\sqrt{\mu + \beta\mu^{\frac{1}{2}}}}\left(e^{-\sqrt{\mu+\beta\mu^{\frac{1}{2}}}\,x} - e^{\sqrt{\mu+\beta\mu^{\frac{1}{2}}}\,x}\right). \quad (2.55)$$

Passing the limit in formula (2.55) when $x \to \infty$ and using formula (2.49), we get

$$u^2(\mu, \infty) = \gamma_2(\mu)\frac{1}{2\sqrt{\mu + \beta\mu^{\frac{1}{2}}}}\lim_{x\to\infty} e^{\sqrt{\mu+\beta\mu^{\frac{1}{2}}}\,x} = 0.$$

From that it follows

$$\gamma_2(\mu) = 0. \quad (2.56)$$

Applying the formulas (2.54), (2.55) and (2.56), we get

$$(\mu + \beta\mu^{\frac{1}{2}} - s^2)u^2(\mu, s) = \frac{2}{\mu^3}\frac{(\mu + \beta\mu^{\frac{1}{2}} - s^2)}{(s+1)}.$$

Then

$$u^2(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)}. \quad (2.57)$$

Similarly, we obtain $u^3(\mu, s)$. Applying formula (2.53) in the third equation of the system (2.48) and making some elimination, we have

$$u^3(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}-\gamma_3(\mu)\frac{1}{(\mu+\delta\mu^{\frac{1}{2}}-s^2)}.$$

Applying the formula

$$\frac{1}{(\mu+\delta\mu^{\frac{1}{2}}-s^2)}=\left(\frac{1}{\sqrt{\mu+\delta\mu^{\frac{1}{2}}}-s}+\frac{1}{\sqrt{\mu+\delta\mu^{\frac{1}{2}}}+s}\right)\frac{1}{2\sqrt{\mu+\delta\mu^{\frac{1}{2}}}},$$

we get

$$u^3(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}-\gamma_3(\mu)\frac{1}{2\sqrt{\mu+\delta\mu^{\frac{1}{2}}}}\left(\frac{1}{s+\sqrt{\mu+\delta\mu^{\frac{1}{2}}}}+\frac{1}{s-\sqrt{\mu+\delta\mu^{\frac{1}{2}}}}\right). \quad (2.58)$$

Taking the inverse Laplace transform with respect to x, we get

$$u^3(\mu,x)=\frac{2}{\mu^3}e^{-x}+\gamma_3(\mu)\frac{1}{2\sqrt{\mu+\delta\mu^{\frac{1}{2}}}}\left(e^{-\sqrt{\mu+\delta\mu^{\frac{1}{2}}}x}-e^{\sqrt{\mu+\delta\mu^{\frac{1}{2}}}x}\right). \quad (2.59)$$

Passing the limit in formula (2.59) when $x\to\infty$ and using formula (2.49), we obtain

$$u^3(\mu,\infty)=\gamma_3(\mu)\frac{1}{2\sqrt{\mu+\delta\mu^{\frac{1}{2}}}}\lim_{x\to\infty}e^{\sqrt{\mu+\delta\mu^{\frac{1}{2}}}x}=0.$$

From that it follows

$$\gamma_3(\mu)=0. \quad (2.60)$$

Applying formulas (2.58), (2.59) and (2.60), we get

$$(\mu+\delta\mu^{\frac{1}{2}}-s^2)u^3(\mu,s)=\frac{2}{\mu^3}\frac{(\mu+\delta\mu^{\frac{1}{2}}-s^2)}{(s+1)}.$$

Then

$$u^3(\mu,s)=\frac{2}{\mu^3}\frac{1}{(s+1)}. \quad (2.61)$$

Applying formula (2.57) and (2.61) in the fourth equation of the system (2.48) and making some computation, we have

$$(\mu+d\mu^{\frac{1}{2}}-s^2)u^4(\mu,s)-d_1u^3(\mu,s)-d_2u^2(\mu,s)=\left(\frac{2}{\mu^2}-d_1\frac{2}{\mu^3}-d_2\frac{2}{\mu^3}+d\frac{2}{\mu^{\frac{5}{2}}}\right)\frac{1}{(s+1)}-s\frac{2}{\mu^3}+\frac{2}{\mu^3}-\gamma_4(\mu).$$

Then

$$(\mu + d\mu^{\frac{1}{2}} - s^2)u^4(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)}(\mu + d\mu^{\frac{1}{2}} - s^2) - \gamma_4(\mu)$$

or

$$u^4(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)} - \gamma_4(\mu)\frac{1}{(\mu + d\mu^{\frac{1}{2}} - s^2)}.$$

Applying the formula

$$\frac{1}{(\mu + d\mu^{\frac{1}{2}} - s^2)} = \left(\frac{1}{\sqrt{\mu + d\mu^{\frac{1}{2}}} - s} + \frac{1}{\sqrt{\mu + d\mu^{\frac{1}{2}}} + s}\right)\frac{1}{2\sqrt{\mu + d\mu^{\frac{1}{2}}}},$$

we get

$$u^4(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)}$$
$$-(\gamma_4(\mu))\frac{1}{2\sqrt{\mu + d\mu^{\frac{1}{2}}}}\left(\frac{1}{s + \sqrt{\mu + d\mu^{\frac{1}{2}}}} + \frac{1}{s - \sqrt{\mu + d\mu^{\frac{1}{2}}}}\right). \quad (2.62)$$

Taking the inverse Laplace transform with respect to x, we get

$$u^4(\mu, x) = \frac{2}{\mu^3}e^{-x} + \gamma_4(\mu)\frac{1}{2\sqrt{\mu + d\mu^{\frac{1}{2}}}}\left(e^{-\sqrt{\mu + d\mu^{\frac{1}{2}}}x} - e^{\sqrt{\mu + d\mu^{\frac{1}{2}}}x}\right). \quad (2.63)$$

Passing the limit in formula (2.63) when $x \to \infty$ and using formula (2.49), we get

$$u^4(\mu, \infty) = \gamma_4(\mu)\frac{1}{2\sqrt{\mu + d\mu^{\frac{1}{2}}}}\lim_{x\to\infty} e^{\sqrt{\mu + d\mu^{\frac{1}{2}}}x} = 0.$$

From that it follows

$$\gamma_4(\mu) = 0. \quad (2.64)$$

Applying formulas (2.62), (2.63) and (2.64), we get

$$(\mu + d\mu^{\frac{1}{2}} - s^2)u^4(\mu, s) = \frac{2}{\mu^3}\frac{(\mu + d\mu^{\frac{1}{2}} - s^2)}{(s+1)}.$$

Then

$$u^4(\mu, s) = \frac{2}{\mu^3}\frac{1}{(s+1)}. \quad (2.65)$$

Finally, applying formulas (2.53), (2.57), (2.61), (2.65) and using the inverse Laplace transform with respect to t and x, we get

$$u^1(t,x) = u^2(t,x) = u^3(t,x) = u^4(t,x) = t^2 e^{-x^2}.$$

Using similar procedure we can get the solution of the following initial boundary value problem

$$
\begin{cases}
\frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^1(t,x)}{\partial x_r^2} = f_1(t,x), \\
\frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} \ u^2(t,x) - \beta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^2(t,x)}{\partial x_r^2} = (f_2(t,x), \\
\frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} \ u^3(t,x) - \delta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^3(t,x)}{\partial x_r^2} = f_3(t,x), \\
\frac{\partial u^4(t,x)}{\partial t} + d \ D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^4(t,x)}{\partial x_r^2} \\
= f_4(t,x), \\
\quad x = (x_1, ..., x_n) \in \overline{\Omega}^+, \ 0 < t < T, \\
u^1(0,x) = \varphi(x), \ u^2(0,x) = \psi(x), \ u^3(0,x) = \xi(x), \ u^4(0,x) = \lambda(x), \\
\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, \\
u^1(t,x) = \alpha_1(t,x), \quad u^1_{x_r}(t,x) = \beta_1(t,x), \\
u^2(t,x) = \alpha_2(t,x), \quad u^2_{x_r}(t,x) = \beta_2(t,x), \\
u^3(t,x) = \alpha_3(t,x), \quad u^3_{x_r}(t,x) = \beta_3(t,x), \\
u^4(t,x) = \alpha_4(t,x), \quad u^4_{x_r}(t,x) = \beta_4(t,x), \\
1 \le r \le n, \ 0 \le t \le T, \ x \in S^+
\end{cases}
\tag{2.66}
$$

for the system of multidimensional fractional partial differential equations. Note that $a_r > a_0 > 0$ and $f_k(t,x), \ k = 1,2,3,4 \left(t \in (0,T), x \in \overline{\Omega}^+\right), \varphi(x), \psi(x),$

$\xi(x)$, $\lambda(x)$, $\left(x \in \overline{\Omega}^{+}\right)$, $\alpha_k(t,x)$, $\beta_k(t,x)$, $k = 1,2,3,4$ $(t \in [0,T], x \in s^{+})$ are given smooth functions. Here and in future Ω^{+} is the open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty,\ 1 \leq k \leq n)$ with the boundary S^{+} and

$$\overline{\Omega}^{+} = \Omega^{+} \cup S^{+}.$$

Note that Laplace transform method described in solving (2.66) can be used only in the case when (2.66) has $a_r(x)$ constant or polynomials coefficients.

2.3 Fourier Transform Method

Now, we obtain the Fourier transform solution of the initial-value problem for the system of fractional partial differential equations.

Example 2.6. Consider the initial-value problem for the system of fractional partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} \\ = (2t + \alpha \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + 2t^2 - 4x^2t^2\ )e^{-x^2}, \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}}\ u^2(t,x) - \beta_1 u^1(t,x) - \frac{\partial^2 u^2(t,x)}{\partial x^2} \\ = (2t + \beta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \beta_1 t^2 + 2t^2 - 4x^2t^2)\ e^{-x^2}, \\ \frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \frac{\partial^2 u^3(t,x)}{\partial x^2} \\ = (2t + \delta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \delta_1 t^2 + 2t^2 - 4x^2t^2)e^{-x^2}, \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1\ u^3(t,x) - d_2\ u^2(t,x) - \frac{\partial^2 u^4(t,x)}{\partial x^2} \\ = (2t + d \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + 2t^2 - 4x^2t^2)e^{-x^2}, \\ 0 < t < 1,\ \ -\infty < x < \infty, \\ u^1(0,x) = u^2(0,x) = u^3(0,x) = u^4(0,x) = e^{-x^2},\ \ -\infty < x < \infty. \end{cases} \tag{2.67}$$

Solution. Here we assume

$$F\{u(t,x)\} = u(t,s),$$
$$F\left\{e^{-x^2}\right\} = n(s).$$

We have that

$$F\left\{\frac{\partial}{\partial t} u(t,x)\right\} = u_t(t,s),$$

$$F\left\{\frac{\partial^2}{\partial x^2}u(t,x)\right\} = -s^2 u(t,s).$$

Taking the Fourier transform of both sides of the system of fractional partial differential equation (2.67), we have the following system of ordinary differential equations

$$\begin{cases} u_t^1(t,s) + \alpha D_t^{\frac{1}{2}} u^1(t,s) - s^2 u^1(t,s) \\ = (2t + \alpha \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + 2t^2 - 4s^2t^2\ )n(s), \\ u_t^2(t,s) + \beta D_t^{\frac{1}{2}} u^2(t,s) - \beta_1 u^1(t,s) - s^2 u^2(t,s) \\ = (2t + \beta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \beta_1 t^2 + 2t^2 - 4s^2t^2) n(s), \\ u_t^3(t,s) + \delta D_t^{\frac{1}{2}} u^3(t,s) - \delta_1 u^1(t,s) - s^2 u^3(t,s) \\ = (2t + \delta \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \delta_1 t^2 + 2t^2 - 4s^2t^2) n(s), \\ u_t^4(t,s) + d D_t^{\frac{1}{2}} u^4(t,s) - d_1 u^3(t,s) - d_2 u^2(t,s) - s^2 u^4(t,s) \\ = (2t + d \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - d_1 t^2 - d_2 t^2 + 2t^2 - 4s^2t^2) n(s), \\ 0 < t < 1, \\ u^1(0,s) = u^2(0,s) = u^3(0,s) = u^4(0,s) = n(s). \end{cases} \tag{2.68}$$

Taking Laplace transform with respect to t in the system (2.68), we obtain the following system of algebraic equations

$$
\begin{cases}
\mu u^1(\mu,s)+\alpha\mu^{\frac{1}{2}}u^1(\mu,s)+s^2u^1(\mu,s)\\
=(\frac{2}{\mu^2}+\alpha\frac{2}{\mu^{\frac{5}{2}}}-\frac{2}{\mu^3}s^2)n(s),\\
\mu u^2(\mu,s)+\beta\mu^{\frac{1}{2}}u^2(\mu,s)-\beta_1u^1(\mu,s)-s^2u^2(\mu,s)\\
=(\frac{2}{\mu^2}+\beta\frac{2}{\mu^{\frac{5}{2}}}-\beta_1\frac{2}{\mu^3}-\frac{2s^2}{\mu^3})n(s),\\
\mu u^3(\mu,s)+\delta\mu^{\frac{1}{2}}u^3(\mu,s)-\delta_1u^1(\mu,s)-s^2u^3(\mu,s)\\
=(\frac{2}{\mu^2}+\delta\frac{2}{\mu^{\frac{5}{2}}}-\delta_1\frac{2}{\mu^3}-\frac{2s^2}{\mu^3})n(s),\\
\mu u^4(\mu,s)+d\mu^{\frac{1}{2}}u^4(\mu,s)-d_1u^3(\mu,s)-d_2u^2(\mu,s)\\
-s^2u^4(\mu,s)=(\frac{2}{\mu^2}+d\frac{2}{\mu^{\frac{5}{2}}}-d_1\frac{2}{\mu^3}-d_2\frac{2}{\mu^3}-\frac{2s^2}{\mu^3})n(s).
\end{cases}
\tag{2.69}
$$

For finding $u^1(\mu,s)$, $u^2(\mu,s)$, $u^3(\mu,s)$ and $u^4(\mu,s)$, we use the system (2.69).

Using the first equation of the system (2.69), we get

$$u^1(\mu,s)=\frac{2}{\mu^3}n(s). \tag{2.70}$$

Applying inverse Laplace transform with respect to t and Fourier transform with respect to x, we get

$$u^1(t,x)=t^2F^{-1}\left\{F\left\{e^{-x^2}\right\}\right\}=t^2e^{-x^2}. \tag{2.71}$$

Putting formula (2.70) in the second equation of the system (2.69), we obtain

$$\left(\mu+\beta\mu^{\frac{1}{2}}-s^2\right)u^2(\mu,s)-\beta_1\frac{2}{\mu^3}n(s)=\frac{2}{\mu^3}\left(\mu+\beta\mu^{\frac{1}{2}}-\beta_1-s^2\right)n(s).$$

Then

$$u^2(\mu, s) = \frac{2}{\mu^3} n(s). \tag{2.72}$$

Applying inverse Laplace transform with respect to t and Fourier transform with respect to x, we get

$$u^2(t, x) = t^2 F^{-1}\left\{F\left\{e^{-x^2}\right\}\right\} = t^2 e^{-x^2}. \tag{2.73}$$

Similarly, putting formula (2.70) in the third equation of the system (2.69), we obtain

$$\left(\mu + \delta\mu^{\frac{1}{2}} - s^2\right) u^3(\mu, s) - \delta_1 \frac{2}{\mu^3} n(s) = \frac{2}{\mu^3}\left(\mu + \delta\mu^{\frac{1}{2}} - s^2 - \delta_1\right) n(s).$$

Then

$$u^3(\mu, s) = \frac{2}{\mu^3} n(s). \tag{2.74}$$

Applying inverse Laplace transform with respect to t and Fourier transform with respect to x, we get

$$u^3(t, x) = t^2 F^{-1}\left\{F\left\{e^{-x^2}\right\}\right\} = t^2 e^{-x^2}. \tag{2.75}$$

Putting formula (2.72) and (2.74) in the fourth equation of the system (2.69), we get

$$(\mu + d\mu^{\frac{1}{2}} - s^2) u^4(\mu, s) - d_1 \frac{2}{\mu^3} n(s) - d_2 \frac{2}{\mu^3} n(s) = \frac{2}{\mu^3}(\mu + d\mu^{\frac{1}{2}} - d_1 - d_2 - s^2) n(s).$$

Then

$$u^4(\mu, s) = \frac{2}{\mu^3} n(s). \tag{2.76}$$

Applying inverse Laplace transform with respect to t and Fourier transform with respect to x, we get

$$u^4(t, x) = t^2 F^{-1}\left\{F\left\{e^{-x^2}\right\}\right\} = t^2 e^{-x^2}. \tag{2.77}$$

Finally, applying formulas (2.71), (2.73), (2.75) and (2.77), we obtain

$$u^1(t, x) = u^2(t, x) = u^3(t, x) = u^4(t, x) = t^2 e^{-x^2}.$$

Using similar procedure we can get the solution of the following initial-value problem

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha D_t^{\frac{1}{2}} u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^1(t,x)}{\partial x_r^2} = f_1(t,x), \\ \frac{\partial u^2(t,x)}{\partial t} + \beta D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^2(t,x)}{\partial x_r^2} = f_2(t,x), \\ \frac{\partial u^3(t,x)}{\partial t} + \delta D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^3(t,x)}{\partial x_r^2} = f_3(t,x), \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1 u^3(t,x) - d_2 u^2(t,x) - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u^4(t,x)}{\partial x_r^2} \\ = f_4(t,x), \\ x = (x_1, ..., x_n) \in \mathbb{R}^n,\ 0 < t < T, \\ u^1(0,x) = \varphi(x),\ u^2(0,x) = \psi(x),\ u^3(0,x) = \xi(x),\ u^4(0,x) = \lambda(x), \\ x = (x_1, ..., x_n) \in \mathbb{R}^n \end{cases} \tag{2.78}$$

for the system of multidimensional fractional partial differential equations. Note that $a_r > a_0 > 0$ and $f_k(t,x),\ k = 1,2,3,4\ (t \in (0,T), x \in \mathbb{R}^n), \varphi(x), \psi(x)$, $\xi(x), \lambda(x)$ $(x \in \mathbb{R}^n)$ are given smooth functions. Note that Fourier transform method described in solving (2.67) can be used only in the case when (2.67) has constant coefficients.

So, all analytical methods described above, namely the Fourier series method, Laplace transform method and the Fourier transform method can be used only in the case when the system of differential equations has constant coefficients or polynomial coefficients. It is well-known that the most general method for solving the system of fractional partial differential equations with dependent in t and in the space variables are finite difference method.

In the last section, we consider the initial-value problem for the system of one-dimensional fractional partial differential equations. The first and second-order of accuracy difference schemes for the numerical solution of this problem are presented. Numerical analysis and discussions are presented.

CHAPTER 3

FINITE DIFFERENCE METHOD OF THE SOLUTION OF SYSTEM OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

When the analytical methods do not work correctly, we can use the numerical method to get approximate solutions of the local and non-local problems for the system of fractional partial differential equations take an important role in applied mathematics. In this section, we present the numerical solution of the initial-boundary-value problem

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} + \alpha\, D_t^{\frac{1}{2}} u^1(t,x) - \frac{\partial^2 u^1(t,x)}{\partial x^2} \\ = (2t + t^2 + \mu \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}) \sin x, \\ \\ \frac{\partial u^2(t,x)}{\partial t} + \beta\, D_t^{\frac{1}{2}} u^2(t,x) - \beta_1 u^1(t,x) \\ - \frac{\partial^2 u^2(t,x)}{\partial x^2} = (2t + \beta \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - \beta_1 t^2 + t^2) \sin x, \\ \\ \frac{\partial u^3(t,x)}{\partial t} + \delta\, D_t^{\frac{1}{2}} u^3(t,x) - \delta_1 u^1(t,x) \\ - \frac{\partial^2 u^3(t,x)}{\partial x^2} = (2t + (1-\delta_1) t^2 + \delta \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}) \sin x, \\ \\ \frac{\partial u^4(t,x)}{\partial t} + d\, D_t^{\frac{1}{2}} u^4(t,x) - d_1\, u^3(t,x) - \\ d_2\, u^2(t,x) - \frac{\partial^2 u^4(t,x)}{\partial x^2} \\ = (2t + d \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} + (1 - d_1 - d_2)\, t^2) \sin x_n, \\ \\ 0 < t < 1, 0 < x < \pi, \\ u^m(0,x) = 0, 0 \le x \le \pi; u^m(t,0) = u^m(t,\pi) = 0, \\ 0 \le t \le 1,\ m = 1,2,3,4 \end{cases} \tag{3.1}$$

for the system of one dimensional partial differential equations. The exact solution of problem (3.1) is $u^m(t,x) = t^2 \sin x,\ m = 1,2,3,4$. For the approximate solutions of the problem (3.1), we consider the first order of accuracy difference scheme in t

$$\begin{cases} \frac{(u^1)_n^k-(u^1)_n^{k-1}}{\tau} + \alpha D_{1,\tau}^{\frac{1}{2}} \left(u^1\right)_n^k - \frac{(u^1)_{n+1}^k-2(u^1)_n^k+(u^1)_{n-1}^k}{h^2} \\ = \left(2t_k + t_k^2 + \mu\frac{8}{3\sqrt{\pi}}t_k^{\frac{3}{2}}\right)\sin x_n, \\ \frac{(u^2)_n^k-(u^2)_n^{k-1}}{\tau} + \beta D_{1,\tau}^{\frac{1}{2}} \left(u^2\right)_n^k - \beta_1 \left(u^1\right)_n^k - \frac{(u^2)_{n+1}^k-2(u^2)_n^k+(u^2)_{n-1}^k}{h^2} \\ = \left(2t_k + \beta\frac{8}{3\sqrt{\pi}}t_k^{\frac{3}{2}} - \beta_1 t_k^2 + t_k^2\right)\sin x_n, \\ \frac{(u^3)_n^k-(u^3)_n^{k-1}}{\tau} + \delta D_{1,\tau}^{\frac{1}{2}} \left(u^3\right)_n^k - \delta_1 \left(u^1\right)_n^k - \frac{(u^3)_{n+1}^k-2(u^3)_n^k+(u^3)_{n-1}^k}{h^2} \\ = \left(2t_k + (1-\delta_1)t_k^2 + \delta\frac{8}{3\sqrt{\pi}}t_k^{\frac{3}{2}}\right)\sin x_n, \\ \frac{(u^4)_n^k-(u^4)_n^{k-1}}{\tau} + d D_{1,\tau}^{\frac{1}{2}} \left(u^4\right)_n^k - d_1 \left(u^3\right)_n^k - d_2 \left(u^2\right)_n^k \\ - \frac{(u^4)_{n+1}^k-2(u^4)_n^k+(u^4)_{n-1}^k}{h^2} \\ = \left(2t_k + d\frac{8}{3\sqrt{\pi}}t_k^{\frac{3}{2}} + (1-d_1-d_2)\,t_k^2\right)\sin x_n, \\ t_k = k\tau,\ x_n = nh,\ 1 \le k \le N,\ 1 \le n \le M-1,\ N\tau = 1,\ Mh = \pi, \\ (u^m)_n^0 = 0,\ 0 \le n \le M; (u^m)_0^k = (u^m)_M^k = 0,\ 0 \le k \le N, \\ m = 1,2,3,4 \end{cases} \tag{3.2}$$

and the second order of accuracy difference scheme in t

$$\begin{cases}
\frac{(u^1)_n^k-(u^1)_n^{k-1}}{\tau}+\alpha D_\tau^{\frac{1}{2}}\left(u^1\right)_n^k- \\
\frac{1}{2}\left[\frac{(u^1)_{n+1}^k-2(u^1)_n^k+(u^1)_{n-1}^k}{h^2}+\frac{(u^1)_{n+1}^{k-1}-2(u^1)_n^{k-1}+(u^1)_{n-1}^{k-1}}{h^2}\right] \\
=\left(2(t_k-\frac{\tau}{2})+(t_k-\frac{\tau}{2})^2+\alpha\frac{8}{3\sqrt{\pi}}(t_k-\frac{\tau}{2})^{\frac{3}{2}}\right)\sin x_n, \\
\\
\frac{(u^2)_n^k-(u^2)_n^{k-1}}{\tau}+\beta D_\tau^{\frac{1}{2}}\left(u^2\right)_n^k-\beta_1\frac{(u^1)_n^k+(u^1)_n^{k-1}}{2} \\
-\frac{1}{2}\left[\frac{(u^2)_{n+1}^k-2(u^2)_n^k+(u^2)_{n-1}^k}{h^2}+\frac{(u^2)_{n+1}^{k-1}-2(u^2)_n^{k-1}+(u^2)_{n-1}^{k-1}}{h^2}\right] \\
=\left(2(t_k-\frac{\tau}{2})+\beta\frac{8}{3\sqrt{\pi}}(t_k-\frac{\tau}{2})^{\frac{3}{2}}+(1-\beta_1)(t_k-\frac{\tau}{2})^2\right)\sin x_n, \\
\\
\frac{(u^3)_n^k-(u^3)_n^{k-1}}{\tau}+\delta D_\tau^{\frac{1}{2}}\left(u^3\right)_n^k-\delta_1\frac{(u^1)_n^k+(u^1)_n^{k-1}}{2} \\
-\frac{1}{2}\left[\frac{(u^3)_{n+1}^k-2(u^3)_n^k+(u^3)_{n-1}^k}{h^2}+\frac{(u^3)_{n+1}^{k-1}-2(u^3)_n^{k-1}+(u^3)_{n-1}^{k-1}}{h^2}\right] \\
=\left(2(t_k-\frac{\tau}{2})+(1-\delta_1)(t_k-\frac{\tau}{2})^2+\delta\frac{8}{3\sqrt{\pi}}(t_k-\frac{\tau}{2})^{\frac{3}{2}}\right)\sin x_n, \\
\\
\frac{(u^4)_n^k-(u^4)_n^{k-1}}{\tau}+dD_\tau^{\frac{1}{2}}\left(u^4\right)_n^k-d_1\frac{(u^3)_n^k+(u^3)_n^{k-1}}{2}-d_2\frac{(u^2)_n^k+(u^2)_n^{k-1}}{2} \\
-\frac{1}{2}\left[\frac{(u^4)_{n+1}^k-2(u^4)_n^k+(u^4)_{n-1}^k}{h^2}+\frac{(u^4)_{n+1}^{k-1}-2(u^4)_n^{k-1}+(u^4)_{n-1}^{k-1}}{h^2}\right] \\
=\left(2(t_k-\frac{\tau}{2})+d\frac{8}{3\sqrt{\pi}}(t_k-\frac{\tau}{2})^{\frac{3}{2}}+(1-d_1-d_2)\,(t_k-\frac{\tau}{2})^2\right)\sin x_n, \\
\\
t_k=k\tau,\ x_n=nh,\ 1\le k\le N,\ 1\le n\le M-1,\ N\tau=1,\ Mh=\pi, \\
(u^m)_n^0=0,\ 0\le n\le M;\ (u^m)_0^k=(u^m)_M^k=0,\ 0\le k\le N, \\
m=1,2,3,4.
\end{cases} \tag{3.3}$$

Here

$$D_{1,\tau}^{\frac{1}{2}}u^k=\frac{1}{\tau^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\sum_{m=1}^{k-1}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}(u^m-u^{m-1}),\ \Gamma(p)=\int_0^\infty t^{p-1}e^{-t}\,dt,$$

$$D_\tau^{\frac{1}{2}} u^k = \begin{cases} d\,[\frac{(6-\frac{1}{2})}{(1-\frac{1}{2})(2-\frac{1}{2})\,2^{(2-\frac{1}{2})}}\,u^0 + \frac{(10-2)}{(1-\frac{1}{2})(2-\frac{1}{2})\,2^{(2-\frac{1}{2})}}\,u^1 + \frac{(3-\frac{1}{2})}{(1-\frac{1}{2})(2-\frac{1}{2})\,2^{(2-\frac{1}{2})}}\,u^2],\ k=1, \\ d\,[(\frac{3}{2})^{\frac{1}{2}}\,\frac{(15-3)}{(2-\frac{1}{2})}\,u^0 + (\frac{3}{2})^{\frac{1}{2}}\,\frac{2(11-2)}{(2-\frac{1}{2})}\,u^1 + (\frac{3}{2})^{\frac{1}{2}}\,\frac{(7-1)}{2-\frac{1}{2}}\,u^2],\ k=2, \\ d\,[\frac{2}{2^{\frac{3}{2}}} + 2(-(k-1)(k-\frac{1}{2})^{\frac{1}{2}} + \frac{2}{3}(k-\frac{1}{2})^{\frac{3}{2}})\,u_n^k + (-4)(-(k-1)(k-\frac{1}{2})^{\frac{1}{2}} \\ +\frac{2}{3}(k-\frac{1}{2})^{\frac{3}{2}})\,u_n^{k-1} + \frac{(-2)}{2^{\frac{3}{2}}} + 2(-(k-1)(k-\frac{1}{2})^{\frac{1}{2}} + \frac{2}{3}(k-\frac{1}{2})^{\frac{3}{2}})\,u_n^{k-2}],\ k=3,4, \\ d\,[\sum_{m=1}^{k-4}\left\{\left[b_1\,(k-m) - \frac{(k-m+\frac{1}{2})^{\frac{1}{2}}}{\frac{1}{2}} + \frac{4}{3}\,b_2\,(k-m)\right] u_n^{m+1} + [-(k-m+\frac{1}{2})^{\frac{1}{2}}\right. \\ +\frac{4}{3}b_2\,(k-m)]\,u_n^m + [-b_1(k-m) - 2(k-m+\frac{1}{2})^{\frac{1}{2}} + \frac{4}{3}b_2(k-m)]\,u_n^{m-1} \\ +[-(4-\frac{1}{2})^{\frac{1}{2}} + 4(4-\frac{1}{2})^{\frac{1}{2}} + \frac{4}{3}(4-\frac{1}{2})^{\frac{3}{2}}]\,u_n^{k-2} - 2[4(4-\frac{1}{2})^{\frac{1}{2}} + \frac{4}{3}(4-\frac{1}{2})^{\frac{1}{2}} \\ \left. +\frac{4}{3}(4-\frac{1}{2})^{\frac{3}{2}}]\,u_n^{k-1} + [(4-\frac{1}{2})^{\frac{1}{2}} + 4(4-\frac{1}{2})^{\frac{1}{2}} + \frac{4}{3}(4-\frac{1}{2})^{\frac{3}{2}}]\,u_n^k\right\}\Big],\ k \geq 5. \end{cases}$$

$$d = \frac{2}{\sqrt{\pi}\,\sqrt{\tau}},\ d_1 = \frac{\sqrt{2}}{6\,\sqrt{\pi}\,\sqrt{\tau}},\ b_1\,(r) = b(r) - d(r),\ b_2\,(r) = -\frac{1}{3}\left(b^3(r) - d^3(r)\right),$$
$$b(r) = \sqrt{r+1/2}, d(r) = \sqrt{r-1/2}.$$

For any $m = 1,2,3,4$, we can write (3.2) and (3.3) in the matrix form

$$\begin{cases} A^m u_{n+1}^m + B^m u_n^m + C^m u_{n-1}^m = D\varphi_n^m,\ 1 \leq n \leq M-1, \\ u_0^m = \overrightarrow{0},\ \ u_M^m = \overrightarrow{0}, \end{cases} \tag{3.4}$$

where A^m, B^m, C^m are $(N+1)\times(N+1)$ matrices and $D = I_{N+1}$ is the identity matrix φ_n^m and u_n^m are $(N+1)\times 1$ column vectors. Therefore, for the solution of the matrix equation (3.4), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n^m = \alpha_{n+1}^m u_{n+1}^m + \beta_{n+1}^m, \quad n = M-1, ..., 1, \tag{3.5}$$

where $u_M^m = \overrightarrow{0}$, α_j^m $(j = 1, ..., M-1)$are $(N+1)\times(N+1)$ square matrices, β_j^m $(j = 1, ..., M-1)$ are $(N+1)\times 1$ column matrices α_1^m, β_1^m are zero matrices and

$$\begin{cases} \alpha_{n+1}^m = -(B^m + C^m\alpha_n^m)^{-1}A^m, \\ \beta_{n+1}^m = (B^m + C^m\alpha_n^m)^{-1}(D\varphi_n^m + C^m\beta_n^m),\ n = 1, ..., M-1. \end{cases}$$

$$A^1 = C^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b_{N-21} & b_{N-22} & b_{N-23} & b_{N-24} & \cdots & b_{N-2N-2} & 0 & 0 & 0 \\ b_{N-11} & b_{N-12} & b_{N-13} & b_{N-14} & \cdots & b_{N-1N-2} & b_{N-1N-1} & 0 & 0 \\ b_{N1} & b_{N2} & b_{N3} & b_{N4} & \cdots & b_{NN-2} & b_{NN-1} & b_{NN} & 0 \\ b_{N+11} & b_{N+12} & b_{N+13} & b_{N+14} & \cdots & b_{N+1N-2} & b_{N+1N-1} & b_{N+1N} & b_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^1 = \begin{bmatrix} (2t_0 + t_0^2 + \alpha \frac{8}{3\sqrt{\pi}} t_0^{\frac{3}{2}}) \sin x_n \\ (2t_1 + t_1^2 + \alpha \frac{8}{3\sqrt{\pi}} t_1^{\frac{3}{2}}) \sin x_n \\ . \\ (2t_{N-1} + t_{N-1}^2 + \alpha \frac{8}{3\sqrt{\pi}} t_{N-1}^{\frac{3}{2}}) \sin x_n \\ (2t_N + t_N^2 + \alpha \frac{8}{3\sqrt{\pi}} t_N^{\frac{3}{2}}) \sin x_n \end{bmatrix}_{(N+1)\times 1},$$

$$A^2 = C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a' & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a' & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a' & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a' & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a' \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{21} & b'_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{31} & b'_{32} & b'_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{41} & b'_{42} & b'_{43} & b'_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b'_{N-21} & b'_{N-22} & b'_{N-23} & b'_{N-24} & \cdots & b'_{N-2N-2} & 0 & 0 & 0 \\ b'_{N-11} & b'_{N-12} & b'_{N-13} & b'_{N-14} & \cdots & b'_{N-1N-2} & b'_{N-1N-1} & 0 & 0 \\ b'_{N1} & b'_{N2} & b'_{N3} & b'_{N4} & \cdots & b'_{NN-2} & b'_{NN-1} & b'_{NN} & 0 \\ b'_{N+11} & b'_{N+12} & b'_{N+13} & b'_{N+14} & \cdots & b'_{N+1N-2} & b'_{N+1N-1} & b'_{N+1N} & b'_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^2 = \begin{bmatrix} (2t_0 + \beta\frac{8}{3\sqrt{\pi}}t_0^{\frac{3}{2}} - \beta_1 t_0^2 + t_0^2)\sin x_n + \beta_1\left(u^1\right)_n^0 \\ (2t_1 + \beta\frac{8}{3\sqrt{\pi}}t_1^{\frac{3}{2}} - \beta_1 t_1^2 + t_1^2)\sin x_n + \beta_1\left(u^1\right)_n^1 \\ . \\ (2t_{N-1} + \beta\frac{8}{3\sqrt{\pi}}t_{N-1}^{\frac{3}{2}} - \beta_1 t_{N-1}^2 + t_{N-1}^2)\sin x_n + \beta_1\left(u^1\right)_n^{N-1} \\ (2t_N + \beta\frac{8}{3\sqrt{\pi}}t_N^{\frac{3}{2}} - \beta_1 t_N^2 + t_N^2)\sin x_n + \beta_1\left(u^1\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A^3 = C^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a^* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a^* & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^* & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^* \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{21} & b^*_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{31} & b^*_{32} & b^*_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{41} & b^*_{42} & b^*_{43} & b^*_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b^*_{N-21} & b^*_{N-22} & b^*_{N-23} & b^*_{N-24} & \cdots & b^*_{N-2N-2} & 0 & 0 & 0 \\ b^*_{N-11} & b^*_{N-12} & b^*_{N-13} & b^*_{N-14} & \cdots & b^*_{N-1N-2} & b^*_{N-1N-1} & 0 & 0 \\ b^*_{N1} & b^*_{N2} & b^*_{N3} & b^*_{N4} & \cdots & b^*_{NN-2} & b^*_{NN-1} & b^*_{NN} & 0 \\ b^*_{N+11} & b^*_{N+12} & b^*_{N+13} & b^*_{N+14} & \cdots & b^*_{N+1N-2} & b^*_{N+1N-1} & b^*_{N+1N} & b^*_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi^3_n = \begin{bmatrix} (2t_0 + (1-\delta_1)t_0^2 + \delta\frac{8}{3\sqrt{\pi}}t_0^{\frac{3}{2}})\sin x_n + \delta_1\left(u^1\right)_n^0 \\ (2t_1 + (1-\delta_1)t_1^2 + \delta\frac{8}{3\sqrt{\pi}}t_1^{\frac{3}{2}})\sin x_n + \delta_1\left(u^1\right)_n^1 \\ . \\ (2t_{N-1} + (1-\delta_1)t_{N-1}^2 + \delta\frac{8}{3\sqrt{\pi}}t_{N-1}^{\frac{3}{2}})\sin x_n + \delta_1\left(u^1\right)_n^{N-1} \\ (2t_N + (1-\delta_1)t_N^2 + \delta\frac{8}{3\sqrt{\pi}}t_N^{\frac{3}{2}})\sin x_n + \delta_1\left(u^1\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A^4 = C^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \tilde{a} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{a} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \tilde{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{a} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{a} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{21} & \tilde{b}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{41} & \tilde{b}_{42} & \tilde{b}_{43} & \tilde{b}_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \tilde{b}_{N-21} & \tilde{b}_{N-22} & \tilde{b}_{N-23} & \tilde{b}_{N-24} & \cdots & \tilde{b}_{N-2N-2} & 0 & 0 & 0 \\ \tilde{b}_{N-11} & \tilde{b}_{N-12} & \tilde{b}_{N-13} & \tilde{b}_{N-14} & \cdots & \tilde{b}_{N-1N-2} & \tilde{b}_{N-1N-1} & 0 & 0 \\ \tilde{b}_{N1} & \tilde{b}_{N2} & \tilde{b}_{N3} & \tilde{b}_{N4} & \cdots & \tilde{b}_{NN-2} & \tilde{b}_{NN-1} & \tilde{b}_{NN} & 0 \\ \tilde{b}_{N+11} & \tilde{b}_{N+12} & \tilde{b}_{N+13} & \tilde{b}_{N+14} & \cdots & \tilde{b}_{N+1N-2} & \tilde{b}_{N+1N-1} & \tilde{b}_{N+1N} & \tilde{b}_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^4 = \begin{bmatrix} (2t_0 + d\frac{8}{3\sqrt{\pi}}t_0^{\frac{3}{2}} + (1-d_1-d_2)\,t_0^2)\sin x_n + d_1\left(u^3\right)_n^0 + d_2\left(u^2\right)_n^0 \\ (2t_1 + d\frac{8}{3\sqrt{\pi}}t_1^{\frac{3}{2}} + (1-d_1-d_2)\,t_1^2)\sin x_n + d_1\left(u^3\right)_n^1 + d_2\left(u^2\right)_n^1 \\ . \\ (2t_{N-1} + d\frac{8}{3\sqrt{\pi}}t_{N-1}^{\frac{3}{2}} + (1-d_1-d_2)\,t_{N-1}^2)\sin x_n + d_1\left(u^3\right)_n^{N-1} + d_2\left(u^2\right)_n^{N-1} \\ (2t_N + d\frac{8}{3\sqrt{\pi}}t_N^{\frac{3}{2}} + (1-d_1-d_2)\,t_N^2)\sin x_n + d_1\left(u^3\right)_n^N + d_2\left(u^2\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$u_s^m = \begin{bmatrix} (u^m)_s^0 \\ (u^m)_s^1 \\ . \\ (u^m)_s^{N-1} \\ (u^m)_s^{N-1} \end{bmatrix}_{(N+1)\times 1}, \; m = 1,2,3,4$$

for the first order of accuracy difference scheme (3.2).

Here

$$A^1 = C^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b_{N-21} & b_{N-22} & b_{N-23} & b_{N-24} & \cdots & b_{N-2N-2} & 0 & 0 & 0 \\ b_{N-11} & b_{N-12} & b_{N-13} & b_{N-14} & \cdots & b_{N-1N-2} & b_{N-1N-1} & 0 & 0 \\ b_{N1} & b_{N2} & b_{N3} & b_{N4} & \cdots & b_{NN-2} & b_{NN-1} & b_{NN} & 0 \\ b_{N+11} & b_{N+12} & b_{N+13} & b_{N+14} & \cdots & b_{N+1N-2} & b_{N+1N-1} & b_{N+1N} & b_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^1 = \begin{bmatrix} (2(t_0 - \frac{\tau}{2}) + (t_0 - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_0 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n \\ (2(t_1 - \frac{\tau}{2}) + (t_1 - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_1 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n \\ . \\ (2(t_{N-1} - \frac{\tau}{2}) + (t_{N-1} - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_{N-1} - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n \\ (2(t_N - \frac{\tau}{2}) + (t_N - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_N - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n \end{bmatrix}_{(N+1)\times 1},$$

$$A^2 = C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a' & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a' & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a' & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a' & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a' \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{21} & b'_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{31} & b'_{32} & b'_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b'_{41} & b'_{42} & b'_{43} & b'_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b'_{N-21} & b'_{N-22} & b'_{N-23} & b'_{N-24} & \cdots & b'_{N-2N-2} & 0 & 0 & 0 \\ b'_{N-11} & b'_{N-12} & b'_{N-13} & b'_{N-14} & \cdots & b'_{N-1N-2} & b'_{N-1N-1} & 0 & 0 \\ b'_{N1} & b'_{N2} & b'_{N3} & b'_{N4} & \cdots & b'_{NN-2} & b'_{NN-1} & b'_{NN} & 0 \\ b'_{N+11} & b'_{N+12} & b'_{N+13} & b'_{N+14} & \cdots & b'_{N+1N-2} & b'_{N+1N-1} & b'_{N+1N} & b'_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^2 = \begin{bmatrix} (2(t_0 - \frac{\tau}{2}) + (t_0 - \frac{\tau}{2})^2 + \alpha \frac{8}{3\sqrt{\pi}}(t_0 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \beta_1 \left(u^1\right)_n^0 \\ (2(t_1 - \frac{\tau}{2}) + (t_1 - \frac{\tau}{2})^2 + \alpha \frac{8}{3\sqrt{\pi}}(t_1 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \beta_1 \left(u^1\right)_n^1 \\ . \\ (2(t_{N-1} - \frac{\tau}{2}) + (t_{N-1} - \frac{\tau}{2})^2 + \alpha \frac{8}{3\sqrt{\pi}}(t_{N-1} - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \beta_1 \left(u^1\right)_n^{N-1} \\ (2(t_N - \frac{\tau}{2}) + (t_N - \frac{\tau}{2})^2 + \alpha \frac{8}{3\sqrt{\pi}}(t_N - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \beta_1 \left(u^1\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A^3 = C^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a^* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a^* & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^* & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^* \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{21} & b^*_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{31} & b^*_{32} & b^*_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ b^*_{41} & b^*_{42} & b^*_{43} & b^*_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b^*_{N-21} & b^*_{N-22} & b^*_{N-23} & b^*_{N-24} & \cdots & b^*_{N-2N-2} & 0 & 0 & 0 \\ b^*_{N-11} & b^*_{N-12} & b^*_{N-13} & b^*_{N-14} & \cdots & b^*_{N-1N-2} & b^*_{N-1N-1} & 0 & 0 \\ b^*_{N1} & b^*_{N2} & b^*_{N3} & b^*_{N4} & \cdots & b^*_{NN-2} & b^*_{NN-1} & b^*_{NN} & 0 \\ b^*_{N+11} & b^*_{N+12} & b^*_{N+13} & b^*_{N+14} & \cdots & b^*_{N+1N-2} & b^*_{N+1N-1} & b^*_{N+1N} & b^*_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^3 = \begin{bmatrix} (2(t_0 - \frac{\tau}{2}) + (t_0 - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_0 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \delta_1 \left(u^1\right)_n^0 \\ (2(t_1 - \frac{\tau}{2}) + (t_1 - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_1 - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \delta_1 \left(u^1\right)_n^1 \\ . \\ (2(t_{N-1} - \frac{\tau}{2}) + (t_{N-1} - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_{N-1} - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \delta_1 \left(u^1\right)_n^{N-1} \\ (2(t_N - \frac{\tau}{2}) + (t_N - \frac{\tau}{2})^2 + \alpha\frac{8}{3\sqrt{\pi}}(t_N - \frac{\tau}{2})^{\frac{3}{2}}) \sin x_n + \delta_1 \left(u^1\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A^4 = C^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \tilde{a} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{a} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \tilde{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{a} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{a} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{21} & \tilde{b}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \tilde{b}_{41} & \tilde{b}_{42} & \tilde{b}_{43} & \tilde{b}_{44} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \tilde{b}_{N-21} & \tilde{b}_{N-22} & \tilde{b}_{N-23} & \tilde{b}_{N-24} & \cdots & \tilde{b}_{N-2N-2} & 0 & 0 & 0 \\ \tilde{b}_{N-11} & \tilde{b}_{N-12} & \tilde{b}_{N-13} & \tilde{b}_{N-14} & \cdots & \tilde{b}_{N-1N-2} & \tilde{b}_{N-1N-1} & 0 & 0 \\ \tilde{b}_{N1} & \tilde{b}_{N2} & \tilde{b}_{N3} & \tilde{b}_{N4} & \cdots & \tilde{b}_{NN-2} & \tilde{b}_{NN-1} & \tilde{b}_{NN} & 0 \\ \tilde{b}_{N+11} & \tilde{b}_{N+12} & \tilde{b}_{N+13} & \tilde{b}_{N+14} & \cdots & \tilde{b}_{N+1N-2} & \tilde{b}_{N+1N-1} & \tilde{b}_{N+1N} & \tilde{b}_{N+1N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\phi_n^4 = \begin{bmatrix} (2(t_0-\frac{\tau}{2})+(t_0-\frac{\tau}{2})^2+\alpha\frac{8}{3\sqrt{\pi}}(t_0-\frac{\tau}{2})^{\frac{3}{2}})\sin x_n + d_1\left(u^3\right)_n^0 + d_2\left(u^2\right)_n^0 \\ (2(t_1-\frac{\tau}{2})+(t_1-\frac{\tau}{2})^2+\alpha\frac{8}{3\sqrt{\pi}}(t_1-\frac{\tau}{2})^{\frac{3}{2}})\sin x_n + d_1\left(u^3\right)_n^1 + d_2\left(u^2\right)_n^1 \\ . \\ (2(t_{N-1}-\frac{\tau}{2})+(t_{N-1}-\frac{\tau}{2})^2+\alpha\frac{8}{3\sqrt{\pi}}(t_{N-1}-\frac{\tau}{2})^{\frac{3}{2}})\sin x_n + d_1\left(u^3\right)_n^{N-1} + d_2\left(u^2\right)_n^{N-1} \\ (2(t_N-\frac{\tau}{2})+(t_N-\frac{\tau}{2})^2+\alpha\frac{8}{3\sqrt{\pi}}(t_N-\frac{\tau}{2})^{\frac{3}{2}})\sin x_n + d_1\left(u^3\right)_n^N + d_2\left(u^2\right)_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$u_s^m = \begin{bmatrix} (u^m)_s^0 \\ (u^m)_s^1 \\ . \\ (u^m)_s^{N-1} \\ (u^m)_s^{N-1} \end{bmatrix}_{(N+1)\times 1}, \quad m = 1,2,3,4$$

for the second order of accuracy difference scheme (3.3).

NUMERICAL ANALYSIS

The errors are computed by

$$(mE)_M^N = \max_{1\le k\le N, 1\le n\le M-1} \left| u^m(t_k, x_n) - (u^m)_n^k \right|, m = 1,2,3,4 \tag{3.6}$$

of the numerical solutions, where $u^m(t_k, x_n)$, $m = 1,2,3,4$ represents the exact solution and $(u^m)_n^k$, $m = 1,2,3,4$ represents the numerical solution at (t_k, x_n). Numerical results are given in Tables 1 and 2. Note that if N and M are doubled, the values of errors decrease by a factor of approximately $1/2^m$ for the m-th order of accuracy difference schemes in t (3.2) and (3.3), respectively. Moreover, the second order of accuracy difference scheme increases faster than the first order of accuracy difference scheme.

Table 3.1: Error analysis for the first order of accuracy difference scheme (3.2).

$\mathbf{E} \cdot_{\mathbf{M}}^{\mathbf{N}}$ /N, M	N = 20	N = 40	N = 80
$E\left(u^1\right)_M^N$	0.1617	0.0805	0.0401
$E\left(u^2\right)_M^N$	0.1673	0.0833	0.0416
$E\left(u^3\right)_M^N$	0.1673	0.0833	0.0416
$E\left(u^4\right)_M^N$	0.1766	0.0884	0.0442

Table 3.2: Error analysis for second order of accuracy difference scheme (3.3).

$\mathbf{E} \cdot_{\mathbf{M}}^{\mathbf{N}}$ /N, M	N = 20	N = 40	N = 80
$E\left(u^1\right)_M^N$	$0.242 * 10^{-3}$	$6.05 * 10^{-5}$	$1.5125 * 10^{-5}$
$E\left(u^2\right)_M^N$	$0.345 * 10^{-3}$	$8.625 * 10^{-5}$	$2.1562 * 10^{-5}$
$E\left(u^3\right)_M^N$	$0.401 * 10^{-3}$	$1.0025 * 10^{-4}$	$2.50625 * 10^{-5}$
$E\left(u^4\right)_M^N$	$0.5213 * 10^{-3}$	$1.30325 * 10^{-4}$	$3.2581 * 10^{-5}$

CHAPTER 4
CONCLUSION

In this thesis, we investigated the system of fractional differential equations can be solved by Fourier series, Laplace transform and Fourier transform methods are used for the solution of several systems of fractional differential equations. Difference schemes are presented for the numerical solution of the initial boundary value problem for the system of one dimensional fractional differential equations. The Matlab implementation of the numerical solution is presented.

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APPENDICES

APPENDIX 1

MATLAB PROGRAMMING

In this part, Matlab programs are presented for the first and second orders of accuracy difference schemes.

1. Matlab Implementation of the First Order of Accuracy Difference Scheme of Problem (3.1)

```
scheme for M=N
clear all; clc; close all;delete '*.asv';
N=20; M= N;
a=1; bb=2; bb1=1; cc=2; cc1=1; dd=4; dd1=1; dd2=2;
h=pi/M;tau=1/N;
%%%%%%%%%%%%%% solution for u¹(t,x)%%%%%%%%%%%%%%%
a1=(-1/(h^2));
for i=2:N+1;
A1(i,i)=a1;
end;A1;B1(1,1)=1;
B1(2,1)=(-1/tau)+(a/(pi*tau)^(1/2))*(gamma(3/2)-gamma(1/2));
B1(2,2)=(1/tau)+(a/((pi*tau)^(1/2)))*gamma(1/2)+(2/(h^2));
B1(3,1)=(a/(pi*tau)^(1/2))*((gamma(5/2)/factorial(2))-gamma(3/2));
for i=4:N+1;
for j=2:i-2;
B1(i,j)=(a/(tau*pi)^(1/2))*((gamma(i-j+(1/2))/factorial(i-j))-(gamma(i-j-(1/2))/factorial(i-j-
1)));
end;
B1(i,1)=(a/(tau*pi)^(1/2))*((gamma(i-1+(1/2))/factorial(i-1))-(gamma(i-2+(1/2))/factorial(i-
2)));
end;
for i=3:N+1;
B1(i,i)=(1/tau)+(a/(tau*pi)^(1/2))*(gamma(0.5))+(2/(h^2));
B1(i,i-1)=(-1/tau)+(a/(tau*pi)^(1/2))*(gamma(3/2)-gamma(1/2));
```

```
end;B1;
C1=A1;C1;
for i=2:N+1;
for j=1:M+1;
t1=(i-1)*tau; x1=(j-1)*h;
phy1(i,j:j)=(2*( t1)+((t1)^2)+(a*8*(t1^(3/2)))/(3*(pi)^(1/2)))*sin(x1) ;
end;
end;
for j=1:M+1;
%x1=(j-1)*h;
phy1(1,j:j)=0;
end;phy1;
for i=1:N+1;
D1(i,i)=1;
end; D1;D1;
I=eye(N+1,N+1);
alpha1{1}=zeros(N+1,N+1);
betha1{1}=zeros(N+1,1);
for j=1:M;
alpha1{j+1}=inv(B1+C1*alpha1{j})*(-A1);
betha1{j+1}=inv(B1+C1*alpha1{j})*(D1*phy1(:,j:j)-C1*betha1{j});
end;
U1{M+1}=zeros(N+1,1);
for Z=M:-1:1;
U1{Z}=alpha1{Z+1}*U1{Z+1}+betha1{Z+1};end;
for Z=1:M;
p1(:,Z+1)=U1{Z};
end;
p1(:,1)=zeros(N+1,1);
for i=1:N+1;
for j=1:M+1;
```

```
t1=(i-1)*tau;
x1=(j-1)*h;
es1(i,j:j)=t1^2*sin(x1);
end;
end;
abs(es1-p1);
maxes1=max(max(es1));
maxapp1=max(max(p1));
maxerror1=max(max(abs(es1-p1)))
%%%%%%%%%%%%%% solution for u²(t,x)%%%%%%%%%%%%%%%
a2=(-1/(h^2));
for i=2:N+1;
A2(i,i)=a2;
end;A2;
B2(1,1)=1;
B2(2,1)=(-1/tau)+(bb/(pi*tau)^(1/2))*(gamma(3/2)-gamma(1/2));
B2(2,2)=(1/tau)+(bb/((pi*tau)^(1/2)))*gamma(1/2)+(2/(h^2));
B2(3,1)=(bb/(pi*tau)^(1/2))*((gamma(5/2)/factorial(2))-gamma(3/2));
for i=4:N+1;
for j=2:i-2;
B2(i,j)=(bb/(tau*pi)^(1/2))*((gamma(i-j+(1/2))/factorial(i-j))-(gamma(i-j-(1/2))/factorial(i-j-1)));
end;
B2(i,1)=(bb/(tau*pi)^(1/2))*((gamma(i-1+(1/2))/factorial(i-1))-(gamma(i-2+(1/2))/factorial(i-2)));
end;
for i=3:N+1;
B2(i,i)=(1/tau)+(bb/(tau*pi)^(1/2))*(gamma(0.5))+(2/(h^2));
B2(i,i-1)=(-1/tau)+(bb/(tau*pi)^(1/2))*(gamma(3/2)-gamma(1/2));
end;B2;
C2=A2;C2;
```

```
for i=2:N+1;
for j=1:M+1;
t2=(i-1)*tau; x2=(j-1)*h;
phy2(i,j:j)=(2*( t2)+((1-bb1)*(t2)^2)+(bb*8*(t2^(3/2)))/(3*(pi)^(1/2)))*sin(x2)+bb1*p1(i-1,j)
;
end;
end;
for j=1:M+1;
%x2=(j-1)*h;
phy2(1,j:j)=0;
end;phy2;
for i=1:N+1;
D2(i,i)=1;
end; D2;D2;
I=eye(N+1,N+1);
alpha2{1}=zeros(N+1,N+1);
betha2{1}=zeros(N+1,1);
for j=1:M;
alpha2{j+1}=inv(B2+C2*alpha2{j})*(-A2);
betha2{j+1}=inv(B2+C2*alpha2{j})*(D2*phy2(:,j:j)-C2*betha2{j});
end;
U2{M+1}=zeros(N+1,1);
for Z=M:-1:1;
U2{Z}=alpha2{Z+1}*U2{Z+1}+betha2{Z+1};
end;
for Z=1:M;
p2(:,Z+1)=U2{Z};
end;
p2(:,1)=zeros(N+1,1);
for i=1:N+1;
for j=1:M+1;
```

```
t2=(i-1)*tau;
x2=(j-1)*h;
es2(i,j:j)=t2^2*sin(x2);
end;
end;
abs(es2-p2);
maxes2=max(max(es2));
maxapp2=max(max(p2));
maxerror2=max(max(abs(es2-p2)))
%%%%%%%%%%%%%% solution for u³(t,x)%%%%%%%%%%%%%%%
a3=(-1/(h^2));
for i=2:N+1;
A3(i,i)=a3;
end;A3;
B3(1,1)=1;
B3(2,1)=(-1/tau)+(cc/(pi*tau)^(1/2))*(gamma(3/2)-gamma(1/2));
B3(2,2)=(1/tau)+(cc/((pi*tau)^(1/2)))*gamma(1/2)+(2/(h^2));
B3(3,1)=(cc/(pi*tau)^(1/2))*((gamma(5/2)/factorial(2))-gamma(3/2));
for i=4:N+1;
for j=2:i-2;
B3(i,j)=(cc/(tau*pi)^(1/2))*((gamma(i-j+(1/2))/factorial(i-j))-(gamma(i-j-(1/2))/factorial(i-j-1)));
end;
B3(i,1)=(cc/(tau*pi)^(1/2))*((gamma(i-1+(1/2))/factorial(i-1))-(gamma(i-2+(1/2))/factorial(i-2)));
end;
for i=3:N+1;
B3(i,i)=(1/tau)+(cc/(tau*pi)^(1/2))*(gamma(0.5))+(2/(h^2));
B3(i,i-1)=(-1/tau)+(cc/(tau*pi)^(1/2))*(gamma(3/2)-gamma(1/2));
end;B3;
C3=A3;C3;
```

```
for i=2:N+1;
for j=1:M+1;
t3=(i-1)*tau; x3=(j-1)*h;
phy3(i,j:j)=((2*t3)+((1-cc1)*(t3)^2)+(cc*8*(t3^(3/2)))/(3*(pi)^(1/2)))*sin(x3)+cc1*p1(i-1,j) ;
end;
end;
for j=1:M+1;
%x3=(j-1)*h;
phy3(1,j:j)=0;
end;phy3;
for i=1:N+1;
D3(i,i)=1;
end; D3;D3;
I=eye(N+1,N+1);
alpha3{1}=zeros(N+1,N+1);
betha3{1}=zeros(N+1,1);
for j=1:M;
alpha3{j+1}=inv(B3+C3*alpha3{j})*(-A3);
betha3{j+1}=inv(B3+C3*alpha3{j})*(D3*phy3(:,j:j)-C3*betha3{j});end;
U3{M+1}=zeros(N+1,1);
for Z=M:-1:1;
U3{Z}=alpha3{Z+1}*U3{Z+1}+betha3{Z+1};end;
for Z=1:M;
p3(:,Z+1)=U3{Z};
end;
p3(:,1)=zeros(N+1,1);
for i=1:N+1;
for j=1:M+1;
t3=(i-1)*tau;
x3=(j-1)*h;
es3(i,j:j)=t3^2*sin(x3);
```

```
end;
end;
abs(es3-p3);
maxes3=max(max(es3));
maxapp3=max(max(p3));
maxerror3=max(max(abs(es3-p3)))
%%%%%%%%%%%%%%% solution for u⁴(t,x)%%%%%%%%%%%%%%%%
a4=(-1/(h^2));
for i=2:N+1;
A4(i,i)=a4;
end;A4;
B4(1,1)=1;
B4(2,1)=(-1/tau)+(dd/(pi*tau)^(1/2))*(gamma(3/2)-gamma(1/2));
B4(2,2)=(1/tau)+(dd/((pi*tau)^(1/2)))*gamma(1/2)+(2/(h^2));
B4(3,1)=(dd/(pi*tau)^(1/2))*((gamma(5/2)/factorial(2))-gamma(3/2));
for i=4:N+1;
for j=2:i-2;
B4(i,j)=(dd/(tau*pi)^(1/2))*((gamma(i-j+(1/2))/factorial(i-j))-(gamma(i-j-(1/2))/factorial(i-j-
1)));
end;
B4(i,1)=(dd/(tau*pi)^(1/2))*((gamma(i-1+(1/2))/factorial(i-1))-(gamma(i-2+(1/2))/factorial(i-
2)));
end;
for i=3:N+1;
B4(i,i)=(1/tau)+(dd/(tau*pi)^(1/2))*(gamma(0.5))+(2/(h^2));
B4(i,i-1)=(-1/tau)+(dd/(tau*pi)^(1/2))*(gamma(3/2)-gamma(1/2));
end;B3;
C4=A4;C4;
for i=2:N+1;
for j=1:M+1;
t4=(i-1)*tau; x4=(j-1)*h;
```

```
phy4(i,j:j)=((2*t4)+((1-dd1-dd2)*(t4)^2)+(dd*8*(t4^(3/2)))/(3*(pi)^(1/2)))*sin(x4)+dd2*p2(i-1,j)+dd1*p3(i-1,j) ;
end;
end;
for j=1:M+1;
%x4=(j-1)*h;
phy4(1,j:j)=0;
end;phy4;
for i=1:N+1;
D4(i,i)=1;
end; D4;D4;
I=eye(N+1,N+1);
alpha4{1}=zeros(N+1,N+1);
betha4{1}=zeros(N+1,1);
for j=1:M;
alpha4{j+1}=inv(B4+C4*alpha4{j})*(-A3);
betha4{j+1}=inv(B4+C4*alpha4{j})*(D4*phy4(:,j:j)-C4*betha4{j});
end;
U4{M+1}=zeros(N+1,1);
for Z=M:-1:1;
U4{Z}=alpha4{Z+1}*U4{Z+1}+betha4{Z+1};
end;
for Z=1:M;
p4(:,Z+1)=U4{Z};
end;
p4(:,1)=zeros(N+1,1);
for i=1:N+1;
for j=1:M+1;
t4=(i-1)*tau;
x4=(j-1)*h;
es4(i,j:j)=t4^2*sin(x4);
```

```
end;
end;
abs(es4-p4);
maxes4=max(max(es4));
maxapp4=max(max(p4));
maxerror4=max(max(abs(es4-p4)))
```